1. M finite energy wavefoms can be represented by a weighted linear combination of orthonormal functions $N \leq M$

Gram-Schmidt procedure: A set of orthonormal waveforms $f_i(t)$ for k-signal $s_i(t)$, $i = 1,\ldots,k$ is formed as:

$$\begin{cases} f_i'(t) = \begin{cases} s_i(t), & i=1 \\ s_i(t) - \sum\limits_{k=1}^{i-1} \langle s_i(t), f_{i-k}(t) \rangle f_{i-k}(t), & i = 2,\ldots,k \end{cases} \\ f_i(t) = \dfrac{f_i'(t)}{\|f_i'(t)\|} = \dfrac{f_i'(t)}{\sqrt{\varepsilon_i}} \end{cases},$$

where

$\langle s_i(t), f_{i-k}(t) \rangle = \int\limits_{-\infty}^{\infty} s_i(t) f_{i-k}^*(t)dt$ is inner product and

$\|f_i'(t)\| = \sqrt{\int\limits_{-\infty}^{\infty} f_i'(t) f_i'^*(t)dt} = \sqrt{\varepsilon_i} =$ is f_i''s norm (length), which is square root of f_i''s energy.

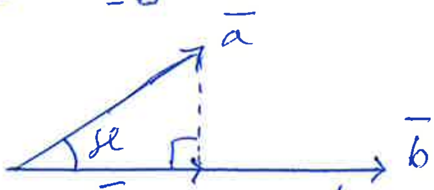
If $\|f_i'(t)\| = 0$, then the i-th signal waveform is linearly dependent, i.e. it is linear combination of the previous signal waveforms. ➔ Number of orthonormal functions/waveforms can be less than number of signals.

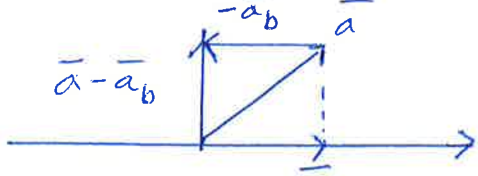
Let $f_1'(t) = 1$ $= s_1(t)$

$\|f_1'(t)\|^2 = \int\limits_{-\infty}^{\infty} |s_1(t)|^2 dt = \int\limits_0^1 1^2 dt = 1$ $\quad t\Big|_0^1 = 1 - 0 = 1$

$\Rightarrow f_1(t) = \dfrac{1}{\sqrt{1}} = 1, \quad 0 \leq t \leq 1$

normalization $\quad = \bar{b}$





$\bar{a}_b = \langle \bar{a}, \bar{b} \rangle \cdot \bar{b}$

By using Gran-Schmidt

$\bar{a}$ $\langle \bar{a}, \bar{b} \rangle \bar{b} = \bar{a}_b$ Have to be orthogonal comparing to $f_1(t)$

$$f_2'(t) = s_2(t) - \langle s_2(t), f_1(t) \rangle f_1(t)$$

$$\langle s_2(t), f_1(t) \rangle = \int_{-\infty}^{\infty} s_2(t) f_1^*(t)dt = \int_0^1 (\cos 2\pi t)dt = \frac{1}{2\pi} \Big/_0^1 \sin 2\pi t = \frac{1}{2\pi}(0-0) = 0$$

$$\Rightarrow f_2'(t) = s_2(t)$$

$$\|f_2'(t)\|^2 = \int_0^1 (\cos^2 2\pi t)dt \qquad (\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x)$$

$$= \int_0^1 \left(\frac{1}{2} + \frac{1}{2}\cos 4\pi t\right)dt = \Big/_0^1 \frac{1}{2}t + \frac{1}{8\pi}\Big/_0^1 \sin 4\pi t = \frac{1}{2}$$

$$\Rightarrow f_2(t) = \frac{s_2(t)}{\sqrt{\frac{1}{2}}} = \sqrt{2}\cos 2\pi t$$

Have to be orthogonal comparing to $f_1(t)$ AND $f_2(t)$

$$f_3'(t) = s_3(t) - \langle s_3(t), f_2(t) \rangle f_2(t) - \langle s_3(t), f_1(t) \rangle f_1(t)$$

$$\langle s_3(t), f_2(t) \rangle = \int_0^1 \left(\cos^2 \pi t \sqrt{2}\cos 2\pi t\right)dt$$

$$= \int_0^1 \left[\left(\frac{1}{2} + \frac{1}{2}\cos 2\pi t\right)\sqrt{2}\cos 2\pi t\right]tdt$$

$$= \int_0^1 (\frac{\sqrt{2}}{2}\cos 2\pi t + \frac{\sqrt{2}}{2}\cos^2 2\pi t)dt$$

$$= \int_0^1 (\frac{\sqrt{2}}{2}\cos 2\pi t + \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}\cos 4\pi t)dt$$

$$= \Big/_0^1 \left(\frac{\sqrt{2}}{4\pi}\sin 2\pi t + \frac{\sqrt{2}}{4}t + \frac{\sqrt{2}}{16\pi}\sin 4\pi t\right) = \frac{\sqrt{2}}{4}$$

$$\langle s_3(t), f_1(t) \rangle = \int_0^1 (\cos^2 \pi t)dt = \int_0^1 (\frac{1}{2} + \frac{1}{2}\cos 2\pi t)dt = \Big/_0^1 (\frac{1}{2}t + \frac{1}{4\pi}\sin 2\pi t) = \frac{1}{2}$$

$$f_3'(t)=\cos^2\pi t-\frac{\sqrt{2}}{4}\sqrt{2}\cos 2\pi t-\frac{1}{2}\cdot 1=\cos^2\pi t-\frac{1}{2}\cos 2\pi t-\frac{1}{2}$$

$$\|f_3'(t)\|=\int_0^1(\cos^4\pi t-\frac{1}{2}\cos^2\pi t\cos 2\pi t-\frac{1}{2}\cos^2\pi t$$

