Definitions in MoCafe

1. Coordinate System in MoCafe

- History: written on 2016-01-10

Consider two reference frames, G and O, and a vector $\boldsymbol{v}$ whose components are represented as $\{\boldsymbol{v}\}_G = (v_{x_1}, v_{y_1}, v_{z_1})^T$ in G. We want to find the representation of the same vector in O, or $\{\boldsymbol{v}\}_O = (v_{x_2}, v_{y_2}, v_{z_2})^T$.

In the following, the axes of the original frame are denoted by x,y,z and the axes of the rotated frame are denoted as X,Y,Z. The coordinate rotation in MoCafe is defined by intrinsic rotations that occur about the axes of the rotating coordinate system, which change its orientation after each elemental rotation. The rotation sequence adopted in MoCafe is denoted 3-1-3, or $z \to x' \to z''$.

- The XYZ system rotates by α about the Z axis (which coincides with the z axis).
- The XYZ system rotates about the rotated X axis by β.
- The XYZ system rotates a third time about the new Z axis by γ.

For the galaxy model, the Euler angles $(\alpha, \beta, -\gamma)$ correspond to the phase(?), inclination, and position angles. Note that the position angle in MoCafe is measured counter-clock wise from X axis in the detector plane. The first the rotation matrix about Z defined by $\{\boldsymbol{v}\}' = R_z(\alpha)\{\boldsymbol{v}\}_G$ is

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the second rotation about X', we get $\{\boldsymbol{v}\}'' = R_x(\beta)\{\boldsymbol{v}\}'$ in which

$$R_x(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix}.$$

From the rotation about Z'', finally, $\{\boldsymbol{v}\}_O = R_z(\gamma)\{\boldsymbol{v}\}''$ with

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now have $\{\boldsymbol{v}\}_O = R\{\boldsymbol{v}\} = R_z(\gamma)R_x(\beta)R_z(\alpha)\{\boldsymbol{v}\}_G$ in which

$$R = \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma & \sin\alpha\cos\gamma + \cos\alpha\cos\beta\sin\gamma & \sin\beta\sin\gamma \\ -\cos\alpha\sin\gamma - \sin\alpha\cos\beta\cos\gamma & -\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma & \sin\beta\cos\gamma \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{pmatrix}.$$

The inverse transform of R is

$$R^{-1} = \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma & -\cos\alpha\sin\gamma - \sin\alpha\cos\beta\cos\gamma & \sin\alpha\sin\beta \\ \sin\alpha\cos\gamma + \cos\alpha\cos\beta\sin\gamma & -\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma & -\cos\alpha\sin\beta \\ \sin\beta\sin\gamma & \sin\beta\cos\gamma & \cos\beta \end{pmatrix}.$$

The coordinates of the observer's system at a distance d, rotated by the Euler angles (α, β, γ), is $(0, 0, d)$ in the observer's system. Then the observer's location vector in the dust cloud system are give by $(x_{\rm obs}, y_{\rm obs}, z_{\rm obs})^T = R^{-1}(0, 0, d)^T$, i.e.,

$$\begin{aligned} x_{\rm obs} &= d\sin\alpha\sin\beta \\ y_{\rm obs} &= -d\cos\alpha\sin\beta \\ z_{\rm obs} &= d\cos\beta \end{aligned}$$

The detector plane coincides with the XY plane of the observer's system. If the coordinates of the observer and photon are $(x_{\rm obs}, y_{\rm obs}, z_{\rm obs})$ and $(x_{\rm p}, y_{\rm p}, z_{\rm p})$, respectively, in the dust cloud system, the photon direction vector to the observer is $\{\boldsymbol{v}\} = (x_{\rm obs} - x_{\rm p}, y_{\rm obs} - y_{\rm p}, z_{\rm obs} - z_{\rm p})^T/|\boldsymbol{v}|$ in the dust grid system. Here, the normalization factor $|\boldsymbol{v}| = \left[(x_{\rm obs} - x_{\rm p})^2 + (y_{\rm obs} - y_{\rm p})^2 + (z_{\rm obs} - z_{\rm p})^2\right]^{1/2}$.

The vector should be transformed to the observer's system. The transformed vector $\{\boldsymbol{v}\}_O = R\{\boldsymbol{v}\}_G = (v_x, v_y, v_z)^T$ can be used to calculate the angular coordinates (α, δ) of the photon direction toward the observer in the detector XY plane.

$$\begin{aligned} \alpha &= \mathrm{atan2}(-v_x, v_z), \\ \delta &= \mathrm{atan2}(-v_y, v_z). \end{aligned}$$

The array indices on the image plane with angular bins $(\Delta\alpha, \Delta\delta)$ are then given

$$\begin{aligned} i &= \mathrm{int}\left(\frac{\alpha}{\Delta\alpha} + \frac{N_i - 1}{2}\right), \\ j &= \mathrm{int}\left(\frac{\delta}{\Delta\delta} + \frac{N_j - 1}{2}\right). \end{aligned}$$

2. Old definition of Coordinate System

- History: written on 2012-04-27

Consider two reference frames, F_1 and F_2, and a vector $\boldsymbol{v}$ whose components are known in F_1, represented as $\{\boldsymbol{v}\}_1 = (v_{x_1}, v_{y_1}, v_{z_1})^T$. We want to determine the representation of the same vector in F_2, or $\{\boldsymbol{v}\}_2 = (v_{x_2}, v_{y_2}, v_{z_2})^T$.

