Interstellar Medium (ISM)

Week 5
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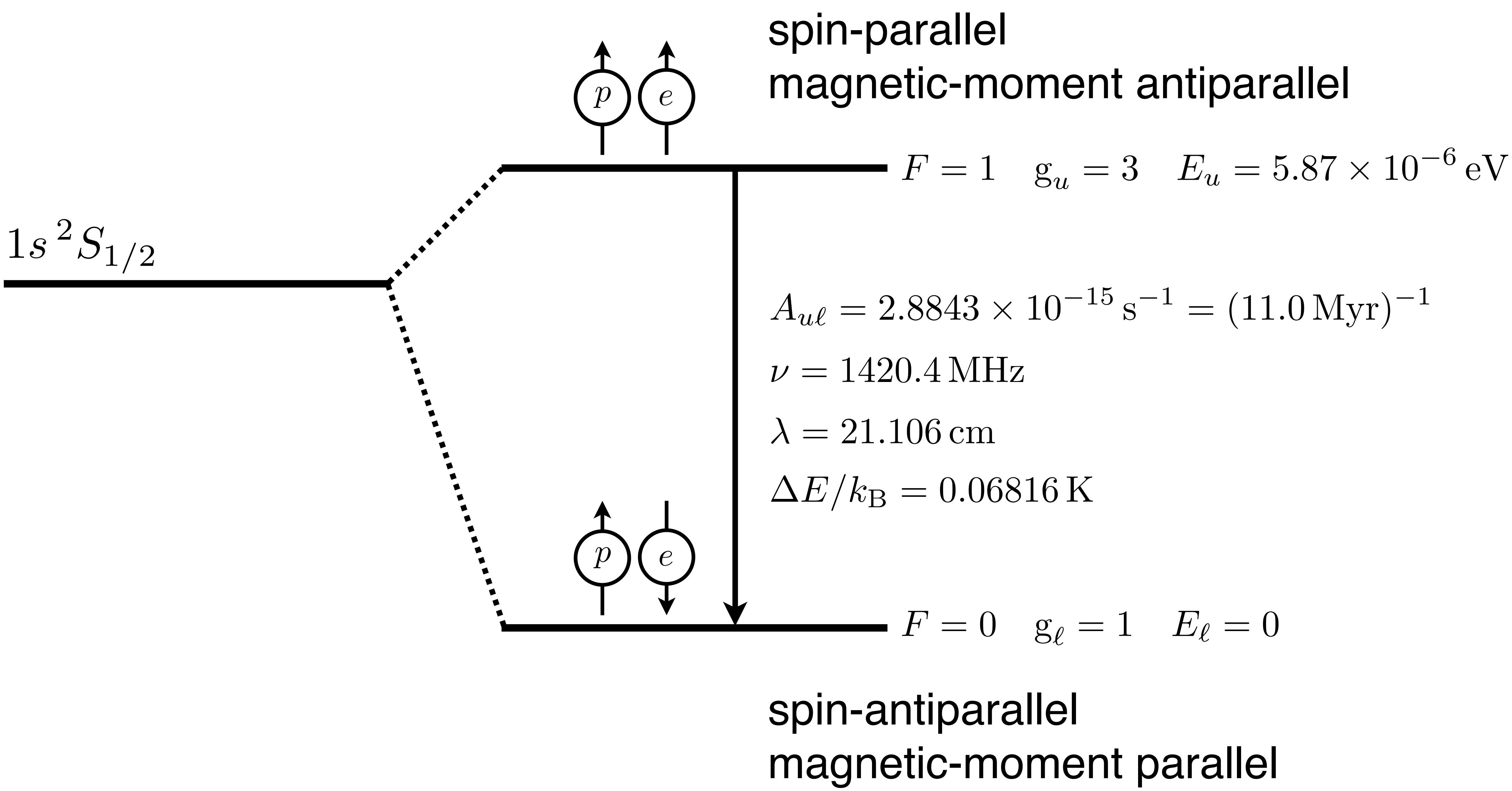
Neutral Medium 3

- H I 21 cm line
- Warm Neutral Medium

21 cm hyperfine line

- The CNM and WNM, taken together, provide over half the mass of the ISM.
  - H is the most abundant element in the universe. In the CNM and WNM, most of the hydrogen is in the form of neutral atoms.
  - The Lyα line of H provides a useful probe of the properties of the CNM and WNM. However, at its wavelength the Earth's atmosphere is highly opaque, and thus observing Lyα absorption requires orbiting UV satellites. In addition, Lyα can be seen in absorption only along those lines of sight toward sources with a high UV flux.
  - To do a global survey of atomic hydrogen in the galaxy, we need some way of easily detecting radiation from hydrogen, regardless of its kinetic temperature or number density.
  - Such a way was first found in 1944, by Henk van de Hulst. He attempted to find emission lines at the wavelengths ~ 1 cm to 20 m, at which the Earth's atmosphere is transparent. He then realized that the hyperfine structure line resulting from a flip of the electron spin within a hydrogen atom should have a wavelength of 21 cm. This was confirmed by Harold Ewen and Edward Purcell in 1951, when they first dectected 21 cm emission from the Milky Way.

Hyperfine splitting of the 1s ground state of atomic H



Difference between Lyα and 21 cm transitions

- The excitation energy for Lyα (E = 10.2 eV, E/k = 118,000 K) is much higher than the kinetic temperature of the neutral ISM.

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{118,000\,\mathrm{K}}{T}\right) \approx 1.7 \times 10^{-51} \text{ at } T = 1000\,\mathrm{K}$$

  - Collisional excitation is unimportant, and most hydrogen atoms are in the lower level of the Lyα transition.
  - The Lyα has a higher energy by a factor of $1.7\text{x}10^6$ than the 21 cm.

- The excitation energy for 21 cm is ~ 6μm, and its equivalent temperature E/k = 0.068 K is much lower than the temperature of the cosmic microwave background.
  - Even the CMB is able to populate the upper level.
  - Thus, there is ample opportunity to populate the upper energy level of the 21 cm hyperfine transition. In excitation equilibrium, the level populations for the 21 cm levels.

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} e^{-h\nu/kT} = 3\,e^{-0.068\,\mathrm{K}/T} \approx 3 \longrightarrow n_u \approx \frac{3}{4} n_\mathrm{H}, \;\; n_\ell \approx \frac{1}{4} n_\mathrm{H}$$

  - In fact, the hyperfine levels may not be in excitation equilibrium. Radio astronomers use the term ***spin temperature*** for 21 cm rather than the “excitation temperature.”

Emissivity and Optical Depth

- ***Emissivity***: The upper level contains ~ 75% of the H I under all conditions of interest, and thus the 21-cm emissivity is effectively independent of the spin temperature.

$$j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell}\phi_\nu \approx \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} n_{\mathrm{H}} \phi_\nu \qquad \left(n_u \approx \frac{3}{4} n_{\mathrm{H}} \right)$$

- ***Optical depth***

$$\kappa_\nu = n_\ell \sigma_{\ell u} - n_u \sigma_{u\ell} = n_\ell \sigma_{\ell u} \left(1 - e^{-h\nu_{u\ell}/kT_{\mathrm{spin}}} \right)$$

Because $h\nu_{u\ell}/kT_{\mathrm{spin}} \ll 1$ for all conditions of interest, the correction for stimulated emission is very important!

$$\kappa_\nu \approx n_\ell \sigma_{\ell u} \frac{h\nu_{u\ell}}{kT_{\mathrm{spin}}} \ll n_\ell \sigma_{\ell u} \qquad \longleftarrow \qquad e^{-h\nu_{u\ell}/kT_{\mathrm{spin}}} \approx 1 - k\nu_{u\ell}/kT_{\mathrm{spin}}$$

$$\begin{aligned} \kappa_\nu &\approx \left(\frac{1}{4} n_{\mathrm{H}} \right) \left(\frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_\ell \phi_\nu \right) \frac{h\nu_{u\ell}}{kT_{\mathrm{spin}}} \qquad \left(n_u \approx \frac{1}{4} n_{\mathrm{H}} \right) \\ &= \frac{3}{32\pi} A_{u\ell} \frac{hc\,\lambda_{u\ell}}{kT_{\mathrm{spin}}} n_{\mathrm{H}} \phi_\nu \end{aligned}$$

- The damping constant of the 21 cm line profile is extremely small, and thus we can assume that the line profile is a Gaussian.

$$a = \frac{\gamma_{u\ell}}{4\pi}\frac{\lambda_{u\ell}}{b} = 4.844\times10^{-20}\left(\frac{\gamma_{u\ell}}{2.8843\times10^{-15}\,\mathrm{s^{-1}}}\right)\left(\frac{\lambda_{u\ell}}{21.106\,\mathrm{cm}}\right)\left(\frac{1\,\mathrm{km\,s^{-1}}}{b}\right)$$

- Hence,

$$\phi_\nu = \frac{1}{\sqrt{\pi}\Delta\nu_D}H(u,a) \approx \frac{c}{\sqrt{\pi}\nu_{\ell u}b}e^{-u^2} \qquad \left(u = v/b,\ \ b = \sqrt{2}\sigma_V = \sqrt{2kT_{\rm gas}/m_{\rm H}}\right)$$

$$\tau_\nu = \kappa_\nu s = \frac{3}{32\pi}A_{u\ell}\frac{hc\,\lambda_{u\ell}}{kT_{\rm spin}}N_{\rm H\,I}\phi_\nu$$

$N_{\rm H\,I} \equiv \int n_{\rm H}ds$ is the column density of H I.

$\sim 10^{21}\,\mathrm{cm^{-21}}$ toward the Galactic disk.

$$= \frac{3}{32\pi}\frac{1}{\sqrt{\pi}}\frac{A_{u\ell}\lambda_{u\ell}^2}{b}\frac{hc}{kT_{\rm spin}}N_{\rm H\,I}e^{-u^2}$$

$$= 3.111\left(\frac{N_{\rm H\,I}}{10^{21}\,\mathrm{cm^{-2}}}\right)\left(\frac{100\,\mathrm{K}}{T_{\rm spin}}\right)\left(\frac{1\,\mathrm{km\,s^{-1}}}{b}\right)e^{-u^2}$$

or $$\tau_\nu = 2.201\left(\frac{N_{\rm H\,I}}{10^{21}\,\mathrm{cm^{-2}}}\right)\left(\frac{100\,\mathrm{K}}{T_{\rm spin}}\right)\left(\frac{1\,\mathrm{km\,s^{-1}}}{b/\sqrt{2}}\right)e^{-u^2}$$

Some lines of sight through our galaxy (at high galactic latitude) are optically thin and other lines of sight (at low galactic latitude) are optically thick at 21 cm.

- ***Self-absorption in the 21-cm line can be important*** in many sightlines in the ISM.
- ***The optical depth is inversely proportional to the spin temperature.***

[1] Column Density Determination

- ***Radioastronomers express the line profile as a function of radial velocity rather than of frequency.*** This is logical because line broadening is only caused by the Doppler effect, its natural width being extremely narrow since the lifetime of the upper level is only limited by collisions which is rare in the diffuse medium.
- We first define the column density per velocity interval.

$$\frac{dN_{\rm HI}}{dv} = N_{\rm HI}\phi_v = N_{\rm HI}\frac{1}{\lambda_{u\ell}}\phi_\nu \qquad\qquad \phi_\nu = \phi_v\left|\frac{dv}{d\nu}\right| = \frac{1}{\lambda_{u\ell}}\phi_\nu$$

  - The column density can be written:

$$\tau_\nu = \frac{3}{32\pi}A_{u\ell}\frac{hc\,\lambda_{u\ell}}{kT_{\rm spin}}N_{\rm H\,I}\phi_\nu \longrightarrow \tau(v) = \frac{3}{32\pi}A_{u\ell}\frac{hc\,\lambda_{u\ell}^2}{kT_{\rm spin}(v)}\frac{dN_{\rm HI}}{dv}$$