$$-\frac{1}{2}\cos 2\pi t\cos^2\pi t+\frac{1}{4}\cos^2 2\pi t+\frac{1}{4}\cos 2\pi t-\frac{1}{2}\cos^2\pi t+\frac{1}{4}\cos 2\pi t+\frac{1}{4})dt$$

$$=\int_0^1(\left(\frac{1}{2}+\frac{1}{2}\cos 2\pi t\right)^2-\left(\frac{1}{2}+\frac{1}{2}\cos 2\pi t\right)\cos 2\pi t-\left(\frac{1}{2}+\frac{1}{2}\cos 2\pi t\right)$$

$$+\frac{1}{2}\cos 2\pi t+\frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\cos 4\pi t\right)+\frac{1}{4})dt$$

$$=\int_0^1(\frac{1}{4}+\frac{1}{4}\cos 2\pi t+\frac{1}{4}\cos^2 2\pi t-\frac{1}{2}\cos 2\pi t-\frac{1}{2}\cos^2 2\pi t$$

$$-\frac{1}{2}-\frac{1}{2}\cos 2\pi t+\frac{1}{2}\cos 2\pi t+\frac{1}{8}+\frac{1}{8}\cos 4\pi t+\frac{1}{4})dt$$

$$=\int_0^1\left(\frac{1}{8}+\frac{1}{8}\cos 4\pi t-\frac{1}{4}-\frac{1}{4}\cos 4\pi t+\frac{1}{8}+\frac{1}{8}\cos 4\pi t-\frac{1}{4}\cos 2\pi t\right)dt$$

$$=\frac{1}{8}-\frac{1}{4}+\frac{1}{8}=0$$

So $s_3(t)$ is linear combination of the $f_1(t)$ and $f_2(t)$. To see this

$$s_3(t)=\cos^2\pi t=\frac{1}{2}+\frac{1}{2}\cos 2\pi t=\frac{1}{2}f_1(t)+\frac{1}{2\sqrt{2}}f_2(t) \quad \#$$

Signals $s_1(t),s_2(t),s_3(t)$ have **orthonormal set of functions**

$$f_1(t)=1,\ 0\le t\le 1$$
$$f_2(t)=\sqrt{2}\cos 2\pi t,\ 0\le t\le 1$$

$$\#\quad s_3(t)=\frac{1}{2}\cdot 1+\frac{1}{2\sqrt{2}}\cdot\sqrt{2}\cos 2\pi t$$

$$=\frac{1}{2}+\frac{1}{2}\cos 2\pi t=\cos^2\pi t$$

Vector representation

We can express the M signals as linear combinations of the $\{f_n(t)\}$.

The vector representations of the 3 signals are:

$$s_k(t) = \sum_{n=1}^{N} s_{kn} f_n(t) \quad k = 1,2,\ldots,M\,, \qquad [2],(4.2\text{–}39)$$

where (inner product)

$$s_{kn} = \langle s_k(t), f_n(t) \rangle \quad n = 1,2,\ldots,N$$

and where M is the number of the signals and N is the number of the orthonormal waveforms and $N \leq M$.

By using (4.2–39)

$$\mathbf{s}_k = \left[s_{k1} s_{k2} \cdots s_{kN} \right] \qquad [2],(4.2\text{–}41)$$

or

$$\{ s_{ki}, i = 1,2,\ldots,N \}$$

Each signal may be represented by the Vector (bold) or equivalently, as a point in the N-dimensional signal space with coordinates s_{ki}.

So ($N = 2$, $M = 3$)

$$s_{11} = \langle s_1(t), f_1(t) \rangle = \int_{-\infty}^{\infty} s_1(t) f_1^*(t)dt = \int_0^1 (1 \cdot 1)dt = 1$$

$$s_{12} = \langle s_1(t), f_2(t) \rangle = \int_{-\infty}^{\infty} s_1(t) f_2^*(t)dt = \int_0^1 (1 \cdot \sqrt{2}\cos 2\pi t)dt = \frac{\sqrt{2}}{2\pi} \Big/_0^1 (\sin 2\pi t) = \frac{\sqrt{2}}{2\pi}(0-0) = 0$$

$$\Rightarrow \mathbf{s}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

$$s_{21} = \langle s_2(t), f_1(t) \rangle = \int_{-\infty}^{\infty} s_2(t) f_1^*(t) dt = \int_0^1 (\cos 2\pi t \cdot 1) dt = \frac{1}{2\pi} \Big/_0^1 (\sin 2\pi t) = \frac{1}{2\pi}(0-0) = 0$$

$$s_{22} = \langle s_2(t), f_2(t) \rangle = \int_{-\infty}^{\infty} s_2(t) f_2^*(t) dt = \int_0^1 (\cos 2\pi t \cdot \sqrt{2}\cos 2\pi t) dt = \int_0^1 (\sqrt{2}\cos^2 2\pi t) dt$$

$$= \sqrt{2}\int_0^1 \left(\frac{1}{2} + \frac{1}{2}\cos 4\pi t dt \right) dt = \sqrt{2}\left(\Big/_0^1 \frac{1}{2} t + \frac{1}{8\pi} \Big/_0^1 \sin 4\pi t \right) = \sqrt{2}\left(\frac{1}{2} + 0 \right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \mathbf{s}_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

$$s_{31} = \langle s_3(t), f_1(t) \rangle = \int_{-\infty}^{\infty} s_3(t) f_1^*(t) dt = \int_0^1 (\cos^2 \pi t \cdot 1) dt$$

$$= \int_0^1 \left(\frac{1}{2} + \frac{1}{2}\cos 2\pi t \right) dt = \Big/_0^1 \left(\frac{1}{2} t + \frac{1}{4\pi}\sin 2\pi t \right) = \frac{1}{2}$$

$$s_{32} = \langle s_3(t), f_2(t) \rangle = \int_{-\infty}^{\infty} s_3(t) f_2^*(t) dt = \int_0^1 (\cos^2 \pi t \sqrt{2}\cos 2\pi t) dt$$

