Astrophysics

Lecture 06
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Resonance Line

- Resonance Line

  A spectral line caused by an electron jumping between ***the ground state*** and ***the first energy level*** in an atom or ion. It is the longest wavelength line produced by a jump to or from the ground state.

  Because the majority of electrons are in the ground state in many astrophysical environments, and because the energy required to reach the first level is the least needed for any transition, resonance lines are usually the strongest lines in the spectrum for any given atom or ion.

Resonance Lines

[Draine, Physics of the interstellar and intergalactic medium]

Table 9.4 Selected Resonance Lines[a] with $\lambda < 3000$ Å

	Configurations	ℓ	u	$E_\ell/hc(\mathrm{cm}^{-1})$	$\lambda_{\rm vac}(\text{Å})$	$f_{\ell u}$
C IV	$1s^2 2s - 1s^2 2p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1550.772	0.0962
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1548.202	0.190
N V	$1s^2 2s - 1s^2 2p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1242.804	0.0780
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1242.821	0.156
O VI	$1s^2 2s - 1s^2 2p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1037.613	0.066
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1037.921	0.133
C III	$2s^2 - 2s2p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	977.02	0.7586
C II	$2s^2 2p - 2s2p^2$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{D}^o_{3/2}$	0	1334.532	0.127
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{D}^o_{5/2}$	63.42	1335.708	0.114
N III	$2s^2 2p - 2s2p^2$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{D}^o_{3/2}$	0	989.790	0.123
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{D}^o_{5/2}$	174.4	991.577	0.110
C I	$2s^2 2p^2 - 2s^2 2p3s$	$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	0	1656.928	0.140
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	16.40	1656.267	0.0588
		$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	43.40	1657.008	0.104
N II	$2s^2 2p^2 - 2s2p^3$	$^3\mathrm{P}_0$	$^3\mathrm{D}^o_1$	0	1083.990	0.115
		$^3\mathrm{P}_1$	$^3\mathrm{D}^o_2$	48.7	1084.580	0.0861
		$^3\mathrm{P}_2$	$^3\mathrm{D}^o_3$	130.8	1085.701	0.0957
N I	$2s^2 2p^3 - 2s^2 2p^2 3s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1199.550	0.130
		$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{3/2}$	0	1200.223	0.0862
O I	$2s^2 2p^4 - 2s^2 2p^3 3s$	$^3\mathrm{P}_2$	$^3\mathrm{S}^o_1$	0	1302.168	0.0520
		$^3\mathrm{P}_1$	$^3\mathrm{S}^o_1$	158.265	1304.858	0.0518
		$^3\mathrm{P}_0$	$^3\mathrm{S}^o_1$	226.977	1306.029	0.0519
Mg II	$2p^6 3s - 2p^6 3p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	2803.531	0.303
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	2796.352	0.608
Al III	$2p^6 3s - 2p^6 3p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1862.790	0.277
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1854.716	0.557

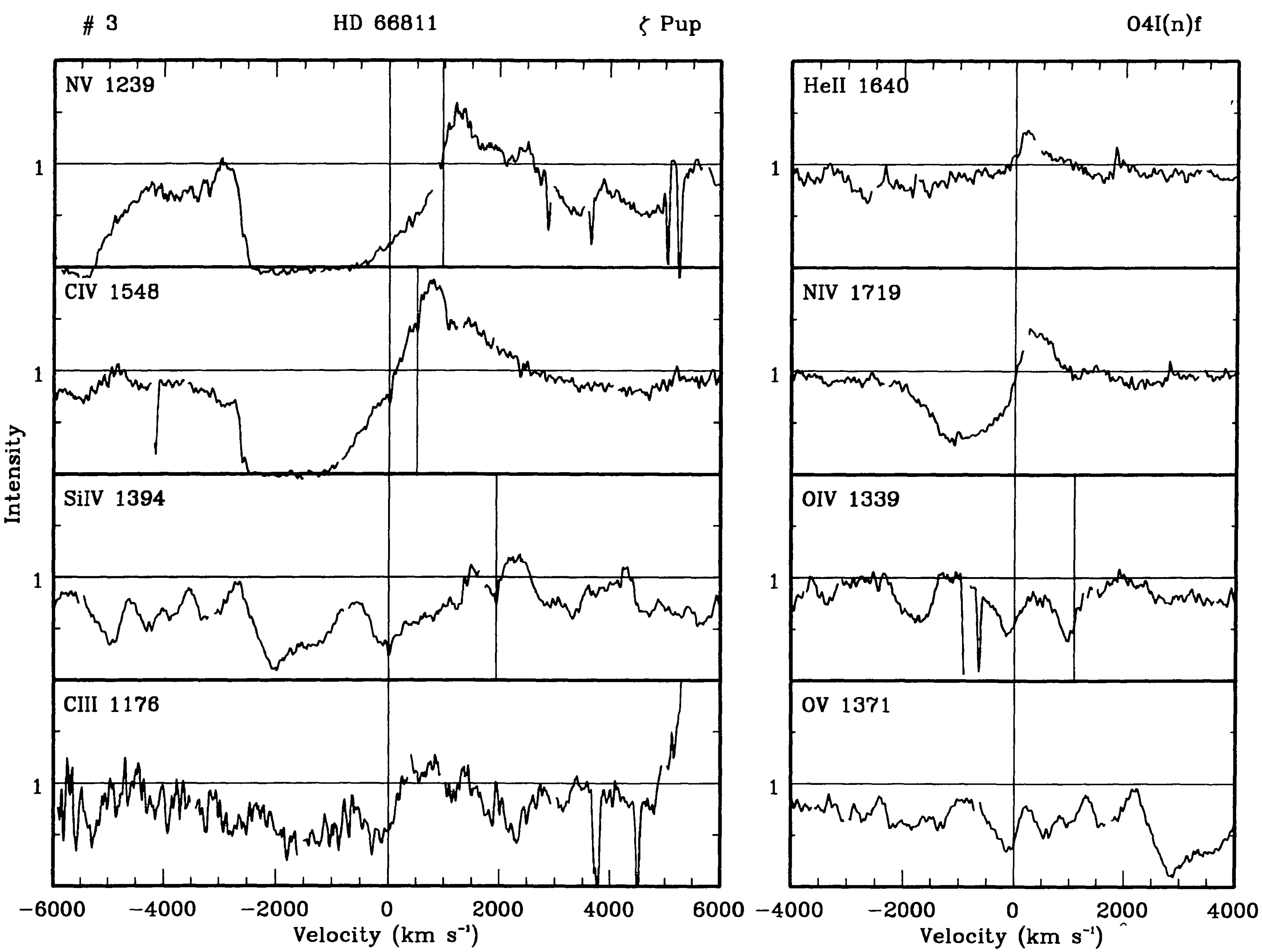
Table 9.4 contd.

	Configurations	ℓ	u	$E_\ell/hc\,(\mathrm{cm}^{-1})$	$\lambda_{\rm vac}(\text{Å})$	$f_{\ell u}$
Mg I	$2p^6 3s^2 - 2p^6 3s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	2852.964	1.80
Al II	$2p^6 3s^2 - 2p^6 3s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	1670.787	1.83
Si III	$2p^6 3s^2 - 2p^6 3s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	1206.51	1.67
P IV	$2p^6 3s^2 - 2p^6 3s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	950.655	1.60
Si II	$3s^2 3p - 3s^2 4s$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{S}_{1/2}$	0	1526.72	0.133
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{S}_{1/2}$	287.24	1533.45	0.133
P III	$3s^2 3p - 3s3p^2$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{D}_{3/2}$	0	1334.808	0.029
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{D}_{5/2}$	559.14	1344.327	0.026
Si I	$3s^2 3p^2 - 3s^2 3p4s$	$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	0	2515.08	0.17
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	77.115	2507.652	0.0732
		$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	223.157	2516.870	0.115
P II	$3s^2 3p^2 - 3s3p^3$	$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	0	1301.87	0.038
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	164.9	1305.48	0.016
		$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	469.12	1310.70	0.115
S III	$3s^2 3p^2 - 3s3p^3$	$^3\mathrm{P}_0$	$^3\mathrm{D}^o_1$	0	1190.206	0.61
		$^3\mathrm{P}_1$	$^3\mathrm{D}^o_2$	298.69	1194.061	0.46
		$^3\mathrm{P}_2$	$^3\mathrm{D}^o_3$	833.08	1200.07	0.51
Cl IV	$3s^2 3p^2 - 3s3p^3$	$^3\mathrm{P}_0$	$^3\mathrm{D}^o_1$	0	973.21	0.55
		$^3\mathrm{P}_1$	$^3\mathrm{D}^o_2$	492.0	977.56	0.41
		$^3\mathrm{P}_2$	$^3\mathrm{D}^o_3$	1341.9	984.95	0.47
P I	$3s^2 3p^3 - 3s^2 3p^2 4s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1774.951	0.154
S II	$3s^2 3p^3 - 3s^2 3p^2 4s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1259.518	0.12
Cl III	$3s^2 3p^3 - 3s^2 3p^2 4s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1015.019	0.58
S I	$3s^2 3p^4 - 3s^2 3p^3 4s$	$^3\mathrm{P}_2$	$^3\mathrm{S}^o_1$	0	1807.311	0.11
		$^3\mathrm{P}_1$	$^3\mathrm{S}^o_1$	396.055	1820.343	0.11
		$^3\mathrm{P}_0$	$^3\mathrm{S}^o_1$	573.640	1826.245	0.11
Cl II	$3s^2 3p^4 - 3s3p^5$	$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	0	1071.036	0.014
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	696.00	1079.080	0.00793
		$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	996.47	1075.230	0.019
Cl I	$3s^2 3p^5 - 3s^2 3p^4 4s$	$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{P}_{3/2}$	0	1347.240	0.114
		$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{P}_{3/2}$	882.352	1351.657	0.0885
Ar II	$3s^2 3p^5 - 3s3p^6$	$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{S}_{1/2}$	0	919.781	0.0089
		$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{S}_{1/2}$	1431.583	932.054	0.0087
Ar I	$3p^6 - 3p^5 4s$	$^1\mathrm{S}_0$	$^2[1/2]^o$	0	1048.220	0.25

