Interstellar Medium (ISM)

Week 2
2025 March 10 (Monday), 9AM

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- Heating -

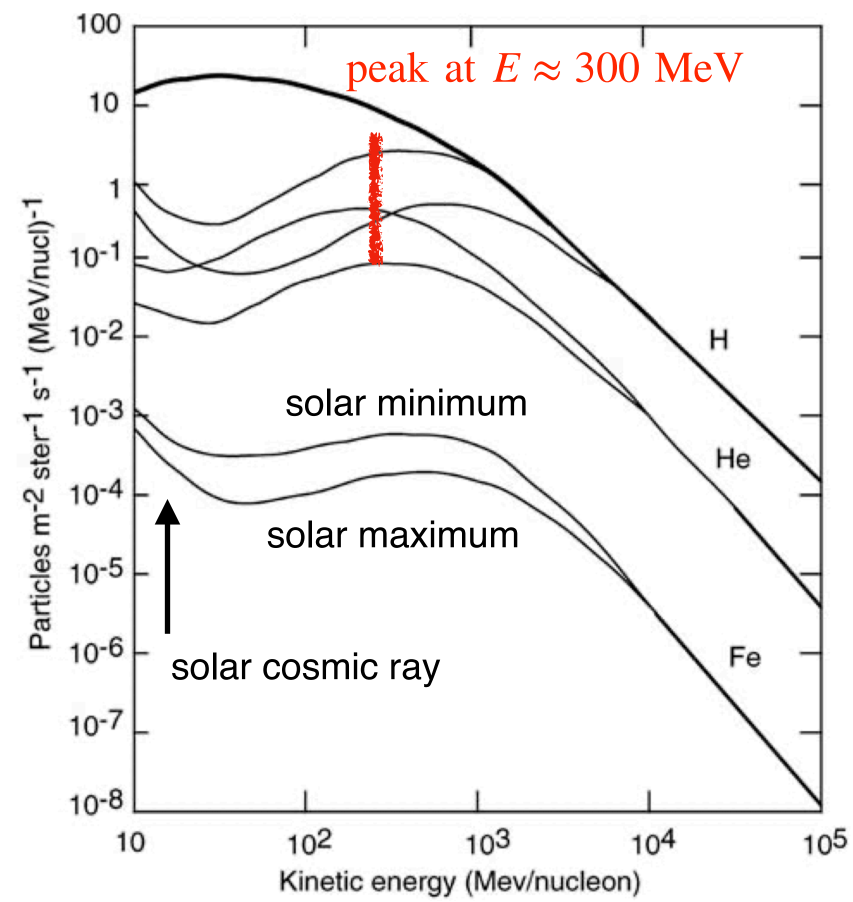
- Heating processes
  - The primary heating mechanisms of the ISM involve free electrons with high energies. Through collisions, the **fast free electrons share their kinetic energy with other particles.**
  - Through further collisions, the distribution of velocities approaches a Maxwellian distribution.
  - ***Sources of free electrons:*** The fast electrons may have been ejected from atoms by cosmic rays, from dust grains by photons or from atoms by photons or they may have been accelerated by shocks.
    - ✦ **Ionization by cosmic rays**
    - ✦ **Photoionization of dust grains by starlight UV.**
    - ✦ **Photoionization of atoms (H, He, C, Mg, Si, Fe, etc) by starlight UV or X-rays.**
  - Other heating sources:
    - ✦ Heating by shock waves and other MHD phenomena.

(1) Cosmic Ray Heating

- Cosmic Rays
  - Cosmic rays consist primarily of protons (~90%) and helium nuclei (~8%), with heavier nuclei (~1%), electrons, positrons, and antiprotons making a small contribution (others ~1%). They have relativistic energies as high as 10^{20} eV.
  - Cosmic rays are of galactic origin (supernova explosions and other high-energy events), and not universal except perhaps at very high energies.
  - The energy distribution of cosmic rays peaks at $E \approx 300\,\mathrm{MeV}$.
  - The solar wind expels cosmic rays of low energy.
- The relatively low-energy cosmic rays ($1 < E < 50$ MeV) can collisional ionize hydrogen atoms:

$$H + p \rightarrow p + p + e^-$$

  - The cosmic ray ionization produces electrons with a spectrum of energies. The ejected electron carries away a mean energy of $\langle E \rangle \approx 35$ eV.
  - Some of this kinetic energy will go into secondary ionization and excitation of H, H_2, and He that will then deexcite radiatively, but a fraction of the secondary electron energy will ultimately end up as thermal kinetic energy.



Differential energy spectra of cosmic-ray protons (H), α particles (He), and iron, near solar minimum and maximum (Silberberg et al. 1988)

- The heating efficiency depends on the fractional ionization.
    - ✦ If the ionization is high, then the primary electron has a high probability of losing its energy by long-range Coulomb scattering off free electrons, and ~ 100% of the initial kinetic energy will be converted to heat.
    - ✦ When the gas is neutral, a fraction of the primary electron energy goes into secondary ionization or excitation of bound states.
    - ✦ Heat per primary ionization (Dalgarno & McCray 1972):

$$E_h \approx 6.5\,\mathrm{eV} + 26.4\,\mathrm{eV}\left(\frac{x_e}{x_e + 0.07}\right)^{1/2}, \quad x_e \equiv \frac{n_e}{n_\mathrm{H}} \quad \text{(ionization fraction)}$$

    - ✦ Heating rate due to cosmic ray ionization:

$$\begin{aligned} G_\mathrm{CR} &\approx \left(n_{\mathrm{H}^0} + n_{\mathrm{He}^0}\right)\zeta_\mathrm{CR} E_h \\ &\approx 1.03 \times 10^{-27}\left(\frac{\zeta_\mathrm{CR}}{10^{-16}\ \mathrm{s}^{-1}}\right)\left[1 + 4.06\left(\frac{x_e}{x_e + 0.07}\right)^{1/2}\right] \quad \mathrm{erg\ s^{-1}} \end{aligned}$$

Here, ζ_CR is the primary cosmic ray ionization rate (the average rate at which a hydrogen atom is ionized by cosmic rays).

- Primary ionization rate by cosmic rays
  - From the observed spectrum of cosmic rays and the definition of the cosmic ray ionization rate:

$$\zeta_{\rm CR} \gtrsim 7 \times 10^{-18}\ {\rm s}^{-1}$$ with a substantially larger rate being allowed by uncertainties

  - The observations of H_3^+ appear to indicate a cosmic ray ionization rate, in diffuse molecular gas,

$$\zeta_{\rm CR} \approx (0.5 - 3) \times 10^{-16}\ {\rm s}^{-1}$$

  - Note that $\zeta_{\rm CR} \approx 10^{-16}\ {\rm s}^{-1} \sim 3\,{\rm Gyr}^{-1}$.

- ***Heating by cosmic rays is the dominant heating mechanism in molecular clouds***, where the dust opacity prevents high-energy photons from entering.

(2) Photoelectric Heating by Dust

- UV and X-ray photons can knock electrons free from dust grains. The ejected electrons carry kinetic energy, which can be effective at heating the surrounding gas.
- ***Photoelectrons emitted by dust grains dominate the heating of the diffuse neutral ISM (CNM and WNM) in the Milky Way.***
- The work function, analogous to the ionization energy of an atom, for graphite is $4.50 \pm 0.05\,\mathrm{eV}$. Therefore, UV photons with $h\nu \gtrsim 5\,\mathrm{eV}$ can kick out photoelectrons from dust grains. The photoelectric heating by dust is dominated by photons with $h\nu \gtrsim 8\,\mathrm{eV}$.

$$G_{\rm pe} \approx 2 \times 10^{-26} \frac{n_{\rm ph}(8 - 13.6\,\mathrm{eV})}{4.3 \times 10^{-3}\,\mathrm{cm}^{-3}} \frac{\langle \sigma_{\rm abs} \rangle}{10^{-21}\,\mathrm{cm}^2} \frac{\langle Y \rangle}{0.1} \frac{\langle E_{\rm pe} \rangle - \langle E_c \rangle}{1\,\mathrm{eV}} \quad \mathrm{erg\ s}^{-1}$$

The gain is independent of temperature.

Here,

$n_{\rm ph}(8 - 13.6\,\mathrm{eV})$ = number density of $8 < h\nu < 13.6\,\mathrm{eV}$ photons
$\langle \sigma_{\rm abs} \rangle$ = total dust photo absorption cross section per H nucleon, averaged over the photon spectrum.
$\langle Y \rangle$ = photoelectric yield averaged over the spectrum of 8 to 13.6 eV photons absorbed by the interstellar grain mixture.
$\langle E_{\rm pe} \rangle$ = mean kinetic energy of escaping photoelectrons.
$\langle E_{\rm c} \rangle$ = mean kinetic energy of electrons captured from the plasma by grains.

- ***Photoelectric heating from dust may be an order of magnitude larger than the cosmic ray heating rate.***

(3) Photoionization Heating

- Photons in the energy range $11.26\,\mathrm{eV} < h\nu < 13.60\,\mathrm{eV}$ $(911.6\mathrm{\AA} < \lambda < 1101\mathrm{\AA})$ are likely to end up ionizing a carbon atom. When carbon is photo-ionized, a free electron is released.

$$C^0 + h\nu \rightarrow C^+ + e^-$$

Ionization energy of C = 11.26 eV.

- The released electron (photoelectron) carries away the energy between 0—2.34 eV.
- If there aren't many photons with $h\nu > 13.6\,\mathrm{eV}$, hydrogen is predominantly in neutral form and most of positively charged ions are C II (C^+).
- The heating gain from the photoionization of carbon is approximately:

$$\begin{aligned} G_{\mathrm{CII}} &= 2.2 \times 10^{-22} f(\mathrm{CI}) \mathscr{A}_C \chi_0 \ \mathrm{erg\ s^{-1}} \\ &\approx 10^{-29}\ \mathrm{erg\ s^{-1}} \end{aligned}$$

Eq (3.8) of The Physics and Chemistry of the Interstellar Medium (A. G. G. M. Tielens)

The gain is independent of temperature.

Here,

$f(\mathrm{CI})$ = neutral fraction of carbon ($\sim 3.3 \times 10^{-4}$)
$\mathscr{A}_C$ = atomic carbon abundance in the gas phase ($\sim 2.70 \times 10^{-4} \times 0.5$)
χ_0 = intensity of the radiation field in units of the average interstellar radiation field.

- In the dusty ISM, the heating by carbon photoionization can't compete with the heating by electrons ejected from dust grains.
- H II regions and the diffuse IGM are the regions where photoionization becomes an important heating source.

(4) Shock Heating

- Shocks are propagating disturbances, characterized by abrupt, nearly discontinuous change in the temperature, pressure, and density.

  - ✦ In the ISM, shocks can be created by a supernova explosion or by the collision between molecular clouds.
  - ✦ On larger scales, shocks can be created by the collision of two galaxy clusters.
  - ✦ Shocks convert the kinetic energy of bulk flow into the thermal energy associated with random particle motion.
  - ✦ By a supernova shock, the temperature can rise to more than 10^{6-7} K.

- Shock heating is the dominant heating mechanism in the HIM of the ISM and in the warm-hot intergalactic medium (WHIM).

- Cooling -

- Decreasing the average kinetic energy of particles in the ISM is usually done by ***radiative cooling*** (emission of radiation).
  - In the CNM, cooling is performed by infrared photons emitted by carbon and oxygen.
    - ✦ ***Oxygen is nearly all in the form of neutral O I.*** (ionization energy = 13.62 eV)
    - ✦ ***Carbon will be nearly all in the form of singly ionized C II.*** (ionization energy = 11.26 eV) The background starlight in our galaxy has enough photons in the relevant energy range $11.26\,\mathrm{eV} < h\nu < 13.60\,\mathrm{eV}$ to keep the C atoms ionized.

- [C II] 158μm (collisionally excited line emission)
  - The electronic ground state of C II is split into two fine levels, separated by an energy $E_{u\ell} = 7.86 \times 10^{-3}\,\mathrm{eV}$, which corresponds to $\lambda = 158\,\mu\mathrm{m}$ and $T = E_{u\ell}/k = 91.2\,\mathrm{K}$.
  - The upper level is populated by collisions with hydrogen atoms and free electrons.
  - *If C II is excited by collisions with free electrons*, the cooling function is given by, for a C abundance $n_{\mathrm{C}}/n_{\mathrm{H}} = 2.7 \times 10^{-4}$,

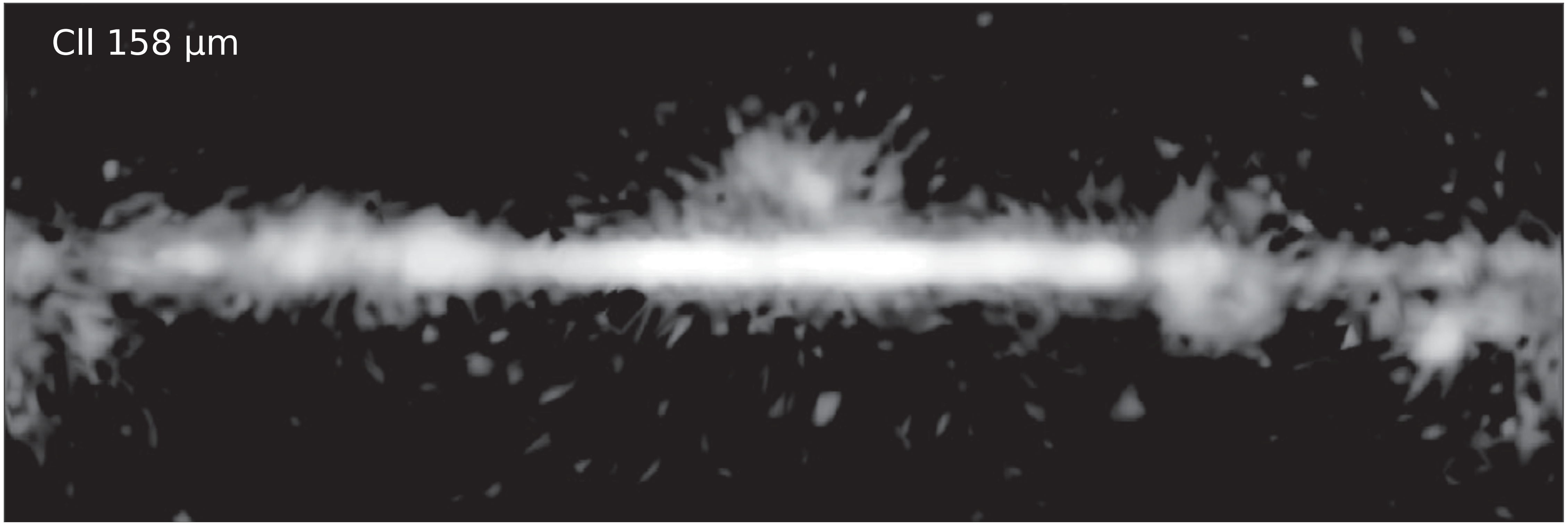
$$\frac{\Lambda^{e}_{[\mathrm{CII}]}}{10^{-27}\,\mathrm{erg\ cm^3\ s^{-1}}} \approx 3.1 \left(\frac{x}{10^{-3}}\right) \left(\frac{T}{100\,\mathrm{K}}\right)^{-1/2} \exp\left(-\frac{91.2\,\mathrm{K}}{T}\right)$$

Here, $x = n_e/n$ is the ionization fraction.

- *If the C II is excited by collisions with hydrogen atoms*, the cooling function is

$$\frac{\Lambda^{\mathrm{H}}_{[\mathrm{CII}]}}{10^{-27}\,\mathrm{erg\ cm^3\ s^{-1}}} \approx 5.2 \left(\frac{T}{100\,\mathrm{K}}\right)^{0.13} \exp\left(-\frac{91.2\,\mathrm{K}}{T}\right)$$

- In the CNM, both contribute significantly to the excitation of C II.



C II 158 μm line emission in the Galaxy. The map size is -180$^{\mathrm{o}}$ to 180$^{\mathrm{o}}$ in Galactic longitude and -60$^{\mathrm{o}}$ and 60$^{\mathrm{o}}$ in Galactic latitude. The data is from all-sky maps created by the Cosmic Microwave Background Explorer.

[Fig. 5.5. Introduction to the Interstellar Medium, J. P. Williams]

- [O I] 63.2μm (collisionally excited emission line)
  - The electronic ground state of O I has a fine splitting of $E_{u\ell}/k = 228\,\mathrm{K}$.
  - The upper level is populated *primarily by collisions with hydrogen atoms*.
  - The resulting cooling function due to the emission of 63.2μm is, for an abundance of $n_{\mathrm{O}}/n_{\mathrm{H}} = 6.0 \times 10^{-4}$,

$$\frac{\Lambda^{\mathrm{H}}_{[\mathrm{OI}]}}{10^{-27}\,\mathrm{erg\ cm^3\ s^{-1}}} \approx 4.1 \left(\frac{T}{100\,\mathrm{K}}\right)^{0.42} \exp\left(-\frac{228\,\mathrm{K}}{T}\right)$$

At $n_O/n_C = 2.2$, cooling by O I doesn't surpass cooling by C II until *T* reaches ~800 K.

- Note:
  - [C II] and [O I] are the dominant form of cooling in molecular clouds and the CNM.
  - Molecular clouds can also cool by emission from the vibrational and rotational transitions of molecules.
  - The critical densities for [C II] and [O I] are $\sim 4 \times 10^3\,\mathrm{cm^{-3}}$ and $\sim 10^5\,\mathrm{cm^{-3}}$, respectively, implying that collisional deexcitation of these levels is unimportant in the diffuse ISM of the Milky Way.

