Definitions in MoCafe/LaRT

1. Coordinate System in MoCafe/LaRT

- History: written on 2016-01-10, modified by Hyunmi Song on 2017-08-21, modified by KI Seon 2017–09-01, 2020-08-16

1.1 Rotation Matrix and Peeling-off

MoCafe/LaRT can set the positions of a galaxy/cloud and an observer depending on which side of that galaxy the observer wants to see (i.e. face-on/edge-on in terms of inclination, ...). Consider two reference frames: the observer frame O and the galaxy/cloud frame G. The target galaxy is at the origin with its disk on the xy-plane in G. Initially, O and G are coincident and the observer is on the z-axis (so, the galaxy is seen as face-on by the observer by default). Hence, the observer position is $(0, 0, d)$ in O. Now, let's rotate the observer and the observer frame so as to have a certain set of (α, β, γ) when seen by the (not-rotated) observer. Our goal is to find the representation in O of a vector represented in G and to find the coordinate of the observer in G.

In the above setting, a set of rotations of the observer's axes that changes the original face-on view to a galaxy with (phase angle α, inclination angle β, position angle γ) is, in a consecutive order,

1. to rotate by α about the z-axis, and
2. to rotate by β about the new y'-axis, and then
3. to rotate by γ about the new z''-axis.

Let's call each matrix $R_z(\alpha)$, $R_y(\beta)$, and $R_z(\gamma)$, respectively, where

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & & \cos\beta \end{pmatrix},$$

and

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Finally, the matrix $R = R_z(\gamma)\, R_y(\beta)\, R_z(\alpha)$ rotates the frame O relative to the frame G. Note that the matrix R is to rotate the axes, not to rotate a vector in a coordinate frame. Therefore, to get a new representation of a vector in the new rotated frame (by R), one needs to multiply that vector by R. So, the matrix that gives a new coordinate of a vector in the rotated frame O is given by

$$R = R_z(\gamma)\, R_x(\beta)\, R_z(\alpha)$$

and, putting the matrices from above altogether, we get

$$R = \begin{pmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{pmatrix}.$$

The inverse matrix, which corresponds to the transformation from the observer's frame to the galaxy frame is given by

$$R^{-1} = R^T = R_z(-\alpha)R_y(-\beta)R_z(-\gamma).$$

Using this matrix, the observer's position vector in G (the fixed frame), $\{\mathrm{k}\}_G$, is given by $R^{-1}\{\boldsymbol{k}\}_O$ where $\{\mathbf{k}\}_O = (0, 0, d)$. Each component of $\{\boldsymbol{k}\}_G$ is as follow:

$$\begin{aligned} x_{\rm obs} &= d\sin\beta\cos\alpha, \\ y_{\rm obs} &= d\sin\beta\sin\alpha, \\ z_{\rm obs} &= d\cos\beta. \end{aligned}$$

In order to apply the peeling-off technique, we need to calculate the direction vector from the photon position to the observer. The detector plane coincides with the xy plane of the observer's system. If the coordinates of the observer and photon are $(x_{\rm obs}, y_{\rm obs}, z_{\rm obs})$ and $(x_{\rm p}, y_{\rm p}, z_{\rm p})$, respectively, in the galaxy frame G, the photon direction vector toward the observer is $\{\boldsymbol{v}\} = (x_{\rm obs} - x_{\rm p}, y_{\rm obs} - y_{\rm p}, z_{\rm obs} - z_{\rm p})^T/|\boldsymbol{v}|$ in the galaxy frame G. Here, the normalization factor $|\boldsymbol{v}| = \left[(x_{\rm obs} - x_{\rm p})^2 + (y_{\rm obs} - y_{\rm p})^2 + (z_{\rm obs} - z_{\rm p})^2\right]^{1/2}$. The direction vector in the observer frame $\{\boldsymbol{v}\}_O = R\{\boldsymbol{v}\}_G = (v_x, v_y, v_z)^T$ can be used to calculate the angular coordinates (θ_x, θ_y) of the photon direction toward the observer in the detector XY plane.

$$\begin{aligned} \theta_x &= \frac{180}{\pi}\mathrm{atan2}(-v_x, v_z), \\ \theta_y &= \frac{180}{\pi}\mathrm{atan2}(-v_y, v_z). \end{aligned}$$

In Fortran, the array indices on the image plane with angular bins $(\Delta\theta_x, \Delta\theta_y)$ are then given by

$$\begin{aligned} i &= \mathrm{floor}\left(\frac{\theta_x}{\Delta\theta_x} + \frac{N_x}{2}\right) + 1 \quad (i = 1, \cdots, N_x), \\ j &= \mathrm{floor}\left(\frac{\theta_y}{\Delta\theta_y} + \frac{N_y}{2}\right) + 1 \quad (j = 1, \cdots, N_y), \end{aligned}$$

where N_x and N_y are the number of image bins, respectively, in x and y dimensnion.