The rotation sequence used in the Monte-Carlo simulation code is the 321, or $z \to y \to x$. Considering a rotation from F_1 to F_2 the first rotation is about z_1 through an angle ϕ_z which is positive according to the right hand rule about the z_1 axis. With two rotations to go, the resulting alignment in general is oriented with neither F_1 or F_2, but some intermediate reference frame (the first of two) denoted F'. Since the rotation was about z_1, z' is parallel to it but neither of the other two primed axes is. The next rotation is through an angle ϕ_y about the axis y' of the first intermediate reference frame to the second intermediate reference frame, F''. Note that $y'' = y'$, and neither y'' or z'' are necessarily axes of either F_1 or F_2. The final rotation is about x'' through angle ϕ_x and the final alignment is parallel to the axes of F_2.

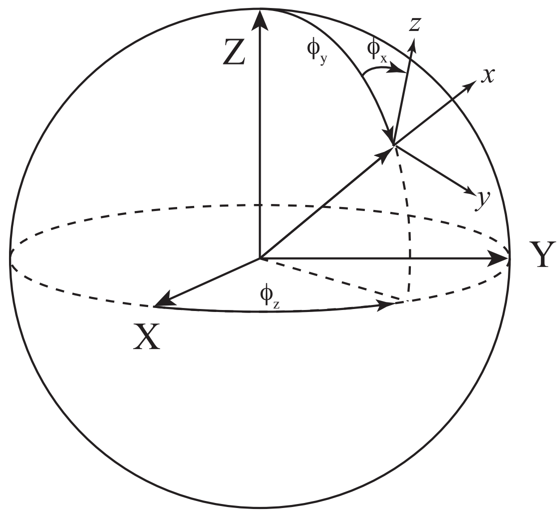

Figure 1: Coordinate system transfrom from (X, Y, Z) to (x, y, z).

Consider first the rotation about z_1, $\{\boldsymbol{v}\}' = T_{F'1}\{\boldsymbol{v}\}_1$ in which

$$T_{F'1} = \begin{pmatrix} \cos\phi_z & \sin\phi_z & 0 \\ -\sin\phi_z & \cos\phi_z & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From the rotation about y' we get $\{\boldsymbol{v}\}'' = T_{F''F'}\{\boldsymbol{v}\}'$ in which

$$T_{F''F'} = \begin{pmatrix} \cos\phi_y & 0 & -\sin\phi_y \\ 0 & 1 & 0 \\ \sin\phi_y & 0 & \cos\phi_y \end{pmatrix}.$$

From the rotation about x'', finally, $\{\boldsymbol{v}\}_2 = T_{2F''}\{\boldsymbol{v}\}''$ with

$$T_{2F''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_x & \sin\phi_x \\ 0 & -\sin\phi_x & \cos\phi_x \end{pmatrix}.$$

We now have $\{\boldsymbol{v}\}_2 = T_{21}\{\boldsymbol{v}\}_1 = T_{2F''}T_{F''F'}T_{F'1}\{\boldsymbol{v}\}_1$ in which

$$T_{21} = \begin{pmatrix} \cos\phi_y\cos\phi_z & \cos\phi_y\sin\phi_z & -\sin\phi_y \\ (\sin\phi_x\sin\phi_y\cos\phi_z - \cos\phi_x\sin\phi_z) & (\sin\phi_x\sin\phi_y\sin\phi_z + \cos\phi_x\cos\phi_z) & \sin\phi_x\cos\phi_y \\ (\cos\phi_x\sin\phi_y\cos\phi_z + \sin\phi_x\sin\phi_z) & (\cos\phi_x\sin\phi_y\sin\phi_z - \sin\phi_x\cos\phi_z) & \cos\phi_x\cos\phi_y \end{pmatrix}.$$

The inverse transform of T_{21} is

$$T_{12} = T_{21}^T = \begin{pmatrix} \cos\phi_y\cos\phi_z & (\sin\phi_x\sin\phi_y\cos\phi_z - \cos\phi_x\sin\phi_z) & (\cos\phi_x\sin\phi_y\cos\phi_z + \sin\phi_x\sin\phi_z) \\ \cos\phi_y\sin\phi_z & (\sin\phi_x\sin\phi_y\sin\phi_z + \cos\phi_x\cos\phi_z) & (\cos\phi_x\sin\phi_y\sin\phi_z - \sin\phi_x\cos\phi_z) \\ -\sin\phi_y & \sin\phi_x\cos\phi_y & \cos\phi_x\cos\phi_y \end{pmatrix}.$$

In the code, the coordinate system F_1 is the base coordinate system ((X, Y, Z) in Figure 1) to represent the dust density grid. The coordinate system F_2 is the coordinate system ((x, y, z) in Figure 1) of the observer who measure the photons from the dust cloud. We note here that ϕ_y and ϕ_x are closely related to the inclination angle of the cloud system and position angle, respectively. However, they differ from the conventionally-defined inclination and position angles by $\pi/2$. In most cases, only the rotation angle about y axis ϕ_y may be needed to vary.

