Astrophysics

Lecture 07
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Four-velocity

The (infinitesimally small) difference between the coordinates of two events is also a four-vector. Dividing by the proper time yields a four-vector, the four-velocity:

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \longrightarrow \quad U^0 = \frac{cdt}{d\tau} = c\gamma_u$$

$$U^i = \frac{dx^i}{d\tau} = \gamma_u u^i$$

$$\text{or} \quad \boxed{\vec{U} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}} \quad \text{where} \quad \gamma_u \equiv \left(1 - u^2/c^2\right)^{-1/2}$$

$$u \equiv \left|\frac{d\mathbf{x}}{dt}\right|$$

$$\boxed{d\tau = dt/\gamma}$$

length of the four-velocity : $\quad \boxed{\vec{U} \cdot \vec{U} = U^\mu U_\mu = -\left(\gamma_u c\right)^2 + \left(\gamma_u \mathbf{u}\right)^2 = -c^2}$

Transformation of the four-velocity:

$$U'^0 = \gamma\left(U^0 - \beta U^1\right)$$
$$U'^1 = \gamma\left(-\beta U^0 + U^1\right)$$
$$U'^2 = U^2$$
$$U'^3 = U^3$$

$$\longrightarrow \quad \gamma_{u'}c = \gamma\left(c\gamma_u - \beta\gamma_u u^1\right)$$
$$\gamma_{u'}u'^1 = \gamma\left(-\beta c\gamma_u + \gamma_u u^1\right)$$
$$\gamma_{u'}u'^2 = \gamma_u u^2$$
$$\gamma_{u'}u'^3 = \gamma_u u^3$$

$\longrightarrow$

The first two equations become:

$$\gamma_{u'} = \gamma\gamma_u\left(1 - vu'/c^2\right)$$
$$\gamma_{u'}u'^1 = \gamma\gamma_u\left(u^1 - v\right)$$

Note: γ denotes the factor for the relative velocity between two frames. γ_u and $\gamma_{u'}$ are the factors for a velocity vector measured in K and K', respectively.

$\longrightarrow$

velocity component: $\quad \boxed{\begin{array}{c} u'^1 = \dfrac{u^1 - v}{1 - vu^1/c^2} \\[2ex] \gamma_{u'} = \gamma\gamma_u\left(1 - \dfrac{vu^1}{c^2}\right) \end{array}}$

speed:

This is the previously derived formula.

This is the transform for speed.

Here, $u^1 = u\cos\theta$ and $u'^1 = u'\cos\theta'$

Momentum and Energy

- Four-momentum of a particle with a mass m_0 is defined by

$$P^\mu \equiv m_0 U^\mu$$

$$P^0 = m_0 c \gamma_v$$

$$P^i = \gamma_v m_0 \mathbf{v}$$

- In the nonrelativistic limit,

$$P^0 c = m_0 c^2 \gamma = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = m_0 c^2 + \frac{1}{2} m_0 v^2 + \cdots$$

Therefore, we interpret $E \equiv P^0 c = \gamma_v m_0 c^2$ as the total energy of the particle.

The quantity $m_0 c^2$ is interpreted as the rest energy of the particle.

Then, $\boxed{\mathbf{p} \equiv \gamma_v m_0 \mathbf{v}, \quad P^\mu = (E/c, \ \mathbf{p})}$ Here, $\mathbf{p}$ is the spatial component of the four-momentum.

Since $\vec{U}^2 = -c^2$, we obtain $\vec{P}^2 = -m_0^2 c^2$. Comparing with $\vec{P}^2 = -\frac{E^2}{c^2} + |\mathbf{p}|^2$, we obtain

$$\boxed{E^2 = m_0^2 c^4 + c^2 |\mathbf{p}|^2}$$

- Photons are massless, but we can still define

$$\boxed{P^\mu = (E/c, \ \mathbf{p}), \quad E = |\mathbf{p}| c \quad \rightarrow \quad \vec{P}^2 = 0} \quad \text{for photons}$$

Wavenumber vector and frequency

- Quantum relations:

$$E = h\nu = \hbar\omega$$
$$p = E/c = \hbar k$$

$$\begin{pmatrix} \omega & = 2\pi\nu \\ k & = 2\pi/\lambda \end{pmatrix}$$

We can define four wavenumber vector:

$$\vec{k} = \frac{1}{\hbar}\vec{P} = \left(\frac{\omega}{c}, \ \mathbf{k}\right)$$

Note that it's a null vector:
$$\vec{k} \cdot \vec{k} = |\mathbf{k}|^2 - \omega^2/c^2 = 0$$

Then, we obtain an invariant:

$$\vec{k} \cdot \vec{x} = k_\mu x^\mu = \mathbf{k} \cdot \mathbf{x} - \omega t$$

Therefore, **the phase of the plane wave is an invariant.**

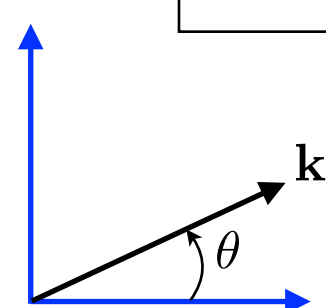$$e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} = e^{i\left(\mathbf{k}'\cdot\mathbf{x}' - \omega t\right)}$$

- Transform of $\vec{k}$ gives the Doppler shift formula.

$$k'^0 = \gamma\left(k^0 - \beta k^1\right)$$
$$k'^1 = \gamma\left(-\beta k^0 + k^1\right)$$
$$k'^2 = k^2$$
$$k'^3 = k^3$$

$$\longrightarrow \quad \boxed{\omega' = \gamma\left(\omega - \beta c k^1\right) = \omega\gamma\left(1 - \frac{v}{c}\cos\theta\right)}$$

$$k^1 = (\omega/c)\cos\theta$$

* Tensor Analysis *

- Definition:

  zeroth-rank tensor : Lorentz invariant (scalar)

  first-rank tensor : four-vector

  second-rank tensor:

$$s' = s$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\tau} T^{\sigma\tau}$$

- Covariant components and mixed components:

$$T_{\mu\nu} = \eta_{\mu\sigma} \eta_{\nu\tau} T^{\sigma\tau} \qquad T^{\mu}{}_{\nu} = \eta_{\nu\tau} T^{\mu\tau} \qquad T_{\mu}{}^{\nu} = \eta_{\mu\sigma} T^{\sigma\nu}$$

- Transformation rules:

$$\begin{aligned} T'_{\mu\nu} &= \eta_{\mu\alpha} \eta_{\nu\beta} T'^{\alpha\beta} \\ &= \eta_{\mu\alpha} \eta_{\nu\beta} \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} T^{\gamma\delta} \\ &= \eta_{\mu\alpha} \eta_{\nu\beta} \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} \eta^{\gamma\sigma} \eta^{\delta\tau} T_{\sigma\tau} \\ &= \widetilde{\Lambda}_{\mu}{}^{\sigma} \widetilde{\Lambda}_{\nu}{}^{\tau} T_{\sigma\tau} \end{aligned}$$

$$\begin{aligned} T'^{\mu}{}_{\nu} &= \eta_{\nu\alpha} T'^{\mu\alpha} \\ &= \eta_{\nu\alpha} \Lambda^{\mu}{}_{\sigma} \Lambda^{\alpha}{}_{\delta} T^{\sigma\delta} \\ &= \eta_{\nu\alpha} \Lambda^{\mu}{}_{\sigma} \Lambda^{\alpha}{}_{\delta} \eta^{\delta\tau} T^{\sigma}{}_{\tau} \\ &= \Lambda^{\mu}{}_{\sigma} \widetilde{\Lambda}_{\nu}{}^{\tau} T^{\sigma}{}_{\tau} \end{aligned}$$

$$\begin{aligned} T'_{\mu}{}^{\nu} &= \eta_{\mu\alpha} T'^{\alpha\nu} \\ &= \eta_{\mu\alpha} \Lambda^{\alpha}{}_{\beta} \Lambda^{\nu}{}_{\tau} T^{\beta\tau} \\ &= \eta_{\mu\alpha} \Lambda^{\alpha}{}_{\beta} \Lambda^{\nu}{}_{\tau} \eta^{\beta\sigma} T_{\sigma}{}^{\tau} \\ &= \widetilde{\Lambda}_{\mu}{}^{\beta} \Lambda^{\nu}{}_{\tau} T_{\sigma}{}^{\tau} \end{aligned}$$