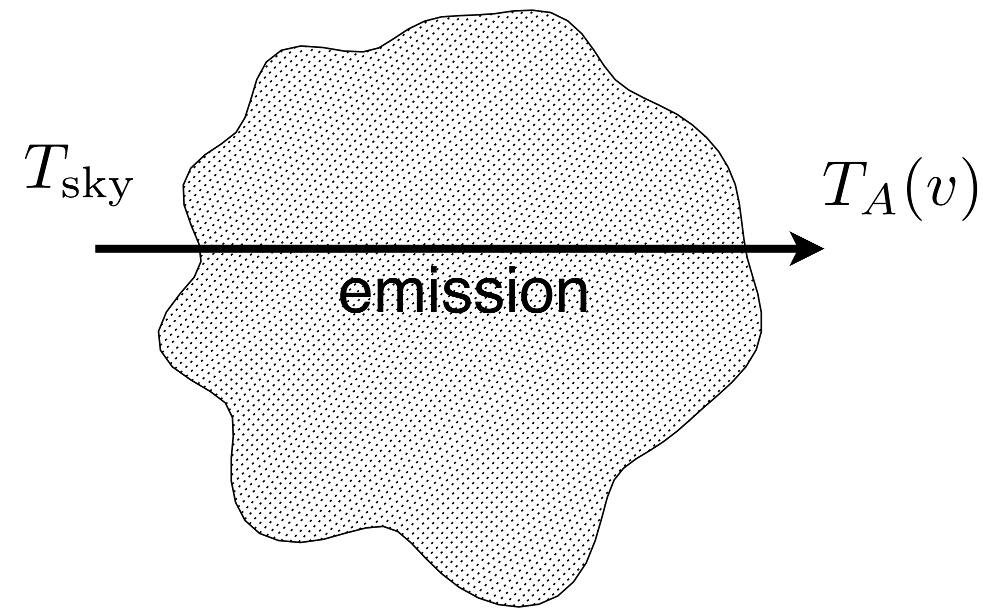
$$\begin{aligned}\frac{dN_{\rm HI}}{dv} &= \frac{32\pi}{3}\frac{k}{A_{u\ell}hc\lambda_{u\ell}^2}T_{\rm spin}(v)\tau(v)\\ &= 1.813\times10^{18}\,\frac{T_{\rm spin}(v)\tau(v)}{\rm K}\quad\left[\frac{\rm cm^{-2}}{\rm km\,s^{-1}}\right]\end{aligned}\qquad N_{\rm HI} = \int dv\frac{dN_{\rm HI}}{dv}$$

  - This indicates that ***we need to know not only the optical depth but also the spin temperature to evaluate the column density***. However, in an optically thin limit, we will show that the dependency on the spin temperature is removed.

- ***Optically thin case:*** Suppose we are looking through an optically thin layer of neutral hydrogen toward a ***"dark sky"***, which is fainter than our hydrogen cloud, with an antenna temperature $T_{\rm sky}$.
  - In the optically thin limit, the RT equation becomes

$$T_A(v) = T_{\rm sky} e^{-\tau_v} + T_{\rm spin}(v)\left(1 - e^{-\tau_v}\right)$$
$$\approx T_{\rm sky} + T_{\rm spin}(v)\tau_v$$

$$\tau(v) \approx \frac{T_A(v) - T_{\rm sky}}{T_{\rm spin}(v)}$$



We measure the antenna temperature of the dark sky from the continuum at frequencies well above and below the 21-cm emission feature.

  - The column density per unit velocity interval is

$$\frac{dN_{\rm HI}}{dv} \approx \frac{32\pi}{3}\frac{k}{A_{u\ell}hc\lambda_{u\ell}^2}\left[T_A(v) - T_{\rm sky}\right]$$
$$= 1.813 \times 10^{18}\,\frac{T_A(v) - T_{\rm sky}}{\rm K}\quad\left[\frac{\rm cm^{-2}}{\rm km\,s^{-1}}\right]$$
$$N_{\rm HI} = \int dv \frac{dN_{\rm HI}}{dv}$$

  - Therefore, the intensity integrated over the line profile gives us the total H I column density without need to know $T_{\rm spin}$, provided that self-absorption is not important.

- ***Alternative approach:***
  - If we now neglect absorption, then

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu \approx j_\nu \longrightarrow I_\nu = I_\nu(0) + \int j_\nu ds = I_\nu(0) + \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} \phi_\nu N_{\mathrm{HI}}$$

  - Now suppose that $I_\nu(0)$ is known independently. We can then integrate the intensity over the line

$$\int [I_\nu - I_\nu(0)]\, d\nu = \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} N_{\mathrm{HI}}$$

  - This can be expressed in terms of antenna temperature T_A and relative velocity $v = [(\nu - \nu_{u\ell})/\nu_{u\ell}]\, c$

$$\begin{aligned}
\int [T_A - T_A(0)]\, dv &= \int \frac{c^2}{2k\nu^2} [I_\nu - I_\nu(0)] \frac{c}{\nu_{u\ell}} d\nu \\
&\approx \frac{c^3}{2k\nu_{u\ell}^3} \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} N_{\mathrm{HI}} \\
&= \frac{3}{32\pi} \frac{hc\lambda_{u\ell}^2}{k} A_{u\ell} N_{\mathrm{HI}}
\end{aligned}$$

- We, then, obtain the same equation as before:

$$N_{\rm HI} \approx \frac{32\pi}{3} \frac{k}{A_{u\ell} hc \lambda_{u\ell}^2} \int \left[T_A - T_A(0)\right] dv$$
$$= 1.813 \times 10^{18} \int \frac{T_A - T_A(0)}{\rm K\,km\,s^{-1}} dv \ \ \left[{\rm cm}^{-2}\right]$$

- Here, we did not use the relation between the optical depth and column density.
  - In the first method, we approximate the RT equation as follows:

$$I_\nu = I_\nu(0) e^{-\tau_\nu} + S_\nu \left(1 - e^{-\tau_\nu}\right)$$
$$\approx I_\nu(0) + S_\nu \tau_\nu$$

Hmm! the first order approximate solution should be

$$I_\nu \approx I_\nu(0)\left(1 - \tau_\nu\right) + S_\nu \tau_\nu$$

  - In the second method, we completely ignored the absorption.

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu$$
$$\approx j_\nu$$

This is a zeroth order approximation.

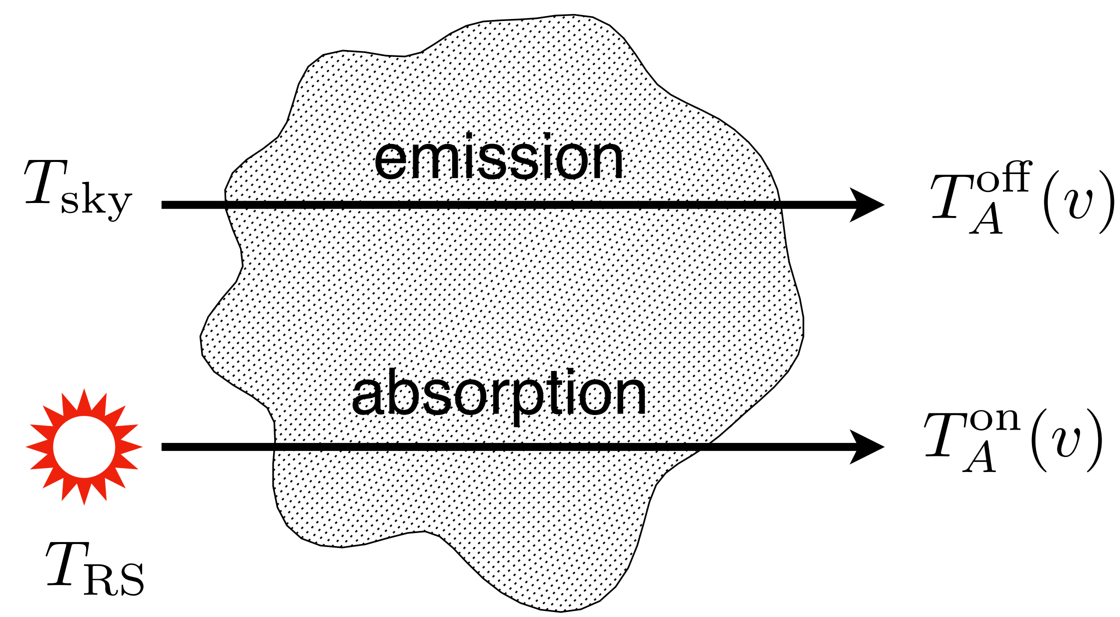
  - Therefore, the second method is self-consistent, but the first method is not.

[2] Spin Temperature Determination

- In cases where we have a ***"bright" background radio source*** with a continuum spectrum (a typical radio-loud quasar or an active galactic nucleus, or a radio galaxy), we can study both emission and absorption by the foreground ISM in our galaxy by comparing ***"on-source" and "off-source" observations***.

- The spectra measured on the blank sky and on the radio source are, respectively,

$$T_A^{\rm on}(v) = T_{\rm RS} e^{-\tau_v} + T_{\rm spin}(v)\left(1 - e^{-\tau_v}\right)$$
$$T_A^{\rm off}(v) = T_{\rm sky} e^{-\tau_v} + T_{\rm spin}(v)\left(1 - e^{-\tau_v}\right)$$



- These two equation can be solved for the two unknowns, $\tau(v)$ and $T_{\rm spin}(v)$.

$$\tau(v) = \ln\left[\frac{T_{\rm RS} - T_{\rm sky}}{T_A^{\rm on}(v) - T_A^{\rm off}(v)}\right]$$

$$T_{\rm spin}(v) = \frac{T_A^{\rm off}(v)T_{\rm RS} - T_A^{\rm on}(v)T_{\rm sky}}{(T_{\rm RS} - T_{\rm sky}) - \left(T_A^{\rm on}(v) - T_A^{\rm off}(v)\right)}$$

The solution gives, in general, the spin temperature as a function of velocity.

- We can also derive the column density from these two quantities for an optically thick cloud.

- ***Optically thin case:*** We now consider an optically thin case where the radio source is much brighter than the spin temperature of the intervening hydrogen cloud, and assume that the spin temperature is independent of velocity.
  - The RT equation for the "on-source" and "off-source" measurements can be written:

$$\boxed{\text{assumptions} : T_{\rm RS} \gg T_{\rm spin}, \;\; \tau_v \ll 1, \;\; \text{and} \; T_{\rm spin} = \text{constant}}$$

$$\begin{aligned} T_A^{\rm on}(v) &= T_{\rm RS} e^{-\tau_v} + T_{\rm spin}(v)\left(1 - e^{-\tau_v}\right) \\ T_A^{\rm off}(v) &= T_{\rm sky} e^{-\tau_v} + T_{\rm spin}(v)\left(1 - e^{-\tau_v}\right) \end{aligned} \quad \longrightarrow \quad \begin{aligned} T_A^{\rm on}(v) &= T_{\rm RS}\, e^{-\tau_v} \\ T_A^{\rm off}(v) &= T_{\rm sky} + T_{\rm spin}\left(1 - e^{-\tau_v}\right) \end{aligned}$$

  - Using the absorption spectrum from the "on-source" observation, we can obtain the "velocity equivalent width."

$$\begin{aligned} W_v &= \int dv \left(1 - e^{-\tau_v}\right) \qquad \Longrightarrow \qquad c \int \frac{d\nu}{\nu_{u\ell}} \left(1 - e^{-\tau_\nu}\right) = cW \\ &= \int dv \left[\frac{T_A^{\rm on}(v) - T_{\rm RS}}{T_{\rm RS}}\right] \end{aligned}$$

- Rearranging the equation for the “off-source” observation, we obtain

$$\int dv\,\left[T_A^{\rm off}(v) - T_{\rm sky}\right] = T_{\rm spin}\int dv\,\left(1 - e^{-\tau_v}\right)$$

- Combining this with the definition of the velocity equivalent width, we now obtain the equation for the spin temperature:

$$T_{\rm spin} = \frac{1}{W_v}\int dv\,\left[T_A^{\rm off}(v) - T_{\rm sky}\right]$$

- This equation can be converted to the column density:

$$T_{\rm spin} = \frac{3}{32\pi}\frac{A_{u\ell}hc\lambda_{u\ell}^2}{k}\frac{1}{W_v}\int dv\frac{dN_{\rm HI}}{dv}$$

$$\frac{dN_{\rm HI}}{dv} \approx \frac{32\pi}{3}\frac{k}{A_{u\ell}hc\lambda_{u\ell}^2}\left[T_A(v) - T_{\rm sky}\right]$$

- In summary, in an optically thin limit,

$$T_{\rm spin} \approx \frac{3}{32\pi}\frac{A_{u\ell}hc\lambda_{u\ell}^2}{k}\frac{N_{\rm HI}}{W_v} = 0.5516\,\frac{N_{\rm HI}/10^{18}\,{\rm cm^{-2}}}{W_v/{\rm km\,s^{-1}}}\ \ [{\rm K}]$$

