Astrophysics

Lecture 06
October 12 (Thur.), 2023

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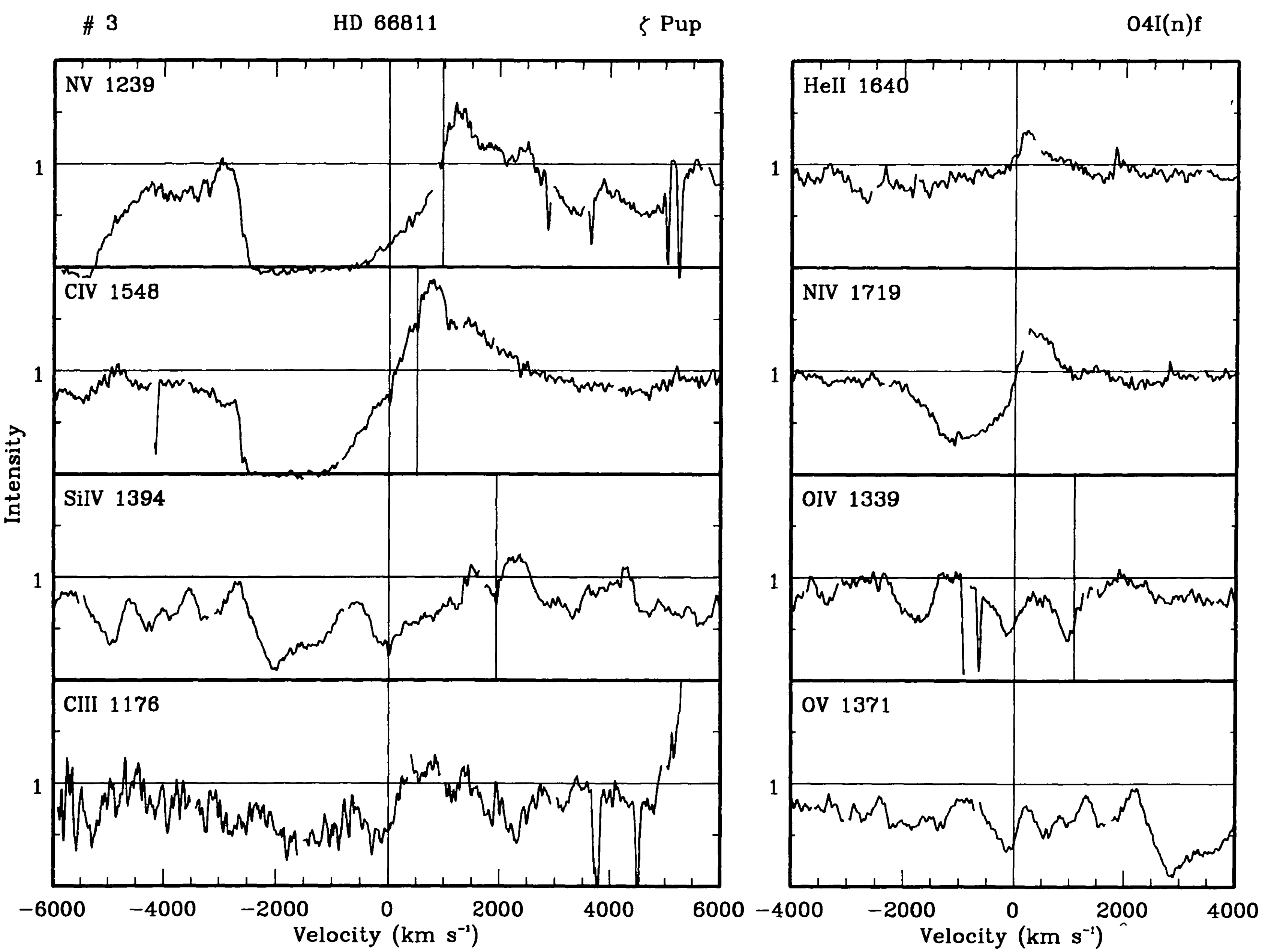
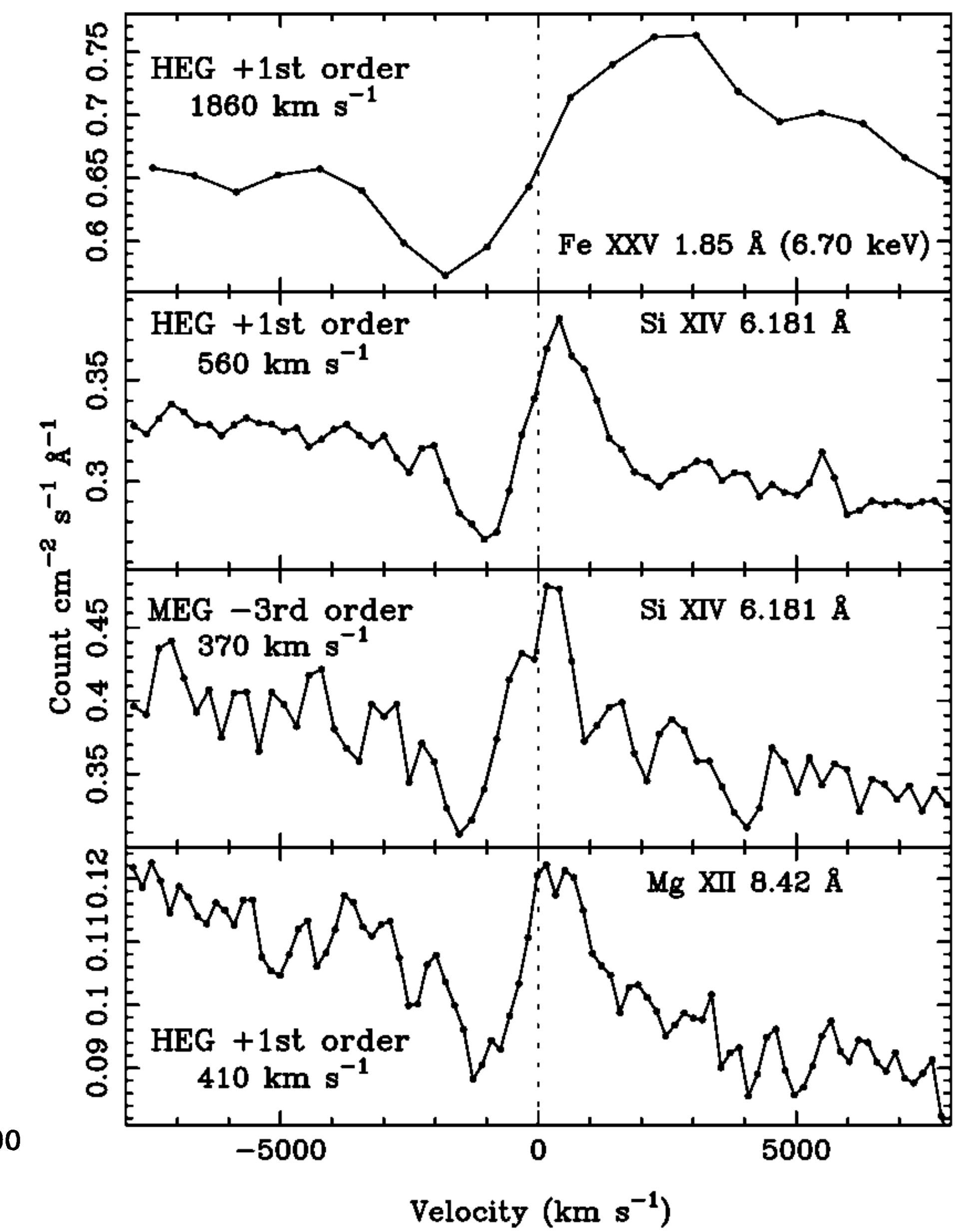
선광일 (Kwang-Il Seon)
UST / KASI

P Cygni Profile

- The P Cygni profile is characterized by **strong (redshifted) emission lines** with corresponding **blueshifted absorption line**.



zeta Puppis (Snow et al., 1994, ApJS, 95, 163)

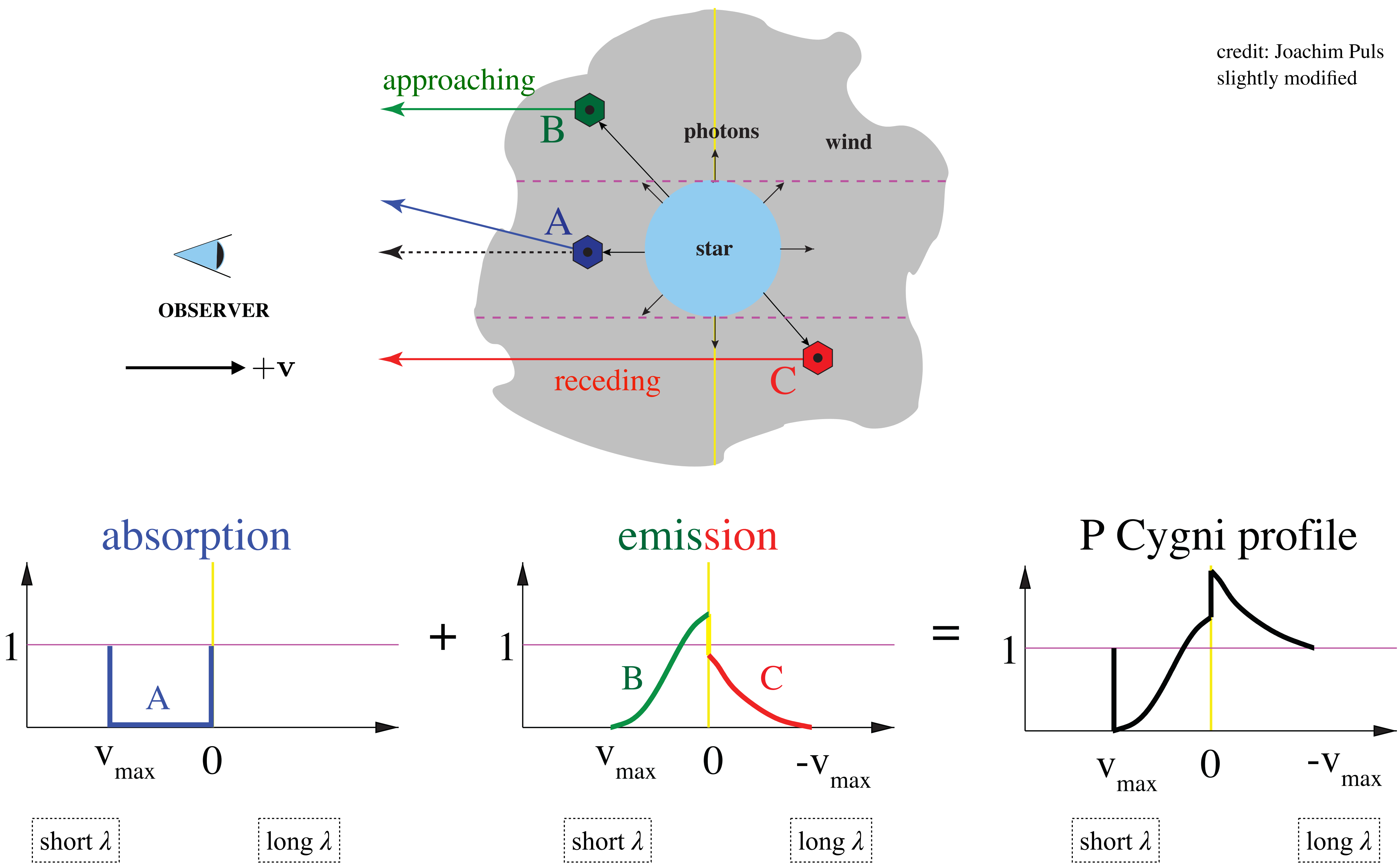


Circinus X-1
(Brandt & Schulz, 2000, ApJ, 544, L123)