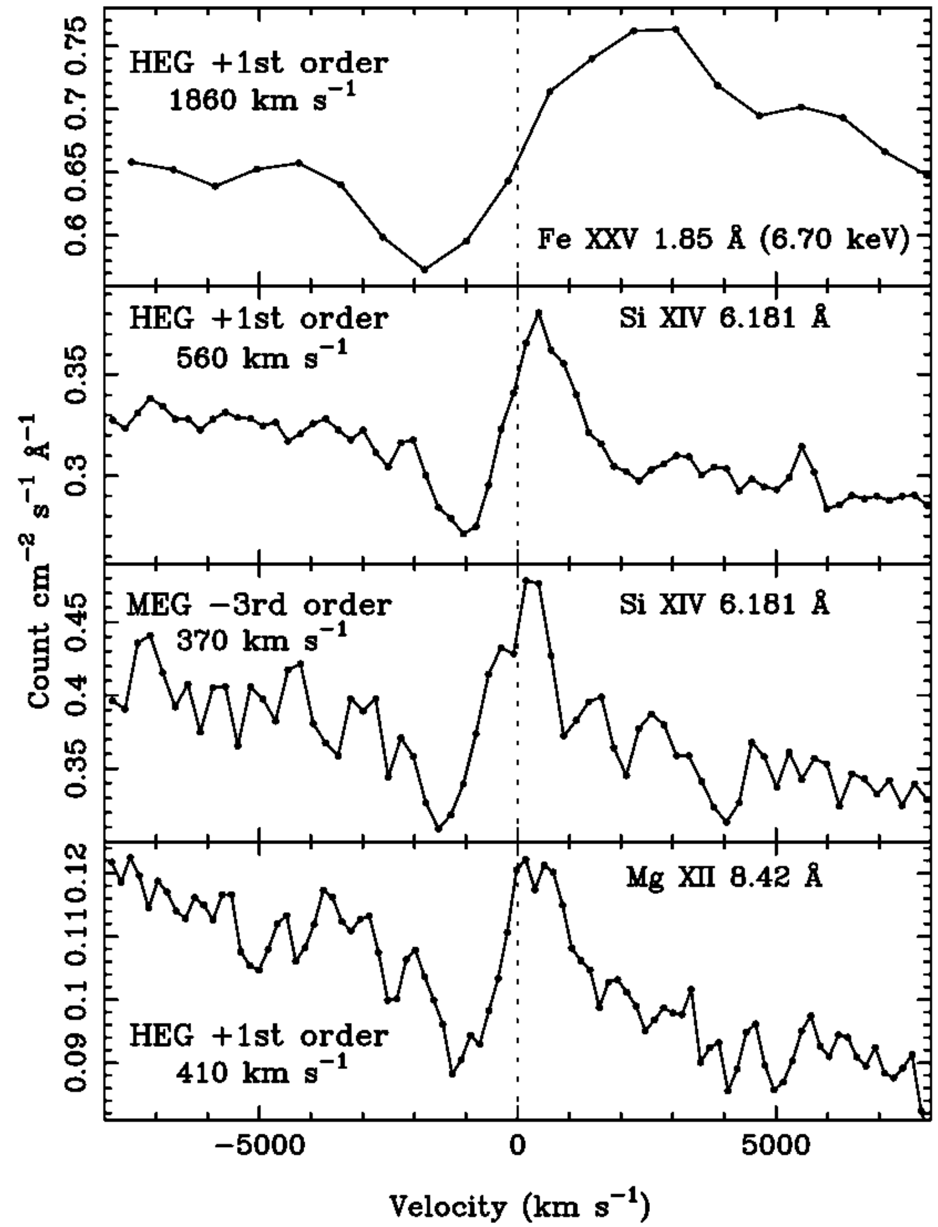
[a] Transition data from NIST Atomic Spectra Database v4.0.0 (Ralchenko et al. 2010)

P Cygni Profile

- The P Cygni profile is characterized by strong emission lines with corresponding blueshifted absorption (+ redshifted emission) line.



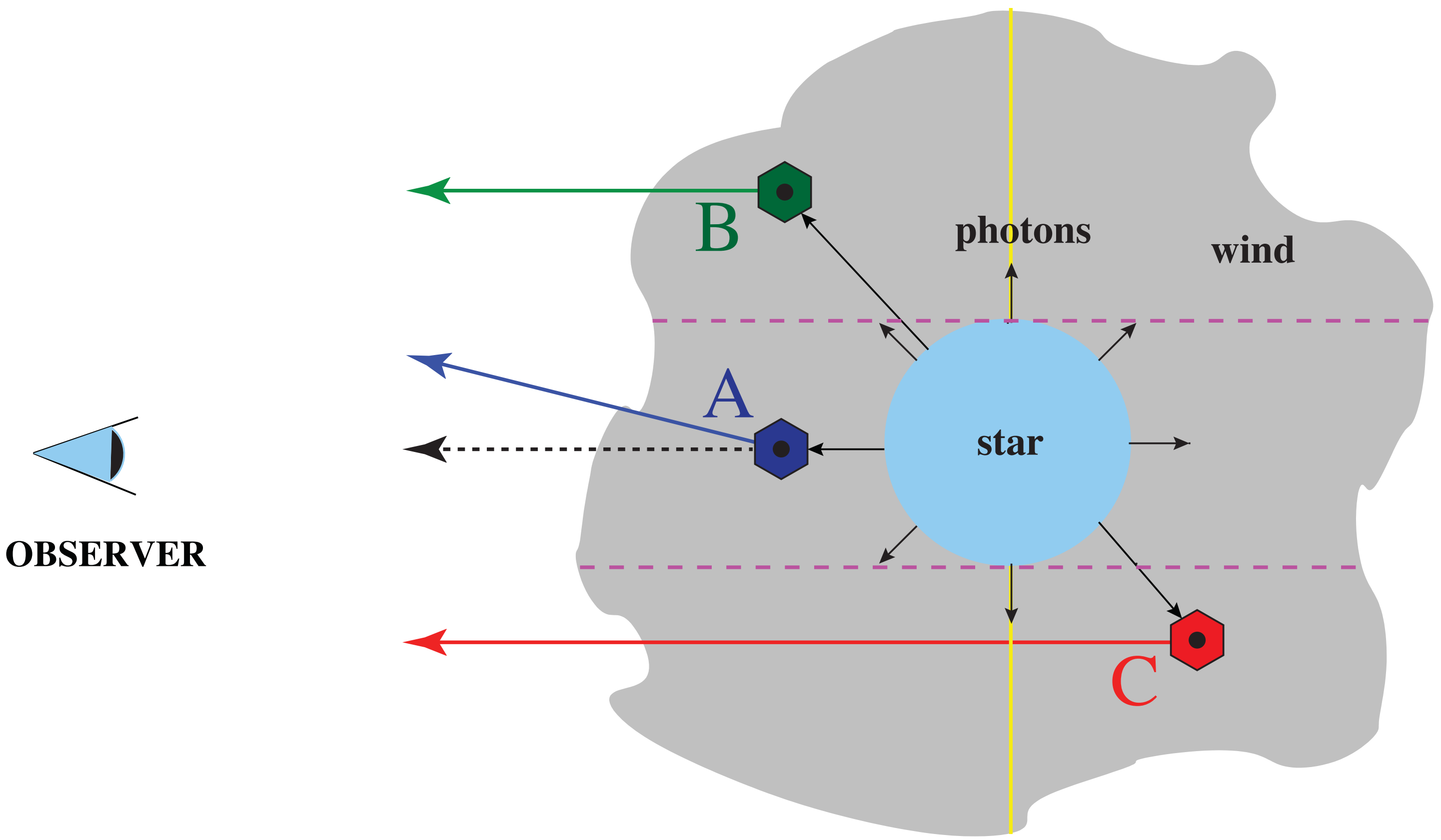
zeta Puppis (Snow et al., 1994, ApJS, 95, 163)



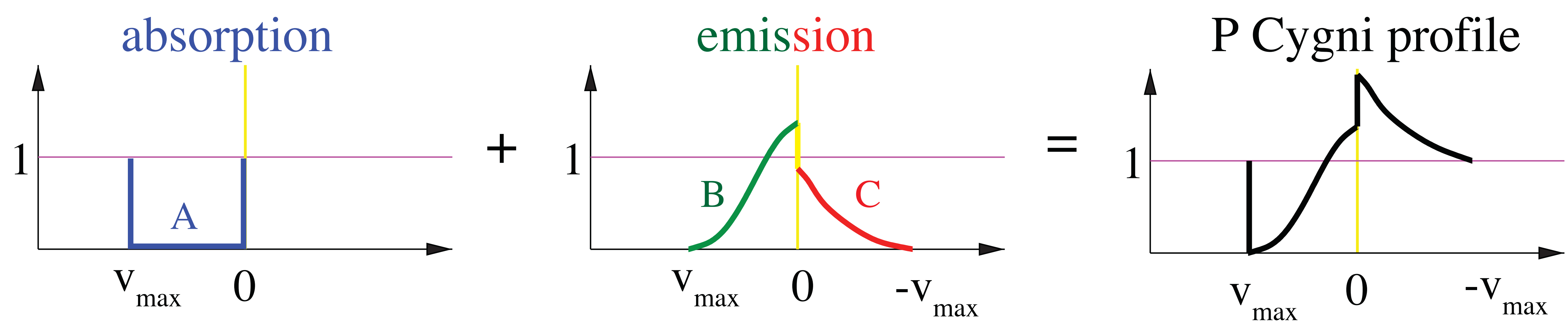
Circinus X-1
(Brandt & Schulz, 2000, ApJ, 544, L123)

P Cygni profile formation

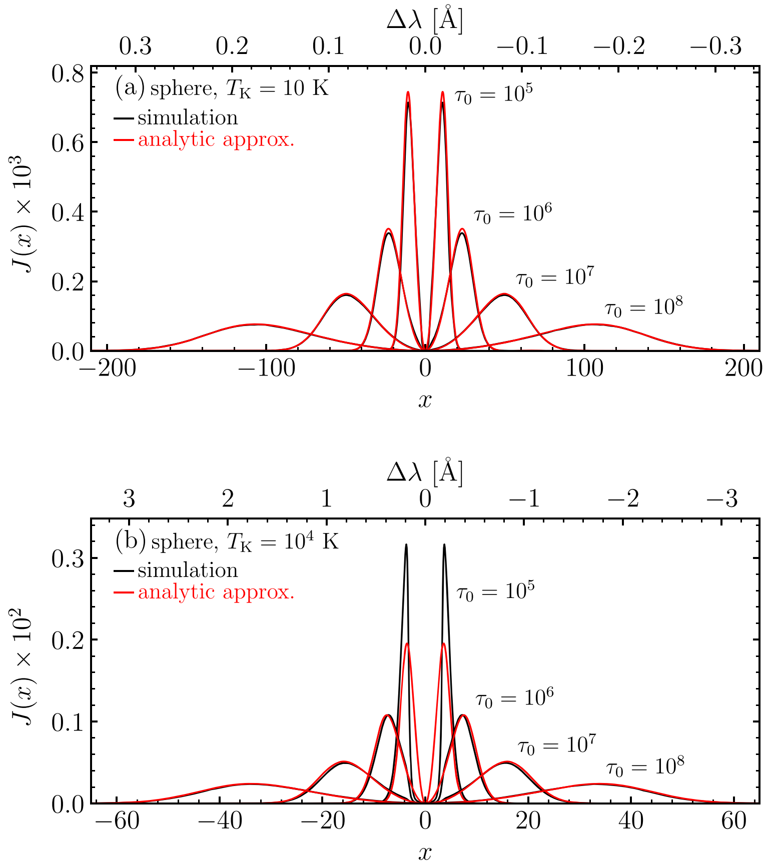
- The blueshifted absorption line is produced by material moving away from the star and toward us, whereas the emission come from other parts of the expanding shell.



