Astrophysics

Lecture 04
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Radiation from Moving Charges

Note on the Dirac delta function.

$$\delta(\mathbf{r}-\mathbf{r}_0)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

$$\int f(x)\delta(x-x_0)dx=f(x_0) \quad \text{if } x_0 \text{ is not a function of } x.$$

$$\int f(x)\delta(g(x))dx=\int f(x)\delta(y)\frac{dy}{(dg/dx)} \qquad \leftarrow \qquad \begin{aligned} y &\equiv g(x') \\ dy &= (dg/dx')dx' \\ dx' &= \frac{dy}{(dg/dx')} \end{aligned}$$

$$=\sum_{x_j}\frac{f(x_j)}{dg/dx|_{x_j}}$$

where x_j are roots of the equation $y=g(x)=0$

A single moving charge: Potentials

- Recall the retarded potentials:

$$\phi(\mathbf{r},t) = \int d^3\mathbf{r}' \int dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$
$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}' \int dt' \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

- Consider a particle of charge q that moves along a trajectory $\mathbf{r} = \mathbf{r}_0(t)$. Its velocity is then $\mathbf{u}(t) = \dot{\mathbf{r}}_0(t)$. The charge and current densities are given by

$$\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{r}_0(t)), \qquad \mathbf{j}(\mathbf{r},t) = q\mathbf{u}(t)\delta(\mathbf{r}-\mathbf{r}_0(t))$$

The δ-function has the property of localizing the charge and current. Let us calculate the retarded potentials due to this charge and current density. Using the property of the δ -function, the potentials become

$$\phi(\mathbf{r},t) = q\int \frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|}\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)$$
$$\mathbf{A}(\mathbf{r},t) = \frac{q}{c}\int \frac{dt'\,\mathbf{u}(t')}{|\mathbf{r}-\mathbf{r}_0(t')|}\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)$$

This is now an integral over the single variable t'.

- We now introduce the notations:

$$\mathbf{R}(t') \equiv \mathbf{r} - \mathbf{r}_0(t') \quad \rightarrow \quad R(t') = |\mathbf{r} - \mathbf{r}_0(t')|$$

We then have

$$\phi(\mathbf{r}, t) = q \int \frac{dt'}{R(t')} \delta(t' - t + R(t')/c)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{c} \int \frac{\mathbf{u}(t')dt'}{R(t')} \delta(t' - t + R(t')/c)$$

These equation can be simplified still further. Let us change variables:

(1) $$t'' = t' - t + R(t')/c \quad \rightarrow \quad dt'' = \left[1 + \frac{1}{c}\dot{R}(t')\right] dt'$$ (Here, *t* is a constant.)

(2)
$$R^2(t') = \mathbf{R}(t') \cdot \mathbf{R}(t')$$
$$2R(t')\dot{R}(t') = -2\mathbf{R}(t') \cdot \mathbf{u}(t') \qquad \leftarrow \dot{\mathbf{R}}(t') = -\mathbf{u}(t')$$
$$\dot{R}(t') = -\frac{\mathbf{R}(t')}{R(t')} \cdot \mathbf{u}(t')$$

(3)
$$\dot{R}(t') = -\mathbf{n}(t') \cdot \mathbf{u}(t')$$
$$dt'' = [1 - \mathbf{n}(t') \cdot \boldsymbol{\beta}(t')]\, dt'$$
where $\mathbf{n}(t') \equiv \dfrac{\mathbf{R}(t')}{R(t')}$ and $\boldsymbol{\beta} \equiv \dfrac{\mathbf{u}}{c}$

(4) $$dt'' = \kappa(t')dt'$$ where $\kappa(t') = 1 - \mathbf{n}(t') \cdot \boldsymbol{\beta}(t')$

A single moving charge: The Lienard-Wiechart Potential

Finally, we obtain

$$\phi(\mathbf{r},t) = q\int \frac{dt''}{\kappa(t')R(t')}\delta(t'')$$

$$\mathbf{A}(\mathbf{r},t) = q\int \frac{dt''\boldsymbol{\beta}(t')}{\kappa(t')R(t')}\delta(t'')$$

Now the integration over the δ-function can be performed by setting $t'' = 0$ or $t' = t_{\text{ret}} \equiv t - R(t_{\text{ret}})/c$.

$$\boxed{\begin{aligned}\phi(\mathbf{r},t) &= \frac{q}{\kappa(t_{\text{ret}})R(t_{\text{ret}})}\\ \mathbf{A}(\mathbf{r},t) &= \frac{q\boldsymbol{\beta}(t_{\text{ret}})}{c\kappa(t_{\text{ret}})R(t_{\text{ret}})}\end{aligned}}$$

$Li\acute{e}nard - Wiechart\ potentials$

(리에나르-비헤르트)

(French-German)

These potentials are called the Lienard-Wiechart potentials.

$$\boxed{\begin{aligned}\phi(\mathbf{r},t) &= \frac{q}{\kappa(t_{\mathrm{ret}})R(t_{\mathrm{ret}})} \\ \mathbf{A}(\mathbf{r},t) &= \frac{q\boldsymbol{\beta}(t_{\mathrm{ret}})}{c\kappa(t_{\mathrm{ret}})R(t_{\mathrm{ret}})}\end{aligned}}$$

$Li\acute{e}nard - Wiechart\ potentials$
(리에나르-비헤르트)
(French-German)

These potentials differ from those of static electromagnetic theory in two ways:

- **Beaming effect**:
  - First, there is the factor $\kappa(t_{\mathrm{ret}}) = 1 - \mathbf{n}(t_{\mathrm{ret}}) \cdot \boldsymbol{\beta}(t_{\mathrm{ret}})$.
  - This factor becomes very important at velocities close to the speed of light, where it tends to concentrate the potentials into a narrow cone about the particle velocity. It is related to the beaming effect found in the Lorentz transformation of photon direction or propagation.
- **Retardation makes it possible for a particle to radiate**:
  - The second difference is that the quantities are all to be evaluated at the retarded time t_{ret}. The major consequence of retardation is that it makes it possible for a particle to radiate.
  - The potentials roughly decrease as $1/r$ so that differentiation to find the fields would give a $1/r^2$ decrease if this differentiation acted solely on the $1/r$ factor.
  - In addition to this, the implicit dependence of the retarded time on on position gives $1/r$ behavior in the fields. We will see that this allows radiation energy to flow to infinite distances.

A single moving charge: Electromagnetic Fields

- The differentiation of the potentials gives the electromagnetic field. The calculation is straightforward but lengthy (see 14.1 of Classical Electrodynamics, Jackson).
- Note that **E** and **B** are always perpendicular, and $|\mathbf{E}| = |\mathbf{B}|$. However, **E** is not in general perpendicular to **n**.

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

$\longrightarrow$

velocity field | acceleration field

$$\mathbf{E}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]_{\mathrm{ret}} + \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\right]_{\mathrm{ret}}$$
$$\mathbf{B}(\mathbf{r},t) = [\mathbf{n}\times\mathbf{E}(\mathbf{r},t)]_{\mathrm{ret}}$$

- The electric field appears as composed of two terms:

  (1) The first, the **velocity field**, falls off as $1/R^2$ and is just the generalization of the Coulomb law to moving particles.

  - ✦ For $u \ll c$ this becomes precisely Coulomb's law.
  - ✦ When the particle moves with constant velocity it is only this term that contributes to the fields.

  (2) The second term, the **acceleration field**, falls off as $1/R$, is proportional to the particle's acceleration and is perpendicular to **n**.

  - ✦ This electric field, together with the corresponding magnetic field, constitutes the radiation field:

Here, [ ] denotes the quantities calculated at the retarded position $\mathbf{r}(t_{\mathrm{ret}})$ and time t_{ret}.

where
$$\mathbf{u} \equiv \dot{\mathbf{r}}_0(t_{\mathrm{ret}})$$
$$\boldsymbol{\beta} \equiv \frac{\mathbf{u}(t_{\mathrm{ret}})}{c} = \frac{\dot{\mathbf{r}}_0(t_{\mathrm{ret}})}{c}$$
$$\dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{u}}(t_{\mathrm{ret}})}{c} = \frac{\ddot{\mathbf{r}}_0(t_{\mathrm{ret}})}{c}$$
$$\mathbf{R} \equiv \mathbf{r} - \mathbf{r}_0(t_{\mathrm{ret}})$$
$$\mathbf{n} \equiv \frac{\mathbf{R}}{R} = \frac{\mathbf{r}-\mathbf{r}_0(t_{\mathrm{ret}})}{|\mathbf{r}-\mathbf{r}_0(t_{\mathrm{ret}})|}$$
$$\kappa \equiv 1 - \mathbf{n}\cdot\boldsymbol{\beta}$$

“Velocity” Field

- The first term depends only on position and velocity.

$$\mathbf{E}_{\mathrm{vel}}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]_{\mathrm{ret}} \qquad \mathbf{B}_{\mathrm{vel}}(\mathbf{r},t) = 0$$

