KAIST Astrophysics (PH481) - Part 1

Week 1b
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[Question 1] Formation of disk galaxies

- Eggen, Lynden-Bell & Sandage (1962)
  - Approximately 10 Gyrs ago, the protogalaxy started to fall together out of intergalactic material. It was either already rotating or acquired its angular momentum from the couples expected by nearby condensations.
  - As the material fell together, the collapse of the galaxy in the radial direction was eventually stopped by rotation, but that in the Z-direction continued, giving rise to a thin disk.
  - With the increased density, the rate of star formation increased… etc.

- Consider a uniform cloud with a density ρ, rotating with an angular speed ω about z-axis.
- The gravitational and centrifugal forces on a test particle with a mass m are, respectively,

$$\mathbf{F}_{\rm grav} = -\frac{GmM}{R^2}\hat{\mathbf{R}} \qquad \mathbf{F}_{\rm cent} = m\omega^2 r\hat{\mathbf{r}}_\perp$$

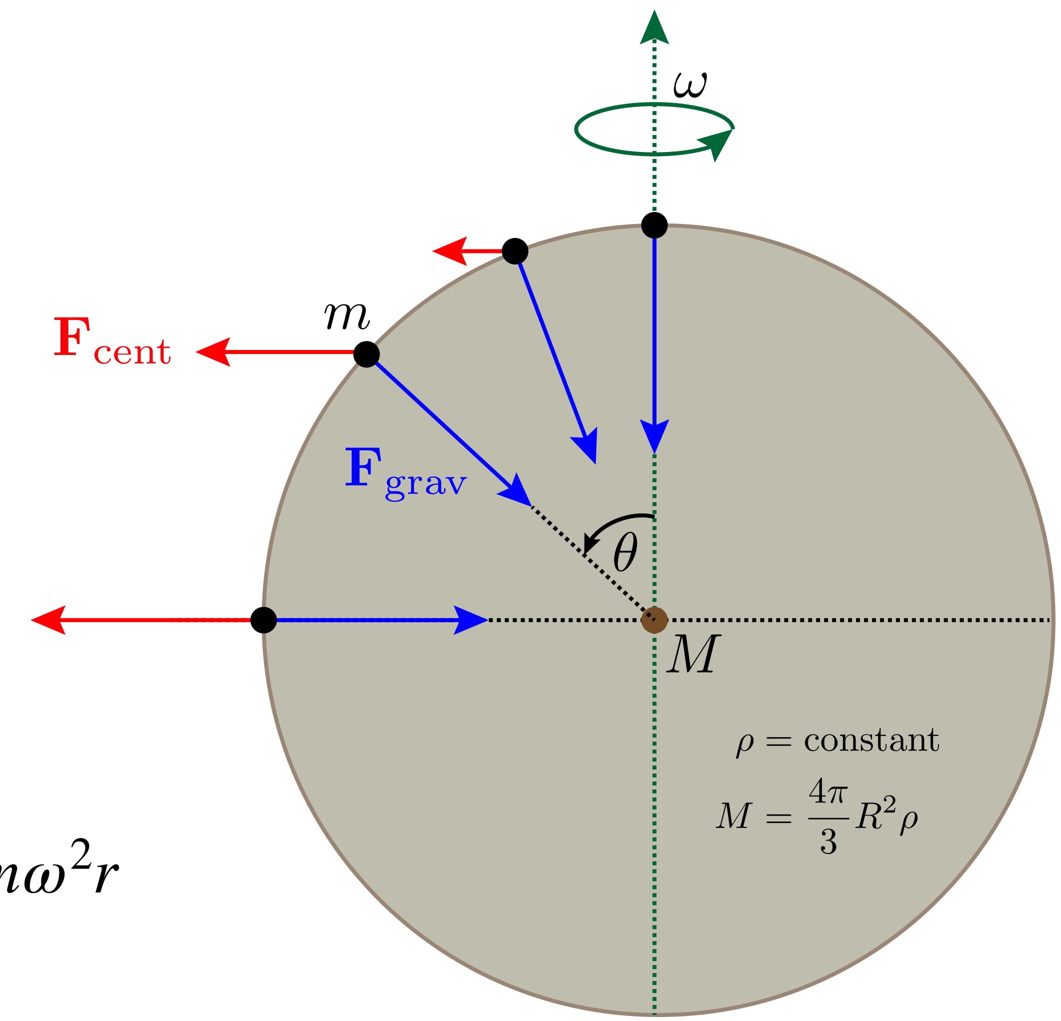
$$\hat{\mathbf{R}} = \cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{r}}_\perp \qquad r = R\sin\theta$$

- If $\left|\mathbf{F}_{\rm grav}\right| = \left|\mathbf{F}_{\rm cent}\right|$ at $\theta = 90^\circ$, we can show that

$$\frac{GmM}{R^2} = m\omega^2 R \longrightarrow \mathbf{F}_{\rm grav}\cdot\hat{\mathbf{r}}_\perp = \frac{GmM}{R^2}\sin\theta = m\omega^2 r = \left|\mathbf{F}_{\rm cent}\right|$$

- This implies that the $\hat{\mathbf{r}}_\perp$ component of the gravitational force component is canceled by the centrifugal force. Only the z-component is left over.
- Then, the disk will be eventually formed.