$$= \int_0^1 \left[\left(\frac{1}{2} + \frac{1}{2}\cos 2\pi t \right)\left(\sqrt{2}\cos 2\pi t \right) \right] dt = \int_0^1 \left(\frac{\sqrt{2}}{2}\cos 2\pi t + \frac{\sqrt{2}}{2}\cos^2 2\pi t \right) dt$$

$$= \int_0^1 \left(\frac{\sqrt{2}}{2}\cos 2\pi t + \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}\cos 4\pi t \right) dt = \Big/_0^1 \left(\frac{\sqrt{2}}{4\pi}\sin 2\pi t + \frac{\sqrt{2}}{4} t + \frac{\sqrt{2}}{16\pi}\sin 4\pi t \right)$$

$$= \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \mathbf{s}_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} \end{bmatrix}.$$

$$\left[\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x \right] \, !$$

$$S_3(t) = \sum_{n=1}^{2} S_{3n} \cdot f_n(t) = S_{31} \cdot f_1(t) + S_{32} \cdot f_2(t)$$

$$= \frac{1}{2} \cdot 1 + \frac{1}{2\sqrt{2}} \cdot \sqrt{2} \cos 2\pi t$$

$$= \frac{1}{2} + \frac{1}{2}\cos 2\pi t = \cos^2 \pi t \quad \text{OK!}$$

$f_1(t) = 1$

$f_2(t) = \sqrt{2}\cos 2\pi t$

Energies

Energy is the square of the length of the vector or the square of the Euclidean distance from the origin to the point in the N-dimensional space.

$$\varepsilon_k = \int_{-\infty}^{\infty} \left[s_k(t)\right]^2 dt = \sum_{n=1}^{N} s_{kn}^2 = \left\|\mathbf{s}_k\right\|^2 = \left[s_{k1} s_{k2} \cdots s_{kn}\right] \begin{bmatrix} s_{k1} \\ s_{k2} \\ \vdots \\ s_{kn} \end{bmatrix} \qquad [2],(4.2\text{–}40)$$

-0

$$\varepsilon_1 = \int_{-\infty}^{\infty} \left[s_k(t)\right]^2 dt = \int_0^1 1^2 dt = 1$$

$$\varepsilon_2 = \sum_{n=1}^{N} s_{kn}^2 = \sum_{n=1}^{2} s_{2n}^2 = s_{21}^2 + s_{22}^2 = 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

$$\varepsilon_3 = \left\|\mathbf{s}_k\right\|^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}.$$

Energy calculations for every signal can be done by using every method.

Euclidean distances

Euclidean distance between two signals (a pair of signals) is defined

$$d_{km} = \left\|\mathbf{s}_m - \mathbf{s}_k\right\| = \left\{ \int_{-\infty}^{\infty} \left[s_m(t) - s_k(t)\right]^2 dt \right\}^{1/2} \qquad [2],(4.2\text{–}48)$$

The length of the difference vector. Euclidean distance is an alternative measure of the similarity of the signal waveforms or vectors.

$$d_{21} = \|\mathbf{s}_1 - \mathbf{s}_2\| = \sqrt{\begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}} = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$$

$$d_{31} = \|\mathbf{s}_1 - \mathbf{s}_3\| = \sqrt{\begin{bmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2\sqrt{2}} \end{bmatrix}} = \sqrt{\frac{1}{4} + \frac{1}{8}} = \sqrt{\frac{3}{8}}$$

$$d_{32} = \|\mathbf{s}_2 - \mathbf{s}_3\| = \sqrt{\begin{bmatrix} -\frac{1}{2} & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} \end{bmatrix}} = \sqrt{\frac{1}{4} + \frac{1}{8}} = \sqrt{\frac{3}{8}}$$

$\bar{S}_1 = [1 \quad 0]$

$\bar{S}_2 = [0 \quad \frac{1}{\sqrt{2}}]$

$\bar{S}_3 = [\frac{1}{2} \quad \frac{1}{2\sqrt{2}}]$

②

Calculate the energy of the FSK-signal. Does it depend on data frequency $m \cdot \Delta f$? Draw the spectrum of the individual FSK-signals if $M = 4$. Start from the following form of the signal

$$s_m^{FSK} = A\cos 2\pi(f_c + m \cdot \Delta f)t,$$

where A is amplitude and $m = \pm 1, \pm 2, ..., \pm \frac{M}{2}$.

$$\varepsilon_m = \int_0^T [A\cos(2\pi f_c t + 2\pi m\Delta f t)]^2 dt, \quad 0 \le t \le T$$

T is symboltime

$$= \int_0^T A^2 \cos^2(2\pi f_c t + 2\pi m\Delta f t)dt$$

$$= \int_0^T \frac{1}{2}A^2 + \frac{A^2}{2}\cos(4\pi(f_c + m\Delta f)t)dt$$

$$= \Big/_0^T \left[\frac{1}{2}A^2 t + \frac{A^2}{2}\frac{1}{4\pi(f_c + m\Delta f)}\sin(4\pi(f_c + m\Delta f)t)\right]$$

$$= \frac{1}{2}A^2 T + \frac{A^2}{2}\frac{1}{4\pi(f_c + m\Delta f)}\sin(4\pi(f_c + m\Delta f)T)$$

look at also Figure 4-1-2 [2]

max $\sin(x) = 1$ ja $f_c >> 1$, ➔ $\dfrac{1}{4\pi(f_c + m\Delta f)} \approx 0$.

$\Rightarrow \varepsilon_m = \frac{1}{2}A^2 T = \varepsilon$, not depend on data frequency $m\Delta f$.