- Collisionally excited Lyα 1216Å
  - The first excited level of atomic hydrogen is $E_{21} = 10.20\,\mathrm{eV}$ above the ground state.
  - The first excited level will not be highly populated by collisions until the temperature reaches $T \sim E_{21}/k = 118{,}000\,\mathrm{K}$. However, there are ~1700 H atoms for every O atom. In addition, Lyα photon carries away ~520 times as much energy as an [O I] photon. Therefore, the cooling by Lyα can compete with cooling by IR fine-structure lines at temperature as low as $T \approx 10^4$ K.
  - The cooling function for H excited by collisions with free electrons is

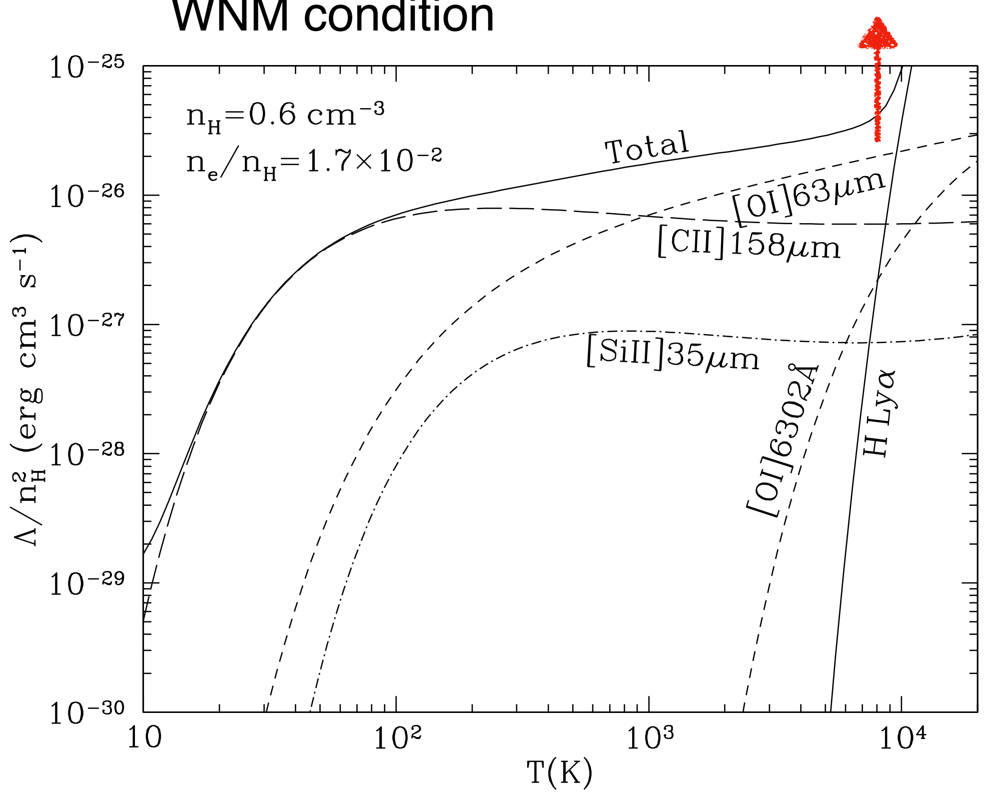
$$\frac{\Lambda^{\mathrm{e}}_{[\mathrm{Ly}\alpha]}}{10^{-27}\,\mathrm{erg\ cm^3\ s^{-1}}} \approx 6 \times 10^5 \left(\frac{x}{10^{-3}}\right)\left(\frac{T}{100\,\mathrm{K}}\right)^{-0.5} \exp\left(-\frac{118{,}000\,\mathrm{K}}{T}\right)$$

- When T > 20,000 K,
  - Atomic hydrogens can be collisionally ionized, followed by radiative recombination to a high energy level, and followed by a cascade down to the ground state.
  - The recombination lines from hydrogen are an important cooling mechanism in the WNM and WIM.
  - These phases are also cooled by line emission from more highly ionized atoms such as O III, C IV, and O VI.

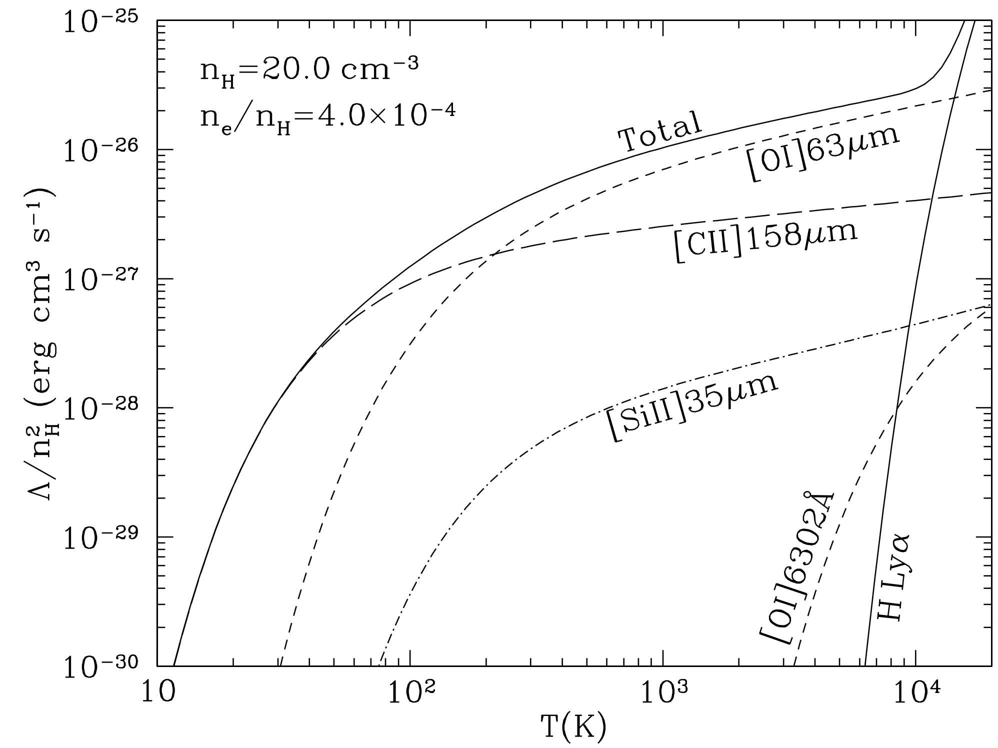
- Free-free emission (Thermal Bremsstrahlung)
  - In the HIM at T > 10^6 K, the "braking radiation" emitted by electrons when they are accelerated by other charged particles can be a significant cooling mechanism.
  - The cooling function is $\Lambda \propto T^{1/2}$.
- Cooling Function

Onset of Lyα at T ~ 8,000 K

WNM condition



CNM condition



  - For 10 < T < 10^4 K, [C II] 158µm and [O I] 63µm lines are major coolants. The [O I] 63µm line is important for T > 100 K. Lyα cooling dominates only at T > 10^4 K.

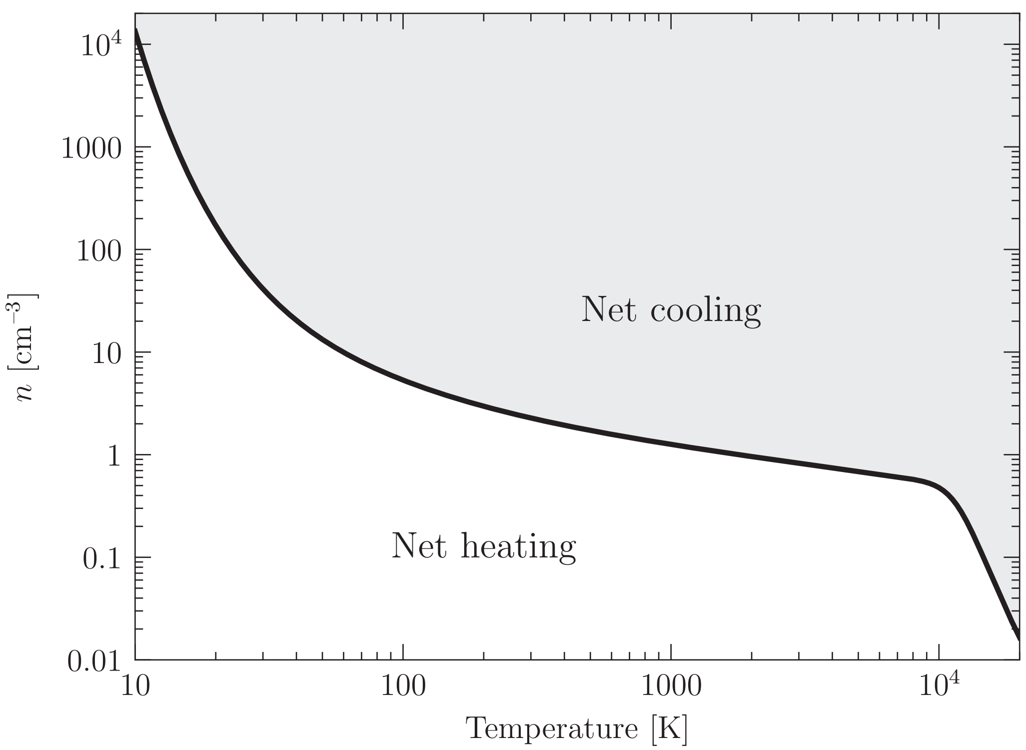
Stable & Unstable Equilibrium

- A thermal equilibrium must have heating and cooling balanced: $g = \ell$.
  - We assume ***photoelectric heating by dust*** and ***cooling by [C II], [O I], and Lyα***. Then, the equilibrium density is obtained by

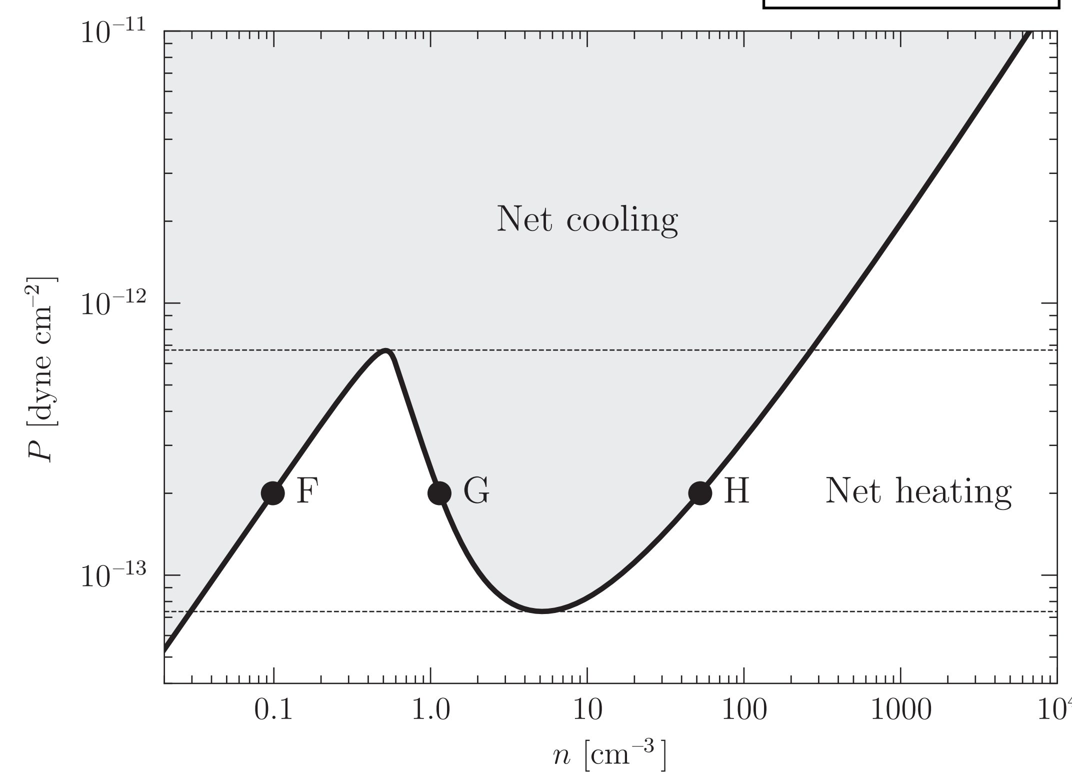
$$n_{\rm eq} G = n_{\rm eq}^2 \Lambda \quad \rightarrow \quad n_{\rm eq}(T) = \frac{G}{\Lambda(T)}$$

Note that G is a (nearly) constant.

n vs T plane



P vs n plane

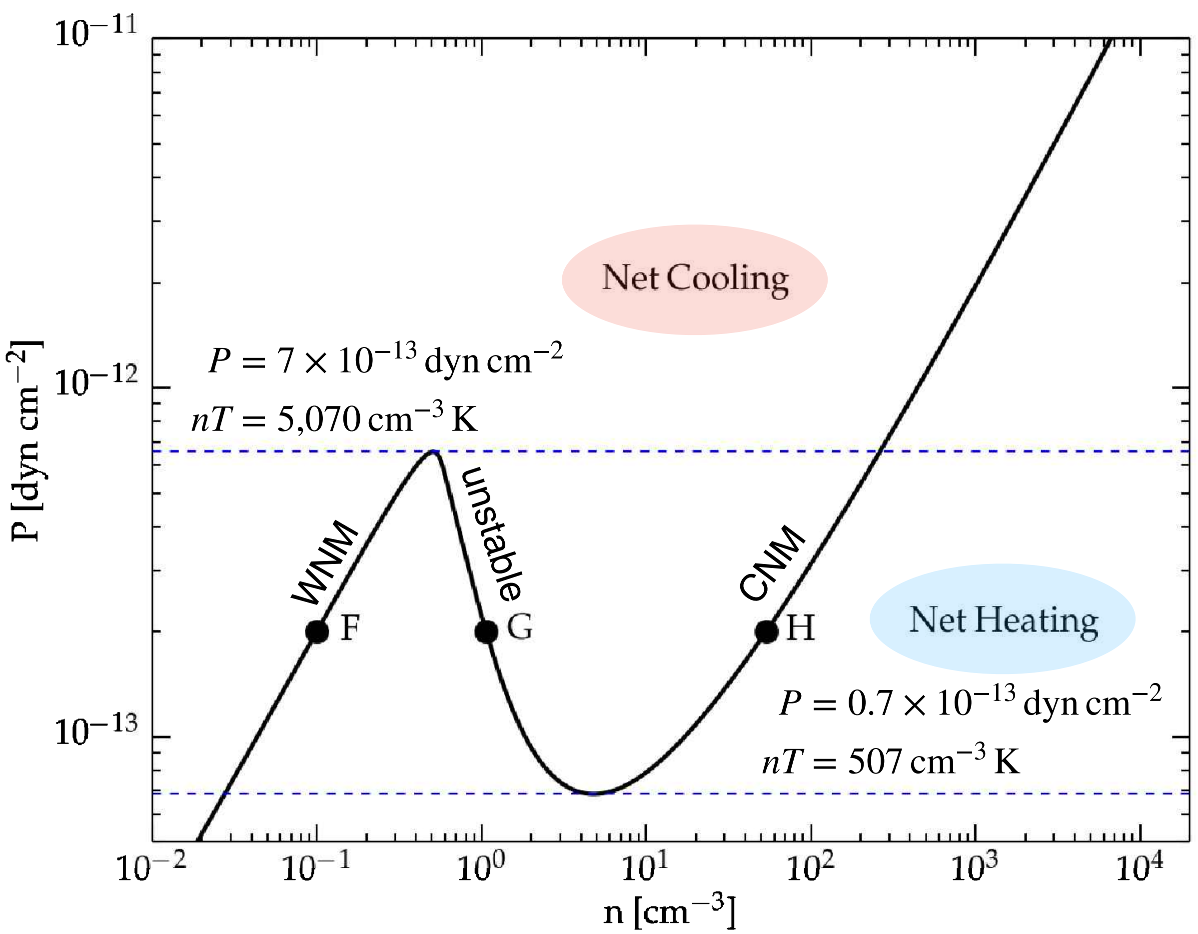


  - If every point along the above equilibrium line represented a stable equilibrium, then there could be a continuous distribution of temperatures, and thus of number densities.
  - However, it's not the case. Not every equilibrium point is a stable equilibrium.
  - The presence of distinct phases in the ISM results from the distinction between stable and unstable equilibrium.

- Pressure Equilibrium

- Let's assume that the interstellar gas is in pressure equilibrium.
  - For pressures in the range $0.7 \times 10^{-13}\,\mathrm{dyn\,cm^{-2}} < P < 7 \times 10^{-13}\,\mathrm{dyn\,cm^{-2}}$, bounded by the dashed lines, **there are three possible values of $n_{\rm eq}$ at a fixed pressure.**

- Consider what happens at a point, for instance F, if you slightly change the temperature while keeping the pressure fixed.
  - If *T* increases, *n* must decrease, and you must move left from point F. This moves you into the net cooling portion, and *T* consequently decreases.
  - If *T* decrease, *n* must increase, and this moves you rightward into the net heating portion, and *T* consequently increases.
  - Thus, a negative feedback restores the original temperature.
- A similar negative feedback maintains temperature stability at point H.
- However, now consider what happens at G.
  - If *T* increases, *n* must decrease, and you must move left from point G. This moves you into the net heating portion, and *T* increases further, until you reach F.
  - If *T* decrease, *n* must increase, and this moves you rightward into the net cooling portion, and *T* decrease further, until you reach H.
  - Thus, a positive feedback makes the point unstable.
- **Consequently, we have two stable equilibrium points (F and H). F = WNM, H = CNM**



- History: Two-Phase Model & Three-Phase Model

- As a result of their analysis, ***Field, Goldsmith, and Habing (1969)*** created a two-phase model of the ISM, consisting of Cold Neutral Clouds, with $n \sim 10\ \mathrm{cm^{-3}}$ and $T \sim 100$ K, embedded within a Warm Intercloud Medium, with $n \sim 0.1\ \mathrm{cm^{-3}}$ and $T \sim 10{,}000$ K.
  - ✦ They were unaware of the role played by dust in heating the ISM, assumed that ***collisional ionization by cosmic rays provided the bulk of the heating***.
  - ✦ FGH (1969) advocated a two-phase model. However, they also speculated “an existence of a third stable phase at $T > 10^6$ K, with bremsstrahlung the chief cooling process.”