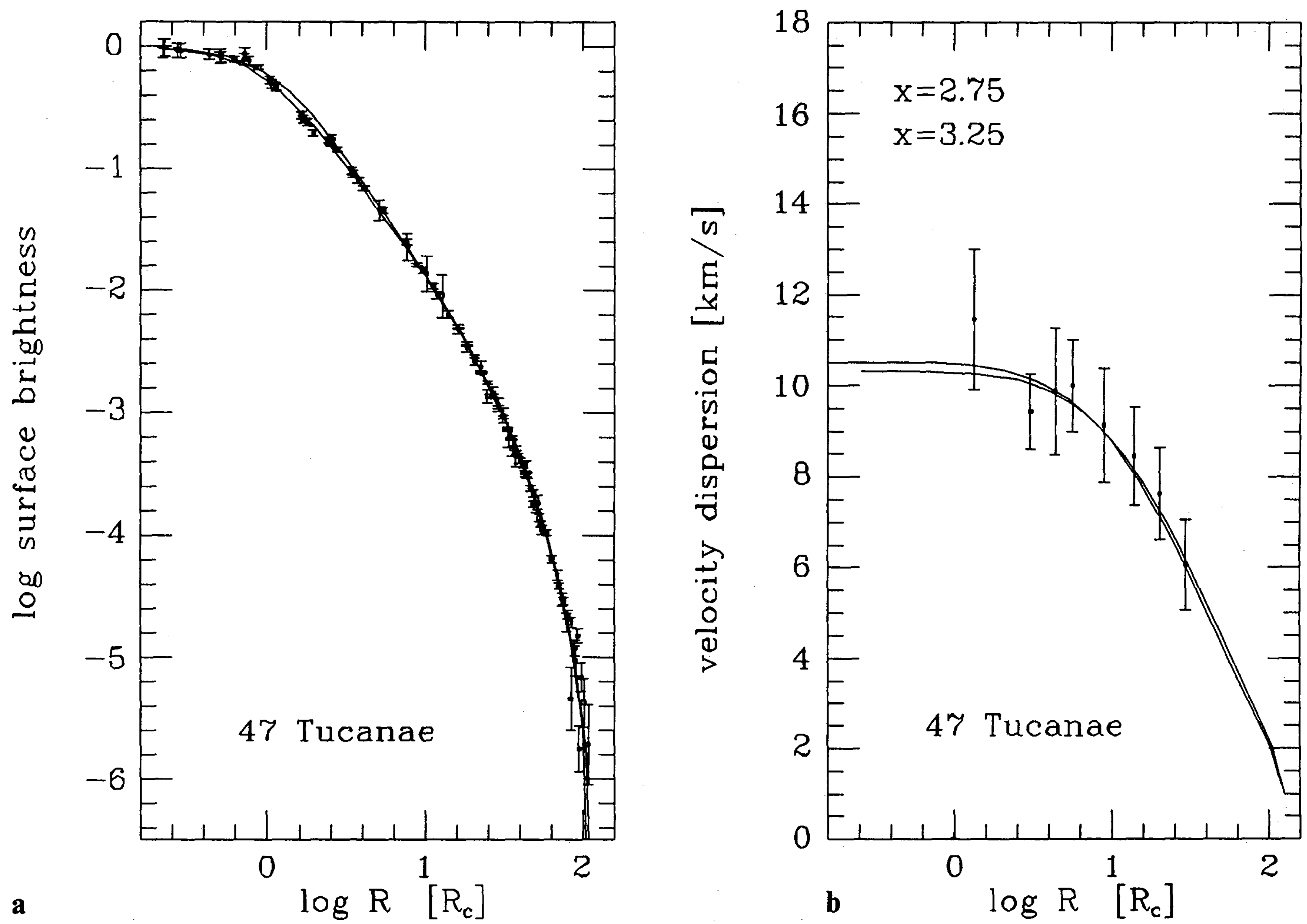
A toy model



[Question 2] No dark matter in globular clusters

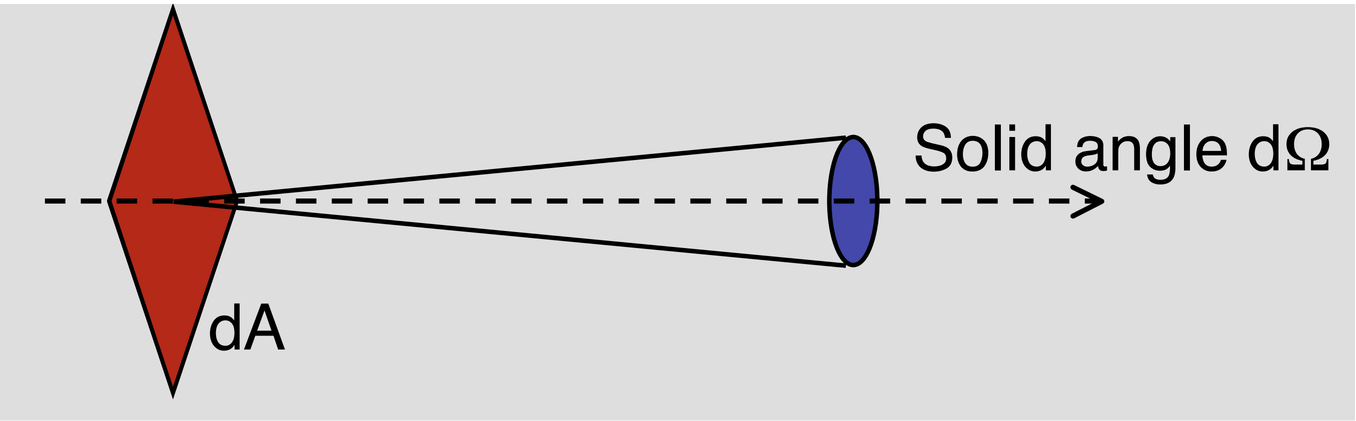
- It is generally assumed that all stars in a GC formed by the collapse and subsequent fragmentation of a single giant molecular cloud. As a consequence, all stars are assumed to have the same age and chemical composition with only negligible spread in these quantities.
- Velocity dispersion: the statistical dispersion of velocities about mean velocity for a group of objects.
  - By measuring the radial velocities of its members, the velocity dispersion of a cluster can be estimated and used to derive the cluster's mass from the viral theorem or the dynamical models.
  - Radial velocity is found by measuring the Doppler width of spectral lines.

- For example in the case of 47 Tuc (Meylan 1989), ***models can be constructed in which much of the inferred mass can be accounted for only by observed stars.*** The total mass of dark matter is less than 30% of the total mass of visible stars.



(Specific) Intensity or (Surface) Brightness

- Recall that ***flux is a measure of the energy carried by all rays passing through a given area***
- ***Intensity is the energy carried along by individual rays.***



$\mathbf{k}$ = direction of propagation

- Let dE_ν be the amount of radiant energy which crosses the area dA in a direction $\mathbf{k}$ within solid angle $d\Omega$ about in a time interval dt with photon frequency between ν and $\nu + d\nu$.
- The monochromatic specific intensity I_ν is then defined by the equation.

$$dE_\nu = I_\nu(\mathbf{k}, \mathbf{x}, t) dA_\mathbf{k} d\Omega d\nu dt$$

- Unit: $\mathrm{erg\ s^{-1}\ cm^{-2}\ sr^{-1}\ Hz^{-1}}$

- Another (more intuitive) name for the specific intensity is ***(surface) brightness***.

Relation between the flux and the specific intensity

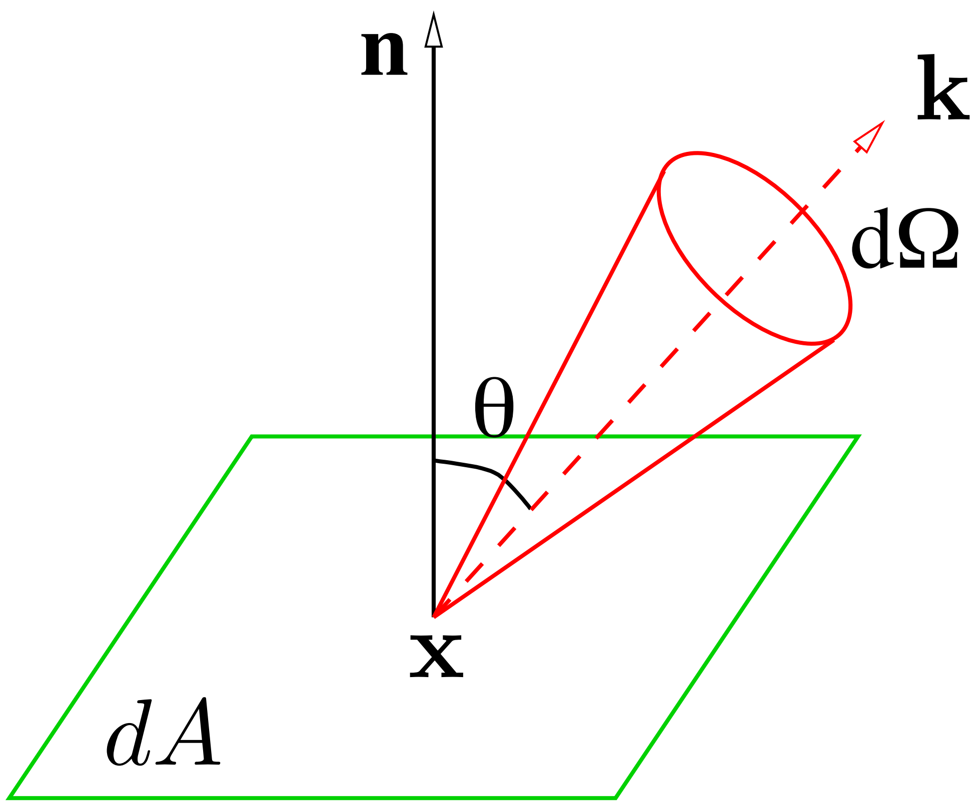
- Let's consider a small area dA, with light rays passing through it at all angles to the normal vector $\mathbf{n}$ of the surface.

- For rays centered about $\mathbf{k}$, the area normal to $\mathbf{k}$ is

$$dA_{\mathbf{k}} = dA\cos\theta$$

  - If $\theta = 90^\circ$, then light rays in that direction contribute zero flux through area dA.

$$F_\nu dA d\nu dt = \int I_\nu(\mathbf{k}, \mathbf{x}, t) dA_{\mathbf{k}} d\Omega d\nu dt$$



- Hence, net flux in the direction of $\mathbf{n}$ is given by integrating over all solid angles:

$$\boxed{F_\nu = \int I_\nu \cos\theta d\Omega = \int_0^{2\pi}\int_0^{\pi} I_\nu \cos\theta \sin\theta d\theta d\phi}$$

Note

- Intensity can be defined as per wavelength interval.

$$I_\nu |d\nu| = I_\lambda |d\lambda| \qquad \leftarrow \left|\frac{d\nu}{d\lambda}\right| = \frac{c}{\lambda^2} = \frac{\nu}{\lambda}$$

$$\nu I_\nu = \lambda I_\lambda$$

- Integrated intensity is defined as the intensity over all frequencies.

$$I = \int_0^\infty I_\nu d\nu = \int_0^\infty I_\lambda d\lambda$$

- In astrophysics, we plot the spectral energy distribution (SED) as νI_ν versus ν or λI_λ versus λ.
  - An example of SED will be shown after the slides for Kirchhoff's law.

How does specific intensity changes along a ray in free space

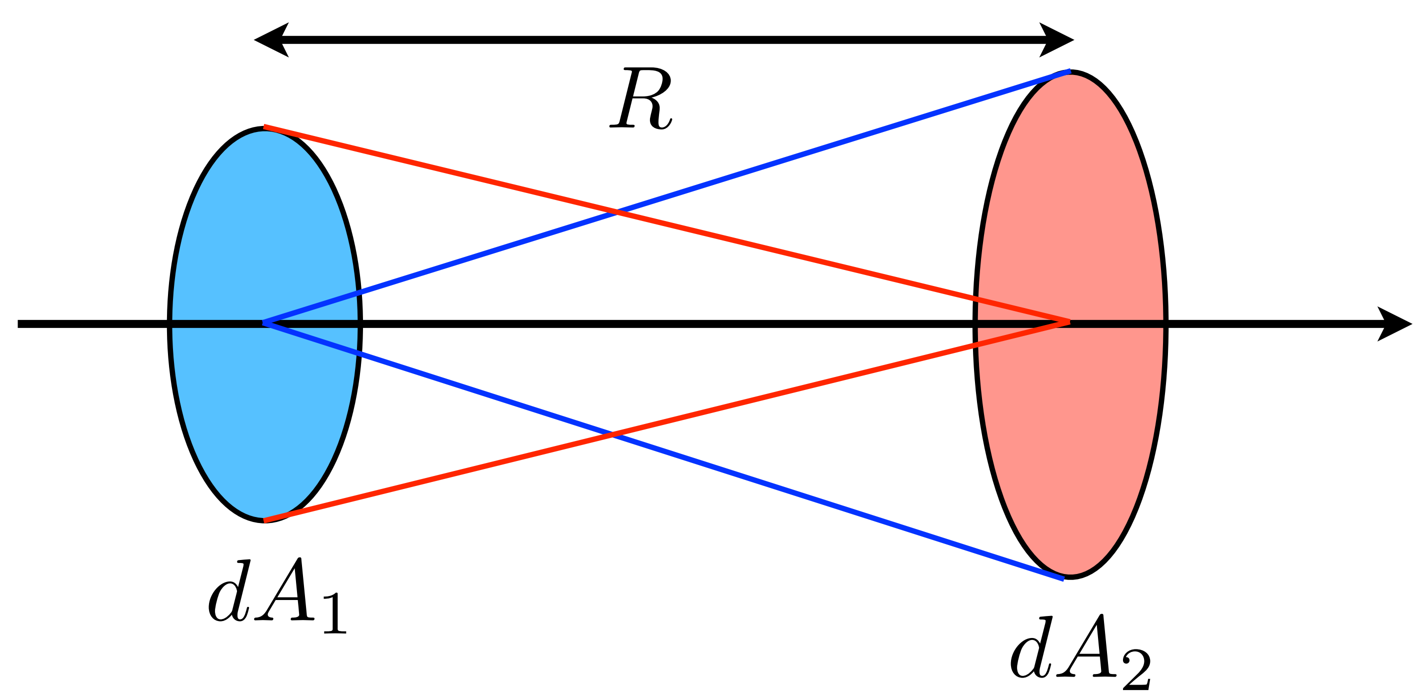
- Consider a bundle of rays and any two points along the rays. Construct areas dA_1 and dA_2 normal to the rays at these points.
  - Consider the energy carried by the rays passing through both areas. Because energy is conserved,