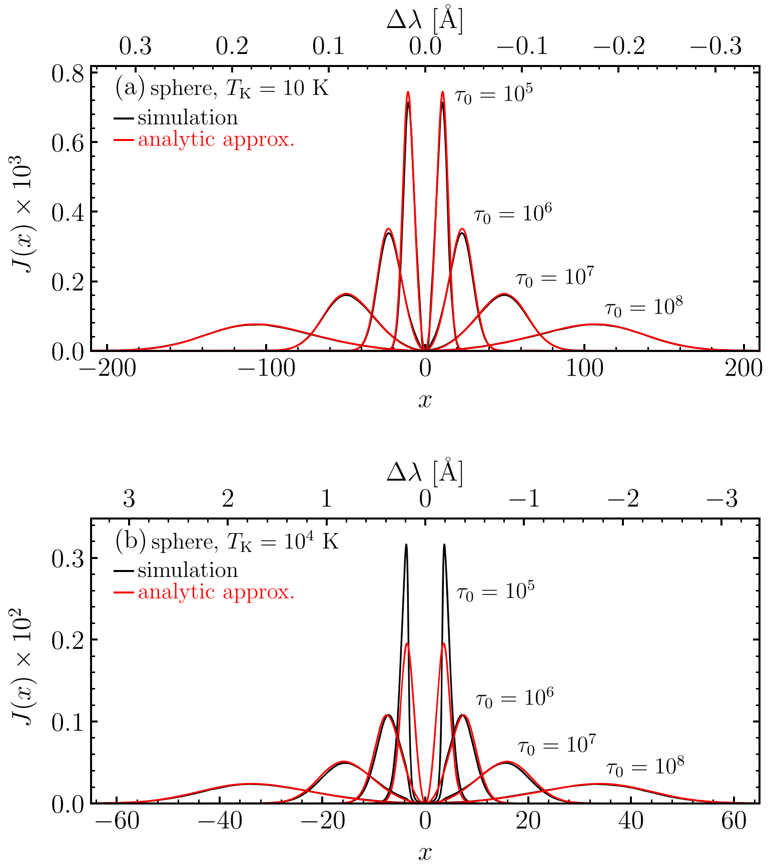
P Cygni profile formation

- The blueshifted absorption line is produced by material moving away from the star and toward us, whereas the emission come from other parts of the expanding shell.

credit: Joachim Puls
slightly modified

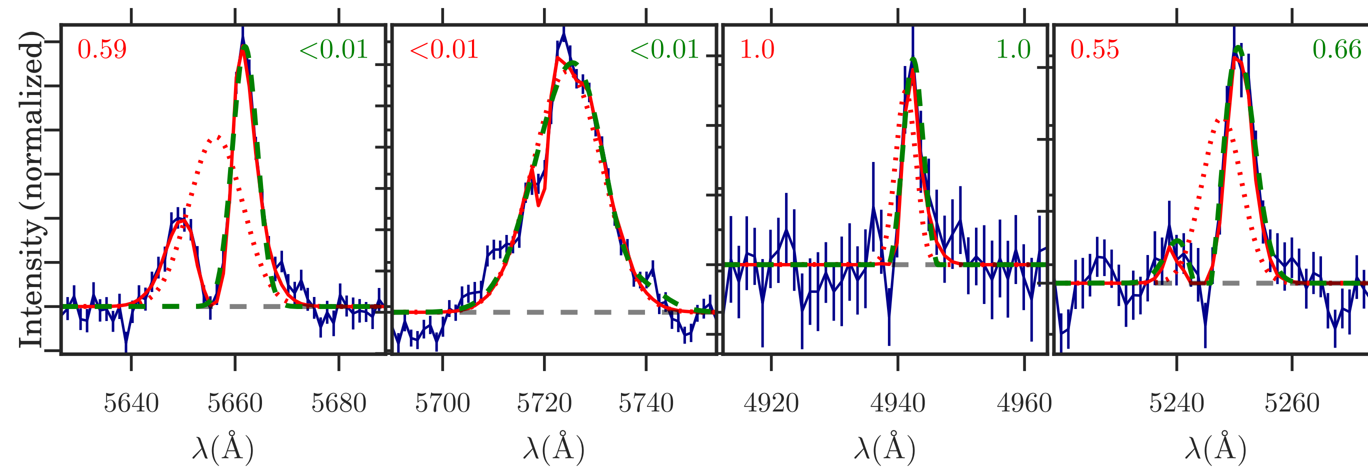


Lyα Resonance Scattering



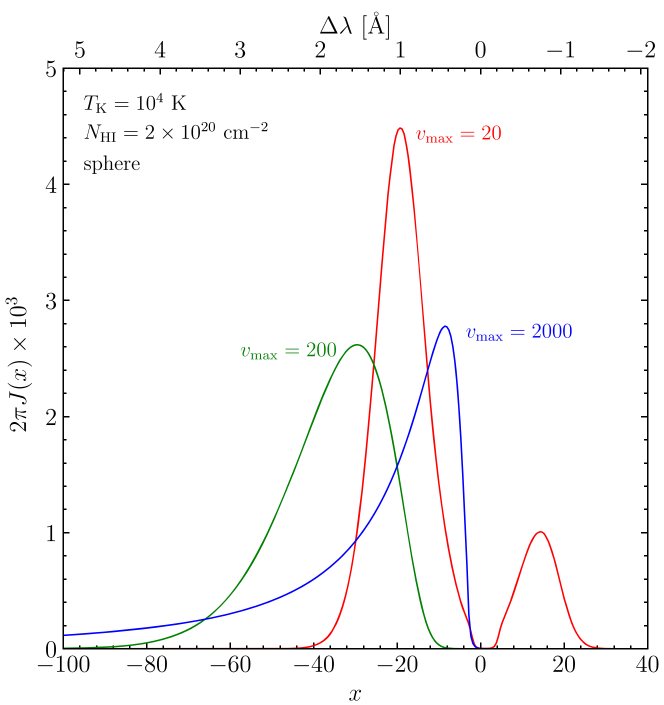
Seon & Kim (2020)

Emergent Lyα spectra from a static, homogeneous sphere at (a) $T = 10$ K and (b) 10^4 K, with different optical depths ($\tau_0 = 10^5 - 10^8$). The black curves are line profiles calculated with LaRT. The red curves denote an analytic series solution, which was derived from the series solution of Dijkstra et al. (2006).



Example spectral fits of the "shell" and "Gaussian-minus-Gaussian" models with all combinations of fitting failures/successes. The blue lines show the data, the red solid line the shell model fit (with the intrinsic spectrum as dotted red line), and the green dashed line shows the "Gaussian-minus-Gaussian" best fit. The numbers in the panel show the $p(\chi^2)$ values of the best fits in the corresponding colors.

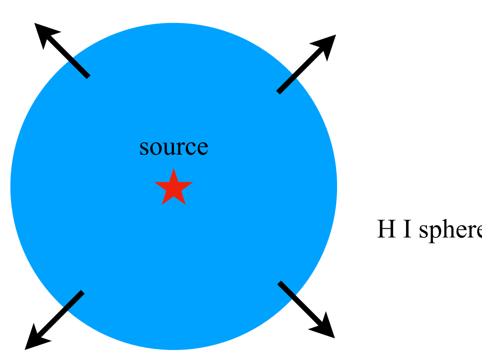
Gronke (2017)



Emergent Lyα for the dynamic motion test cases, in which the gas expands isotropically and has a temperature of $T = 10^4$ K and a column density of $N_{HI} = 2 \times 10^{20}\,\mathrm{cm}^{-2}$. The maximum velocity V_{max} of the Hubble-like outflow is denoted in units of km s^{-1}. The ordinate is the mean intensity integrated over the solid angle outgoing from the spherical surface.

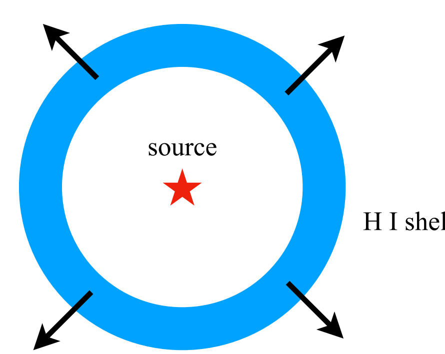
Seon & Kim (2020)

Hubble-like outflow



$$V = V_{max} \frac{r}{r_{max}}$$

spherically expanding shell model



$$V = V_{shell} \ \text{(constant)}$$

Semiclassical (Weissokpf-Woolley) Picture of Quantum Levels

- In the semiclassical picture, each level is viewed as a continuous distribution of sublevels with energies close to the energy of the level (E_n).

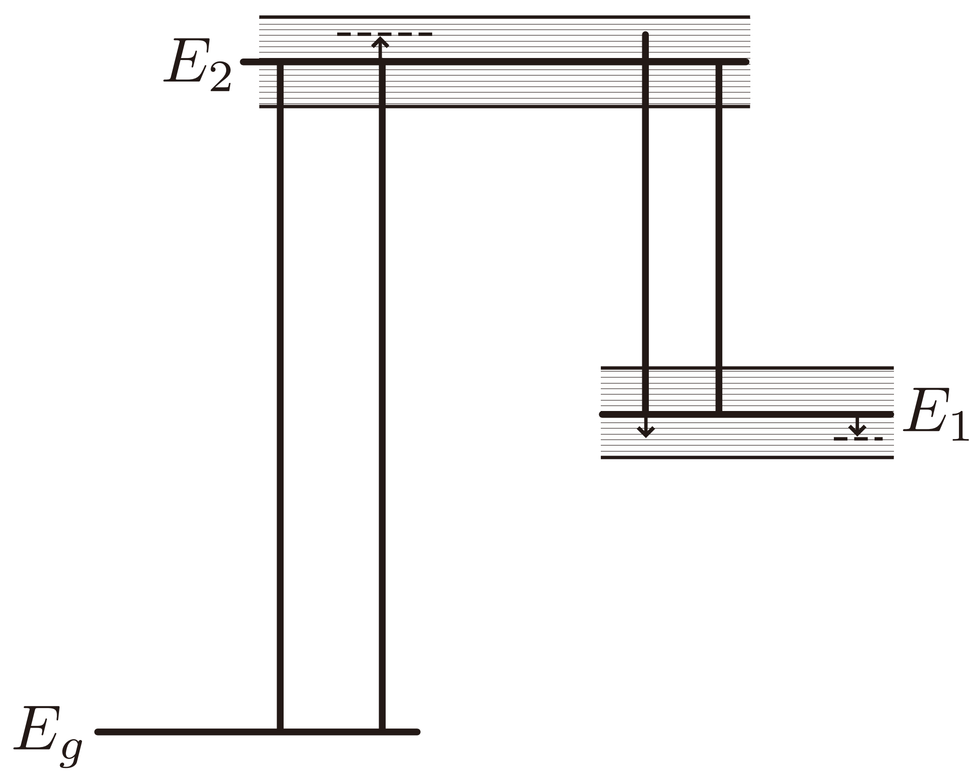
  The distribution of sublevels are explained by the Heisenberg Uncertainty Principle. The level has a lifetime $\Delta t = 1/A$ (A = Einstein A coefficient) and a spread in energy about $\Delta E \approx \hbar/\Delta t = \hbar A$.

$$\Delta E \Delta t \approx \hbar$$

The ground level has no spread in energy

because $\Delta t = \infty$.

The atom is in a definite sublevel of some level.



A transition in a spectral line is considered to be an instantaneous transition between a definite sublevel of an initial level to a definite sublevel of a final level.

The energy spread of a sublevel is described by a Lorentzian profile.

Raman Scattering*

- Raman scattering or the Raman effect is the inelastic scattering of a photon.

  When photons are scattered from an atom or molecule, **most photons are elastically scattered (i.e., Rayleigh scattering)**, such that the scattered photons have the same energy (frequency and wavelength) as the incident photons. However, a small fraction of the scattered photons (approximately 1 in 10 million) are scattered by an excitation, with the scattered photons having a frequency different from, and usually lower than, that of the incident photons.