credit: Joachim Puls
slightly modified

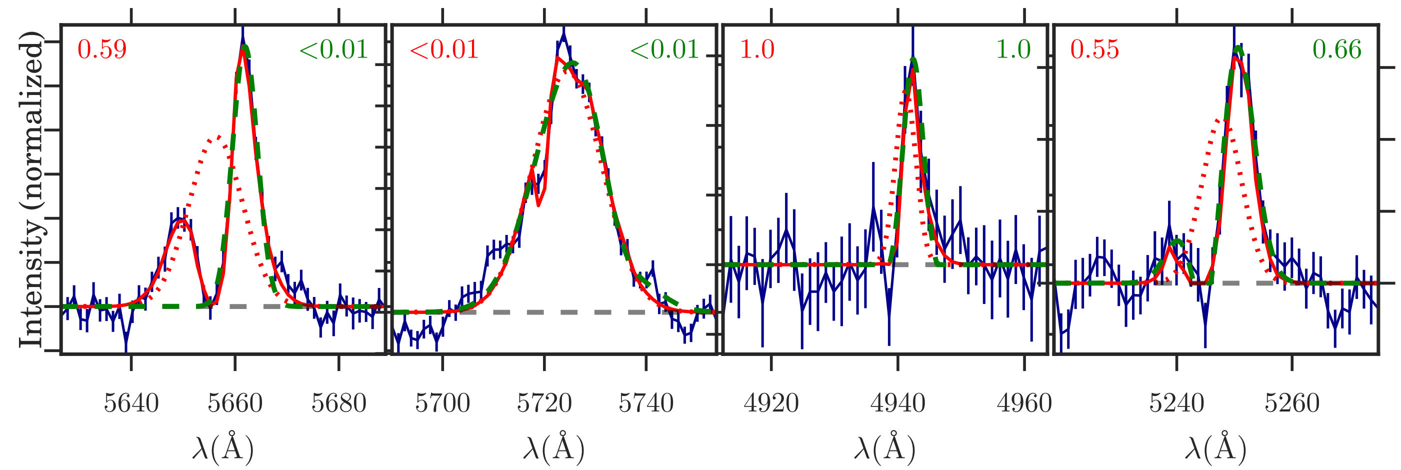


Lyα Resonance Scattering



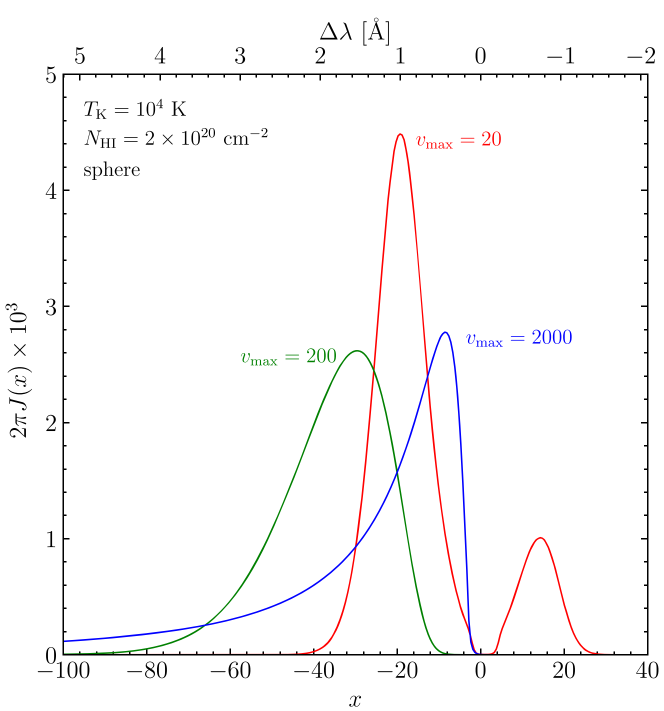
Seon & Kim (2020)

Emergent Lyα spectra from a static, homogeneous sphere at (a) $T = 10$ K and (b) 10^4 K, with different optical depths ($\tau_0 = 10^5 - 10^8$). The black curves are line profiles calculated with LaRT. The red curves denote an analytic series solution, which was derived from the series solution of Dijkstra et al. (2006).



Example spectral fits of the "shell" and "Gaussian-minus-Gaussian" models with all combinations of fitting failures/successes. The blue lines show the data, the red solid line the shell model fit (with the intrinsic spectrum as dotted red line), and the green dashed line shows the "Gaussian-minus-Gaussian" best fit. The numbers in the panel show the $p(\chi^2)$ values of the best fits in the corresponding colors.

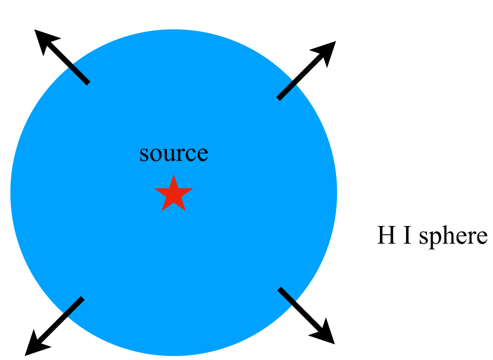
Gronke (2017)



Emergent Lyα for the dynamic motion test cases, in which the gas expands isotropically and has a temperature of $T = 10^4$ K and a column density of $N_{HI} = 2 \times 10^{20}\,\mathrm{cm}^{-2}$. The maximum velocity V_{max} of the Hubble- like outflow is denoted in units of km s^{-1}. The ordinate is the mean intensity integrated over the solid angle outgoing from the spherical surface.

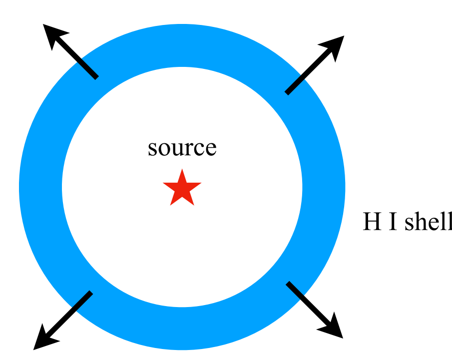
Seon & Kim (2020)

Hubble-like outflow



$$V = V_{max}\frac{r}{r_{max}}$$

spherically expanding shell model



$$V = V_{shell} \text{ (constant)}$$

Semiclassical (Weissokpf-Woolley) Picture of Quantum Levels

- In the semiclassical picture, each level is viewed as a continuous distribution of sublevels with energies close to the energy of the level (E_n).

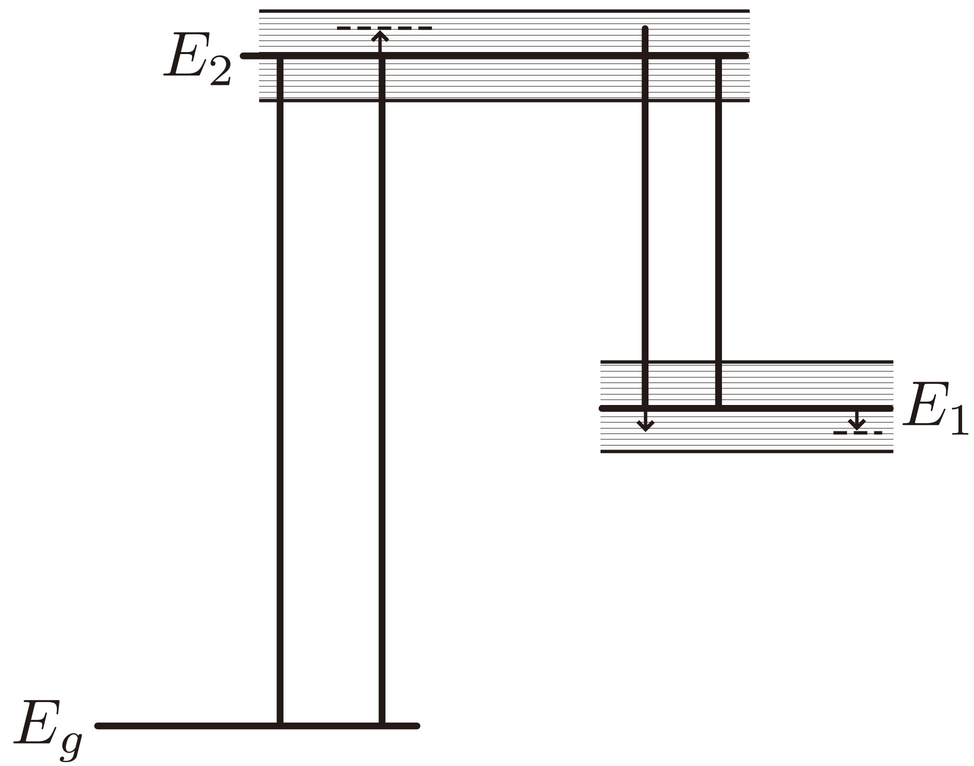
  The distribution of sublevels are explained by the Heisenberg Uncertainty Principle. The level has a lifetime $\Delta t = 1/A$ (A = Einstein A coefficient) and a spread in energy about $\Delta E \approx \hbar/\Delta t = \hbar A$.

$$\Delta E \Delta t \approx \hbar$$

The ground level has no spread in energy

because $\Delta t = \infty$.

The atom is in a definite sublevel of some level.



A transition in a spectral line is considered to be an instantaneous transition between a definite sublevel of an initial level to a definite sublevel of a final level.

The energy spread of a sublevel is described by a Lorentzian profile.

Raman Scattering*

- Raman scattering or the Raman effect is the inelastic scattering of a photon.

  When photons are scattered from an atom or molecule, **most photons are elastically scattered (i.e., Rayleigh scattering)**, such that the scattered photons have the same energy (frequency and wavelength) as the incident photons. However, a small fraction of the scattered photons (approximately 1 in 10 million) are scattered by an excitation, with the scattered photons having a frequency different from, and usually lower than, that of the incident photons.

  Typically this effect involves vibrational energy being gained by a molecule as incident photons from a visible laser are shifted to lower energy.

- Astrophysical Example: **Scattering of O VI doublet (λλ1038, 1032) by neutral hydrogen.**

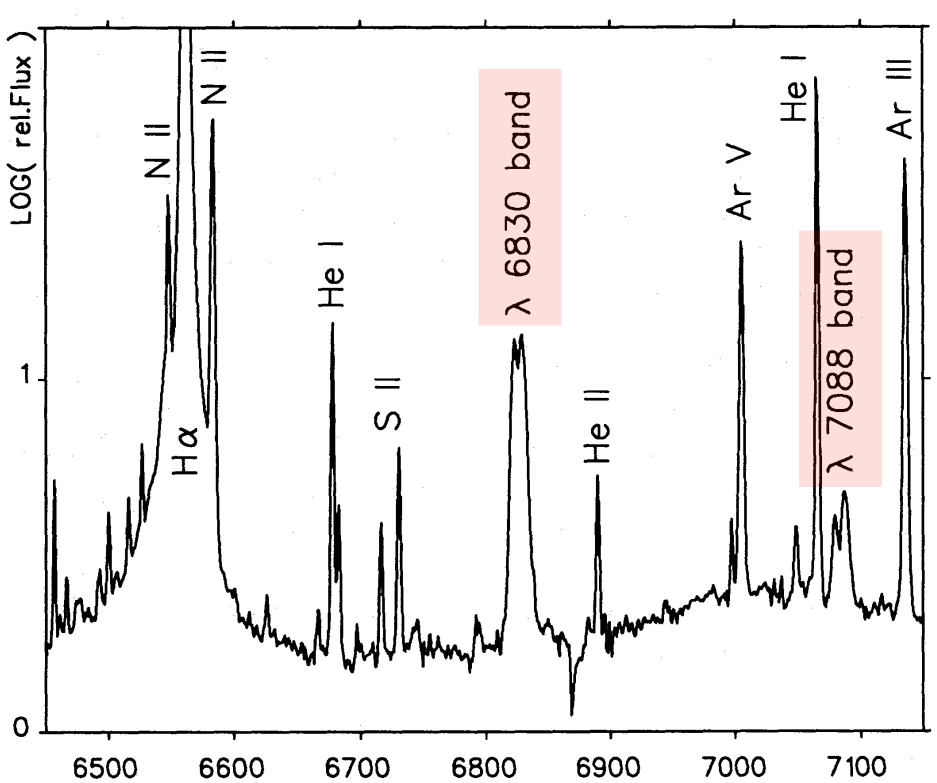
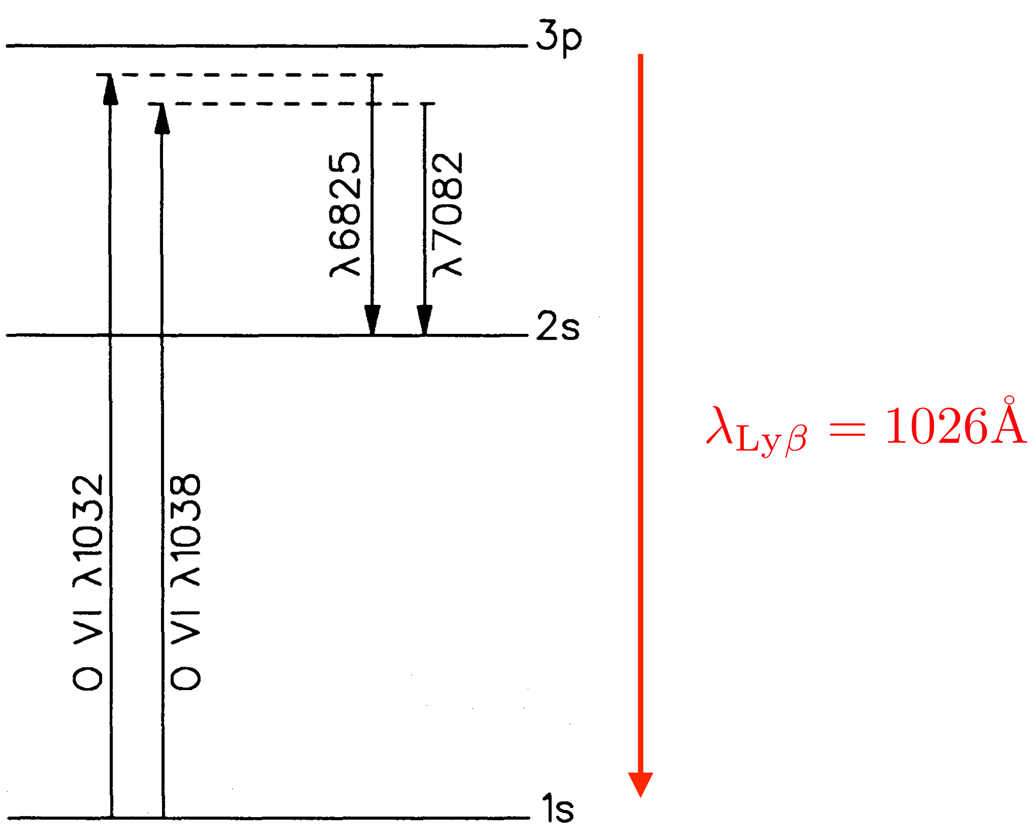