$$dE_1 = I_1 dA_1 d\Omega_1 d\nu dt = dE_2 = I_2 dA_2 d\Omega_2 d\nu dt$$

  - Here, $d\Omega_1$ is the solid angle subtended by dA_2 at the location 1 and so forth.

$$d\Omega_1 = \frac{dA_2}{R^2}$$
$$d\Omega_2 = \frac{dA_1}{R^2}$$

$$\rightarrow \boxed{I_1 = I_2}$$



- Conclusion (the constancy of intensity):
  - the specific intensity remains the same as radiation propagates through free space.

$$I_1 = I_2$$

  - If we measure the distance along a ray by variable s, we can express the result equivalently in differential form:

$$\boxed{\frac{dI}{ds} = 0}$$

- But, this looks a bit strange. Is this property consistent with the inverse square law for the flux?

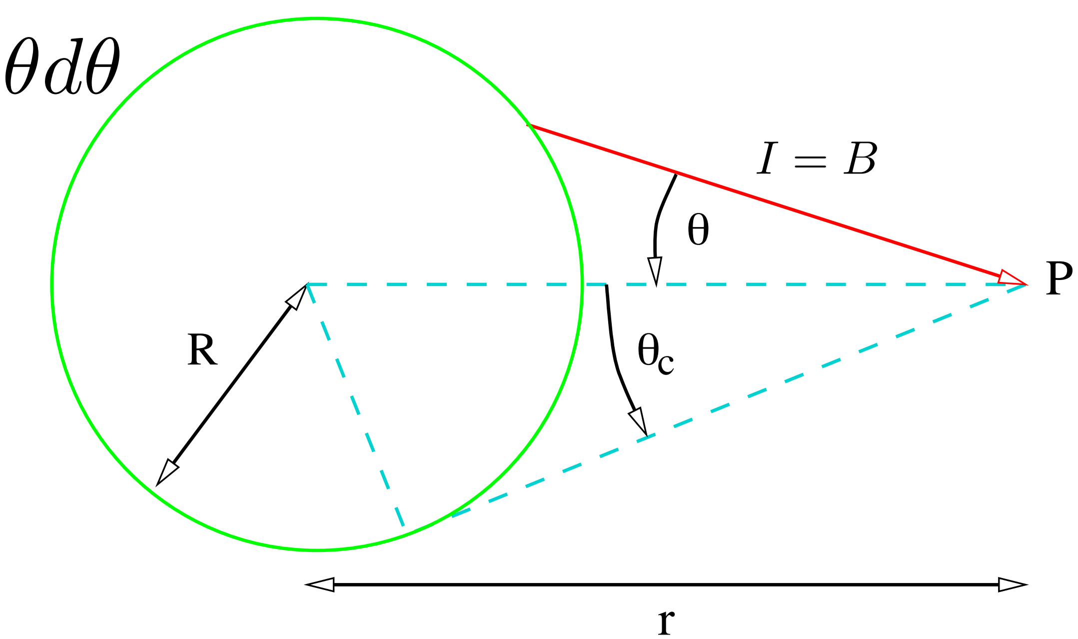
Inverse Square Law for a Uniformly Bright Sphere

- How can we explain the inverse square law from the point of intensity?
  - Let's calculate the flux at P from a sphere of uniform brightness B.

$$F = \int I \cos\theta d\Omega = B \int_0^{\pi} d\phi \int_0^{\theta_c} \cos\theta \sin\theta d\theta$$

$$= \pi B (1 - \cos^2\theta_c) = \pi B \sin^2\theta_c$$

$$\sin\theta_c = \frac{R}{r} \rightarrow \boxed{F = \pi B \left(\frac{R}{r}\right)^2}$$



  - Therefore, there is ***no conflict between the constancy of intensity and the inverse square law***.
- Note
  - The flux at a surface of uniform brightness B is $F = \pi B$.
  - For stellar atmosphere, the ***astrophysical flux*** is defined by F/π .

(Specific) Energy Density

- Consider a bundle of rays passing through a volume element dV in a direction Ω .
  - Then, the energy density per unit solid angle is defined by

$$dE = u_\nu(\Omega) dV d\Omega d\nu$$

  - Since radiation travels at velocity c, the volume element is

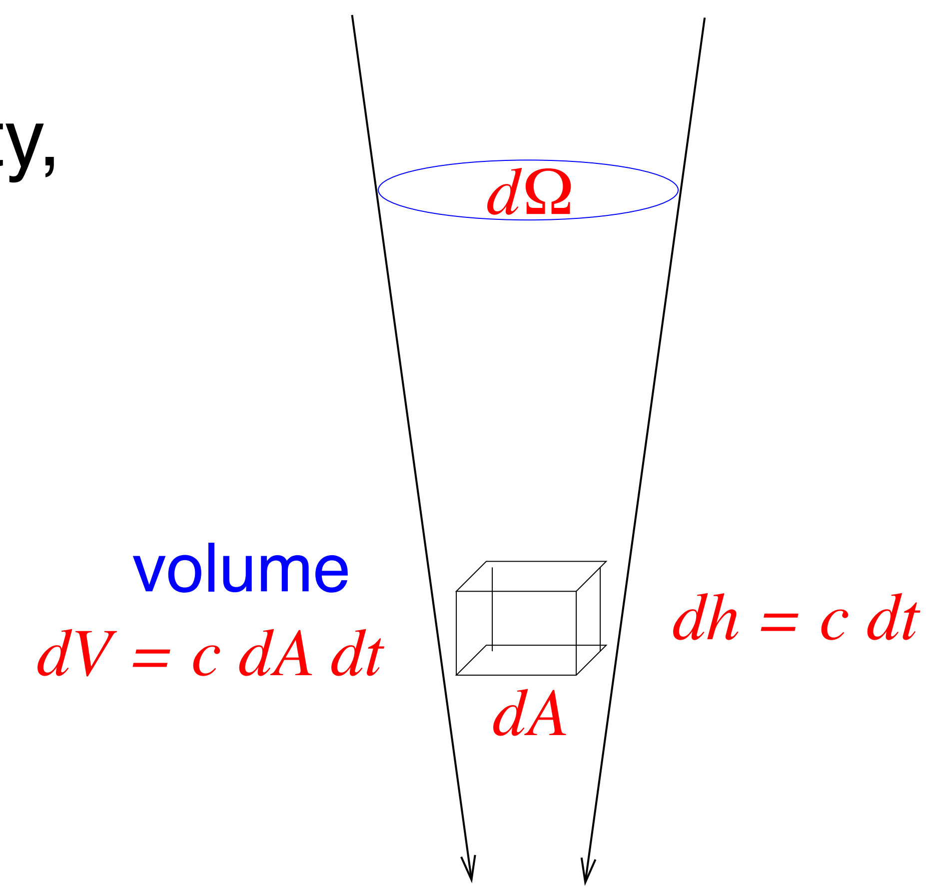
$$dV = dA(cdt)$$

  - According to the definition of the intensity,

$$dE = I_\nu dA dt d\Omega d\nu$$

  - Then, we have

$$\boxed{u_\nu(\Omega) = I_\nu(\Omega)/c}$$



Energy Density and Mean Intensity

- Integrating over all solid angle, we obtain

$$u_\nu = \int u_\nu(\Omega)d\Omega = \frac{1}{c}\int I_\nu d\Omega$$

- ***Mean intensity*** is defined by

$$\boxed{J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega}$$

- Then, the energy density is

$$u_\nu = \frac{4\pi}{c}J_\nu$$

- Total energy density is obtained by integrating over all frequencies.

$$u = \int u_\nu d\nu = \frac{4\pi}{c}\int J_\nu d\nu$$

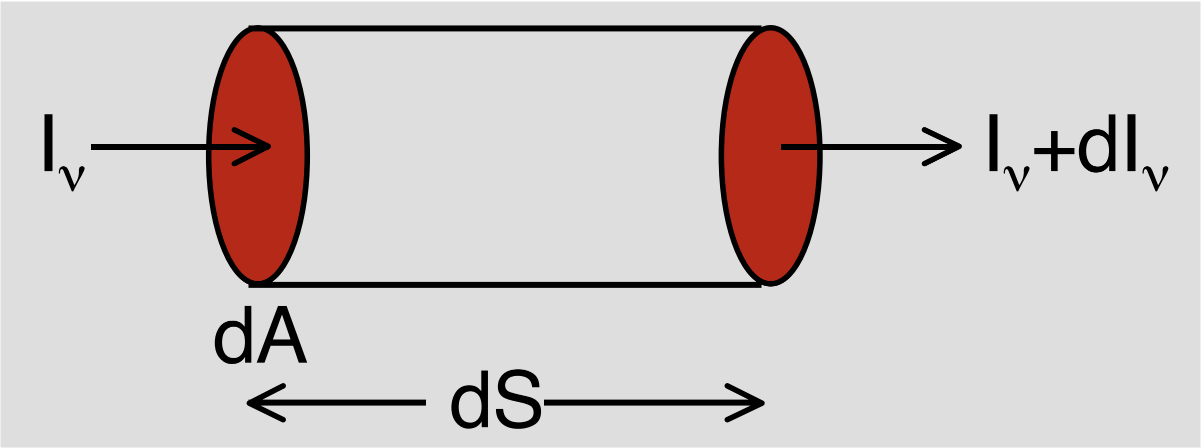
Radiative Transfer Equation

- In reality, as a ray passes through matter, energy may be added, subtracted, or scattered from it by emission, absorption, or scattering.