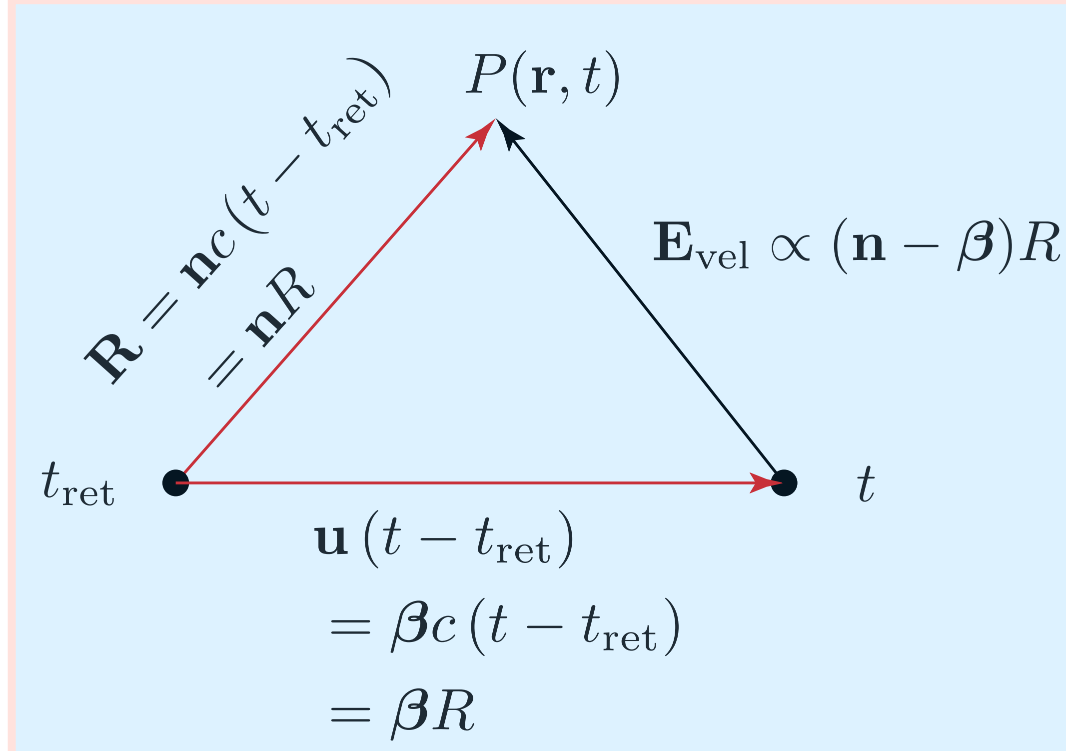
- A remarkable fact is that **the “velocity” electric field always points along the line toward the “current” position of the particle.**

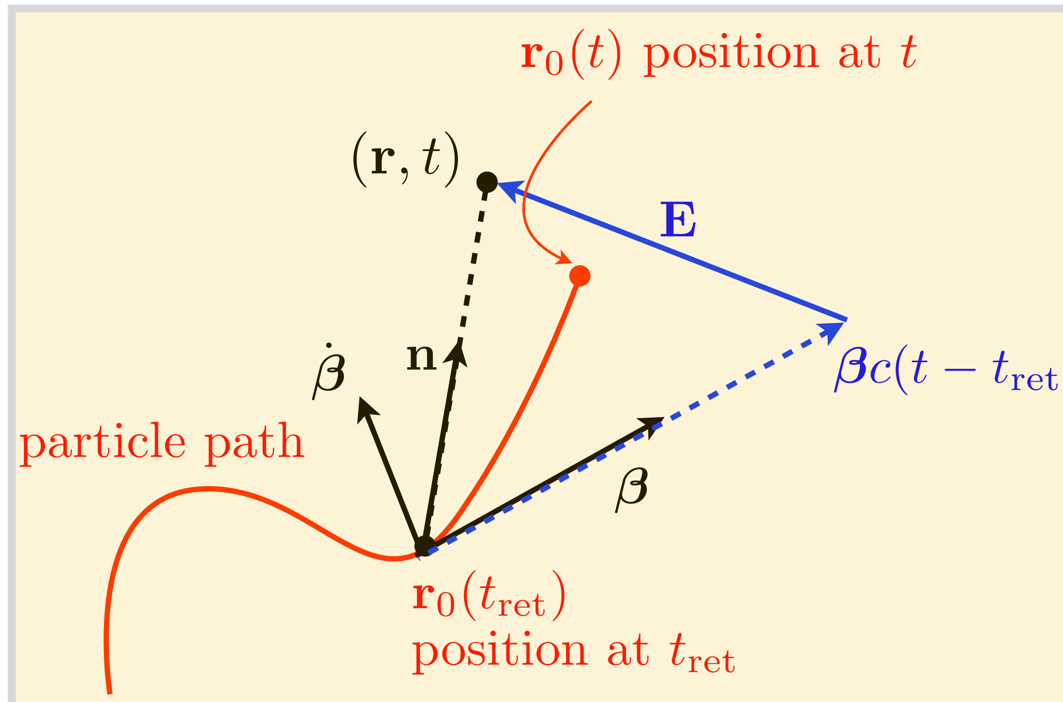
  The displacement of the photon from the retarded point $\mathbf{r}_0(t_{\mathrm{ret}})$ (point at t_{ret}) to the field point $\mathbf{r}$ during the light travel time = $\mathbf{n}c(t - t_{\mathrm{ret}})$.

  In the same time, the particle undergoes a displacement $\boldsymbol{\beta}c(t - t_{\mathrm{ret}})$.

  The displacement between the field point and the current position of the particle is given by $(\mathbf{n} - \boldsymbol{\beta})c(t - t_{\mathrm{ret}})$ which is the direction of the velocity field.

Note that, if the velocity is not a constant, the ***true displacement*** *of the particle* $\neq \boldsymbol{\beta}c(t - t_{\mathrm{ret}})$.





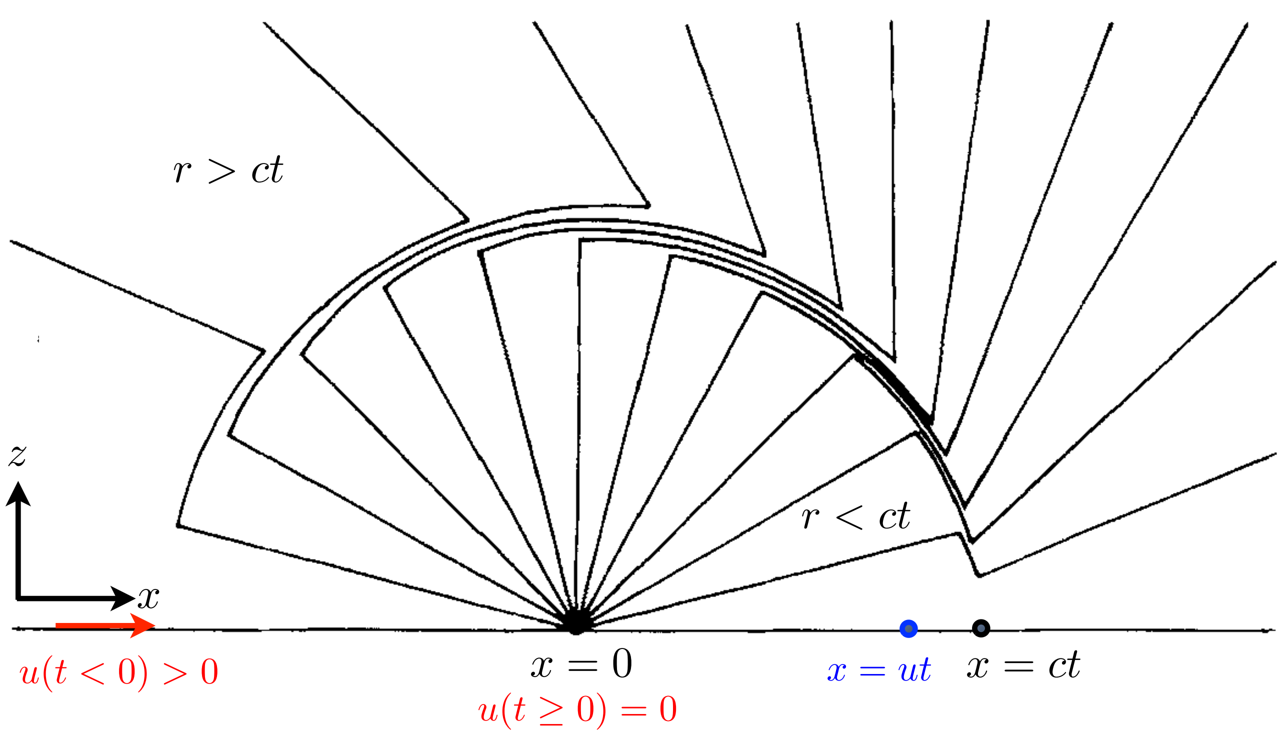
Geometry for calculation of the radiation field at a point $(\mathbf{r}, t)$ in spacetime.

“Acceleration” (or “radiation”) Field

- The second term (a) falls off as $1/R$, (b) is proportional to the particle’s acceleration, and (c) is perpendicular to $\mathbf{n}$.

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) = \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\right]_{\rm ret}$$
$$\mathbf{B}_{\rm rad}(\mathbf{r},t) = [\mathbf{n}\times\mathbf{E}_{\rm rad}]_{\rm ret}$$

- How an acceleration can give rise to a transverse field that decreases as $1/R$: Consider a particle, which originally moved with a constant velocity along the x-axis and stopped at $x = 0$ at time $t = 0$. At time t (> 0), the field outside radius ct is radial and points to the position ($x = ut$) where the particle would have been if there had been no deceleration, since no information of the deceleration has yet propagated. On the other hand, the field inside radius ct is “informed” and is radially directed to the true position ($x = 0$) of the particle.