- Symmetric tensor = a tensor that is invariant under a permutation of its indices.

$$T^{\mu\nu} = T^{\nu\mu}$$

- Antisymmetric tensor : if it alternates sign when any two indices of the subset are interchanged.

$$T^{\mu\nu} = -T^{\nu\mu}$$

- Examples of the second-rank tensors

A product of two vectors: $A^\mu B^\nu$

$$A'^\mu B'^\nu = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau A^\sigma B^\tau$$

The Minkowski metric:

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Kronecker-delta:

$$\delta^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **Higher-rank tensors**

  - Addition: $A^\mu + B^\mu$, $F^{\mu\nu} + G^{\mu\nu}$

  - Multiplication: $A^\mu B^\nu$, $F^{\mu\nu} G_{\sigma\tau}$

  - Raising and Lowering Indices: The metric can be used to change contravariant indices into covariant ones, and vice versa, by the processes of raising and lowering.

  - Contraction: $\quad A^\mu B_\nu \quad \rightarrow \quad A^\mu B_\mu \quad$ scalar

    $\qquad\qquad T^{\mu\nu}{}_\sigma \quad \rightarrow \quad T^{\mu\nu}{}_\nu \qquad$ vector

    $$T'^{\mu\nu}{}_\nu = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \widetilde{\Lambda}_\nu{}^\tau T^{\alpha\beta}{}_\tau = \Lambda^\mu{}_\alpha \delta^\tau{}_\beta T^{\alpha\beta}{}_\tau = \Lambda^\mu{}_\alpha T^{\alpha\beta}{}_\beta$$

  - Gradients of Tensor Fields: A tensor field is a tensor that is a function of the spacetime coordinates in Cartesian coordinate systems. The <u>gradient operation $\partial/\partial x^\mu \equiv \partial_\mu$</u> acting on such a field produces a tensor field of on higher rank with <u>μ as a new covariant index</u>.

    $$\lambda \quad \rightarrow \quad \frac{\partial \lambda}{\partial x^\mu} \equiv \partial_\mu \lambda \equiv \lambda,_\mu \quad \text{vector (gradient)} \qquad A^\mu \quad \rightarrow \quad \frac{\partial A^\mu}{\partial x^\mu} \equiv \partial_\mu A^\mu \equiv A^\mu,_\mu \quad \text{scalar (divergence)}$$

  - **Invariance of form or Lorentz covariance or covariance**: A fundamental property of a tensor equation is that if it is true in one Lorentz frame, then it is true in all Lorentz frames. Covariance plays a powerful role in helping decide what the proper equations of physics are.

[Covariance of Electromagnetic Phenomena]

- Equation of charge conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

The above equation can be written as a tensor equation,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\partial t}, \ \frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ \frac{\partial}{\partial z} \right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(-\frac{\partial}{c\partial t}, \ \frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ \frac{\partial}{\partial z} \right)$$

$$\boxed{\frac{\partial j^\mu}{\partial x^\mu} = 0,} \quad j^\mu{}_{,\,\mu} = 0 \quad \text{or} \quad \partial_\mu j^\mu = 0$$

if the **four-current** is defined by

$$j^\mu = \begin{pmatrix} \rho c \\ \mathbf{j} \end{pmatrix} \qquad\qquad j_\mu = \begin{pmatrix} -\rho c \\ \mathbf{j} \end{pmatrix}$$

- Note that the Jacobian (determinant) of the transformation from x_μ to x'_μ is simply the determinant of Λ, which is unity. Therefore, the **four-volume element is an invariant.**

$$\boxed{dx'_0 dx'_1 dx'_2 dx'_3 = \det \Lambda \, dx_0 dx_1 dx_2 dx_3 = dx_0 dx_1 dx_2 dx_3}$$

Since ρ is the zeroth component of the four-current, **the charge element within a three-volume element is an invariant.**

$$\boxed{de = \rho \, dx_1 dx_2 dx_3} \qquad \boxed{de' = de}$$

It is also an empirical fact that e is invariant.

- The set of vector and scalar wave equations in the Lorentz gauge is

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{j}$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{4\pi}{c} (\rho c)$$

If we define the **four-potential**

$$A^\mu = \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix}, \quad A_\mu = \begin{pmatrix} -\phi \\ \mathbf{A} \end{pmatrix},$$

then the wave equations can be written as the tensor equations

$$\frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} = -\frac{4\pi}{c} j^\mu, \quad \partial_\nu \partial^\nu A^\mu = -\frac{4\pi}{c} j^\mu, \quad A^{\mu,\nu}_{,\nu} = -\frac{4\pi}{c} j^\mu$$

d'Alembertian operator: $\quad \Box \equiv \dfrac{\partial^2}{\partial x^\nu \partial x_\nu} \quad \rightarrow \quad \boxed{\Box A^\mu = -\dfrac{4\pi}{c} j^\mu}$

- The Lorentz gauge should be preserved under Lorentz transformations.

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \quad \rightarrow \quad \boxed{\frac{\partial A^\mu}{\partial x^\mu} = 0} \text{ or } \quad A^\mu_{\ \mu} = 0$$

- **Electromagnetic field tensor**:

  The fields are expressed in terms of the potentials as

  $$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

  $$\mathbf{B} = \nabla \times \mathbf{A}$$

  The x components of the electric and magnetic fields are explicitly

  $$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial\phi}{\partial x} = \partial^0 A^1 - \partial^1 A^0$$

  $$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \partial^2 A^3 - \partial^3 A^2$$

  These equations imply that the electric and magnetic fields, six components in all, are the elements of a **second-rank, antisymmetric field-strength tensor**, because a rank two antisymmetric tensor has exactly six independent components.

  $$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \longrightarrow$$

  $$F^{0i} = E_i$$

  $$F^{i0} = -E_i$$

  $$F^{12} = -F^{21} = B_3, \quad \cdots$$

covariant field-strength tensor

$$F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}$$

$$\longrightarrow$$

$$F_{0i} = \eta_{0\alpha}\eta_{i\beta}F^{\alpha\beta} = -F^{0i}$$

$$F_{i0} = \eta_{i\alpha}\eta_{0\beta}F^{\alpha\beta} = -F^{i0}$$

$$F_{ij} = \eta_{i\alpha}\eta_{j\beta}F^{\alpha\beta} = F^{ij}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$\longrightarrow$$

$$F_{0i} = -E_i$$

$$F_{i0} = E_i$$

$$F_{12} = -F_{21} = B_3, \ \cdots$$

- The two Maxwell equations containing sources (inhomogeneous equations):

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}$$

$$\longrightarrow$$

$$\sum_{i=1}^{3} \partial_i E_i = \frac{4\pi}{c}j^0$$

$$\partial_2 B_3 - \partial_3 B_2 - \partial_0 E_1 = \frac{4\pi}{c}j^1$$

$$\longrightarrow$$

$$-\sum_{i=1}^{3} \partial_i F^{i0} = \frac{4\pi}{c}j^0$$

$$-\partial_0 F^{01} - \partial_2 F^{21} - \partial_3 F^{31} = \frac{4\pi}{c}j^1$$

$$\boxed{\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c}j^\nu \ \text{ or } \ \partial_\nu F^{\mu\nu} = \frac{4\pi}{c}j^\mu}$$

$$\partial^\mu F_{\mu\nu} = -\frac{4\pi}{c}j_\nu \ \text{ or } \ \partial^\nu F_{\mu\nu} = \frac{4\pi}{c}j_\mu$$

- The conservation of charge easily follows from the above equation and the asymmetric property.