  - The intensity will not in general remain constant.

  - We need to derive the radiative transfer equation.

- If the radiation travels through a medium which is itself emitting radiation, that will add to the energy:



  - Spontaneous emission coefficient j_ν is the amount of energy emitted per unit time, per unit solid angle, per unit frequency, and per unit volume:

$$dE = j_\nu dV d\Omega dt d\nu \quad (j_\nu : \mathrm{erg\ cm^{-3}\ s^{-1}\ sr^{-1}\ Hz^{-1}})$$

  - In going a distance ds, a beam of cross section dA travels through a volume $dV = dAds$. Thus the intensity added to the beam is by ds is

$$dI_\nu = j_\nu ds \quad \longleftarrow \quad dE = (dI_\nu) dA d\Omega dt d\nu$$

- Therefore, the equation of radiative transfer for pure emission becomes:

$$\frac{dI_\nu}{ds} = j_\nu$$

  - If we know what j_ν is, we can integrate this equation to find the change in specific intensity as radiation propagates through the medium:

$$I_\nu(s) = I_\nu(0) + \int_0^s j_\nu(s')ds'$$

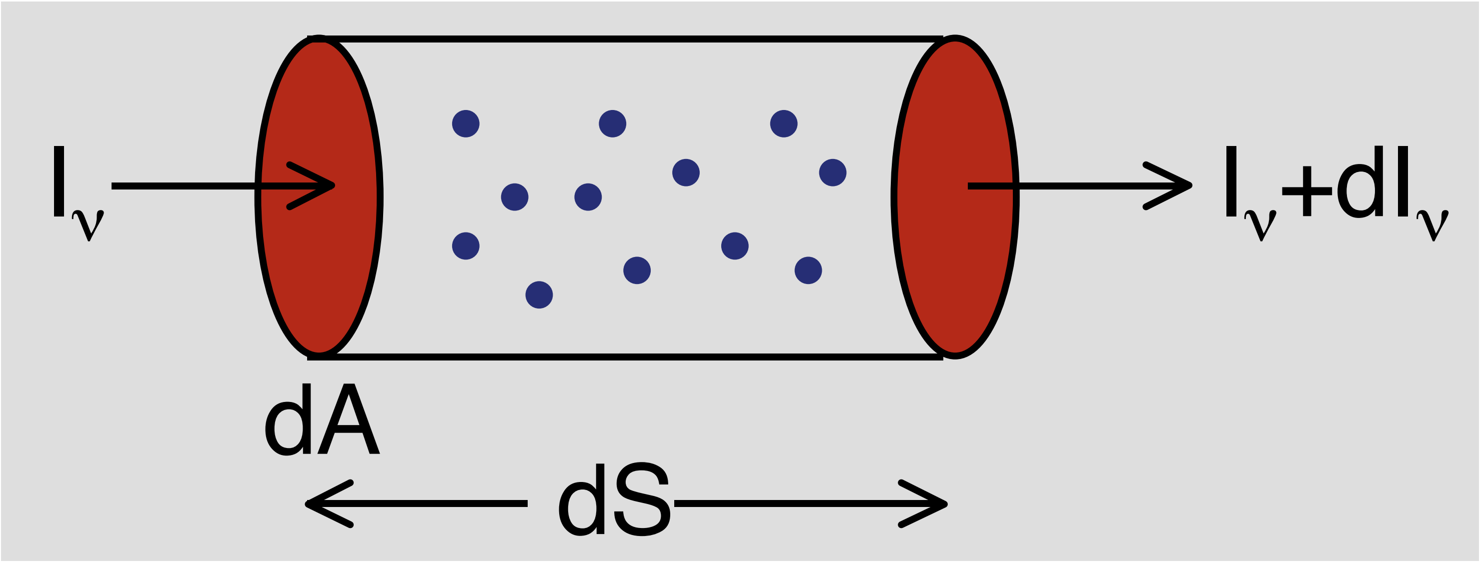
- We also define (angle-averaged, mass) emissivity ϵ_ν to be the energy emitted spontaneously per unit frequency, per unit time, and per unit mass. For isotropic emission,

$$dE = \epsilon_\nu \rho dV dt d\nu \frac{d\Omega}{4\pi}$$

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi} \quad \text{or} \quad \int j_\nu d\Omega = \epsilon_\nu \rho \qquad (\epsilon_\nu : \mathrm{erg\ g^{-1}\ s^{-1}\ Hz^{-1}})$$

Absorption

- If the radiation travels through a medium which absorbs radiation, the energy in the beam will be reduced:



  - Let n denote the number density of absorbers (particles per unit volume).
  - Assume that each absorber has cross-sectional area = σ_ν (in units of cm^2).
  - If a beam travels through ds, total area of absorbers is

$$\text{number of absorbers} \times \text{cross} - \text{section} = n \times dA \times ds \times \sigma_\nu$$

Fraction of radiation absorbed = fraction of area blocked:

$$\frac{dI_\nu}{I_\nu} = -\frac{ndAds\sigma_\nu}{dA} = -n\sigma_\nu ds$$

$$dI_\nu = -n\sigma_\nu I_\nu ds \equiv -\alpha_\nu I_\nu ds$$

- ***Absorption coefficient*** is defined as $\boxed{\alpha_\nu \equiv n\sigma_\nu \text{ (units: cm}^{-1})}$, meaning the ***total cross-sectional area per unit volume***.
- We can write the absorption coefficient in terms of mass:

$$\alpha_\nu \equiv \rho\kappa_\nu$$

- κ_ν is called the ***mass absorption coefficient*** or the ***opacity***.
- Opacity has units of $\mathrm{cm}^2\ \mathrm{g}^{-1}$ (i.e., the cross-section of a gram of material, gas or dust).

- Rearranging the previous equation, we obtain the equation of radiative transfer for pure absorption:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

  - The amount of reduced energy depends on how much radiation we already have.
  - Integrate to find how radiation changes along path:

$$\int_0^s \frac{dI_\nu}{I_\nu} = -\int_0^s \alpha_\nu(s')ds'$$

$$\left[\ln I_\nu\right]_0^s = -\int_0^s \alpha_\nu(s')ds'$$

$$I_\nu(s) = I_\nu(0)\exp\left[-\int_0^s \alpha_\nu(s')ds'\right]$$

- If the absorption coefficient is a constant (example: a uniform density gas of ionized hydrogen), then we obtain

$$I_\nu(s) = I_\nu(0)e^{-\alpha_\nu s}$$

specific intensity after distance s

initial intensity at $s = 0$.

radiation exponentially absorbed with distance

- ***Optical depth***:
  - Imagine radiation traveling into a cloud of absorbing gas, exponential defines a scale over which radiation is attenuated.
  - When $\int_0^s \alpha_\nu(s')ds' = 1$, the intensity will be reduced to $1/e$ of its original value.

- We define the optical depth τ_ν as:

$$\tau_\nu(s) = \int_0^s \alpha_\nu(s')ds' \quad \text{or} \quad d\tau_\nu = \alpha_\nu ds$$

- A medium is said to be ***optically thick*** at a frequency ν if the optical depth for a typical path through the medium satisfies:

$$\tau_\nu(s) > 1$$

- The medium is ***optically thin*** if, instead:

$$\tau_\nu(s) < 1$$

- An optically thin medium is one which a typical photon of frequency ν can pass through without being (significantly) absorbed.

Radiative Transfer Equation

- ***Radiative transfer equation*** with both absorption and emission is

$$\frac{dI_\nu}{ds} = -\underbrace{\alpha_\nu I_\nu}_{\text{absorption}} + \underbrace{j_\nu}_{\text{emission}}$$

  - We can rewrite the radiative transfer equation using the optical depth as a measure of \`distance' rather than s:

$$\frac{dI_\nu}{\alpha_\nu ds} = -I_\nu + \frac{j_\nu}{\alpha_\nu}$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

  - where $S_\nu \equiv j_\nu/\alpha_\nu$ ***is called the source function***. This is an alternative and sometimes more convenient way to write the equation.