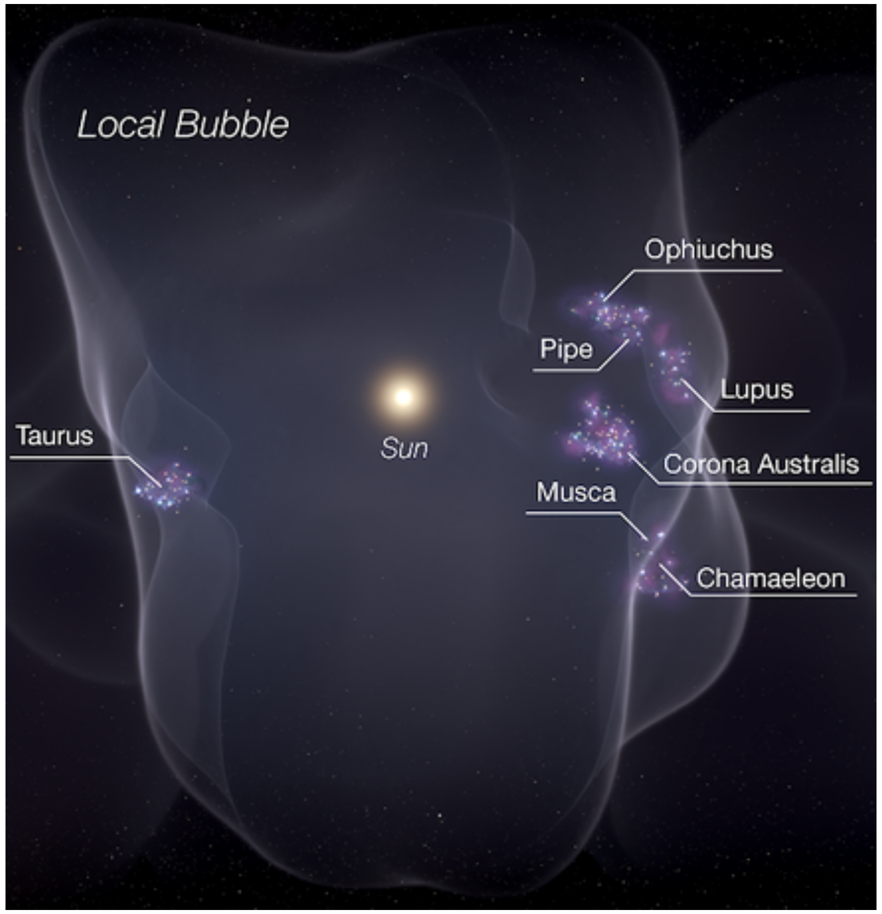
- In the 1970s, detection of a diffuse soft X-ray background and of emission lines such as O VI 1032, 1038Å hinted at the existence of interstellar gas with $T \sim 10^6$ K. In fact, the Sun resides in a “***Local Bubble***” of hot gas, with $T \sim 10^6$ K and $n \sim 0.004\ \mathrm{cm^{-3}}$.

- Cox & Smith (1974) suggested that supernova remnants could produce a bubbly hot phase, and that the bubbles blown by supernovae would have a porosity factor (volume fraction of the ISM occupied by hot bubbles):

porosity factor: $$q > 0.1 \left(\frac{r_{\rm SN}}{10^{-13}\,\mathrm{pc^{-3}\,yr^{-1}}} \right)$$ $r_{\rm SN}$ is the supernova rate per unit volume.

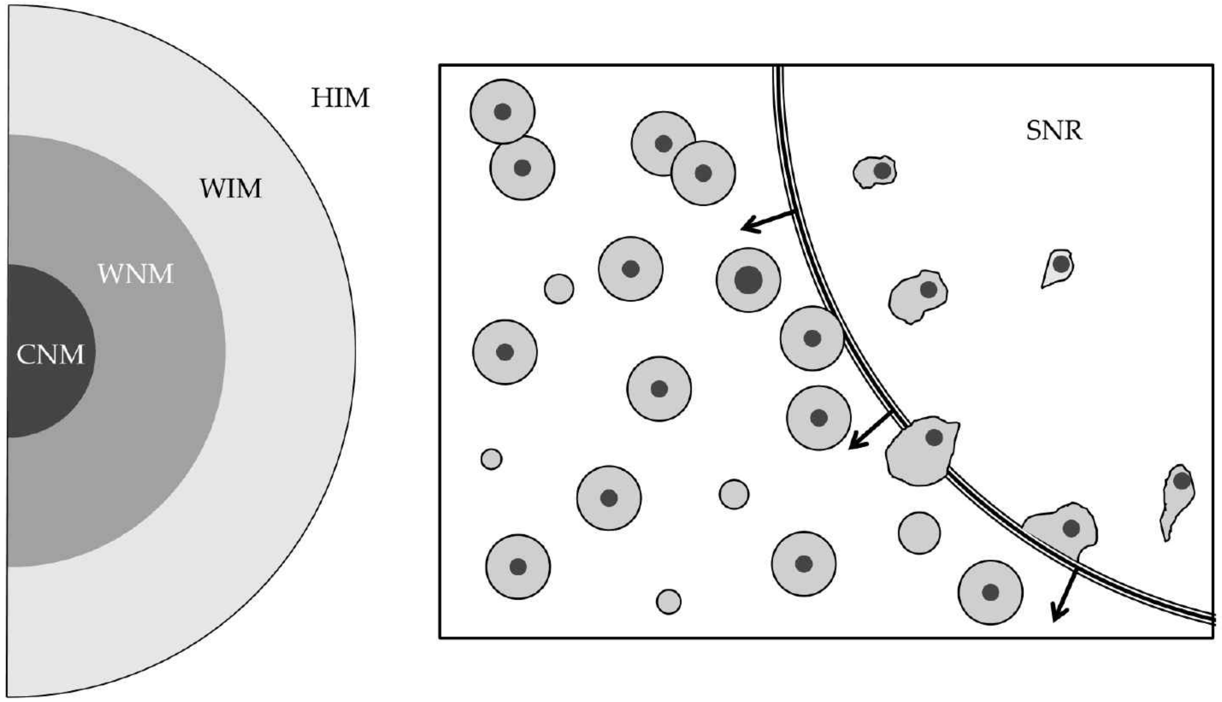
- If $0.1 < q < 0.5$, the expanding supernova remnants can join to form supersized bubbles and elongated tunnels of hot gas.

- A superbubble or supershell is a cavity which is ~100 pc across and is populated with hot (10^6 K) gas atoms, less dense than the surrounding ISM, blown against that medium and carved out by multiple supernovae and stellar winds.



- History: McKee & Ostriker's Three-Phase Model

- McKee & Ostriker (1977)
  - They made a more elaborate argument for three phases within the ISM.
  - ***Cold Neutral Medium***, with $T \sim 80\,\mathrm{K}$, $n \sim 40\,\mathrm{cm}^{-3}$, and a low fractional ionization $x = n_e/n \sim 0.001$.
  - ***Warm Medium***, containing both ionized and neutral components, $T \sim 8000\,\mathrm{K}$ and $n \sim 0.3\,\mathrm{cm}^{-3}$, the ionization fraction ranging from $x \sim 0.15$ in the neutral component (WNM) to $x \sim 0.7$ in the ionized component (WIM).
  - ***Hot Ionized Medium***, consisting of the overlapping supernova bubbles, with $T \sim 10^6\,\mathrm{K}$ and $n \sim 0.002\,\mathrm{cm}^{-3}$, and $x \sim 1$ (nearly complete ionization).



The left panel shows a typical cold neutral cloud, surrounded by the warm medium (WNM and WIM).

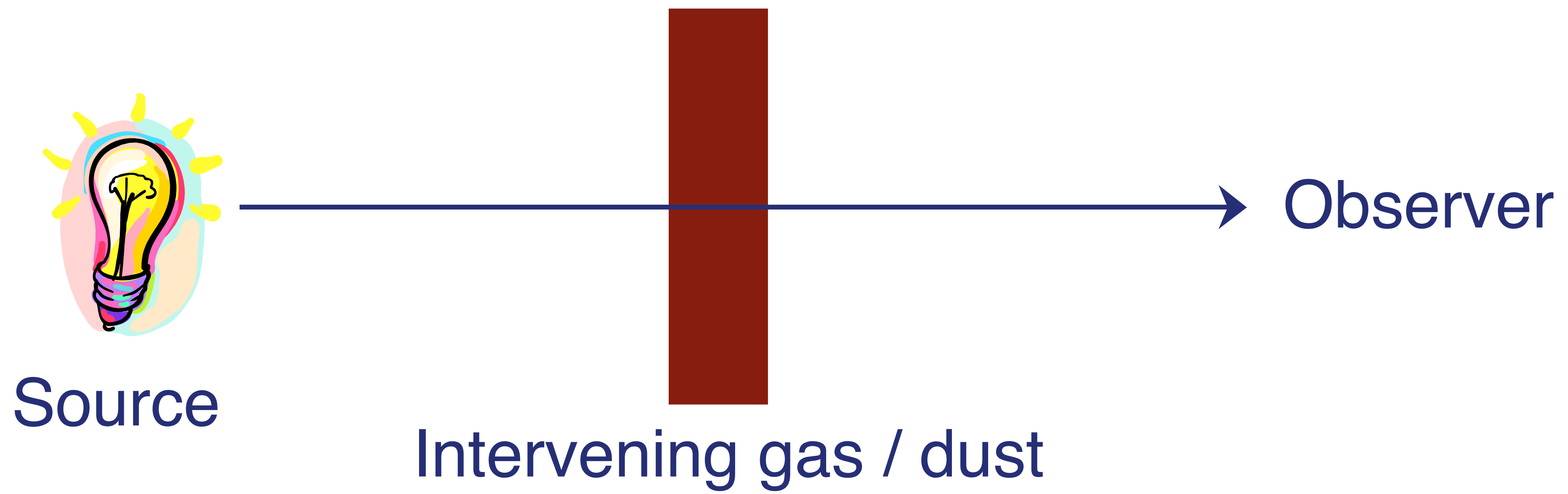
The right panel shows an expanding supernova blastwave overtaking a population of cold clouds.

- However, in many ways, the ISM is a dynamic, turbulent, dusty, magnetized place.
- The five-phase model is largely empirical (not relying on assumptions about thermal pressure equilibrium).

Radiative Transfer

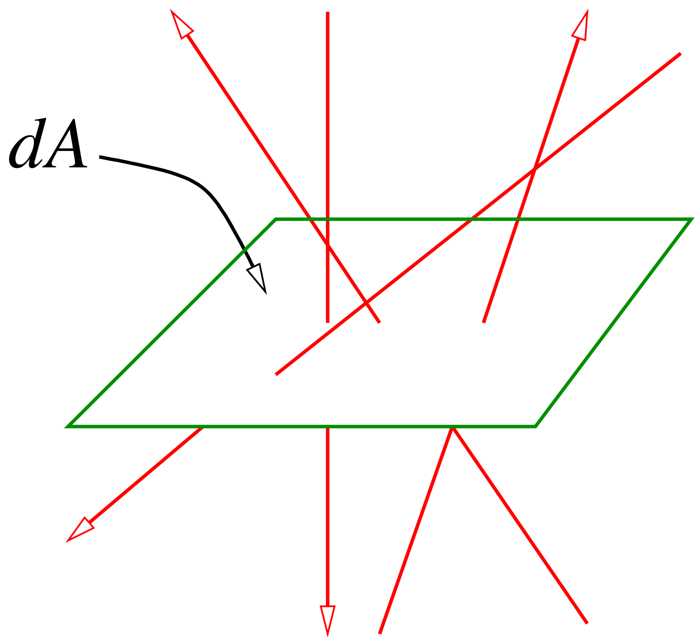
Radiative Transfer

- How is radiation affected as it propagates through intervening gas and dust media to the observer?



Simplication & Complexity

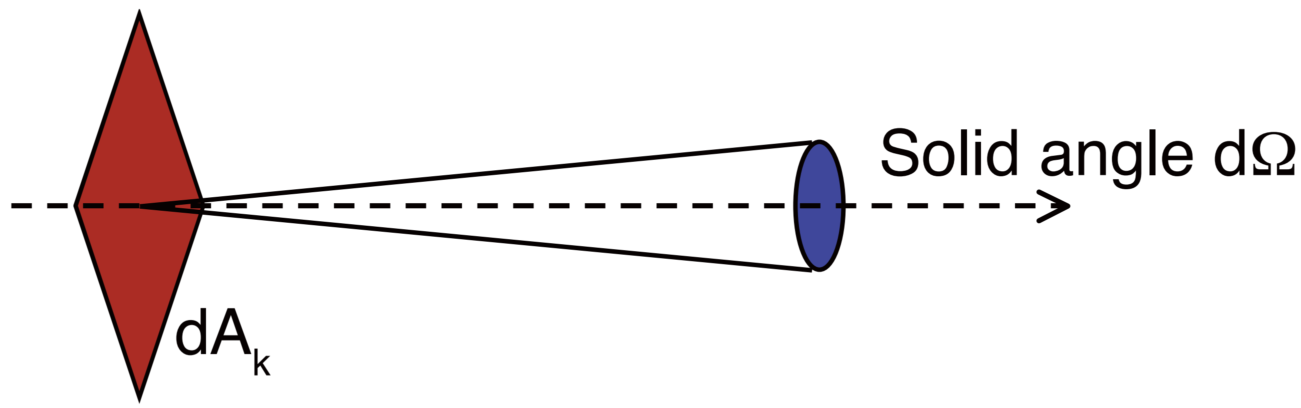
- Simplification:
  - Astronomical objects are normally much larger than the wavelength of radiation they emit.
  - Diffraction can be neglected.
  - Light rays travel to us along straight lines.



- Complexity:
  - At one point, photons can be traveling in several different directions.
  - For instance, at the center of a star, photons are moving equally in all directions. (However, radiation from a star seen by a distant observer is moving almost exactly radially.)
  - Full specification of radiation needs to say how much radiation is moving in each direction at every point. Therefore, we are dealing with the five- or six-dimensional problem. ($[x, y, z] + [\theta, \phi] + [t]$)

‘Specific’ Intensity (Surface Brightness)

- ***Intensity is the energy carried along by individual rays.***



$\mathbf{k}$ = direction of propagation

- Let dE_ν be the amount of radiant energy which crosses the area dA_k perpendicular to a direction $\mathbf{k}$ within solid angle $d\Omega$ about in a time interval dt with photon frequency between ν and $\nu + d\nu$.
- The monochromatic specific intensity I_ν is then defined by the equation.

$$\boxed{dE_\nu = I_\nu(\mathbf{k},\, \mathbf{x},\, t) dA_\mathbf{k} d\Omega d\nu dt} \quad \Rightarrow \quad I_\nu(\mathbf{k}, \mathbf{x}, t) = \frac{dE_\nu}{dA_\mathrm{k} d\Omega d\nu dt}$$

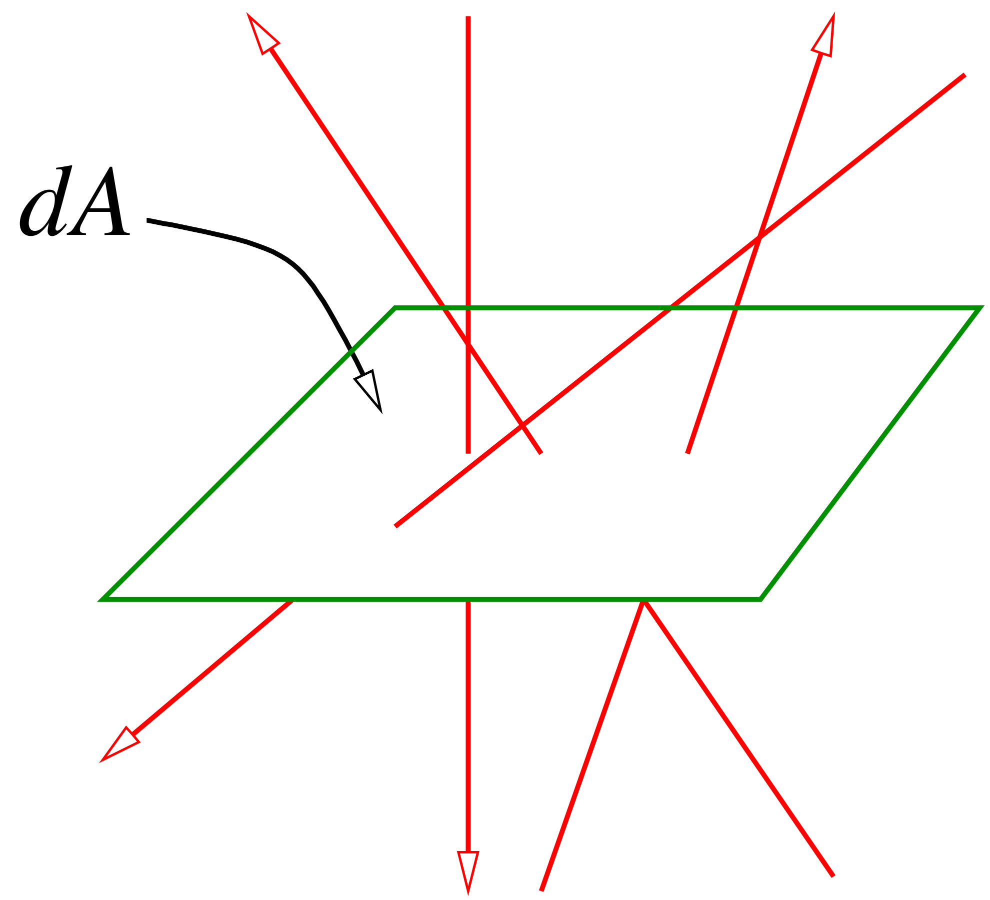
- Unit: $\mathrm{erg\ s^{-1}\ cm^{-2}\ sr^{-1}\ Hz^{-1}}$

- From the view point of an observer, the specific intensity is called ***surface brightness***.