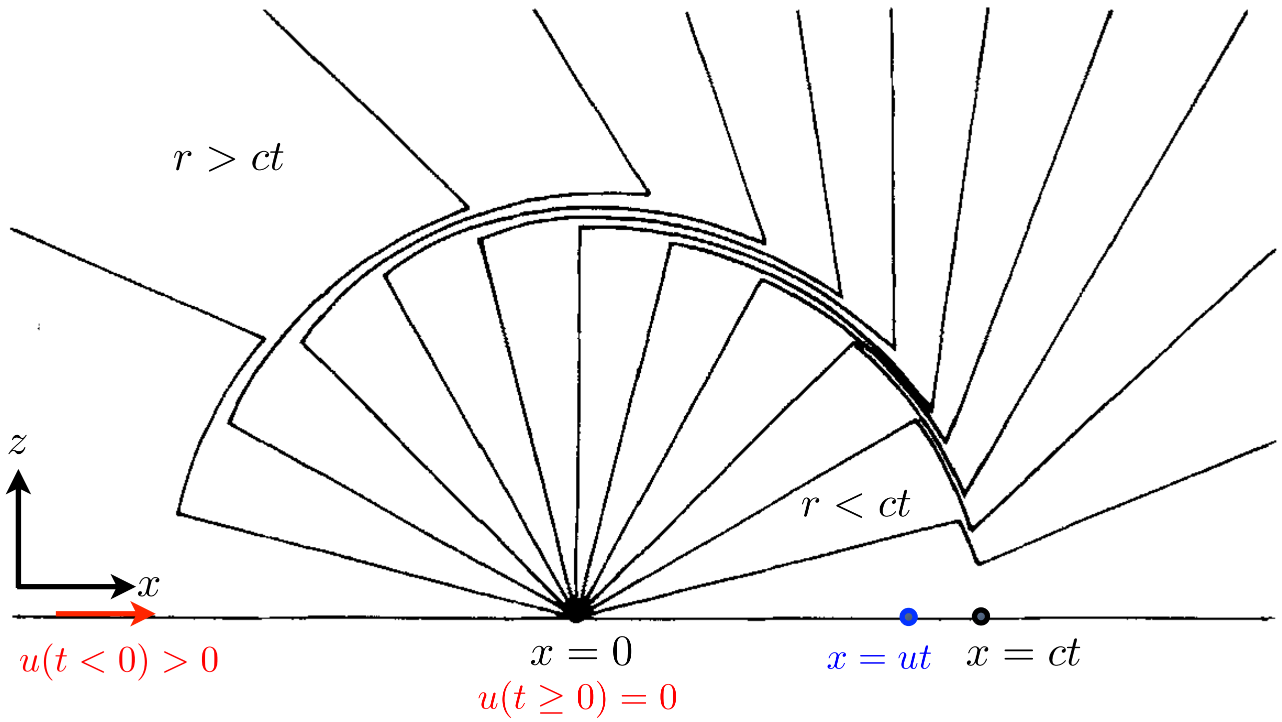


The fields at $x > ct$ were made when $t < 0$, while those at $x < ct$ were made when $t > 0$.

These two fields must be connected to be consistent with Gauss's law and flux conservation.

- The transition zone between them will propagate outward.
- The electric field in the transition (shell) zone is transverse.
- The radial thickness of the shell would be the light travel distance during the time interval over which the deceleration occurs, and thus is constant.
- However, the radius of the shell (or ring) increases as *R*.
- Since the total number of flux lines (in *xy*-plane) must be conserved, the strength of the field varies as 1/R.

$$\boxed{E(\delta x)(2\pi R) = \text{constant} \ \rightarrow \ E \propto \frac{1}{R}}$$



A single moving charge: Radiation Power*

- Power per unit frequency per unit solid angle of the radiation field of a single particle

Recall:

$$\frac{dW}{dAd\omega} = c\left|E(\hat{\omega})\right|^2$$

$$d\Omega = \frac{dA}{R^2}$$

$$\begin{aligned}\frac{dW}{d\omega d\Omega} &= \frac{R^2 dW}{d\omega dA} = R^2 c\left|\bar{E}(\omega)\right|^2 \\ &= \frac{c}{4\pi^2}\left|\int [R\mathbf{E}(t)]_{\mathrm{ret}}\, e^{i\omega t} dt\right|^2 \\ &= \frac{q^2}{4\pi^2 c}\left|\int \left[\mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\kappa^{-3}\right]_{\mathrm{ret}} e^{i\omega t} dt\right|^2\end{aligned}$$

Note: the expression in the brackets is evaluated at the retarded time $t' = t - R(t')/c$.

Now, changing variables from t to $t' = t - R(t')/c$ in the integral.

$$\begin{aligned}dt &= dt' + \frac{1}{c}\frac{dR(t')}{dt'}dt' = \kappa(t')dt' \\ &= (1 - \mathbf{n}(t')\cdot\boldsymbol{\beta}(t'))\, dt'\end{aligned}$$

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\kappa^{-2} e^{i\omega\left(t'+R(t')/c\right)} dt'\right|^2$$

We are only interested in the electric field measured at a far distance. Thus, we consider the canse where $\left|\mathbf{r}_0\right| \ll |\mathbf{r}| = r$.

(1) $R(t') = |\mathbf{r} - \mathbf{r}_0(t')| = [(\mathbf{r} - \mathbf{r_0}) \cdot (\mathbf{r} - \mathbf{r}_0)]^{1/2}$

$$= \left[r^2 - 2(\mathbf{r} \cdot \mathbf{r}_0) + r_0^2\right]^{1/2} = r\left[1 - \frac{2(\mathbf{r} \cdot \mathbf{r}_0)}{r^2} + \frac{r_0^2}{r^2}\right]^{1/2}$$

$$\approx r\left(1 - \frac{\mathbf{r} \cdot \mathbf{r}_0}{r^2}\right)$$

$$= r - \mathbf{n} \cdot \mathbf{r}_0$$

Here, we use the following relation. We also note that $\mathbf{n}$ is now independent of t' in our approximation.

$$\mathbf{n} \equiv \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|} \approx \frac{\mathbf{r}}{r}$$

(2) $\kappa(t') = 1 - \mathbf{n}(t') \cdot \boldsymbol{\beta}(t') \approx 1 - \mathbf{n} \cdot \boldsymbol{\beta}(t')$. Here, again $\mathbf{n}$ is independent of t'.

(3) We note that $e^{i\omega\left(t' + R(t')/c\right)} = e^{i\omega r/c} e^{i\omega\left(t' - \mathbf{n}\cdot\mathbf{r}_0(t')/c\right)}$ and $\left|e^{i\omega r/c}\right| = 1$

Then we obtain $$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \mathbf{n} \times \left\{(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right\} \kappa^{-2} e^{i\omega\left(t' - \mathbf{n}\cdot\mathbf{r}_0(t')/c\right)} dt'\right|^2$$

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\kappa^{-2}e^{i\omega\left(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c\right)}dt'\right|^2$$

We can integrate the above equation by parts to obtain an expression without $\dot{\boldsymbol{\beta}}$. We first note the following relation (which is proved in the next slide).

$$\kappa^{-2}\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\} = \frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right]$$

With the rule of integration by parts $\int f'g\,dt = fg - \int fg'\,dt$, we obtain

This term vanishes under the assumption of a finite wave train.

$$\begin{aligned}
&\int \mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-2}e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}dt' \\
&= \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\kappa^{-1}e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}\Big|_{-\infty}^{\infty} - \int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\kappa^{-1}\{i\omega(1-\mathbf{n}\cdot\dot{\mathbf{r}}_0(t')/c)\}e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}dt' \\
&= -i\omega\int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\kappa^{-1}(1-\mathbf{n}\cdot\boldsymbol{\beta})e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}dt' \\
&= -i\omega\int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}dt' \qquad \leftarrow \quad \kappa = 1-\mathbf{n}\cdot\boldsymbol{\beta}
\end{aligned}$$

$$\boxed{\frac{dW}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\exp\left[i\omega\left(t'-\frac{\mathbf{n}\cdot\mathbf{r}_0(t')}{c}\right)\right]dt'\right|^2}$$

This formula will be used later.

- Proof of the relation:

$$\kappa^{-2}\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}=\frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right]$$

note the vector identity: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{C}\cdot\mathbf{A})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$$\frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right]=\kappa^{-2}\left[-\frac{d\kappa}{dt'}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})+\kappa\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right]$$

$$\text{Here, use the relations}: \kappa=1-\mathbf{n}\cdot\boldsymbol{\beta},\quad \frac{d\kappa}{dt'}=-\mathbf{n}\cdot\dot{\boldsymbol{\beta}}$$

$$\begin{aligned}
\frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right]&=\kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\}+(1-\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\}\right]\\
&=\kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\}+\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})-(\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\}\right]\\
&=\kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta})-\boldsymbol{\beta}\}+\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})-(\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})-\dot{\boldsymbol{\beta}}\}\right]\\
&=\kappa^{-2}\left[-(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta}+(\mathbf{n}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}}+\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right]\\
&=\kappa^{-2}\left[-\mathbf{n}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})+\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right]\\
&=\kappa^{-2}\left[\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]
\end{aligned}$$

Radiation from Nonrelativistic Particles

- Using the above formulae we can discuss many radiation processes. The previous formulae is fully relativistic. However, for the moment, we will discuss nonrelativistic particles:

$$\beta = \frac{u}{c} \ll 1$$

- Order of magnitude comparison of the two fields:

$$E_{\rm rad} \approx \frac{q}{c}\frac{\dot{\beta}}{\kappa^3 R}, \quad E_{\rm vel} \approx \frac{q}{\kappa^3 R^2} \quad \rightarrow \quad \frac{E_{\rm rad}}{E_{\rm vel}} \approx \frac{R\dot{u}}{c^2}$$