Fig.1. Raman scattered emission bands in the symbiotic star V1016 Cyg. The spectrum was obtained on the 1.93m telescope at the Observatoire de Haute Provence.



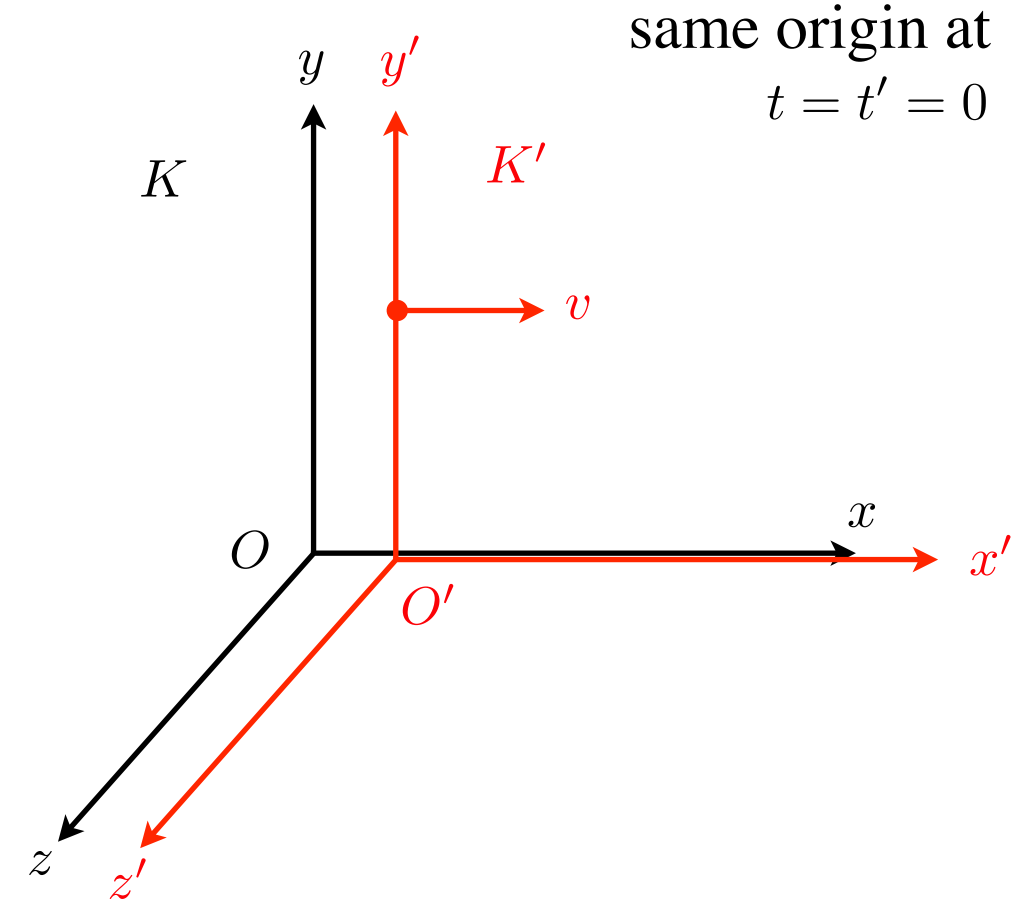
Schmid (1989, A&A, 211, L31)

Relativistic Covariance and Kinematics

Galilean Transformation/Relativity

- ***Galilean transformation*** is used to transform between the coordinates of two **inertial frames of reference** which differ only by constant relative motion within the constructs of Newtonian physics.

same origin at
$t = t' = 0$



$$
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

Newton's law is invariant under the Galilean transformation.

However, Maxwell's equations are not invariant under the Galilean transformation.

Let us consider two frames K and K', as shown above, with a relative uniform velocity v.

- ***Lorentz transformation*** **is the result of attempts by Lorentz and others to explain how the speed of light was observed to be independent of the reference frame, and to understand the symmetries of the Maxwell's equations.**

* Review of Lorentz Transformations *

- **Postulates in the special theory of relativity**

  (1) The laws of nature are the same in two frames of reference in uniform relative motion with no rotation.

  (2) The speed of light is c in all such frames.

- **space-time event**: an event that takes place at a location in space and time.

- **Derivation of Lorentz transforms**:

  If a pulse of light is emitted at the origin at $t = 0$, each observer will see an expanding sphere centered on his own origin. Therefore, we have the equations of the expanding sphere in each frame.

$$x^2 + y^2 + x^2 - c^2t^2 = 0, \quad x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \qquad (1)$$

  Since space is assumed to be homogeneous, the transformation must be linear.

$$x' = a_1 x + a_2 t, \quad y' = y, \quad z' = z, \quad t' = b_1 x + b_2 t$$

  We note that the origin of K' $(x' = 0)$ is a point that moves with speed v as seen in K. Its location in K is given by $x = vt$. Therefore, we have

$$\begin{aligned} x' &= a_1 x + a_2 t \\ 0 &= a_1(vt) + a_2 t \end{aligned} \qquad \frac{a_2}{a_1} = -v$$

$$\begin{aligned} x' &= a_1(x - vt) \\ y' &= y \\ z' &= z \\ t' &= b_1 x + b_2 t \end{aligned} \qquad (2)$$

Substitute Eq. (2) into Eq. (1): $x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$

$$a_1^2(x - vt)^2 + y^2 + z^2 - c^2(b_1x + b_2t)^2 = x^2 + y^2 + z^2 - c^2t^2$$
$$\left(a_1^2 - c^2b_1^2\right)x^2 - 2\left(a_1^2v + c^2b_1b_2\right)xt + \left(a_1^2v^2 - c^2b_2^2\right)t^2 = x^2 - c^2t^2$$

(Note: we didn't assume that $x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$)

Therefore, the following equations should be satisfied.

$$\begin{aligned} a_1^2 - c^2b_1^2 &= 1 \quad (a) \\ \left(a_1^2v + c^2b_1b_2\right) &= 0 \quad (b) \\ a_1^2v^2 - c^2b_2^2 &= -c^2 \quad (c) \end{aligned} \longrightarrow \begin{aligned} (a)\ \ b_1^2 &= \frac{a_1^2 - 1}{c^2} \\ (c)\ \ b_2^2 &= 1 + \frac{v^2}{c^2}a_1^2 \end{aligned}$$

$$(b)\ \ a_1^4v^2 = c^4b_1^2b_2^2 = c^2a_1^2 + v^2a_1^4 - c^2 - v^2a_1^2$$
$$\rightarrow\ a_1 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma$$

a_1 should be positive because $x' > 0$ when $x > 0$ at $t = 0$.

$$\longrightarrow \boxed{(a)\ \ b_1 = -\frac{v}{c^2}\gamma, \quad (c)\ \ b_2 = \gamma}$$

We take a positive b_2 because $t' > 0$ when $t > 0$. Then, it is clear that b_1 is negative from (b).

Finally, we obtain the Lorentz transformation (and its inverse):

The inverse has the same form as the original except that the primed and unprimed variables are interchanged, and v is replaced by $-v$.

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c}x\right) \end{aligned} \qquad \begin{aligned} x &= \gamma(x' + vt') \\ y &= y \\ z &= z \\ t &= \gamma\left(t' + \frac{v}{c}x'\right) \end{aligned}$$

where $\gamma \equiv \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(1 - \beta^2\right)^{-1/2}; \quad \beta \equiv \dfrac{v}{c}$

Lorentz factor $1 \le \gamma \le \infty; \quad 0 \le \beta \le 1$

Length Contraction

- **Length contraction** (Lorentz-Fitzgerald contraction): Suppose a rigid rod of length $L_0 = x_2{}' - x_1{}'$ is carried at rest in K'. What is the length as measured in K? The positions (x_2 and x_1) of the ends of the rod are marked at the same time in K.

$$\boxed{\begin{aligned} &L_0 = x_2' - x_1' = \gamma\,(x_2 - x_1) = \gamma L \\ &L = L_0/\gamma \end{aligned}}$$

Therefore, the rod appears shorter by a factor $1/\gamma$ in K.

If both carry rods (of the same length when compared at rest) each thinks the other's rod has shrunk!

It would appear to K' that the two ends of the moving stick were not marked at the same time by the other observer (in K). (Since the Lorentz transformation of time depends on position, temporal simultaneity is not Lorentz invariant.)

Time Dilation

- **Time dilation**: Suppose a clock at rest at the origin of K' measures off a time interval $T_0 = t_2{}' - t_1{}'$. What is the time interval measured in K? Note that the clock is at rest at the origin of K' so that $x_2{}' = x_1{}' = 0$.

$$\boxed{\begin{aligned} T &= t_2 - t_1 = \gamma\left(t_2' - t_1'\right) = \gamma T_0 \\ T &= \gamma T_0 \end{aligned}}$$

  The time interval has increased by a factor γ, so that the moving clock appears to have slowed down, as measured in K. By symmetry, K' thinks clocks in K have slowed down, too.

  The resolution of this apparent contradiction is a result of looking at the manner of measuring an interval of time between two events separated in space. K measures t_1 as the moving clock passes x_1, then measures t_2 as it passes x_2; he/she simply subtracts $t_2 - t_1$ on the assumption that his/her own two clocks at x_1 and x_2 are synchronized. K' will object to this, since according to his/her observations the two clocks in K are not synchronized at all.