Sometimes, we want to calculate an optical depth along the line of sight from a pixel on the detector plane toward the sky (galaxy). In this case, the direction vector toward the galaxy system from a pixel (i, j) on the detector is given by

$$\begin{aligned} v_x &= \tan\left[\left(i - \frac{N_x + 1}{2}\right)\Delta\theta_x \frac{\pi}{180}\right] \Big/ v \\ v_y &= \tan\left[\left(j - \frac{N_y + 1}{2}\right)\Delta\theta_y \frac{\pi}{180}\right] \Big/ v \\ v_z &= -1/v, \end{aligned}$$

where $v = \left(v_x^2 + v_y^2 + v_z^2\right)^{1/2}$. Here, $\Delta\theta_x$ and $\Delta\theta_y$ are in units of degree. Note that this direction vector is calculated in the detector coordinate system and thus should be transformed into the galaxy coordinate system.

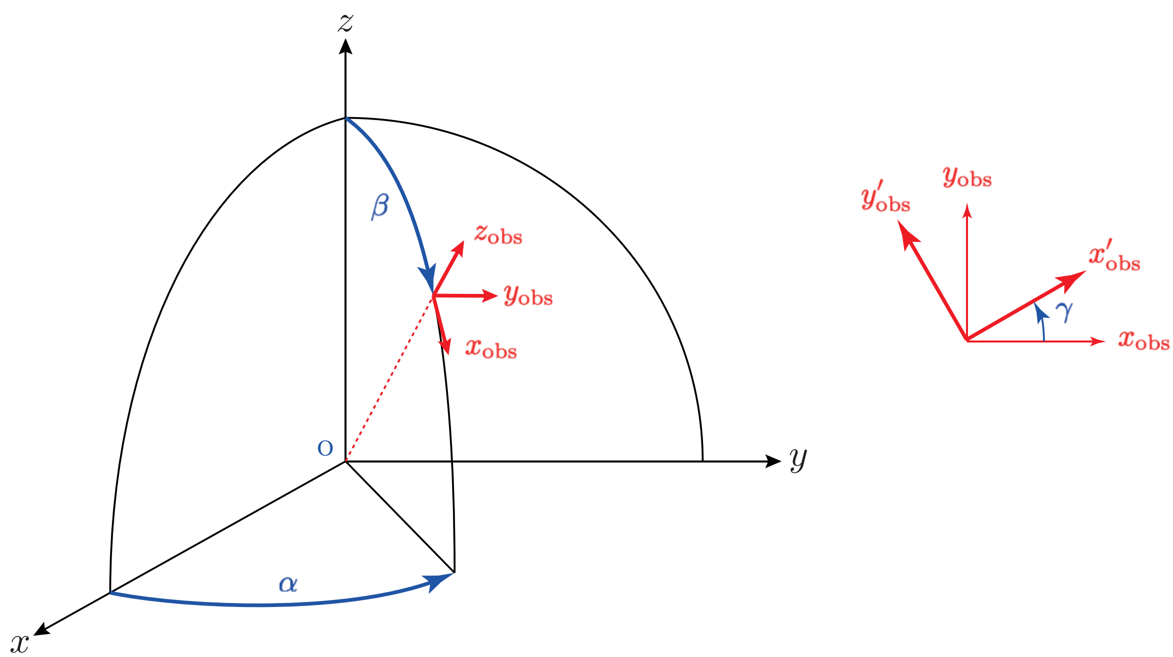

Figure 1: Coordinate system transfrom from the galaxy coordinate system (X, Y, Z) to the observer frame (x, y, z).

1.2 Edge-on galaxy

When we model an edge-on galaxy, we assume that the galactic disk lies on the xy plane. Then, the phase angle for the spiral pattern, inclination angle of the disk, and position angle on the sky can be defined as:

$$\begin{aligned} \text{phase angle} &= -\alpha \\ \text{inclination angle} &= -\beta \\ \text{position angle} &= -\gamma. \end{aligned}$$

2. Scattering Geometry

- History: written on 2014-12-15, updated on 12-18
- This part will be updated later (2020.08.16).

The following definitions and equations are mostly the same as those of Bianchi et al. (1996, ApJ, 465, 127). See also Chandrasekhar (1960, Radiative Transfer), Code & Whitney (1995, ApJ) and Whitney (2011, BASI).

2.1 Transformation of Stokes parameters

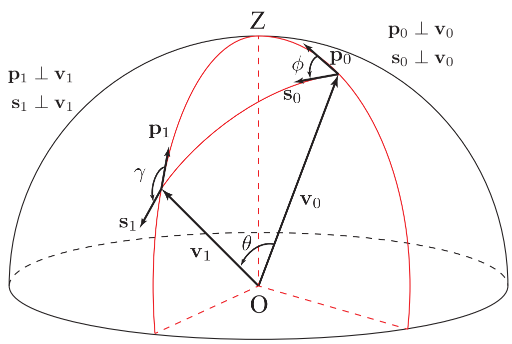

Figure 2: Geometry for scattering. A photon propagating into direction $\mathbf{v}_0$ scatters through angle θ into direction $\mathbf{v}_1$.