The coordinate of the observer at a distance d, rotated by (ϕ_x, ϕ_y, ϕ_z) angles, is $(d, 0, 0)$ in the observer's coordinate system. Then the observer's coordinates in the dust cloud system are give by

$$\begin{aligned} x_{\rm obs} &= d\cos\phi_y\cos\phi_z \\ y_{\rm obs} &= d\cos\phi_y\sin\phi_z \\ z_{\rm obs} &= -d\sin\phi_y \end{aligned}$$

The detector plane coincides with the y_2z_2 plane of the observer's coordinate system. If the coordinates of the observer and photon are $(x_{\rm obs}, y_{\rm obs}, z_{\rm obs})$ and $(x_{\rm p}, y_{\rm p}, z_{\rm p})$, respectively, in the dust cloud system, the photon direction vector to the observer is $\{\boldsymbol{v}\}_1 = (x_{\rm obs} - x_{\rm p}, y_{\rm obs} - y_{\rm p}, z_{\rm obs} - z_{\rm p})^T/|\boldsymbol{v}|$ in the dust cloud system. Here, the normalization factor $|\boldsymbol{v}| = \left[(x_{\rm obs} - x_{\rm p})^2 + (y_{\rm obs} - y_{\rm p})^2 + (z_{\rm obs} - z_{\rm p})^2\right]^{1/2}$. The angular coordinates (α, δ) of the photon direction toward the observer in the y_2z_2 plane are

$$\begin{aligned} \alpha &= \mathrm{atan2}(-v_{y_2}, v_{x_2}), \\ \delta &= \mathrm{atan2}(-v_{z_2}, v_{x_2}). \end{aligned}$$

The array indices on the image plane with angular bins $(\Delta\alpha, \Delta\delta)$ are then given

$$\begin{aligned} i &= \mathrm{int}\left(\frac{\alpha}{\Delta\alpha} + \frac{N_i - 1}{2}\right), \\ j &= \mathrm{int}\left(\frac{\delta}{\Delta\delta} + \frac{N_j - 1}{2}\right). \end{aligned}$$

3. Scattering Geometry

- History: written on 2014-12-15, updated on 12-18

The following definitions and equations are mostly the same as those of Bianchi et al. (1996, ApJ, 465, 127). See also Chandrasekhar (1960, Radiative Transfer), Code & Whitney (1995, ApJ) and Whitney (2011, BASI).

3.1 Transformation of Stokes parameters

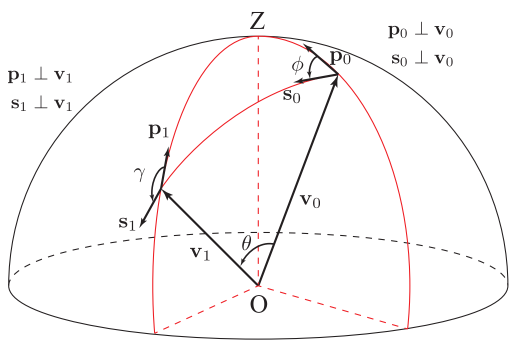

Figure 2: Geometry for scattering. A photon propagating into direction $\mathbf{v}_0$ scatters through angle θ into direction $\mathbf{v}_1$.

We use the Stokes vector (or parameters) $\mathbf{S}$ to describe the polarization:

$$\mathbf{S} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix},$$

where I is the intensity, Q the linear polarization aligned parallel to the z-axis, U is the linear polarization aligned $\pm 45°$ to the z-axis and V is the circular polarization. We note that the polarization (i.e., the Stokes vector) depends on the frame of reference. A scattering diagram is shown in Figure 2 (see also Chandrasekhar 1960). In Figure 2, $\mathbf{v}_0$ and $\mathbf{v}_1$ are unit vectors defining the propagation direction before and after scattering, respectively. The scattering plane is defined by two unit vectors $\mathbf{v}_0$ and $\mathbf{v}_1$. Two angles define $\mathbf{v}_1$ with respect to $\mathbf{v}_0$: the polar angle θ between $\mathbf{v}_0$ and $\mathbf{v}_1$, which is the actual angle of scattering, and the azimuthal angle $\mathbf{p}_0$ and $\mathbf{s}_0$. The reference directions for the Stokes parameters of the incoming and scattered photons are $\mathbf{p}_0$ and $\mathbf{p}_1$, respectively. The unit vector $\mathbf{s}_0$ is the projection of $\mathbf{v}_1$ onto the plane normal to $\mathbf{v}_0$. The vector $\mathbf{s}_0$ also lies in the scattering plane defined by $\mathbf{v}_0$ and $\mathbf{v}_1$.