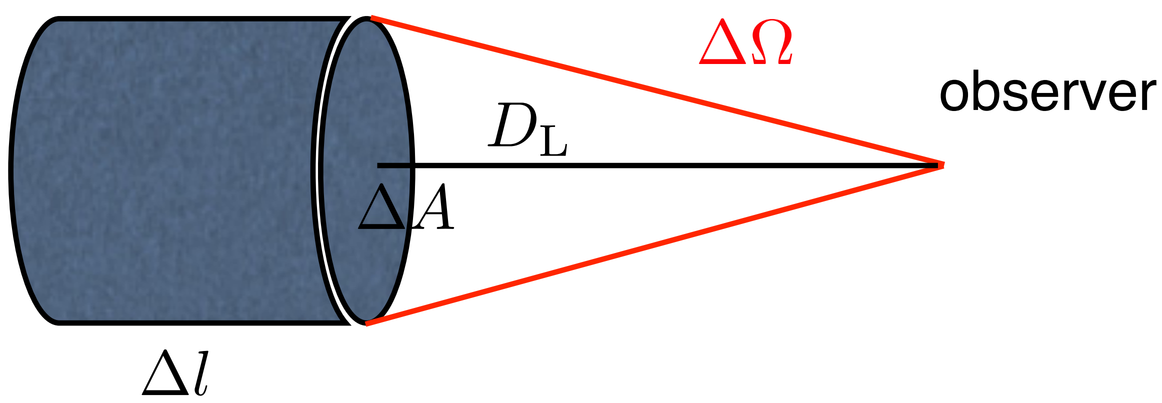
$$N_{\rm HI} \approx \frac{32\pi}{3}\frac{k}{A_{u\ell}hc\lambda_{u\ell}^2}\int dv\,\left[T_A^{\rm off}(v) - T_{\rm sky}\right] = 1.813\times 10^{18}\int dv\left[\frac{T_A^{\rm off}(v) - T_{\rm sky}}{{\rm K\,km\,s^{-1}}}\right]\ \ [{\rm cm^{-2}}]$$

$$T_{\rm spin} \approx \frac{1}{W_v}\int dv\,\left[T_A^{\rm off}(v) - T_{\rm sky}\right]$$

$$W_v = \int dv\left[\frac{T_A^{\rm on}(v) - T_{\rm RS}}{T_{\rm RS}}\right]$$

H I mass of an External Galaxy

- With the assumption that the emitting regions are optically thin, the total mass $M_{\rm HI}$ of H I in an external galaxy can be determined from the observed flux in the 21-cm line:



$F_\nu =$ observed flux density

$$F_{\rm obs} = \int F_\nu d\nu_{\rm obs} = I\Delta\Omega$$

$$I = \int I_\nu d\nu_{\rm obs} = \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} N_{\rm HI}$$

- Here, $D_{\rm L}$ is the luminosity distance to the galaxy.

$$n_{\rm H}\Delta l = N_{\rm HI}$$

$$\Delta A = D_{\rm L}^2 \Delta\Omega = D_{\rm L}^2 \frac{F_{\rm obs}}{I}$$

$$M_{\rm HI} = m_{\rm H} n_{\rm H} \Delta V = m_{\rm H} n_{\rm H} \Delta l \Delta A$$

$$= m_{\rm H} N_{\rm HI} D_{\rm L}^2 \frac{F_{\rm obs}}{I}$$

$$\therefore \; M_{\rm HI} = \frac{16\pi}{3} \frac{m_{\rm H}}{A_{u\ell} h\nu_{u\ell}} D_{\rm L}^2 F_{\rm obs}$$

$$= 4.945 \times 10^7 M_\odot \left(\frac{D_{\rm L}}{\rm Mpc}\right)^2 \left(\frac{F_{\rm obs}}{\rm Jy\,MHz}\right)$$

- If the redshift of the galaxy is z :

$$\nu_{\rm obs} = \nu/(1+z)$$
$$d\nu_{\rm obs} = \frac{\nu_{u\ell}}{(1+z)}\frac{dv}{c}$$

$$\begin{aligned}
M_{\rm HI} &= \frac{16\pi}{3}\frac{m_{\rm H}}{A_{u\ell}h\nu_{u\ell}}D_{\rm L}^2\int F_\nu d\nu_{\rm obs} \\
&= \frac{16\pi}{3}\frac{m_{\rm H}}{A_{u\ell}h\nu_{u\ell}}D_{\rm L}^2\frac{\nu_{u\ell}}{c}\frac{1}{1+z}\int F_\nu dv \\
&= \frac{16\pi m_{\rm H}}{3A_{u\ell}hc}D_{\rm L}^2(1+z)^{-1}\int F_\nu dv \\
&= 2.343\times 10^5\, M_\odot(1+z)^{-1}\left(\frac{D_{\rm L}}{\rm Mpc}\right)^2\frac{\int F_\nu dv}{\rm Jy\,km\,s^{-1}}
\end{aligned}$$

- Radio astronomers often report the integrated flux in “Jy km s^{-1}.”

Observation: Example 1

- All-sky map of H I 21-cm line intensity from the LAB survey (Kalberla et al. 2005), with angular resolution ~ 0.6 deg.
  - Scale gives $\log_{10} N(\mathrm{HI})$ [cm^{-2}]. The LMC and SMC are visible, with a connecting H I "bridge".

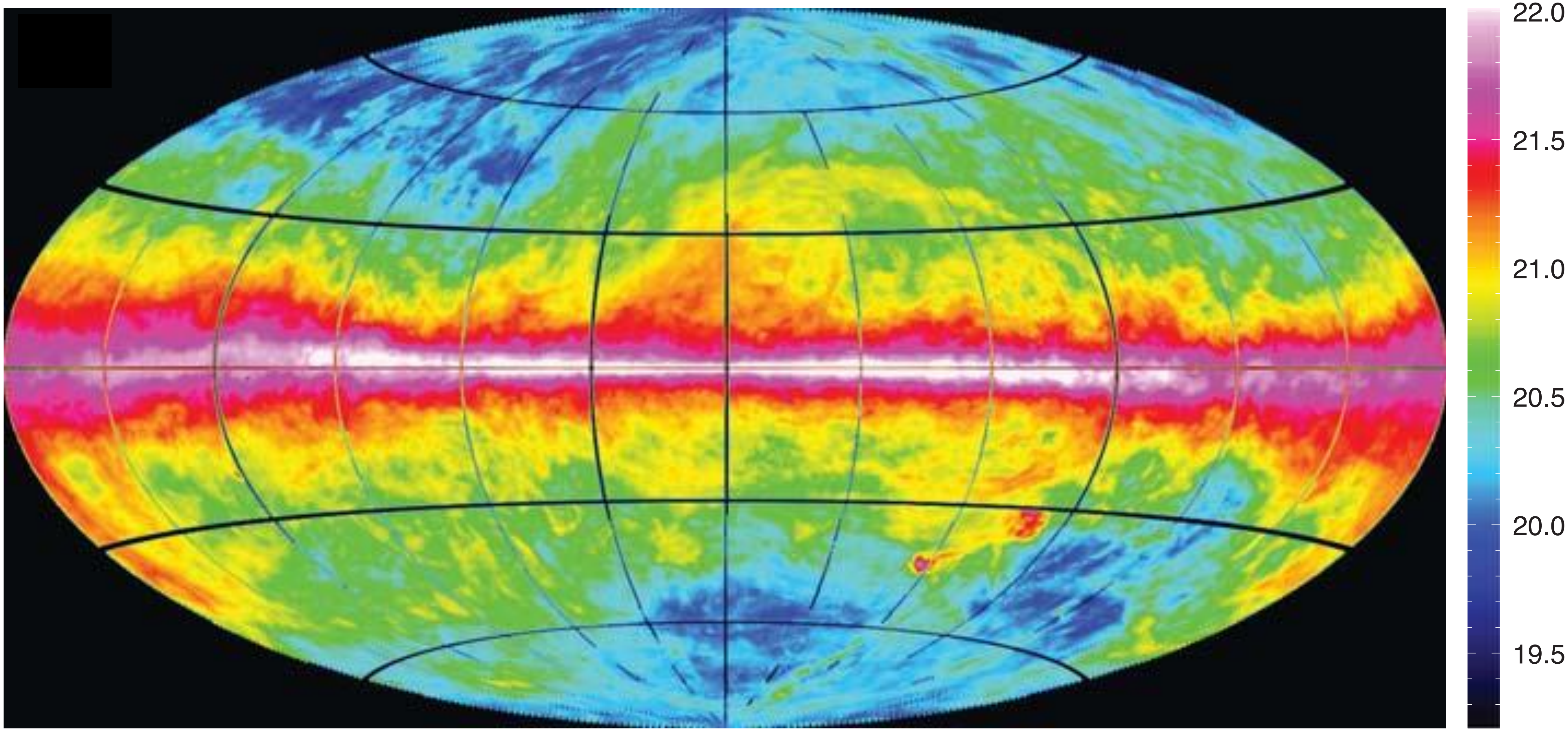


Plate 3 in [Draine]