If we focus on a particular Fourier component of frequency ν or the particle has a characteristic frequency of oscillation $\nu \sim 1/T$, then $\dot{u} = u\nu$, and the above equation becomes:

$$\frac{E_{\rm rad}}{E_{\rm vel}} \sim \frac{Ru\nu}{c^2} = \frac{u}{c}\frac{R}{\lambda}$$

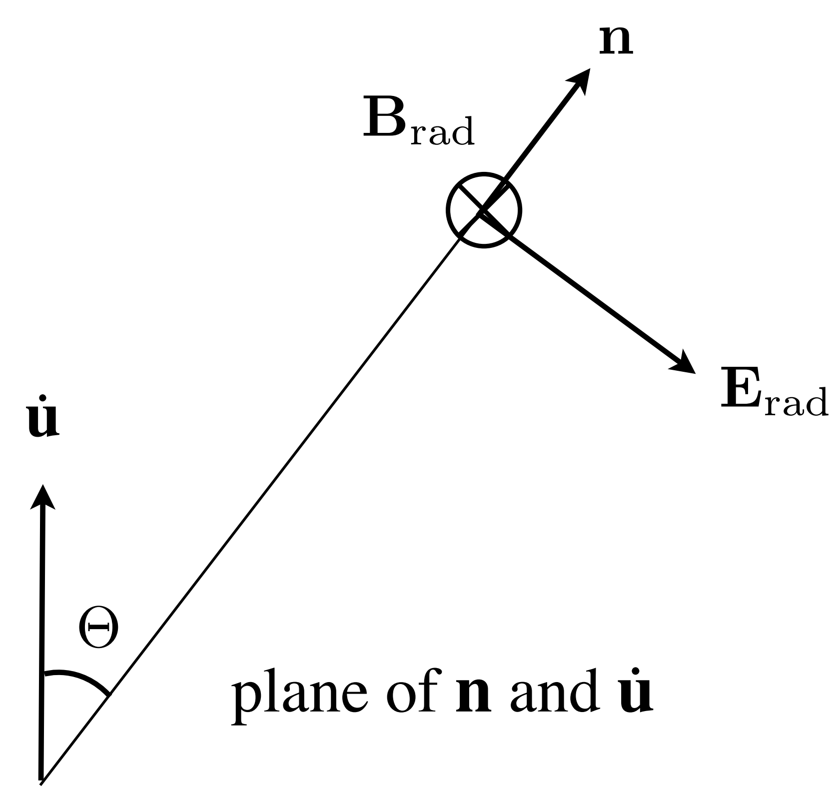
For field points inside the "near zone", $R \lesssim \lambda$, the velocity field is stronger than the radiation field by a factor $c/u = 1/\beta$.

For field points sufficiently far in the "far zone", $R \gg \lambda(c/u) = \lambda/\beta$, the radiation field dominates and increases its domination linearly with R. In astronomy, we are only interested in the "far zone". Therefore, let's consider only the radiation field.

Larmor's Formula

- When $\beta \ll 1$, the EM fields can be simplified to

$$\mathbf{E}_{\mathrm{rad}} = \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R} \times \left\{(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right\}\right]_{\mathrm{ret}}$$
$$\approx \left[\frac{q}{Rc^2}\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}})\right]_{\mathrm{ret}}$$
$$\mathbf{B}_{\mathrm{rad}} = [\mathbf{n} \times \mathbf{E}_{\mathrm{rad}}]_{\mathrm{ret}}$$



The $\mathrm{E}_{\mathrm{rad}}$ field is in the plane of ($\mathbf{n}$, $\dot{\mathbf{u}}$).

- As shown in the figure, $\mathbf{E}_{\mathrm{rad}}$ is in the plane defined by $\mathbf{n}$ and $\dot{\mathbf{u}}$.

  Note that $\mathbf{n}$ and $\mathbf{n} \times \dot{\mathbf{u}}$ are perpendicular and $\left|\mathbf{n} \times \dot{\mathbf{u}}\right| = \left|\dot{u}\right| \sin\Theta$, where Θ is the angle between $\mathbf{n}$ and $\dot{\mathbf{u}}$.

  Therefore, the magnitudes of $\mathbf{E}_{\mathrm{rad}}$ and $\mathbf{B}_{\mathrm{rad}}$ are

$$\therefore \ |\mathbf{E}_{\mathrm{rad}}| = |\mathbf{B}_{\mathrm{rad}}| = \frac{q\dot{u}}{Rc^2}\sin\Theta$$

Also, note that

$$\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}) = \mathbf{n}(\mathbf{n} \cdot \dot{\mathbf{u}}) - \dot{\mathbf{u}}$$
$$\{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}})\}^2 = (\mathbf{n} \cdot \dot{\mathbf{u}})^2 + (\dot{\mathbf{u}})^2 - 2(\mathbf{n} \cdot \dot{\mathbf{u}})^2$$
$$= \dot{u}^2 \cos^2\Theta + \dot{u}^2 - 2\dot{u}^2 \cos^2\Theta$$
$$= \dot{u}^2(1 - \cos^2\Theta)$$
$$= \dot{u}^2 \sin^2\Theta$$

- The Poynting vector is in direction of **n** and has a magnitude.

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}E_{\rm rad}^2\mathbf{n}$$

$$S = \frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^4}\sin^2\Theta \equiv \frac{dW}{dtdA} \quad (\text{erg s}^{-1}\ \text{cm}^{-2})$$

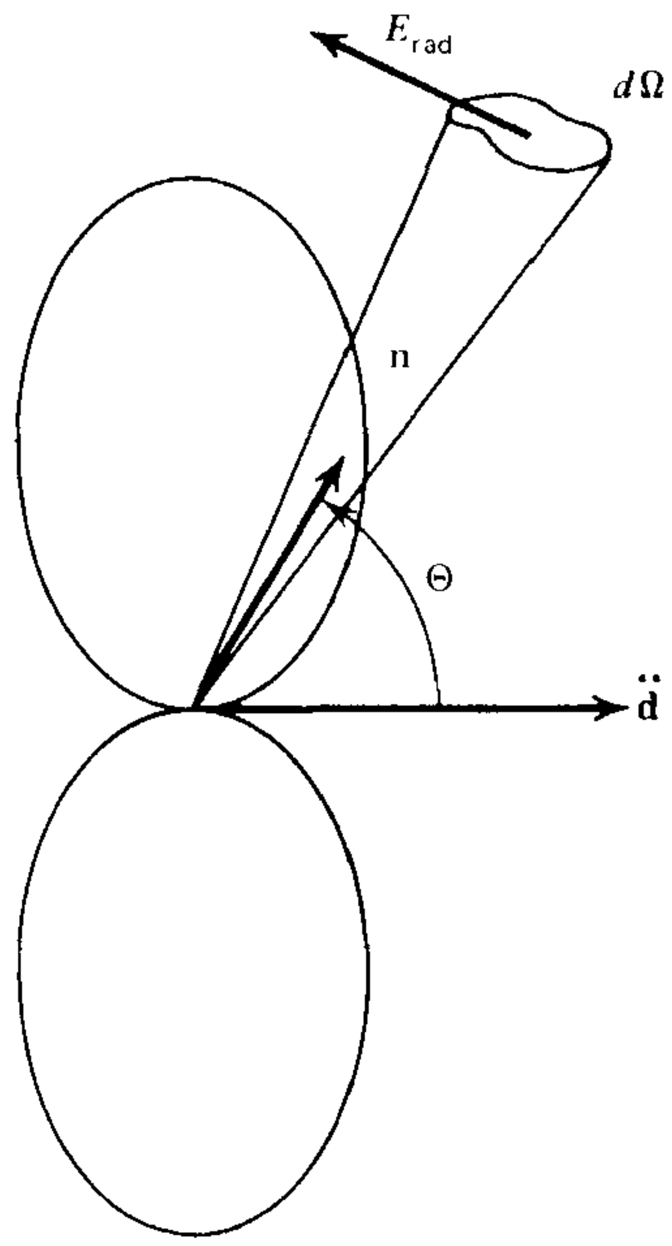
This is an outward flow of energy (per unit time and per unit area), along the direction **n**.