Example: Thomson Scattering

- A free electron has a cross-section to radiation given by

$$\sigma_\nu^{\mathrm{T}} = 6.7 \times 10^{-25}\ \mathrm{cm}^2$$

independent of frequency. Then, the opacity is

$$\kappa_\nu = \frac{n}{\rho}\sigma_\nu = \frac{N_A}{1g}\sigma_\nu = 0.4\ \mathrm{cm}^2\ \mathrm{g}^{-1}$$

$$(N_A = 6.022 \times 10^{23}\ \text{particles/mole} \quad : \quad \text{Avogardo constant})$$

if the gas is pure hydrogen (protons and electrons only).

Note: The mass of one mole of a chemical compound, in grams, is numerically equal to the average mass of one molecule of the compound.

Mean Free Path

- From the exponential absorption law, the probability of a photon absorbed between optical depths τ_ν and $\tau_\nu + d\tau_\nu$ is

$$|dI_\nu| = \left|\frac{dI_\nu}{d\tau_\nu}\right| d\tau_\nu \quad \& \quad |dI_\nu| \propto P(\tau_\nu)d\tau_\nu \quad \rightarrow \quad \boxed{P(\tau_\nu) = e^{-\tau_\nu}}$$

  - The mean optical depth traveled is thus equal to unity:

$$\langle\tau_\nu\rangle = \int_0^\infty \tau_\nu P(\tau_\nu)d\tau_\nu = \int_0^\infty \tau_n u e^{-\tau_\nu} d\tau_\nu = 1$$

  - ***The mean free path is defined as the average distance a photon can travel through an absorbing material without being absorbed.*** In a homogeneous medium, the mean free path is determined by

$$\langle\tau_\nu\rangle = \alpha_\nu \ell_\nu = 1 \quad \rightarrow \quad \boxed{\ell_\nu = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}}$$

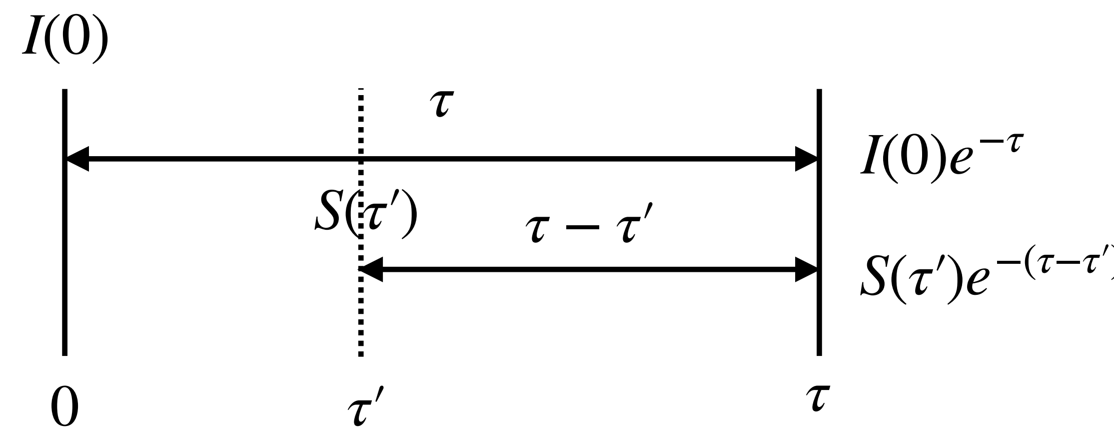
  - A local mean path at a point in an inhomogeneous material can be also defined.

Formal Solution of the RT equation

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

$$\frac{d}{d\tau_\nu}\left(e^{\tau_\nu} I_\nu\right) = e_\nu^{\tau} S_\nu$$



$$\boxed{I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')} S_\nu(\tau_\nu') d\tau_\nu'}$$

- The solution is easily interpreted as the sum of two terms:
  - the initial intensity diminished by absorption
  - the integrated source diminished by absorption.
- For a constant source function, the solution becomes

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right) \end{aligned}$$

Relaxation

- "Relaxation" $\dfrac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$

$$I_\nu > S_\nu \quad \rightarrow \quad \frac{dI_\nu}{d\tau_\nu} < 0, \quad \text{then } I_\nu \text{ tends to decrease along the ray}$$

$$I_\nu < S_\nu \quad \rightarrow \quad \frac{dI_\nu}{d\tau_\nu} > 0, \quad \text{then } I_\nu \text{ tends to increase along the ray}$$

- ***The source function is the quantity that the specific intensity tries to approach***, and does approach if given sufficient optical depth.

$$\text{As} \quad \tau_\nu \rightarrow \infty, \quad I_\nu \rightarrow S_\nu$$

The RT Equation including scattering

- Including scattering term, we obtain an integrodifferential equation.

$$\boldsymbol{\Omega} \cdot \nabla I_\nu = -\alpha_\nu^{\rm ext} I_\nu + j_\nu + \alpha_\nu^{\rm sca} \int \phi_\nu(\boldsymbol{\Omega}, \boldsymbol{\Omega}') I_\nu(\boldsymbol{\Omega}') d\Omega'$$

  - scattering coefficient: $\alpha_\nu^{\rm scatt}$ $({\rm cm}^{-1})$
  - extinction coefficient: $\alpha_\nu^{\rm ext} = \alpha_\nu^{\rm abs} + \alpha_\nu^{\rm scatt}$
  - scattering phase function: $\int \phi_\nu(\boldsymbol{\Omega}, \boldsymbol{\Omega}') d\Omega = 1$
  - for isotropic scattering $\phi_\nu(\boldsymbol{\Omega}, \boldsymbol{\Omega}') = \dfrac{1}{4\pi}$

- Stimulated emission:
  - We consider "absorption" to include both "true absorption" and stimulated emission, because both are proportional to the intensity of the incoming beam (unlike spontaneous emission).

Thermal equilibrium

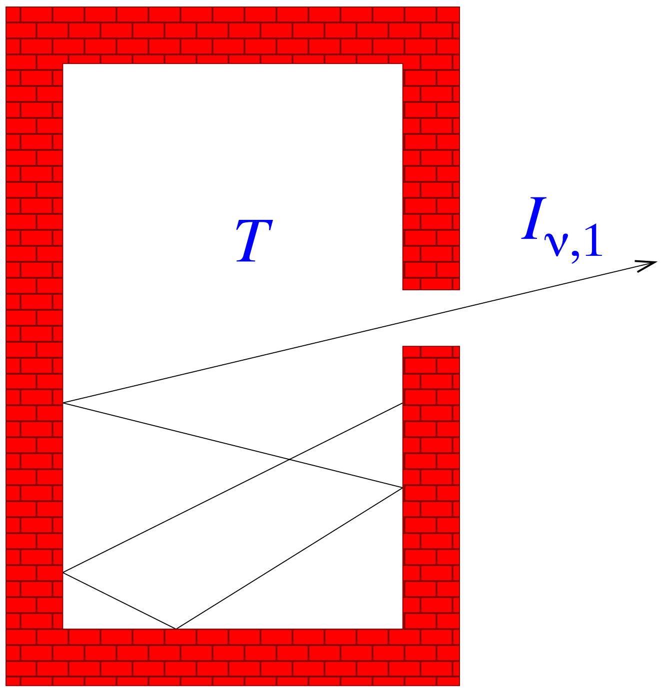
- In general, equilibrium means a state of balance.
- ***Thermal equilibrium* refers to steady states of temperature, which defines the average energy of material or photons.**

- In a state of (complete) ***thermodynamic equilibrium (TE)***, no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system. ***In TE, matter and radiation are in equilibrium at the same temperature T.***

- If the material is (locally) in thermodynamic equilibrium at a well-defined temperature *T*, ***it is said to be in local thermodynamic equilibrium (LTE) even if it is not in equilibrium with the radiation field.***

Blackbody

- Imagine a container bounded by opaque walls with a very small hole.
  - ***Photons will be scattered and absorbed many times, (and eventually trapped and completely absorbed in the box).*** Under such conditions, the particles and photons continually share their kinetic energies. In perfect thermal equilibrium, ***the average particle kinetic energy will equal to the average photon energy, and a unique temperature T can be defined.***
  - The intensity and spectrum of the radiation emerging from the hole would be independent of the wall material (e.g., wood, copper, or concrete, etc) and any absorbing material that may be inside the cavity.
  - A blackbody is an idealized physical body that absorbs all incident radiation regardless of frequency or angle of incidence (i.e., perfect absorber). The above cavity can be regarded to be a blackbody.

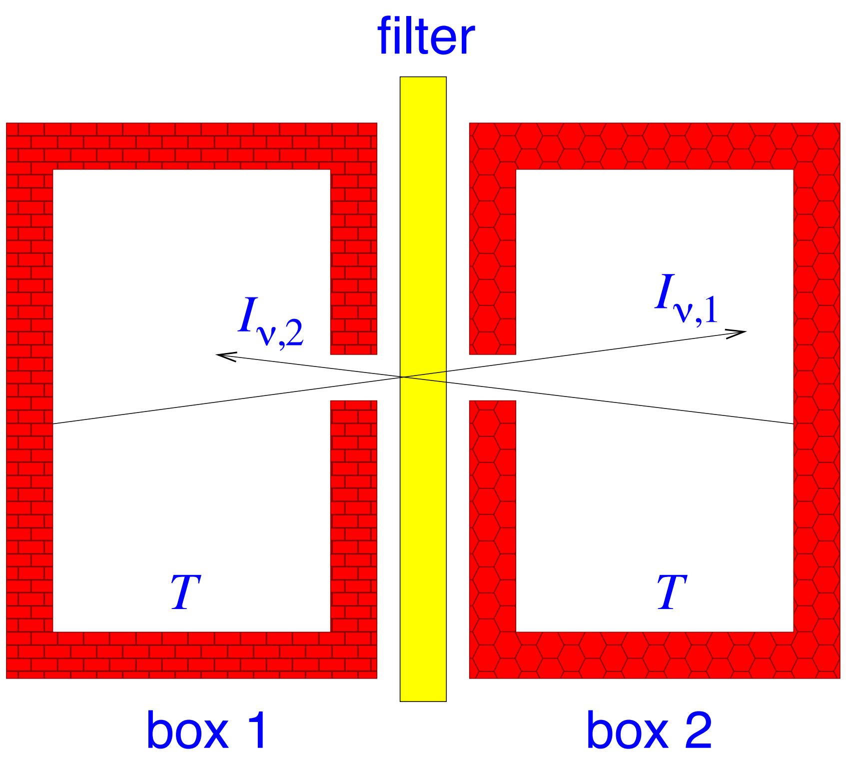


box 1

- Radiation from a blackbody in thermal equilibrium is called the blackbody radiation.