  Typically this effect involves vibrational energy being gained by a molecule as incident photons from a visible laser are shifted to lower energy.

- Astrophysical Example: **Scattering of O VI doublet (λλ1038, 1032) by neutral hydrogen.**

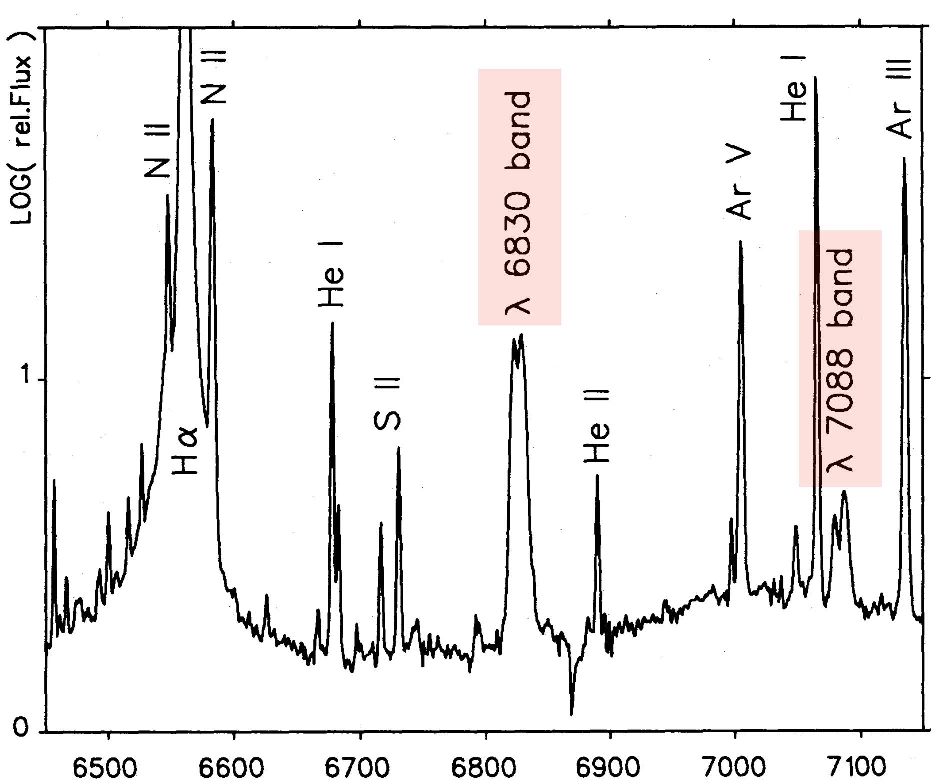
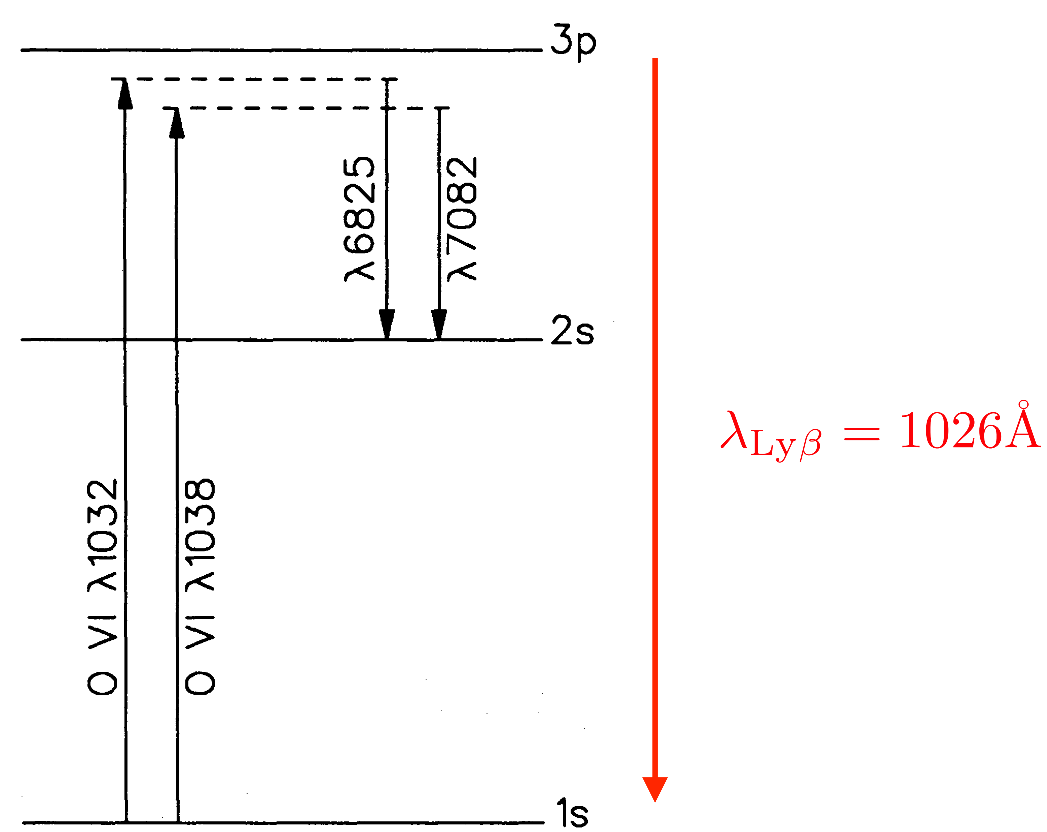


Fig.1. Raman scattered emission bands in the symbiotic star V1016 Cyg. The spectrum was obtained on the 1.93m telescope at the Observatoire de Haute Provence.



Schmid (1989, A&A, 211, L31)

Relativistic Covariance and Kinematics

Galilean Transformation/Relativity

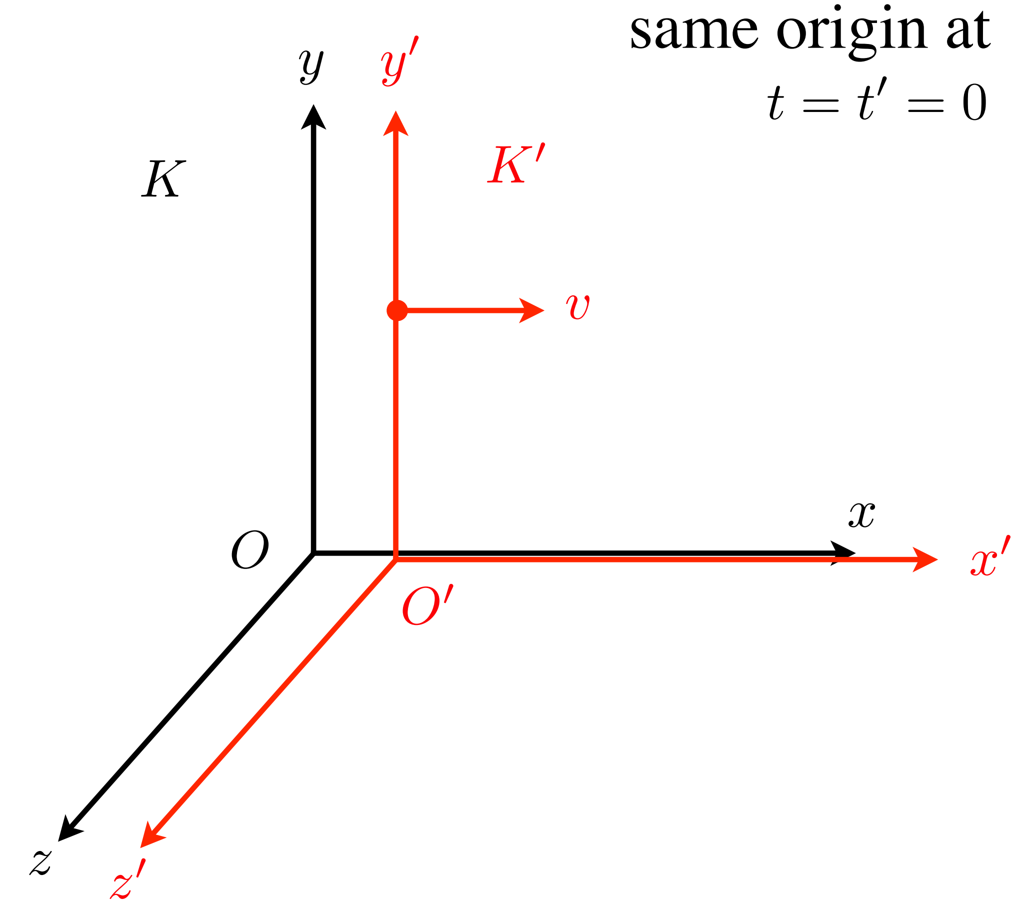
- ***Galilean transformation*** is used to transform between the coordinates of two **inertial frames of reference** which differ only by constant relative motion within the constructs of Newtonian physics.

$$
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

Newton's law is invariant under the Galilean transformation.

However, *Maxwell's equations are not invariant under the Galilean transformation.*

- ***Lorentz transformation*** **is the result of attempts by Lorentz and others to explain how the speed of light was observed to be independent of the reference frame, and to understand the symmetries of the Maxwell's equations.**



Let us consider two frames K and K', as shown above, with a relative uniform velocity v.

* Review of Lorentz Transformations *

- **Postulates in the special theory of relativity**

  (1) The laws of nature are the same in two frames of reference in uniform relative motion with no rotation.

  (2) The speed of light is c in all such frames.

- **space-time event**: an event that takes place at a location in space and time.

- **Derivation of Lorentz transforms**:

  If a pulse of light is emitted at the origin at $t = 0$, each observer will see an expanding sphere centered on his own origin. Therefore, we have the equations of the expanding sphere in each frame.

$$x^2 + y^2 + z^2 - c^2t^2 = 0, \quad x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \qquad (1)$$

Since space is assumed to be homogeneous, the transformation must be linear.

$$x' = a_1 x + a_2 t, \quad y' = y, \quad z' = z, \quad t' = b_1 x + b_2 t$$

We note that the origin of K' $(x' = 0)$ is a point that moves with speed v as seen in K. Its location in K is given by $x = vt$. Therefore, we have

$$\boxed{\begin{aligned} x' &= a_1 x + a_2 t \\ 0 &= a_1(vt) + a_2 t \end{aligned} \qquad \frac{a_2}{a_1} = -v}$$

$$\begin{aligned} x' &= a_1(x - vt) \\ y' &= y \\ z' &= z \\ t' &= b_1 x + b_2 t \end{aligned} \qquad (2)$$

Substitute Eq. (2) into Eq. (1): $x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$

$$a_1^2(x - vt)^2 + y^2 + z^2 - c^2(b_1x + b_2t)^2 = x^2 + y^2 + z^2 - c^2t^2$$
$$\left(a_1^2 - c^2b_1^2\right)x^2 - 2\left(a_1^2v + c^2b_1b_2\right)xt + \left(a_1^2v^2 - c^2b_2^2\right)t^2 = x^2 - c^2t^2$$

(Note: we didn't assume that $x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$)

Therefore, the following equations should be satisfied.

$$\begin{aligned} a_1^2 - c^2b_1^2 &= 1 \quad (a) \\ \left(a_1^2v + c^2b_1b_2\right) &= 0 \quad (b) \\ a_1^2v^2 - c^2b_2^2 &= -c^2 \quad (c) \end{aligned} \longrightarrow \begin{aligned} &(a)\ \ b_1^2 = \frac{a_1^2 - 1}{c^2} \\ &(c)\ \ b_2^2 = 1 + \frac{v^2}{c^2}a_1^2 \end{aligned}$$

$$(b)\ \ a_1^4v^2 = c^4b_1^2b_2^2 = c^2a_1^2 + v^2a_1^4 - c^2 - v^2a_1^2$$

$$\rightarrow\ a_1 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma$$

a_1 should be positive because $x' > 0$ when $x > 0$ at $t = 0$.

$$\longrightarrow \boxed{(a)\ \ b_1 = -\frac{v}{c^2}\gamma, \quad (c)\ \ b_2 = \gamma}$$

We take a positive b_2 because $t' > 0$ when $t > 0$. Then, it is clear that b_1 is negative from (b).