Right (11)

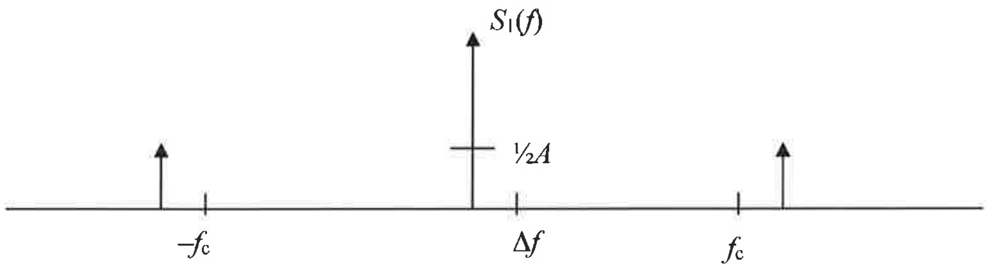
$M = 4 \rightarrow$ FSK signals are: $\left[\cos 2\pi f_c t \leftrightarrow \frac{1}{2}\delta(f+f_c)+\frac{1}{2}\delta(f-f_c)\right]$

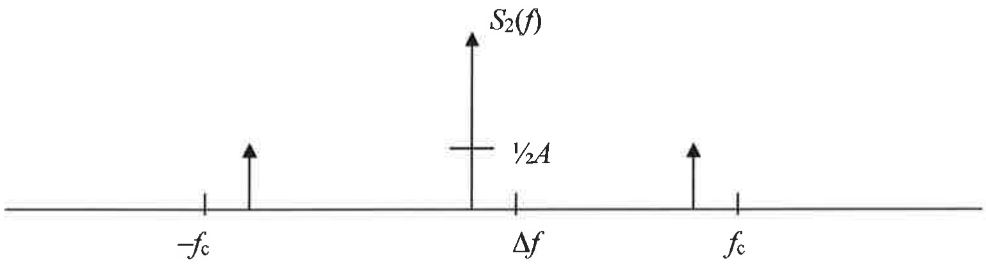
$$m=1,\ s_1(t) = A\cos 2\pi(f_c+\Delta f)t \leftrightarrow S_1(f) = \frac{1}{2}A\delta(f+f_c+\Delta f)+\frac{1}{2}A\delta(f-f_c-\Delta f)$$

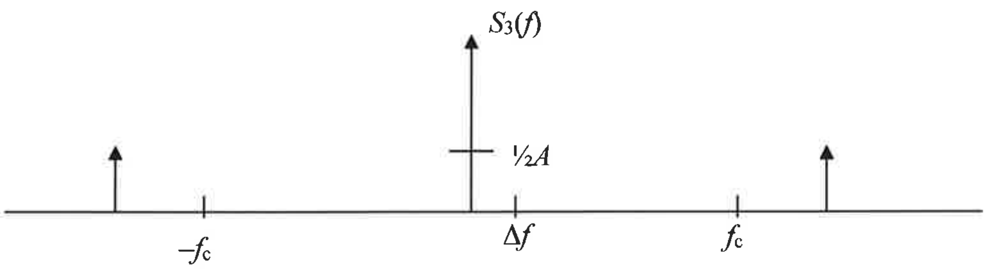
$$m=-1,\ s_2(t) = A\cos 2\pi(f_c-\Delta f)t \leftrightarrow S_2(f) = \frac{1}{2}A\delta(f+f_c-\Delta f)+\frac{1}{2}A\delta(f-f_c+\Delta f)$$

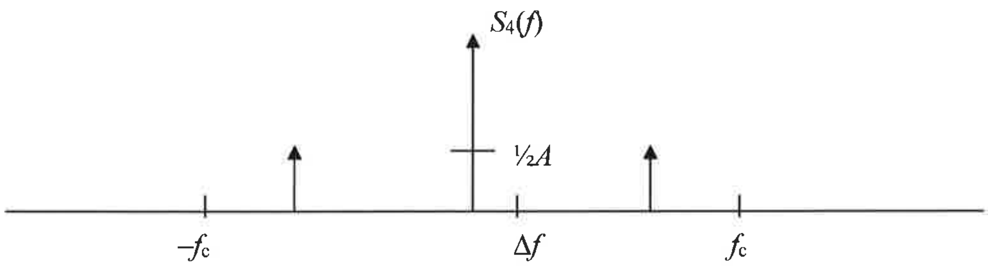
$$m=2,\ s_3(t) = A\cos 2\pi(f_c+2\Delta f)t \leftrightarrow S_3(f) = \frac{1}{2}A\delta(f+f_c+2\Delta f)+\frac{1}{2}A\delta(f-f_c-2\Delta f)$$

$$m=-2,\ s_4(t) = A\cos 2\pi(f_c-2\Delta f)t \leftrightarrow S_4(f) = \frac{1}{2}A\delta(f+f_c-2\Delta f)+\frac{1}{2}A\delta(f-f_c+2\Delta f)$$









3 [*Every equation and pictures in this exercise are from book [2] = Proakis*]

a) $S_{\text{QPSK}} = g(t)\cos(\omega_c t + \theta_i) \qquad 0 \le t \le T \quad T$ is symbol time (5.2–47)

The bandwidth of the bandpass channel is

$W' = 3000 - 600 = 2400\text{ Hz} = 2W$.

Since each symbol of the QPSK constellation conveys 2 bits of information, the symbol rate of transmission is

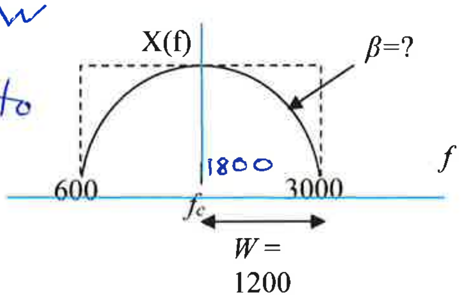
Transmission symbol rate $R = \frac{1}{T} = \frac{2400\,(\text{bit/s})}{2(\text{bit/symbol})} = 1200$ symbol/s