Flux 'Density'

- Definition
  - ***Flux is a measure of the energy carried by all rays passing through a given area***
  - Consider a small area dA, exposed to radiation for a time dt.
  - Flux F_ν is defined as ***the total (net) energy passing through a unit area in all directions within a unit time interval.***

$$\boxed{dE = F \times dA \times dt} \Rightarrow F_\nu = \frac{dE_\nu}{dA d\nu dt}$$

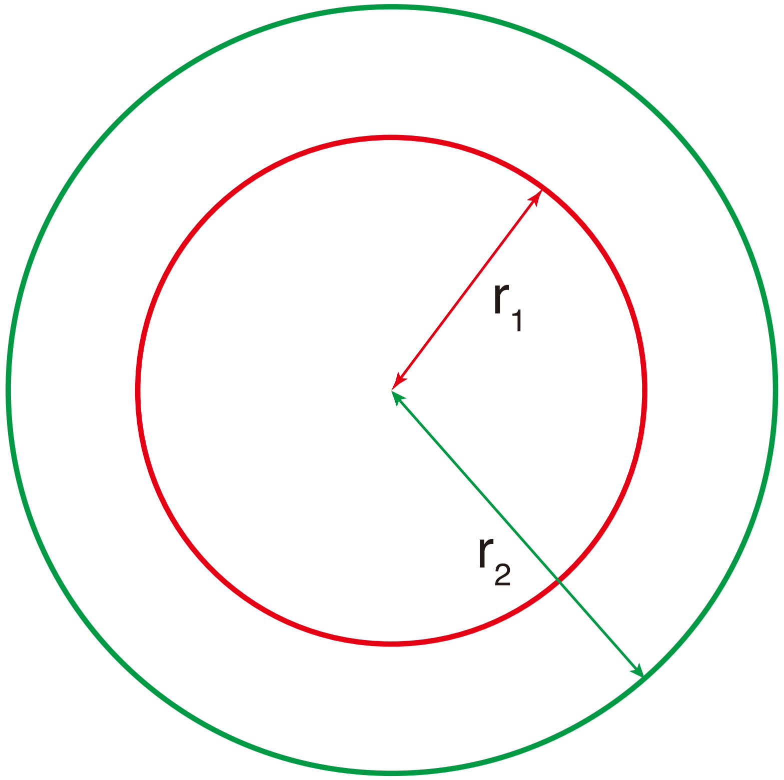


  - Note that F_ν ***depends on the orientation of the area element*** dA.
  - Unit: $\mathrm{erg\ cm^{-2}\ s^{-1}}$

- F_ν is often called the "flux density."
- Radio astronomers use a special unit to define the flux density: $1\ \mathrm{Jansky\ (Jy)} = 10^{-23}\ \mathrm{erg\ s^{-1}\ cm^{-2}\ Hz^{-1}}$

Inverse Square Law

- Flux from an isotropic radiation source, i.e., a source emitting equal amounts of energy in all directions.



  - Because of energy conservation, flux through two shells around the source must be the same.

$$4\pi r_1^2 F(r_1) = 4\pi r_2^2 F(r_2)$$

  - Therefore, we obtain the inverse square law.

$$F = \frac{\text{const.}}{r^2}$$

Energy Flux Density

- Real detectors are sensitive to a limited range of wavelengths. We need to consider how the incident radiation is distributed over frequency.

Total energy flux: $F = \int F_\nu d\nu$ Integral of F_ν over all frequencies

Units: $\mathrm{erg\ s^{-1}\ cm^{-2}\ Hz^{-1}}$

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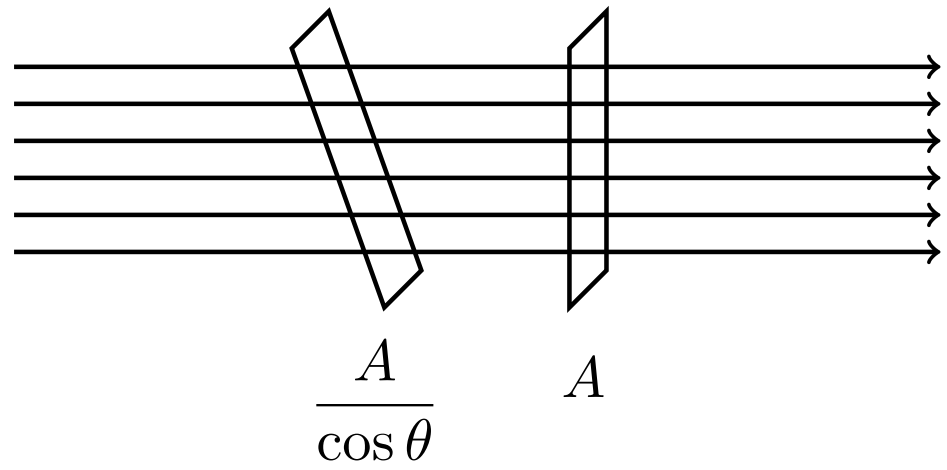
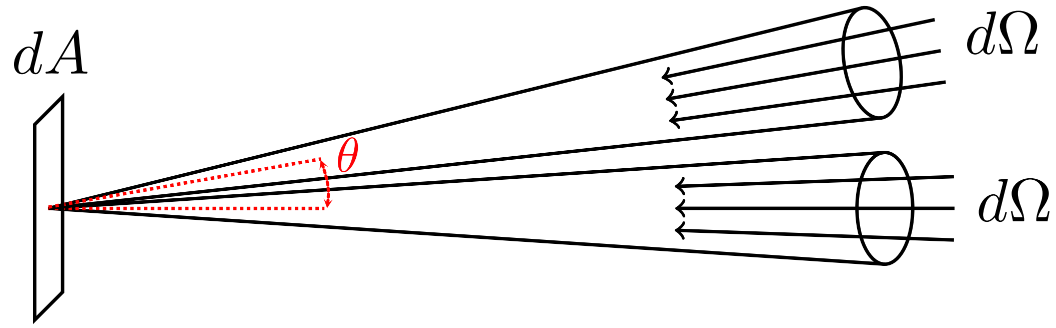
Flux vs. Intensity



The power delivered to the two surfaces are equal though there areas differ.

The flux is **the power per unit area** so the tilted surface gets less flux.



Two intensities are equal.
The upper set or rays delivers less flux.

The rate that energy is delivered to a surface from light traveling around a direction θ is $I\cos\theta d\Omega$.

Relation between the flux and the specific intensity

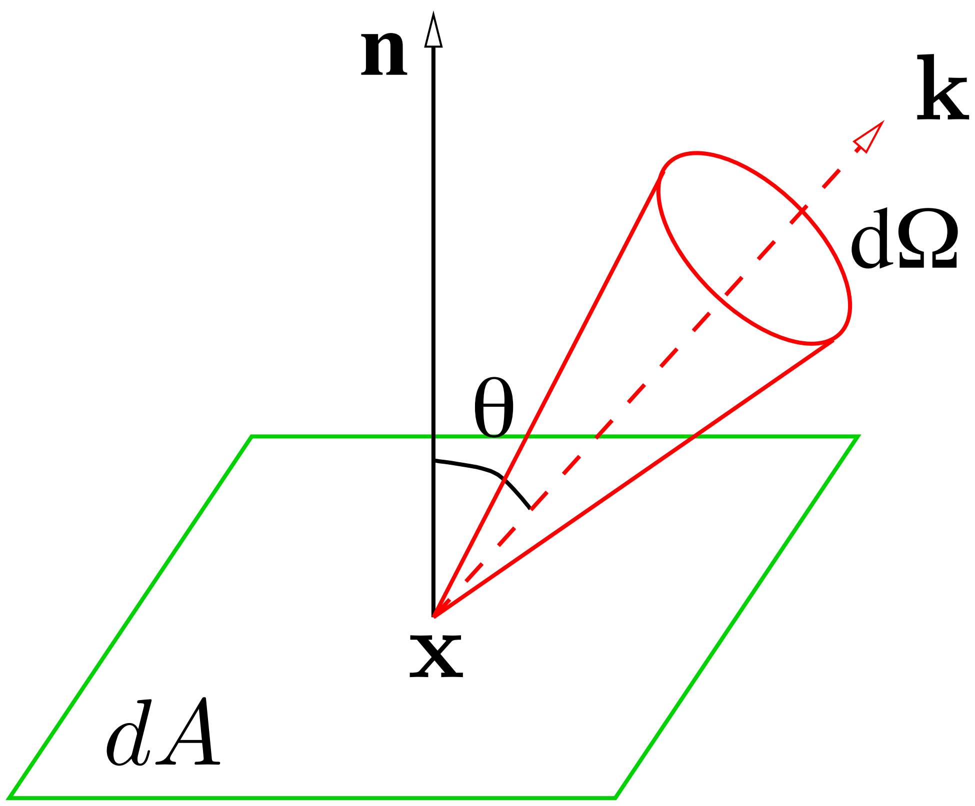
- Let's consider a small area dA, with light rays passing through it at all angles to the normal vector $\mathbf{n}$ of the surface.

- For rays centered about $\mathbf{k}$, the area normal to $\mathbf{k}$ is

$$dA_{\mathbf{k}} = dA\cos\theta$$

  - By the definition,

$$F_\nu dA d\nu dt = \int I_\nu(\mathbf{k}, \mathbf{x}, t) dA_{\mathbf{k}} d\Omega d\nu dt$$



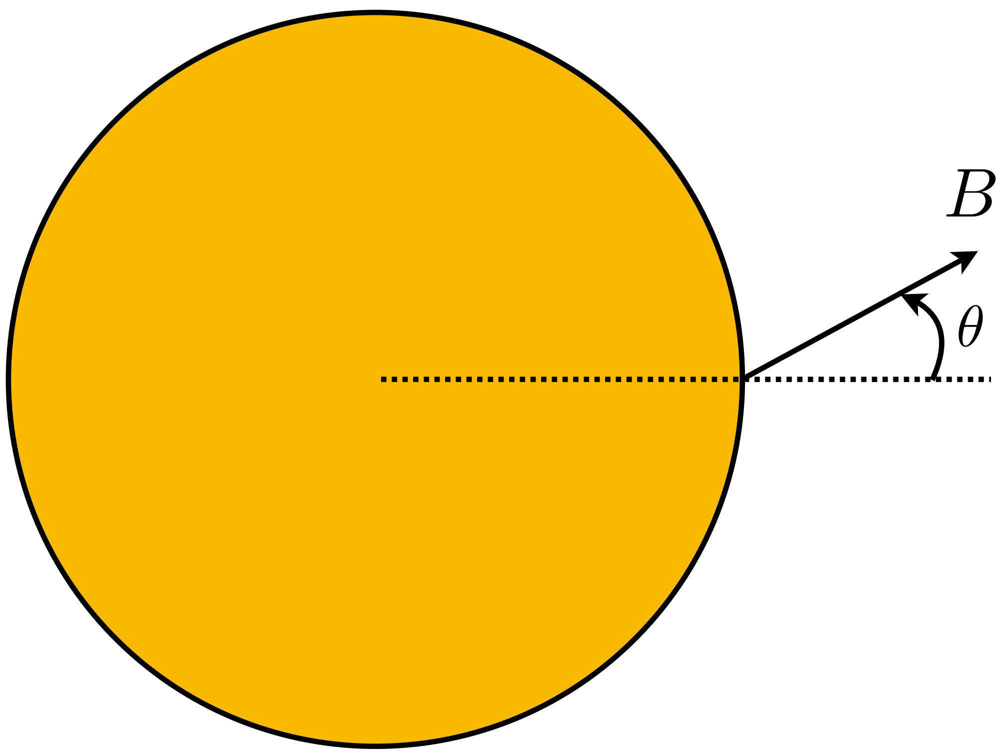
- Hence, net flux in the direction of $\mathbf{n}$ is given by integrating over all solid angles:

$$\boxed{F_\nu = \int I_\nu \cos\theta d\Omega = \int_0^{2\pi}\int_0^{\pi} I_\nu \cos\theta \sin\theta d\theta d\phi}$$

[Note] **flux** **= sum of all ray vectors projected onto a normal vector**
intensity **= absolute value of a single ray vector**

Flux from the surface of a uniformly bright sphere

- Let's calculate the flux at P on a sphere of uniform brightness B



$$F = \int B \cos\theta d\Omega = \int_0^1 \int_0^{2\pi} B\mu d\mu d\phi$$

$$F = \pi B$$

The total luminosity from the sphere is then

$$\boxed{L = (4\pi R^2)F = (4\pi R^2)\pi B}$$

- In stellar atmosphere, the **astrophysical flux** is defined by F/π .

Note — SED

- Intensity can be defined as per wavelength interval.

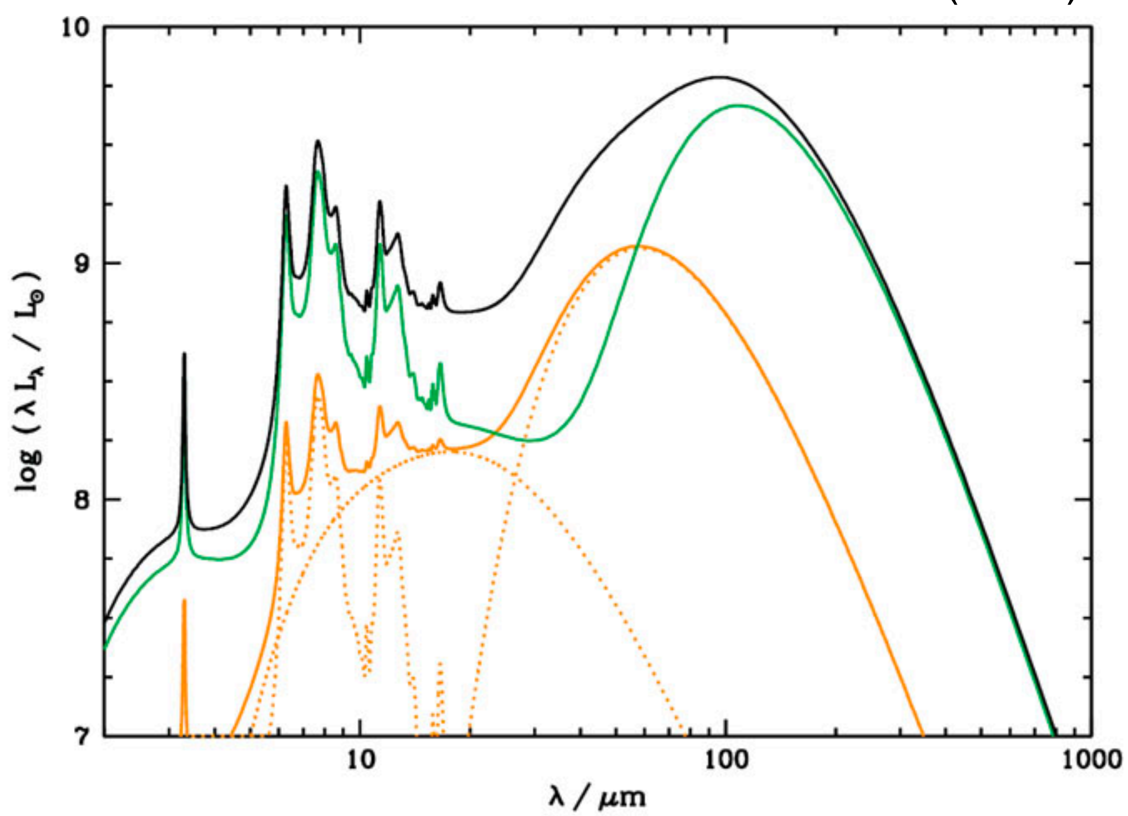
$$\begin{aligned} I_\nu |d\nu| &= I_\lambda |d\lambda| \\ \nu I_\nu &= \lambda I_\lambda \end{aligned} \quad \leftarrow \left| \frac{d\nu}{d\lambda} \right| = \frac{c}{\lambda^2} = \frac{\nu}{\lambda} \quad \longleftarrow \quad \nu = \frac{c}{\lambda}$$

- Integrated intensity is defined as the intensity over all frequencies.

$$I = \int_0^\infty I_\nu d\nu = \int_0^\infty I_\lambda d\lambda$$

- In astrophysics, we plot the ***spectral energy distribution (SED)*** as νI_ν versus ν or λI_λ versus λ.