Observations: Example 2

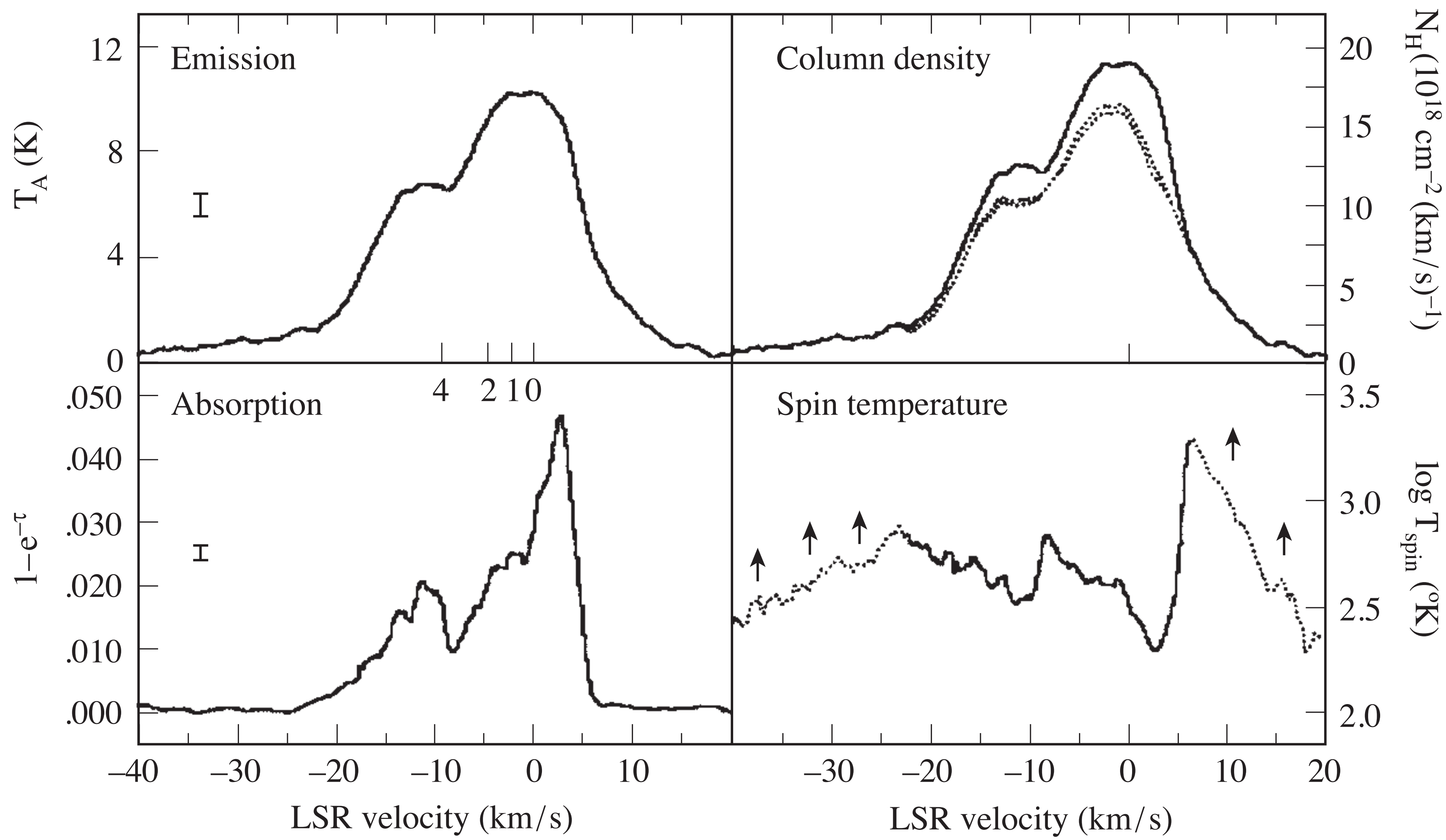
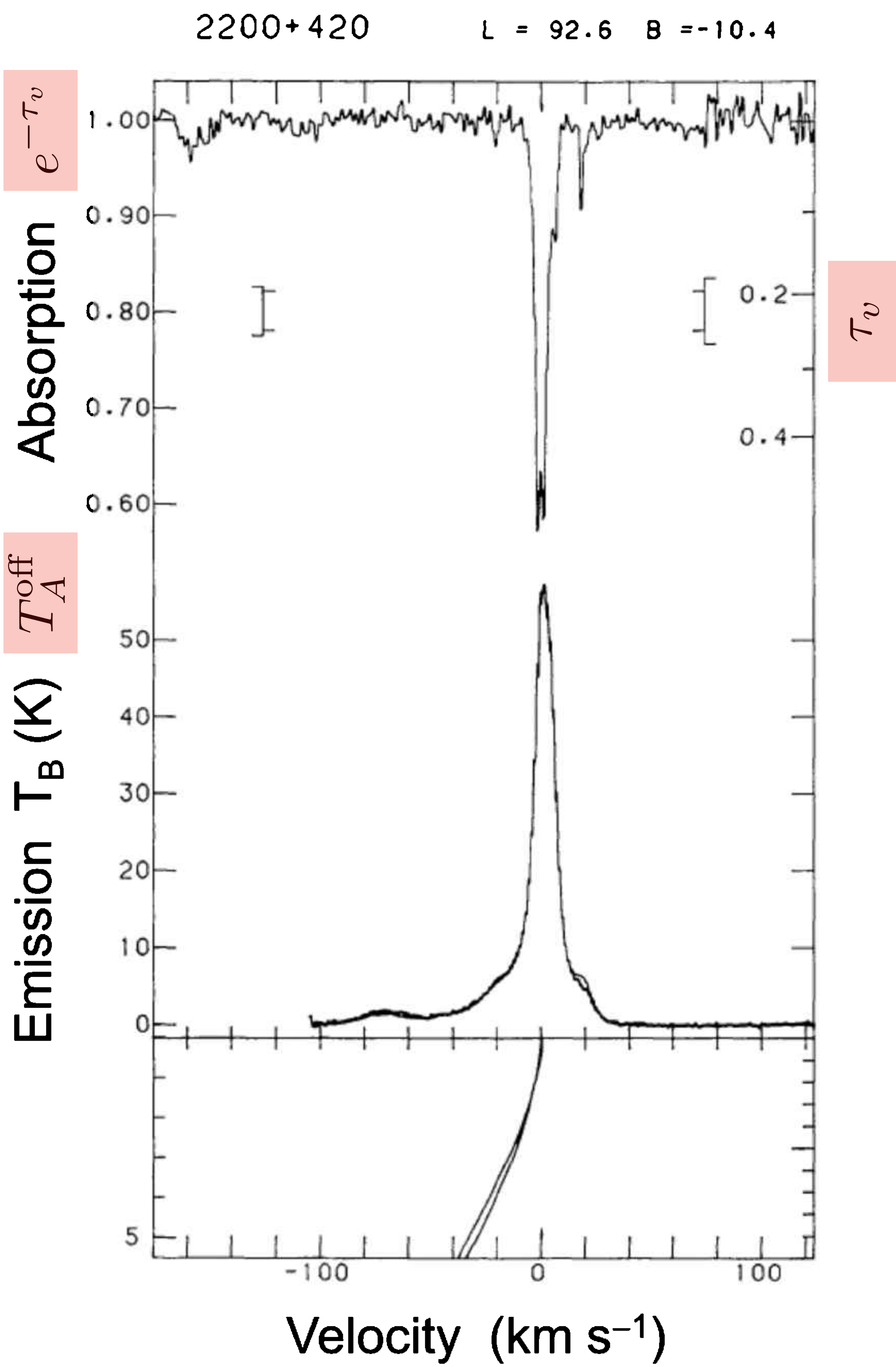


Figure 29.1 Left panels: Observed H I emission (off the quasar 3C48) and absorption (toward 3C48, at $\ell = 134°$, $b = -28.7°$). Lower right: spin temperature $T_{\rm spin}(v)$ as a function of LSR velocity. Tick marks labeled 0, 1, 2, and 4 on abscissa of left panels show the LSR velocity expected for gas at a distance of 0,1,2,4 kpc (for an assumed Galactic rotation curve). Upper right: $dN(\mathrm{H\,I})/dv$ for different assumptions regarding the relative (foreground/background) locations of cold absorbing gas and warm gas seen only in emission. From Dickey et al. (1978).

[Figure 29.1 in Draine]

Observations: Example 3



H I 21-cm absorption and emission along the line of sight towards BL Lacertae
[Dickey et al. 1983; Figure 3.3 in Ryden]

- maximum optical depth :

$$\tau_v \sim 0.5$$

- equivalent width of the absorption line :

$$W_v = 7\,\mathrm{km\,s^{-1}}$$

- integrated line intensity of the emission line :

$$\int \Delta T_A^{\rm off}(v)dv \approx 930\,\mathrm{K\,km\,s^{-1}}$$

- column density from the emission line :

$$N_{\rm HI} \approx 1.69\times 10^{21}\,\mathrm{cm^{-2}}$$

- Now, the spin temperature is

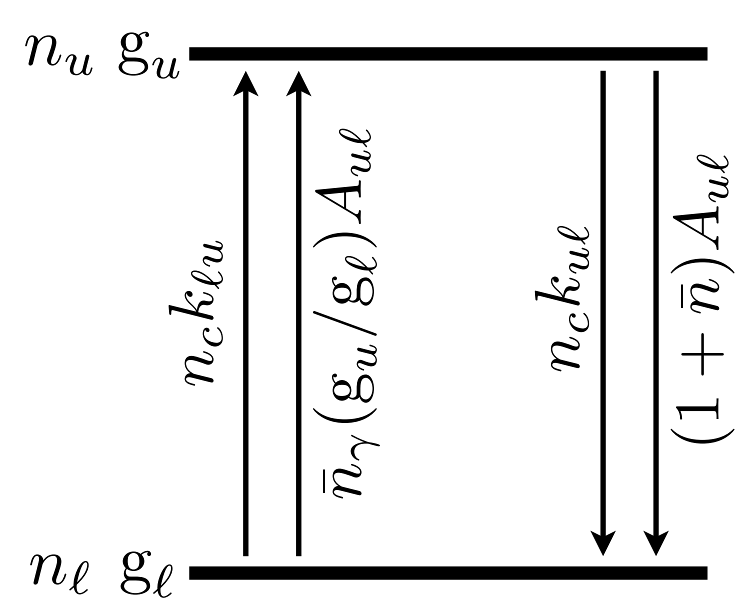
$$T_{\rm spin} = \frac{\int \Delta T_A^{\rm off}(v)dv}{W_v} = \frac{930\,\mathrm{K},\ \mathrm{km\,s^{-1}}}{7\,\mathrm{km\,s^{-1}}} \approx 133\,\mathrm{K}.$$

Observation: CNM + WNM

- The most recent emission-absorption surveys (Heiles & Troland 2003) support the idea that, in the solar neighborhood (i.e., within ~ 500 pc of the Sun), interstellar H I is found primarily in two distinct phases: the CNM and the WNM. About 40% of the H I (by mass) is in the CNM, with a median spin temperature T ~ 70 K. The remaining 60% of the H I is in the WNM phase, which appears to have a volume filling factor ~ 50% near the disk midplane.
- Because warm H I absorbs very weakly, for some of the WNM material it is only possible to determine a lower bound on T_{spin}. Heiles & Troland (2003) conclude that > 48% of the WNM has $500 < T_{spin} < 5000$ K, at these temperatures the gas is expected to be thermally unstable.

Level Population

- In some cases, it is sufficient to consider only the ground state and the first excited state.
  - Consider collisional excitation and de-excitation by some species (e.g., electrons) with density n_c, and suppose that radiation with the energy density u_ν .
  - The population of the excited state must satisfy:



$$\frac{dn_u}{dt} = n_\ell \left[n_c k_{\ell u} + \bar{n}_\gamma \frac{\mathrm{g}_u}{\mathrm{g}_\ell} A_{u\ell} \right] - n_u \left[n_c k_{u\ell} + (1+\bar{n}_\gamma)\, A_{u\ell} \right]$$

photon occupation number

$$\left(\bar{n}_\gamma \equiv \frac{c^3}{8\pi h \nu^3} u_\nu \right)$$

  - The steady-state solution with radiation and collision present is

$$\frac{n_u}{n_\ell} = \frac{n_c k_{\ell u} + \bar{n}_\gamma (\mathrm{g}_u/\mathrm{g}_\ell) A_{u\ell}}{n_c k_{u\ell} + (1+\bar{n}_\gamma) A_{u\ell}}$$

Using this equation, we can calculate the excitation temperature between the two levels.

Here, by the principle of detailed balance, the upward collisional rate coefficient is given in term of the downward rate coefficient by

$$k_{\ell u} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} k_{u\ell} e^{-E_{u\ell}/kT_{\mathrm{gas}}} \qquad (T_{\mathrm{gas}} = \text{gas kinetic energy})$$

- It is instructive to examine the population equation in various limits:
  - In the limit of $n_c \to \infty$ and no radiation field $\bar{n}_\gamma = 0$:

$$\frac{n_u}{n_\ell} = \frac{n_c k_{\ell u}}{n_c k_{u\ell} + A_{u\ell}} = \frac{k_{\ell u}}{k_{u\ell}} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} e^{-E_{u\ell}/kT_{\mathrm{gas}}} \qquad \therefore\ T_{\mathrm{exc}} = T_{\mathrm{gas}}$$

  - If $n_c = 0$ and the radiation field has a brightness temperature of $T_b = T_{\mathrm{rad}}$ at the frequency $\nu = E_{u\ell}/h$:

$$\frac{n_u}{n_\ell} = \frac{\bar{n}_\gamma (\mathrm{g}_u/\mathrm{g}_\ell)}{(1+\bar{n}_\gamma)} = \frac{\mathrm{g}_u}{g_\ell} e^{-E_{u\ell}/kT_{\mathrm{rad}}} \qquad \therefore\ T_{\mathrm{exc}} = T_{\mathrm{rad}}$$

$$\bar{n}_\gamma = \left(e^{E_{u\ell}/kT_{\mathrm{rad}}} - 1\right)^{-1} \implies 1 + \bar{n}_\gamma = \bar{n}_\gamma\, e^{E_{u\ell}/kT_{\mathrm{rad}}}$$

  - If we have a radiation with the brightness temperature $T_b = T_{\mathrm{rad}} = T_{\mathrm{gas}}$, then we can show that

$$\frac{n_u}{n_\ell} = \frac{n_c k_{\ell u} + \bar{n}_\gamma (\mathrm{g}_u/\mathrm{g}_\ell) A_{u\ell}}{n_c k_{u\ell} + (1+\bar{n}_\gamma) A_{u\ell}} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} e^{-E_{u\ell}/kT_{\mathrm{gas}}} \qquad \therefore\ T_{\mathrm{exc}} = T_{\mathrm{rad}} = T_{\mathrm{gas}}$$

Critical Density

- For a collision partner c, we define the critical density $n_{\mathrm{crit},u}$ for an excited state *u* to be the density for which collisional de-excitation equals radiative de-excitation, including stimulated emission:

$$n_{\mathrm{crit},u}(c) \equiv \frac{\sum_{\ell<u} \left[1+(n_\gamma)_{u\ell}\right] A_{u\ell}}{\sum_{\ell<u} k_{u\ell}(c)}$$

- Note that this definition applies to multilevel systems, but each excited level *u* may have a different critical density. The critical density depends on the intensity of ambient radiation. For many transitions, the correction is unimportant, but for 21-cm line, it is important.