Since the Mueller matrix elements are more easily defined for Stokes vectors referred to the scattering plane, we first have to rotate the Stokes vector about $\mathbf{v}_0$ from the initial reference direction $\mathbf{p}_0$ to the new reference direction $\mathbf{s}_0$. The stokes parameters of the incoming photon refereed to the scattering plane (the reference direction $\mathbf{s}_0$) are given by then

$$\mathbf{S}_0' = \mathbf{L}(-\phi)\mathbf{S}_0,$$

where

$$\mathbf{L}(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\varphi & \sin 2\varphi & 0 \\ 0 & -\sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the rotation matrix is defined for the clockwise rotation as described in Chandrasekhar (1960; Chap 1.)

The updated Stokes vector after scattering (but still referred to the scattering plane, or the reference direction $\mathbf{s}_1$), is then given by applying the scattering Mueller matrix $\mathbf{R}(\theta)$:

$$\mathbf{S}_1' = \mathbf{R}(\theta)\mathbf{L}(-\phi)\mathbf{S}_0,$$

where

$$\mathbf{R}(\theta) = \begin{pmatrix} S_{11}(\theta) & S_{12}(\theta) & 0 & 0 \\ S_{12}(\theta) & S_{11}(\theta) & 0 & 0 \\ 0 & 0 & S_{33}(\theta) & S_{34}(\theta) \\ 0 & 0 & -S_{34}(\theta) & S_{33}(\theta) \end{pmatrix}.$$

The matrix elements are calculated using a Mie code, such as bhmie or MIEV0.

The new Stokes parameters are defined with respect to $\mathbf{s}_1$, the unit vector that lies in the scattering plane and in the plane perpendicular to the new direction of propagation $\mathbf{v}_1$. We now have to calculate the Stokes parameters referred to a reference direction $\mathbf{p}_1$. The final Stokes parameters are then given by

$$\mathbf{S}_1 = \mathbf{L}(\gamma)\mathbf{R}(\theta)\mathbf{L}(-\phi)\mathbf{S}_0.$$

$$\begin{aligned}\mathbf{L}(\gamma)\mathbf{R}(\theta)\mathbf{L}(-\phi) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\gamma & \sin 2\gamma & 0 \\ 0 & -\sin 2\gamma & \cos 2\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{12} & S_{11} & 0 & 0 \\ 0 & 0 & S_{33} & S_{34} \\ 0 & 0 & -S_{34} & S_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\phi & -\sin 2\phi & 0 \\ 0 & \sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\gamma & \sin 2\gamma & 0 \\ 0 & -\sin 2\gamma & \cos 2\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12}\cos 2\phi & -S_{12}\sin 2\phi & 0 \\ S_{12} & S_{11}\cos 2\phi & -S_{11}\sin 2\phi & 0 \\ 0 & S_{33}\sin 2\phi & S_{33}\cos 2\phi & S_{34} \\ 0 & -S_{34}\sin 2\phi & -S_{34}\cos 2\phi & S_{33} \end{pmatrix}\end{aligned}$$

$$
= \begin{pmatrix} S_{11} & S_{12}\cos 2\phi & -S_{12}\sin\phi & 0 \\ S_{12}\cos 2\gamma & S_{11}\cos 2\gamma\cos 2\phi + S_{33}\sin 2\gamma \sin 2\phi & -S_{11}\cos 2\gamma \sin 2\phi + S_{33}\sin 2\gamma\cos 2\phi & S_{34}\sin 2\gamma \\ -S_{12}\sin 2\gamma & -S_{11}\sin 2\gamma\cos 2\phi + S_{33}\cos 2\gamma\sin 2\phi & S_{11}\sin 2\gamma\sin 2\phi + S_{33}\cos 2\gamma\cos 2\phi & S_{34}\cos 2\gamma \\ 0 & -S_{34}\sin 2\phi & -S_{34}\cos 2\phi & S_{33} \end{pmatrix}
$$

3.2 Polarization Handedness and Angle

Linear polarization map is obtained by calculating, for each pixel, the degree of linear polarization and the polarization angle,

$$
\begin{aligned} P &= \sqrt{Q^2 + U^2}/I \\ \tan 2\psi &= U/Q \end{aligned}
$$

or

$$
\begin{aligned} Q/I &= P\cos 2\psi \\ U/I &= P\sin 2\psi \end{aligned}
$$

We adopt the traditional convection according to which an elliptically polarized wave is designated as right-handed if the vibration ellipse is rotating in the clockwise sense as viewed by an observer looking toward the source of light (Figure 3). The opposite convention seems to be favored by van de Hulst (1957). The sign of V specifies the handedness of the vibration ellipse: positive denotes right-handed and negative denotes left-handed. The polarization angle ψ is defined to be a clockwise angle between $\mathbf{e}_{\|}$ and $\mathbf{e}_{\perp}$.