$$\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu \partial_\mu \left(-F^{\nu\mu}\right) = -\partial_\mu \partial_\nu F^{\mu\nu} \qquad \therefore \boxed{\begin{array}{l} \partial_\mu \partial_\nu F^{\mu\nu} = 0 \\[2em] \partial_\nu j^\nu = -\dfrac{c}{4\pi} \partial_\nu \partial_\mu F^{\mu\nu} = 0 \end{array}}$$

index exchange

- The "internal" Maxwell equations (homogeneous equations):

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\longrightarrow$$

$$\sum_{i=1}^{3} \partial_i B_i = 0$$
$$\partial_2 E_3 - \partial_3 E_2 + \partial_0 B_1 = 0$$

$$\longrightarrow$$

$$\partial_1 F^{23} + \partial_2 F^{31} + \partial_3 F^{12} = 0$$
$$\partial_2 F^{30} + \partial_3 F^{20} + \partial_0 F^{23} = 0$$

$$\boxed{\partial_\mu F^{\nu\sigma} + \partial_\nu F^{\sigma\mu} + \partial_\sigma F^{\mu\nu} = 0} \quad \text{or} \quad \partial^\mu F_{\nu\sigma} + \partial^\nu F_{\sigma\mu} + \partial^\sigma F_{\mu\nu} = 0$$

The equation can be written concisely as $F^{[\mu\nu,\sigma]} = 0$ or $F_{[\mu\nu,\sigma]} = 0$, where [ ] around indices denote all permutations of indices, with even permutation contributing with a positive sign and odd permutation with a negative sign, for example,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{[\nu,\mu]}$$

- Transformation of Electromagnetic Fields

- Since $F^{\mu\nu}$ is a second-rank tensor, its components transform in the usual way:

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} F^{\alpha\beta} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

For a pure boost along the *x*-axis:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow$$

$$E'_x = F'^{01} = \Lambda^0{}_0 \Lambda^1{}_1 F^{01} + \Lambda^0{}_1 \Lambda^1{}_0 F^{10} = \gamma^2 E_x - \beta^2 \gamma^2 E_x = E_x$$

$$E'_y = F'^{02} = \Lambda^0{}_0 \Lambda^2{}_2 F^{02} + \Lambda^0{}_1 \Lambda^2{}_2 F^{12} = \gamma E_y - \beta\gamma B_z$$

$$E'_z = F'^{03} = \Lambda^0{}_0 \Lambda^3{}_3 F^{03} + \Lambda^0{}_1 \Lambda^3{}_3 F^{13} = \gamma E_z + \beta\gamma B_y$$

$$B'_x = F'^{23} = \Lambda^2{}_2 \Lambda^3{}_3 F^{23} = B_x$$

$$B'_y = F'^{31} = \Lambda^3{}_3 \left(\Lambda^1{}_0 F^{30} - \Lambda^1{}_1 F^{31} \right) = \beta\gamma E_z + \gamma B_y$$

$$B'_z = F'^{12} = \Lambda^1{}_0 \Lambda^2{}_2 F^{03} + \Lambda^1{}_1 \Lambda^2{}_2 F^{12} = -\beta\gamma E_y + \gamma B_z$$

- In general,

$$
\begin{array}{ll}
\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} & \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \\
\mathbf{E}'_{\perp} = \gamma \left(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B} \right) & \mathbf{B}'_{\perp} = \gamma \left(\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E} \right)
\end{array}
$$

The concept of a pure electric or pure magnetic is not Lorentz invariant.

- Lorenz invariants:

  - The dot product of F with itself or "square" of F is

$$F^{\mu\nu}F_{\mu\nu} = \sum_{i=1}^{3} F^{0i}F_{0i} + \sum_{i=1}^{3} F^{i0}F_{i0} + \sum_{i\neq j} F^{ij}F_{ij}$$
$$= 2\left(\mathbf{B}^2 - \mathbf{E}^2\right)$$

Therefore, $\mathbf{B}^2 - \mathbf{E}^2$ is invariant under Lorentz transformations.

  - The determinant of F is invariant.

$$\det F = \left(\mathbf{E}\cdot\mathbf{B}\right)^2$$

Thus $\mathbf{E}\cdot\mathbf{B}$ is also an invariant.

  - The determinant of any second-rank tensor is scalar, since

$$\det A_{\alpha\beta} = \det \tilde{\Lambda}_\alpha{}^\mu \tilde{\Lambda}_\beta{}^\nu A'_{\mu\nu}$$
$$= \left(\det \tilde{\Lambda}\right)^2 \det A'_{\mu\nu}$$
$$= \det A'_{\mu\nu}$$

[Relativistic Mechanics and the Lorentz Four-Force]

- We can define a **four-acceleration** a^μ in exactly the same way as we obtained the four-velocity.

$$a^\mu \equiv \frac{dU^\mu}{d\tau}$$

Note that the four-acceleration and four-velocity are orthogonal:

$$\vec{a} \cdot \vec{U} \equiv \frac{dU^\mu}{d\tau} U_\mu = \frac{1}{2} \frac{d}{d\tau} \left(U^\mu U_\mu \right) = \frac{1}{2} \frac{d}{d\tau} \left(-c^2 \right) = 0$$

- We can also define the **four-force** F^μ from the Lorentz force, so as to obtain a relativistic form of Newton's equation.

$$F^\mu \equiv m_0 a^\mu = \frac{dP^\mu}{d\tau}$$

$$\vec{F} = \frac{d\vec{P}}{d\tau} = \gamma \frac{d\vec{P}}{dt} = \gamma \left(\frac{1}{c} \frac{dE}{dt}, \ \frac{d\mathbf{p}}{dt} \right)$$

- Since $\mathbf{F}_{\text{Lorentz}} = q \left[\mathbf{E} + \frac{1}{c} \left(\mathbf{v} \times \mathbf{B} \right) \right]$, the **Lorentz four-force** should involve (1) the electromagnetic field tensor and (2) the four-velocity and should also be (3) a four-vector and (4) proportional to the charge of the particle. Therefore, the simplest possibility is

$$F^\mu_{\text{Lorentz}} = \frac{q}{c} F^{\mu\nu} U_\nu$$

- Let's check to see if it is indeed what we want.

$$\boxed{\frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v}}$$

$$F^0_{\text{Lorentz}} = \frac{q}{c} F^{0\nu} U_\nu = \frac{q}{c} \sum_{i=1}^{3} E_i \gamma v_i = \frac{q}{c} \gamma \left(\mathbf{E} \cdot \mathbf{v} \right) \longrightarrow \frac{dE}{dt} = q\mathbf{E} \cdot \mathbf{v} \quad \text{: conservation of energy}$$

The rate of change of particle energy is the mechanical work done on the particle by the field.

$$F^1_{\text{Lorentz}} = \frac{q}{c} F^{1\nu} U_\nu = \frac{q}{c} \left(F^{10} (-\gamma c) + F^{12} \gamma v_2 + F^{13} \gamma v_3 \right)$$
$$= \frac{q}{c} \gamma \left(E_1 c + B_3 v_2 - B_2 v_3 \right)$$

$$\longrightarrow \frac{d\mathbf{p}}{dt} = q \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right)$$

Therefore, we obtained the desired expression for the four-Lorentz force.

- Note that **the four-force is always orthogonal to the four-velocity**:

$$\vec{F} \cdot \vec{U} = m_0 \left(\vec{a} \cdot \vec{U} \right) = 0$$

It implies that **every four-force must have some velocity dependence**, although this dependence might become negligible in the nonrelativistic limit.

For the Lorentz four-force, in particular, we find

$$\vec{F}_{\text{Lorentz}} \cdot \vec{U} = \frac{q}{c} F^{\mu\nu} U_\mu U_\nu = 0$$

because $F^{\mu\nu}$ is antisymmetric and $U_\mu U_\nu$ is symmetric.