- Simultaneity is relative: Simultaneous events at two different spatial points in the primed frame is not simultaneous in the unprimed frame.

  Many of the apparent contradictions of special relativity are simply a result of the relativity of simultaneity between two events separated in space.

- Time dilation is detected in the increased half-lives of unstable particles moving rapidly in an accelerator or in the cosmic-ray flux.

Transformation of Velocities

- If a point has a velocity $\mathbf{u}'$ in frame K', what is its velocity $\mathbf{u}$ in frame K. Writing Lorentz transformations for differentials

$$dx = \gamma\left(dx' + v dt'\right), \quad dy = dy', \quad dz = dz'$$
$$dt = \gamma\left(dt' + \frac{v}{c^2} dx'\right)$$

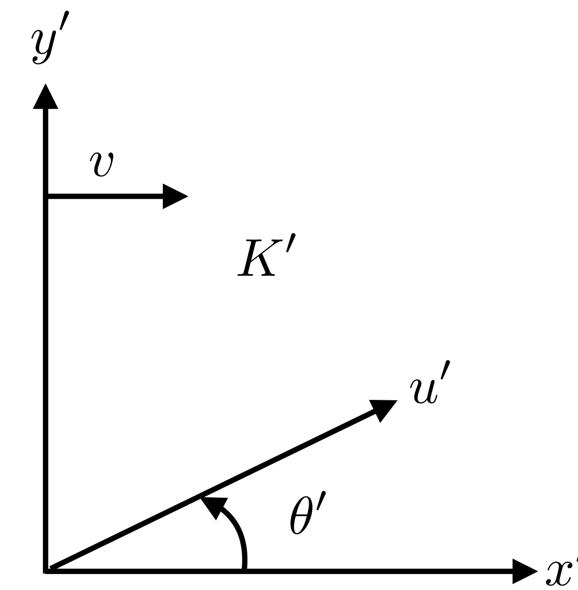
We then have the relations

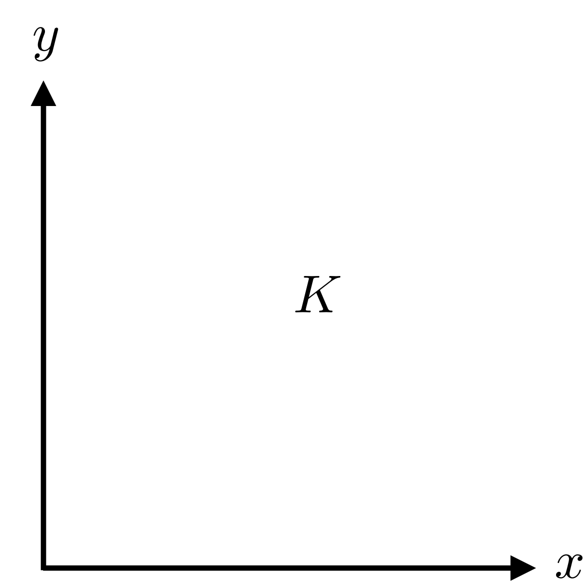
$$u_x = \frac{dx}{dt} = \frac{\gamma\left(dx' + v dt'\right)}{\gamma\left(dt' + v dx'/c^2\right)} = \frac{u'_x + v}{1 + v u'_x/c^2}$$
$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + v dx'/c^2\right)} = \frac{u'_y}{\gamma\left(1 + v u'_x/c^2\right)}$$
$$u_z = \frac{dz}{dt} = \frac{u'_z}{\gamma\left(1 + v u'_x/c^2\right)}$$

or

$$u_\parallel = \frac{u'_\parallel + v}{1 + v u'_\parallel/c^2}$$
$$u_\perp = \frac{u'_\perp}{\gamma\left(1 + v u'_\parallel/c^2\right)}$$

in terms of the components of $\mathbf{u}$ perpendicular to and parallel to $\mathbf{v}$





- **Aberration formula**: the directions of the velocities in the two frames are related by

$$\tan\theta = \frac{u_\perp}{u_\parallel} = \frac{u'_\perp}{\gamma\left(u'_\parallel + v\right)} = \frac{u'\sin\theta'}{\gamma\left(u'\cos\theta' + v\right)} \qquad \text{where } u' \equiv |\mathbf{u}'|$$

- **Aberration of light**

For the case of light: $u' = c$

$$\tan\theta = \frac{\sin\theta'}{\gamma\left(\cos\theta' + v/c\right)} = \frac{\sin\theta'}{\gamma\left(\cos\theta' + \beta\right)}$$

$$\cos\theta = \frac{\gamma\left(\cos\theta' + v/c\right)}{\sqrt{\gamma^2\left(\cos\theta' + v/c\right)^2 + \sin^2\theta'}} = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$\sin\theta = \frac{\sin\theta'}{\sqrt{\gamma\left(\cos\theta' + v/c\right)^2 + \sin^2\theta'}} = \frac{\sin\theta'}{\gamma\left(1 + \beta\cos\theta'\right)}$$

Using the identity, $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{\sin\theta}{1 + \cos\theta}$

The aberration formula can be written as: $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{(1/\gamma)\sin\theta'}{1 + \beta\cos\beta' + \cos\theta' + \beta} = \dfrac{(1/\gamma)\sin\theta'}{(1+\beta)\left(1 + \cos\theta'\right)}$

$$\boxed{\tan\left(\frac{\theta}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^2 \tan\left(\frac{\theta'}{2}\right) \quad \rightarrow \quad \theta < \theta'}$$

- **Beaming ("headlight") effect**:

  If photons are emitted isotropically in K', then half will have $\theta' < \pi/2$ and half $\theta' > \pi/2$.

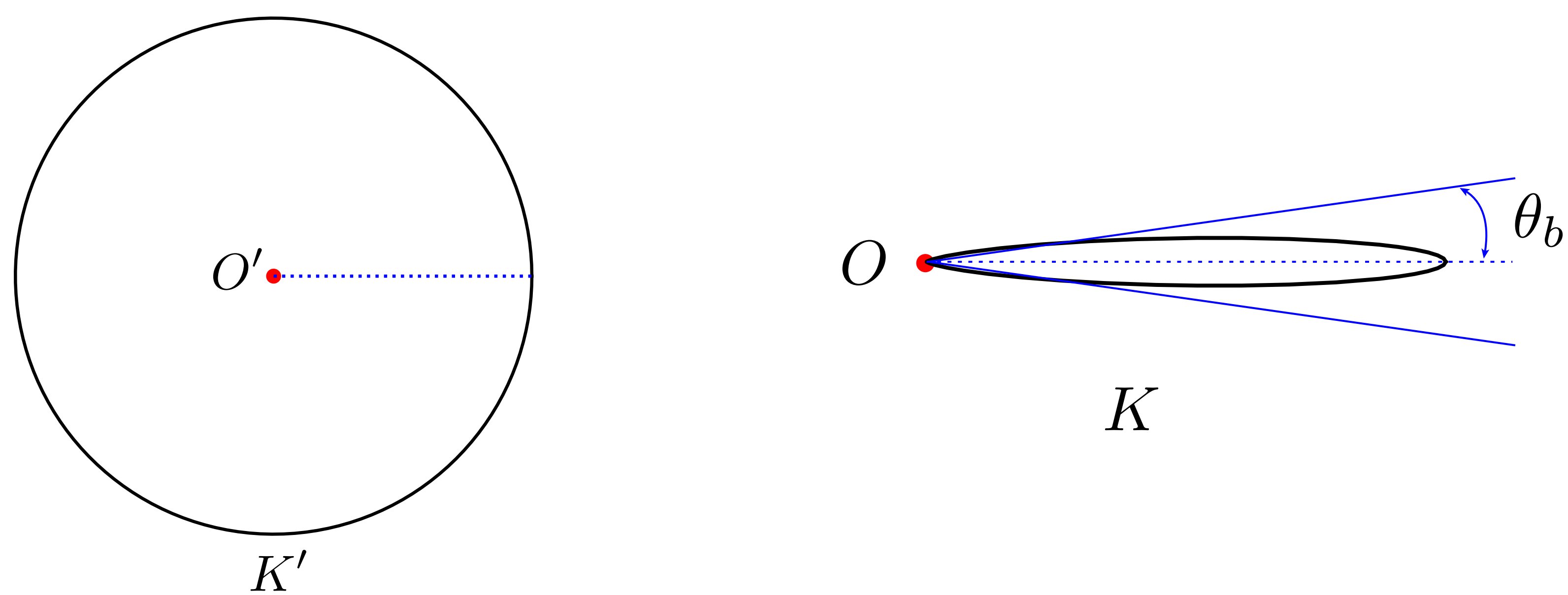
  Consider a photon emitted at right angles to v in K'. Then we have

  beam half-angle: $$\boxed{\sin\theta_b = \frac{1}{\gamma}, \quad \cos\theta_b = \beta, \quad \text{or} \quad \tan\left(\frac{\theta_b}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}}$$

  For highly relativistic speeds, $\gamma \gg 1$, θ_b becomes small:

  $$\sin\theta_b \approx \theta_b \longrightarrow \boxed{\theta_b \approx \frac{1}{\gamma}}$$

  Therefore, in frame K, photons are concentrated in the forward direction, with half of them lying within a cone of half-angle $1/\gamma$. Very few photons will be emitted with $\theta \gg 1/\gamma$.

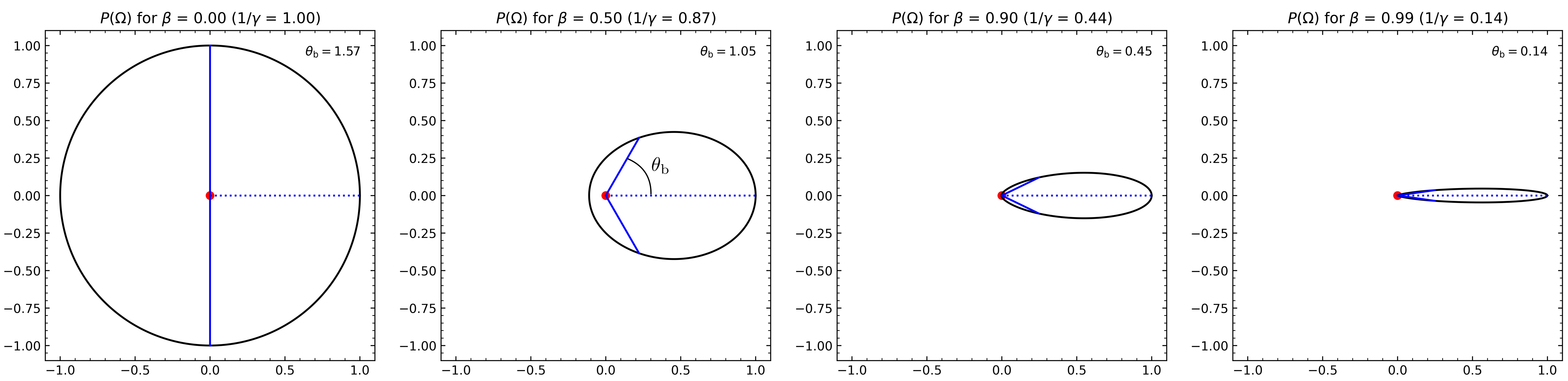


Power density emitted along the direction θ

$$P(\mu) d\mu d\phi = P'(\mu') d\mu' d\phi \quad \rightarrow \quad P(\mu) = P'(\mu') \left| \frac{d\mu'}{d\mu} \right|$$