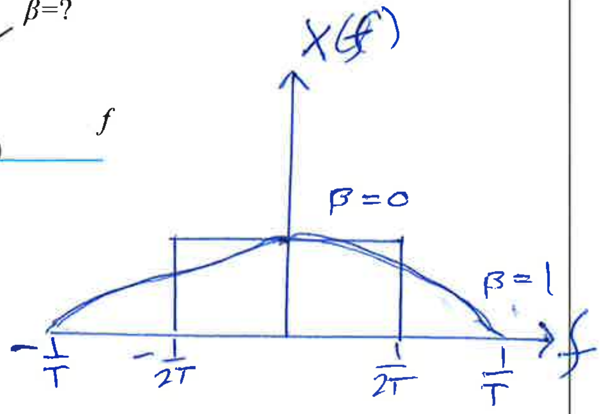
(fits two times in the bandwidth in use)

3) $T > \frac{1}{2W}$ i.e. $\frac{1}{T} < 2W$

Numerous choises to obtain $B(f) = T$

If $\beta = 1 \Rightarrow \frac{1}{T} = W$





$\frac{1+\beta}{2T} = 1200 \;(= W)$

$\Rightarrow 1 + \beta = 2T \cdot 1200 = 2/1200 \cdot 1200 \Rightarrow \beta = 2 - 1 = 1 \qquad$ (9.2–26)

$\beta = 1$

Frequency characteristic if β = 1 will be found (9.2–26) $\Rightarrow \frac{1}{T} = 1200 = W$ or 1

$$X_{rc}(f) = \begin{cases} T & \left(0 \le |f| \le \frac{1-\beta}{2T}\right) \\ \frac{T}{2}\left\{1 + \cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right\} & \left(\frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T}\right) \\ 0 & \left(|f| > \frac{1+\beta}{2T}\right) \end{cases}$$

(3. $T > \frac{1}{2W}$

$\Rightarrow \frac{1}{T} < 2W$)

$$X_{rc}(f)=\begin{cases}T & f=0\\ \frac{T}{2}(1+\cos\pi T|f|) & -1200\le f\le 1200\\ 0 & \text{otherwise}\end{cases}$$

$[\cos^2 x = \frac{1}{2}(1+\cos 2x)]$

$$=\frac{1}{2400}(1+\cos\frac{\pi|f|}{1200})\quad |f|\le 1200$$

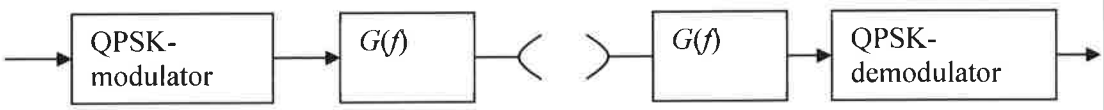
$T\cos^2\left(\frac{\pi T|f|}{2}\right) = T\cdot\left[\frac{1}{2}(1+\cos\pi T|f|)\right]$

If the desired spectral characteristic is split evenly between the transmitting filter $G_T(f)$ and the receiving filter $G_R(f)$, then $X_{rc}(f)$ is divided between transmitter and receiver

$= \frac{1}{1200}\cos^2\left(\frac{\pi|f|}{2400}\right)$

$$G(f)=\sqrt{X_{rc}(f)}\qquad (9.2\text{–}28 \text{ and } 29)$$

$$=\sqrt{\frac{1}{1200}}\cos\left(\frac{\pi|f|}{2400}\right).$$



$2W = 2400$

2) $\frac{1}{T} = 2W \Rightarrow$ one possibility to obtain $B(f)=T$

b)

$$R=\frac{1}{T}=\frac{4800\text{ bit/s}}{2\text{ bit/symb.}}=2400\text{ symbols/s } = 2W$$

Fits 1 time to the channel in use $\Rightarrow$

$\beta = 0 \Rightarrow \frac{1}{T} = 2W$

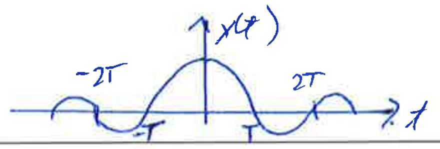
$$\frac{1+\beta}{2T}=1200\Rightarrow 1+\beta=\frac{2}{2400}\cdot 1200\Rightarrow\beta=0$$

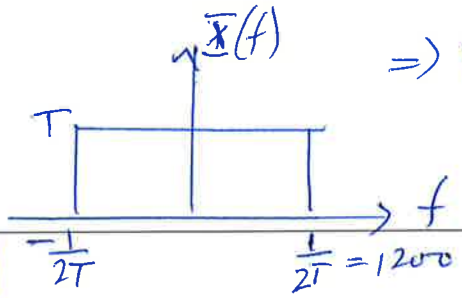
$$\Rightarrow\begin{cases}X_{rc}(f)=T & |f|\le\frac{1}{2T}=1200\text{ Hz}\\ 0 & \text{otherwise}\end{cases}\qquad (9.2-24)$$

$\frac{1}{T} = 2400 = 2\cdot 1200$

$$\Rightarrow G(f)=\sqrt{T}.$$

$\Rightarrow W = \frac{1}{2T} = 1200$





The pulse is $\mathrm{sinc}\left(\frac{\pi t}{T}\right)$

4.

Log-likelihood function for the equivalent low-pass signal is

Decision-directed parameter estimation

$$\Lambda_L(\phi) = \mathrm{Re}\left\{\left[\frac{1}{N_0}\int_{T_0} r_l(t)s_l^*(t)dt\right]e^{j\phi}\right\}, \qquad [2],(6.2\text{–}35)$$

where

Received equivalent low-pass signal

$$r_l(t) = s_l(t)e^{-j\phi} + z(t) \qquad [2],(6.2-33)$$

$$s_l(t) = \sum_n I_n g(t-nT) + j\sum_n J_n g\left(t-nT-\frac{T}{2}\right) \qquad [2],(6.2-43)$$

We assume that the information seq. $\{I_n\}$ is known.