Galaxy SEDs
da Cunha et al. (2008)



Luminosity

- To determine the energy per unit time, we integrate flux over area.
  - ***Monochromatic luminosity***: Considering a sphere centered on a source with radius *R*, the monochromatic luminosity is

$$L_\nu = R^2 \int d\Omega F_\nu$$
$$= 4\pi R^2 F_\nu \qquad \text{for an isotropic source}$$

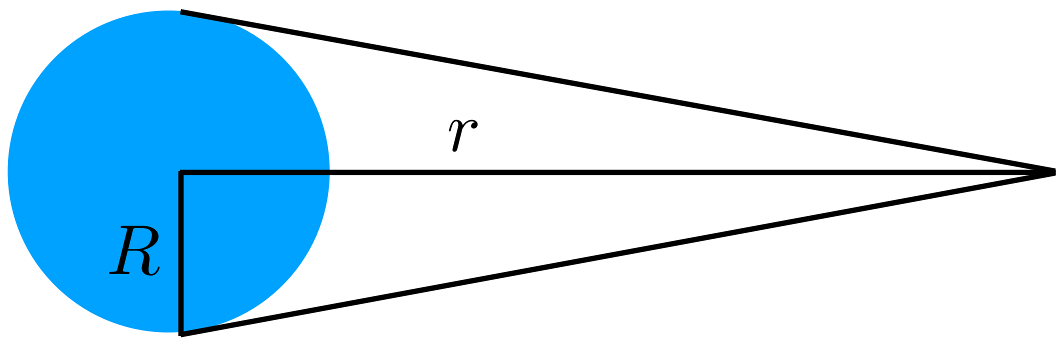
  - The ***bolometric luminosity*** is

$$L_{\text{bol}} = \int L_\nu d\nu = \int L_\lambda d\lambda = 4\pi R^2 \int F_\nu d\nu$$

- Flux and Luminosity of an extended source

$$F = \pi I \left(\frac{R}{r}\right)^2 = I \frac{A}{r^2}$$
$$= I\Omega_{\text{source}}$$

$$L = (4\pi r^2)F = (4\pi r^2) I\Omega_{\text{source}}$$



$A = \pi R^2$

Specific Energy Density

- Consider a bundle of rays passing through a volume element dV in a direction Ω.
  - Then, the energy density per unit solid angle is defined by

$$dE = u_\nu(\Omega) dV d\Omega d\nu$$

  - Since radiation travels at velocity c, the volume element is

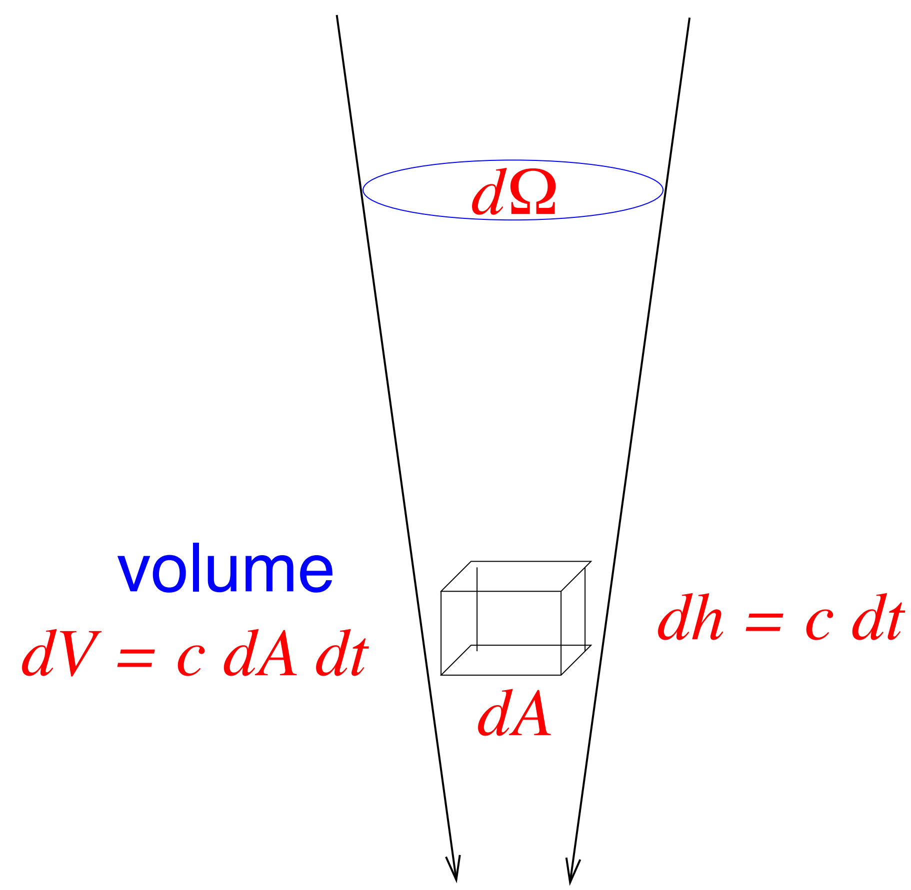
$$dV = dA(cdt)$$

  - According to the definition of the intensity,

$$dE = I_\nu dA dt d\Omega d\nu$$

  - Then, we have

$$\boxed{u_\nu(\Omega) = I_\nu(\Omega)/c}$$



Energy Density and Mean Intensity

- Integrating over all solid angle, we obtain

$$u_\nu = \int u_\nu(\Omega)d\Omega = \frac{1}{c}\int I_\nu d\Omega$$

- ***Mean intensity*** is defined by

$$\boxed{J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega}$$

- Then, the energy density is

$$\boxed{u_\nu = \frac{4\pi}{c}J_\nu}$$

- Total energy density is obtained by integrating over all frequencies.

$$u = \int u_\nu d\nu = \frac{4\pi}{c}\int J_\nu d\nu$$

< Radiative Transfer Equation > — in free space

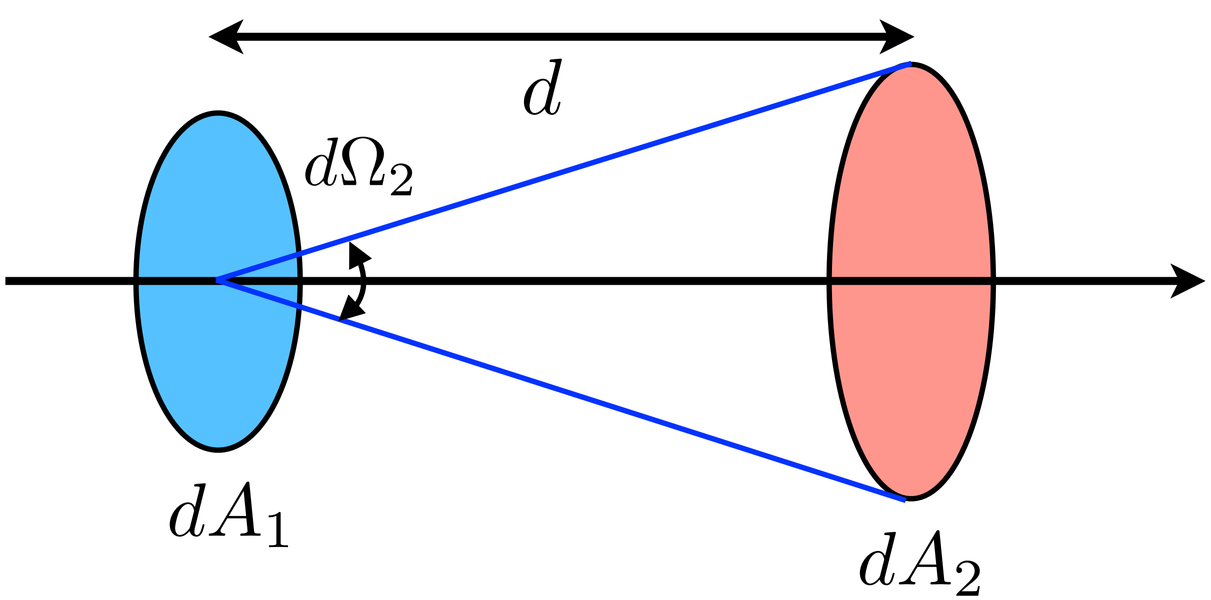
- How does intensity changes along a ray in free space
  - Suppose a bundle of rays and any two points along the rays and construct two "infinitesimal" areas dA_1 and dA_2 normal to the rays at these points.
  - What are the energies carried by the rays passing through both areas?

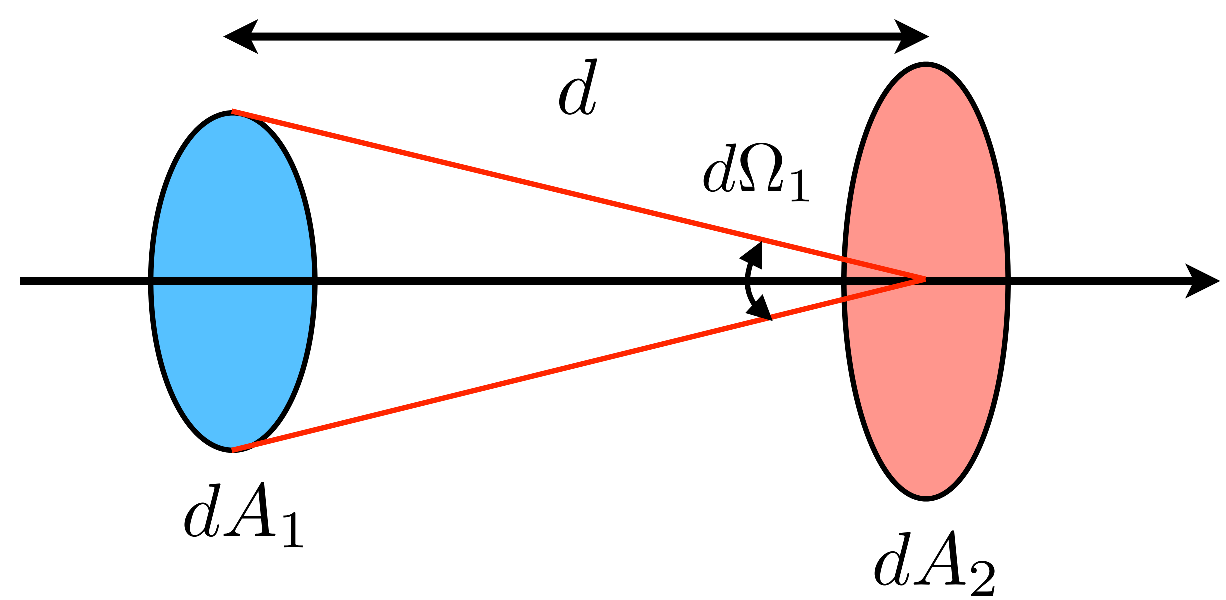
energy passing through 1

$$dE_1 = I_1 dA_1 d\Omega_2 d\nu dt$$

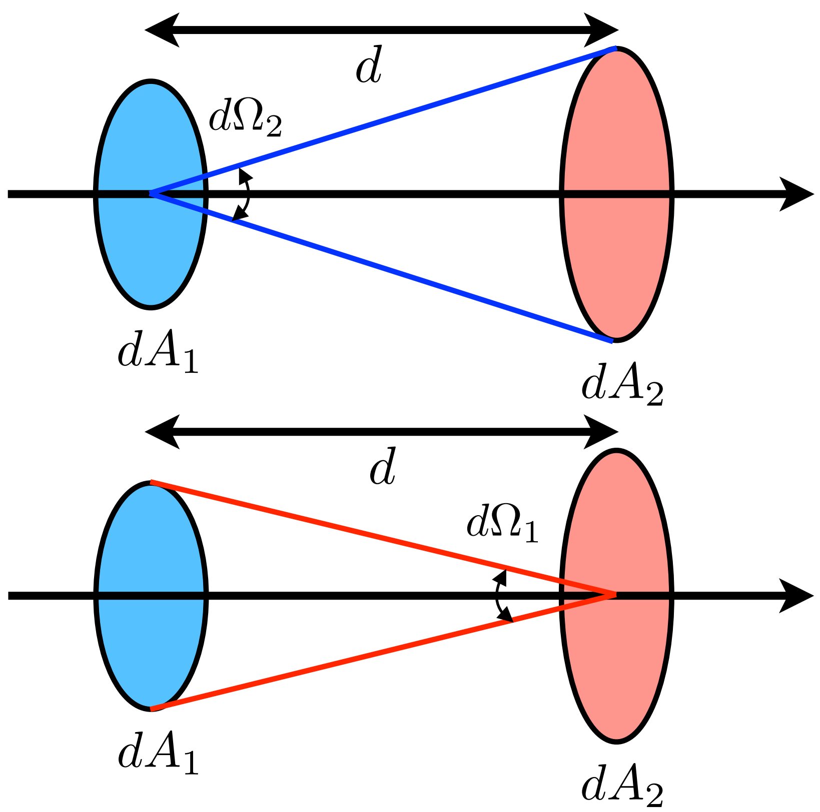
energy passing through 2

$$dE_2 = I_2 dA_2 d\Omega_1 d\nu dt$$





- Here, $d\Omega_2$ is the solid angle subtended by dA_2 at the location 1 and $d\Omega_1$ is the solid angle subtended by dA_1 at the location 2.



$$d\Omega_2 = \frac{dA_2}{d^2}$$

$$d\Omega_1 = \frac{dA_1}{d^2}$$

Conservation of energy:
Because energy is conserved,

$$dE_1 = dE_2 \quad \rightarrow \quad I_1 = I_2$$

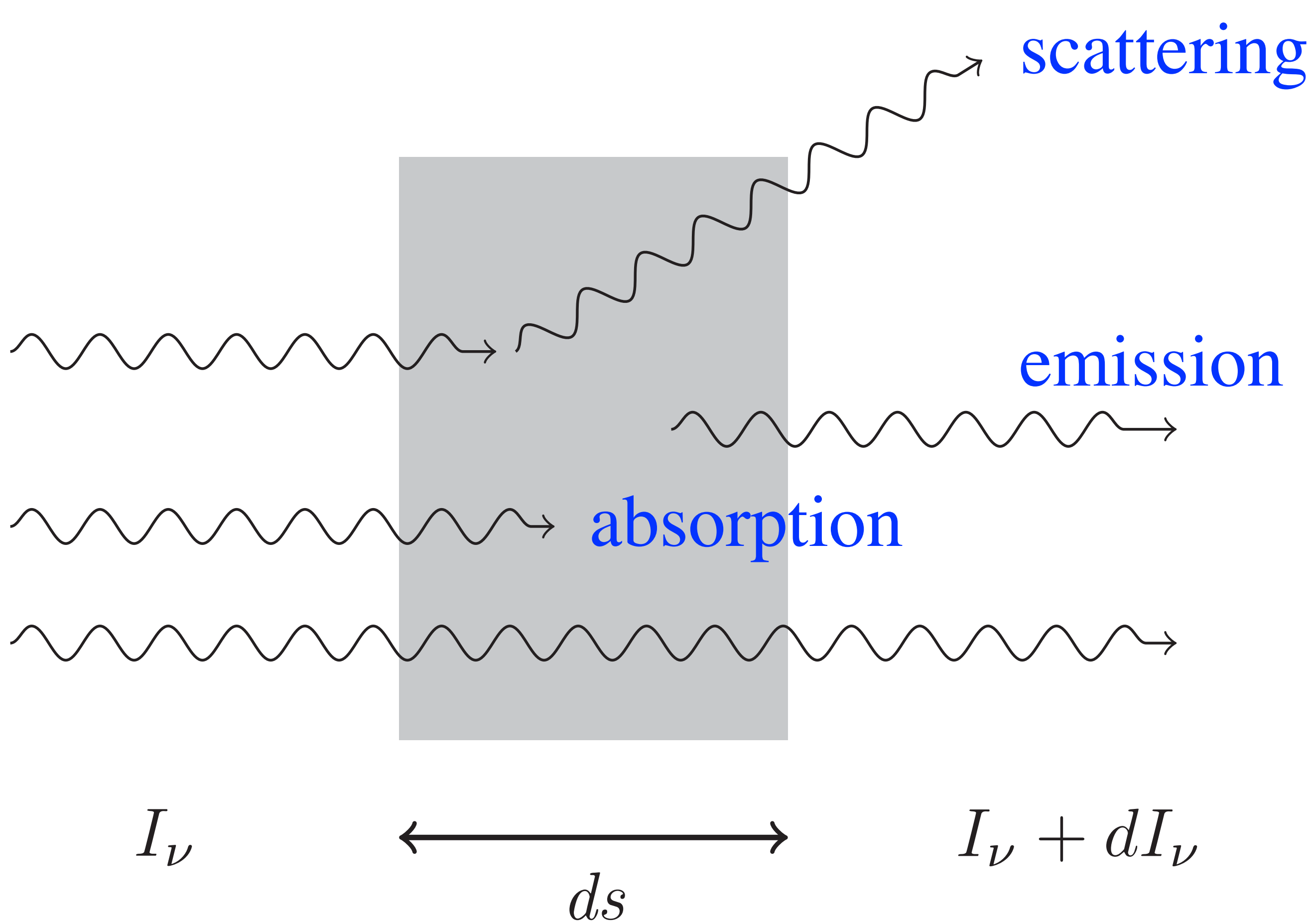
- Conclusion (***the constancy of intensity***): $I_1 = I_2 \longrightarrow \boxed{\dfrac{dI}{ds} = 0}$
  - the specific intensity remains the same as radiation propagates through free space.

- We receive the same specific intensity at the telescope as is emitted at the source.
  - Imagine looking at a uniformly lit wall and walking toward it. As you get closer, a field-of-view with fixed angular size will see a progressively smaller region of the wall, but this is exactly balanced by the inverse square law describing the spreading of the light rays from the wall.

< Radiative Transfer Equation > — Emission & Absorption

- In reality, as a ray passes through matter, energy may be added, subtracted, or scattered from it by emission, absorption, or scattering.

  - The intensity will not in general remain constant.
  - These interactions are described by the ***radiative transfer equation***.