- Radiation patter: The energy emitted per unit time into solid angle $d\Omega$ about **n** can be obtained by multiplying the Poynting vector by R^2.

$$\frac{dW}{dtd\Omega} = R^2\frac{dW}{dtdA} = R^2 S$$

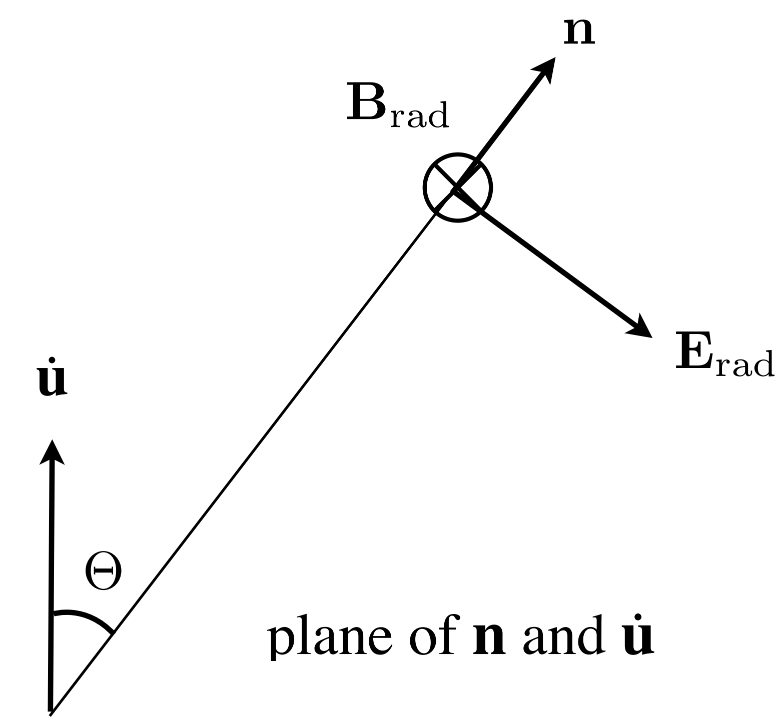
$$= \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta$$

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta}$$



- Total power emitted into all angles:

$$P = \frac{dW}{dt} = \int d\Omega \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2\Theta = \frac{q^2 \dot{u}^2}{2c^3} \int_{-1}^{1} (1-\mu^2) d\mu$$



$$\boxed{\frac{dP}{d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2\Theta} \qquad \boxed{P = \frac{2q^2 \dot{u}^2}{3c^3}}$$

This is the Larmor's Formula for emission from a single accelerated charge q.

The emission from a single accelerated charge has the following properties:

1. The Power emitted is proportional to the square of the charge and the square of the acceleration.
2. We have the characteristic dipole pattern $\sin^2\Theta$: no radiation is emitted along the direction of acceleration, and the maximum is emitted perpendicular to acceleration. (see the figure in the previous slide)
3. The instantaneous direction of $\mathbf{E}_{\rm rad}$ is determined by $\dot{\mathbf{u}}$ and $\mathbf{n}$. If the particle accelerates along a line, the radiation will be 100% linearly polarized in the plane of $\dot{\mathbf{u}}$ and $\mathbf{n}$.

Dipole Approximation (the radiation from many particles)

- Consider many particles with positions $\mathbf{r}_i$, velocities $\mathbf{u}_i$, and charges q_i $(i = 1,2,3,\cdots,N)$. The radiation field at large distances can be found by adding together the $\mathbf{E}_{\rm rad}$ from each particle.
- However, the radiation field equations refer to conditions at retarded time, and the retarded times will differ for each particle. Therefore, we must keep track of the phase relations between the particles.

  There are situations in which it is possible to ignore this difficulty:

  Let L = typical size of the system
  τ = typical time scale for variations within the system

  If $\tau \gg L/c$ (light-travel-time), the differences in retarded time across the source are negligible.

  Note that τ can represent the time scale over which significant changes in the radiation field, and this in turn determines typical characteristic frequency of the emitted radiation. ***This condition is equivalent to the condition for the characteristic frequency (or characteristic wavelength) :***

$$\nu \approx \frac{1}{\tau} \ll \frac{c}{L} \quad \text{or} \quad \lambda = \frac{c}{\nu} \gg L$$

  In other words, ***the differences in retarded times can be ignored when the system size is much smaller than the characteristic wavelength.***

- We may also characterize τ as the time a particle takes to change its motion substantially.

  Let ℓ be a characteristic scale of the particle's orbit and u be a typical velocity, then $\tau \sim \ell/u$. The above condition $\tau \gg L/c$ then imply $u/c = \ell/(\tau c) \ll \ell/L$

  But since $\ell < L$, **the condition for dipole approximation is simply equivalent to the nonrelativistic condition:**

$$u \ll c$$

  With the above conditions met we can use the nonrelativistic form of the radiation fields:

$$\mathbf{E}_{\mathrm{rad}} = \sum_i \frac{q_i}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}_i)}{R_i}$$

- Let R_0 be the distance from some point in the system to the field point. Then, $R_i = R_0 + \ell_i \approx R_0$ as $R_0 \gg \ell_i$. Finally, we have

$$\mathbf{E}_{\mathrm{rad}} \approx \frac{1}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \sum_i q_i \dot{\mathbf{u}}_i)}{R_0} \quad \rightarrow \quad \mathbf{E}_{\mathrm{rad}} = \frac{\mathbf{n} \times \left(\mathbf{n} \times \ddot{\mathbf{d}}\right)}{c^2 R_0}$$

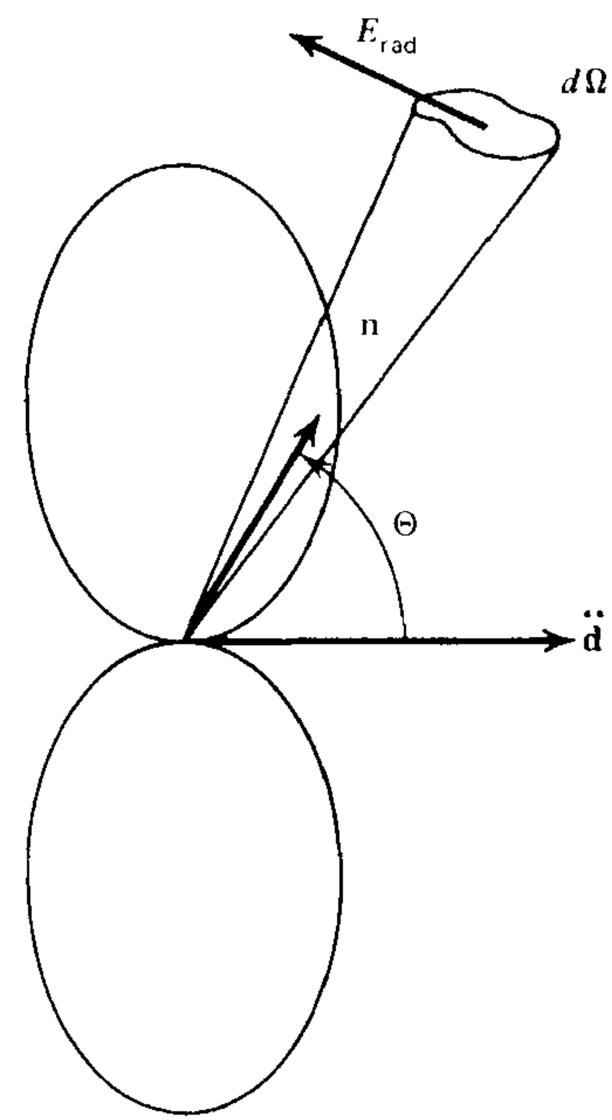
  where the electric dipole moment is defined as $\mathbf{d} = \sum_i q_i \mathbf{r}_i$

  Note that <u>the right-hand side of the above equations must still be evaluated at a retarded time</u>, but using any point within the region, say, the position used to define R_0.

- As before, for a single particle, we find the generalized formulas for the radiation patter and the total power, which are called the dipole approximation:

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2 \Theta, \qquad P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}$$

Note that the instantaneous polarization of **E** lines in the plane of $\ddot{\mathbf{d}}$ and **n**.



- **Spectrum of radiation in the dipole approximation**:

For simplicity we assume that **d** always lies in a single direction. Then, the magnitude of the electric field is given by

$$E(t) = \ddot{d}(t)\frac{\sin\Theta}{c^2 R_0}$$

where $d(t)$ is the magnitude of the dipole moment.

Fourier transform of $d(t)$ is defined as $d(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\bar{d}(\omega)d\omega$

Then, $\ddot{d}(t) = -\int_{-\infty}^{\infty} \omega^2 e^{-i\omega t}\bar{d}(\omega)d\omega$

$$\bar{E}(\omega) = -\frac{1}{c^2 R_0}\omega^2 \bar{d}(\omega)\sin\Theta$$

- The energy per unit solid angle per frequency range in the dipole approximation is given by

$$\begin{aligned} \frac{dW}{d\omega d\Omega} &= R_0^2 \frac{dW}{d\omega dA} \\ \frac{dW}{d\omega dA} &= c \left| \bar{E}(\omega) \right|^2 \end{aligned} \longrightarrow \frac{dW}{d\omega d\Omega} = \frac{\omega^4}{c^3} \left| \bar{d}(\omega) \right|^2 \sin^2 \Theta$$

The total energy per frequency range is

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} \left| \bar{d}(\omega) \right|^2$$

- The above formulas describe an interesting property of dipole radiation, namely, that the spectrum of the emitted radiation is related directly to the frequencies of oscillation of the dipole moment. However, this property is not true for particles with relativistic velocities.

- It is also worthwhile to note the dependence of $\omega^4 \propto \lambda^{-4}$ in the power spectrum.