Mathematical Formulae

- **Gamma function**

$$\Gamma(x) \equiv \int_0^\infty t^{x-1} e^{-t} dt, \quad \Gamma(x) = (x-1)! = (x-1)\Gamma(x-2), \quad \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

- **Euler-Mascheroni constant**

$$\gamma \equiv \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \ln(n)\right) = -\int_0^\infty e^{-x}\ln x\, dx = 0.577215664901532$$

- **Modified Bessel function of the second kind**

$$K_n(x) \equiv \frac{\Gamma(n+1/2)(2x)^n}{\sqrt{\pi}} \int_0^\infty \frac{\cos t}{(t^2+x^2)^{n+1/2}} dt$$

(1) $\;0 < x < \sqrt{n+1}$

$$K_n(x) \approx \begin{cases} -\ln\left(\dfrac{x}{2}\right) - \gamma & \text{if } n = 0 \\[2ex] \dfrac{\Gamma(n)}{2}\left(\dfrac{2}{x}\right)^n & \text{if } n > 0 \end{cases}$$

(2) $\;x \gg \left|n^2 - 1/4\right|$

$$K_n(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}\left[1 + \frac{(4n^2-1)}{8x}\right]$$

Recurrence formulae

$$K_{n-1}(x) - K_{n+1}(x) = -\frac{2n}{x}K_n(x)$$

$$K_{n-1}(x) + K_{n+1}(x) = -2K_n'(x)$$

Integral formula

$$\int x K_n^2(x)dx = \frac{1}{2}x^2\left[K_n^2(x) - K_{n-1}(x)K_{n+1}(x)\right]$$

$$= -xK_{n-1}(x)K_n(x) +$$

$$\frac{1}{2}x^2\left[K_n^2(x) - K_{n-1}^2(x)\right]$$

[Fields of a Uniformly Moving Charge]

- Recall that the electromagnetic field from a single moving charge is given by

$$
\mathbf{E}(\mathbf{r},t) = q \underbrace{\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2} \right]_{\text{ret}}}_{\text{velocity field}} + \frac{q}{c} \underbrace{\left[\frac{\mathbf{n}}{\kappa^3 R} \times \left\{ (\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right\} \right]_{\text{ret}}}_{\text{acceleration field}}
$$

$$
\mathbf{B}(\mathbf{r},t) = \left[\mathbf{n} \times \mathbf{E}(\mathbf{r},t) \right]_{\text{ret}}
$$

- The velocity field is caused by a motion of the charge relative to the observer frame. Hence, the velocity field would be obtained by Lorentz transforming the static electromagnetic field (Coulomb field).

- Let's find the fields of a charge moving with constant velocity v along the x axis. In the rest frame K' of the particle the fields are given by
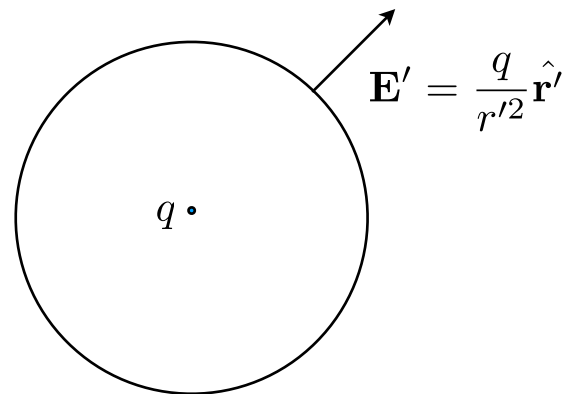
$\mathbf{E}' = \frac{q}{r'^2} \hat{\mathbf{r}}'$ **Coulomb field in K'**

$$\mathbf{E}' = \left(E_x', \, E_y', \, E_z' \right) = \frac{q}{r'^3} \left(x', \, y', \, z' \right) \quad \text{where} \quad r' = \left(x'^2 + y'^2 + z'^2 \right)^{1/2}$$

$$\mathbf{B}' = (0, \, 0, \, 0)$$

Inverse transformation of the previous one:

$$E_x = \frac{q x'}{r'^3} \qquad B_x = 0$$

$$\mathbf{E}_\parallel = \mathbf{E}_\parallel' \qquad \mathbf{B}_\parallel = \mathbf{B}_\parallel'$$

$$\mathbf{E}_\perp = \gamma \left(\mathbf{E}_\perp' - \boldsymbol{\beta} \times \mathbf{B}' \right) \qquad \mathbf{B}_\perp = \gamma \left(\mathbf{B}_\perp' + \boldsymbol{\beta} \times \mathbf{E}' \right)$$

$$\longrightarrow \qquad E_y = \gamma \frac{q y'}{r'^3} \qquad B_y = -\gamma \beta \frac{q z'}{r'^3}$$

$$E_z = \gamma \frac{q z'}{r'^3} \qquad B_z = \gamma \beta \frac{q y'}{r'^3}$$

Now, we need to express these equations in terms of the unprimed coordinates. Since $x' = \gamma(x - vt)$, $y' = y$, $z' = z$, we obtain

$$E_x = \gamma \frac{q \, (x - vt)}{r^3} \qquad B_x = 0$$

$$E_y = \gamma \frac{q y}{r^3} \qquad B_y = -\gamma \beta \frac{q z}{r^3}$$

$$E_z = \gamma \frac{q z}{r^3} \qquad B_z = \gamma \beta \frac{q y}{r^3}$$

where $\quad r = \left[\gamma^2 \left(x - vt \right)^2 + y^2 + z^2 \right]^2 \quad (= r')$

Question:

Is this equivalent to the velocity field given by the Lienard-Wiechert potentials?

- Velocity field from the retarded potential

- For simplicity, assume $z = 0$.

$$\mathbf{E} = (E_x,\, E_y,\, E_z) = \gamma \frac{q}{r^3}\, (x - vt,\, y,\, z)$$

$$= \gamma \frac{q}{\left(\gamma^2 \bar{x}^2 + y^2\right)^{3/2}}\, (\bar{x},\, y,\, 0) \qquad \text{where } \bar{x} \equiv x - vt$$

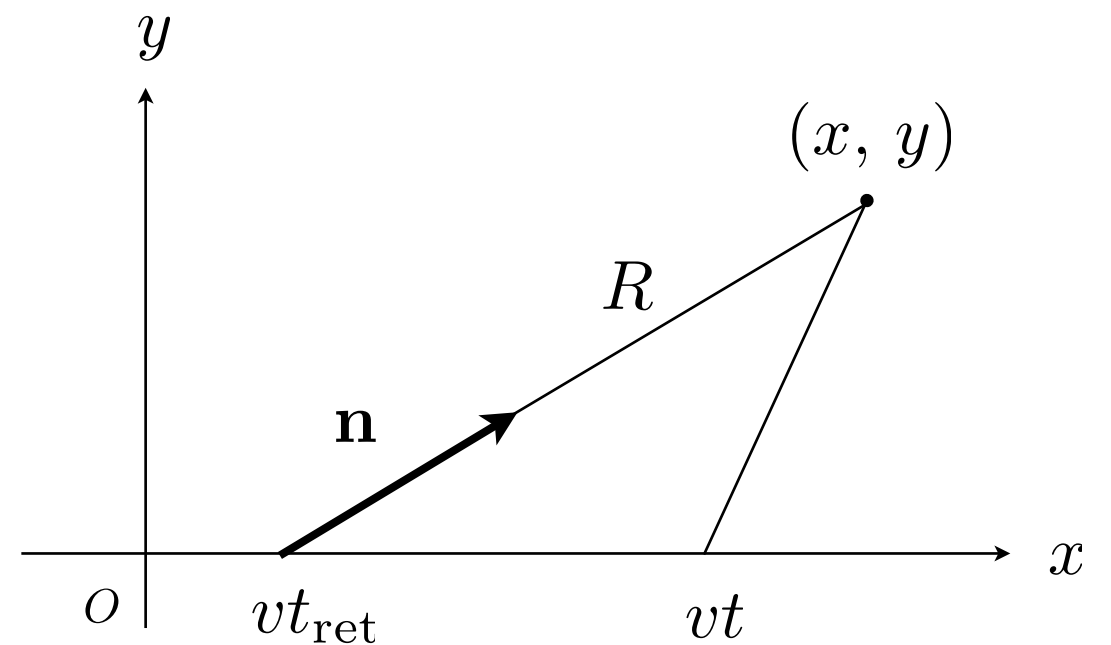Let us first find where the **retarded position** of the particle is.