We use the Stokes vector (or parameters) $\mathbf{S}$ to describe the polarization:

$$\mathbf{S} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix},$$

where I is the intensity, Q the linear polarization aligned parallel to the z-axis, U is the linear polarization aligned $\pm 45°$ to the z-axis and V is the circular polarization. We note that the polarization (i.e., the Stokes vector) depends on the frame of reference. A scattering diagram is shown in Figure 2 (see also Chandrasekhar 1960). In Figure 2, $\mathbf{v}_0$ and $\mathbf{v}_1$ are unit vectors defining the propagation direction before and after scattering, respectively. The scattering plane is defined by two unit vectors $\mathbf{v}_0$ and $\mathbf{v}_1$. Two angles define $\mathbf{v}_1$ with respect to $\mathbf{v}_0$: the polar angle θ between $\mathbf{v}_0$ and $\mathbf{v}_1$, which is the actual angle of scattering, and the azimuthal angle $\mathbf{p}_0$ and $\mathbf{s}_0$. The reference directions for the Stokes parameters of the incoming and scattered photons are $\mathbf{p}_0$ and $\mathbf{p}_1$, respectively. The unit vector $\mathbf{s}_0$ is the projection of $\mathbf{v}_1$ onto the plane normal to $\mathbf{v}_0$. The vector $\mathbf{s}_0$ also lies in the scattering plane defined by $\mathbf{v}_0$ and $\mathbf{v}_1$.

Since the Mueller matrix elements are more easily defined for Stokes vectors referred to the scattering plane, we first have to rotate the Stokes vector about $\mathbf{v}_0$ from the initial reference direction $\mathbf{p}_0$ to the new reference direction $\mathbf{s}_0$. The stokes parameters of the incoming photon refereed to the scattering plane (the reference direction $\mathbf{s}_0$) are given by then

$$\mathbf{S}_0' = \mathbf{L}(-\phi)\mathbf{S}_0,$$

where

$$\mathbf{L}(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\varphi & \sin 2\varphi & 0 \\ 0 & -\sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the rotation matrix is defined for the clockwise rotation as described in Chandrasekhar (1960; Chap 1.)

The updated Stokes vector after scattering (but still referred to the scattering plane, or the reference direction $\mathbf{s}_1$), is then given by applying the scattering Mueller matrix $\mathbf{R}(\theta)$:

$$\mathbf{S}_1' = \mathbf{R}(\theta)\mathbf{L}(-\phi)\mathbf{S}_0,$$

where

$$\mathbf{R}(\theta) = \begin{pmatrix} S_{11}(\theta) & S_{12}(\theta) & 0 & 0 \\ S_{12}(\theta) & S_{11}(\theta) & 0 & 0 \\ 0 & 0 & S_{33}(\theta) & S_{34}(\theta) \\ 0 & 0 & -S_{34}(\theta) & S_{33}(\theta) \end{pmatrix}.$$

The matrix elements are calculated using a Mie code, such as bhmie or MIEV0.

The new Stokes parameters are defined with respect to $\mathbf{s}_1$, the unit vector that lies in the scattering plane and in the plane perpendicular to the new direction of propagation $\mathbf{v}_1$. We now have to calculate the Stokes parameters referred to a reference direction $\mathbf{p}_1$. The final Stokes parameters are then given by

$$\mathbf{S}_1 = \mathbf{L}(\gamma)\mathbf{R}(\theta)\mathbf{L}(-\phi)\mathbf{S}_0.$$

$$
\begin{aligned}
\mathbf{L}(\gamma)\mathbf{R}(\theta)\mathbf{L}(-\phi) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\gamma & \sin 2\gamma & 0 \\ 0 & -\sin 2\gamma & \cos 2\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{12} & S_{11} & 0 & 0 \\ 0 & 0 & S_{33} & S_{34} \\ 0 & 0 & -S_{34} & S_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\phi & -\sin 2\phi & 0 \\ 0 & \sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\gamma & \sin 2\gamma & 0 \\ 0 & -\sin 2\gamma & \cos 2\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12}\cos 2\phi & -S_{12}\sin 2\phi & 0 \\ S_{12} & S_{11}\cos 2\phi & -S_{11}\sin 2\phi & 0 \\ 0 & S_{33}\sin 2\phi & S_{33}\cos 2\phi & S_{34} \\ 0 & -S_{34}\sin 2\phi & -S_{34}\cos 2\phi & S_{33} \end{pmatrix} \\
&= \begin{pmatrix} S_{11} & S_{12}\cos 2\phi & -S_{12}\sin\phi & 0 \\ S_{12}\cos 2\gamma & S_{11}\cos 2\gamma\cos 2\phi + S_{33}\sin 2\gamma\sin 2\phi & -S_{11}\cos 2\gamma\sin 2\phi + S_{33}\sin 2\gamma\cos 2\phi & S_{34}\sin 2\gamma \\ -S_{12}\sin 2\gamma & -S_{11}\sin 2\gamma\cos 2\phi + S_{33}\cos 2\gamma\sin 2\phi & S_{11}\sin 2\gamma\sin 2\phi + S_{33}\cos 2\gamma\cos 2\phi & S_{34}\cos 2\gamma \\ 0 & -S_{34}\sin 2\phi & -S_{34}\cos 2\phi & S_{33} \end{pmatrix}
\end{aligned}
$$