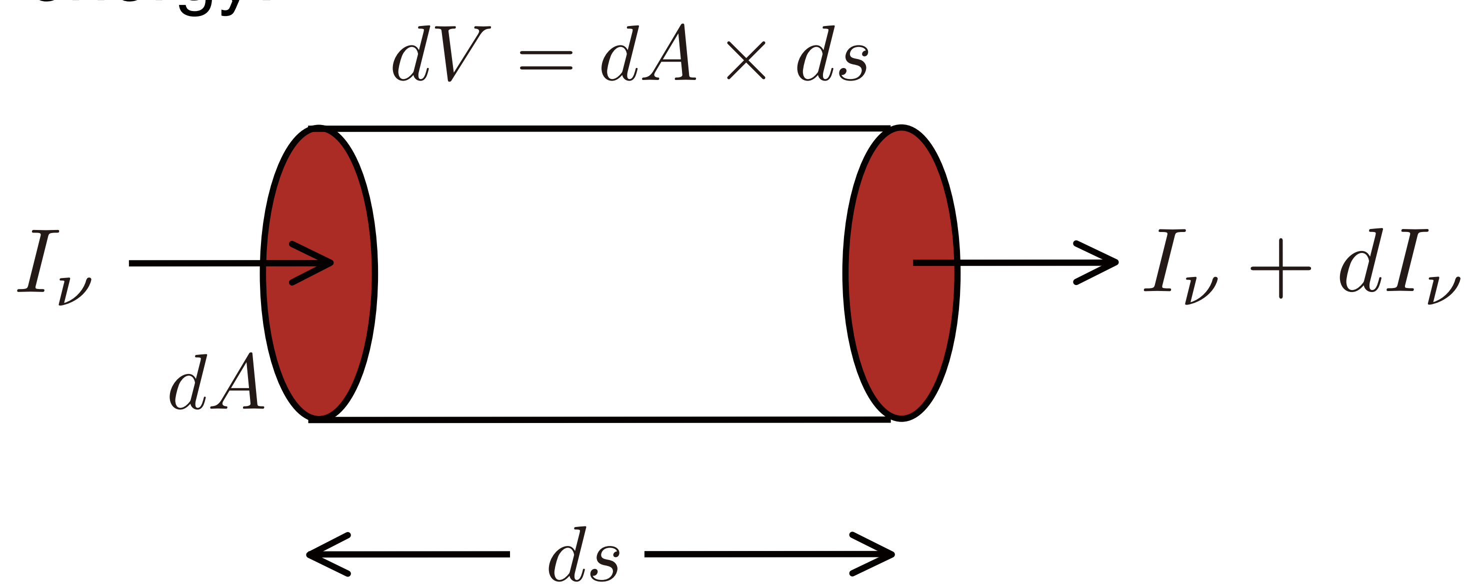
loss: scattering & absorption

gain: emission

[J. P. Williams]
Introduction to the Interstellar Medium

Emission

- If the radiation travels through a medium which is itself emitting radiation, that will add to the energy:



- Spontaneous ***"emission coefficient" or "emissivity"*** j_ν is the amount of energy emitted per unit time, per unit solid angle, per unit frequency, and per unit volume:

$$dE = j_\nu dV d\Omega dt d\nu \quad (j_\nu : \mathrm{erg\ cm^{-3}\ s^{-1}\ sr^{-1}\ Hz^{-1}})$$

- In going a distance ds, a beam of cross section dA travels through a volume $dV = dAds$. Thus the intensity added to the beam is by ds is

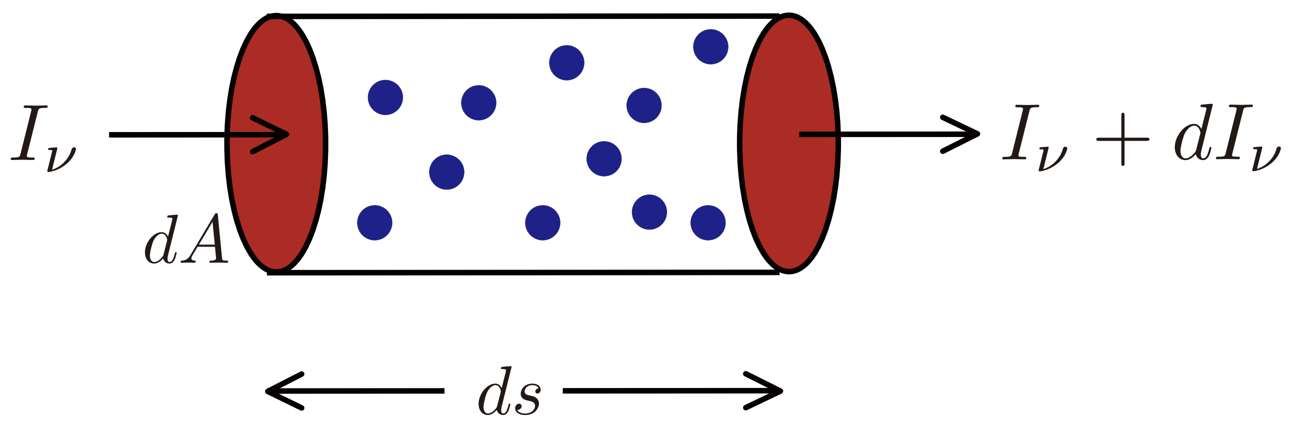
$$dI_\nu = j_\nu ds \longleftarrow dE = (dI_\nu) dA d\Omega dt d\nu$$

- Therefore, the equation of radiative transfer for pure emission becomes:

$$\frac{dI_\nu}{ds} = j_\nu$$

Absorption

- If the radiation travels through a medium which absorbs radiation, the energy in the beam will be reduced:



- Let n denote the number density of absorbers (particles per unit volume).
- Assume that each absorber has a cross-sectional area of σ_ν (in units of cm^2).

'geometric' cross section: $\sigma = \pi r^2$ for a spherical particle with a radius r

- If a beam travels through ds, total area of absorbers is

$$\text{number of absorbers} \times \text{cross section} = \left(n \times dA \times ds\right) \times \sigma_\nu$$

Fraction of radiation absorbed = Fraction of area blocked

$$\frac{dI_\nu}{I_\nu} = -\frac{n dA ds \sigma_\nu}{dA} = -n\sigma_\nu ds$$

$$dI_\nu = -n\sigma_\nu I_\nu ds \equiv -\alpha_\nu I_\nu ds$$

$$\longrightarrow \quad \boxed{\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu}$$

- ***Absorption coefficient*** is defined as $\alpha_\nu \equiv n\sigma_\nu$ (units: cm^{-1}), meaning the ***total cross-sectional area per unit volume***.

$$\boxed{\begin{aligned} \alpha_\nu &= n\sigma_\nu \quad [\mathrm{cm}^{-1}] \\ &= \rho\kappa_\nu \end{aligned}}$$

where ρ is the mass density and κ_ν is called the **mass absorption coefficient** or the **opacity coefficient**.

Emission + Absorption

- ***Radiative transfer equation*** with both absorption and emission is

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

absorption emission

  - We can rewrite the radiative transfer equation using the optical depth as a measure of \`distance' rather than s:

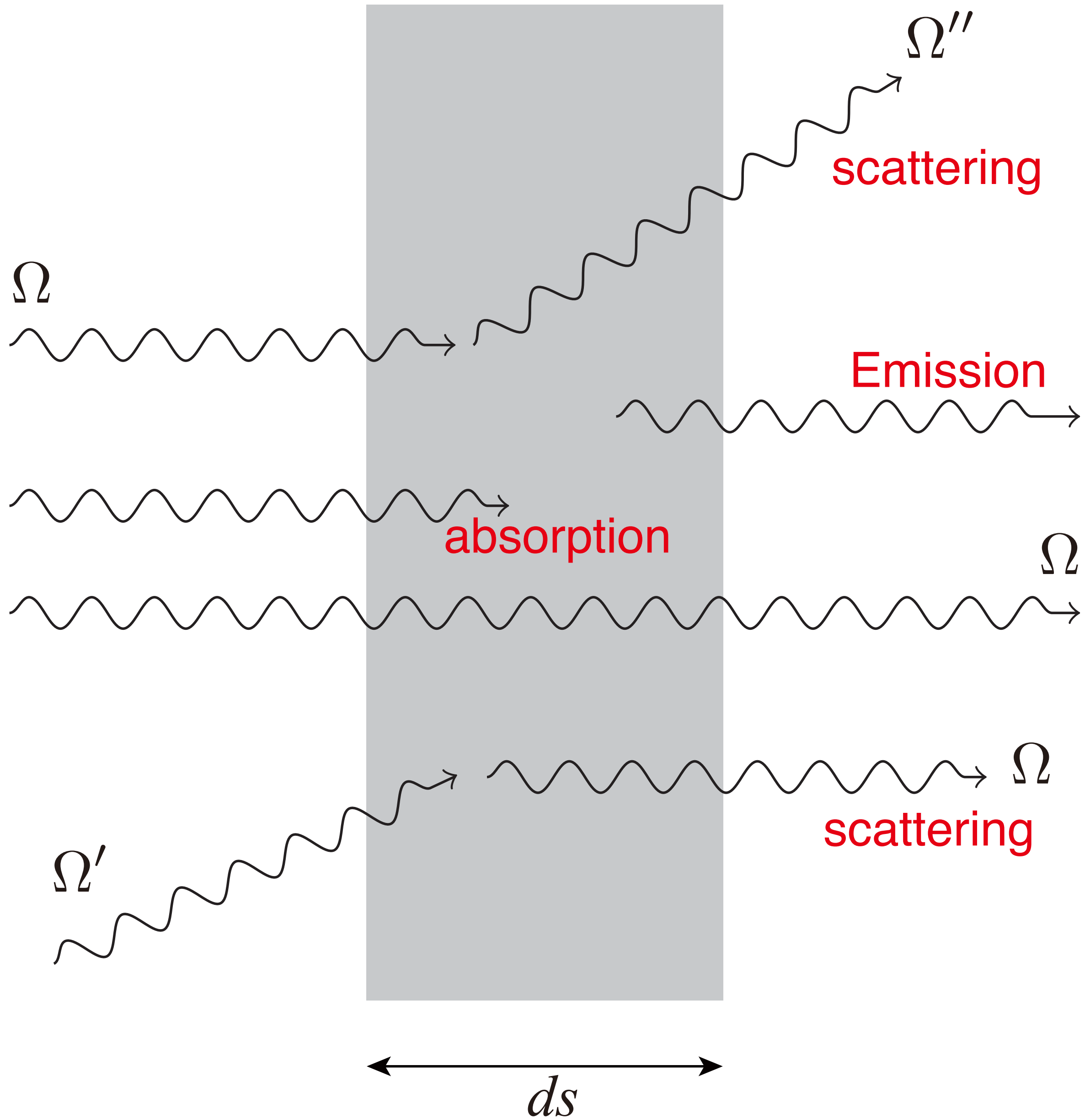
$$\frac{dI_\nu}{\alpha_\nu ds} = -I_\nu + \frac{j_\nu}{\kappa_\nu}$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

  - where $S_\nu \equiv j_\nu/\alpha_\nu$ ***is called the source function***. This is an alternative and sometimes more convenient way to write the equation.

Emission + Absorption + Scattering

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\rm ext} I_\nu + j_\nu + \underbrace{\alpha_\nu^{\rm scatt} \int \Phi_\nu(\Omega' \to \Omega) I_\nu(\Omega') d\Omega'}_{\Rightarrow\ j_\nu^{\rm scatt}}$$



- extinction cross section

$$\sigma_\nu^{\rm ext} = \sigma_\nu^{\rm abs} + \sigma_\nu^{\rm scatt}$$

- extinction coefficient

$$\alpha_\nu^{\rm ext} = \alpha_\nu^{\rm abs} + \alpha_\nu^{\rm scatt}$$
$$= n\sigma_\nu^{\rm abs} + n\sigma_\nu^{\rm scatt}$$

- scattering phase function

$$\Phi_\nu(\Omega' \to \Omega)$$

$$\int \Phi_\nu(\Omega' \to \Omega) d\Omega = 1$$

$$\int \Phi_\nu(\Omega' \to \Omega) d\Omega' = 1$$

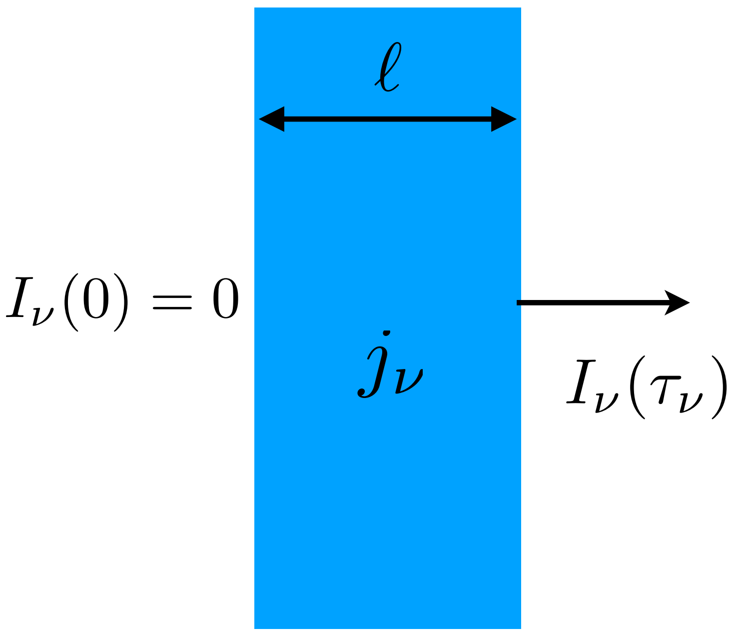
Solution: Emission Only

- For pure emission, $\alpha_\nu = 0$

$$\frac{dI_\nu}{ds} = j_\nu$$

$$\boxed{I_\nu(s) = I_\nu(0) + \int_0^s j_\nu(s')ds'}$$

  - The brightness increase is equal to the emission coefficient integrated along the line of sight.



$$\boxed{\begin{array}{l} I_\nu = j_\nu \ell \\ \qquad \text{if } I_\nu(0) = 0 \text{ and } j_\nu = \text{constant} \end{array}}$$

Solution: Absorption Only

- Pure absorption: $j_\nu = 0$

  Rearranging the previous equation, we obtain the equation of radiative transfer for pure absorption:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

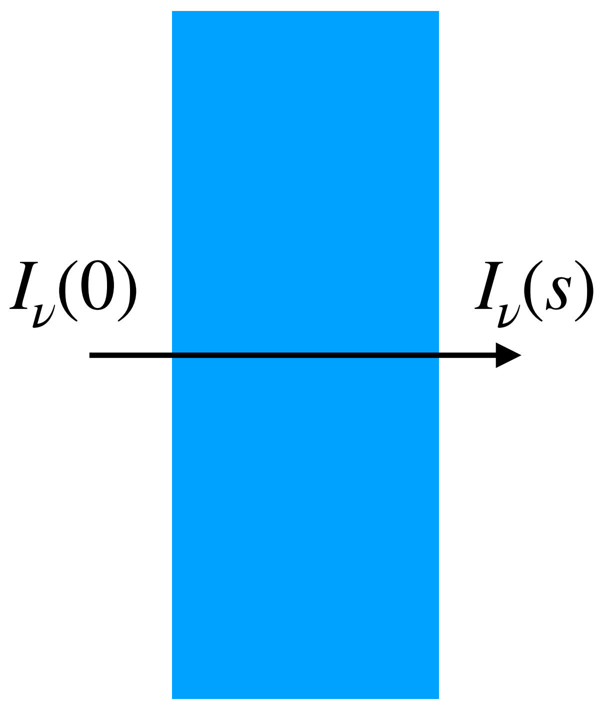
  - The amount of reduced energy depends on how much radiation we already have.
  - Integrate to find how radiation changes along path:

$$\int_0^s \frac{dI_\nu}{I_\nu} = -\int_0^s \alpha_\nu(s')ds'$$

$$[\ln I_\nu]_0^s = -\int_0^s \alpha_\nu(s')ds'$$

$$I_\nu(s) = I_\nu(0)\exp\left[-\int_0^s \alpha_\nu(s')ds'\right]$$

$$I_\nu(s) = I_\nu(0)\exp\left[-\int_0^s \alpha_\nu(s')ds'\right]$$



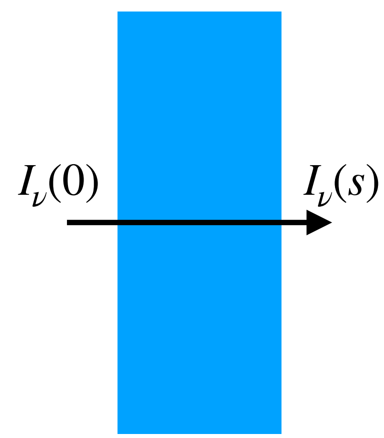
- The brightness decreases along the ray by the exponential of the absorption coefficient integrated along the line of sight.

- ***Optical depth***:

  Imagine radiation traveling into a cloud of absorbing gas, the exponential defines a scale over which radiation is attenuated.

  We define the optical depth τ_ν as:

$$\tau_\nu(s) = \int_0^s \alpha_\nu(s')ds' \quad \text{or} \quad d\tau_\nu = \alpha_\nu ds$$



Transmitted Light

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu}$$

Absorbed Light

$$I_\nu^{\mathrm{abs}}(\tau_\nu) = I_\nu(0)(1 - e^{-\tau_\nu})$$

- A medium is said to be ***optically thick*** at a frequency ν if the optical depth for a typical path through the medium satisfies:

$$\tau_\nu(s) > 1 \qquad I_\nu(\tau_\nu) \rightarrow 0 \qquad I_\nu^{\mathrm{abs}}(\tau_\nu) \rightarrow I_\nu(0)$$

- The medium is ***optically thin*** if, instead:

$$\tau_\nu(s) < 1 \qquad I_\nu(\tau_\nu) \rightarrow I_\nu(0) \qquad I_\nu^{\mathrm{abs}}(\tau_\nu) \rightarrow 0$$

An optically thin medium is one which a typical photon of frequency ν can pass through without being (significantly) absorbed.

- If the absorption coefficient is a constant (example: a uniform density gas of ionized hydrogen), then we obtain

$$I_\nu(s) = I_\nu(0)e^{-\alpha_\nu s}$$

specific intensity after distance s

initial intensity at $s = 0$.

radiation exponentially absorbed with distance

- ***Attenuation***
  - Imagine radiation traveling into a cloud of absorbing gas, exponential defines a scale over which radiation is attenuated.
  - When $\int_0^s \alpha_\nu(s')ds' = 1$, the intensity will be reduced to $1/e$ of its original value.

Mean Free Path

- From the exponential absorption law, the ***probability of a photon absorbed*** between optical depths τ_ν and $\tau_\nu + d\tau_\nu$ is

$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu}$

probability = $\boxed{\dfrac{|dI_\nu|}{I_\nu(0)} = \left|\dfrac{dI_\nu}{\tau_\nu}\right| d\tau_\nu = e^{-\tau_\nu} d\tau_\nu}$ $\rightarrow$ $\boxed{P(\tau_\nu) = e^{-\tau_\nu}}$

= **probability density function** for the absorption at an optical depth τ_ν

- The mean optical depth traveled is thus equal to unity:

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu P(\tau_\nu) d\tau_\nu = \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1$$

- ***The mean free path is defined as the average distance a photon can travel through an absorbing material until it is absorbed.*** In a homogeneous medium, the mean free path is determined by

$$\langle \tau_\nu \rangle = \alpha_\nu \ell_\nu = 1 \quad \rightarrow \quad \boxed{\ell_\nu = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}}$$

- A local mean path at a point in an inhomogeneous material can be also defined.