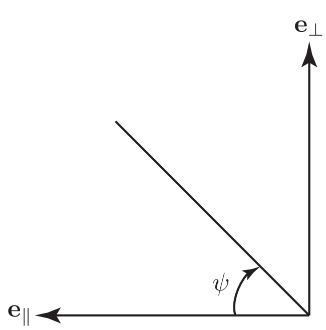

Table 2.2 Stokes Parameters for Polarized Light

Linearly Polarized				
0° ↔	90° ↕	+45° ↘	−45° ↙	γ
$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$	$\begin{pmatrix}1\\\cos 2\gamma\\\sin 2\gamma\\0\end{pmatrix}$
Circularly Polarized				
	Right		Left	
	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$		$\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$	

Figure 3: (Left panel) Definition of the polarization angle and handedness. (Right panel) Illustration of Stokes parameters for polarized light (adopted from Bohren & Huffman 1983). Note $\mathbf{e}_{\|}$ and $\mathbf{e}_{\perp}$ corresponds to $\mathbf{y}$ and $\mathbf{x}$ axes, respectively.

3.3 Phase Function

3.3.1 Scattering Angles

The phase function is obtained from the equation of $\mathbf{S}_1' = \mathbf{R}(\theta)\mathbf{L}(-\phi)\mathbf{S}_0$:

$$\begin{aligned}\Phi(\theta,\phi) &= \frac{I_1'(\theta,\phi)}{I_0} = S_{11}(\theta) + S_{12}(\theta)\frac{Q_0'}{I_0'} \\ &= S_{11}(\theta) + S_{12}(\theta)\left(\frac{Q_0}{I_0}\cos 2\phi - \frac{U_0}{I_0}\sin 2\phi\right).\end{aligned}$$

The angles θ and ϕ are distributed according to the above equation. The phase function $\Phi(\theta,\phi)$ is normalized so that

$$\begin{aligned}1 &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \Phi(\theta,\phi) \\ &= \int_0^{\pi} d\theta \sin\theta \tilde{\Phi}(\theta),\end{aligned}$$

where

$$\tilde{\Phi}(\theta) = \int_0^{2\pi} \Phi(\theta,\phi) d\phi = S_{11}(\theta).$$

$\tilde{\Phi}(\theta)$ depends only on θ, so θ can be derived from the Monte Carlo method, independently both of ϕ and of the polarization of the incident radiation. The scattering polar angle θ can be calculated from a random number ξ_1, inverting

$$\int_0^{\bar{\theta}} \tilde{\Phi}(\theta)\sin\theta d\theta = \xi_1.$$

Substituting this value of θ into equation, we can find $\bar{\phi}$ from

$$\int_0^{\bar{\phi}} \Phi(\bar{\theta},\phi)d\phi \bigg/ \int_0^{2\pi} \Phi(\bar{\theta},\phi)d\phi = \xi_2,$$

or

$$\frac{1}{2\pi}\left\{\bar{\phi} + \frac{S_{12}(\bar{\theta})}{2S_{11}(\bar{\theta})}\left[\frac{Q_0}{I_0}\sin 2\bar{\phi} - \frac{U_0}{I_0}\left(1-\cos 2\bar{\phi}\right)\right]\right\} = \xi_2.$$

In terms of the polarization angle ψ_0 ($Q/I = \cos 2\psi_0$ and $U/I = \sin 2\psi_0$), the phase function and the equation for $\bar{\phi}$ are expressed as follows, respectively:

$$\Phi(\theta,\phi) = S_{11}(\theta) + S_{12}(\theta)P_0\cos 2(\phi+\psi_0)$$

$$f\left(\bar{\phi}\right) \equiv \frac{1}{2\pi}\left\{\bar{\phi} + \frac{S_{12}(\bar{\theta})}{2S_{11}(\bar{\theta})}P_0\left[\sin 2\left(\bar{\phi}+\psi_0\right) - \sin 2\psi_0\right]\right\} = \xi_2.$$

How we solve the above equation for given $\bar{\theta}$ and ξ_2 values ? Note that $|S_{12}| \le |S_{11}|$ and $0 \le P_0 \le 1$. The degree of polarization S_{12}/S_{11} can be 100% at $\bar{\theta} = \pi/2$ for Rayleigh scattering at long wavelengths ($\lambda \gtrsim 10\mu$m). However, the polarization degree is usually $S_{12}/S_{11} < 1$. Letting the left hand side $f(\bar{\phi})$, we find the derivative of the function:

$$f'(\phi) = \frac{1}{2\pi}\left\{1 + \frac{S_{12}}{S_{11}}P_0\cos 2\left(\phi+\psi_0\right)\right\} \ge 0.$$

Therefore, there exists a ϕ value that gives $f'(\phi) = 0$ as $|S_{12}/S_{11}| = P_0 = 1$, in which Newton-Raphson method is inapplicable. However, in most cases, $|S_{12}/S_{11}| < 1$ and $P_0 < 1$ and thus the Newton-Raphson method can be used in most cases to solve the equation for $\bar{\phi}$. We can start the Newton-Raphson method from $\bar{\phi} = 2\pi\xi_2$. If the case of $f'(\phi) = 0$ happens, we need to use Brent Method to solve the equation.