2.2 Scattering Matrix for Resonance Scattering

Scattering matrix for resonance line scattering in the scattering plane is given by Equations (259) and (260) in Chadrasekhar (1960),

$$
\begin{pmatrix} I'_l \\ I'_r \\ U' \\ V' \end{pmatrix} = R \begin{pmatrix} I_l \\ I_r \\ U \\ V \end{pmatrix}
$$

$$
\begin{aligned}
R &= \frac{3}{2}E_1 \begin{pmatrix} \cos^2\theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}E_2 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{3}{2}E_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{2}E_1\cos^2\theta + \frac{1}{2}E_2 & \frac{1}{2}E_2 & 0 & 0 \\ \frac{1}{2}E_2 & \frac{3}{2}E_1 + \frac{1}{2}E_2 & 0 & 0 \\ 0 & 0 & \frac{3}{2}E_1\cos\theta & 0 \\ 0 & 0 & 0 & \frac{3}{2}E_3\cos\theta \end{pmatrix}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_l' &= \left(\frac{3}{2}E_1\cos^2\theta + \frac{1}{2}E_2\right) I_l + \frac{1}{2}E_2 I_r \\
I_r' &= \frac{1}{2}E_2 I_l + \left(\frac{3}{2}E_1 + \frac{1}{2}E_2\right) I_r
\end{aligned}
$$

From the definitions,

$$
\begin{aligned}
I &= I_l + I_r \\
Q &= I_l - I_r
\end{aligned}
$$

$$
\begin{aligned}
I' &= \left(\frac{3}{2}E_1\cos^2\theta + E_2\right)\frac{1}{2}(I+Q) + \left(\frac{3}{2}E_1 + E_2\right)\frac{1}{2}(I-Q) \\
Q' &= \left(\frac{3}{2}E_1\cos^2\theta\right)\frac{1}{2}(I+Q) - \left(\frac{3}{2}E_1\right)\frac{1}{2}(I-Q)
\end{aligned}
$$

$$
\begin{aligned}
I' &= \left[\frac{3}{4}E_1\left(\cos^2\theta + 1\right) + E_2\right] I + \frac{3}{4}E_1\left(\cos^2\theta - 1\right) Q \\
Q' &= \frac{3}{4}E_1\left(\cos^2\theta - 1\right) I + \frac{3}{4}E_1\left(\cos^2\theta + 1\right) Q
\end{aligned}
$$

Therefore, the scattering matrix for the conventional Stokes vector defined in the scattering plane is given by

$$
\begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix} = \mathbf{R}(\theta) \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}
$$

$$
\mathbf{R}(\theta) = \frac{3}{4}E_1 \begin{pmatrix} \cos^2\theta + 1 & \cos^2\theta - 1 & 0 & 0 \\ \cos^2\theta - 1 & \cos^2\theta + 1 & 0 & 0 \\ 0 & 0 & 2\cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + E_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{3}{2}E_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta \end{pmatrix}
$$

$$
\mathbf{R}(\theta) = \begin{pmatrix} \frac{3}{4}E_1\left(\cos^2\theta + 1\right) + E_2 & \frac{3}{4}E_1\left(\cos^2\theta - 1\right) & 0 & 0 \\ \frac{3}{4}E_1\left(\cos^2\theta - 1\right) & \frac{3}{4}E_1\left(\cos^2\theta + 1\right) & 0 & 0 \\ 0 & 0 & \frac{3}{2}E_1\cos\theta & 0 \\ 0 & 0 & 0 & \frac{3}{2}E_3\cos\theta \end{pmatrix}.
$$

The scattering matrix in the comoving frame is given by

$$\mathbf{R}(\theta)\mathbf{L}(-\phi) = \begin{pmatrix} \frac{3}{4}E_1\left(\cos^2\theta+1\right)+E_2 & \frac{3}{4}E_1\left(\cos^2\theta-1\right) & 0 & 0 \\ \frac{3}{4}E_1\left(\cos^2\theta-1\right) & \frac{3}{4}E_1\left(\cos^2\theta+1\right) & 0 & 0 \\ 0 & 0 & \frac{3}{2}E_1\cos\theta & 0 \\ 0 & 0 & 0 & \frac{3}{2}E_3\cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\phi & \sin 2\phi & 0 \\ 0 & -\sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4}E_1\left(\cos^2\theta+1\right)+E_2 & \frac{3}{4}E_1\left(\cos^2\theta-1\right)\cos 2\phi & \frac{3}{4}E_1\left(\cos^2\theta-1\right)\sin 2\phi & 0 \\ \frac{3}{4}E_1\left(\cos^2\theta-1\right) & \frac{3}{4}E_1\left(\cos^2\theta+1\right)\cos 2\phi & \frac{3}{4}E_1\left(\cos^2\theta+1\right)\sin 2\phi & 0 \\ 0 & -\frac{3}{2}E_1\cos\theta\sin 2\phi & \frac{3}{2}E_1\cos\theta\cos 2\phi & 0 \\ 0 & 0 & 0 & \frac{3}{2}E_3\cos\theta \end{pmatrix}$$

$$\begin{aligned} I' &= \left[\frac{3}{4}E_1\left(\cos^2\theta+1\right)+E_2\right]I + \left[\frac{3}{4}E_1\left(\cos^2\theta-1\right)\cos 2\phi\right]Q + \left[\frac{3}{4}E_1\left(\cos^2\theta-1\right)\sin 2\phi\right]U \\ Q' &= \left[\frac{3}{4}E_1\left(\cos^2\theta-1\right)\right]I + \left[\frac{3}{4}E_1\left(\cos^2\theta+1\right)\cos 2\phi\right]Q + \left[\frac{3}{4}E_1\left(\cos^2\theta+1\right)\sin 2\phi\right]U \\ U' &= \left[\frac{3}{2}E_1\cos\theta\sin 2\phi\right]Q - \left[\frac{3}{2}E_1\cos\theta\cos 2\phi\right]U \\ V' &= \left[\frac{3}{2}E_3\cos\theta\right]V \end{aligned}$$