$$t_{\mathrm{ret}} \equiv t - R/c$$

$$R^2 = (x - vt_{\mathrm{ret}})^2 + y^2 = (\bar{x} + \beta R)^2 + y^2$$

$R \longrightarrow$
$$\left(1 - \beta^2\right) R^2 - 2\bar{x}\beta R - \bar{x}^2 - y^2 = 0$$
$$R^2 - 2\bar{x}\gamma^2 \beta R - \gamma^2 \left(\bar{x}^2 + y^2\right) = 0$$

$$R = \gamma^2 \beta \bar{x} \pm \left[\gamma^4 \beta^2 \bar{x}^2 + \gamma^2 \left(\bar{x}^2 + y^2\right)\right]^{1/2}$$

$$= \gamma^2 \beta \bar{x} \pm \gamma \left[\gamma^2 \beta^2 \bar{x}^2 + \left(\bar{x}^2 + y^2\right)\right]^{1/2}$$

$$= \gamma^2 \beta \bar{x} \pm \gamma \left(\gamma^2 \bar{x}^2 + y^2\right)^{1/2}$$

$$\text{positive solution} \quad \rightarrow \quad R = \gamma^2 \beta \bar{x} + \gamma \left(\gamma^2 \bar{x}^2 + y^2\right)^{1/2}$$

- Let us first find where the **retarded position** of the particle is.

(1)
$$\mathbf{n} = \frac{(x - vt + vR/c)\,\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{R}$$

$$= \frac{(\bar{x} + \beta R)}{R}\hat{\mathbf{x}} + \frac{y}{R}\hat{\mathbf{y}}$$

$$= \left(\frac{\bar{x}}{R} + \beta\right)\hat{\mathbf{x}} + \frac{y}{R}\hat{\mathbf{y}}$$

$$\boxed{\boldsymbol{\beta} = \beta\hat{\mathbf{x}}}$$

$$\mathbf{n} - \boldsymbol{\beta} = \frac{\bar{x}}{R}\hat{\mathbf{x}} + \frac{y}{R}\hat{\mathbf{y}}$$

$$\mathbf{E} = \gamma\frac{qR}{(\gamma^2\bar{x}^2 + y^2)^{3/2}}(\mathbf{n} - \boldsymbol{\beta})$$

$$\boxed{\mathbf{n} \cdot \boldsymbol{\beta} = \beta^2 + \frac{\bar{x}}{R}\beta}$$

(2)
$$(\gamma^2\bar{x}^2 + y^2)^{1/2} = \frac{R - \gamma^2\beta\bar{x}}{\gamma} = R\gamma\left(\frac{1}{\gamma^2} - \frac{\beta\bar{x}}{R}\right)$$

$$= R\gamma\left(1 - \beta^2 - \frac{\beta\bar{x}}{R}\right)$$

$$= R\gamma\left[1 - \beta\left(\frac{\bar{x}}{R} + \beta\right)\right]$$

$$= R\gamma(1 - \mathbf{n} \cdot \boldsymbol{\beta}) = R\gamma\kappa$$

$$\boxed{R = \gamma^2\beta\bar{x} + \gamma\left(\gamma^2\bar{x}^2 + y^2\right)^{1/2}}$$

$$\therefore \boxed{\begin{aligned}\mathbf{E} &= q\frac{(\mathbf{n} - \boldsymbol{\beta})}{\gamma^2\kappa^3 R^2} \\ &= q\frac{(\mathbf{n} - \boldsymbol{\beta})(1 - \beta^2)}{\kappa^3 R^2}\end{aligned}}$$

This is identical to the velocity field component!

- Time-dependence of the electric field at a point

- For simplicity, let us choose the field point to be at $(0, b, 0)$. This involves no loss in generality. Then,

$$E_x = -\frac{q\gamma vt}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = -\frac{q}{b^2}\frac{\gamma vt/b}{(\gamma^2 v^2 t^2/b^2 + 1)^{3/2}}$$

$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = \frac{q\gamma}{b^2}\frac{1}{(\gamma^2 v^2 t^2/b^2 + 1)^{3/2}}$$
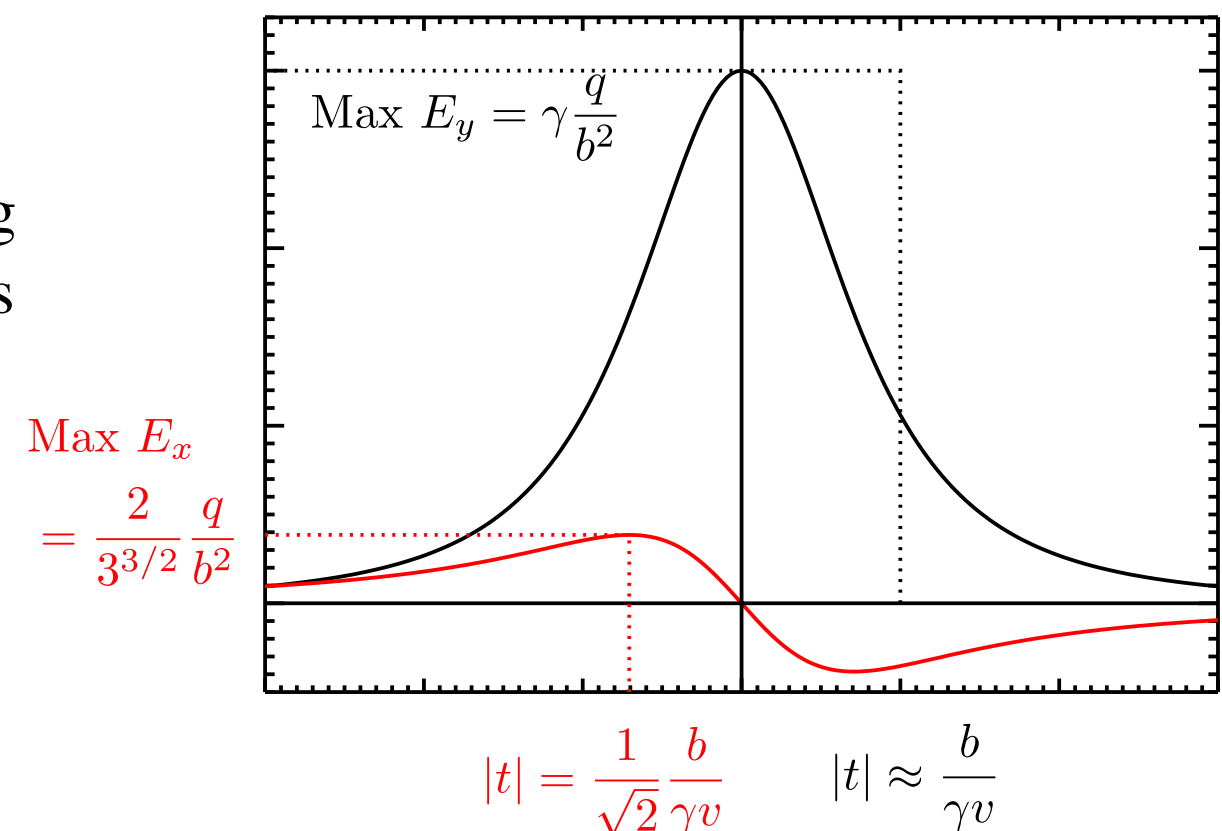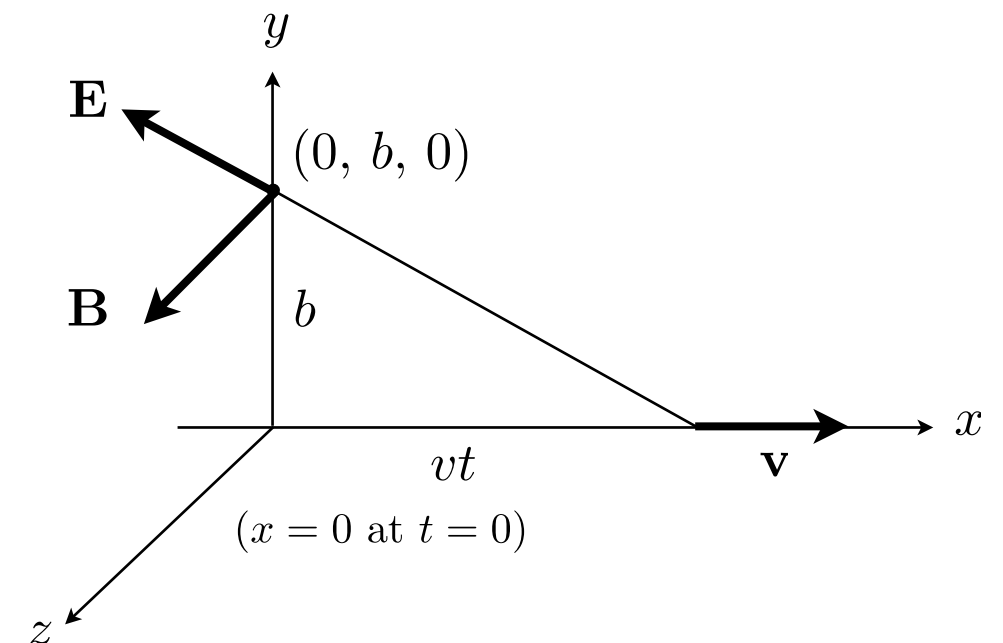
$$E_z = 0$$