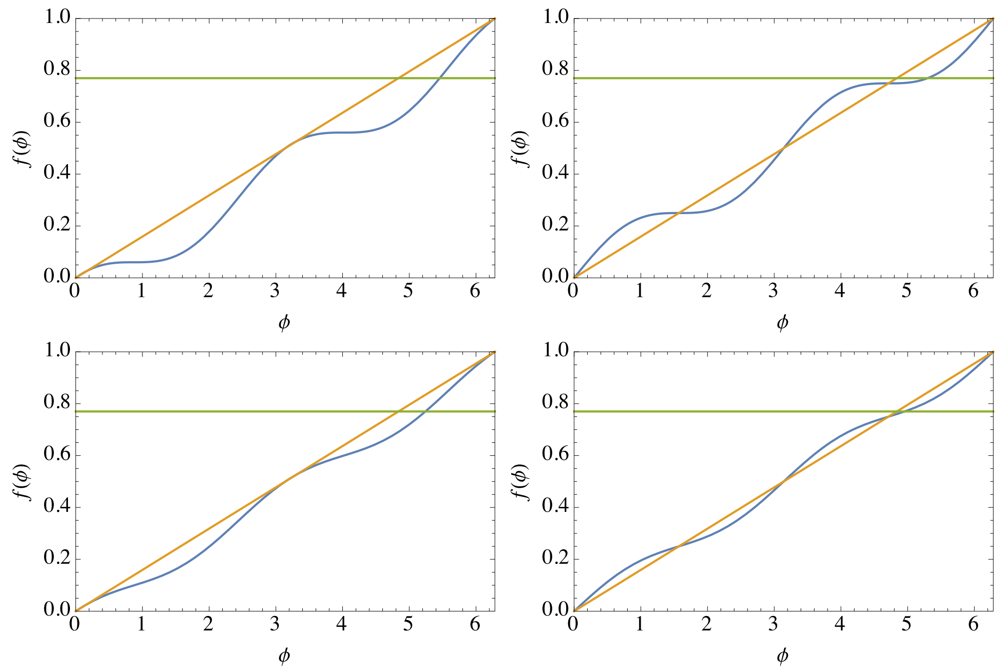

Figure 4: The equation $f(\bar{\phi})$ to calculate the azimuthal scattering angle $\bar{\phi}$. Blue and orange curves denote the equations $y = f(\bar{\phi})$ and $y = \bar{\phi}/2\pi$, respectively. Green curves denote $y = \xi_2 = 0.77$. Top panels are for $|S_{12}/S_{11}|\, P_0 = 1$ and bottom panels are for $|S_{12}/S_{11}|\, P_0 = 0.5$. The polarization angle was assumed to be $\psi_0 = 0.7$ and $\psi_0 = 0$ for the left and right panels, respectively. These plots can be reproduced with the Mathematica code "phase_phi.nb."

3.3.2 New Direction Cosine

Using the values of $\bar{\theta}$ and $\bar{\phi}$, we obtain the direction cosines of $\mathbf{v}_1 = (v_x^1, v_y^1, v_z^1)$:

Let us introduce a unit vector $\mathbf{w}_0$, which lies in $x - y$ plane and form an orthonormal triad together with $\mathbf{v}_0$ and $\mathbf{p}_0$ (see Left panel of Figure 5): i.e., $\mathbf{w}_0 \cdot \mathbf{v}_0 = 0$ and $\mathbf{w}_0 \cdot \hat{\mathbf{z}} = 0$

$$\begin{aligned} \mathbf{w}_0 &= \frac{1}{\rho_0}\mathbf{v}_0 \times \hat{\mathbf{z}} \\ &= \left(v_y^0\hat{\mathbf{x}} - v_x^0\hat{\mathbf{y}}\right)/\rho_0 \end{aligned}$$

where $\rho_0 = |\mathbf{v}_0 \times \hat{\mathbf{z}}| = \sqrt{(v_x^0)^2 + (v_y^0)^2} = \sqrt{1 - (v_z^0)^2}$. Then the unit vector $\mathbf{p}_0$ is given by

$$\mathbf{p}_0 = \mathbf{w}_0 \times \mathbf{v}_0 = \frac{1}{\rho_0}\left(\mathbf{v}_0 \times \hat{\mathbf{z}}\right) \times \mathbf{v}_0$$

$$
\begin{aligned}
&= \frac{1}{\rho_0}\left\{(\mathbf{v}_0\cdot\mathbf{v}_0)\,\hat{\mathbf{z}} - (\hat{\mathbf{z}}\cdot\mathbf{v}_0)\,\mathbf{v}_0\right\} \\
&= \frac{1}{\rho_0}\left\{\hat{\mathbf{z}} - (\hat{\mathbf{z}}\cdot\mathbf{v}_0)\,\mathbf{v}_0\right\} \\
&= (-v_x^0 v_z^0/\rho_0)\hat{\mathbf{x}} + (-v_y^0 v_z^0/\rho_0)\hat{\mathbf{y}} + \rho_0\hat{\mathbf{z}}
\end{aligned}
$$