- The probability of a photon being absorbed within an optical depth τ_ν is

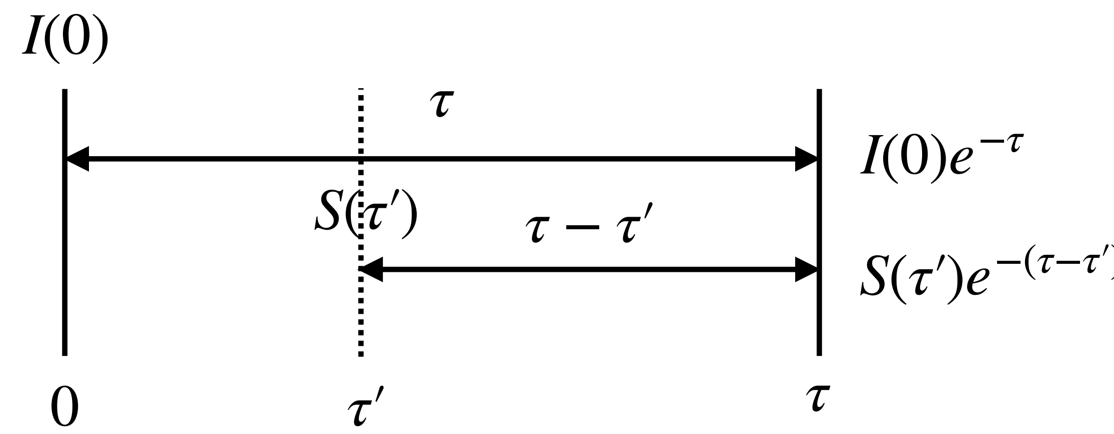
$$\int_0^{\tau_\nu} P(\tau'_\nu) d\tau'_\nu = 1 - e^{-\tau_\nu}$$

Formal Solution of the RT equation

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

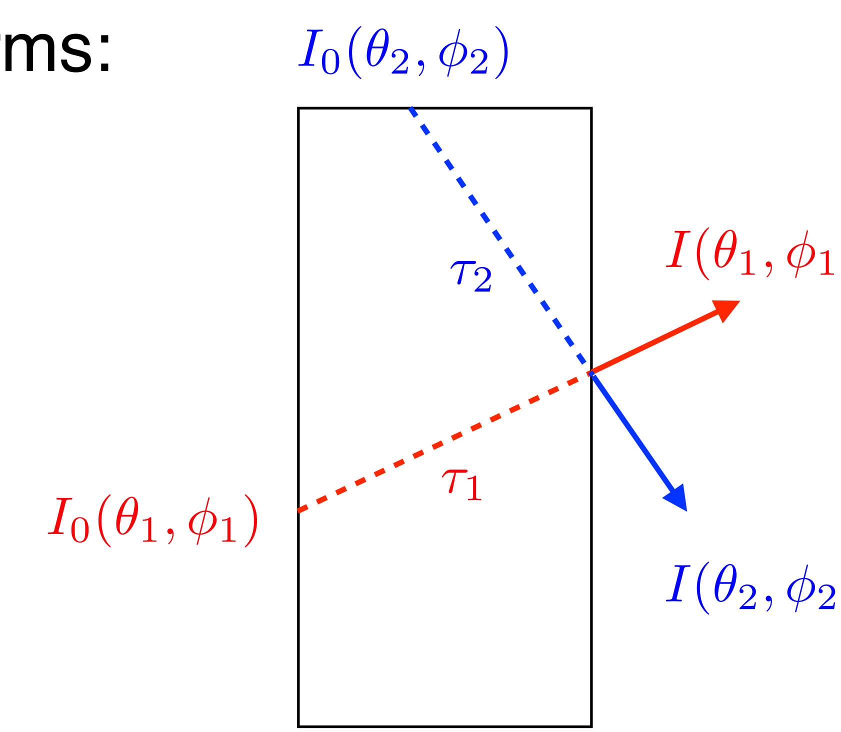
$$e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

$$\frac{d}{d\tau_\nu}\left(e^{\tau_\nu} I_\nu\right) = e^{\tau}_{\nu} S_\nu$$



$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu)d\tau'_\nu$$

- The solution is easily interpreted as the sum of two terms:
  - the initial intensity diminished by absorption
  - the integrated source diminished by absorption.
- For a constant source function, the solution becomes



$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right) \end{aligned}$$

Relaxation

- "Relaxation"

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$I_\nu > S_\nu \quad \rightarrow \quad \frac{dI_\nu}{d\tau_\nu} < 0, \quad \text{then } I_\nu \text{ tends to decrease along the ray}$$

$$I_\nu < S_\nu \quad \rightarrow \quad \frac{dI_\nu}{d\tau_\nu} > 0, \quad \text{then } I_\nu \text{ tends to increase along the ray}$$

- ***The source function is the quantity that the specific intensity tries to approach***, and does approach if given sufficient optical depth.

$$\text{As} \;\; \tau_\nu \rightarrow \infty, \quad I_\nu \rightarrow S_\nu$$

Thermal equilibrium

- In general, equilibrium means a state of balance.
  - ***Thermal equilibrium* refers to steady states of temperature, which defines the average energy of material or photons.**
  - In a state of (complete) ***thermodynamic equilibrium (TE)***, no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system. ***In TE, matter and radiation are in equilibrium at the same temperature T.***
  - If the material is (locally) in thermodynamic equilibrium at a well-defined temperature *T*, ***it is said to be in local thermodynamic equilibrium (LTE) even if it is not in equilibrium with the radiation field.***
- ***Note that thermal equilibrium differ from thermodynamic equilibrium.***

The state of LTE

- Macroscopically, LTE is characterized by the following three equilibrium distributions:
  - **Maxwellian velocity distribution** of particles, written here in terms of distribution for the absolute values of velocity,

$$f(v)dv = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) 4\pi v^2 dv$$

    where m is the particle mass and k the Boltzmann constant.

  - **Boltzmann excitation equation**,

$$\frac{n_i}{N_I} = \frac{g_i}{U_I} e^{-E_i/kT}$$

    where n_i is the population of level i, g_i is its statistical weight, and E_i is the level energy, measured from the ground state; N_I and U_I are the total number density and the partition function of the ionization state I to which level i belongs, respectively.

  - **Saha ionization equation**,

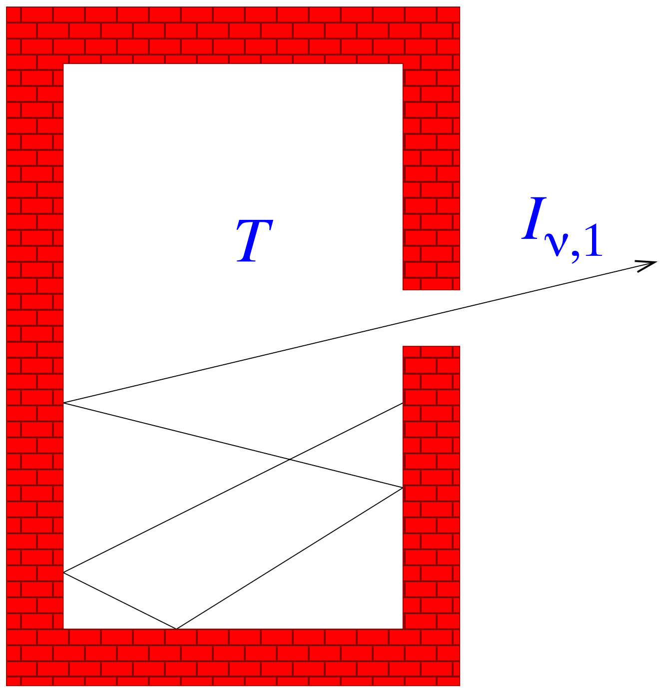
$$\frac{N_I}{N_{I+1}} = n_e \frac{U_I}{U_{I+1}} \left(\frac{h^2}{2\pi m_e kT}\right)^{3/2} e^{\chi_I/kT}$$

    where χ_I is the ionization potential of ion I.

- Microscopically, LTE holds if all atomic processes are in detailed balance, i.e., if the number of processes $A \rightarrow B$ is exactly balanced by the number of inverse processes $B \rightarrow A$.

Blackbody

- Imagine a container bounded by opaque walls with a very small hole.
  - ***Photons will be scattered and absorbed many times, (and eventually trapped and completely absorbed in the box).*** Under such conditions, the particles and photons continually share their kinetic energies. In perfect thermal equilibrium, ***the average particle kinetic energy will be equal to the average photon energy, and a unique temperature T can be defined.***



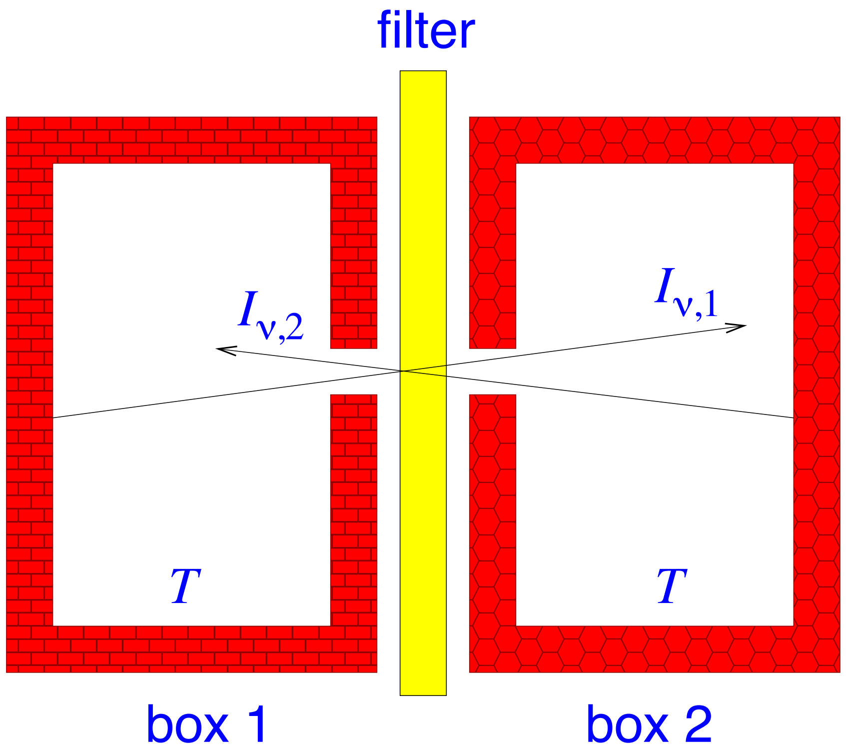
box 1

  - A **blackbody** is an idealized physical body that absorbs all incident radiation regardless of frequency or angle of incidence (i.e., perfect absorber). The above cavity can be regarded to be a blackbody.

  - Radiation from a blackbody in thermal equilibrium is called the **blackbody radiation**.

Blackbody radiation is the universal function.

- ***Now, consider another cavity (box 2), also at the same temperature***, but made of different material or shape and connect two cavities with a filter transparent only in the narrow frequency range ν and $\nu + d\nu$.
  - ***In equilibrium at T, radiation should transfer no net energy from one cavity to the other.*** Otherwise, one cavity will cool down and the other heats up; this violates the second law of thermodynamics.
  - Therefore, the intensity or spectrum that passes through the holes should be a universal function of *T* and should be isotropic.
  - The intensity and spectrum of the radiation emerging from the hole should be independent of the wall material (e.g., wood, copper, or concrete, etc) and any absorbing material that may be inside the cavity.



  - The universal function is called the Planck function $B_\nu(T)$.
  - This is the blackbody radiation.

Kirchhoff's Law in TE

- In (full) thermodynamic equilibrium at temperature *T*, by definition, we know that

$$\frac{dI_\nu}{ds} = 0 \quad \text{and} \quad I_\nu = B_\nu(T)$$

- We also note that

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

- Then, we can obtain ***the Kirchhoff's law for a system in TE***:

$$\boxed{\frac{j_\nu(T)}{\alpha_\nu(T)} = B_\nu(T)}$$

- This is remarkable because it connects the properties $j_\nu(T)$ and $\alpha_\nu(T)$ of any kind of matter to the single universal spectrum $B_\nu(T)$.

Kirchhoff’s Law in LTE

- Recall that Kirchhoff’s law was derived for a system in thermodynamic equilibrium.

- ***Kirchhoff’s law applies not only in TE but also in LTE***:
  - Recall that $B_\nu(T)$ is independent of the properties of the radiating /absorbing material.
  - In contrast, both $j_\nu(T)$ and $\alpha_\nu(T)$ depend only on the materials in the cavity and on the temperature of that material; they do not depend on the ambient radiation field or its spectrum.
  - Therefore, the Kirchhoff’s law should be true even for the case of LTE.
  - ***In LTE, the ratio of the emission coefficient to the absorption coefficient is a function of temperature and frequency only.***
  - This generalized version of Kirchhoff’s law is an exceptionally valuable tool for calculating the emission coefficient from the absorption coefficient or vice versa.

Implications of Kirchhoff's Law

- A good absorber is a good emitter, and a poor absorber is a poor emitter. (In other words, a good reflector must be a poor absorber, and thus a poor emitter.)

$$j_\nu = \alpha_\nu B_\nu(T) \quad \rightarrow \quad j_\nu \text{ increases as } \alpha_\nu \text{ increases}$$

  - It is not possible to thermally radiate more energy than a blackbody, at equilibrium.

$$j_\nu < B_\nu(T) \quad \text{because} \quad \alpha_\nu < 1$$

  - The radiative transfer equation in LTE can be rewritten:

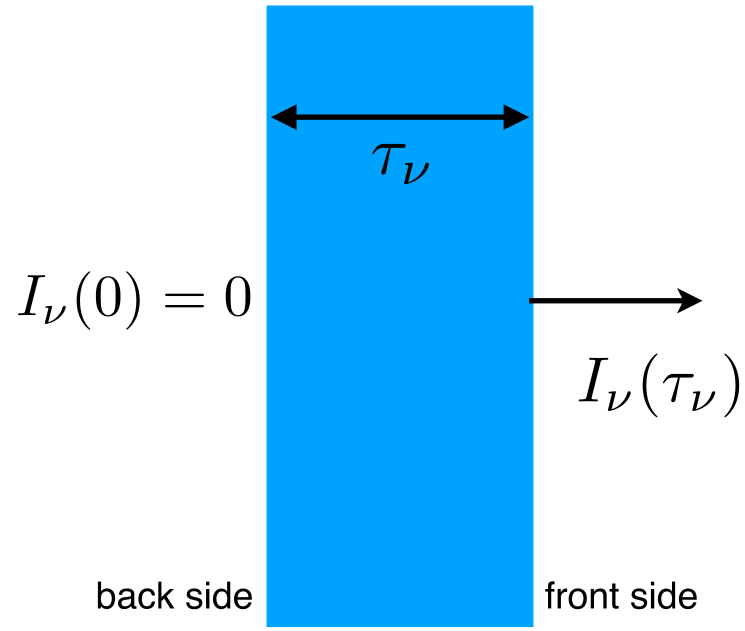
$$\boxed{\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T)}$$

- ***Blackbody radiation vs. Thermal radiation***
  - ***Blackbody radiation*** means $I_\nu = B_\nu(T)$. An object for which the intensity is the Planck function is emitting blackbody radiation.
  - ***Thermal radiation is defined to be radiation emitted by "matter" in LTE***. Thermal radiation means $S_\nu = B_\nu(T)$. An object for which the source function is the Planck function is emitting thermal radiation.
  - ***Thermal radiation becomes blackbody radiation only for optically thick media.***

Blackbody Radiation vs. Thermal Radiation

- To see the difference between thermal and blackbody radiation,
  - Consider a slab of material with optical depth τ_ν that is producing thermal radiation.
  - If no light is falling on the back side of the slab, the intensity that is measured on the front side of the slab is

$$\begin{array}{l} I_\nu(0) = 0 \\ S_\nu = B_\nu \end{array} \longrightarrow \quad \begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right) \\ &= B_\nu\left(1 - e^{-\tau_\nu}\right) \end{aligned}$$



  - If the slab is optical thick at frequency ν $(\tau_\nu \gg 1)$, then

$$\boxed{I_\nu = B_\nu \quad \text{as } \tau_\nu \to \infty}$$

  - If the slab is optically thin $(\tau_\nu \ll 1)$, then

$$\boxed{I_\nu \approx \tau_\nu B_\nu \ll B_\nu \quad \text{as } \tau_\nu \ll 1}$$

This indicates that the radiation, although it is thermal, will not be blackbody radiation.

Thermal radiation becomes blackbody radiation only for optical thick media.

< Planck Spectrum (Spectrum of blackbody radiation) >

- There is no perfect blackbody.
  - However, the cosmic microwave background comes quite close; stars can sometimes be usefully approximated as blackbodies.
  - By the end of the 19th century, the blackbody spectrum was fairly well known empirically, from laboratory studies. In 1900, Max Planck, using his idea of quantized energies, derived the blackbody spectrum.

See Radiative Transfer (Rybicki & Lightman) for the derivation of the Planck function "Fundamentals of Statistical and Thermal Physics" (Federick Reif) or "Astrophysical Concepts" (Harwit) for more details.

Spectrum of blackbody radiation

- The frequency dependence of blackbody radiation is given by the ***Planck function***:

$$B_\nu d\nu = B_\lambda d\lambda$$

$$\boxed{B_\nu(T) = \frac{2h\nu^3/c^2}{\exp\left(h\nu/k_\mathrm{B}T\right) - 1} \quad \text{or} \quad B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp\left(hc/\lambda k_\mathrm{B}T\right) - 1}}$$

$$h = 6.63 \times 10^{-27} \text{ erg s (Planck's constant)}$$

$$k_\mathrm{B} = 1.38 \times 10^{-16} \text{ erg K}^{-1} \text{ (Boltzmann's constant)}$$

- ***Energy density:***

$$u_\nu(T) = \frac{4\pi}{c} B_\nu(T) = \frac{8\pi h\nu^3/c^3}{\exp\left(h\nu/k_\mathrm{B}T\right) - 1}$$

Note that the textbook Ryden's "Interstellar and Intergalactic Medium" use the symbol $\varepsilon_\nu(T)$ to denote the energy density.