Critical densities for fine-structure excitation [Table 17.1 in Draine, revised for both H and e$^-$, errata]

Ion	ℓ	u	E_ℓ/k (K)	E_u/k (K)	$\lambda_{u\ell}$ (μm)	$n_{\mathrm{crit},u}(\mathrm{H})$ $T=100$ K (cm^{-3})	$n_{\mathrm{crit},u}(\mathrm{H})$ $T=5000$ K (cm^{-3})	$n_{\mathrm{crit},u}(e^-)$ $T=100$ K (cm^{-3})	$n_{\mathrm{crit},u}(e^-)$ $T=5000$ K (cm^{-3})
C II	$^2\mathrm{P}^{\mathrm{o}}_{1/2}$	$^2\mathrm{P}^{\mathrm{o}}_{3/2}$	0	91.21	157.74	2.7×10^3	1.5×10^3	6.8	40.
C I	$^3\mathrm{P}_0$	$^3\mathrm{P}_1$	0	23.60	609.7	620	170	76.	6.4
	$^3\mathrm{P}_1$	$^3\mathrm{P}_2$	23.60	62.44	370.37	720	150	75.	6.3
O I	$^3\mathrm{P}_2$	$^3\mathrm{P}_1$	0	227.71	63.185	2.5×10^5	4.9×10^4	1.8×10^5	4.8×10^4
	$^3\mathrm{P}_1$	$^3\mathrm{P}_0$	227.71	326.57	145.53	2.4×10^4	8.6×10^3	2.3×10^4	5.8×10^3
Si II	$^2\mathrm{P}^{\mathrm{o}}_{1/2}$	$^2\mathrm{P}^{\mathrm{o}}_{3/2}$	0	413.28	34.814	2.5×10^5	1.2×10^5	140.	1.5×10^3
Si I	$^3\mathrm{P}_0$	$^3\mathrm{P}_1$	0	110.95	129.68	4.8×10^4	2.8×10^4	2.9×10^4	830.
	$^3\mathrm{P}_1$	$^3\mathrm{P}_2$	110.95	321.07	68.473	9.9×10^4	3.6×10^4	4.4×10^4	1.9×10^3

Interstellar Radiation Fields

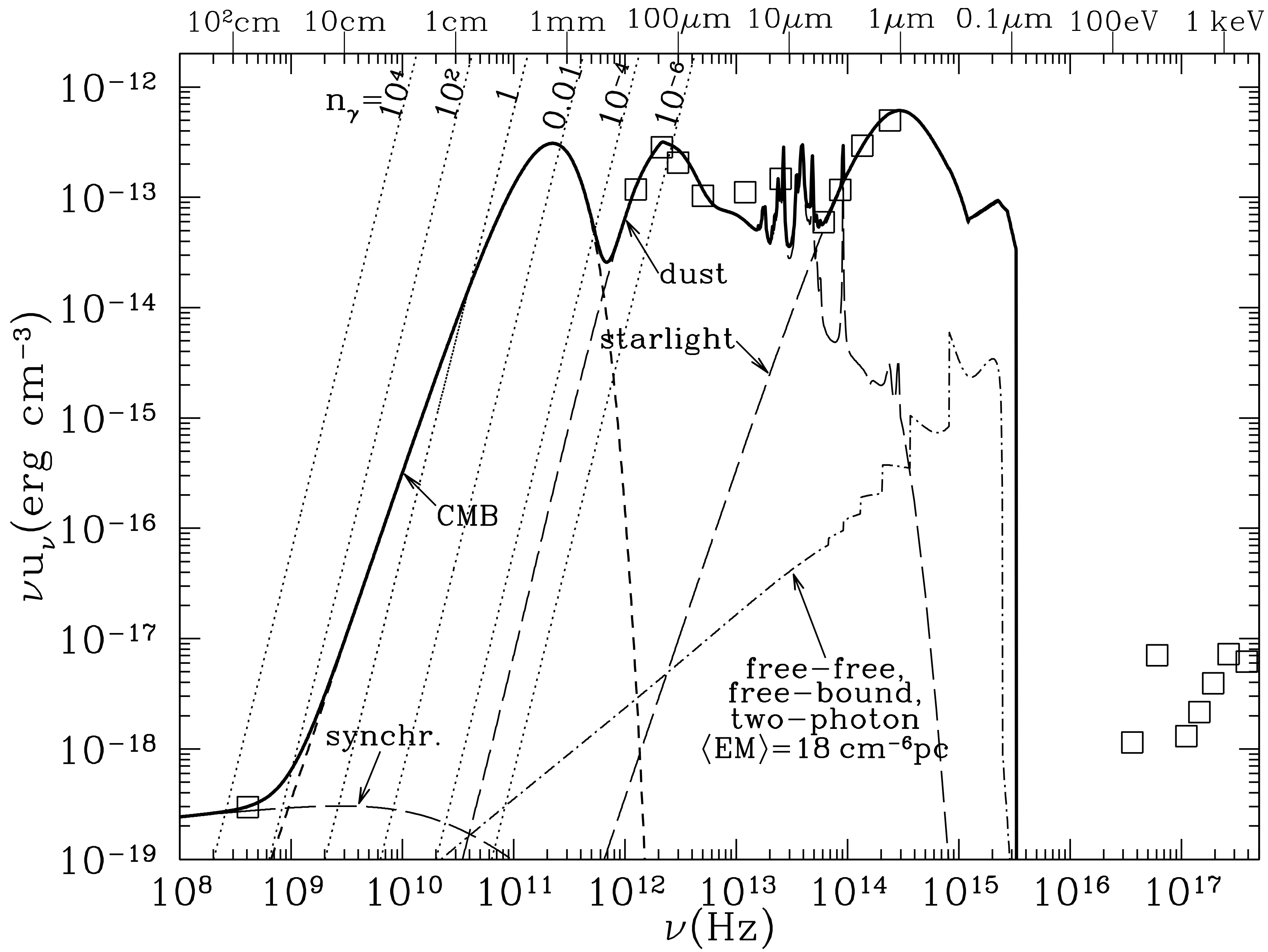


Figure 12.1 Interstellar continuum radiation field in an H I cloud in the solar neighborhood (see text). Spectral lines are not included. Solid line is the sum of all components for $h\nu \leq 13.6\,\mathrm{eV}$. Squares show the measured sky brightness at 408 MHz (Haslam et al. 1982), the all-sky measurements by COBE-DIRBE in 10 broad bands from 240 μm to 1.25 μm (Arendt et al. 1998), and all-sky measurements by ROSAT between 150 eV and 2 keV (Snowden 2005, private communication). Dotted lines are contours of constant photon occupation number n_γ.

Example: H I Spin Temperature

- Colliosional rate coefficients:

$$\frac{n_1}{n_0} = \frac{n_c k_{01} + \bar{n}_\gamma (\mathrm{g}_1/\mathrm{g}_0) A_{10}}{n_c k_{10} + (1+\bar{n}_\gamma) A_{10}}$$

- Collision with other H atoms

(Allison & Dalgarno 1969; Zygelman 2005)

$$k_{10}(\mathrm{H}) \approx \begin{cases} 1.19\times 10^{-10} T_2^{0.74-0.20\ln T_2}\,\mathrm{cm^3\,s^{-1}} & (20\,\mathrm{K} < T < 300\,\mathrm{K}) \\ 2.24\times 10^{-10} T_2^{0.207} \mathrm{e}^{-0.876/T_2}\,\mathrm{cm^3\,s^{-1}} & (300\,\mathrm{K} < T < 10^3\,\mathrm{K}) \end{cases}$$

$$k_{01}(\mathrm{H}) \approx 3k_{10}(\mathrm{H}) e^{-0.0682\,\mathrm{K}/T}$$

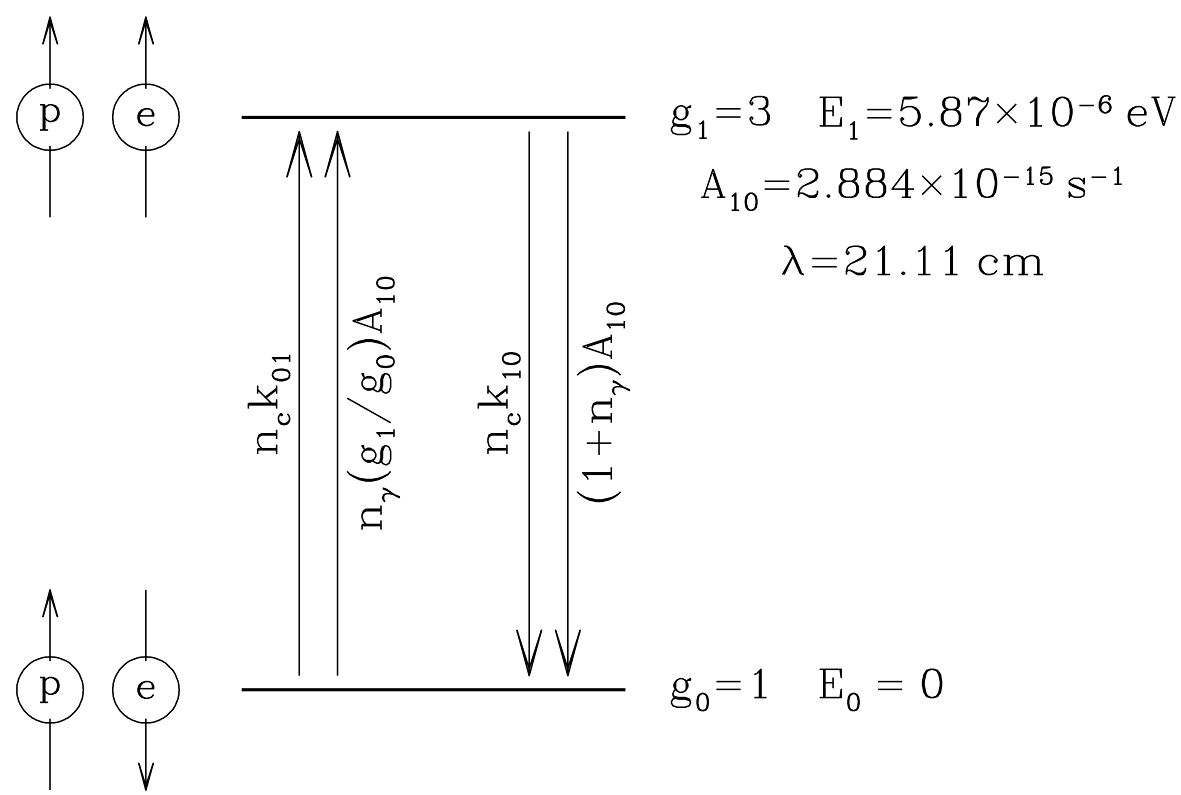
- Collision with electrons

(Furlanetto & Furlanetto 2007)

$$k_{10}(e^-) \approx 2.26\times 10^{-9}\,(T/100\,\mathrm{K})^{0.5}\,\mathrm{cm^3\,s^{-1}} \quad (1 \lesssim T \lesssim 500\,\mathrm{K})$$

$$k_{01}(e^-) \approx 3k_{10}(e^-) e^{-0.0682\,\mathrm{K}/T}$$

- This is a factor ~ 10 larger than that for H atoms. However, ***electrons will be minor importance in regions with a fractional ionization*** $x_e \lesssim 0.03$, such as the CNM and WNM.