Blackbody radiation if the universal function.

- ***Now, consider another cavity (box 2), also at the same temperature***, but made of different material or shape and connect two cavities with a filter transparent only in the narrow frequency range ν and $\nu + d\nu$.
  - ***In equilibrium at T, radiation should transfer no net energy from one cavity to the other.*** Otherwise, one cavity will cool down and the other heats up; this violates the second law of thermodynamics.
  - Therefore, the intensity or spectrum that passes through the holes should be a universal function of *T* and should be isotropic.
  - The universal function is called the Planck function $B_\nu(T)$.
  - This is the blackbody radiation.



Kirchhoff's Law in TE

- In (full) thermodynamic equilibrium at temperature *T*, by definition, we know that

$$\frac{dI_\nu}{ds} = 0 \quad \text{and} \quad I_\nu = B_\nu(T)$$

- We also note that

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

- Then, we can obtain ***the Kirchhoff's law for a system in TE***:

$$\boxed{\frac{j_\nu(T)}{\alpha_\nu(T)} = B_\nu(T)}$$

- This is remarkable because it connects the properties $j_\nu(T)$ and $\alpha_\nu(T)$ of any kind of matter to the single universal spectrum $B_\nu(T)$.

Kirchhoff’s Law in LTE

- Note that Kirchhoff’s law was derived for a system in thermodynamic equilibrium.

- ***Kirchhoff’s law applies in LTE as well as in TE***:
  - Recall that $B_\nu(T)$ is independent of the properties of the radiating / absorbing material.
  - In contrast, both $j_\nu(T)$ and $\alpha_\nu(T)$ depend only on the materials in the cavity and on the temperature of that material; they do not depend on the ambient radiation field or its spectrum.
  - Therefore, the Kirchhoff’s law should be true even for the case of LTE.
  - ***In LTE, the ratio of the emission coefficient to the absorption coefficient is a function of temperature and frequency only.***
  - This generalized version of Kirchhoff’s law is an exceptionally valuable tool for calculating the emission coefficient from the absorption coefficient or vice versa.

Implications of Kirchhoff's Law

- A good absorber is a good emitter, and a poor absorber is a poor emitter. (In other words, a good reflector must be a poor absorber, and thus a poor emitter.)

$$j_\nu = \alpha_\nu B_\nu(T) \quad \rightarrow \quad j_\nu \text{ increases as } \alpha_\nu \text{ increases}$$

  - It is not possible to thermally radiate more energy than a blackbody, at equilibrium.

$$j_\nu < B_\nu(T) \quad \text{because} \quad \alpha_\nu < 1$$

  - The radiative transfer equation in LTE can be rewritten:

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T)$$

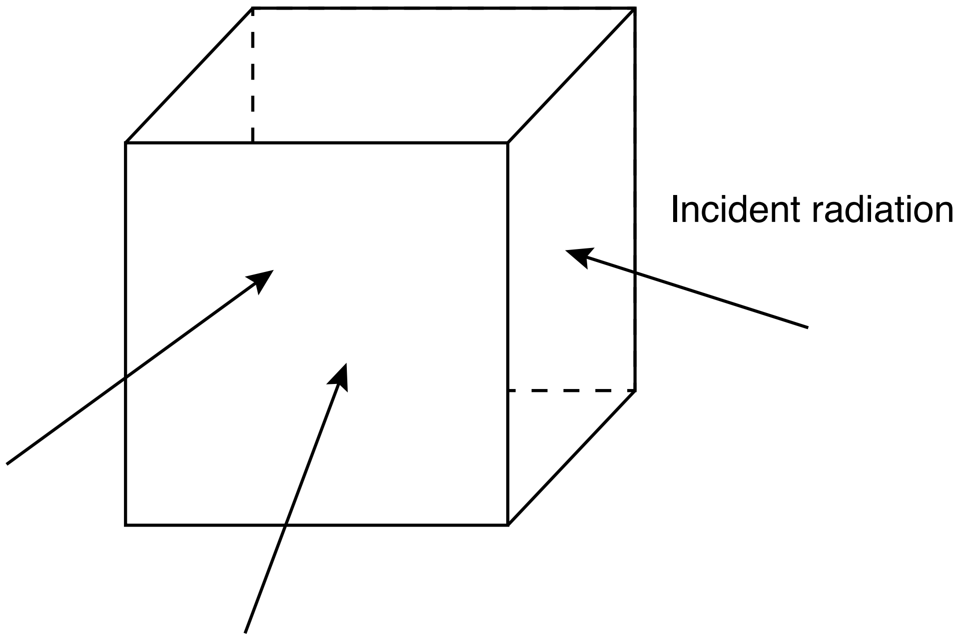
  - Note:
    - Blackbody radiation means $I_\nu = B_\nu(T)$.
    - Thermal radiation is defined to be radiation emitted by "matter" in LTE. Thermal radiation means $S_\nu = B_\nu(T)$.
    - Thermal radiation becomes blackbody radiation only for optically thick media.

Application of Kirchhoff's Law: Dust Emission

- Consider a dusty cloud with a volume V through which external radiation passes.

(1) The total absorbed energy should be balanced by the energy emitted by dust grains.

$$\int L_\nu^{\mathrm{abs}} d\nu = \int L_\nu^{\mathrm{em}} d\nu$$



(2) The energy emitted at frequency ν can be express in terms of the emission coefficient:

$$L_\nu^{\mathrm{em}} = 4\pi V j_\nu^{\mathrm{em}}$$

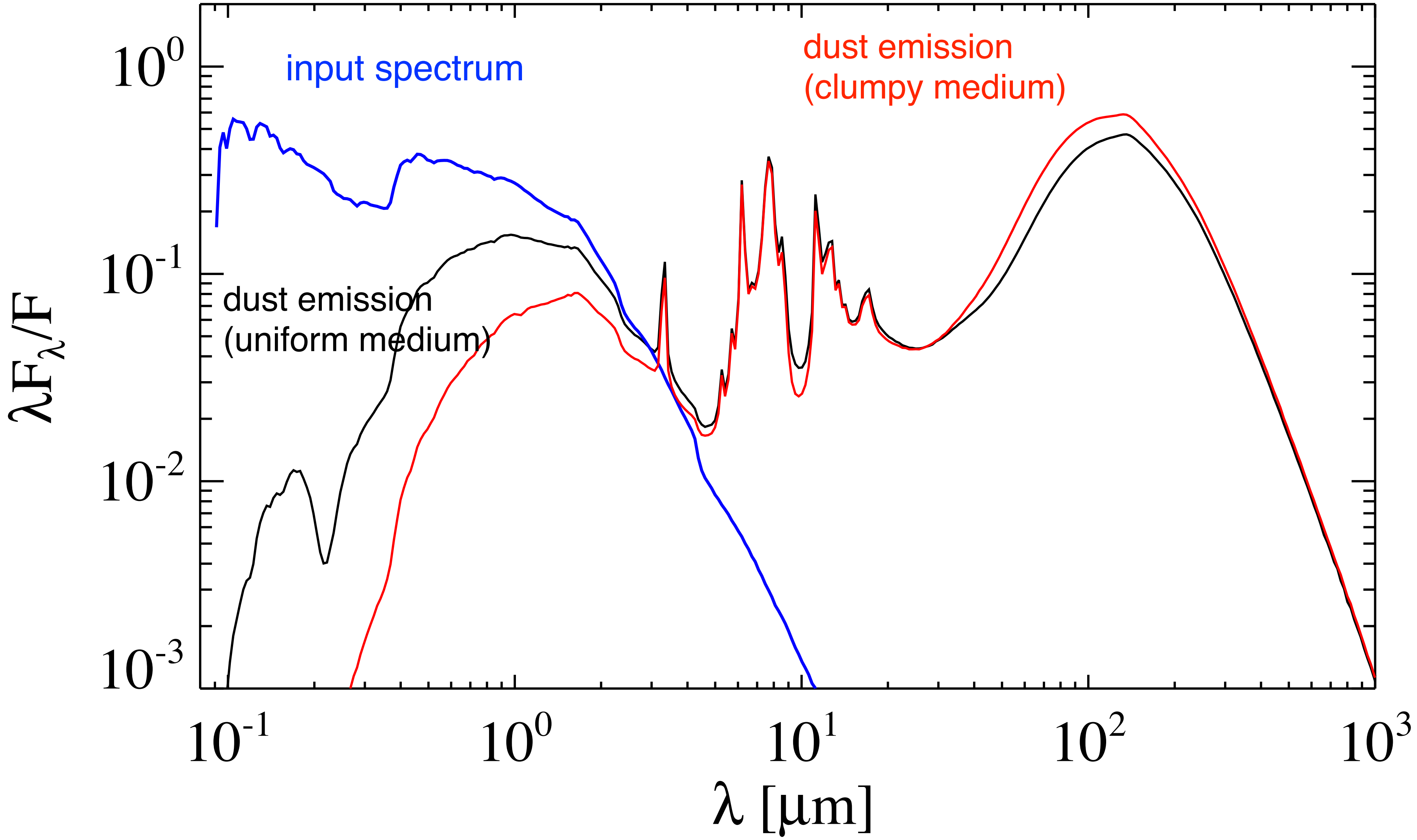
(3) Using Kirchhoff's law ($j_\nu^{\mathrm{em}} = \rho \kappa_\nu^{\mathrm{abs}} B_\nu(T)$), we can calculate the temperature of dust:

$$\int L_\nu^{\mathrm{abs}} d\nu = 4\pi\rho V \int \kappa_\nu^{\mathrm{abs}} B_\nu(T) d\nu$$