The vector $\mathbf{v}_1$ can be expressed in terms of the orthonormal triad $(\mathbf{p}_0,\ \mathbf{w}_0,\ \mathbf{v}_0)$:

$$
\mathbf{v}_1 = \sin\bar{\theta}\left(\cos\bar{\phi}\mathbf{p}_0 + \sin\bar{\phi}\mathbf{w}_0\right) + \cos\bar{\theta}\mathbf{v}_0.
$$

By substituting the expressions for $\mathbf{p}_0$, $\mathbf{w}_0$, and $\mathbf{v}_0$, we can readily obtain expressions for the components of $\mathbf{v}_1$ in the stationary coordinate system:

$$
\begin{aligned}
v_x^1 &= \frac{\sin\bar{\theta}}{\rho}\left(-v_x^0 v_z^0\cos\bar{\phi} + v_y^0\sin\bar{\phi}\right) + \cos\bar{\theta}v_x^0 \\
v_y^1 &= \frac{\sin\bar{\theta}}{\rho}\left(-v_y^0 v_z^0\cos\bar{\phi} - v_x^0\sin\bar{\phi}\right) + \cos\bar{\theta}v_y^0 \\
v_z^1 &= \rho\sin\bar{\theta}\cos\bar{\phi} + \cos\bar{\theta}v_z^0
\end{aligned}
$$

The above expressions are exactly the same as Equation (A37) of Bianchi et al. (1996). Note that $\bar{\phi}$ is related to the angle φ' defined in Pozdnyakov et al. (1983, Soviet Scientific Review, p 323) by $\varphi' = \bar{\phi} - \pi/2$.

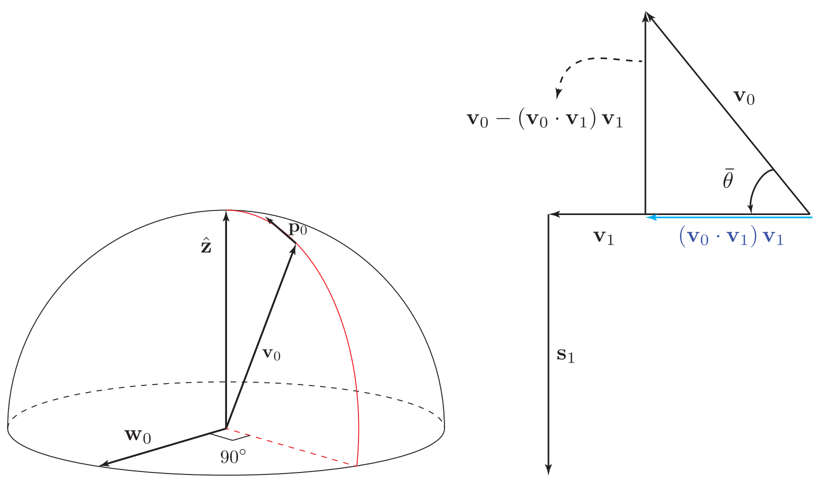

Figure 5: Geometries (a) for $\mathbf{w}_0$ and $\mathbf{p}_0$, and (b) for the vectors $\mathbf{v}_0$, $\mathbf{v}_1$, and $\mathbf{s}_1$.

3.3.3 Transformation Angle for the Stokes Parameters

The expression for $\mathbf{p}_1$ is analogous to that for $\mathbf{p}_0$ provided that we substitute $\mathbf{v}_1$ for $\mathbf{v}_0$.

$$
\begin{aligned}
\mathbf{p}_1 &= \frac{1}{\rho_1}\left(\mathbf{v}_1\times\hat{\mathbf{z}}\right)\times\mathbf{v}_1 \\
&= \frac{1}{\sqrt{1-(v_z^1)^2}}\left\{\hat{\mathbf{z}} - (\hat{\mathbf{z}}\cdot\mathbf{v}_1)\,\mathbf{v}_1\right\}
\end{aligned}
$$

where $\rho_1 = \sqrt{1-(v_z^1)^2}$. Since $\mathbf{s}_1$ lies in the scattering plane, it is parallel to the projection of $\mathbf{v}_0$ onto the plane normal to $\mathbf{v}_1$, but with opposite direction (see Right panel of Figure 5).