- Radiation Field strength
  - The radiation field near 21 cm is dominated by the cosmic microwave background plus Galactic synchrotron emission. The antenna temperature is

$$T_A \approx T_{\rm CMB} + T_{\rm syn} = 2.73\,{\rm K} + 1.04\,{\rm K} = 3.77\,{\rm K}$$

  - Photon occupation number:

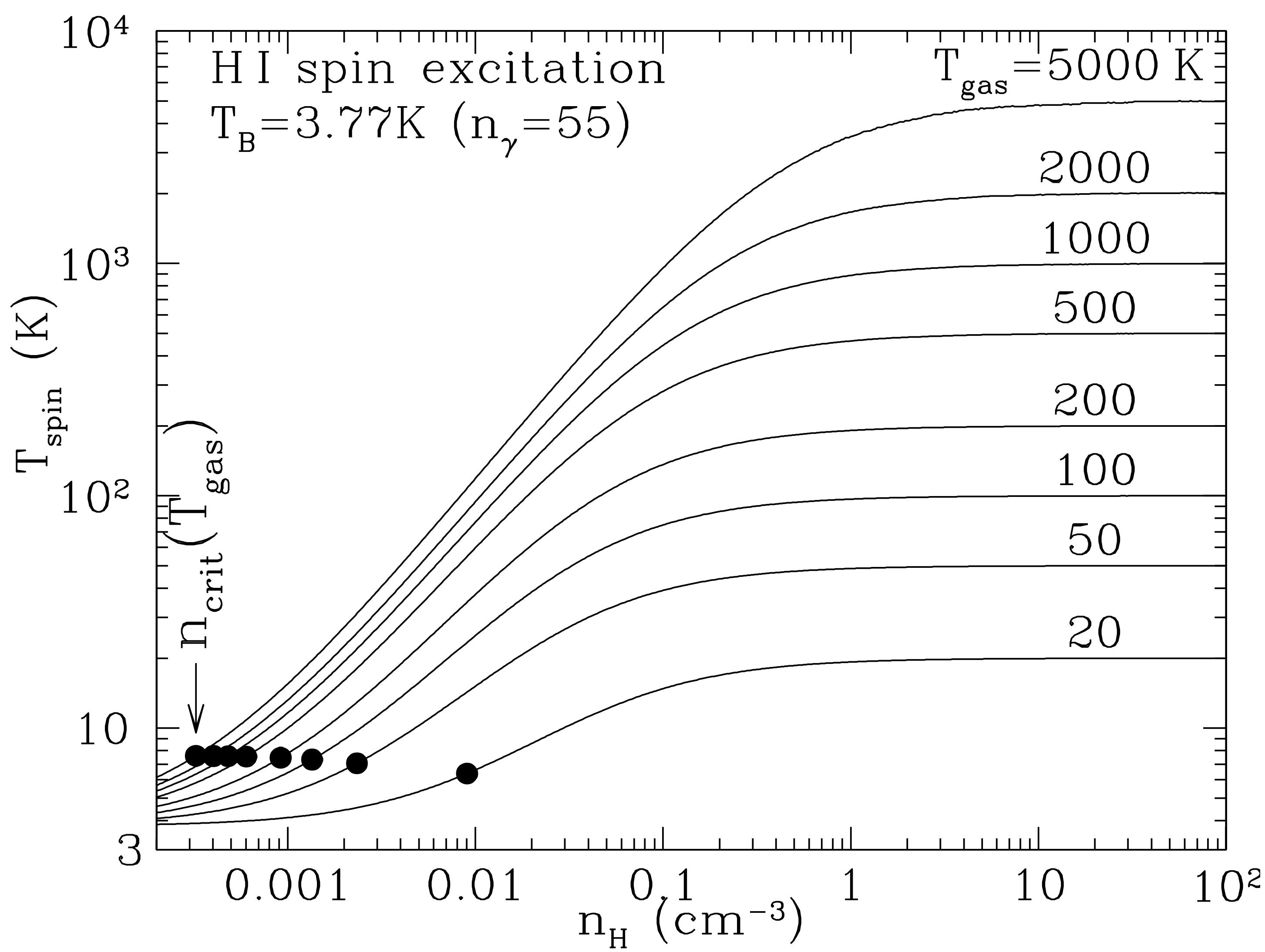
$$\bar{n}_\gamma = \left(e^{h\nu/kT_{\rm B}} - 1\right)^{-1} \approx \frac{kT_A}{h\nu} \approx \frac{3.77\,{\rm K}}{0.0682\,{\rm K}} \approx 55$$

  - The critical density is then

$$n_{\rm crit}(H) \approx 1.7 \times 10^{-3}\,(T/100\,{\rm K})^{-0.66}\,{\rm cm}^{-3}$$

$$\begin{aligned} n_{\rm crit} &\approx 0.07\,{\rm cm}^{-3} \text{ at } T \sim 10\,{\rm K} \\ &\approx 6 \times 10^{-4}\,{\rm cm}^{-3} \text{ at } T \sim 1000\,{\rm K} \end{aligned}$$

- H I spin temperature as a function of density n_H, including only 21 cm continuum radiation and collisions with H atoms. Lyα scattering is not included.
  - Filled circles show $n_{\rm crit}(\rm H)$ for each temperature.
  - It is important to note that one requires $n \gg n_{\rm crit}$ in order to have $T_{\rm spin}$ within, say, 10% of $T_{\rm gas}$, particularly at high temperatures.



[Fig. 17.2 in Draine]

Note that Ryden states that "in the CNM and WNM, we expect the hyperfine levels of atomic hydrogen to be collisionally excited, and to have a spin temperature close to the gas temperature." based on that $n_{\rm crit} \sim 6\times10^{-4}$ cm^{-3} at T ~ 1000 K.

However, this is not true in the WNM.

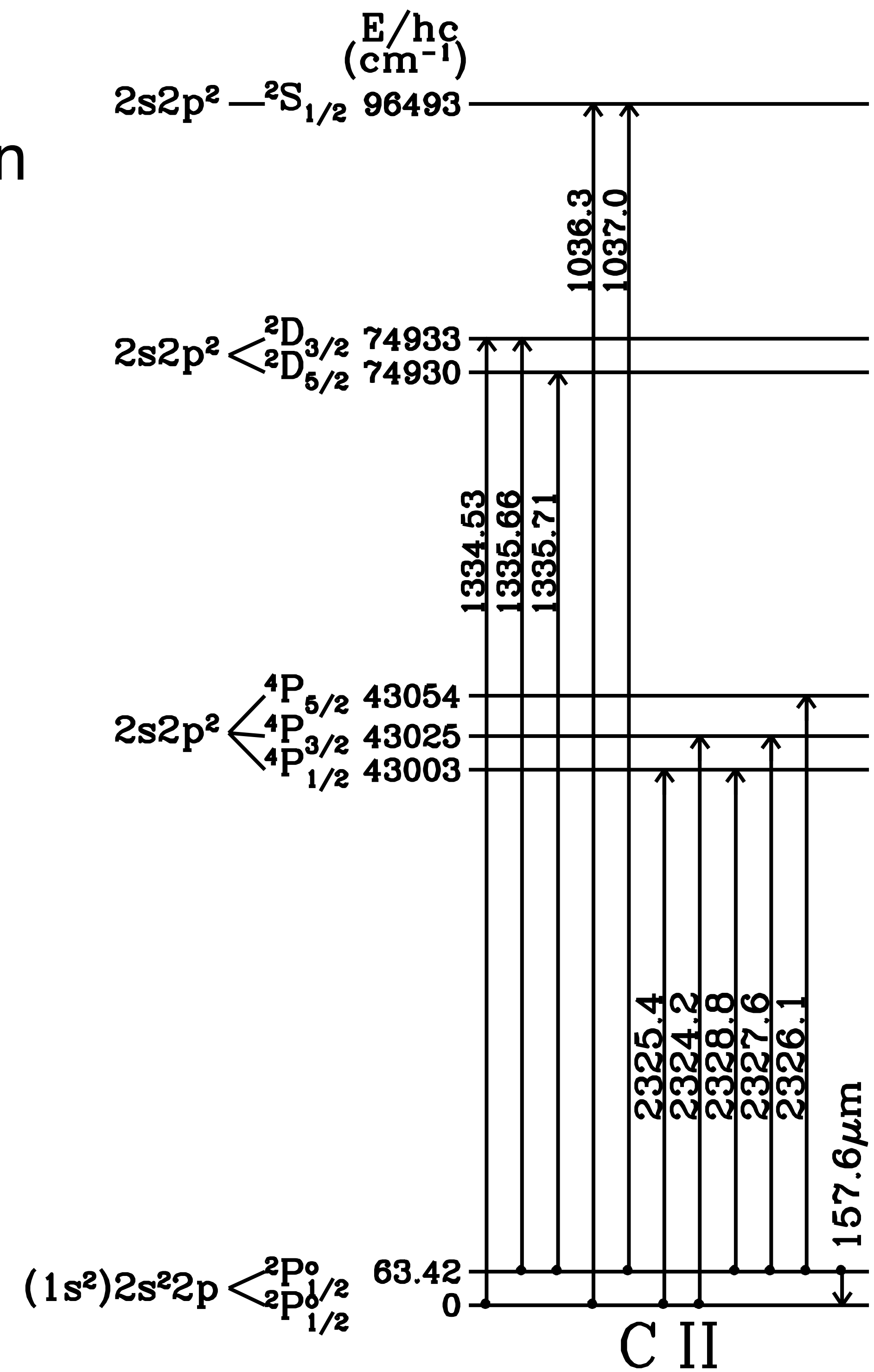
Only in the CNM, the collisional excitation is strong enough to bring the spin temperature close to the gas kinetic temperature.

Example: C II Fine Structure Excitation

- The ground electronic state $1s^22s^22p\ ^2P^o$ of C^+ contains two fine-structure levels.
- The electronically excited states have an excitation energy that is much higher than the kinetic temperature of the CNM.

$$2235\,\text{Å} \rightarrow E_{u\ell} = 0.56\,\text{eV} \rightarrow T = 6440\,\text{K}$$

- We may, therefore, consider the two fine-structure levels in the ground electronic state to be a two level atom.
- Will the populations of these two levels be thermalized in the ISM?

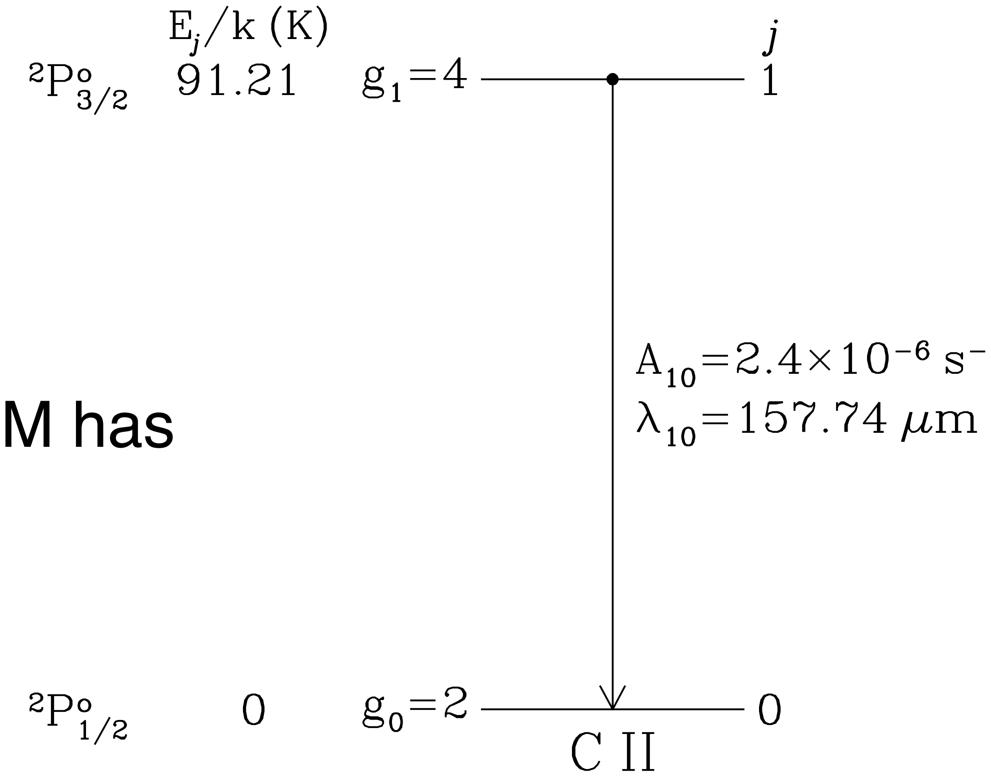


- Rate coefficients for collisional de-excitation:

$$\left\langle \Omega \left({}^2P^o_{1/2}, {}^2P^o_{3/2} \right) \right\rangle \approx 2.1$$

$$k_{10}(e^-) \approx 4.53 \times 10^{-8} \, T_4^{-1/2} \, \mathrm{cm}^3 \, \mathrm{s}^{-1}$$

$$k_{10}(H) \approx 7.58 \times 10^{-10} \, T_2^{0.1281 + 0.0087 \ln T_2} \, \mathrm{cm}^3 \, \mathrm{s}^{-1}$$