Finally, we obtain the Lorentz transformation (and its inverse):

The inverse has the same form as the original except that the primed and unprimed variables are interchanged, and v is replaced by $-v$.

$$\boxed{\begin{aligned} x' &= \gamma(x - vt) & x &= \gamma(x' + vt') \\ y' &= y & y &= y \\ z' &= z & z &= z \\ t' &= \gamma\left(t - \frac{v}{c}x\right) & t &= \gamma\left(t' + \frac{v}{c}x'\right) \end{aligned}}$$

where $\gamma \equiv \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \left(1 - \beta^2\right)^{-1/2}; \quad \beta \equiv \dfrac{v}{c}$

Lorentz factor $1 \leq \gamma \leq \infty; \quad 0 \leq \beta \leq 1$

Length Contraction

- **Length contraction** (Lorentz-Fitzgerald contraction): Suppose a rigid rod of length $L_0 = x_2{}' - x_1{}'$ is carried at rest in K'. What is the length as measured in *K*? The positions (x_2 and x_1) of the ends of the rod are marked at the same time in *K*.

$$x' = \gamma(x - vt) \longrightarrow \boxed{\begin{array}{l} L_0 = x_2' - x_1' = \gamma\left(x_2 - x_1\right) = \gamma L \\ L = L_0/\gamma \end{array}}$$

Therefore, **the rod appears shorter by a factor $1/\gamma$ in *K*.**

If both carry rods (of the same length when compared at rest) each thinks the other's rod has shrunk!

It would appear to K' that the two ends of the moving stick were not marked at the same time by the other observer (in K). (Since the Lorentz transformation of time depends on position, **temporal simultaneity is not Lorentz invariant.**)

Time Dilation

- **Time dilation**: Suppose a clock at rest at the origin of K' measures off a time interval $T_0 = t_2{}' - t_1{}'$. What is the time interval measured in K? Note that the clock is at rest at the origin of K' so that $x_2{}' = x_1{}' = 0$.

$$t = \gamma\left(t' + \frac{v}{c}x'\right) \longrightarrow \boxed{\begin{aligned} &T = t_2 - t_1 = \gamma\left(t_2' - t_1'\right) = \gamma T_0 \\ &T = \gamma T_0 \end{aligned}}$$

  The time interval has increased by a factor γ, so that **the moving clock appears to have slowed down, as measured in *K***. By symmetry, K' thinks clocks in K have slowed down, too.

  The resolution of this apparent contradiction is a result of looking at the manner of measuring an interval of time between two events separated in space. K measures t_1 as the moving clock passes x_1, then measures t_2 as it passes x_2; he/she simply subtracts $t_2 - t_1$ on the assumption that his/her own two clocks at x_1 and x_2 are synchronized. K' will object to this, since according to his/her observations the two clocks in K are not synchronized at all.

- Simultaneity is relative: Simultaneous events at two different spatial points in the primed frame is not simultaneous in the unprimed frame.

  Many of the apparent contradictions of special relativity are simply a result of the relativity of simultaneity between two events separated in space.

- Time dilation is detected in the increased half-lives of unstable particles moving rapidly in an accelerator or in the cosmic-ray flux.

Transformation of Velocities

- If a point has a velocity $\mathbf{u}'$ in frame K', what is its velocity $\mathbf{u}$ in frame K. Writing Lorentz transformations for differentials

$$dx = \gamma\left(dx' + v dt'\right), \quad dy = dy', \quad dz = dz'$$
$$dt = \gamma\left(dt' + \frac{v}{c^2} dx'\right)$$

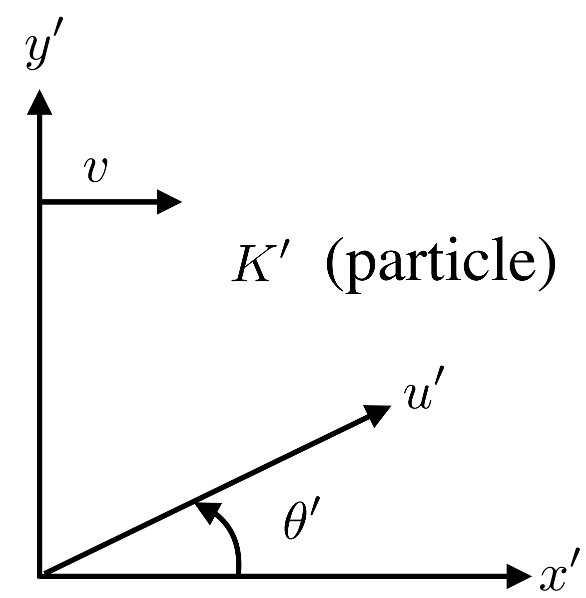
We then have the relations

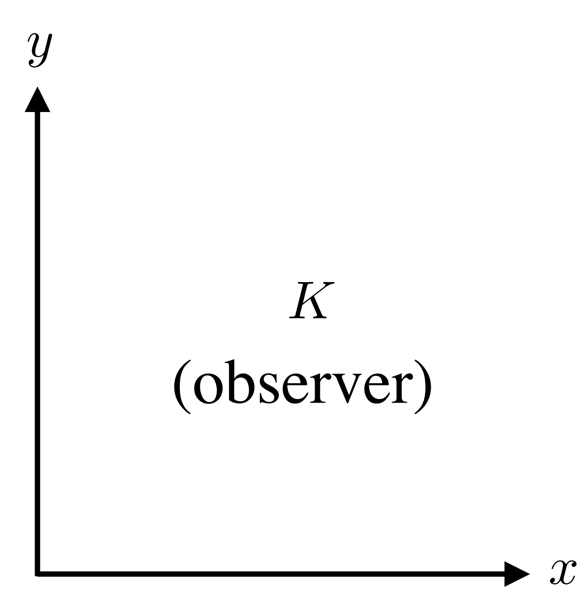
$$u_x = \frac{dx}{dt} = \frac{\gamma\left(dx' + v dt'\right)}{\gamma\left(dt' + v dx'/c^2\right)} = \frac{u'_x + v}{1 + v u'_x/c^2}$$
$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + v dx'/c^2\right)} = \frac{u'_y}{\gamma\left(1 + v u'_x/c^2\right)}$$
$$u_z = \frac{dz}{dt} = \frac{u'_z}{\gamma\left(1 + v u'_x/c^2\right)}$$

or

$$u_\parallel = \frac{u'_\parallel + v}{1 + v u'_\parallel/c^2}$$
$$u_\perp = \frac{u'_\perp}{\gamma\left(1 + v u'_\parallel/c^2\right)}$$

in terms of the components of **u** perpendicular to and parallel to **v**





- **Aberration formula**: the directions of the velocities in the two frames are related by

$$\tan\theta = \frac{u_\perp}{u_\parallel} = \frac{u'_\perp}{\gamma\left(u'_\parallel + v\right)} = \frac{u'\sin\theta'}{\gamma\,(u'\cos\theta' + v)} \qquad \text{where } u' \equiv |\mathbf{u}'|$$

- **Aberration of light**

For the case of light: $u' = c$

$$\tan\theta = \frac{\sin\theta'}{\gamma\,(\cos\theta' + v/c)} = \frac{\sin\theta'}{\gamma\,(\cos\theta' + \beta)}$$

$$\cos\theta = \frac{\gamma\,(\cos\theta' + v/c)}{\sqrt{\gamma^2\,(\cos\theta' + v/c)^2 + \sin^2\theta'}} = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$\sin\theta = \frac{\sin\theta'}{\sqrt{\gamma\,(\cos\theta' + v/c)^2 + \sin^2\theta'}} = \frac{\sin\theta'}{\gamma\,(1 + \beta\cos\theta')}$$

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \qquad \sin\theta = \frac{\sin\theta'}{\gamma\,(1 + \beta\cos\theta')}$$

Using the identity, $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{\sin\theta}{1 + \cos\theta}$

The aberration formula can be written as: $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{(1/\gamma)\sin\theta'}{1 + \beta\cos\beta' + \cos\theta' + \beta} = \dfrac{(1/\gamma)\sin\theta'}{(1+\beta)\,(1 + \cos\theta')}$

$$\boxed{\tan\left(\frac{\theta}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^2 \tan\left(\frac{\theta'}{2}\right) \quad \rightarrow \quad \theta < \theta'}$$

- **Beaming ("headlight") effect**:

  If photons are emitted isotropically in K', then half will have $\theta' < \pi/2$ and half $\theta' > \pi/2$.

  Consider a photon emitted at right angles to v in K'. Then we have

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

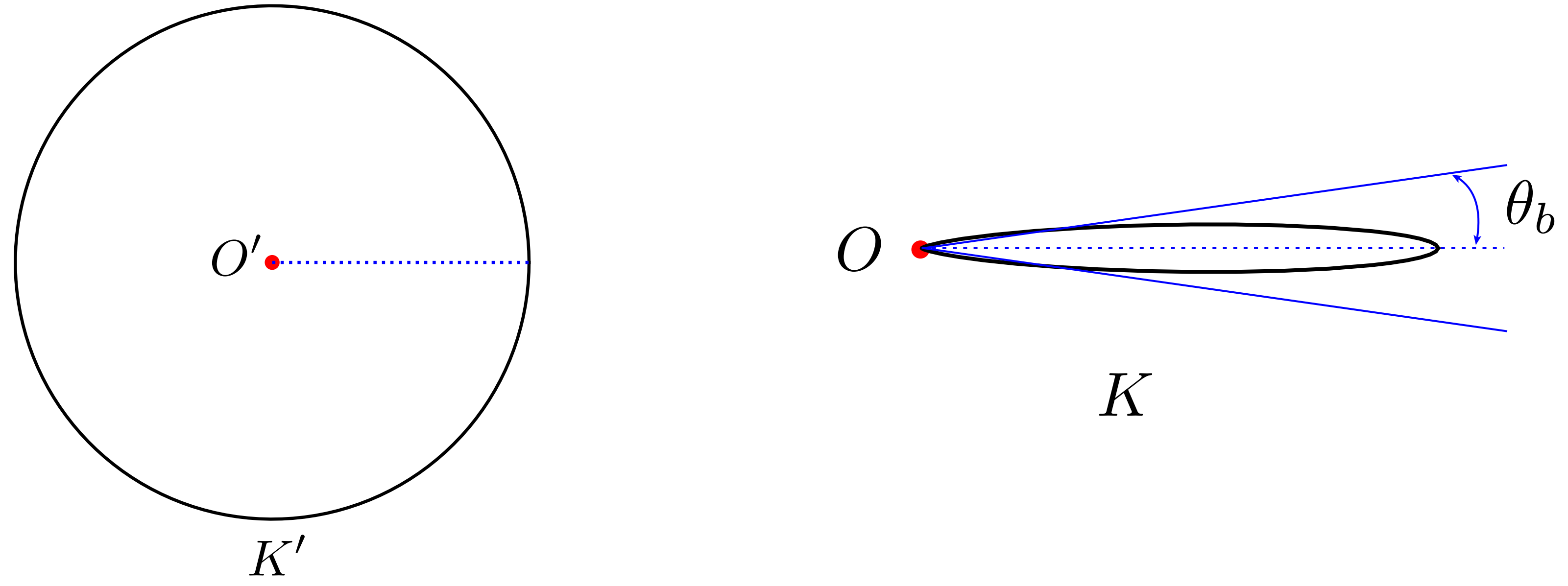
$$\sin\theta = \frac{\sin\theta'}{\gamma\,(1 + \beta\cos\theta')}$$

beam half-angle: $$\boxed{\sin\theta_b = \frac{1}{\gamma}, \quad \cos\theta_b = \beta, \quad \text{or} \quad \tan\left(\frac{\theta_b}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}}$$

For highly relativistic speeds, $\gamma \gg 1$, θ_b becomes small:

$$\sin\theta_b \approx \theta_b \longrightarrow \boxed{\theta_b \approx \frac{1}{\gamma}}$$

Therefore, in frame K, photons are concentrated in the forward direction, with half of them lying within a cone of half-angle $1/\gamma$. Very few photons will be emitted with $\theta \gg 1/\gamma$.