2.3 Polarization Handedness and Angle

Linear polarization map is obtained by calculating, for each pixel, the degree of linear polarization and the polarization angle,

$$\begin{aligned} P &= \sqrt{Q^2+U^2}/I \\ \tan 2\psi &= U/Q \end{aligned}$$

or

$$\begin{aligned} Q/I &= P\cos 2\psi \\ U/I &= P\sin 2\psi \end{aligned}$$

We adopt the traditional convection according to which an elliptically polarized wave is designated as right-handed if the vibration ellipse is rotating in the clockwise sense as viewed by an observer looking toward the source of light (Figure 3). The opposite convention seems to be favored by van de Hulst (1957). The sign of V specifies the handedness of the vibration ellipse: positive denotes right-handed and negative denotes left-handed. The polarization angle ψ is defined to be a clockwise angle between $\mathbf{e}_{\parallel}$ and $\mathbf{e}_{\perp}$.

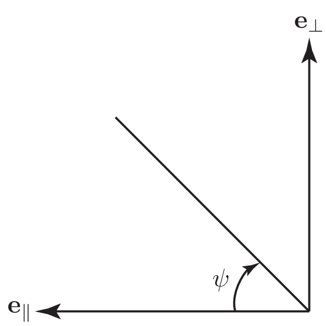

Table 2.2 Stokes Parameters for Polarized Light

Linearly Polarized				
0° ↔	90° ↕	+45° ⤡	−45° ⤢	γ
$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$	$\begin{pmatrix}1\\\cos 2\gamma\\\sin 2\gamma\\0\end{pmatrix}$
Circularly Polarized				
	Right		Left	
	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$		$\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$	

Figure 3: (Left panel) Definition of the polarization angle and handedness. (Right panel) Illustration of Stokes parameters for polarized light (adopted from Bohren & Huffman 1983). Note $\mathbf{e}_\parallel$ and $\mathbf{e}_\perp$ corresponds to $\mathbf{y}$ and $\mathbf{x}$ axes, respectively.

2.4 Phase Function

2.4.1 Scattering Angles

The phase function is obtained from the equation of $\mathbf{S}_1' = \mathbf{R}(\theta)\mathbf{L}(-\phi)\mathbf{S}_0$:

$$\begin{aligned} P(\theta,\phi) &= \frac{I'(\theta,\phi)}{I} = S_{11}(\theta) + S_{12}(\theta)\frac{Q'}{I} \\ &= S_{11}(\theta) + S_{12}(\theta)\left(\frac{Q}{I}\cos 2\phi - \frac{U}{I}\sin 2\phi\right). \end{aligned}$$

The angles θ and ϕ are distributed according to the above equation. The phase function $\Phi(\theta,\phi)$ is normalized so that

$$\begin{aligned} 1 &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta P(\theta,\phi) \\ &= \int_0^{\pi} d\theta \sin\theta P(\theta), \end{aligned}$$

where

$$P(\theta) = \int_0^{2\pi} P(\theta,\phi) d\phi = S_{11}(\theta).$$

$P(\theta)$ depends only on θ, so θ can be derived from one of Monte Carlo techniques, independently both of ϕ and of the polarization of the incident radiation. The scattering polar angle θ can be calculated from a random number ξ_1, inverting

$$\int_0^{\bar{\theta}} P(\theta)\sin\theta d\theta = \xi_1.$$

Substituting the value $\bar{\theta}$ into equation, we can find $\bar{\phi}$ from

$$\int_0^{\bar{\phi}} P(\bar{\theta},\phi) d\phi \bigg/ \int_0^{2\pi} P(\bar{\theta},\phi) d\phi = \xi_2,$$

or

$$\frac{1}{2\pi}\left\{\bar{\phi}+\frac{S_{12}(\bar{\theta})}{2S_{11}(\bar{\theta})}\left[\frac{Q}{I}\sin 2\bar{\phi}-\frac{U}{I}\left(1-\cos 2\bar{\phi}\right)\right]\right\}=\xi_2.$$

In terms of the polarization angle χ ($Q/I=\cos 2\beta\cos 2\chi$ and $U/I=\cos 2\beta\sin 2\chi$), the phase function and the equation for $\bar{\phi}$ are expressed as follows, respectively:

$$\begin{aligned}
P(\theta,\phi) &= S_{11}(\theta)+S_{12}(\theta)\cos 2\beta\cos 2(\phi+\chi)\\
P(\theta) &= S_{11}(\theta)\\
P(\phi) &= \frac{1}{2\pi}\left[1+\frac{S_{12}(\bar{\theta})}{S_{11}(\bar{\theta})}\cos 2\beta\cos 2(\phi+\chi)\right].
\end{aligned}$$

Rejection method

$$P_{\max}=\frac{1}{2\pi}\left[1+\left|\frac{S_{12}}{S_{11}}P_L\right|\right]$$

Note that $\cos 2\beta=1$ for linearly polarized light.