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'} \quad \rightarrow \quad \mu' = \frac{\mu - \beta}{1 - \beta\mu} \quad \rightarrow \quad \frac{d\mu'}{d\mu} = \frac{1}{\gamma\,(1 - \beta\mu)^2}$$

$$\boxed{\begin{aligned} P'(\mu') &= \frac{1}{2} \\ P(\mu) &\propto \frac{1}{(1-\beta\mu)^2} \end{aligned}}$$

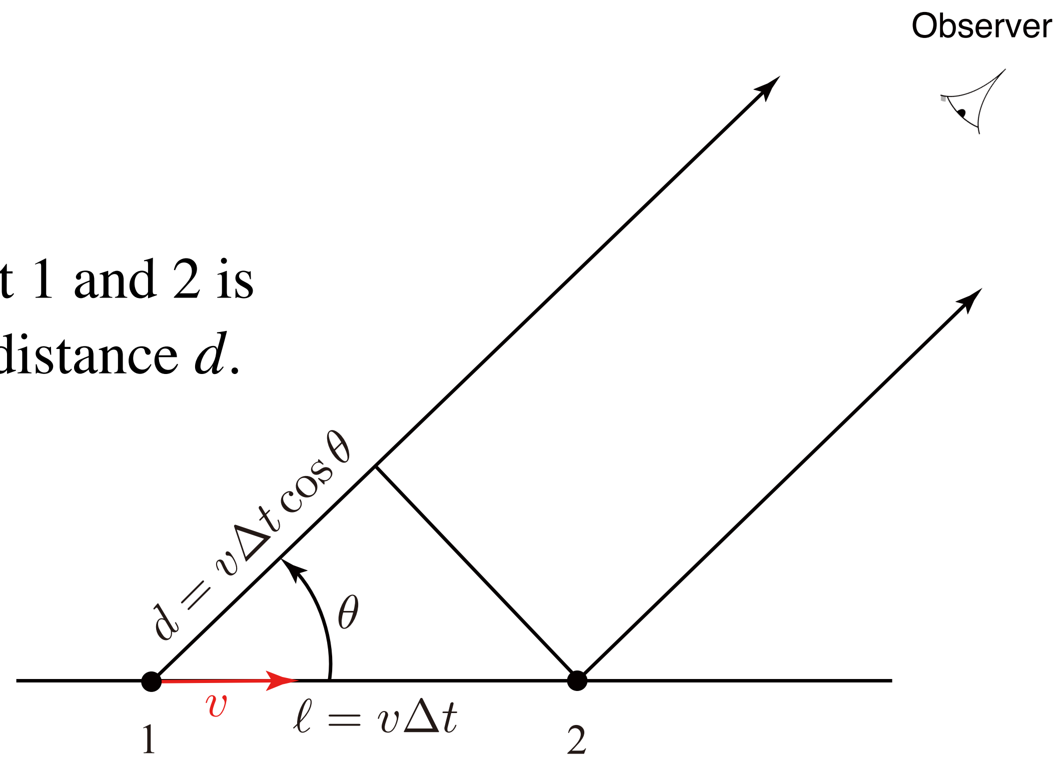


Doppler Effect

- In the rest frame of the observer K, imagine that the moving source emits one period of radiation as it moves from point 1 to point 2 at velocity v.

Let frequency of the radiation in the rest frame K' of the source = ω'. Then the time taken to move from point 1 to point 2 in the observer's frame is given by the time-dilation effect:

$$\Delta t' = \frac{2\pi}{\omega} \quad \rightarrow \quad \Delta t = \Delta t' \gamma = \frac{2\pi}{\omega'}\gamma$$



Now consider the situation of the right hand side figure.

The difference in arrival times Δt_A of the radiation emitted at 1 and 2 is equal to Δt minus the time taken for radiation to propagate a distance d.

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t\left(1 - \frac{v}{c}\cos\theta\right)$$

Therefore, the observed frequency ω will be

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma\left(1-\beta\cos\theta\right)}$$

$$\boxed{\frac{\omega}{\omega'} = \frac{1}{\gamma\left(1-\beta\cos\theta\right)}} \qquad \boxed{\frac{\nu}{\nu'} = \frac{1}{\gamma\left(1-\beta\cos\theta\right)}}$$

Note that the angle θ is measured in the rest frame K.

Note that $1-\beta\cos\theta$ appears even classically. But, the factor $1/\gamma$ is purely a relativistic effect.

- If a source approaches head-on, $\theta = 0$, we obtain

$$\nu = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \nu_0$$

Here, $\nu_0 = \nu'$ is the emitted frequency measured by K'.

- Classical (nonrelativistic) Doppler shift ($\beta \ll 1$):

$$\frac{\nu}{\nu_0} \approx \left(1 - \frac{\beta^2}{2} + \cdots\right)(1 + \beta\cos\theta + \cdots) \approx 1 + \beta\cos\theta \cdots \quad (\beta \ll 1)$$

- Doppler shifts in astronomy
  - The frequencies of spectral lines from celestial sources are often shifted owing to the motions of the emitting objects: gaseous clouds, stars, or galaxies.
  - Astronomical sign convention: In the classical Doppler shift, the observed shift of frequency reflects only the component of the velocity along the line of sight, the radial component v_r. The astronomical convention is that **the radial component be positive if it is directed outward and negative if it is directed inward.** Then, the classical Doppler shift takes the form:

$$\boxed{\frac{\nu}{\nu_0} = 1 - \frac{v_r}{c} \quad \text{or} \quad \frac{\nu - \nu_0}{\nu_0} = -\frac{v_r}{c}}$$

ν and ν_0 are the observed and emitted frequencies, respectively.

  - Optical astronomers work in wavelength units. Then the Doppler shift is given by

$$\boxed{\frac{\lambda}{\lambda_0} = 1 + \frac{v_r}{c} \quad \text{or} \quad \frac{\lambda - \lambda_0}{\lambda_0} = \frac{v_r}{c}}$$

$$\frac{\lambda}{\lambda_0} = \left(\frac{1 + v_r/c}{1 - v_r/c}\right)^{1/2}$$

relativistic version for strictly radial motion.

- **Redshift parameter**: The optical spectra of distant luminous objects called quasars have spectral lines shifted by large amounts of lower frequencies. If these redshifts are interpreted as Doppler shifts, they indicate recession velocities approaching the speed of light. These velocities are due to the expansion of the universe; the expansion is such that the more distant the object, the faster it recedes. Astronomers define the "redshift" parameter z as

$$\boxed{z \equiv \frac{\lambda - \lambda_0}{\lambda_0} = \frac{\lambda}{\lambda_0} - 1}$$

$$z + 1 = \left(\frac{1 + v_r/c}{1 - v_r/c}\right)^{1/2}$$ relativistic version for strictly radial motion.

- The most distant quasars known are at redshifts $z \approx 6$. At this redshift, $\lambda/\lambda_0 = 7$, indicating that the observed wavelength is seven times the rest wavelength in the quasar frame. An ultraviolet emission line at $\lambda_0 = 121.5$ nm (Lyman α) would be shifted almost into the near infrared at 850.5 nm. In this case, the speed factor is $\beta_r = V_r/c = 0.960$. The quasar is receding at 96% the speed of light.

- We remind that special relativity is not really appropriate to our universe with its changing rate of expansion.

- **Transverse (second-order) Doppler effect**:
  - Now consider that a source moves relativistically from left to right. In this case, we find

$$\frac{\omega}{\omega'} = \frac{1}{\gamma} \leq 1 \quad \text{at} \quad \theta = \pi/2$$

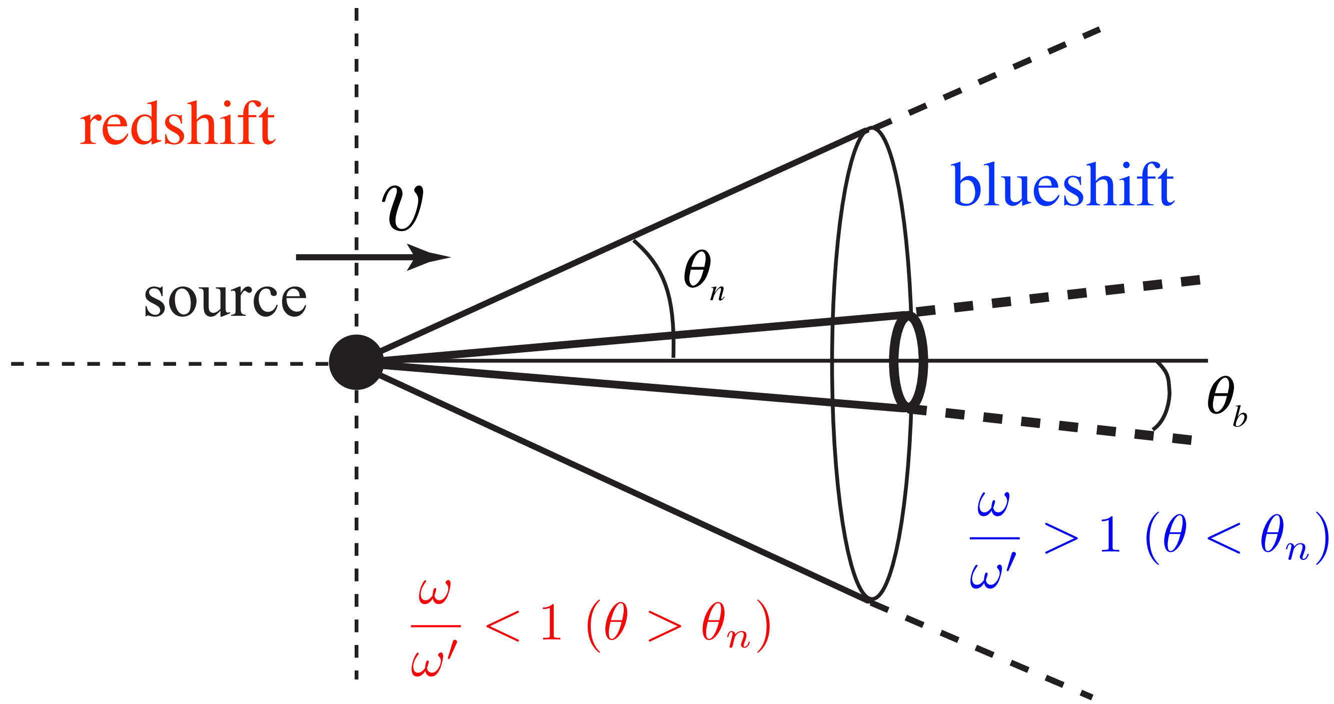
Surprising! A redshift to lower frequency ($\nu/\nu_0 < 1$) in contrast to the classical case, which yields no shift.