(4) The emission spectrum is then obtained by

$$L_\nu^{\mathrm{em}} = 4\pi\rho V \kappa_\nu^{\mathrm{abs}} B_\nu(T)$$

- Spectral Energy Distribution
  - Blue spectrum is the input stellar spectrum from a typical spiral galaxy.
  - Dust emission was calculated using the Kirchhoff's law.
  - This shows a typical SED shape of galaxies.



Spectrum of blackbody radiation

- The frequency dependence of blackbody radiation is given by the ***Planck function***:

$$B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/k_\mathrm{B}T) - 1} \text{ or } B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda k_\mathrm{B}T) - 1}$$

$$h = 6.63 \times 10^{-27} \text{ erg s (Planck's constant)}$$

$$k_\mathrm{B} = 1.38 \times 10^{-16} \text{ erg K}^{-1} \text{ (Boltzmann's constant)}$$

See “Fundamentals of Statistical and Thermal Physics” (Federick Reif) or “Astrophysical Concepts” (Harwit) for the derivation.

Stefan-Boltzmann Law

- Emergent flux is proportional to T^4.

$$F = \pi \int B_\nu(T) d\nu = \pi B(T) \longleftarrow B(T) = \int B_\nu(T) d\nu = \frac{ac}{4\pi} T^4 = \frac{\sigma}{\pi} T^4$$

$$F = \sigma T^4$$

$$\text{Stephan} - \text{Boltzmann constant} : \sigma = \frac{2\pi^5 k_B^4}{15 c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^2 \text{ s}^{-1} \text{ K}^{-4} \text{ sr}^{-1}$$

- Total energy density ***(another form of the Stefan-Boltzmann law)***

$$u = \frac{4\pi}{c} \int B_\nu(T) d\nu = \frac{4\pi}{c} B(T)$$

$$u = aT^4$$

$$u(T) = \left(\frac{T}{3400 \text{ K}} \right)^4 \text{ erg cm}^{-3}$$

$$\text{radiation constant} : a \equiv \frac{4\sigma}{c} = 7.57 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$$

Rayleigh-Jeans Law & Wien Law

- ***Rayleigh-Jeans Law (low-energy limit)***

$$h\nu \ll k_{\rm B}T \;\; (\nu \ll 2\times 10^{10}{\rm Hz}(T/1{\rm K})) \quad \rightarrow \boxed{I_\nu^{RJ}(T) = \frac{2\nu^2}{c^2}k_{\rm B}T}$$

  - ultraviolet catastrophe: if the equation is applied to all frequencies, the total amount of energy would diverge. $\int \nu^2 d\nu \rightarrow \infty$

- ***Wien Law (high-energy limit)***

$$h\nu \gg k_{\rm B}T \quad \rightarrow \boxed{I_\nu^{W}(T) = \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{k_{\rm B}T}\right)}$$

Wien Displacement Law

- Frequency at which the peak occurs:

$$\frac{\partial B_\nu}{\partial \nu}|_{\nu=\nu_{\max}} = 0 \quad \rightarrow \quad x = 3(1-e^{-x}), \text{ where } x = h\nu_{\max}/k_{\mathrm{B}}T$$

$$\boxed{h\nu_{\max} = 2.82 k_{\mathrm{B}} T \quad \text{or} \quad \frac{\nu_{\max}}{T} = 5.88 \times 10^{10} \text{ Hz deg}^{-1}}$$

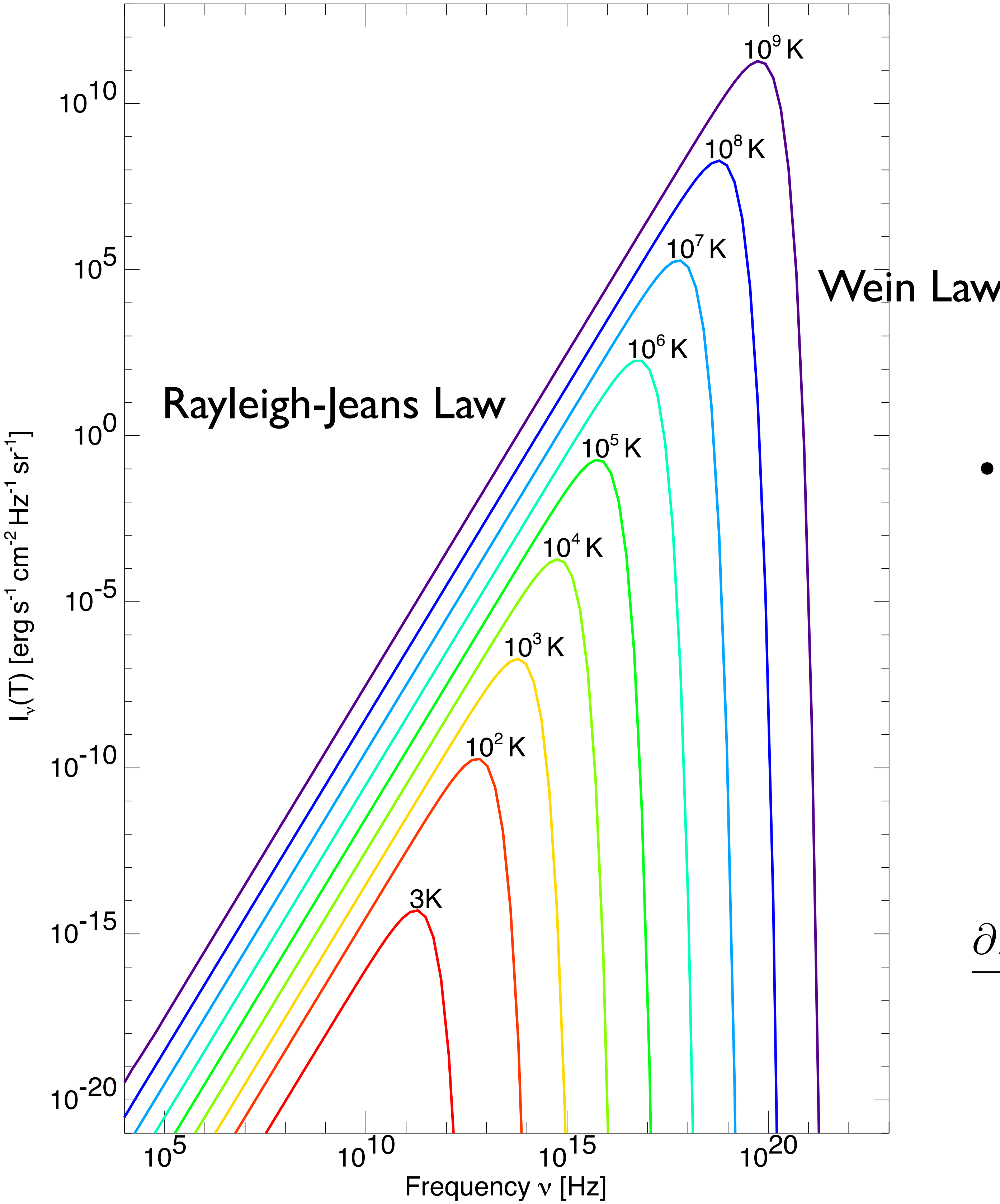
- Wavelength at which the peak occurs:

$$\frac{\partial B_\lambda}{\partial \lambda}|_{\lambda=\lambda_{\max}} = 0 \quad \rightarrow \quad y = 5(1-e^{-y}), \text{ where } y = hc/(\lambda_{\max} k_{\mathrm{B}} T)$$

$$y = 4.97 \quad \text{and} \quad \lambda_{\max} T = 0.290 \text{ cm deg}$$

- Note that $\nu_{\max} \neq c/\lambda_{\max}$

Monotonicity with Temperature



- Monotonicity:
  - Of two blackbody curves, the one with higher temperature lies entirely above the other.

$$\frac{\partial B_\nu(T)}{\partial T} = \frac{2h^2\nu^4}{c^2 k_B T^2} \frac{\exp(h\nu/k_T)}{[\exp(h\nu/k_B T) - 1]^2} > 0$$

Characteristic Temperatures

- ***Brightness Temperature:***

$$I_\nu = B_\nu(T_b)$$

  - The definition is used especially in radio astronomy, where the RJ law is usually applicable. In the RJ limit,

$$T_b = \frac{c^2}{2\nu^2 k_{\mathrm{B}}} I_\nu$$

  - Radiative transfer equation in the RJ limit:

$$\frac{dT_b}{d\tau_\nu} = -T_b + T \quad (T = \text{the temperature of the material})$$

$$T_b = T_b(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu}) \quad \text{if } T \text{ is constant.}$$

  - In the Wien region, the concept is not so useful.