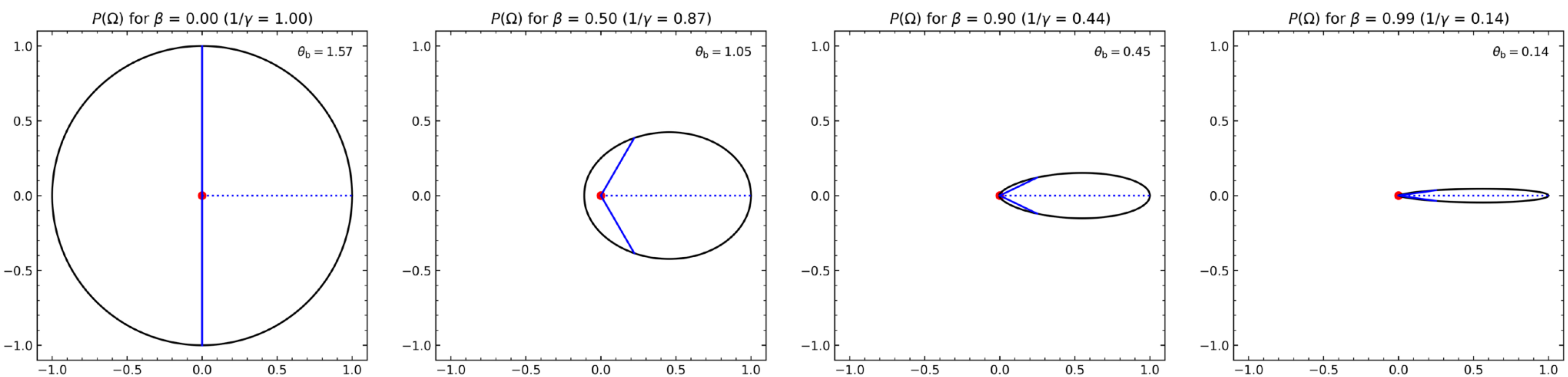
Power density emitted along the direction θ

$$P(\mu)d\mu d\phi = P'(\mu')d\mu' d\phi' \quad \rightarrow \quad P(\mu) = P'(\mu')\left|\frac{d\mu'}{d\mu}\right|$$

$\phi = \phi'$ from the symmetry

$$\boxed{P'(\mu') = \frac{1}{2} \qquad P(\mu) \propto \frac{1}{(1-\beta\mu)^2}}$$

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'} \quad \rightarrow \quad \mu' = \frac{\mu - \beta}{1 - \beta\mu} \quad \rightarrow \quad \frac{d\mu'}{d\mu} = \frac{1}{\gamma\,(1-\beta\mu)^2}$$

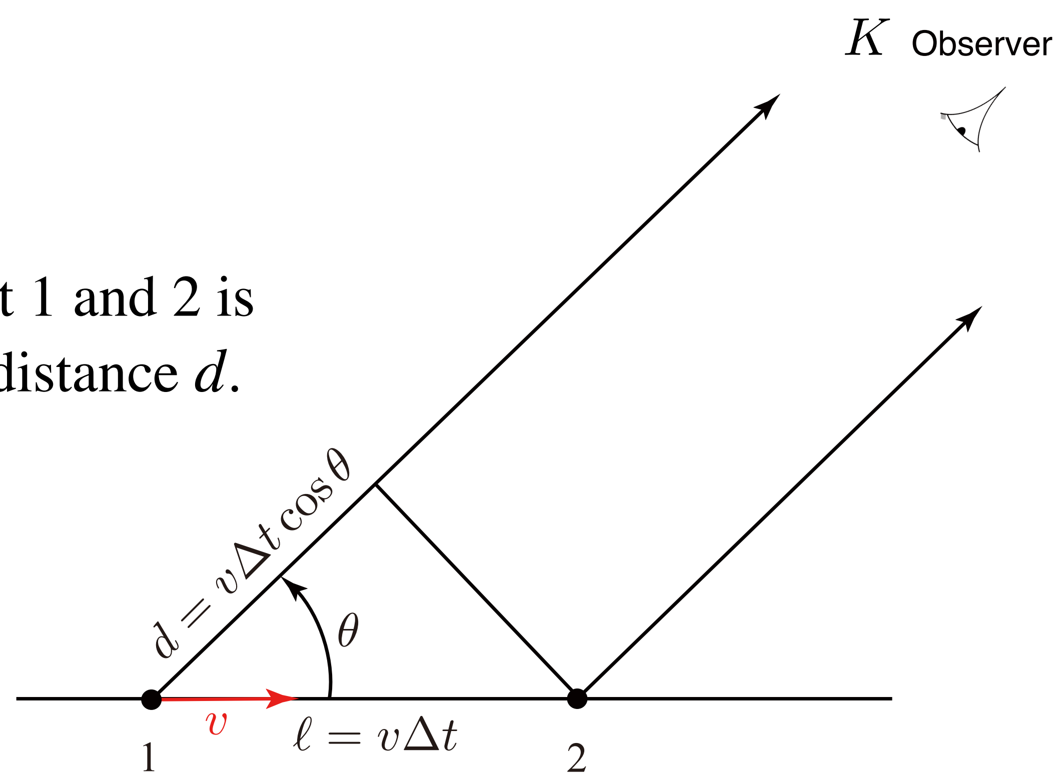


Doppler Effect

- In the rest frame of the observer K, imagine that the moving source emits one period of radiation as it moves from point 1 to point 2 at velocity v.

  Let the frequency of the radiation in the rest frame K' of the source be ω'. Then the time taken to move from point 1 to point 2 in the observer's frame is given by the time-dilation effect:

$$\Delta t' = \frac{2\pi}{\omega'} \quad \rightarrow \quad \Delta t = \Delta t' \gamma = \frac{2\pi}{\omega'}\gamma$$



Now consider the situation of the right hand side figure.

The difference in arrival times Δt_A of the radiation emitted at 1 and 2 is equal to Δt minus the time taken for radiation to propagate a distance d.

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t \left(1 - \frac{v}{c}\cos\theta\right)$$

Therefore, the observed frequency ω will be

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma\,(1 - \beta\cos\theta)}$$

$$\boxed{\frac{\omega}{\omega'} = \frac{1}{\gamma\,(1 - \beta\cos\theta)}} \qquad \boxed{\frac{\nu}{\nu'} = \frac{1}{\gamma\,(1 - \beta\cos\theta)}}$$

Note that the angle θ is measured in the rest frame K.

Note that $1 - \beta\cos\theta$ appears even classically. But, the factor $1/\gamma$ is purely a relativistic effect.

- If a source approaches head-on, $\theta = 0$, we obtain

$$\nu = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \nu_0$$

Here, $\nu_0 = \nu'$ is the emitted frequency measured by K'.

- Classical (nonrelativistic) Doppler shift $(\beta \ll 1)$:

$$\frac{\nu}{\nu_0} \approx \left(1 - \frac{\beta^2}{2} + \cdots\right)(1 + \beta\cos\theta + \cdots) \approx 1 + \beta\cos\theta \cdots \quad (\beta \ll 1)$$

This is the component of the velocity along the line of sight.

- Doppler shifts in astronomy
  - The frequencies of spectral lines from celestial sources are often shifted owing to the motions of the emitting objects: gaseous clouds, stars, or galaxies.
  - **Astronomical sign convention:** In the classical Doppler shift, the observed shift of frequency reflects only the component of the velocity along the line of sight, the radial component v_r. The astronomical convention is that **the radial component be positive if it is directed outward and negative if it is directed inward.** Then, the classical Doppler shift takes the form:

$$\boxed{\frac{\nu}{\nu_0} = 1 - \frac{v_r}{c} \quad \text{or} \quad \frac{\nu - \nu_0}{\nu_0} = -\frac{v_r}{c}}$$

ν and ν_0 are the observed and emitted frequencies, respectively.

  - Optical astronomers work in wavelength units. Then the Doppler shift is given by

$$\boxed{\frac{\lambda}{\lambda_0} = 1 + \frac{v_r}{c} \quad \text{or} \quad \frac{\lambda - \lambda_0}{\lambda_0} = \frac{v_r}{c}}$$

$$\frac{\lambda}{\lambda_0} = \left(\frac{1 + v_r/c}{1 - v_r/c}\right)^{1/2}$$

relativistic version for strictly radial motion.

- **Redshift parameter**: The optical spectra of distant luminous objects called quasars have spectral lines shifted by large amounts of lower frequencies. If these redshifts are interpreted as Doppler shifts, they indicate recession velocities approaching the speed of light. These velocities are due to the expansion of the universe; the expansion is such that the more distant the object, the faster it recedes. Astronomers define the "redshift" parameter z as

$$\boxed{z \equiv \frac{\lambda - \lambda_0}{\lambda_0} = \frac{\lambda}{\lambda_0} - 1}$$

$$z + 1 = \left(\frac{1 + v_r/c}{1 - v_r/c}\right)^{1/2}$$ relativistic version for strictly radial motion.

- The most distant quasars known are at redshifts $z \approx 6$. At this redshift, $\lambda/\lambda_0 = 7$, indicating that the observed wavelength is seven times the rest wavelength in the quasar frame. An ultraviolet emission line at $\lambda_0 = 121.5$ nm (Lyman α) would be shifted almost into the near infrared at 850.5 nm. In this case, the speed factor is $\beta_r = V_r/c = 0.960$. The quasar is receding at 96% the speed of light.
- We remind that special relativity is not really appropriate to our universe with its changing rate of expansion.

- **Transverse (second-order) Doppler effect**:
  - Now consider that a source moves relativistically from left to right. In this case, we find

$$\boxed{\frac{\omega}{\omega'} = \frac{1}{\gamma} \leq 1 \quad \text{at} \quad \theta = \pi/2}$$

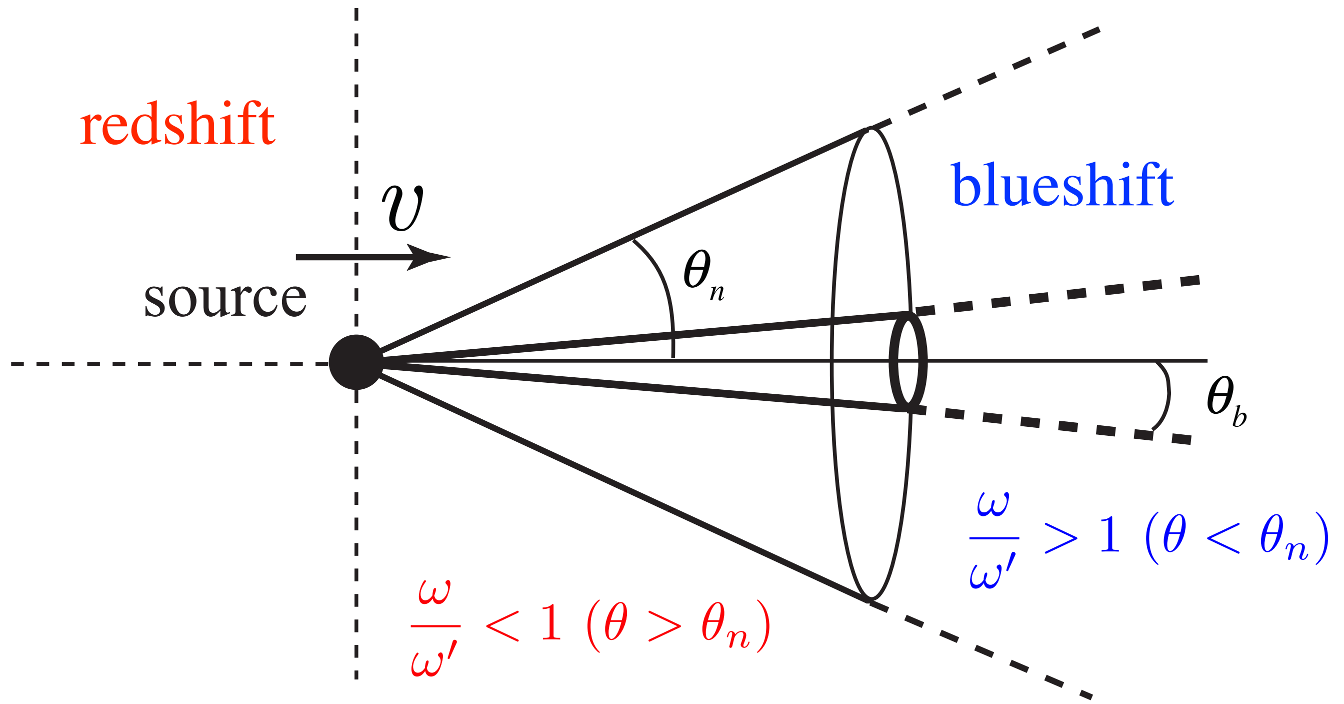
Surprising! A redshift to lower frequency ($\nu/\nu_0 < 1$) in contrast to the classical case, which yields no shift.