↑ O-QPSK

$$I_n = \pm 1$$

$$J_n = \pm 1$$

$$T_0 = KT, \quad K \in \mathbb{Z}_+$$ observation interval

Let's denote (by definition)

$$y_n = \int_{nT}^{(n+1)T} r_l(t)g^*(t-nT)dt \qquad [2],(6.2-37)$$

$$x_n = \int_{(n+½)T}^{(n+3/2)T} r_l(t)g^*\left(t-nT-\frac{T}{2}\right),dt$$

y_n is the output of the matched filter in the n_{th} signal interval

then

$$\Lambda_L(\phi) = \mathrm{Re}\left\{\left[\frac{1}{N_0}\int_{T_0} r_l(t)\left[\sum_n I_n g(t-nT) + j\sum_n J_n g\left(t-nT-\frac{T}{2}\right)\right]^* dt\right]e^{j\phi}\right\}$$

$$= \mathrm{Re}\left\{\frac{1}{N_0}e^{j\phi}\left[\sum_n I_n \int_{T_0} r_l(t)g^*(t-nT)dt - j\sum_n J_n\int_{T_0} r_l(t)g^*\left(t-nT-\frac{T}{2}\right)dt\right]\right\}$$

$$= \mathrm{Re}\left\{\frac{1}{N_0}e^{j\phi}\left[\sum_{n=0}^{K-1} I_n y_n - j\sum_{n=0}^{K-1} J_n x_n\right]\right\}$$

$$= \mathrm{Re}\left\{\frac{1}{N_0}\left[\cos(\phi) + j\sin(\phi)\right]\left[\sum_{n=0}^{K-1} I_n y_n - j\sum_{n=0}^{K-1} J_n x_n\right]\right\}.$$

Let's denote

$$jA = j\sum_{n=0}^{K-1} I_n y_n + \sum_{n=0}^{K-1} J_n x_n$$

$$\mathrm{Re}\{jA\} = \sum_{n=0}^{K-1} J_n x_n = -\mathrm{Im}\{A\}$$

$$A = \sum_{n=0}^{K-1} I_n y_n - j\sum_{n=0}^{K-1} J_n x_n \, ,$$

then

$$\Lambda_L(\phi) = \mathrm{Re}\left\{\frac{1}{N_0} A\cos(\phi) + j\frac{1}{N_0} A\sin(\phi)\right\}$$
$$= \frac{1}{N_0}\mathrm{Re}\{A\}\cos(\phi) - \frac{1}{N_0}\mathrm{Im}\{A\}\sin(\phi).$$

Let's find ϕ, which is the maximum value of the log-likelihood function, differentiating the log-likelihood with respect to ϕ and setting the derivative equal to zero.

$$\frac{\partial\Lambda_L(\phi)}{\partial\phi} = -\frac{1}{N_0}\mathrm{Re}\{A\}\sin(\phi) - \frac{1}{N_0}\mathrm{Im}\{A\}\cos(\phi)$$

$$\frac{\partial\Lambda_L(\phi)}{\partial\phi} = 0 \Rightarrow \mathrm{Re}\{A\}\sin(\hat{\phi}_{ML}) = -\mathrm{Im}\{A\}\cos(\hat{\phi}_{ML}) \Leftrightarrow$$

$$\tan(\hat{\phi}_{ML}) = \frac{\sin(\hat{\phi}_{ML})}{\cos(\hat{\phi}_{ML})} = -\frac{\mathrm{Im}\{A\}}{\mathrm{Re}\{A\}} \Rightarrow$$

$$\hat{\phi}_{ML} = -\arctan\left\{\frac{\mathrm{Im}\left\{\sum_{n=0}^{K-1} I_n y_n - j\sum_{n=0}^{K-1} J_n x_n\right\}}{\mathrm{Re}\left\{\sum_{n=0}^{K-1} I_n y_n - j\sum_{n=0}^{K-1} J_n x_n\right\}}\right\}. \quad \square$$

Decision-directed (or decision-feedback) Carrier phase estimate.

340

6.2.1 Maximum-Likelihood Carrier Phase Estimation

First, we derive the maximum-likelihood carrier phase estimate. For simplicity, we assume that the delay τ is known and, in particular, we set $\tau = 0$. The function to be maximized is the likelihood function given in Equation 6.1–8. With ϕ substituted for ψ, this function becomes

$$\begin{aligned}\Lambda(\phi) &= \exp\left\{-\frac{1}{N_0}\int_{T_0}[r(t) - s(t;\phi)]^2\,dt\right\}\\ &= \exp\left\{-\frac{1}{N_0}\int_{T_0} r^2(t)\,dt + \frac{2}{N_0}\int_{T_0} r(t)s(t;\phi)\,dt - \frac{1}{N_0}\int_{T_0} s^2(t;\phi)\,dt\right\}\end{aligned} \tag{6.2–8}$$

Note that the first term of the exponential factor does not involve the signal parameter ϕ. The third term, which contains the integral of $s^2(t;\phi)$, is a constant equal to the signal energy over the observation interval T_0 for any value of ϕ. Only the second term, which involves the cross correlation of the received signal $r(t)$ with the signal $s(t;\phi)$, depends on the choice of ϕ. Therefore, the likelihood function $\Lambda(\phi)$ may be expressed as

$$\Lambda(\phi) = C\exp\left[\frac{2}{N_0}\int_{T_0} r(t)s(t;\phi)\,dt\right] \tag{6.2–9}$$

where C is a constant independent of ϕ.

The ML estimate $\hat{\phi}_{\mathrm{ML}}$ is the value of ϕ that maximizes $\Lambda(\phi)$ in Equation 6.2–9. Equivalently, the value $\hat{\phi}_{\mathrm{ML}}$ also maximizes the logarithm of $\Lambda(\phi)$, i.e., the log-likelihood function

$$\Lambda_L(\phi) = \frac{2}{N_0}\int_{T_0} r(t)s(t;\phi)\,dt \tag{6.2–10}$$

Note that in defining $\Lambda_L(\phi)$ we have ignored the constant term $\ln C$.