- At $\lambda = 158 \, \mu\mathrm{m}$, the continuum background in the ISM has

$$\bar{n}_\gamma \approx 10^{-5} \ll 1$$

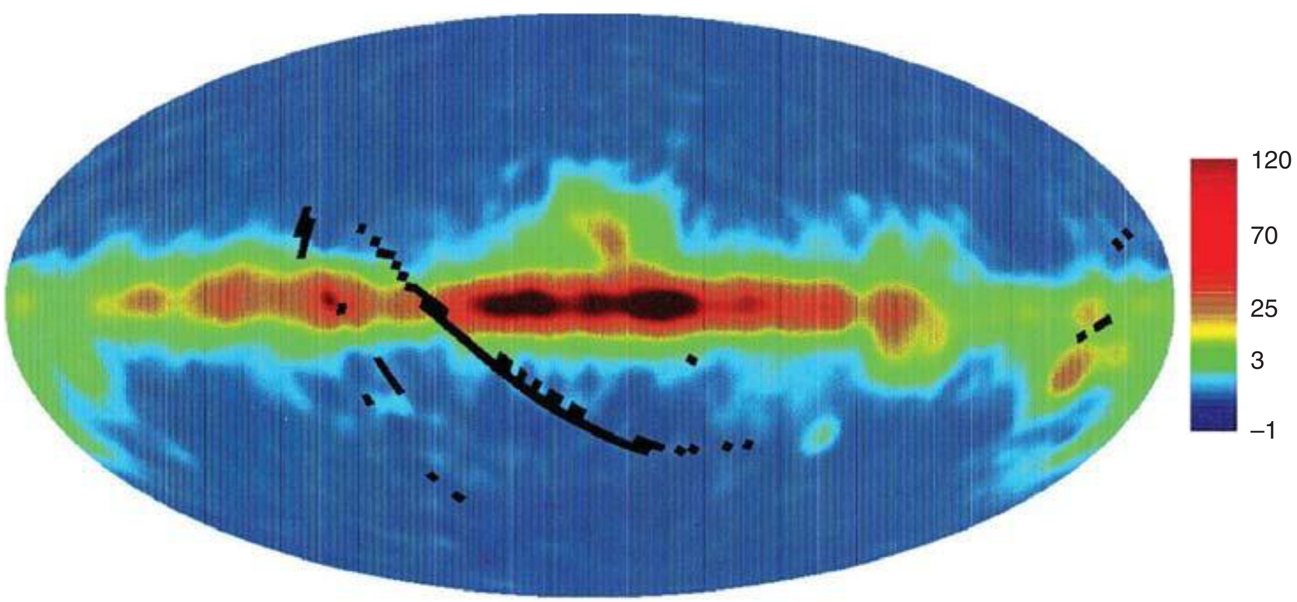
- Critical densities:

$$n_{\mathrm{crit}}(H) \approx 3.2 \times 10^3 \, T_2^{-0.1281 - 0.0087 \ln T_2} \, \mathrm{cm}^{-3}$$

$$n_{\mathrm{crit}}(e^-) \approx 53 \, T_4^{1/2} \, \mathrm{cm}^{-3}$$ (Barinovs et al. 2005)

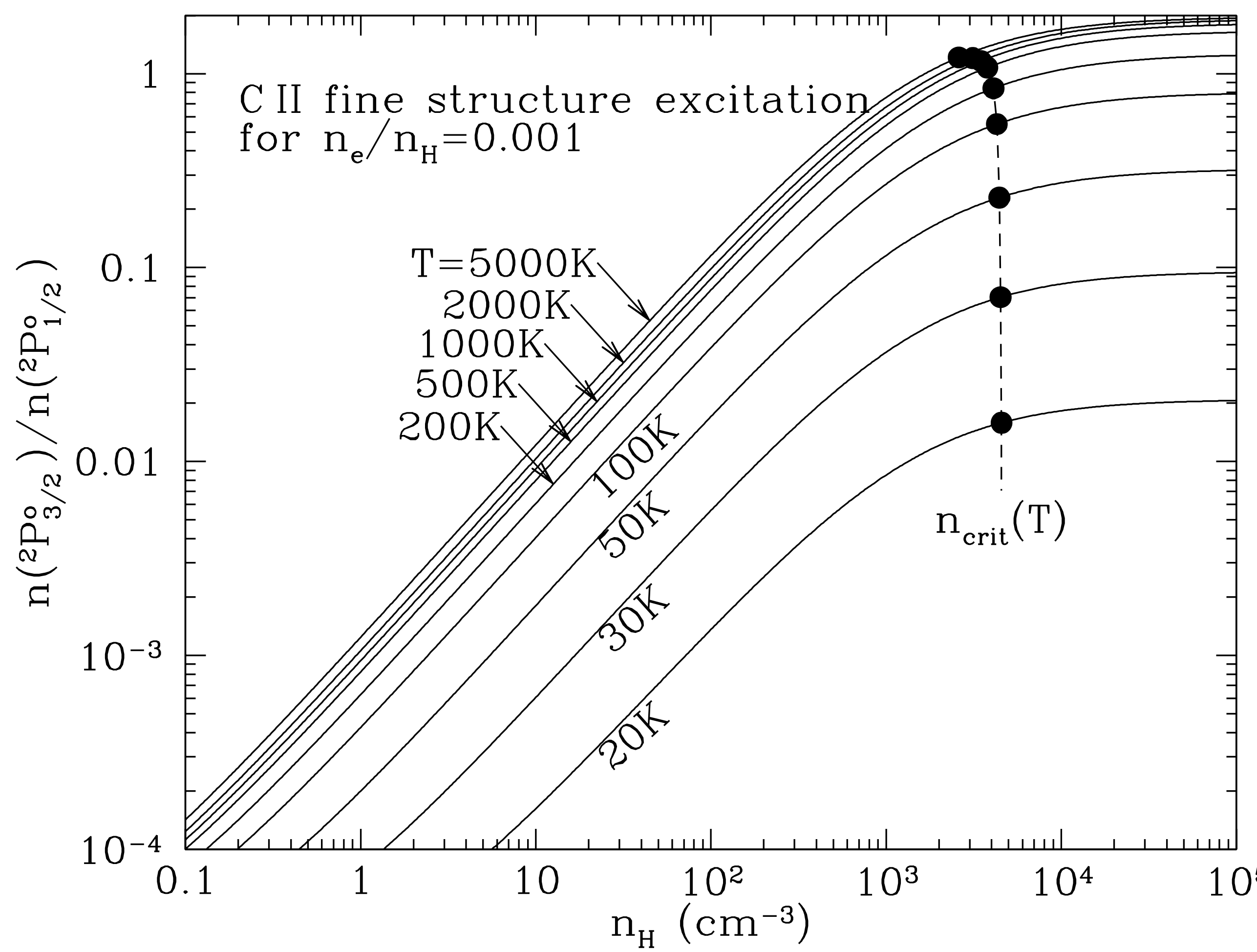
- The critical densities are much higher than the typical densities in the CNM and WNM. Thus, the C II fine-structure levels will be sub-thermally excited.

- The C II fine-structure levels will be sub-thermally excited. It follows that collisional excitations of the upper level will usually be followed by radiative decays, removing energy from the gas.
- The [C II] 158 μm transition is the principal cooling transition for the diffuse gas in star-forming galaxies.



All-sky map of [C II] 158 μm emission, made by Far InfraRed Absolute Spectrophotometer (FIRAS) on the COsmic Background Explorer (COBE) satellite (Fixsen et al. 1999).

[Plate 3 in Draine]



[Fig. 17.4 in Draine]

Equation for the 21-cm Spin Temperature

- We have derived the equation for the level populations in the presence of collision and radiation. Now, we will derive an intuitive equation for the spin temperature of the 21-cm line.

$$\frac{n_1}{n_0} = \frac{n_c k_{01} + \bar{n}_\gamma (g_1/g_0) A_{10}}{n_c k_{10} + (1 + \bar{n}_\gamma) A_{10}}$$

- Let's define the temperature corresponding to the 21-cm transition.

$$T_* = E_{10}/k = 0.0682\,\mathrm{K}$$

- The temperatures of radiation and gas will be much higher than this:

$$T_{\rm gas} \approx 10 - 10^4\,\mathrm{K} \gg T_*, \quad T_{\rm rad} = 3.77\,\mathrm{K} \gg T_*, \quad T_{\rm spin} \gg T_*$$

- The population ratio can be written in terms of the excitation (spin) temperature:

$$\frac{n_1}{n_0} = \frac{g_u}{g_\ell} e^{-T_*/T_{\rm spin}} \simeq \frac{g_u}{g_\ell}\left(1 - \frac{T_*}{T_{\rm spin}}\right)$$

- Similarly,

$$k_{01} = \frac{g_1}{g_0} k_{10} e^{-T_*/T_{\rm gas}} \simeq \frac{g_1}{g_0} k_{10}\left(1 - \frac{T_*}{T_{\rm gas}}\right)$$

$$\bar{n}_\gamma = \frac{1}{e^{T_*/T_{\rm rad}} - 1} \simeq \frac{T_{\rm rad}}{T_*}$$

- Substituting these into the population equation, we obtain

$$1-\frac{T_*}{T_{\rm spin}}=\frac{n_c k_{10}\left(1-T_*/T_{\rm gas}\right)+\left(T_{\rm rad}/T_*\right)A_{10}}{n_c k_{10}+\left(1+T_{\rm rad}/T_*\right)A_{10}}$$

- Finally, we obtain the following equation:

$$T_{\rm spin}=\frac{T_*+T_{\rm rad}+y_c T_{\rm gas}}{1+y_c} \quad \Longleftarrow \quad y_c\equiv\frac{T_*}{T_{\rm gas}}\frac{n_c k_{10}}{A_{10}}$$

- Ignoring T_* term, we obtain an intuitive equation for the spin temperature.

$$T_{\rm spin}=\frac{T_{\rm rad}+y_c T_{\rm gas}}{1+y_c} \quad \Longleftarrow \quad y_c\equiv\frac{T_*}{T_{\rm gas}}\frac{n_c k_{10}}{A_{10}}$$

This equation was first derived by G. Field (1948).

- This equation describes the spin temperature as ***a weighted mean of the radiation and gas temperatures with weights of 1 and*** y_c.

- From the equation, we can show that

$$\begin{aligned} T_{\rm spin} &\simeq T_{\rm rad} \quad \text{if } y_c \ll 1 \\ T_{\rm spin} &\simeq T_{\rm gas} \quad \text{if } y_c \gg 1 \end{aligned}$$

- A new critical density of the colliding particle may be defined:

$$y_c=1 \quad \Longrightarrow \quad n^*_{\rm crit}=\frac{T_{\rm gas}}{T_*}\frac{A_{10}}{k_{10}}$$

compare with the original definition.

$$n_{\rm crit}\equiv\frac{\left[1+(n_\gamma)_{10}\right]A_{10}}{k_{10}}$$

Wouthuysen-Field Effect

- Level population - the spin temperature (Ts) is determined by 3 mechanisms.

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} \exp\left(-\frac{h\nu_{10}}{k_B T_S}\right)$$

$$T_* = h\nu_{10}/k_B = 0.068 \text{ K}$$

(1) Direct Radiative Transitions by the background radiation field (Cosmic Microwave Background + Galactic Synchrotron = 3.77 K)

$$I_\nu = \frac{2k_B T_R}{\lambda_{10}^2} \quad (T_R = \text{brightness temperature}) \qquad T_R = 2.73 \text{ K or } 3.77 \text{ K}$$

(Rayleigh-Jeans Law)

(2) Collisional Transitions (collision with hydrogen and electron)

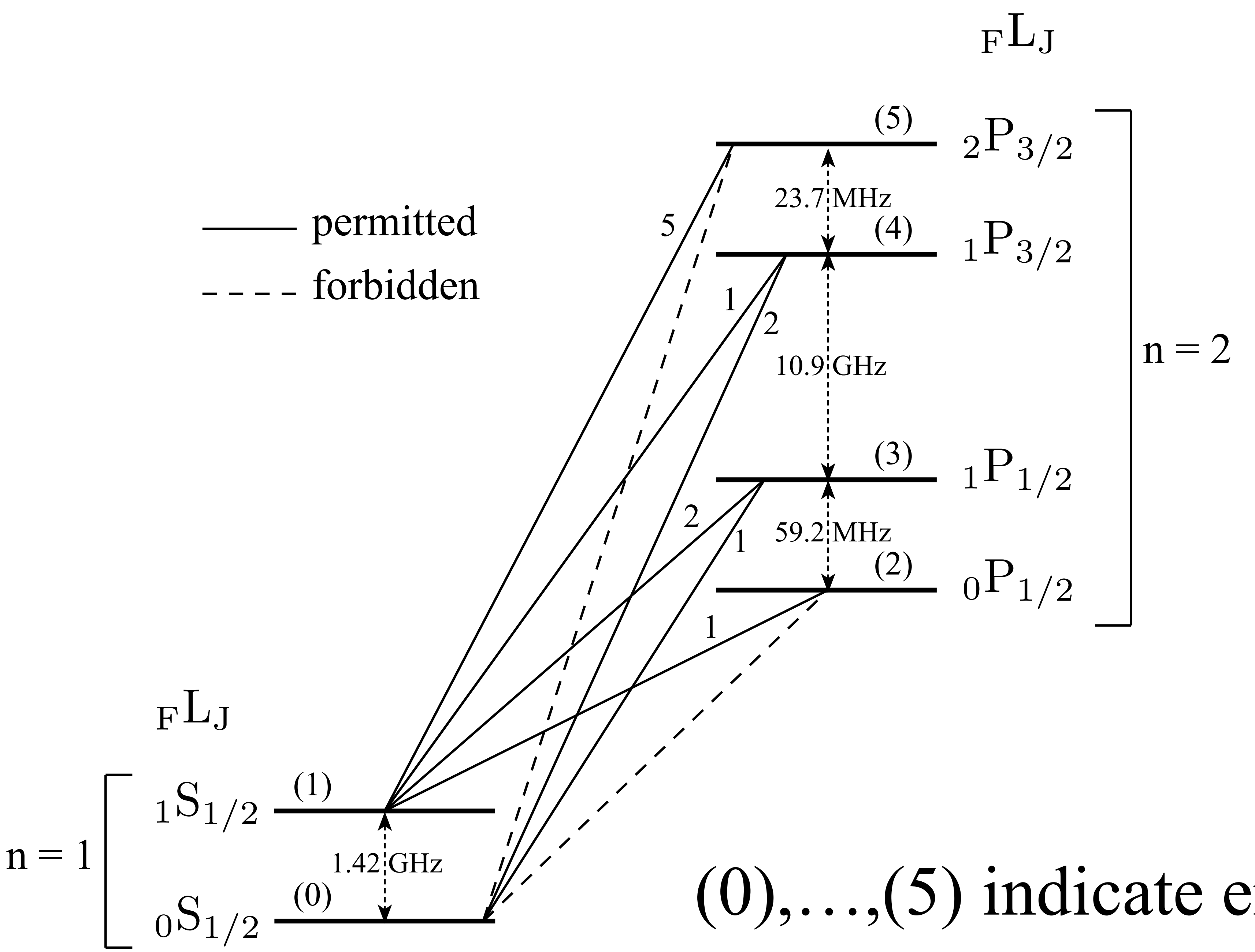
$$\text{gas kinetic temperature} = T_K$$