Second-order Doppler effect

- Recall beam half-angle = $\theta_b = \sin^{-1}\gamma^{-1}$
- Angle for null Doppler shift is defined by:

$$\frac{\omega}{\omega'} = \frac{1}{\gamma\,(1-\beta\cos\theta_n)} = 1$$

$$\rightarrow \quad \cos\theta_n = \frac{1-\gamma^{-1}}{\beta} = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{-1/2}$$

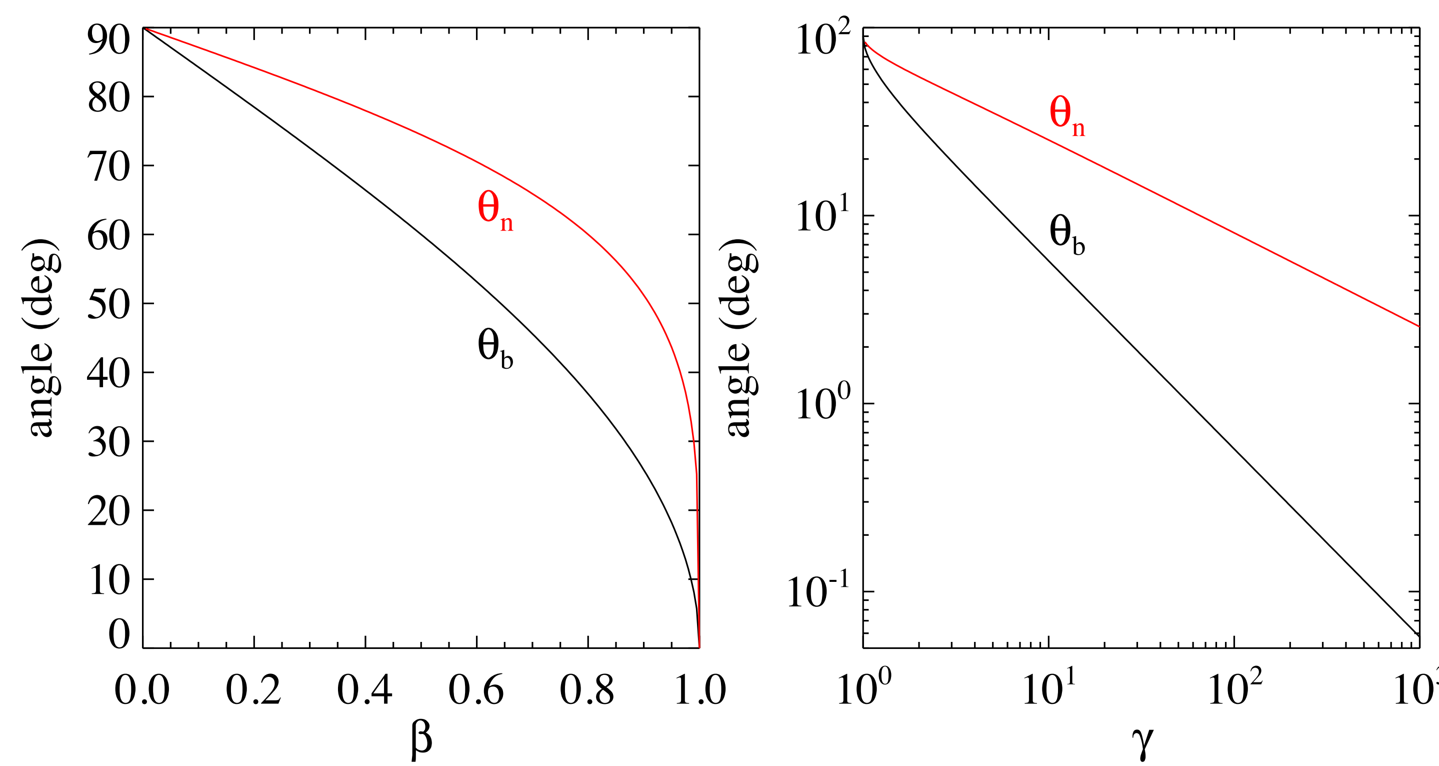


Relativistic Doppler effect can yield redshift even as a source approaches.

$$\cos\theta_n = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{-1/2} \approx 1 - \frac{1}{\gamma} \quad \text{for} \quad \gamma \gg 1$$

$$1 - \frac{\theta_n^2}{2} \approx 1 - \frac{1}{\gamma}$$

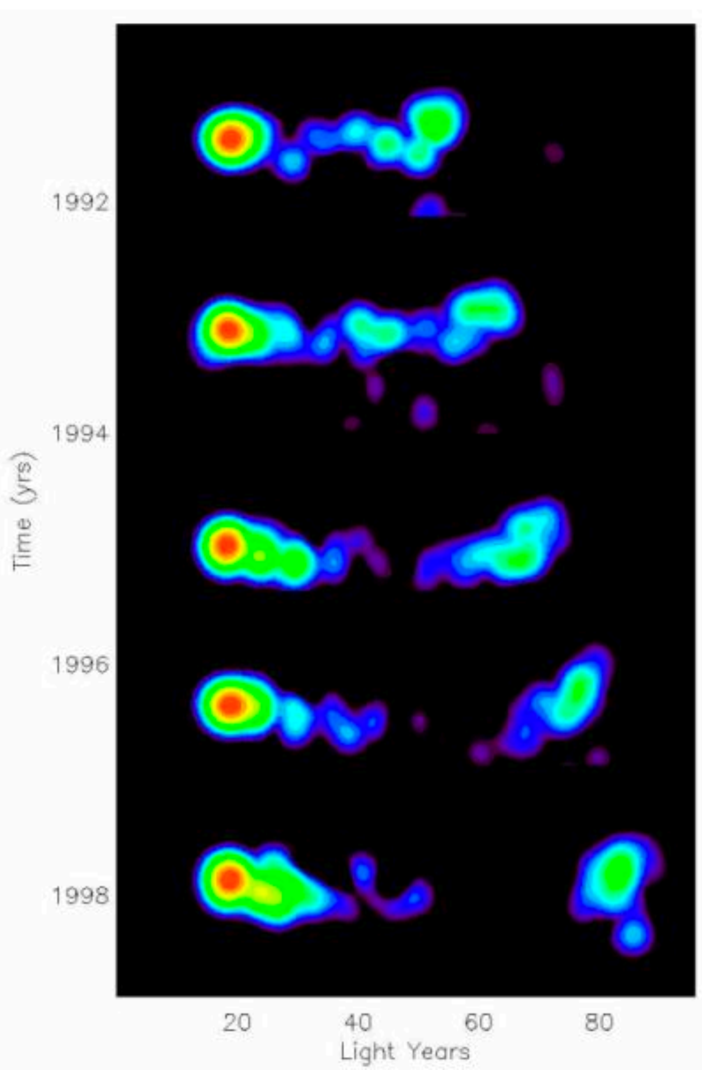
$$\boxed{\theta_n \approx \sqrt{\frac{2}{\gamma}} \approx \sqrt{2\theta_b}}$$



- Note $\theta_b < \theta_n$

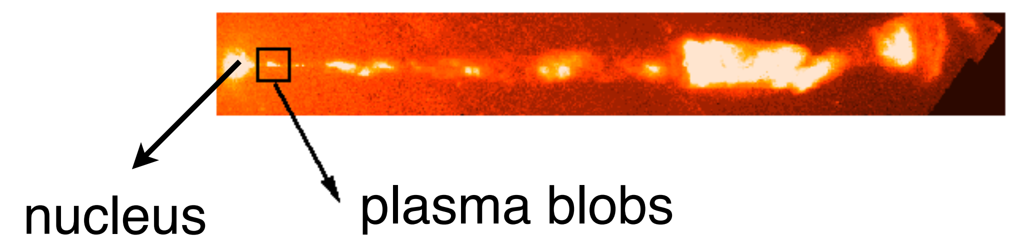
Superluminal motion of relativistic jets

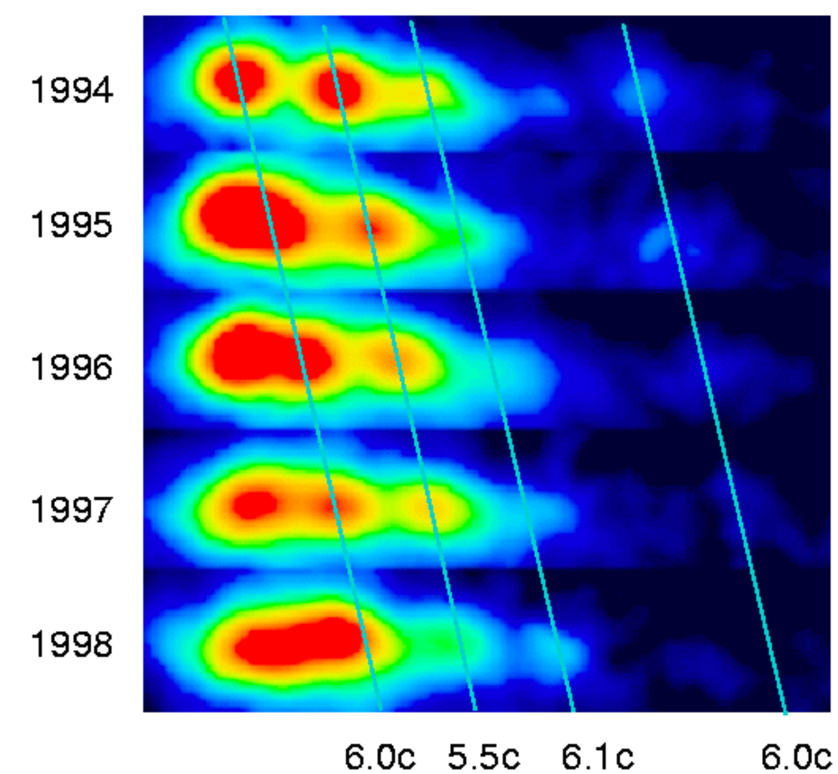
- Background
  - In astronomy, **superluminal motion** is the apparently faster-than-light motion seen in some radio galaxies, quasars, and recently also in some galactic sources called micro quasars. All of these source are thought to contain a black hole, responsible for the ejection of mass at high velocities.
  - In 1966, Martin Rees pointed out that "an object moving relativistically in suitable directions may appear to a distant observer to have a transverse velocity much greater than the speed of light."