EXAMPLE 6.2-1. As an example of the optimization to determine the carrier phase, let us consider the transmission of the unmodulated carrier $A\cos 2\pi f_c t$. The received signal is

$$r(t) = A\cos(2\pi f_c t + \phi) + n(t)$$

where ϕ is the unknown phase. We seek the value ϕ, say $\hat{\phi}_{\mathrm{ML}}$, that maximizes

$$\Lambda_L(\phi) = \frac{2A}{N_0}\int_{T_0} r(t)\cos(2\pi f_c t + \phi)\,dt$$

A necessary condition for a maximum is that

$$\frac{d\Lambda_L(\phi)}{d\phi} = 0$$

Slides 72 →

5.

Received signal is [Book 2, p. 340]

$$\begin{aligned} r(t) &= A\cos(2\pi f_c t + \phi) + n(t), \quad 0 \le t \le T_0 \\ &= s(t;\phi) + n(t) \end{aligned}$$

Likelihood function is

$$\begin{aligned} \Lambda(\phi) &= e^{-\frac{1}{N_0}\int_{T_0}\left[r(t)-s(t;\phi)\right]^2 dt} \qquad [2],(6.2-8) \\ &= e^{-\frac{1}{N_0}\int_{T_0}\left[r^2(t)-2r(t)s(t;\phi)+s^2(t;\phi)\right]dt} \\ &= e^{-\frac{1}{N_0}\left[\int_{T_0} r^2(t)dt-\int_{T_0} 2r(t)s(t;\phi)dt+\int_{T_0} s^2(t;\phi)dt\right]} \\ &\qquad\quad 1. \qquad\qquad 2. \qquad\qquad 3. \\ &= e^{-\frac{1}{N_0}\left[\int_{T_0} r^2(t)dt\right]} e^{\frac{2}{N_0}\left[\int_{T_0} r(t)s(t;\phi)dt\right]} e^{-\frac{1}{N_0}\int_{T_0} s^2(t;\phi)dt} \\ &= Ce^{\frac{2}{N_0}\left[\int_{T_0} r(t)s(t;\phi)dt\right]}. \end{aligned}$$

Where C is a constant, independents of ϕ.

Log-likelihood function is

$$\Lambda_L(\phi) = \ln(C) + \frac{2}{N_0}\left[\int_{T_0} r(t)s(t;\phi)dt\right].$$

So

$$\frac{\Lambda_L(\phi)}{\partial\phi} = \frac{\partial}{\partial\phi}\left[\frac{2}{N_0}\int_{T_0} r(t)s(t;\phi)dt\right]$$

$$= \frac{2}{N_0}\int_{T_0}\frac{\partial}{\partial\phi}\left[r(t)s(t;\phi)\right]dt$$

$$= \frac{2}{N_0}\int_{T_0}\left[\overbrace{\frac{\partial r(t)}{\partial\phi}}^{=0}s(t;\phi) + r(t)\frac{\partial s(t;\phi)}{\partial\phi}\right]dt$$

$$= \frac{2}{N_0}\int_{T_0} r(t)\frac{\partial s(t;\phi)}{\partial\phi}dt$$

$$= \frac{2}{N_0}\int_{T_0} r(t)\left[-A\sin(2\pi f_c t + \phi)\right]dt$$

$$= -\frac{2A}{N_0}\int_{T_0} r(t)\sin(2\pi f_c t + \phi)dt$$

$$\frac{\Lambda_L(\phi)}{\partial\phi} = 0 \Leftrightarrow \int_{T_0} r(t)\sin(2\pi f_c t + \hat{\phi}_{ML})dt = 0 \qquad [2],(6.2\text{–}11)$$

We observe that the optimality condition given by previous equation implies the use of a loop to extract the estimate as illustrated in figure **(6.2–1)**. So, a PLL for obtaining the ML estimate of the phase of an unmodulated carrier.

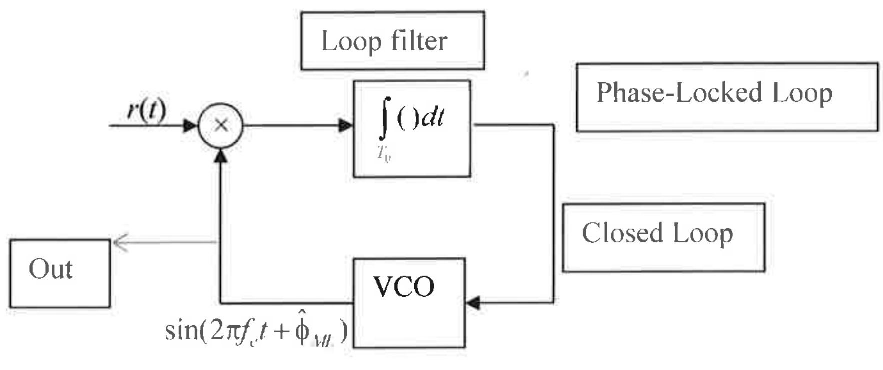


Figure 1.

On the other hand

$$\int_{T_0} r(t)\sin(2\pi f_c t+\hat{\phi}_{ML})dt = \int_{T_0} r(t)\left[\sin(2\pi f_c t)\cos(\hat{\phi}_{ML})+\cos(2\pi f_c t)\sin(\hat{\phi}_{ML})\right]dt$$

$$=\cos(\hat{\phi}_{ML})\int_{T_0} r(t)\sin(2\pi f_c t)dt+\sin(\hat{\phi}_{ML})\int_{T_0} r(t)\cos(2\pi f_c t)dt$$

$$=0\Rightarrow$$

$$\frac{\sin(\hat{\phi}_{ML})}{\cos(\hat{\phi}_{ML})}=\tan(\hat{\phi}_{ML})=-\frac{\int_{T_0} r(t)\sin(2\pi f_c t)dt}{\int_{T_0} r(t)\cos(2\pi f_c t)dt}\Rightarrow$$

$$\hat{\phi}_{ML}=-\arctan\left\{\frac{\int_{T_0} r(t)\sin(2\pi f_c t)dt}{\int_{T_0} r(t)\cos(2\pi f_c t)dt}\right\} \qquad [2],(6.2-12)$$

Implementation of this equation is shown in figure 2 **(6.2–2)**.