A general Multipole Expansion*

- The above treatment was obtained only qualitatively. We would like to be more explicit.
- Recall that the vector potential is

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}' \int dt' \, \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

- Consider a Fourier analysis of the sources and fields:

$$\mathbf{j}_\omega(\mathbf{r}) = \int \mathbf{j}(\mathbf{r},t)e^{i\omega t}dt$$

$$\mathbf{A}_\omega(\mathbf{r}) = \int \mathbf{A}(\mathbf{r},t)t^{i\omega t}dt$$

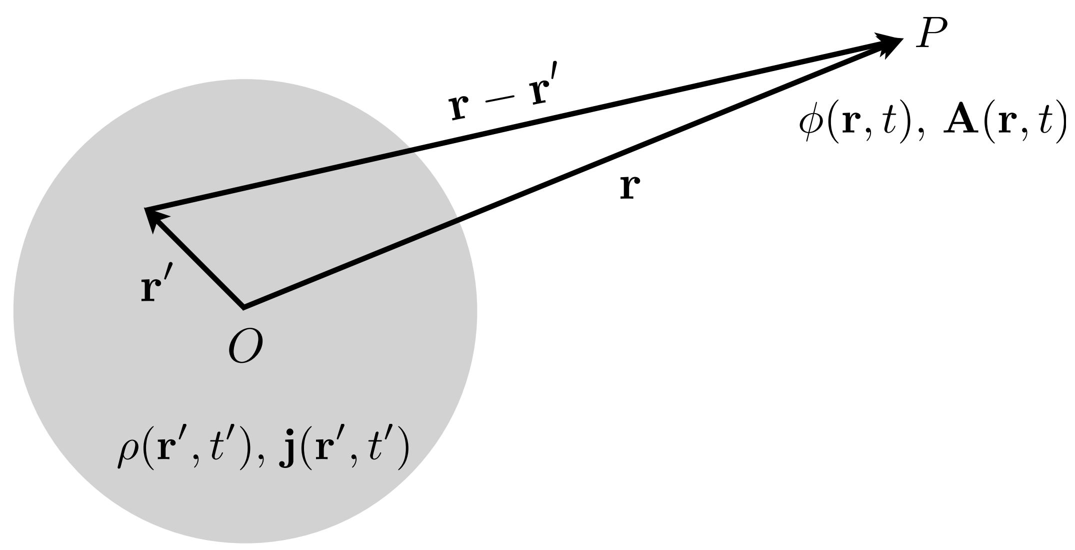
Then, the vector potential becomes

$$\begin{aligned}
\mathbf{A}_\omega(\mathbf{r}) &= \frac{1}{c}\int d^3\mathbf{r}' \int dt' \int dt \, \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}e^{i\omega t}\delta\left(t'-t+|\mathbf{r}-\mathbf{r}'|/c\right) \\
&= \frac{1}{c}\int d^3\mathbf{r}' \int dt' \, \frac{\mathbf{j}(\mathbf{r},t')}{|\mathbf{r}-\mathbf{r}'|}e^{i\omega t'}e^{i\omega|\mathbf{r}-\mathbf{r}'|/c} \\
&= \frac{1}{c}\int d^3\mathbf{r}' \, \frac{\mathbf{j}_\omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|} \qquad \leftarrow \quad k \equiv \omega/c
\end{aligned}$$

Note that this exponential term is caused by the retardation.

This equation now relate single Fourier components of source $\mathbf{j}$ and potential $\mathbf{A}$.

- Let's choose an origin of coordinates inside the source of size L. Then, at field points such that $r \gg L$, we will expand the potential in a power series of kr'.



$$
\begin{aligned}
|\mathbf{r}-\mathbf{r}'| &= \left[(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')\right]^{1/2} = \left[r^2 - 2\,(\mathbf{r}\cdot\mathbf{r}') + r'^2\right]^{1/2} \\
&= r\left[1 - \frac{2\,(\mathbf{r}\cdot\mathbf{r}')}{r^2} + \frac{r'^2}{r^2}\right]^{1/2} \\
&\approx r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \cdots\right) && \leftarrow\ (1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots \\
&= r - \mathbf{n}\cdot\mathbf{r}' + \cdots && \leftarrow\ \text{Here, } \mathbf{n} \equiv \frac{\mathbf{r}}{r} \quad (\mathbf{n}\text{ points toward the field point }\mathbf{r})
\end{aligned}
$$

Similarly,

$$
\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r} + \frac{\mathbf{n}\cdot\mathbf{r}'}{r^2} + \cdots
$$

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{1}{c}\int \frac{\mathbf{j}_\omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{ik|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$
$$\approx \frac{1}{c}\int \mathbf{j}_\omega(\mathbf{r}')\frac{1}{r}\left(1+\frac{\mathbf{n}\cdot\mathbf{r}'}{r}\right) e^{ikr}e^{-ik\mathbf{n}\cdot\mathbf{r}'}d^3\mathbf{r}'$$
$$= \frac{e^{ikr}}{cr}\left[\int \mathbf{j}_\omega(\mathbf{r}')e^{-ik\mathbf{n}\cdot\mathbf{r}'}d^3\mathbf{r}' + \frac{1}{r}\int(\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}_\omega(\mathbf{r}')e^{-ik\mathbf{n}\cdot\mathbf{r}'}d^3\mathbf{r}'\right]$$

(1) The factor $\exp(ikr)$ outside the integral expresses **the effect of retardation from the source as a whole**.

(2) The factor $\exp(-ik\mathbf{n}\cdot\mathbf{r}')$ inside the integral expresses **the relative retardation of each element** of the source.

In our slow motion approximation, $kL = 2\pi L/\lambda \ll 1$, the first and second integrals can be approximated, respectively, to be

$$\int \mathbf{j}_\omega(\mathbf{r}')e^{ik\mathbf{n}\cdot\mathbf{r}'}d^3\mathbf{r}' \approx \int \mathbf{j}_\omega(\mathbf{r}')\left[1-ik\mathbf{n}\cdot\mathbf{r}'+\cdots\right]d^3\mathbf{r}' \qquad \leftarrow \quad e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$
$$\int(\mathbf{n}\cdot\mathbf{r}')\mathbf{j}_\omega(\mathbf{r}')e^{ik\mathbf{n}\cdot\mathbf{r}'}d^3\mathbf{r}' \approx \int(\mathbf{n}\cdot\mathbf{r}')\mathbf{j}_\omega(\mathbf{r}')\left[1+\cdots\right]d^3\mathbf{r}'$$

Then, the vector potential becomes

$$\mathbf{A}_\omega(\mathbf{r}) \approx \frac{e^{ikr}}{cr}\left[\int \mathbf{j}_\omega(\mathbf{r}')d^3\mathbf{r}' + \left(\frac{1}{r}-ik\right)\int(\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}_\omega(\mathbf{r}')d^3\mathbf{r}' + \mathcal{O}\left((\mathbf{n}\cdot\mathbf{r}')^2\right)\right]$$

- The "electric" dipole approximation results from taking just the first term in the above equation:

$$\mathbf{A}_\omega(\mathbf{r})|_{\text{dipole}} \approx \frac{e^{ikr}}{cr}\int \mathbf{j}_\omega(\mathbf{r}')d^3\mathbf{r}'$$

- The second term give the "electric" quadrupole and "magnetic" dipole terms.

$$\mathbf{A}_\omega(\mathbf{r}) \approx \frac{e^{ikr}}{cr}\left(\frac{1}{r} - ik\right)\int (\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}_\omega(\mathbf{r}')d^3\mathbf{r}'$$

The term inside the integral can be expressed in terms of a symmetric and asymmetric terms for $\mathbf{r}'$ and $\mathbf{j}$.

$$\begin{aligned}(\mathbf{n}\cdot\mathbf{r}')\mathbf{j} &= \frac{1}{2}\left[(\mathbf{n}\cdot\mathbf{r})'\mathbf{j} + (\mathbf{n}\cdot\mathbf{j})\mathbf{r}'\right] + \frac{1}{2}\left[(\mathbf{n}\cdot\mathbf{r})'\mathbf{j} - (\mathbf{n}\cdot\mathbf{j})\mathbf{r}'\right] \\ &= \frac{1}{2}\left[(\mathbf{n}\cdot\mathbf{r})'\mathbf{j} + (\mathbf{n}\cdot\mathbf{j})\mathbf{r}'\right] + \frac{1}{2}(\mathbf{r}'\times\mathbf{j})\times\mathbf{n}\end{aligned}$$

The first and second terms correspond to the electric quadrupole and magnetic dipole terms, respectively.

- Lamor's formula is obtained by assuming $kr \gg 1$, or in other words, by taking the far zone approximation in addition to the dipole approximation.

Thomson Scattering ((free) Electron Scattering)

- Recall the dipole formula

$$\boxed{\frac{dP}{d\Omega} = \frac{dW}{dtd\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\sin^2\Theta, \quad P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}}$$

- Let us consider the **process in which a free charged particle (electron) radiates in response to an incident electromagnetic wave**.

  In non-relativistic case, we may neglect magnetic force.

  magnetic/electric force ratio in Lorentz force: $F_B/F_E \sim (v/c)B/E = v/c \ll 1$