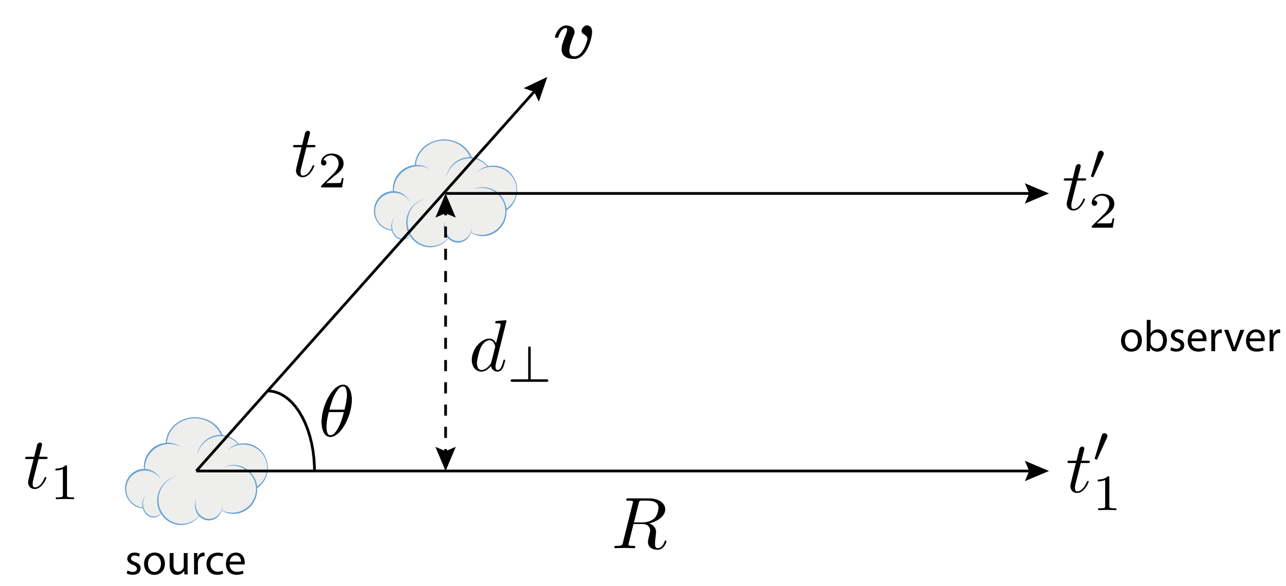
The superluminal expansion of 3C 279 is shown over a six year period.

Superluminal Motion in the M87 Jet





The inner regions of the jet of M87 as observed by the Hubble Space Telescope through the period 1992-1998 (Biretta et al. 1999). The optical knots are observed to move out from the nucleus at speeds up to 6c.



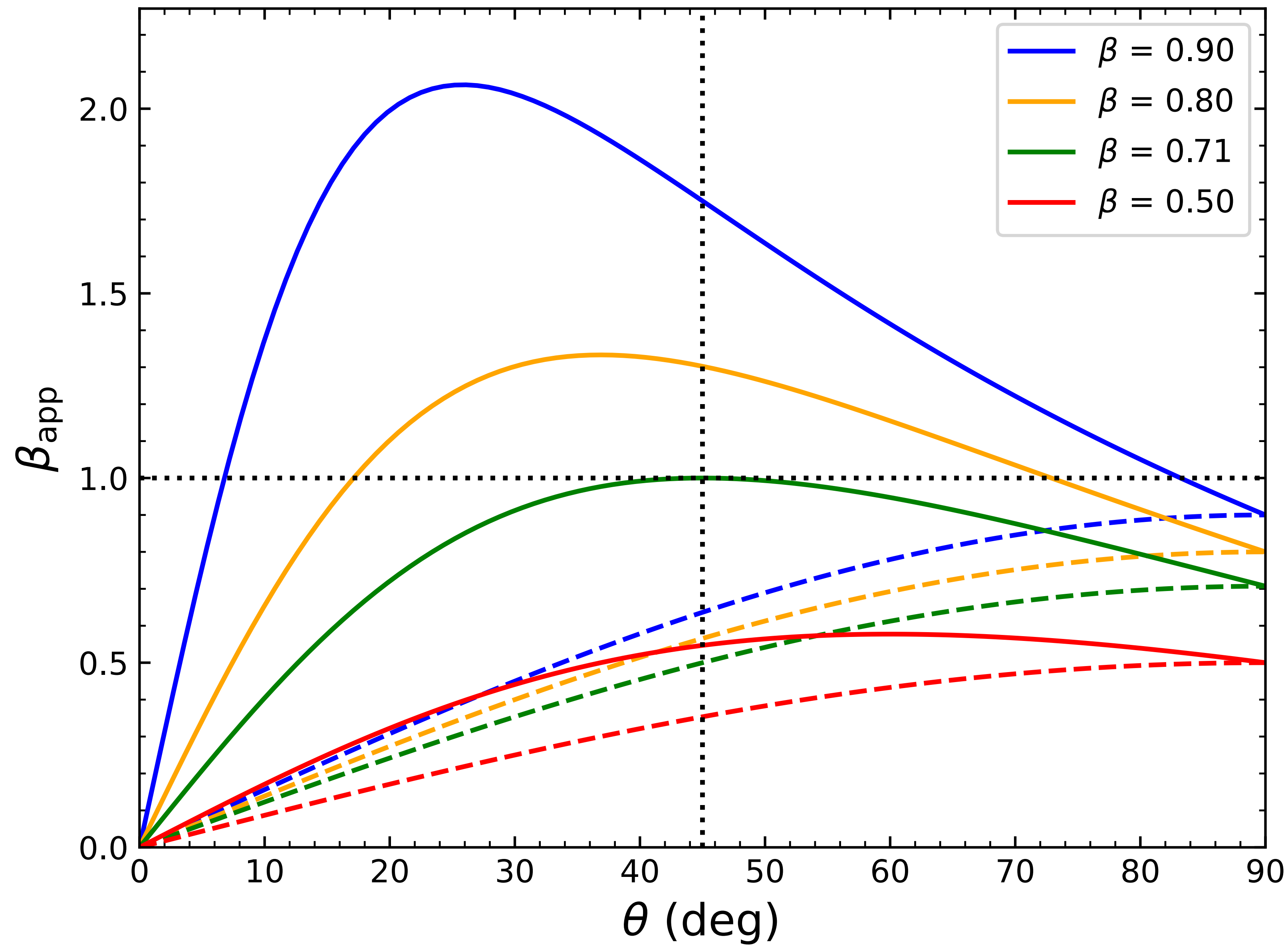
- Let's imagine that a plasma cloud is ejected from a source, moving with a velocity v along a direction that makes an angle θ $(0 \leq \theta \leq \pi/2)$ with line line of sight.
  - Suppose that the cloud starts at an instant t_1; the photons emitted at this time reach Earth at $t_1' = t_1 + R/c$, where R is the distance to the source from Earth.
  - At t_2, the cloud is at a distance $v(t_2 - t_2)$ from the source. Its distance from the source projected on the sky is $d_\perp = v(t_2 - t_1)\sin\theta$, whereas its distance from Earth is $R - v(t_2 - t_1)\cos\theta$.
  - Therefore, the photons that left at time t_2 reach Earth at the time $t_2' = t_2 + \left[R - v(t_2 - t_1)\cos\theta\right]/c$.
  - Thus, to an observer on Earth, the cloud appears to have moved the distance $d_\perp$ in the time interval $t_2' - t_1' = (t_2 - t_1)(1 - v/c\cos\theta)$.
  - The "apparent" velocity $v_{\mathrm{app}} \equiv d_\perp/(t_2' - t_1')$ of the cloud on the sky is

$$\mathrm{Eq.(1):}\quad \boxed{\beta_{\mathrm{app}} = \frac{\beta\sin\theta}{1 - \beta\cos\theta},\quad \text{where } \beta_{\mathrm{app}} = v_{\mathrm{app}}/c, \text{ and } \beta = v/c}$$

$$\beta_{\text{app}} = \frac{\beta \sin\theta}{1 - \beta\cos\theta}$$

This factor reduces the arrival time interval of the signals

In the figure, dashed lines denote $\beta_0 = \beta \sin\theta$.



- When the actual speed of the object is close to the speed of light, the apparent speed can be observed as greater than the speed of light.
- **The superluminal effect arises because of the quickly decreasing path length between a rapidly approaching object and the observer.** The detected signal at t_2 is observed at a greatly reduced interval by the observer because this signal has a much shorter distance to travel.

Lorentz Invariant

- **Lorentz invariant**: A quantity (scalar) that remains unchanged by a Lorentz transform is said to be a "Lorentz invariant." For instance,

$$\begin{aligned} x'^2 + y'^2 + z'^2 - c^2 t'^2 &= \gamma^2 \left(x - \beta ct\right)^2 + y^2 + z^2 - \gamma^2 \left(ct - \beta x\right)^2 \\ &= \gamma^2 \left(1 - \beta^2\right) x^2 + y^2 + z^2 + \gamma^2 \left(\beta^2 c^2 - c^2\right) t^2 \\ &= x^2 + y^2 + z^2 - c^2 t^2 \end{aligned}$$

- **Proper distance**: Since all events are subject to the same transformation, the space-time "interval" between two event is also invariant.

$$ds^2 \equiv dx^2 + dy^2 + dz^2 - c^2 dt^2$$

This is the spatial distance between two events occurring at the same time ($dt = 0$). This is called the proper distance between the two points.

- **Proper time** interval: $c^2 d\tau^2 \equiv -ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

This measures time intervals between events occurring at the same spatial location ($dx = dy = dz = 0$).

If the coordinate differentials refer to the position of the origin of another reference frame traveling with velocity v, then

$$(d\tau)^2 = (dt)^2 - \frac{(dt)^2}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = (dt)^2\left(1 - \frac{v^2}{c^2}\right) \quad \longrightarrow \quad d\tau = dt\left(1 - \beta^2\right)^{1/2} = dt/\gamma$$

This is the time dilation formula in which $d\tau$ is the time interval measured by the observer in motion.

* Four-Vectors *

- **Four-vector**: Invariant in 3D rotations: $dx^2 + dy^2 + dz^2$

  By analogy, the invariance of the space-time interval suggests to define a vector in 4D space (4 dimensional space-time vector or four-vector). The quantities x^μ $(\mu = 0,1,2,3)$ define coordinates of an event in space-time.

$$\vec{x} \equiv x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}$$

Contravariant components

- **Minkowski space**: The fact that the expression for s^2 contains a minus sign in front on c^2t^2 means that space-time is not a Euclidean space; it is a special space called Minkowski space. Such space can be handled in two ways, either by including $\sqrt{-1}$ in the definition of the time component or by introduction of a ***metric***. Once the notational difficulties of the metric approach are mastered, it is not much more complicated than the $\sqrt{-1}$ approach.

  A metric tensor allows defining lengths of curves, angles, and distances in differential geometry. Let's define **Minkowski metric**, which can be presented in the 4×4 matrix:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that this metric is symmetric: $\eta_{\mu\nu} = \eta_{\nu\mu}$

- **Summation convention**:

  The invariant can now be written in terms of the Minkowski metric:

$$s^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\mu x^\nu$$

  An important and beautiful notational advance (originated by Einstein) is the summation convention. In any single term containing **a Greek index repeated twice (between contravariant and covariant indices)**, a summation is implied over that index. This index is often called a dummy index.

  Therefore, we can write the invariant s^2 without the summation signs.

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu$$

  An important point is that an index cannot be repeated more than twice in a single term; for example, the combination $\eta_{\mu\mu} x^\mu$ is regarded as meaningless.