$$B_x = 0, \quad B_y = 0, \quad B_z = \beta E_y$$

y

$\mathbf{E}$

$(0,\ b,\ 0)$

$\mathbf{B}$

b

vt $\mathbf{v}$ x

$(x = 0 \text{ at } t = 0)$

z

(1) The fields are strong only during a time interval of $|vt| \lesssim b/\gamma$, implying that the field of the moving charge are concentrated in the plane transverse to its motion into an angle of order $1/\gamma$.

$$\Delta\theta \approx v\Delta t/b \sim 1/\gamma$$

(2) As $\gamma \gg 1 \ \rightarrow \ |E_x|_{\max} \ll |E_y|_{\max}$

The field of a highly relativistic charge appears to be a pulse of radiation traveling in the same direction as the charge and confined to the transverse plane.

$\text{Max } E_y = \gamma\dfrac{q}{b^2}$

$\text{Max } E_x = \dfrac{2}{3^{3/2}}\dfrac{q}{b^2}$

$|t| = \dfrac{1}{\sqrt{2}}\dfrac{b}{\gamma v}$ $|t| \approx \dfrac{b}{\gamma v}$

see the animation and python code:
https://seoncafe.github.io/Teaching_files/2023b_astrophysics/ani_pulse.mp4
https://seoncafe.github.io/Teaching_files/2023b_astrophysics/plot_pulse.py

- Spectrum of the pulse

- Spectrum of this pulse is given by (if the x-axis component is ignored)

$$\hat{E}(\omega) = \frac{1}{2\pi} \int E_y(t) e^{i\omega t} dt$$

$$= \frac{q\gamma b}{2\pi} \int_{-\infty}^{\infty} \left(\gamma^2 v^2 t^2 + b^2\right)^{-3/2} e^{i\omega t} dt = \frac{q\gamma b}{2\pi} \int_{0}^{\infty} \left(\gamma^2 v^2 t^2 + b^2\right)^{-3/2} \left(e^{i\omega t} + e^{-i\omega t}\right) dt$$

$$= \frac{q\gamma b}{\pi} \int_{0}^{\infty} \left(\gamma^2 v^2 t^2 + b^2\right)^{-3/2} \cos\omega t \, dt$$

This integral can be done in terms of the modified Bessel function:

$$K_n(x) \equiv \frac{\Gamma(n+1/2)(2x)^n}{\sqrt{\pi}} \int_{0}^{\infty} \frac{\cos t}{(t^2+x^2)^{n+1/2}} dt \qquad \text{Gamma function:} \quad \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

$$\hat{E}(\omega) = \frac{q\gamma b}{\pi} \left(\frac{\gamma^2 v^2}{\omega^2}\right)^{-3/2} \frac{1}{\omega} \int_{0}^{\infty} \left(\omega^2 t^2 + \frac{b^2\omega^2}{\gamma^2 v^2}\right)^{-3/2} (\cos\omega t) \, d(\omega t)$$

$$= \frac{q}{\pi b v} \frac{b\omega}{\gamma v} K_1\left(\frac{b\omega}{\gamma v}\right)$$

Thus the spectrum is

$$\frac{dW}{dA d\omega} = c \left|\hat{E}(\omega)\right|^2 = \frac{q^2}{\pi^2 b^2 v^2} \left(\frac{b\omega}{\gamma v}\right)^2 K_1^2\left(\frac{b\omega}{\gamma v}\right)$$
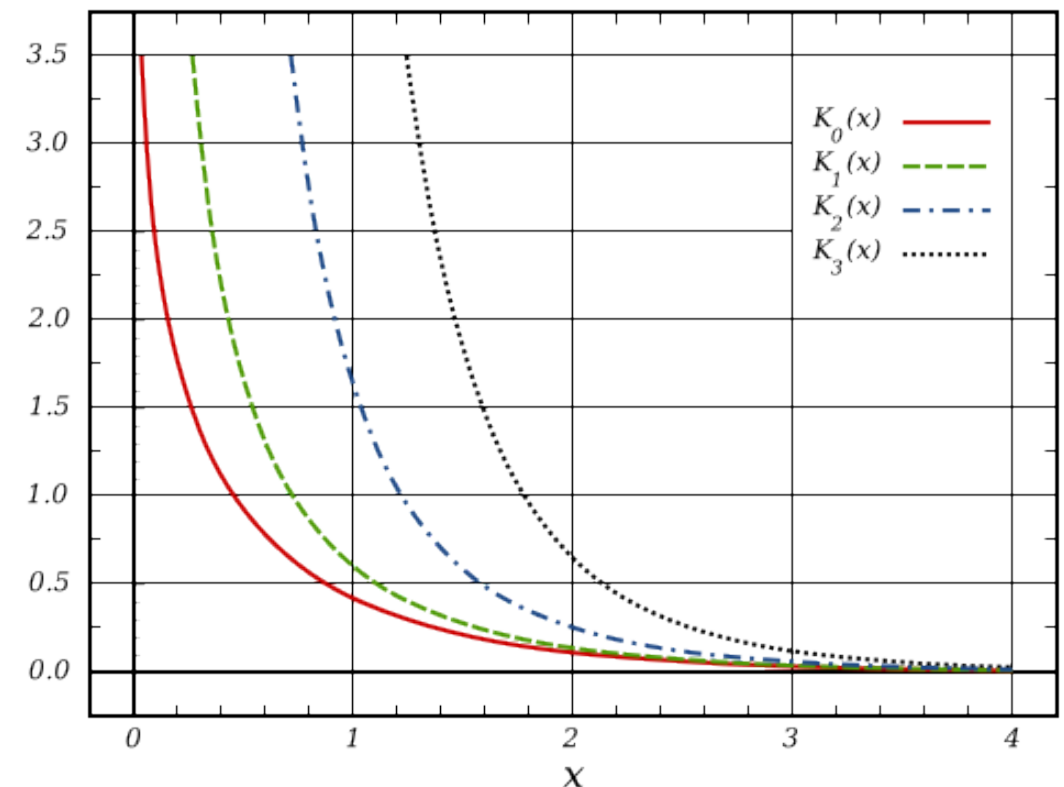
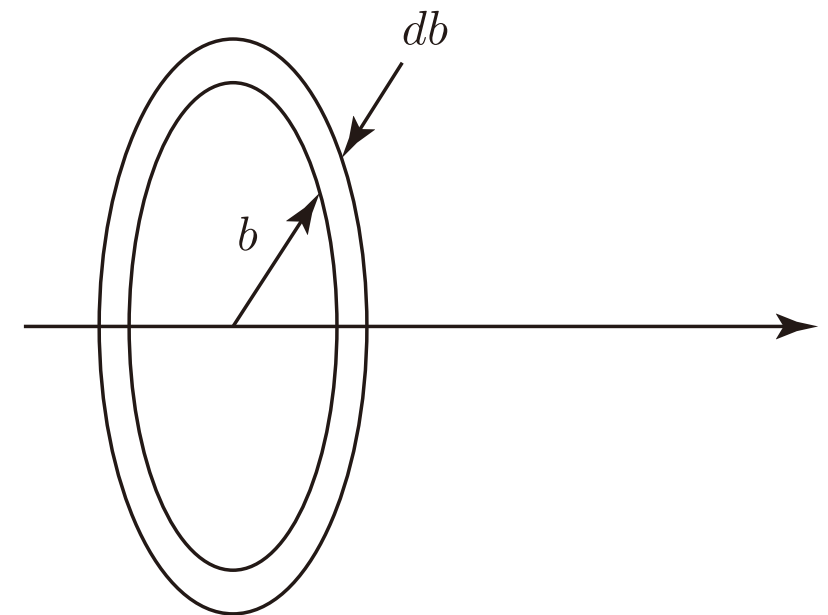- The spectrum starts cut off for $\omega > \gamma v/b$.