  Consider a monochromatic wave with frequency ω_0 and linearly polarized in direction $\hat{\boldsymbol{\epsilon}}$:

$$\mathbf{E} = \hat{\boldsymbol{\epsilon}} E_0 \sin \omega_0 t$$

  Thus the force on a particle with the charge *e* is

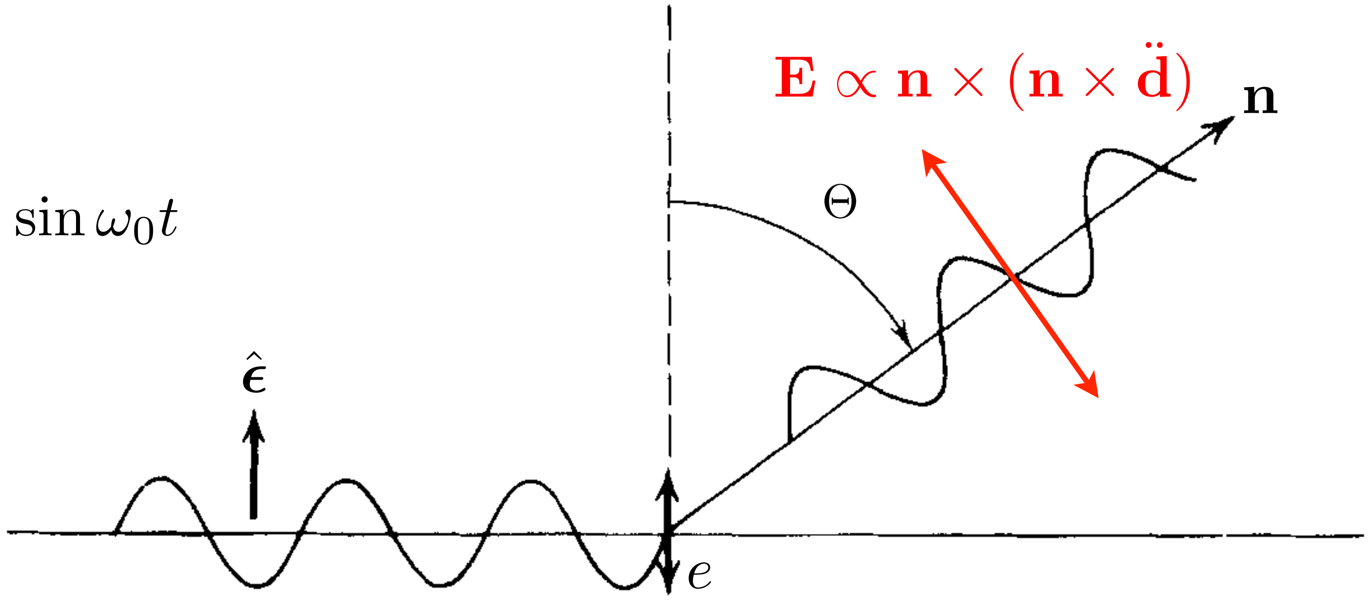
$$\mathbf{F} = e\mathbf{E} = \hat{\boldsymbol{\epsilon}} e E_0 \sin \omega_0 t$$

  the acceleration of the electron is

$$\ddot{\mathbf{r}} = \hat{\boldsymbol{\epsilon}} \frac{eE_0}{m} \sin \omega_0 t, \quad \ddot{\mathbf{d}} = e\ddot{\mathbf{r}} = \hat{\boldsymbol{\epsilon}} \frac{e^2 E_0}{m} \sin \omega_0 t$$

  the dipole moment is

$$\mathbf{d} = -\hat{\boldsymbol{\epsilon}} \left(\frac{e^2 E_0}{m\omega_0^2} \right) \sin \omega_0 t$$



- We obtain the time-averaged power per solid angle $\left(\langle \sin^2 \omega_0 t\rangle = 1/2\right)$:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{< \ddot{\mathbf{d}}^2 >}{4\pi c^3} \sin^2\Theta = \frac{e^4 E_0^2}{8\pi m^2 c^3} \sin^2\Theta, \quad \langle P\rangle = \frac{e^4 E_0^2}{3m^2 c^3}$$

Note that the time-averaged incident flux is

$$\langle S\rangle = \frac{c}{8\pi} E_0^2$$

The **differential cross section**, $\dfrac{d\sigma}{d\Omega}$, for linearly polarized radiation is obtained by

$$\frac{d\sigma}{d\Omega} \equiv \left\langle \frac{dP}{d\Omega} \right\rangle / \langle S\rangle , \quad \boxed{\therefore \frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \sin^2\Theta = r_0^2 \sin^2\Theta, \quad r_0 \equiv \frac{e^2}{mc^2}}$$

where the quantity r_0 gives a measure of the "size" of the point charge. (Note electrostatic potential energy $e\phi = e^2/r_0$).

For an electron, the classical electron radius has a value $r_0 = 2.82 \times 10^{-13}$ cm.

The total cross section is found by integrating over solid angle.

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi r_0^2 \int_{-1}^{1} (1-\mu^2) d\mu = \frac{8\pi}{3} r_0^2$$

For an electron, the scattering process is then called Thomson scattering or electron scattering, and the **Thomson cross section** is

$$\boxed{\sigma_{\mathrm{T}} = \frac{8\pi}{3} r_0^2 = 6.652 \times 10^{-25}\ \mathrm{cm}^2}$$

- Note:

  The total and differential cross sections are frequency independent.

  The scattered radiation is linearly polarized in the plane of the incident polarization vector $\hat{\epsilon}$ and the direction of scattering $\mathbf{n}$.

  $\sigma \propto 1/m^2$: electron scattering is larger than ions by a factor of $(m_\mathrm{p}/m_e)^2 = (1836)^2 \approx 3.4 \times 10^6$.

  We have implicitly assumed that electron recoil is negligible. This is only valid for nonrelativistic energies. For higher energies, the (quantum-mechanical) Klein-Nishina cross section has to be used.

- What is **the cross section for scattering of unpolarized radiation**?

  An unpolarized beam can be regarded as the independent superposition of two linear-polarized beams with perpendicular axes.

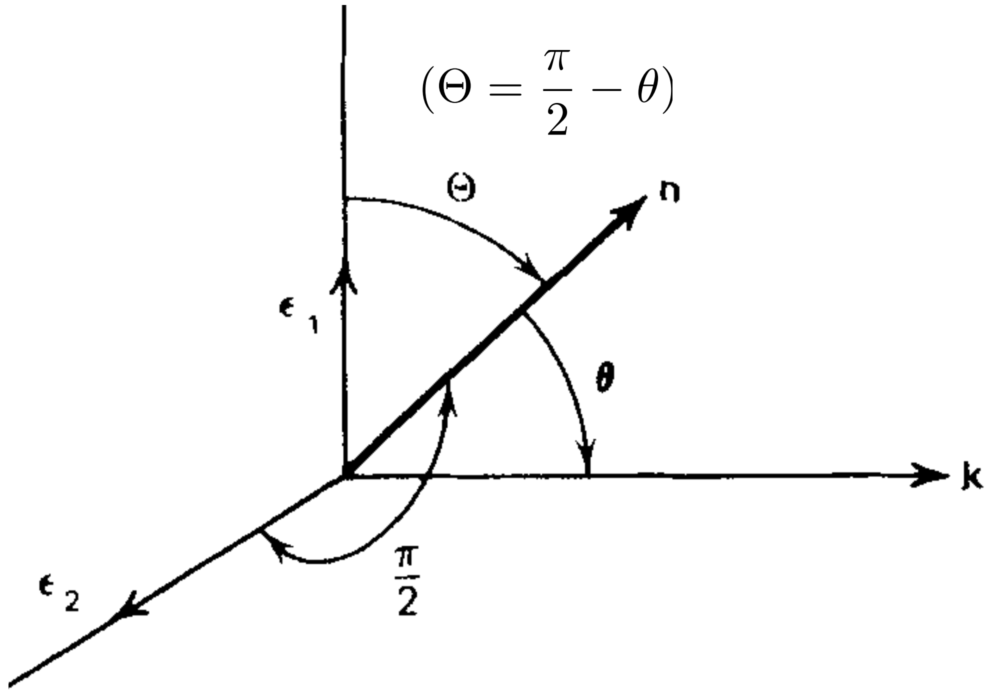
  Let us assume that $\mathbf{n}$ = direction of scattered radiation

  $\mathbf{k}$ = direction of incident radiation

  Choose

  the first electric field along $\hat{\epsilon}_1$, which is in the $\mathbf{n} - \mathbf{k}$ plane

  the second one along $\hat{\epsilon}_2$ orthogonal to this plane and to $\mathbf{n}$



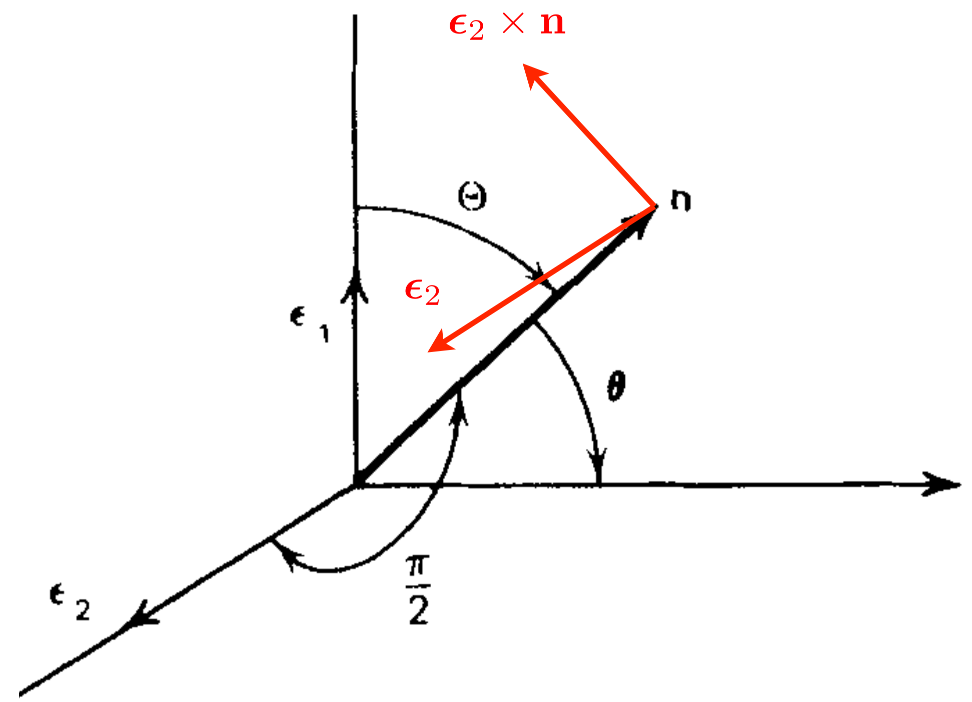
- Let Θ = angle between $\boldsymbol{\epsilon}_1$ and $\mathbf{n}$, and note that angle between $\boldsymbol{\epsilon}_2$ and $\mathbf{n}$ = $\pi/2$.