(3) Lyα pumping: Indirect Radiative Transitions involving intermediate levels by scattered Lyα photons.

$$\text{Ly}\alpha \text{ color temperature} = T_\alpha$$

WF effect - Hydrogen Atom

- see Field (1958)



(0),…,(5) indicate energy levels

Relative line strengths are

$$S_{51} : S_{41} : S_{40} : S_{31} : S_{30} : S_{21} = 5 : 1 : 2 : 2 : 1 : 1$$

$$S_{50} = S_{20} = 0$$

Line strengths

- Let's consider transitions from nJF to n'J'F' over all possible n'J'.
- Note statistical weight

$$g_{\mathrm{F}} = 2F + 1$$

- Applying the sum rule to four sets of downward transitions from n=2 to n'=1, the line strengths S_{ji} are

$$S_{51} : S_{41} + S_{40} : S_{31} + S_{30} : S_{21} = 5 : 3 : 3 : 1$$

- Applying the sum rule to four sets of upward transitions from n=1 to n'=2,

$$S_{04} : S_{14} + S_{15} = 1 : 3$$

$$S_{03} : S_{12} + S_{13} = 1 : 3$$

- Note $S_{ij} = S_{ji}$. Then, solving the above relations gives

$$S_{51} : S_{41} : S_{40} : S_{31} : S_{30} : S_{21} = 5 : 1 : 2 : 2 : 1 : 1$$

Ref: Condon & Shortley, The Theory of Atomic Spectra, Chap. IX, p236

Einstein A-coefficients

- Note

$$S_{ji} \propto g_j A_{ji} \quad \text{or} \quad \frac{S_{ji}}{S_\alpha} = \frac{g_j}{g_{\text{tot}}} \frac{A_{ji}}{A_\alpha}$$

$$S_\alpha = \sum_{j=2}^{5} \sum_{i=0}^{1} S_{ji} \rightarrow \text{total line strength}$$

$$g_{\text{tot}} = \sum_{j=2}^{5} g_j = 12 \rightarrow \text{total statistical weight of n} = 2 \text{ level}$$

$$A_\alpha = \text{A coefficient for Ly}\alpha \text{ transition from n} = 2 \text{ to n} = 1$$

- From the above relations,

$$\frac{A_{20}}{A_\alpha} = \frac{A_{50}}{A_\alpha} = 0$$

$$\frac{A_{21}}{A_\alpha} = \frac{A_{51}}{A_\alpha} = 1$$

$$\frac{A_{30}}{A_\alpha} = \frac{A_{41}}{A_\alpha} = \frac{1}{3}$$

$$\frac{A_{31}}{A_\alpha} = \frac{A_{40}}{A_\alpha} = \frac{2}{3}$$

Ref: Condon & Shortley, The Theory of Atomic Spectra, Chap. IX, p236

Equation for spin temperature (a)

- In stationary state, rate equation for the population of the hyperfine states 0 and 1 can be written

$$\frac{dn_1}{dt} = -\frac{dn_0}{dt} = n_0\left(P_{01}^R + P_{01}^c + P_{01}^\alpha\right) - n_1\left(P_{10}^R + P_{10}^c + P_{10}^\alpha\right) = 0 \qquad \text{Eq (1)}$$

where $P^C, P^R, P^\alpha = \text{transition rates (per sec) caused by collisions, radio (21 cm), and Ly}\alpha$

(1) Definition of spin temperature (T_S):

$$\frac{n_1}{n_0} = \frac{g_1}{g_0}\exp\left(-\frac{h\nu_{10}}{k_\mathrm{B}T_S}\right) \longleftarrow \quad T_* = \frac{h\nu_{10}}{k_B} = 0.0681\ ^\circ\mathrm{K}$$

$$\simeq 3\left(1 - \frac{T_*}{T_S}\right) \qquad \text{Eq (2)}$$

Equation for spin temperature (b)

(2) Definition of brightness temperature of radio (21 cm) radiation (T_R) :

$$I_\nu = \frac{2k_\mathrm{B}T_R}{\lambda_{10}^2}$$

Using this definition, we obtain the radiative transition rates due to 21 cm radiation :

$$P_{01}^R \equiv I_\nu B_{01} = I_\nu \frac{g_1}{g_0}\frac{c^2}{2h\nu_{10}^3}A_{10} = 3\frac{T_R}{T_*}A_{10} \qquad \text{Eq (3)}$$

$$P_{10}^R \equiv A_{10} + I_\nu B_{10} = A_{10} + I_\nu \frac{c^2}{2h\nu_{10}^3}A_{10} = \left(1+\frac{T_R}{T_*}\right)A_{10} \qquad \text{Eq (4)}$$

(3) From the detailed balance between the collisional excitation and de-excitation, we obtain

$$\frac{P_{01}^c}{P_{10}^c} = \frac{g_1}{g_0}\exp\left(-\frac{h\nu_{10}}{k_\mathrm{B}T_K}\right) \simeq 3\left(1-\frac{T_*}{T_K}\right) \qquad \text{Eq (5)}$$

(T_K is the gas kinetic temperature.)

Equation for spin temperature (c)

(4) Indirect excitation and de-excitation rates due to Lyman-alpha photon are given by

$$\begin{aligned}
P_{01}^{\alpha} &= \sum_{j=2}^{5} B_{0j} J(\nu_{0j}) \frac{A_{j1}}{\sum_i A_{ji}} \\
&= \sum_{j=2}^{5} \frac{c^2}{2h\nu_{0j}^3} \frac{g_j A_{j0}}{g_0} J(\nu_{0j}) \frac{A_{j1}}{\sum_i A_{ji}} \\
&= \frac{c^2}{2h\nu_{0\alpha}^3} J(\nu_{0\alpha}) \left(\frac{g_3}{g_0} A_{30} \frac{A_{31}}{A_{30}+A_{31}} + \frac{g_4}{g_0} A_{40} \frac{A_{41}}{A_{40}+A_{41}} \right) \qquad \leftarrow \quad \text{if } \nu_{0j} = \nu_{0\alpha} \\
&= \frac{4}{3} \frac{c^2}{2h\nu_{0\alpha}^3} A_{\alpha} J(\nu_{0\alpha})
\end{aligned}$$

$$\begin{aligned}
P_{10}^{\alpha} &= \sum_{j=2}^{5} B_{1j} J(\nu_{1j}) \frac{A_{j0}}{\sum_i A_{ji}} \\
&= \sum_{j=2}^{5} \frac{c^2}{2h\nu_{1j}^3} \frac{g_j A_{j1}}{g_1} J(\nu_{1j}) \frac{A_{j0}}{\sum_i A_{ji}} \\
&= \frac{c^2}{2h\nu_{1\alpha}^3} J(\nu_{1\alpha}) \left(\frac{g_3}{g_1} A_{31} \frac{A_{30}}{A_{30}+A_{31}} + \frac{g_4}{g_1} A_{41} \frac{A_{40}}{A_{40}+A_{41}} \right) \qquad \leftarrow \quad \text{if } \nu_{1j} = \nu_{1\alpha} \\
&= \frac{4}{9} \frac{c^2}{2h\nu_{1\alpha}^3} A_{\alpha} J(\nu_{1\alpha})
\end{aligned}$$

Equation for spin temperature (d)

$$
\begin{aligned}
\therefore \frac{P_{01}^{\alpha}}{P_{10}^{\alpha}} &= 3\left(\frac{\nu_{1\alpha}}{\nu_{0\alpha}}\right)^{3} \frac{J(\nu_{0\alpha})}{J(\nu_{1\alpha})} \\
&\approx 3\frac{J(\nu_{0\alpha})}{J(\nu_{1\alpha})}
\end{aligned}
$$

Assume that the line profile is approximated by an exponential function:

$$
\text{if } J(\nu) \propto \exp\left(-\frac{h(\nu-\nu_{\alpha})}{k_B T_{\alpha}}\right) \text{ at the line center, then}
$$

$$
\begin{aligned}
\frac{P_{01}^{\alpha}}{P_{10}^{\alpha}} &= 3\exp\left(-\frac{h(\nu_{0\alpha}-\nu_{1\alpha})}{k_B T_{\alpha}}\right) \\
&= 3\exp\left(-\frac{h\nu_{10}}{k_B T_{\alpha}}\right) \\
&\approx 3\left(1-\frac{h\nu_{10}}{k_B T_{\alpha}}\right) \\
&= 3\left(1-\frac{T_{*}}{T_{\alpha}}\right) \quad \text{Eq (6)}
\end{aligned}
$$

Equation for spin temperature (e)

Combining Eq (1) - (6), we obtain the following equation for the spin temperature in terms of the 21 cm brightness temperature, gas kinetic temperature, and Ly-alpha color temperature:

$$1 - \frac{T_*}{T_S} = \frac{\frac{T_R}{T_*}A_{10} + \left(1 - \frac{T_*}{T_K}\right)P_{10}^c + \left(1 - \frac{T_*}{T_\alpha}\right)P_{10}^\alpha}{\left(1 + \frac{T_R}{T_*}\right)A_{10} + P_{10}^c + P_{10}^\alpha}$$

$$= 1 - \frac{A_{10} + \frac{T_*}{T_K}P_{10}^c + \frac{T_*}{T_\alpha}P_{10}^\alpha}{\left(1 + \frac{T_R}{T_*}\right)A_{10} + P_{10}^c + P_{10}^\alpha}$$

$$\frac{T_*}{T_S} = \frac{A_{10} + \frac{T_*}{T_K}P_{10}^c + \frac{T_*}{T_\alpha}P_{10}^\alpha}{\left(1 + \frac{T_R}{T_*}\right)A_{10} + P_{10}^c + P_{10}^\alpha}$$

$$\frac{T_S}{T_*} = \frac{\left(1 + \frac{T_R}{T_*}\right)A_{10} + P_{10}^c + P_{10}^\alpha}{A_{10} + \frac{T_*}{T_K}P_{10}^c + \frac{T_*}{T_\alpha}P_{10}^\alpha}$$

$$T_S = \frac{T_* + T_R + T_*\frac{P_{10}^c}{A_{10}} + T_*\frac{P_{10}^\alpha}{A_{10}}}{1 + \frac{T_*}{T_K}\frac{P_{10}^c}{A_{10}} + \frac{T_*}{T_\alpha}\frac{P_{10}^\alpha}{A_{10}}}$$

$$\boxed{T_S = \frac{T_* + T_R + y_c T_K + y_\alpha T_\alpha}{1 + y_c + y_\alpha}}$$

where
$$y_c \equiv \frac{T_*}{T_K}\frac{P_{10}^c}{A_{10}}$$
$$y_\alpha \equiv \frac{T_*}{T_\alpha}\frac{P_{10}^\alpha}{A_{10}}$$

$$T_* = \frac{h\nu_{10}}{k_B} = 0.0681\ ^\circ\mathrm{K}$$ (This term is negligible in the above equation.)

Homework