Second-order Doppler effect

- Recall beam half-angle = $\theta_b = \sin^{-1}\gamma^{-1}$
- Angle for null Doppler shift is defined by:

$$\frac{\omega}{\omega'} = \frac{1}{\gamma\,(1-\beta\cos\theta_n)} = 1$$

$$\rightarrow \quad \cos\theta_n = \frac{1-\gamma^{-1}}{\beta} = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{-1/2}$$

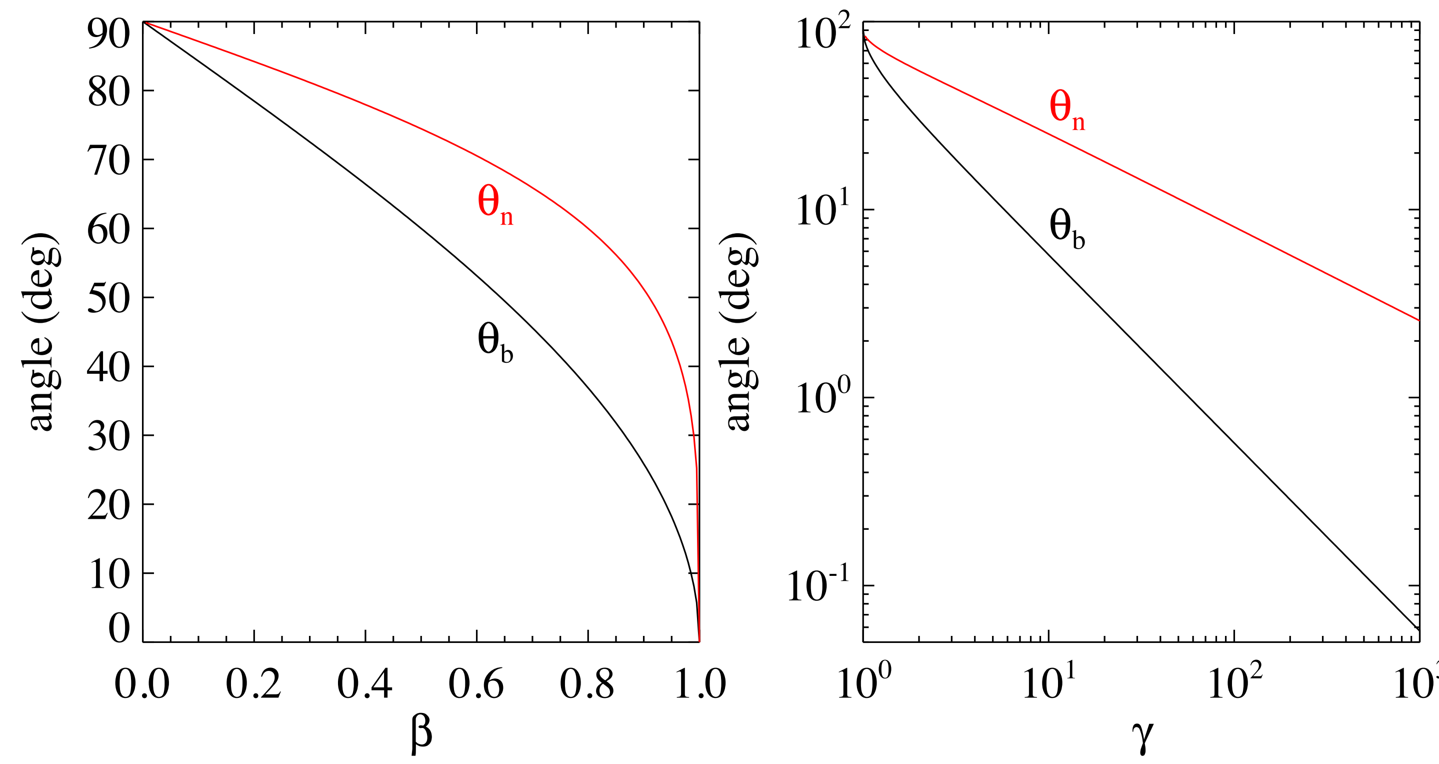


Relativistic Doppler effect can yield redshift even as a source approaches.

$$\cos\theta_n = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{-1/2} \approx 1 - \frac{1}{\gamma} \quad \text{for} \quad \gamma \gg 1$$

$$1 - \frac{\theta_n^2}{2} \approx 1 - \frac{1}{\gamma}$$

$$\boxed{\theta_n \approx \sqrt{\frac{2}{\gamma}} \approx \sqrt{2\theta_b}}$$



- Note $\theta_b < \theta_n$

Lorentz Invariant

- **Lorentz invariant**: A quantity (scalar) that remains unchanged by a Lorentz transform is said to be a "Lorentz invariant." For instance,

$$\begin{aligned} x'^2 + y'^2 + z'^2 - c^2 t'^2 &= \gamma\left(x - \beta ct\right)^2 + y^2 + z^2 - \gamma^2\left(ct - \beta x\right)^2 \\ &= \gamma\left(1 - \beta^2\right)x^2 + y^2 + z^2 + \gamma^2\left(\beta^2 c^2 - c^2\right)t^2 \\ &= x^2 + y^2 + z^2 - c^2 t^2 \end{aligned}$$

- **Proper distance**: Since all events are subject to the same transformation, the space-time "interval" between two event is also invariant.

$$ds^2 \equiv dx^2 + dy^2 + dz^2 - c^2 dt^2$$

This is the spatial distance between two events occurring at the same time. This is called the proper distance between the two points.

- **Proper time** interval: $c^2 d\tau^2 \equiv -ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

This measures time intervals between events occurring at the same spatial location ($dx = dy = dz = 0$).

If the coordinate differentials refer to the position of the origin of another reference frame traveling with velocity v, then

$$(d\tau)^2 = (dt)^2 - \frac{(dt)^2}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = (dt)^2\left(1 - \frac{v^2}{c^2}\right) \quad \longrightarrow \quad d\tau = dt\left(1 - \beta^2\right)^{1/2} = dt/\gamma$$

This is the time dilation formula in which $d\tau$ is the time interval measured by the observer in motion.

* Four-Vectors *

- **Four-vector**: Invariant in 3D rotations: $dx^2 + dy^2 + dz^2$

  By analogy, the invariance of the space-time interval suggests to define a vector in 4D space (4 dimensional space-time vector or four-vector). The quantities $x^\mu \, (\mu = 0,1,2,3)$ define coordinates of an event in space-time.

$$\vec{x} \equiv x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}$$

Contravariant components

- **Minkowski space**: The fact that the expression for s^2 contains a minus sign in front on c^2t^2 means that space-time is not a Euclidean space; it is a special space called Minkowski space. Such space can be handled in two ways, either by including $\sqrt{-1}$ in the definition of the time component or by introduction of a ***metric***. Once the notational difficulties of the metric approach are mastered, it is not much more complicated than the $\sqrt{-1}$ approach.

  A metric tensor allows defining lengths of curves, angles, and distances in differential geometry. Let's define **Minkowski metric**, which can be presented in the 4×4 matrix:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that this metric is symmetric:
$\eta_{\mu\nu} = \eta_{\nu\mu}$

- **Summation convention**:

  The invariant can now be written in terms of the Minkowski metric:

$$s^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\mu x^\nu$$

  An important and beautiful notational advance (originated by Einstein) is the summation convention. In any single term containing **a Greek index repeated twice** (between contravariant and covariant indices), a summation is implied over that inde. This index is often called a dummy index.

  Therefore, we can write the invariant s^2 without the summation signs.

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu$$

  An important point is that an index cannot be repeated more than twice in a single term; for example, the combination $\eta_{\mu\mu} x^\mu$ is regarded as meaningless.

- **Contravariant/Covariant components**

contravariant components: (superscripted) $$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

covariant components: (subscripted) $$x_\mu = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -ct \\ x \\ y \\ z \end{pmatrix}$$

They are related by $\boxed{x_\mu = \eta_{\mu\nu} x^\nu, \quad x^\mu = \eta^{\mu\nu} x_\nu}$.

The metric can be used to raise or lower indices.

Now, the invariant s^2 can be written simply

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu \quad \rightarrow \quad s^2 = x^\mu x_\mu$$

The components of a position (velocity etc.) vector *contra-vary* with a change of basis vectors to compensate. Transformation rules between the following two vector components are inverse. This is the basic idea of "contravariant" and "covariant."

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu, \qquad \frac{\partial A}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial A}{\partial x^\nu}$$

Note that summation on indices occurs only between contravariant and covariant indices.

- **Lorentz transform** (corresponding to a boost along the x axis) can be written in terms of a transformation matrix.

Lorentz transformation:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$$

transformation matrix:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Any arbitrary Lorentz transformation can be written in the above form, since the spatial 3D rotation necessary to align the x axes before and after the boost are also of linear form.

- **Conditions for the Lorentz transformation**:

  From the invariance of s^2, we must have

$$\eta_{\mu\nu}x^\mu x^\nu = \eta_{\sigma\tau}x'^\sigma x'^\tau = \eta_{\sigma\tau}\Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu x^\mu x^\nu$$

  This can be true for arbitrary x^μ only if

$$\eta_{\mu\nu} = \eta_{\sigma\tau}\Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \quad \text{or equivalently} \quad \boldsymbol{\eta} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{\Lambda} \quad \text{in matrix form}$$

  Taking determinants yields

$$\det \boldsymbol{\Lambda} = \pm 1$$

  Proper Lorentz transformations (to keep the right-handness), which rules out reflections such as $x \to -x$.

$$\det \boldsymbol{\Lambda} = 1$$

  Isochronous Lorentz transformations (to ensure that the sense of flow of time is the same in two frames)

$$\Lambda^0{}_0 \geq 1$$

- The **Lorentz transformation** of the covariant component can be obtained as follows:

$$x'_\mu = \eta_{\mu\tau}x'^\tau = \eta_{\mu\tau}\Lambda^\tau{}_\sigma x^\sigma = \eta_{\mu\tau}\Lambda^\tau{}_\sigma \eta^{\sigma\nu}x_\nu$$

$$\boxed{\therefore \quad x'_\mu = \widetilde{\Lambda}_\mu{}^\nu x_\nu \quad \text{where} \quad \widetilde{\Lambda}_\mu{}^\nu \equiv \eta_{\mu\tau}\Lambda^\tau{}_\sigma \eta^{\sigma\nu}, \quad \widetilde{\Lambda}_\mu{}^\nu = \frac{\partial x'_\mu}{\partial x_\nu}}$$

- From the invariance of $s^2 = x^\mu x_\mu$:

$$x'^\sigma x'_\sigma = \Lambda^\sigma{}_\nu x^\nu \widetilde{\Lambda}_\sigma{}^\mu x_\mu = \Lambda^\sigma{}_\nu \widetilde{\Lambda}_\sigma{}^\mu x^\nu x_\mu$$

$$\therefore \; \Lambda^\sigma{}_\nu \widetilde{\Lambda}_\sigma{}^\mu = \delta^\mu{}_\nu$$

$$\therefore \; \widetilde{\Lambda}_\sigma{}^\mu = \left(\Lambda^{-1}\right)^\mu{}_\sigma$$

where we have introduced the Kronecker delta

$$\delta^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ Identity matrix

- For any arbitrary contravariant components,

$$Q^\mu = \delta^\mu{}_\nu Q^\nu$$

- Note that

$$\eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu{}_\nu$$

- Inverse transform

$$\widetilde{\Lambda}_\sigma{}^\mu \times \left(x'^\sigma = \Lambda^\sigma{}_\nu x^\nu\right) \quad \xrightarrow{\text{using } \Lambda^\sigma{}_\nu \widetilde{\Lambda}_\sigma{}^\mu = \delta^\mu{}_\nu} \quad x^\mu = \widetilde{\Lambda}_\sigma{}^\mu x'^\sigma$$

note : $\widetilde{\Lambda}_\sigma{}^\mu = \left(\Lambda^{-1}\right)^\mu{}_\sigma$

Other Four-vectors

- Four-vector: is defined such that the transformation of components between any two frames is given by the same transformation law as applies to x^μ

contravariant | covariant

$$\vec{A} \quad \rightarrow \quad A^\mu = \eta^{\mu\nu} A_\nu \qquad A_\mu = \eta_{\mu\nu} A^\nu$$

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu \qquad A'_\mu = \widetilde{\Lambda}_\mu{}^\nu A_\nu$$

- Let us consider two four-vectors $\overrightarrow{A}$ and $\overrightarrow{B}$. We define the scalar product of them.

$$A'^\mu B'_\mu = \Lambda^\mu{}_\nu \widetilde{\Lambda}_\mu{}^\sigma A^\nu B_\sigma = \delta^\sigma{}_\nu A^\nu B_\sigma = A^\nu B_\nu \quad \longrightarrow \quad \boxed{\vec{A} \cdot \vec{B} = A^\mu B_\mu = A'^\mu B'_\mu}$$

Therefore, the scalar product of any two four-vectors is a Lorentz invariant or scalar. In particular, the "square" of a four vector is an invariant. Thus, our starting point, the invariance of $s^2 = x^\mu x_\mu$, is seen to be a general property of four-vectors.