$$\begin{aligned} \mathbf{s}_1 &\parallel (\mathbf{v}_0 \cdot \mathbf{v}_1)\,\mathbf{v}_1 - \mathbf{v}_0 \\ &\parallel (\mathbf{v}_0 \times \mathbf{v}_1) \times \mathbf{v}_1 \end{aligned}$$

Since the length of the vector $(\mathbf{v}_0 \cdot \mathbf{v}_1)\,\mathbf{v}_1 - \mathbf{v}_0$ is $\sin\bar{\theta} = \sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}$, the vector $\mathbf{s}_1$ is given by

$$\begin{aligned} \mathbf{s}_1 &= \frac{1}{\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}}\left\{(\mathbf{v}_0 \cdot \mathbf{v}_1)\,\mathbf{v}_1 - \mathbf{v}_0\right\} \\ &= \frac{1}{\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}}\,(\mathbf{v}_0 \times \mathbf{v}_1) \times \mathbf{v}_1 \end{aligned}$$

The angle γ between $\mathbf{p}_1$ and $\mathbf{s}_1$ is found as follows:

$$\begin{aligned} \cos\gamma &= \mathbf{p}_1 \cdot \mathbf{s}_1 \\ &= \frac{1}{\sqrt{1-(v_z^1)^2}\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}}\left\{(\mathbf{v}_0 \cdot \mathbf{v}_1)\,(\hat{\mathbf{z}}\cdot\mathbf{v}_1) - \hat{\mathbf{z}}\cdot\mathbf{v}_0\right\} \\ &= \frac{v_z^1\,(\mathbf{v}_0\cdot\mathbf{v}_1) - v_z^0}{\sqrt{1-(v_z^1)^2}\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}} \end{aligned}$$

Since both $\mathbf{s}_1$ and $\mathbf{p}_1$ are perpendicular to $\mathbf{v}_1$, we obtain the following relation:

$$\begin{aligned} (\sin\gamma)\mathbf{v}_1 &= \mathbf{p}_1 \times \mathbf{s}_1 \\ &= \frac{1}{\sqrt{1-(v_z^1)^2}\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}}\left[(\mathbf{v}_1 \times \hat{\mathbf{z}}) \times \mathbf{v}_1\right] \times \left[(\mathbf{v}_0 \times \mathbf{v}_1) \times \mathbf{v}_1\right] \end{aligned}$$

Using the identity $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D}))\,\mathbf{C} - (\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}))\,\mathbf{D}$, we find

$$\begin{aligned} \left[(\mathbf{v}_1 \times \hat{\mathbf{z}}) \times \mathbf{v}_1\right] \times \left[(\mathbf{v}_0 \times \mathbf{v}_1) \times \mathbf{v}_1\right] &= -\left\{(\mathbf{v}_1\times\hat{\mathbf{z}})\cdot\left[\mathbf{v}_1 \times (\mathbf{v}_0\times\mathbf{v}_1)\right]\right\}\mathbf{v}_1 \\ &= -\left\{(\mathbf{v}_1\times\hat{\mathbf{z}})\cdot\left[\mathbf{v}_0 - (\mathbf{v}_0\cdot\mathbf{v}_1)\,\mathbf{v}_1\right]\right\}\mathbf{v}_1 \\ &= -\left[(\mathbf{v}_1\times\hat{\mathbf{z}})\cdot\mathbf{v}_0\right]\mathbf{v}_1 \\ &= -\left[\hat{\mathbf{z}}\cdot(\mathbf{v}_0\times\mathbf{v}_1)\right]\mathbf{v}_1. \end{aligned}$$

Here, we used the identity $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$. We finally obtain

$$\begin{aligned} \sin\gamma &= \frac{-\hat{\mathbf{z}}\cdot(\mathbf{v}_0\times\mathbf{v}_1)}{\sqrt{1-(v_z^1)^2}\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}} \\ &= \frac{v_y^0 v_x^1 - v_x^0 v_y^1}{\sqrt{1-(v_z^1)^2}\sqrt{1-(\mathbf{v}_0\cdot\mathbf{v}_1)^2}} \end{aligned}$$

3.4 Stokes parameters in Monte Carlo simulation

In Monte Carlo simulation, the scattered photon is not distributed over the entire solid angle but, instead, deflected into a single direction and attenuated by a factor equal to the albedo. The Stokes vector after scattering will be

$$\tilde{\mathbf{S}}_1 = \frac{aI_0}{I_1}\mathbf{S}_1 = \frac{aI_0}{I_1}\mathbf{L}(\gamma)\mathbf{R}(\theta)\mathbf{L}(-\phi)\mathbf{S}_0,$$

or

$$\begin{pmatrix} \tilde{I}_1 \\ \tilde{Q}_1 \\ \tilde{U}_1 \\ \tilde{V}_1 \end{pmatrix} = \frac{aI_0}{I_1}\begin{pmatrix} I_1 \\ Q_1 \\ U_1 \\ V_1 \end{pmatrix}.$$