$$f\left(\bar{\phi}\right)\equiv\frac{1}{2\pi}\left\{\bar{\phi}+\frac{S_{12}(\bar{\theta})}{2S_{11}(\bar{\theta})}\cos 2\beta\left[\sin 2\left(\bar{\phi}+\chi\right)-\sin 2\chi\right]\right\}=\xi_2.$$

How we solve the above equation for given $\bar{\theta}$ and ξ_2 values ? Note that $|S_{12}|\le|S_{11}|$ and $0\le p_0\le 1$. The degree of polarization S_{12}/S_{11} can be 100% at $\bar{\theta}=\pi/2$ for Rayleigh scattering at long wavelengths ($\lambda\gtrsim 10\mu$m). However, the polarization degree is usually $S_{12}/S_{11}<1$. Letting the left hand side $f(\bar{\phi})$, we find the derivative of the function:

$$f'(\phi)=\frac{1}{2\pi}\left\{1+\frac{S_{12}}{S_{11}}P_0\cos 2\left(\phi+\psi_0\right)\right\}\ge 0.$$

Therefore, there exists a ϕ value that gives $f'(\phi)=0$ as $|S_{12}/S_{11}|=P_0=1$, in which Newton-Raphson method is inapplicable. However, in most cases, $|S_{12}/S_{11}|<1$ and $P_0<1$ and thus the Newton-Raphson method can be used in most cases to solve the equation for $\bar{\phi}$. We can start the Newton-Raphson method from $\bar{\phi}=2\pi\xi_2$. If the case of $f'(\phi)=0$ happens, we need to use Brent Method to solve the equation.

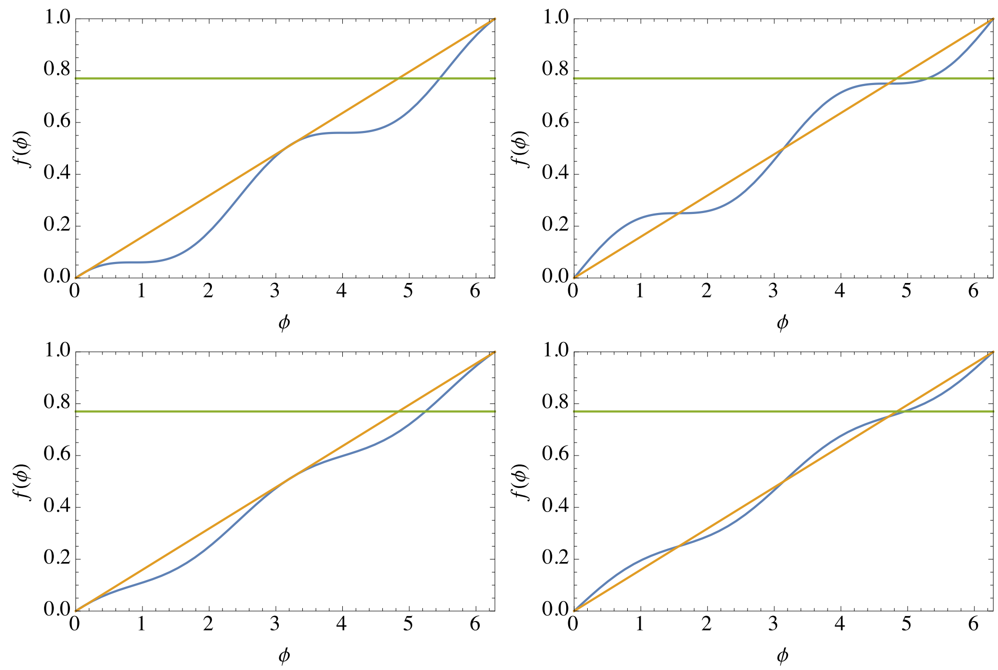

Figure 4: The equation $f(\bar{\phi})$ to calculate the azimuthal scattering angle $\bar{\phi}$. Blue and orange curves denote the equations $y = f(\bar{\phi})$ and $y = \bar{\phi}/2\pi$, respectively. Green curves denote $y = \xi_2 = 0.77$. Top panels are for $|S_{12}/S_{11}|\, P_0 = 1$ and bottom panels are for $|S_{12}/S_{11}|\, P_0 = 0.5$. The polarization angle was assumed to be $\psi_0 = 0.7$ and $\psi_0 = 0$ for the left and right panels, respectively. These plots can be reproduced with the Mathematica code "phase_phi.nb."

2.4.2 New Direction Cosine

Using the values of $\bar{\theta}$ and $\bar{\phi}$, we obtain the direction cosines of $\mathbf{v}_1 = (v_x^1, v_y^1, v_z^1)$:

Let us introduce a unit vector $\mathbf{w}_0$, which lies in $x - y$ plane and form an orthonormal triad together with $\mathbf{v}_0$ and $\mathbf{p}_0$ (see Left panel of Figure 5): i.e., $\mathbf{w}_0 \cdot \mathbf{v}_0 = 0$ and $\mathbf{w}_0 \cdot \hat{\mathbf{z}} = 0$

$$
\begin{aligned}
\mathbf{w}_0 &= \frac{1}{\rho_0}\mathbf{v}_0 \times \hat{\mathbf{z}} \\
&= \left(v_y^0\hat{\mathbf{x}} - v_x^0\hat{\mathbf{y}}\right)/\rho_0
\end{aligned}
$$

where $\rho_0 = |\mathbf{v}_0 \times \hat{\mathbf{z}}| = \sqrt{(v_x^0)^2 + (v_y^0)^2} = \sqrt{1 - (v_z^0)^2}$. Then the unit vector $\mathbf{p}_0$ is given by

$$
\begin{aligned}
\mathbf{p}_0 &= \mathbf{w}_0 \times \mathbf{v}_0 = \frac{1}{\rho_0}\left(\mathbf{v}_0 \times \hat{\mathbf{z}}\right) \times \mathbf{v}_0 \\
&= \frac{1}{\rho_0}\left\{\left(\mathbf{v}_0 \cdot \mathbf{v}_0\right)\hat{\mathbf{z}} - \left(\hat{\mathbf{z}} \cdot \mathbf{v}_0\right)\mathbf{v}_0\right\} \\
&= \frac{1}{\rho_0}\left\{\hat{\mathbf{z}} - \left(\hat{\mathbf{z}} \cdot \mathbf{v}_0\right)\mathbf{v}_0\right\} \\
&= (-v_x^0 v_z^0/\rho_0)\hat{\mathbf{x}} + (-v_y^0 v_z^0/\rho_0)\hat{\mathbf{y}} + \rho_0\hat{\mathbf{z}}
\end{aligned}
$$

The vector $\mathbf{v}_1$ can be expressed in terms of the orthonormal triad $(\mathbf{p}_0,\ \mathbf{w}_0,\ \mathbf{v}_0)$:

$$\mathbf{v}_1 = \sin\bar{\theta}\left(\cos\bar{\phi}\mathbf{p}_0 + \sin\bar{\phi}\mathbf{w}_0\right) + \cos\bar{\theta}\mathbf{v}_0.$$

By substituting the expressions for $\mathbf{p}_0$, $\mathbf{w}_0$, and $\mathbf{v}_0$, we can readily obtain expressions for the components of $\mathbf{v}_1$ in the stationary coordinate system:

$$\begin{aligned}
v_x^1 &= \frac{\sin\bar{\theta}}{\rho}\left(-v_x^0 v_z^0\cos\bar{\phi} + v_y^0\sin\bar{\phi}\right) + \cos\bar{\theta}v_x^0 \\
v_y^1 &= \frac{\sin\bar{\theta}}{\rho}\left(-v_y^0 v_z^0\cos\bar{\phi} - v_x^0\sin\bar{\phi}\right) + \cos\bar{\theta}v_y^0 \\
v_z^1 &= \rho\sin\bar{\theta}\cos\bar{\phi} + \cos\bar{\theta}v_z^0
\end{aligned}$$

The above expressions are exactly the same as Equation (A37) of Bianchi et al. (1996). Note that $\bar{\phi}$ is related to the angle φ' defined in Pozdnyakov et al. (1983, Soviet Scientific Review, p 323) by $\varphi' = \bar{\phi} - \pi/2$.

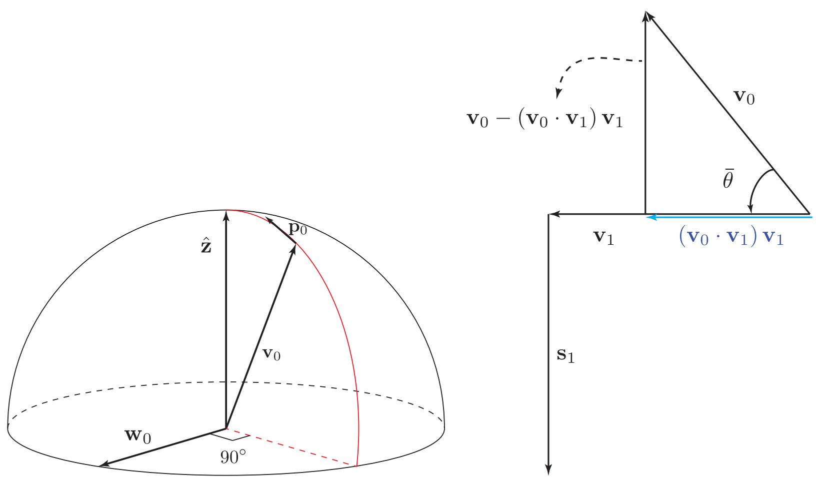

Figure 5: Geometries (a) for $\mathbf{w}_0$ and $\mathbf{p}_0$, and (b) for the vectors $\mathbf{v}_0$, $\mathbf{v}_1$, and $\mathbf{s}_1$.