  This is predicted from the uncertainty principle, since the pulse is confined roughly to a time interval of order $\Delta t \approx b/\gamma v$.

$$\Delta\omega \sim \frac{1}{\Delta t} \sim \gamma v/b$$

- Total energy per unit frequency range is obtained by

$$\frac{dW}{d\omega} = 2\pi \int_{b_{\mathrm{min}}}^{b_{\mathrm{max}}} \frac{dW}{dAd\omega} b\,db$$

$$b_{\mathrm{max}} \to \infty, \quad b_{\mathrm{min}} \neq 0 \ (\text{a finite value})$$

The lower limit has been chosen not as zero but as some minimum distance b_{min}, such that the approximation of the field by means of classical electrodynamics and a point charge is valid.

$$\begin{aligned}\frac{dW}{d\omega} &= \frac{2q^2 c}{\pi v^2} \int_x^\infty y K_1^2(y)\,dy \\ &= \frac{2q^2 c}{\pi v^2} \left[x K_0(x) K_1(x) - \frac{1}{2} x^2 \left(K_1^2(x) - K_0^2(x) \right) \right]\end{aligned}$$

where $\quad y \equiv \dfrac{\omega b}{\gamma v}$ and $x \equiv \dfrac{\omega b_{\mathrm{min}}}{\gamma v}$

- Two limiting cases:

(1) $\omega \ll \dfrac{\gamma v}{b_{\min}}$ $(x \ll 1)$,

$$xK_0(x)K_1(x) - \frac{1}{2}x^2\left(K_1^2(x) - K_0^2(x)\right)$$

$$\approx x\left(-\ln(x/2) - \gamma\right)\frac{1}{x} - \frac{x^2}{2}\left[\frac{1}{x^2} - (\ln(x/2) + \gamma)^2\right]$$

$$\approx \ln\left[\frac{2}{x}e^{-(\gamma+1/2)}\right]$$

$$= \ln\left(\frac{0.68}{x}\right)$$

$\longrightarrow$

$$\boxed{\begin{array}{c} \omega \ll \dfrac{\gamma v}{b_{\min}} \\[2mm] \dfrac{dW}{d\omega} = \dfrac{2q^2c}{\pi v^2}\ln\left(0.68\dfrac{\gamma v}{\omega b_{\min}}\right) \end{array}}$$

(2) $\omega \gg \dfrac{\gamma v}{b_{\min}}$ $(x \gg 1)$,

$$xK_0(x)K_1(x) - \frac{1}{2}x^2\left(K_1^2(x) - K_0^2(x)\right)$$

$$\approx x\frac{\pi}{2x}e^{-2x} - \frac{1}{2}x^2\frac{\pi}{2x}e^{-2x}\left[\left(\frac{3}{8x}\right)^2 - \left(\frac{1}{8x}\right)^2\right]$$

$$= \frac{\pi}{4}e^{-2x}$$

$\longrightarrow$

$$\boxed{\begin{array}{c} \omega \gg \dfrac{\gamma v}{b_{\min}} \\[2mm] \dfrac{dW}{d\omega} = \dfrac{q^2c}{2v^2}\exp\left(-\dfrac{2\omega b_{\min}}{\gamma v}\right) \end{array}}$$

[Emission from Relativistic Particles]

- Total emitted power:

Imagine **an instantaneous rest frame** K', such that the particle has zero velocity at a certain time. We can then calculate the radiation emitted by use of the dipole (Larmor) formula.

Suppose that the particle emits a total amount of energy dW' in this frame in time dt'. The momentum of this radiation is zero, $d\mathbf{p}' = 0$, because the emission is symmetrical in the frame.

The energy in a frame K moving with velocity $-\mathbf{v}$ w.r.t. the particle is:

$$dW = \gamma dW' \qquad \longleftarrow \qquad dE = cdP^0 = c\tilde{\Lambda}^0{}_\mu dP'^\mu = c\tilde{\Lambda}^0{}_0 dP^0 = \gamma dE'$$

The time interval dt is simply

$$dt = \gamma dt'$$

The total power emitted in frames K and K' are given by

$$P = \frac{dW}{dt}, \quad P' = \frac{dW'}{dt'}$$

Thus **the total emitted power is a Lorentz invariant** for any emitter that emits with front-back symmetry in its instantaneous rest frame.

$$P = P'$$

- **The Larmor formula in covariant form:**

  Recall that $\vec{a} \cdot \vec{U} = 0$, and because $\vec{U} = (c, \mathbf{0})$ in the instantaneous rest frame of the particle, we have

  $$a'_0 = 0 \quad \rightarrow \quad |\boldsymbol{a}'|^2 = a'_k a'^k = a'_\mu a'^\mu = \vec{a} \cdot \vec{a}$$

  Therefore,

  $$P' = \frac{2q^2}{3c^3} |\boldsymbol{a}'|^2 \quad \longrightarrow \quad P = \frac{2q^2}{3c^3} \vec{a} \cdot \vec{a}$$

  $\qquad\qquad$ nonrelativistic $\qquad\qquad\qquad\qquad\qquad$ relativistic

- Expression of P in terms of the three-vector acceleration

  Recall

  $$dt = \gamma \left(dt' + \frac{v}{c^2} dx'_\parallel \right)$$

  $$u_\parallel = \frac{u'_\parallel + v}{1 + vu'_\parallel/c^2}$$

  $$u_\perp = \frac{u'_\perp}{\gamma \left(1 + vu'_\parallel/c^2 \right)}$$

  $\longrightarrow$

  $$\sigma \equiv \left(1 + vu'_\parallel/c^2 \right)$$

  $$dt = \gamma dt' \sigma$$

  $$u_\parallel = \frac{u'_\parallel + v}{\sigma}$$

  $$u_\perp = \frac{u'_\perp}{\gamma \sigma}$$

  $\longrightarrow$

  $$dt = \gamma dt' \sigma$$

  $$du_\parallel = \frac{du'_\parallel}{\sigma} - \frac{u'_\parallel + v}{\sigma^2} \frac{v}{c^2} du'_\parallel$$

  $$= \frac{du'_\parallel}{\sigma^2} \left(1 - \frac{v^2}{c^2} \right) = \frac{du'_\parallel}{\gamma^2 \sigma^2}$$

  $$du_\perp = \frac{du'_\perp}{\gamma \sigma} - \frac{u'_\perp}{\gamma \sigma^2} \frac{v}{c^2} du'_\parallel$$

  $$= \frac{1}{\gamma \sigma^2} \left(\sigma du'_\perp - \frac{vu'_\perp}{c^2} du'_\parallel \right)$$

Hence,

$$a_\parallel = \frac{du_\parallel}{dt} = \frac{1}{\gamma^3\sigma^3}\frac{du'_\parallel}{dt'}$$

$$a_\perp = \frac{du_\perp}{dt} = \frac{1}{\gamma^2\sigma^3}\left(\sigma\frac{du'_\perp}{dt'} - \frac{vu'_\perp}{c^2}\frac{du'_\parallel}{dt'}\right)$$