- **Contravariant/Covariant components**

$$\boxed{\mathbf{v} = v^x \mathbf{e}_x + v^y \mathbf{e}_y + v^z \mathbf{e}_z}$$

contravariant components: (superscripted) $x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$

covariant components: (subscripted) $x_\mu = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -ct \\ x \\ y \\ z \end{pmatrix}$

They are related by $\boxed{x_\mu = \eta_{\mu\nu} x^\nu, \quad x^\mu = \eta^{\mu\nu} x_\nu}$.

The metric can be used to raise or lower indices.

Now, the invariant s^2 can be written simply

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu \quad \rightarrow \quad s^2 = x^\mu x_\mu$$

Note that summation on indices occurs only between contravariant and covariant indices.

The components of a position (velocity etc.) vector *contra-vary* with a change of basis vectors to compensate. Transformation rules between the following two vector components are inverse. This is the basic idea of "contravariant" and "covariant."

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu, \qquad \frac{\partial A}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial A}{\partial x^\nu}$$

- **Lorentz transform** (corresponding to a boost along the *x* axis) can be written in terms of a transformation matrix.

Lorentz transformation:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$$

transformation matrix:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Any arbitrary Lorentz transformation can be written in the above form, since the spatial 3D rotation necessary to align the *x* axes before and after the boost are also of linear form.

- **Conditions for the Lorentz transformation**:

  From the invariance of s^2, we must have

$$\eta_{\mu\nu} x^\mu x^\nu = \eta_{\sigma\tau} x'^\sigma x'^\tau = \eta_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu x^\mu x^\nu$$

  This can be true for arbitrary x^μ only if

$$\eta_{\mu\nu} = \eta_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \quad \text{or equivalently} \quad \boldsymbol{\eta} = \boldsymbol{\Lambda}^{\mathrm{T}} \boldsymbol{\eta} \boldsymbol{\Lambda} \quad \text{in matrix form}$$

  Taking determinants yields

$$\det \boldsymbol{\Lambda} = \pm 1$$

  Proper Lorentz transformations (to keep the right-handness), which rules out reflections such as $x \to -x$.

$$\det \boldsymbol{\Lambda} = 1$$

  Isochronous Lorentz transformations (to ensure that the sense of flow of time is the same in two frames)

$$\Lambda^0{}_0 \geq 1$$

- The **Lorentz transformation** of the covariant component can be obtained as follows:

$$x'_\mu = \eta_{\mu\tau} x'^\tau = \eta_{\mu\tau} \Lambda^\tau{}_\sigma x^\sigma = \eta_{\mu\tau} \Lambda^\tau{}_\sigma \eta^{\sigma\nu} x_\nu$$

$$\boxed{\begin{aligned} \therefore \quad x'_\mu = \widetilde{\Lambda}_\mu{}^\nu x_\nu \quad \text{where} \quad & \widetilde{\Lambda}_\mu{}^\nu \equiv \eta_{\mu\tau} \Lambda^\tau{}_\sigma \eta^{\sigma\nu} \\ & \widetilde{\Lambda}_\mu{}^\nu = \frac{\partial x'_\mu}{\partial x_\nu} \end{aligned}}$$

- From the invariance of $s^2 = x^\mu x_\mu$:

$$x'^\sigma x'_\sigma = \Lambda^\sigma{}_\nu x^\nu \widetilde{\Lambda}_\sigma{}^\mu x_\mu = \Lambda^\sigma{}_\nu \widetilde{\Lambda}_\sigma{}^\mu x^\nu x_\mu$$

$$\therefore\ \Lambda^\sigma{}_\nu \widetilde{\Lambda}_\sigma{}^\mu = \delta^\mu{}_\nu$$
$$\therefore\ \widetilde{\Lambda}_\sigma{}^\mu = \left(\Lambda^{-1}\right)^\mu{}_\sigma$$

where we have introduced the Kronecker delta

$$\delta^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ Identity matrix

- For any arbitrary contravariant components,

$$Q^\mu = \delta^\mu{}_\nu Q^\nu$$

- Note that

$$\eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu{}_\nu$$

- Inverse transform

$$\widetilde{\Lambda}_\sigma{}^\mu \times (x'^\sigma = \Lambda^\sigma{}_\nu x^\nu) \xrightarrow{\text{using } \Lambda^\sigma{}_\nu \widetilde{\Lambda}_\sigma{}^\mu = \delta^\mu{}_\nu} x^\mu = \widetilde{\Lambda}_\sigma{}^\mu x'^\sigma \qquad \text{note :}\quad \widetilde{\Lambda}_\sigma{}^\mu = \left(\Lambda^{-1}\right)^\mu{}_\sigma$$

Other Four-vectors

- Four-vector: is defined such that the transformation of components between any two frames is given by the same transformation law as applies to x^μ

contravariant	covariant
$\vec{A} \rightarrow \quad A^\mu = \eta^{\mu\nu} A_\nu$	$A_\mu = \eta_{\mu\nu} A^\nu$
$A'^\mu = \Lambda^\mu{}_\nu A^\nu$	$A'_\mu = \widetilde{\Lambda}_\mu{}^\nu A_\nu$

- Let us consider two four-vectors $\overrightarrow{A}$ and $\overrightarrow{B}$. We define the scalar product of them.

$$A'^\mu B'_\mu = \Lambda^\mu{}_\nu \widetilde{\Lambda}_\mu{}^\sigma A^\nu B_\sigma = \delta^\sigma{}_\nu A^\nu B_\sigma = A^\nu B_\nu \quad \longrightarrow \quad \boxed{\vec{A} \cdot \vec{B} = A^\mu B_\mu = A'^\mu B'_\mu}$$

Therefore, the scalar product of any two four-vectors is a Lorentz invariant or scalar. In particular, the "square" of a four vector is an invariant. Thus, our starting point, the invariance of $s^2 = x^\mu x_\mu$, is seen to be a general property of four-vectors.

- Note

$$\begin{aligned} \vec{A} \cdot \vec{A} > 0 \quad &\rightarrow \quad \text{spacelike four} - \text{vector} \\ = 0 \quad &\rightarrow \quad \text{light} - \text{like (or null) four} - \text{vector} \\ < 0 \quad &\rightarrow \quad \text{time} - \text{like four} - \text{vector} \end{aligned}$$

$$\begin{aligned} A^0 \quad &\rightarrow \quad \text{time component} \\ A^i \quad &\rightarrow \quad \text{space components (ordinary three} - \text{vector)} \end{aligned}$$

$$\vec{A} \cdot \vec{B} = -A^0 B_0 + \mathbf{A} \cdot \mathbf{B} = -A^0 B_0 + A^i B_i \quad (i = 1, 2, 3)$$

Four-velocity

The (infinitesimally small) difference between the coordinates of two events is also a four-vector. Dividing by the proper time yields a four-vector, the four-velocity:

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \quad \longrightarrow \quad U^0 = \frac{cdt}{d\tau} = c\gamma_u, \quad U^i = \frac{dx^i}{d\tau} = \gamma_u u^i$$

$d\tau = dt/\gamma$

or

$$\boxed{\vec{U} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}}$$

where $\gamma_u \equiv \left(1 - u^2/c^2\right)^{-1/2}$, $u \equiv \left|\frac{d\mathbf{x}}{dt}\right|$

length of the four-velocity : $\boxed{\vec{U} \cdot \vec{U} = U^\mu U_\mu = -\left(\gamma_u c\right)^2 + \left(\gamma_u \mathbf{u}\right)^2 = -c^2}$

Transformation of the four-velocity:

$$\begin{aligned} U'^0 &= \gamma\left(U^0 - \beta U^1\right) \\ U'^1 &= \gamma\left(-\beta U^0 + U^1\right) \\ U'^2 &= U^2 \\ U'^3 &= U^3 \end{aligned} \quad \longrightarrow \quad \begin{aligned} \gamma_{u'} c &= \gamma\left(c\gamma_u - \beta\gamma_u u^1\right) \\ \gamma_{u'} u'^1 &= \gamma\left(-\beta c\gamma_u + \gamma_u u^1\right) \\ \gamma_{u'} u'^2 &= \gamma_u u^2 \\ \gamma_{u'} u'^3 &= \gamma_u u^3 \end{aligned}$$

$\longrightarrow$ The first two equations become:

$$\gamma_{u'} = \gamma\gamma_u\left(1 - vu'/c^2\right)$$
$$\gamma_{u'} u'^1 = \gamma\gamma_u\left(u^1 - v\right)$$

Note: γ denotes the factor for the relative velocity between two frames. γ_u and $\gamma_{u'}$ are the factors for a velocity vector measured in K and K', respectively.

$\longrightarrow$ velocity component: $u'^1 = \dfrac{u^1 - v}{1 - vu^1/c^2}$ This is the previously derived formula.

speed: $\gamma_{u'} = \gamma\gamma_u\left(1 - \dfrac{vu^1}{c^2}\right)$ This is the transform for speed.

Here, $u^1 = u\cos\theta$ and $u'^1 = u'\cos\theta'$

Momentum and Energy

- Four-momentum of a particle with a mass m_0 is defined by

$$P^\mu \equiv m_0 U^\mu \qquad \begin{aligned} P^0 &= m_0 c \gamma_v \\ P^i &= \gamma_v m_0 \mathbf{v} \end{aligned}$$

- In the nonrelativistic limit,

$$P^0 c = m_0 c^2 \gamma = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = m_0 c^2 + \frac{1}{2} m_0 v^2 + \cdots$$

Therefore, we interpret $E \equiv P^0 c = \gamma_v m_0 c^2$ as the total energy of the particle.

The quantity $m_0 c^2$ is interpreted as the rest energy of the particle.

Then,

$$\mathbf{p} \equiv \gamma_v m_0 \mathbf{v}, \quad P^\mu = (E/c,\ \mathbf{p})$$ Here, **p** is the spatial component of the four-momentum.

Since $\vec{U}^2 = -c^2$, we obtain $\vec{P}^2 = -m_0^2 c^2$. Comparing with $\vec{P}^2 = -\dfrac{E^2}{c^2} + |\mathbf{p}|^2$, we obtain

$$\boxed{E^2 = m_0^2 c^4 + c^2 |\mathbf{p}|^2}$$

- Photons are massless, but we can still define

$$\boxed{P^\mu = (E/c,\ \mathbf{p}), \quad E = |\mathbf{p}|\, c \quad \rightarrow \quad \vec{P}^2 = 0}$$ for photons

Homework (due data: 10/26)

[Q7] See pages 21-23

(1) From Eq.(1), show that the condition for the apparent velocity to exceed the speed of light ($\beta_{\text{app}} > 1$) is given by Eq. (2).

$$\text{Eq.}(1): \quad \beta_{\text{app}} = \frac{\beta \sin\theta}{1 - \beta\cos\theta}$$

$$\text{Eq.}(2): \quad \sin\theta + \cos\theta \geq \frac{1}{\beta}$$

(2) Take a square on both sides of Eq. (2) and show that the condition for the superluminal motion is

$$\text{Eq.}(3): \quad \sin 2\theta \geq \frac{1}{\beta^2} - 1$$

(5) Note that $\sin 2\theta$ is symmetric about $\theta = \pi/4$. By setting $\theta = \pi/4 + x$, show that Eq.(3) is equivalent to the following condition.

$$\text{Eq.}(4): \quad \left|\theta - \frac{\pi}{4}\right| \leq \frac{1}{2}\cos^{-1}\left(\frac{1}{\beta^2} - 1\right)$$

(6) From Eq.(3) or (4), show that there is a limit on β below which the source will never appear superluminal. The limit is $\beta_{\text{min}} = 1/\sqrt{2}$. This result indicates that the cloud should move relativistically at least at a velocity of ~ 71% of the speed of light to show the superluminal motion.

(7) Differentiate the right hand side of Eq.(1) with respect to θ and calculate the angle θ_{max} at which the apparent velocity is maximal. What is the maximal, apparent velocity $\beta_{\text{app}}^{\text{max}}$?