- Photon occupation number:
  - The photon occupation number is dimensionless, and is simply **the average number of photons per mode per polarization**.

$$n_\gamma(\nu) = \frac{1}{4\pi\rho_s}\frac{u_\nu}{h\nu} = \frac{\langle E\rangle}{h\nu}$$
$$\rho_s = \frac{2\nu^2}{c^3}$$

$$\boxed{n_\gamma(\nu) = \frac{c^2}{2h\nu^3} I_\nu}$$

  - If the radiation field is a blackbody, the photon occupation number is given by

$$\boxed{n_\gamma(\nu;\, T) = \frac{1}{\exp\left(h\nu/k_{\rm B}T\right) - 1}}$$

Bose-Einstein statistics

Stefan-Boltzmann Law

- Emergent flux is proportional to T^4.

$$F = \pi \int_0^\infty B_\nu(T) d\nu = \pi B(T) \quad \longleftarrow \quad B(T) = \int_0^\infty B_\nu(T) d\nu = \frac{ac}{4\pi} T^4 = \frac{\sigma_{\mathrm{SB}}}{\pi} T_4$$

$$F = \sigma_{\mathrm{SB}} T^4$$

$$\text{Stephan} - \text{Boltzmann constant}: \sigma_{\mathrm{SB}} = \frac{2\pi^5 k_{\mathrm{B}}^4}{15 c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^2 \text{ s}^{-1} \text{ K}^{-4} \text{ sr}^{-1}$$

- Total energy density ***(another form of the Stefan-Boltzmann law)***

$$u = \frac{4\pi}{c} \int_0^\infty B_\nu(T) d\nu = \frac{4\pi}{c} B(T) \qquad u(T) = \left(\frac{T}{3400 \text{ K}} \right)^4 \text{ erg cm}^{-3}$$

$$u = aT^4$$

$$\text{radiation constant}: a \equiv \frac{4\sigma_{\mathrm{SB}}}{c} = 7.57 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$$

Rayleigh-Jeans Law & Wien Law

Rayleigh-Jeans Law
(low-energy limit)

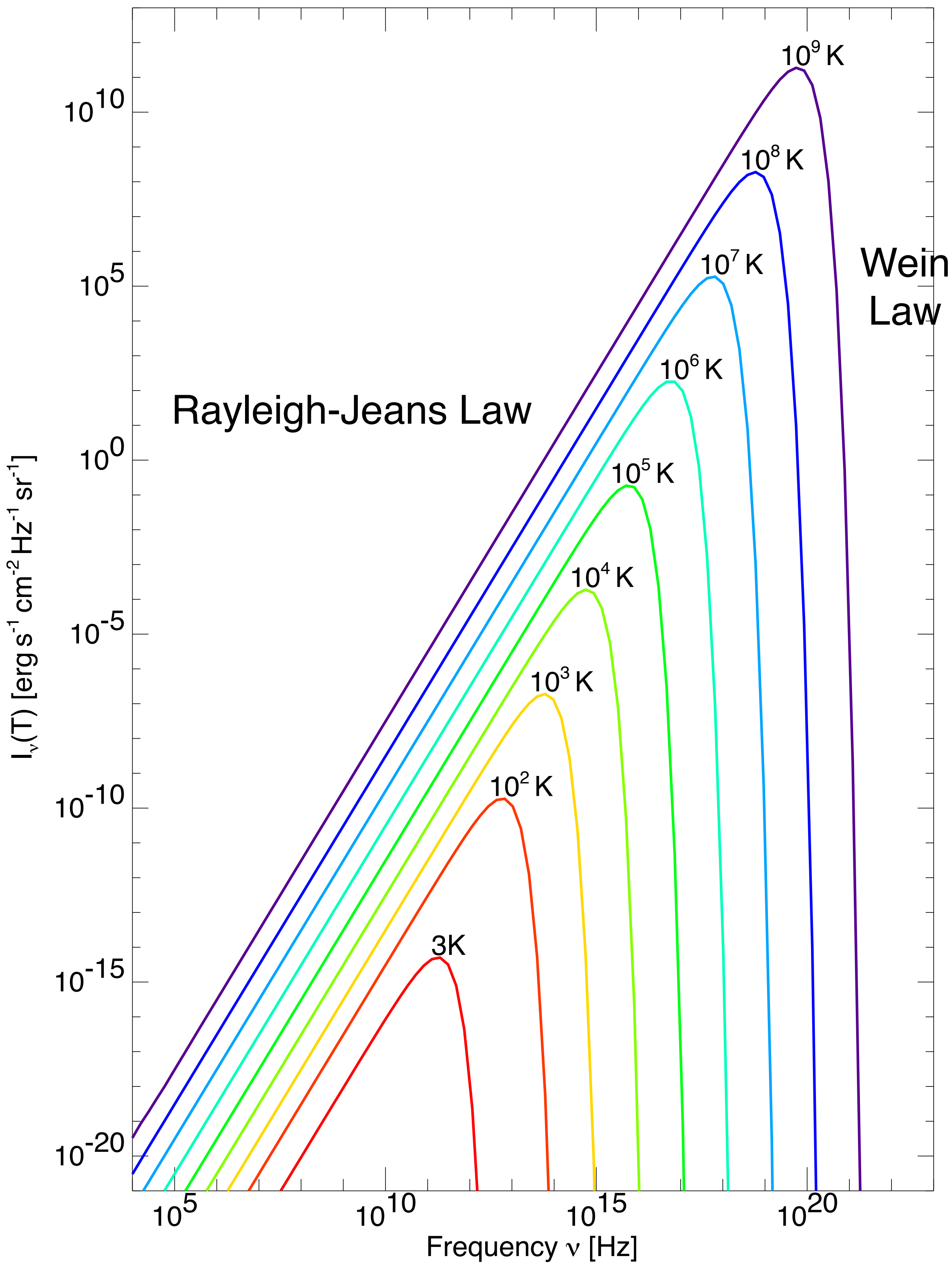
$$h\nu \ll k_{\rm B}T \quad \left(\nu \ll 2\times 10^{10}\,(T/1\,{\rm K})\,{\rm Hz}\right)$$

$$I_\nu^{RJ}(T) = \frac{2\nu^2}{c^2}k_{\rm B}T$$

Wien Law
(high-energy limit)

$$h\nu \gg k_{\rm B}T$$

$$I_\nu^{W}(T) = \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{k_{\rm B}T}\right)$$



Characteristic Temperatures

- ***Brightness Temperature:***
  - The brightness temperature is defined to be the temperature such that a blackbody at that temperature would have specific intensity:

$$I_\nu = B_\nu(T_b) \quad \rightarrow \quad T_b(\nu) = \frac{h\nu/k_\mathrm{B}}{\ln\left[1 + 2h\nu^3/(c^2 I_\nu)\right]}$$

- ***Antenna Temperature:***
  - Radio astronomers are used to working at very low frequencies. They define the antenna temperature as being the brightness temperature in the Rayleigh-Jeans tail.

$$I_\nu = \frac{2\nu^2}{c^2} k_\mathrm{B} T_b \quad \rightarrow \quad T_A \equiv \frac{c^2}{2k_\mathrm{B}\nu^2} I_\nu$$

  - Radiative transfer equation in the RJ limit:
    - If the matter has its energy levels populated according to an excitation temperature $T_\mathrm{exc} \gg h\nu/k_\mathrm{B}$, then the source function is given by $S_\nu(T_\mathrm{exc}) = \left(2\nu^2/c^2\right) k_\mathrm{B} T_\mathrm{exc}$ from the generalized Kirchhoff's law.
    - Then, RT equation becomes

$$\boxed{\frac{dT_A}{d\tau_\nu} = -T_A + T_\mathrm{exc} \quad \text{if } h\nu \ll k_\mathrm{B} T_\mathrm{exc}}$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

$$T_A = T_A(0)e^{-\tau_\nu} + T_\mathrm{exc}(1 - e^{-\tau_\nu}) \quad \text{if } T_\mathrm{exc} \text{ is constant.}$$

- ***Color Temperature:***
  - By fitting the spectrum to a blackbody curve without regarding to vertical scale (absolute intensity scale), a color temperature T_c is obtained.
  - The color temperature correctly gives the temperature of a blackbody source of unknown absolute scale.
- ***Effective Temperature:***
  - The effective temperature of a source is obtained by equating the actual flux F to the flux of a blackbody at temperature $T_{\rm eff}$.

$$F = \int\int I_\nu \cos\theta d\nu d\Omega = \sigma T_{\rm eff}^4$$

Stefan-Boltzmann law

- ***Excitation Temperature:***
  - The excitation temperature of level u relative to level ℓ is defined by

Boltzmann distribution

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{E_{u\ell}}{k_{\rm B} T_{\rm exc}}\right) \quad \rightarrow \quad T_{\rm exc} \equiv \frac{E_{u\ell}/k_{\rm B}}{\ln\left(\frac{n_\ell/g_\ell}{n_u/g_u}\right)} \quad (E_{u\ell} \equiv E_u - E_\ell)$$

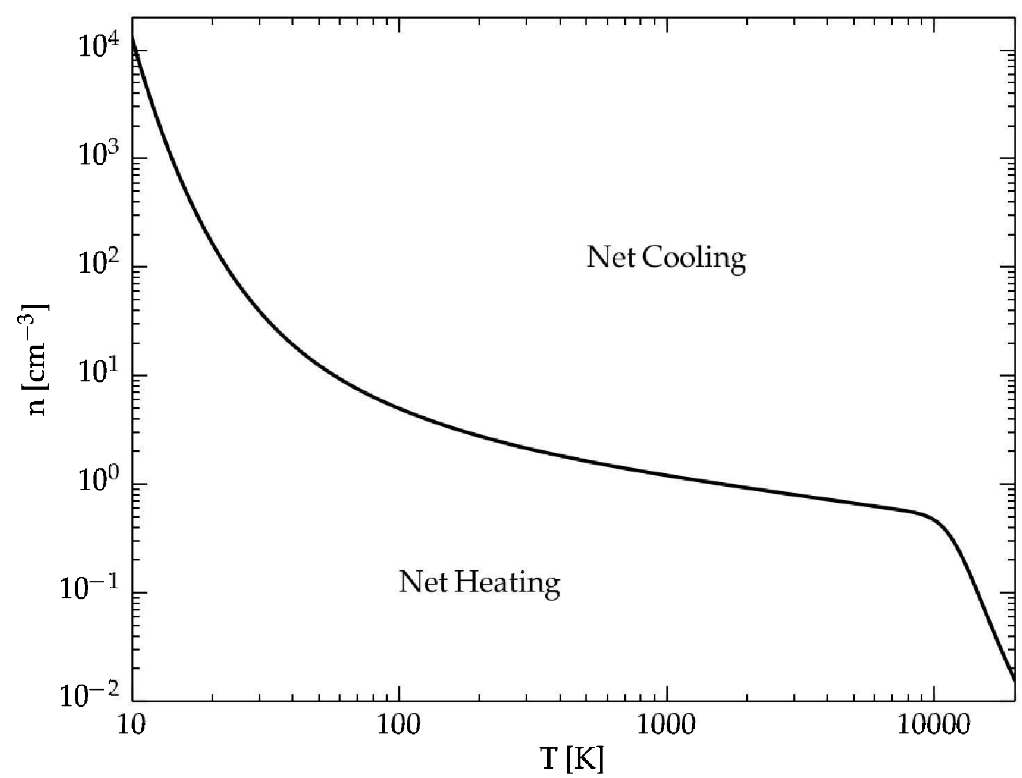
  - Radio astronomers studying the 21 cm line sometimes use the term "spin temperature" $T_{\rm spin}$ for excitation temperature.

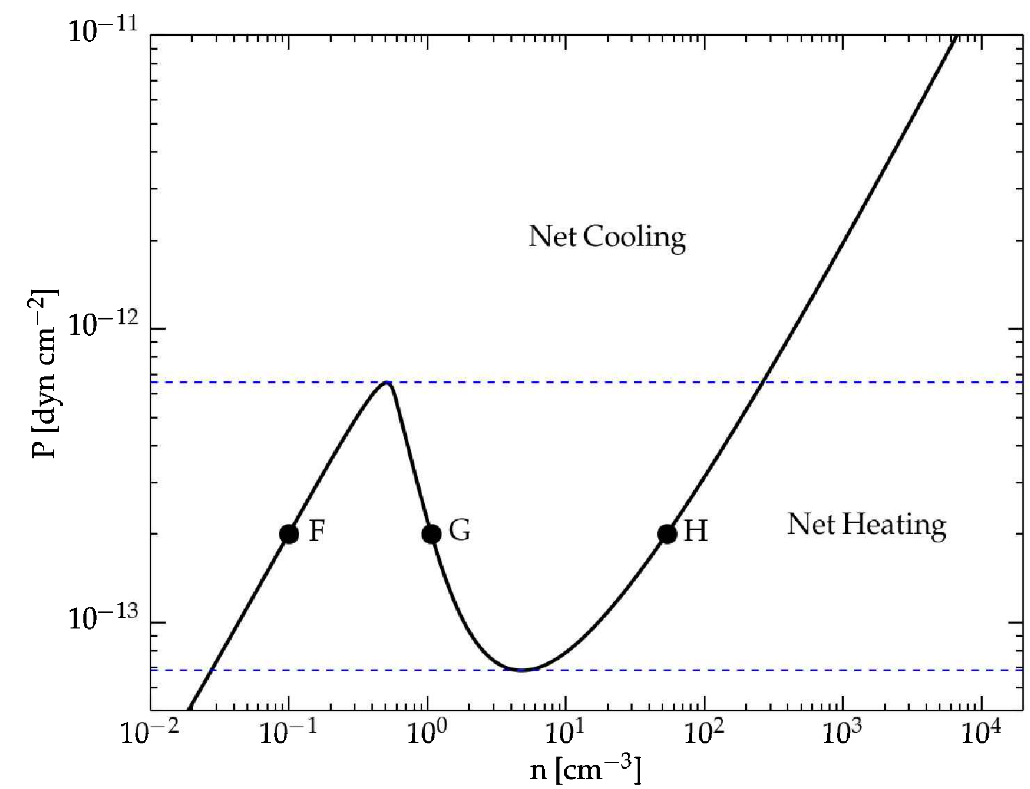
Homework (due date: 03/24)

[Q2] Two stable phases

- Use the formulae for the photoelectric heating rate by dust and the cooling rate by [CII] 158μm, [OI] 63.2μm, and Lyα, described in this lecture note (and textbook).
- Use python or whatever you can use.

1. Reproduce the figures shown below.





2. Make a plot "P versus T," in addition to the above plots.
3. Compute the numerical values of the ***"stable,"*** equilibrium ***densities*** and ***temperatures*** for two pressures $P = 2 \times 10^{-13}$ dyn cm^{-2} and 4×10^{-13} dyn cm^{-2}.

[Q3] Radiative Transfer

Suppose a sphere of gas with a radius R emits a forbidden line, and its emissivity is constant over the sphere, i.e., $j(x, y, z) = j_0$ (erg cm^{-3} s^{-1} sr^{-1}). An observer is located at a distance d from the sphere's center. The observer measures the intensity of the emission toward a direction with an angle θ, as shown in Figure (a). Without the loss of generality, we can assume that the direction vector of the line of sight is in the xy plane. The line of sight intersects the outer boundary of the sphere at two points, a and b.

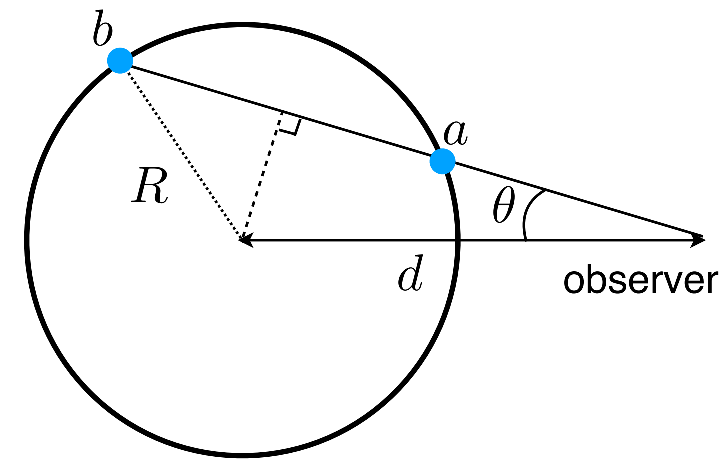


Figure (a)

(1). If we make the assumption that there is an absence of dust, and the emission line is forbidden, then, there will be no absorption. In this case, show that the intensity at the angle θ is given by:

$I(\theta) = j_0 \int_a^b ds = j_0 s_\theta$, where s_θ is the distance between the points a and b.

(2). Show that the distance is $s_\theta = 2\sqrt{R^2 - d^2 \sin^2\theta}$ and therefore the intensity at angle θ is $I(\theta) = 2j_0\sqrt{R^2 - d^2\sin^2\theta}$.

(3). Now, assume the case where $d \to \infty$ and $\theta \to 0$ (see Figure (b)). Then, the intensity at θ can be approximated as the intensity of a parallel ray that passes through the sphere at an impact parameter defined by $p = d\tan\theta$. In this limit, show that $p \approx d\sin\theta$ and $I(p) = 2j_0\sqrt{R^2 - p^2}$.

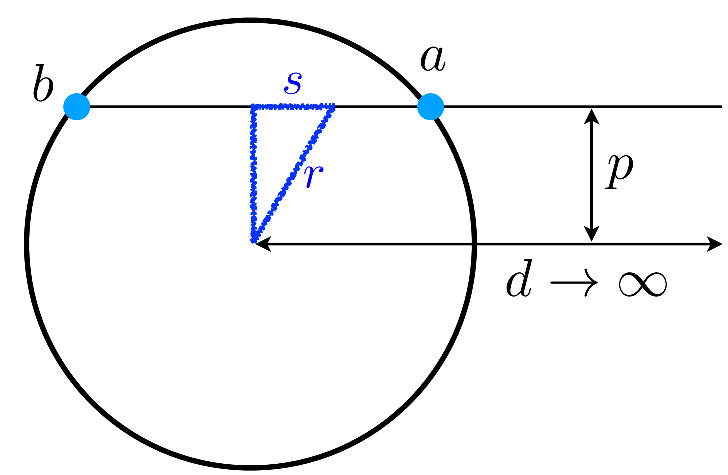


Figure (b)

Assume that the emissivity is a function of the radial distance from the sphere's center r, i.e., $j(r)$.

(4). In the limit of $d \to \infty$ (Figure (b)), show that $I(p) = \int_a^b j(r)ds = 2\int_p^R j(r)\frac{rdr}{\sqrt{r^2 - p^2}}$.

Note that this integral is called the Abel transform of $j(r)$.
(https://en.wikipedia.org/wiki/Abel_transform)

(5). Now, use the equation of (4) and derive the same result $\left(I(p) = 2j_0\sqrt{R^2 - p^2}\right)$ as in (3) if $j(r) = j_0 =$ constant.