  $\theta = \pi/2 - \Theta$ = angle between the scattered wave and incident wave

  Then, the differential cross section for unpolarized radiation

  is the average of the cross sections for scattering of two electric fields.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm unpol} = \frac{1}{2}\left[\left(\frac{d\sigma}{d\Omega}\right)_{\epsilon_2} + \left(\frac{d\sigma}{d\Omega}\right)_{\epsilon_1}\right]$$
$$= \frac{1}{2}\left[\left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\rm pol} + \left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\rm pol}\right]$$
$$= \frac{1}{2}r_0^2\left(1+\sin^2\Theta\right)$$



$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\rm unpol} = \frac{1}{2}r_0^2\left(1+\cos^2\theta\right)}$$

This depends only on the angle between the incident and scattered directions, as it should for unpolarized radiation.

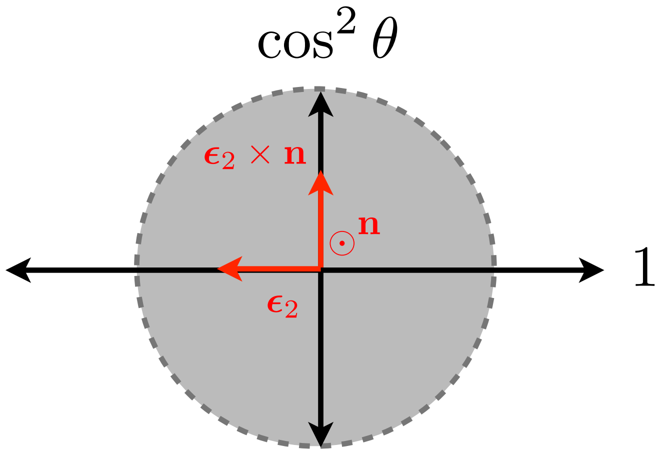
Total cross section: $$\sigma_{\rm unpol} = \int \left(\frac{d\sigma}{d\Omega}\right)_{\rm unpol} d\Omega = \pi r_0^2 \int_{-1}^{1} (1+\mu^2)d\mu$$
$$= \frac{8\pi}{3}r_0^2$$
$$= \sigma_{\rm pol}$$

Properties of Thomson Scattering

- Forward-backward symmetry: differential cross section is symmetric under $\theta \to -\theta$.
- Total cross section of unpolarized incident radiation = total cross section for polarized incident radiation. This is because the electron at rest has no preferred direction defined.
- **Scattering creates polarization**

The scattered intensity is proportional to $1 + \cos^2\theta$, of which 1 arises from the incident electric field along ϵ_2 and $\cos^2\theta$ from the incident electric field along ϵ_1.

"$\cos^2\theta$" of the polarization along ϵ_2 will be cancelled out by the independent polarization along $\epsilon_2 \times \mathbf{n}$.



Therefore, the degree of polarization of the scattered wave:

$$\Pi = \frac{1 - \cos^2\theta}{1 + \cos^2\theta}$$

Electron scattering of a completely unpolarized incident wave produces a scattered wave with some degree of polarization.

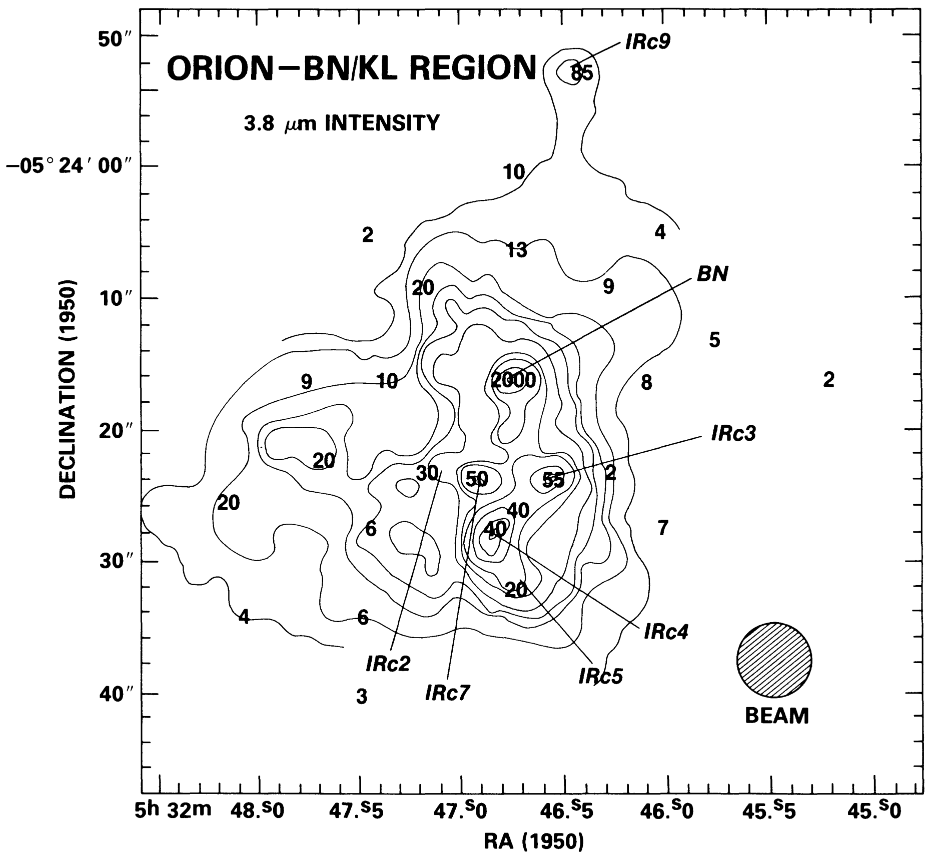
<u>No net polarization along the incident direction</u> $(\theta = 0)$, since, by symmetry, all directions are equivalent.

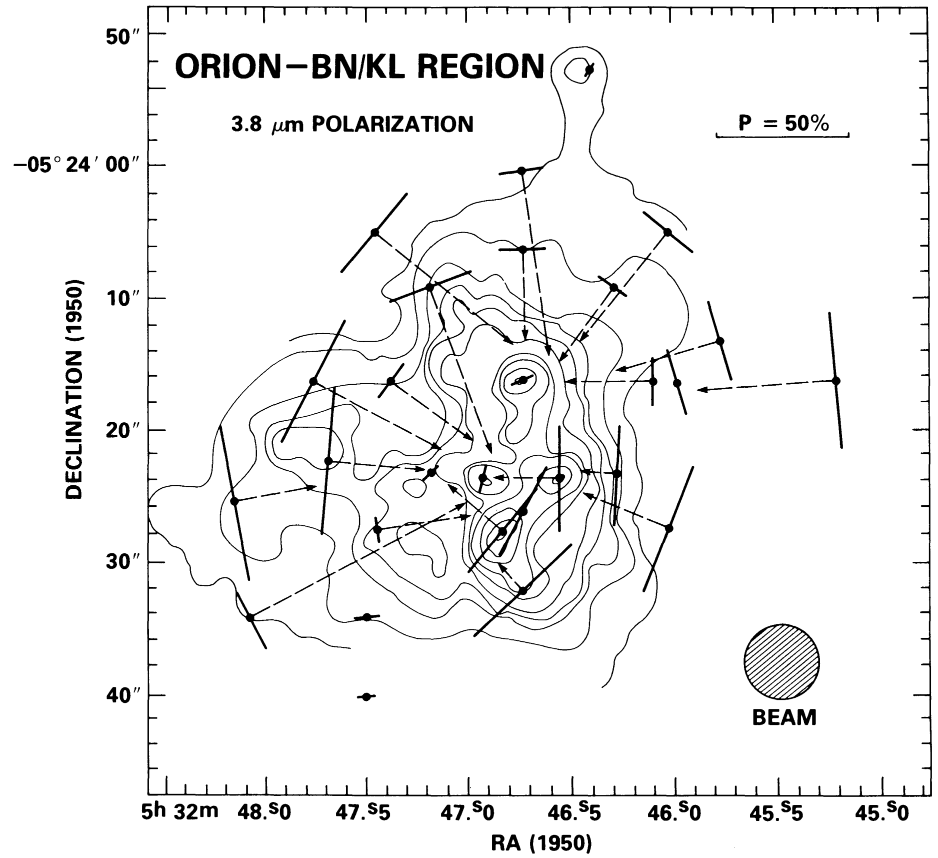
<u>100% polarization perpendicular to the incident direction</u> $(\theta = \pi/2)$, since the electron's motion is confined to a plane normal to the incident direction.

Astrophysical Applications of Polarization by Scattering

- Detection of a concentric pattern of polarization vectors in an extended region indicates that the light comes via scattering from a central point source.

Werner et al. (1983, ApJL, 265, L13)





- Left map shows the IR intensity map at 3.8 um of the Becklin-Neugebauer/Kleinmann-Low region of Orion. It is not easy to identify which bright spots correspond to locations of possible protostars.
- However, the polarization map singles out only two positions of intrinsic luminosity: IRc2 (now known to be an intense protostellar wind) and BN (suspected to be a relatively high-mass star)
- All the other bright spots (IRc3 through 7) correspond to IR reflection nebulae.