- Note

$$\begin{aligned} \vec{A} \cdot \vec{A} > 0 \quad &\rightarrow \quad \text{spacelike four} - \text{vector} \\ = 0 \quad &\rightarrow \quad \text{light} - \text{like (or null) four} - \text{vector} \\ < 0 \quad &\rightarrow \quad \text{time} - \text{like four} - \text{vector} \end{aligned}$$

$$\begin{aligned} A^0 \quad &\rightarrow \quad \text{time component} \\ A^i \quad &\rightarrow \quad \text{space components (ordinary three} - \text{vector)} \end{aligned}$$

$$\vec{A} \cdot \vec{B} = -A^0 B_0 + \mathbf{A} \cdot \mathbf{B} = -A^0 B_0 + A^i B_i \quad (i = 1, 2, 3)$$

Four-velocity

The (infinitesimally small) difference between the coordinates of two events is also a four-vector. Dividing by the proper time yields a four-vector, the four-velocity:

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \quad \longrightarrow \quad U^0 = \frac{c\,dt}{d\tau} = c\gamma_u, \quad U^i = \frac{dx^i}{d\tau} = \gamma_u u^i$$

or

$$\boxed{\vec{U} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}}$$

where $\gamma_u \equiv \left(1 - u^2/c^2\right)^{-1/2}$, $u \equiv \left|\dfrac{d\mathbf{x}}{dt}\right|$

length of the four-velocity : $\boxed{\vec{U} \cdot \vec{U} = U^\mu U_\mu = -\left(\gamma_u c\right)^2 + \left(\gamma_u \mathbf{u}\right)^2 = -c^2}$

Transformation of the four-velocity:

$$\begin{aligned} U'^0 &= \gamma\left(U^0 - \beta U^1\right) \\ U'^1 &= \gamma\left(-\beta U^0 + U^1\right) \\ U'^2 &= U^2 \\ U'^3 &= U^3 \end{aligned} \quad \longrightarrow \quad \begin{aligned} \gamma_{u'} c &= \gamma\left(c\gamma_u - \beta\gamma_u u^1\right) \\ \gamma_{u'} u'^1 &= \gamma\left(-\beta c\gamma_u + \gamma_u u^1\right) \\ \gamma_{u'} u'^2 &= \gamma_u u^2 \\ \gamma_{u'} u'^3 &= \gamma_u u^3 \end{aligned}$$

$\longrightarrow$ The first two equations become:

$$\begin{aligned} \gamma_{u'} &= \gamma\gamma_u\left(1 - vu'/c^2\right) \\ \gamma_{u'} u'^1 &= \gamma\gamma_u\left(u^1 - v\right) \end{aligned}$$

Note: γ denotes the factor for the relative velocity between two frames.
γ_u and $\gamma_{u'}$ are the factors for a velocity vector measured in K and K', respectively.

$\longrightarrow$

velocity component: $$u'^1 = \frac{u^1 - v}{1 - vu^1/c^2}$$ This is the previously derived formula.

speed: $$\gamma_{u'} = \gamma\gamma_u\left(1 - \frac{vu^1}{c^2}\right)$$ This is the transform for speed.

Here, $u^1 = u\cos\theta$ and $u'^1 = u'\cos\theta'$

Momentum and Energy

- Four-momentum of a particle with a mass m_0 is defined by

$$P^\mu \equiv m_0 U^\mu \qquad \begin{aligned} P^0 &= m_0 c \gamma_v \\ P^i &= \gamma_v m_0 \mathbf{v} \end{aligned}$$

- In the nonrelativistic limit,

$$P^0 c = m_0 c^2 \gamma = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = m_0 c^2 + \frac{1}{2} m_0 v^2 + \cdots$$

Therefore, we interpret $E \equiv P^0 c = \gamma_v m_0 c^2$ as the total energy of the particle.

The quantity $m_0 c^2$ is interpreted as the rest energy of the particle.

Then,

$$\mathbf{p} \equiv \gamma_v m_0 \mathbf{v}, \quad P^\mu = (E/c,\ \mathbf{p})$$

Since $\vec{U}^2 = -c^2$, we obtain

$$\vec{P}^2 = -m_0^2 c^2 = -\frac{E^2}{c^2} + |\mathbf{p}|^2$$

$$E^2 = m_0^2 c^4 + c^2 |\mathbf{p}|^2$$

- Photons are massless, but we can still define

$$P^\mu = (E/c,\ \mathbf{p}), \quad E = |\mathbf{p}|\, c \quad \rightarrow \quad \vec{P}^2 = 0$$

Wavenumber vector and frequency

- Quantum relations:

$$E = h\nu = \hbar\omega \qquad \begin{pmatrix} \omega & = 2\pi\nu \\ k & = 2\pi/\lambda \end{pmatrix}$$
$$p = E/c = \hbar k$$

We can define four wavenumber vector:

$$\vec{k} = \frac{1}{\hbar}\vec{P} = \left(\frac{\omega}{c},\ \mathbf{k}\right)$$

Note that it's a null vector:

$$\vec{k}\cdot\vec{k} = |\mathbf{k}|^2 - \omega^2/c^2 = 0$$

Then, we obtain an invariant:

$$\vec{k}\cdot\vec{x} = k_\mu x^\mu = \mathbf{k}\cdot\mathbf{x} - \omega t$$

Therefore, the phase of the plane wave is an invariant.

- Transform for $\overrightarrow{k}$ (Doppler formula):

$$\begin{aligned} k'^0 &= \gamma\left(k^0 - \beta k^1\right) \\ k'^1 &= \gamma\left(-\beta k^0 + k^1\right) \\ k'^2 &= k^2 \\ k'^3 &= k^3 \end{aligned} \quad \longrightarrow \quad \omega' = \gamma\left(\omega - \beta c k^1\right) = \omega\gamma\left(1 - \frac{v}{c}\cos\theta\right)$$

$$k^1 = (\omega/c)\cos\theta$$

* Tensor Analysis *

- Definition:

  zeroth-rank tensor : Lorentz invariant (scalar) $s' = s$

  first-rank tensor : four-vector $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

  second-rank tensor: $T'^{\mu\nu} = \Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau} T^{\sigma\tau}$

- Covariant components and mixed components:

$$T_{\mu\nu} = \eta_{\mu\sigma}\eta_{\nu\tau}T^{\sigma\tau} \qquad T^{\mu}{}_{\nu} = \eta_{\nu\tau}T^{\mu\tau} \qquad T_{\mu}{}^{\nu} = \eta_{\mu\sigma}T^{\sigma\nu}$$

- Transformation rules:

$$\begin{aligned} T'_{\mu\nu} &= \eta_{\mu\alpha}\eta_{\nu\beta}T'^{\alpha\beta} \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}T^{\gamma\delta} \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}\eta^{\gamma\sigma}\eta^{\delta\tau}T_{\sigma\tau} \\ &= \widetilde{\Lambda}_{\mu}{}^{\sigma}\widetilde{\Lambda}_{\nu}{}^{\tau}T_{\sigma\tau} \end{aligned}$$

$$\begin{aligned} T'^{\mu}{}_{\nu} &= \eta_{\nu\alpha}T'^{\mu\alpha} \\ &= \eta_{\nu\alpha}\Lambda^{\mu}{}_{\sigma}\Lambda^{\alpha}{}_{\delta}T^{\sigma\delta} \\ &= \eta_{\nu\alpha}\Lambda^{\mu}{}_{\sigma}\Lambda^{\alpha}{}_{\delta}\eta^{\delta\tau}T^{\sigma}_{\tau} \\ &= \Lambda^{\mu}{}_{\sigma}\widetilde{\Lambda}_{\nu}{}^{\tau}T^{\sigma}{}_{\tau} \end{aligned}$$

$$\begin{aligned} T'_{\mu}{}^{\nu} &= \eta_{\mu\alpha}T'^{\alpha\nu} \\ &= \eta_{\mu\alpha}\Lambda^{\alpha}{}_{\beta}\Lambda^{\nu}{}_{\tau}T^{\beta\tau} \\ &= \eta_{\mu\alpha}\Lambda^{\alpha}{}_{\beta}\Lambda^{\nu}{}_{\tau}\eta^{\beta\sigma}T_{\sigma}{}^{\tau} \\ &= \widetilde{\Lambda}_{\mu}{}^{\beta}\Lambda^{\nu}{}_{\tau}T_{\sigma}{}^{\tau} \end{aligned}$$

- symmetric tensor = a tensor that is invariant under a permutation of its indices.

$$T^{\mu\nu} = T^{\nu\mu}$$

- antisymmetric tensor : if it alternates sign when any two indices of the subset are interchanged.

$$T^{\mu\nu} = -T^{\nu\mu}$$

- Examples of the second-rank tensors

A product of two vectors: $A^\mu B^\nu$

$$A'^\mu B'^\nu = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau A^\sigma B^\tau$$

The Minkowski metric: $\eta^{\mu\nu}$

The Kronecker-delta: $\delta^\mu{}_\nu$

- Higher-rank tensors
  - Addition: $A^{\mu} + B^{\mu},\ F^{\mu\nu} + G^{\mu\nu}$
  - Multiplication: $A^{\mu}B^{\nu},\ F^{\mu\nu}G_{\sigma\tau}$
  - Raising and Lowering Indices: The metric can be used to change contravariant indices into covariant ones, and vice versa, by the processes of raising and lowering.
  - Contraction: $A^{\mu}B_{\nu} \rightarrow A^{\mu}B_{\mu}$ scalar
    $T^{\mu\nu}{}_{\sigma} \rightarrow T^{\mu\nu}{}_{\nu}$ vector

$$T'^{\mu\nu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\widetilde{\Lambda}_{\nu}{}^{\tau}T^{\alpha\beta}{}_{\tau} = \Lambda^{\mu}{}_{\alpha}\delta^{\tau}{}_{\beta}T^{\alpha\beta}{}_{\tau} = \Lambda^{\mu}{}_{\alpha}T^{\alpha\beta}{}_{\beta}$$

  - Gradients of Tensor Fields: A tensor field is a tensor that is a function of the spacetime coordinates in Cartesian coordinate systems. The <u>gradient operation $\partial/\partial x^{\mu} \equiv \partial_{\mu}$</u> acting on such a field produces a tensor field of on higher rank with <u>μ as a new covariant index</u>.

$$\lambda \rightarrow \frac{\partial\lambda}{\partial x^{\mu}} \equiv \partial_{\mu}\lambda \equiv \lambda_{,\,\mu} \quad \text{vector (gradient)} \qquad A^{\mu} \rightarrow \frac{\partial A^{\mu}}{\partial x^{\mu}} \equiv \partial_{\mu}A^{\mu} \equiv A^{\mu}_{,\,\mu} \quad \text{scalar (divergence)}$$

  - **Invariance of form or Lorentz covariance or covariance**: A fundamental property of a tensor equation is that if it is true in one Lorentz frame, then it is true in all Lorentz frames. Covariance plays a powerful role in helping decide what the proper equations of physics are.