2.4.3 Transformation Angle for the Stokes Parameters

The expression for $\mathbf{p}_1$ is analogous to that for $\mathbf{p}_0$ provided that we substitute $\mathbf{v}_1$ for $\mathbf{v}_0$.

$$\begin{aligned}
\mathbf{p}_1 &= \frac{1}{\rho_1}\left(\mathbf{v}_1\times\hat{\mathbf{z}}\right)\times\mathbf{v}_1 \\
&= \frac{1}{\sqrt{1-(v_z^1)^2}}\left\{\hat{\mathbf{z}} - \left(\hat{\mathbf{z}}\cdot\mathbf{v}_1\right)\mathbf{v}_1\right\}
\end{aligned}$$

where $\rho_1 = \sqrt{1-(v_z^1)^2}$. Since $\mathbf{s}_1$ lies in the scattering plane, it is parallel to the projection of $\mathbf{v}_0$ onto the plane normal to $\mathbf{v}_1$, but with opposite direction (see Right panel of Figure 5).

$$\begin{aligned}
\mathbf{s}_1 &\parallel \left(\mathbf{v}_0\cdot\mathbf{v}_1\right)\mathbf{v}_1 - \mathbf{v}_0 \\
&\parallel \left(\mathbf{v}_0\times\mathbf{v}_1\right)\times\mathbf{v}_1
\end{aligned}$$

Since the length of the vector $(\mathbf{v}_0 \cdot \mathbf{v}_1)\,\mathbf{v}_1 - \mathbf{v}_0$ is $\sin\bar{\theta} = \sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}$, the vector $\mathbf{s}_1$ is given by

$$\begin{aligned}
\mathbf{s}_1 &= \frac{1}{\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}}\left\{(\mathbf{v}_0\cdot\mathbf{v}_1)\,\mathbf{v}_1 - \mathbf{v}_0\right\} \\
&= \frac{1}{\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}}\,(\mathbf{v}_0\times\mathbf{v}_1)\times\mathbf{v}_1
\end{aligned}$$

The angle γ between $\mathbf{p}_1$ and $\mathbf{s}_1$ is found as follows:

$$\begin{aligned}
\cos\gamma &= \mathbf{p}_1\cdot\mathbf{s}_1 \\
&= \frac{1}{\sqrt{1-(v_z^1)^2}\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}}\left\{(\mathbf{v}_0\cdot\mathbf{v}_1)\,(\hat{\mathbf{z}}\cdot\mathbf{v}_1) - \hat{\mathbf{z}}\cdot\mathbf{v}_0\right\} \\
&= \frac{v_z^1\,(\mathbf{v}_0\cdot\mathbf{v}_1) - v_z^0}{\sqrt{1-(v_z^1)^2}\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}}
\end{aligned}$$

Since both $\mathbf{s}_1$ and $\mathbf{p}_1$ are perpendicular to $\mathbf{v}_1$, we obtain the following relation:

$$\begin{aligned}
(\sin\gamma)\mathbf{v}_1 &= \mathbf{p}_1\times\mathbf{s}_1 \\
&= \frac{1}{\sqrt{1-(v_z^1)^2}\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}}\left[(\mathbf{v}_1\times\hat{\mathbf{z}})\times\mathbf{v}_1\right]\times\left[(\mathbf{v}_0\times\mathbf{v}_1)\times\mathbf{v}_1\right]
\end{aligned}$$

Using the identity $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D}))\,\mathbf{C} - (\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}))\,\mathbf{D}$, we find

$$\begin{aligned}
\left[(\mathbf{v}_1\times\hat{\mathbf{z}})\times\mathbf{v}_1\right]\times\left[(\mathbf{v}_0\times\mathbf{v}_1)\times\mathbf{v}_1\right] &= -\left\{(\mathbf{v}_1\times\hat{\mathbf{z}})\cdot\left[\mathbf{v}_1\times(\mathbf{v}_0\times\mathbf{v}_1)\right]\right\}\mathbf{v}_1 \\
&= -\left\{(\mathbf{v}_1\times\hat{\mathbf{z}})\cdot\left[\mathbf{v}_0 - (\mathbf{v}_0\cdot\mathbf{v}_1)\,\mathbf{v}_1\right]\right\}\mathbf{v}_1 \\
&= -\left[(\mathbf{v}_1\times\hat{\mathbf{z}})\cdot\mathbf{v}_0\right]\mathbf{v}_1 \\
&= -\left[\hat{\mathbf{z}}\cdot(\mathbf{v}_0\times\mathbf{v}_1)\right]\mathbf{v}_1.
\end{aligned}$$

Here, we used the identity $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$. We finally obtain

$$\begin{aligned}
\sin\gamma &= \frac{-\hat{\mathbf{z}}\cdot(\mathbf{v}_0\times\mathbf{v}_1)}{\sqrt{1-(v_z^1)^2}\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}} \\
&= \frac{v_y^0 v_x^1 - v_x^0 v_y^1}{\sqrt{1-(v_z^1)^2}\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}}
\end{aligned}$$

2.5 Stokes parameters in Monte Carlo simulation

In Monte Carlo simulation, the scattered photon is not distributed over the entire solid angle but, instead, deflected into a single direction and attenuated by a factor equal to the albedo. The Stokes vector after scattering will be

$$\tilde{\mathbf{S}}_1 = \frac{aI_0}{I_1}\mathbf{S}_1 = \frac{aI_0}{I_1}\mathbf{L}(\gamma)\mathbf{R}(\theta)\mathbf{L}(-\phi)\mathbf{S}_0,$$

or

$$\begin{pmatrix} \tilde{I}_1 \\ \tilde{Q}_1 \\ \tilde{U}_1 \\ \tilde{V}_1 \end{pmatrix} = \frac{aI_0}{I_1} \begin{pmatrix} I_1 \\ Q_1 \\ U_1 \\ V_1 \end{pmatrix}.$$