Transformation of three-vector acceleration:

$$a_\parallel = \frac{1}{\gamma^3\sigma^3}a'_\parallel$$

$$a_\perp = \frac{1}{\gamma^2\sigma^3}\left(\sigma a'_\perp - \frac{vu'_\perp}{c^2}a'_\parallel\right)$$

$$\text{where} \quad \sigma \equiv \left(1 + \frac{vu'_\parallel}{c^2}\right)$$

$\longrightarrow$

In an instantaneous rest frame of a particle,

$$u'_\parallel = u'_\perp = 0, \quad \sigma = 1$$

$$a'_\parallel = \gamma^3 a_\parallel$$
$$a'_\perp = \gamma^2 a_\perp$$

Note $\quad \tan\theta'_a \equiv \dfrac{a'_\perp}{a'_\parallel} = \dfrac{1}{\gamma}\dfrac{a_\perp}{a_\parallel} = \dfrac{1}{\gamma}\tan\theta_a$

Thus we can write

$$P = \frac{2q^2}{3c^3}|\boldsymbol{a}'|^2 = \frac{2q^2}{3c^3}\left(a'^2_\parallel + a'^2_\perp\right)$$
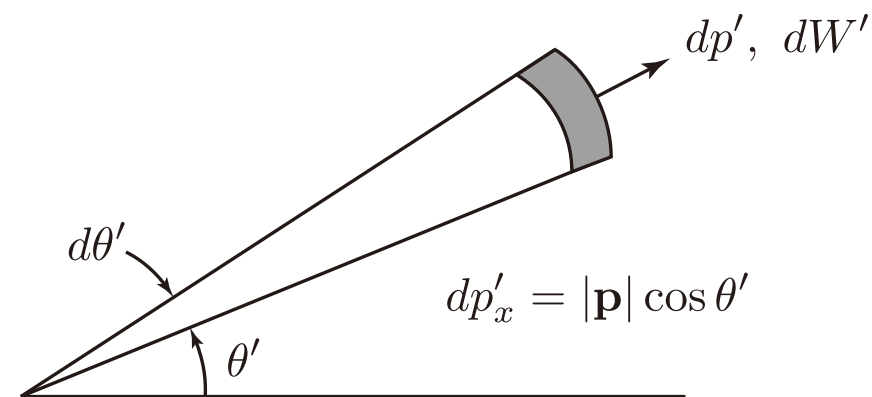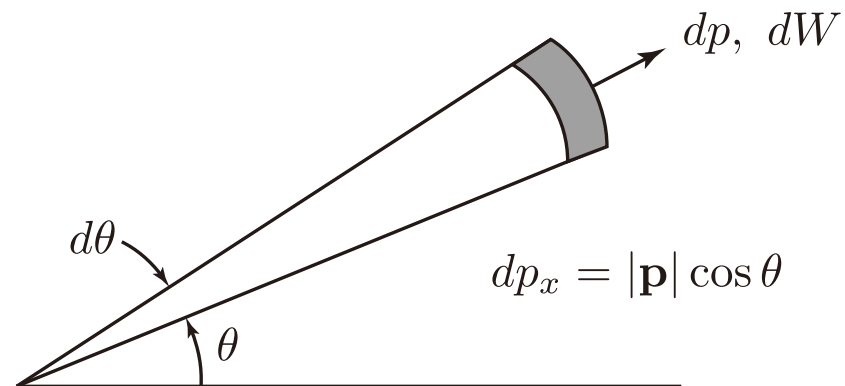
$\longrightarrow$

$$P = \frac{2q^2}{3c^3}\gamma^4\left(\gamma^2 a^2_\parallel + a^2_\perp\right)$$

nonrelativistic

relativistic

Differential Power

- **Angular Distribution of Emitted and Received Power**

Note: $d\phi' = d\phi$

In the instantaneous rest frame of the particle, let us consider an amount of energy dW' that is emitted into the solid angle $d\Omega' = \sin\theta' d\theta' d\phi'$ (see the above figure).

$$\mu \equiv \cos\theta \quad \rightarrow \quad d\Omega = d\mu d\phi$$
$$\mu' \equiv \cos\theta' \quad \rightarrow \quad d\Omega' = d\mu' d\phi'$$

Recall the aberration formula:

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \quad \rightarrow \quad \mu = \frac{\mu' + \beta}{1 + \beta\mu'} \quad \rightarrow \quad \mu' = \frac{\mu - \beta}{1 - \beta\mu}$$

$$d\mu = \frac{d\mu'}{\gamma^2 (1 + \beta\mu')^2} \qquad\qquad d\Omega = \frac{d\Omega'}{\gamma^2 (1 + \beta\mu')^2}$$

$$d\mu' = \frac{d\mu}{\gamma^2 (1 - \beta\mu)^2} \qquad \longrightarrow \qquad d\Omega' = \frac{d\Omega}{\gamma^2 (1 - \beta\mu)^2}$$

Recall that energy and momentum form a four-vector

$$\vec{P} = (E/c,\ \mathbf{p}) \quad \text{and} \quad |\mathbf{p}| = E/c \quad \longrightarrow \quad dW = \gamma(dW' + v\,dp'_x) = \gamma(1 + \beta\mu')dW'$$

$$\therefore \quad dW = \gamma(1 + \beta\mu')dW', \quad dW' = \gamma(1 - \beta\mu)dW$$

$$\boxed{\frac{dW}{d\Omega} = \gamma^3\,(1 + \beta\mu')^3\,\frac{dW'}{d\Omega'}, \quad \frac{dW'}{d\Omega'} = \gamma^3\,(1 - \beta\mu)^3\,\frac{dW}{d\Omega}}$$

In the rest frame, **the power emitted in a unit time interval** is $\qquad \dfrac{dP'}{d\Omega'} \equiv \dfrac{dW'}{dt'\,d\Omega'}$

However, in the observer's frame, there are two possible choices for the time interval to calculate the power.

(1) $\underline{dt = \gamma dt':}$

This is the time interval during which the emission occurs. With this choice we obtain **the emitted power**.

(2) $\underline{dt_A = \gamma(1 - \beta\mu)dt' \quad \text{or} \quad dt_A = \gamma^{-1}(1 + \beta\mu')^{-1}dt':}$

This is the time interval of the radiation as received by a stationary observer in K. With this choice we obtain **the received power**.

- Thus we obtain the two results:

$$\frac{dP_e}{d\Omega} = \gamma^2 \left(1 + \beta\mu'\right)^3 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4 \left(1 - \beta\mu\right)^3} \frac{dP'}{d\Omega'}$$

$$\frac{dP_r}{d\Omega} = \gamma^4 \left(1 + \beta\mu'\right)^4 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4 \left(1 - \beta\mu\right)^4} \frac{dP'}{d\Omega'}$$

P_r is the power actually measured by an observer. It has the expected symmetry property of yielding the inverse transformation by interchanging primed and unprimed variables, along with a change of sign of β.

P_e is used in the discussion of emission coefficient.

In practice, **the distinction between emitted and received power is often not important, since they are equal in an average sense for stationary distributions of particles.**

- Beaming effect:

If the radiation is isotropic in the particle's frame, then the angular distribution in the observer's frame will be highly peaked in the forward direction for highly relativistic velocities.

The factor $\gamma^{-4} \left(1 - \beta\mu\right)^{-4}$ is sharply peaked near $\theta \approx 0$ with an angular scale of order $1/\gamma$.

$$\gamma^{-4} \left(1 - \beta\mu\right)^{-4} \approx \gamma^{-4} \left[1 - \left(1 - \frac{1}{2\gamma^2}\right)\left(1 - \frac{\theta^2}{2}\right)\right]^{-4} = \gamma^{-4} \left(\frac{1}{2\gamma^2} + \frac{\theta^2}{2}\right)^{-4} = \left(\frac{2\gamma}{1 + \gamma^2\theta^2}\right)^4$$