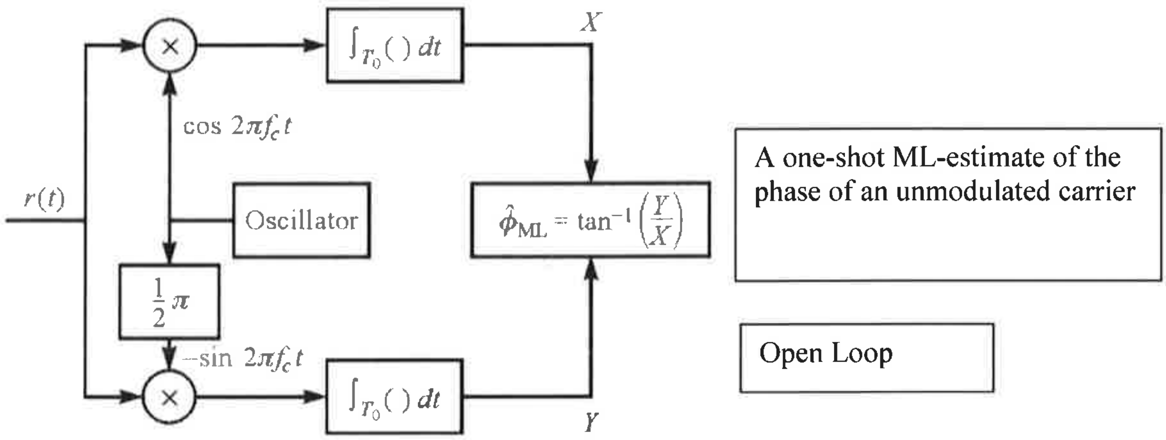


Figure 2.

PLL provides the ML-estimate of the phase of on unmodulated carrier.

6. **[Book 2, pages 368 READ]**
Cramer-Rao lower bound for unbiased estimate is

$$\operatorname{var}\left\{\hat{\phi}\right\} = \sigma_{\hat{\phi}}^2 \geq -\frac{1}{E\left\{\dfrac{\partial^2 \Lambda_L(\phi)}{\partial \phi^2}\right\}} \qquad [2],(6.5\text{–}6)$$

Unbiased => $\mathrm{E}\left[\hat{\phi}(x)\right] - \phi = 0$, where ϕ is the true value of the parameter =>

Estimation = True value = $\mathrm{E}\left[\hat{\phi}(x)\right] = \phi$

It provides a benchmark for comparing the variance of any practical estimate to the lower bound. Any estimate that is unbiased and whose variance attains the lower bound is called an efficient estimate.

σ_ϕ^2 may be difficult to compute

Log-likelihood function from previous problem is

$$\Lambda_L(\phi) = \ln(C) + \frac{2}{N_0}\left[\int_{T_0} r(t)s(t;\phi)dt\right],$$

if derived two times, we get

$$\frac{\partial \Lambda_L(\phi)}{\partial \phi} = -\frac{2A}{N_0}\int_{T_0} r(t)\sin(2\pi f_c t + \phi)dt$$

$$\frac{\partial^2 \Lambda_L(\phi)}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left[-\frac{2A}{N_0}\int_{T_0} r(t)\sin(2\pi f_c t + \phi)dt\right]$$

$$= -\frac{2A}{N_0}\int_{T_0} \frac{\partial}{\partial \phi}\left[r(t)\sin(2\pi f_c t + \phi)\right]dt$$

$$= -\frac{2A}{N_0}\int_{T_0}\left[\overbrace{\frac{\partial r(t)}{\partial \phi}}^{=0}\sin(2\pi f_c t + \phi) + r(t)\frac{\partial \sin(2\pi f_c t + \phi)}{\partial \phi}\right]dt$$

$$= -\frac{2A}{N_0}\int_{T_0} r(t)\cos(2\pi f_c t + \phi)dt$$

Estimated value is

$$E\left\{\frac{\partial^2\Lambda_L(\phi)}{\partial\phi^2}\right\} = E\left\{-\frac{2A}{N_0}\int_{T_0} r(t)\cos(2\pi f_c t+\phi)dt\right\}$$

Noise is zero mean

$$= -\frac{2A}{N_0}\int_{T_0} E\{r(t)\}\cos(2\pi f_c t+\phi)dt$$

$$= -\frac{2A}{N_0}\int_{T_0} A\cos(2\pi f_c t+\phi)\cdot\cos(2\pi f_c t+\phi)dt$$

$$= -\frac{2A^2}{N_0}\int_{T_0} \cos^2(2\pi f_c t+\phi)dt$$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$= -\frac{A^2}{N_0}\int_{T_0}\left[1+\cos(4\pi f_c t+2\phi)\right]dt$$

$$= -\frac{A^2}{N_0}\left[\int_{T_0} dt + \int_{T_0}\cos(4\pi f_c t+2\phi)dt\right]$$

$$= -\frac{A^2}{N_0}\left[T_0 + \Big/_{0}^{T_0}\sin(4\pi f_c t+2\phi)\cdot\frac{1}{4\pi f_c}\right]$$

$$= -\frac{A^2}{N_0}\left[T_0 + \underbrace{\frac{1}{4\pi f_c}\left[\sin(4\pi f_c T_0+2\phi)-\sin(2\phi)\right]}_{\approx 0}\right]$$

$$= -\frac{A^2T_0}{N_0}, \qquad [2],(6.5-10)$$

if $f_c >> 1$. So, the variance of ML estimate is lower-bounded as

$$\sigma^2_{\hat{\phi}_{\mathrm{ML}}} = \mathrm{var}\left\{\hat{\phi}_{\mathrm{ML}}\right\} \geq -\frac{1}{E\left\{\frac{\partial^2\Lambda_L(\phi)}{\partial\phi^2}\right\}} = -\frac{1}{-\frac{A^2T_0}{N_0}} = \frac{N_0}{A^2T_0}$$