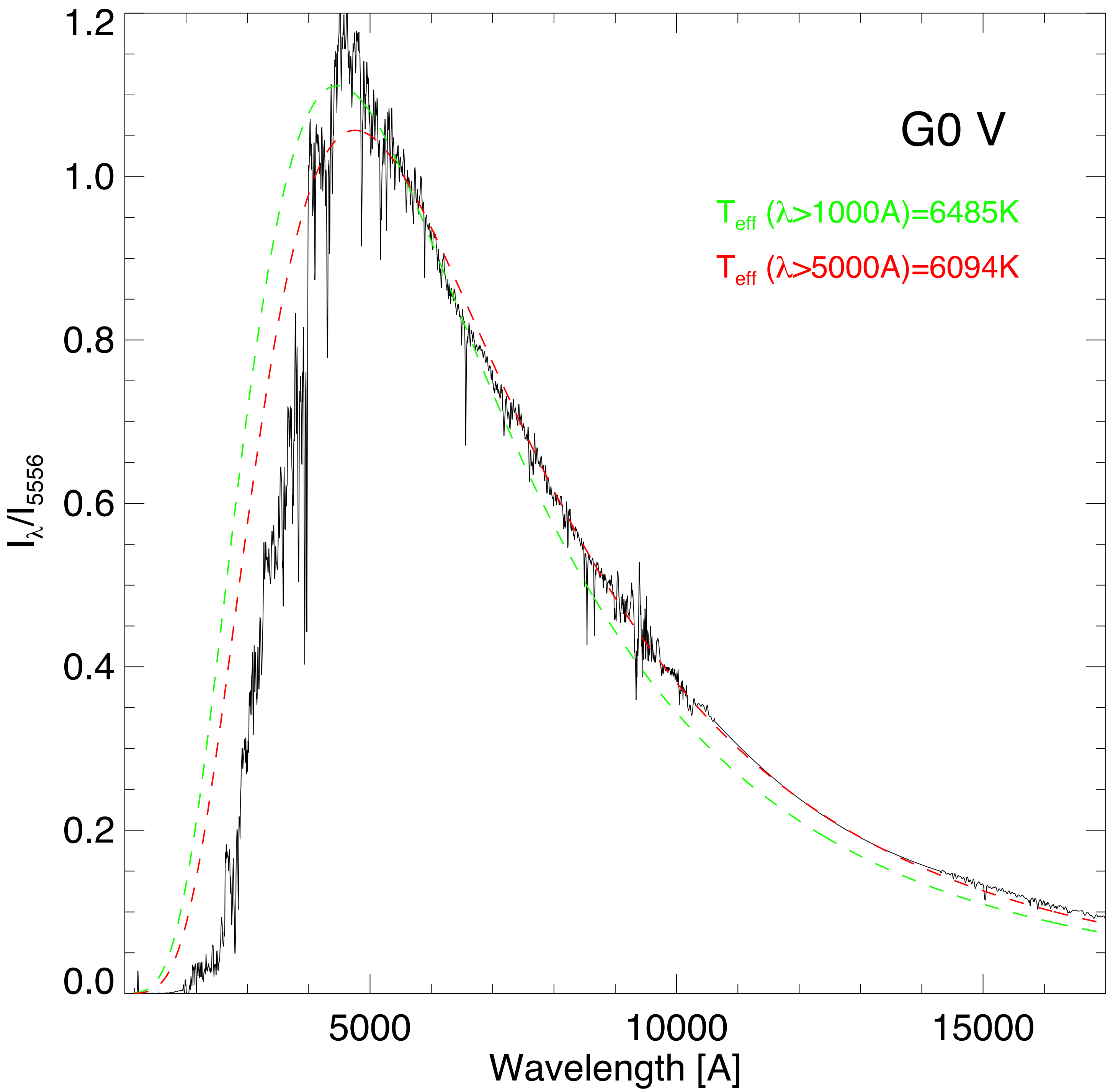
- ***Color Temperature:***
  - By fitting the spectrum to a blackbody curve without regarding to vertical scale (absolute intensity scale), a color temperature T_c is obtained.
  - The color temperature correctly gives the temperature of a blackbody source of unknown absolute scale.

- ***Effective Temperature:***
  - The effective temperature of a source is obtained by equating the actual flux F to the flux of a blackbody at temperature $T_{\rm eff}$.

$$F = \int \cos\theta I_\nu d\nu d\Omega = \sigma T_{\rm eff}^4$$

- ***Excitation Temperature:***

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} \exp\left(-\frac{(E_u - E_l)}{k_{\rm B} T_{\rm ex}} \right)$$

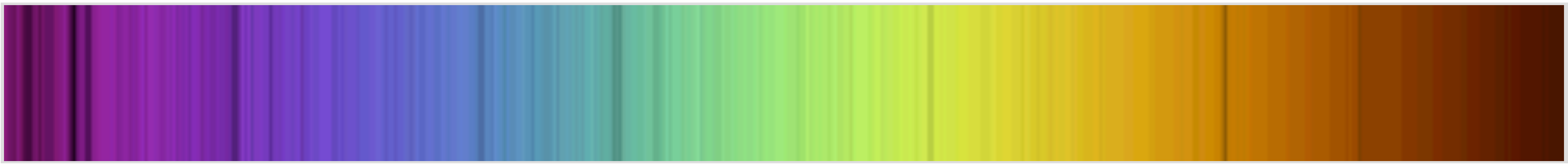


G0V spectrum (Pickles 1998, PASP, 110, 863)

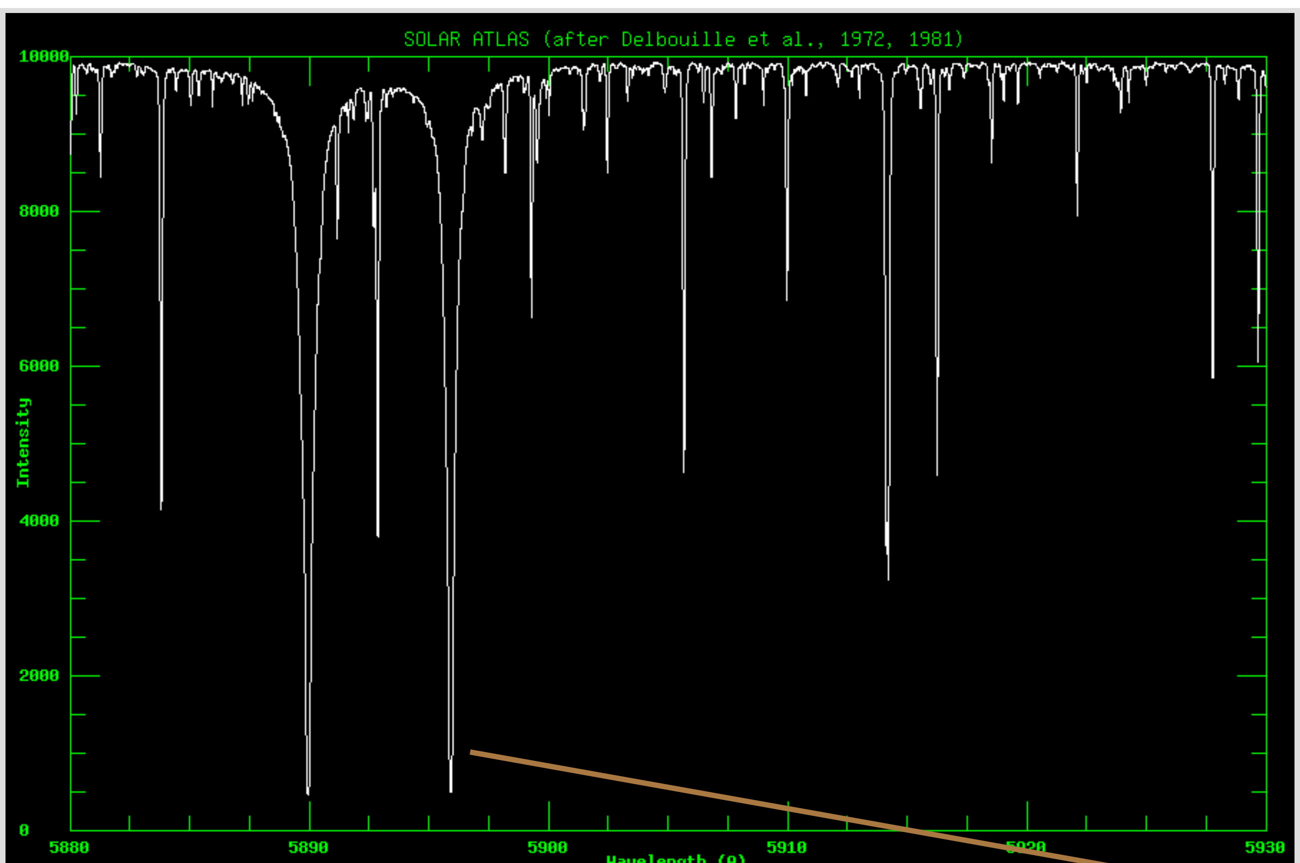
(Note that the solar spectral type is G2V.)

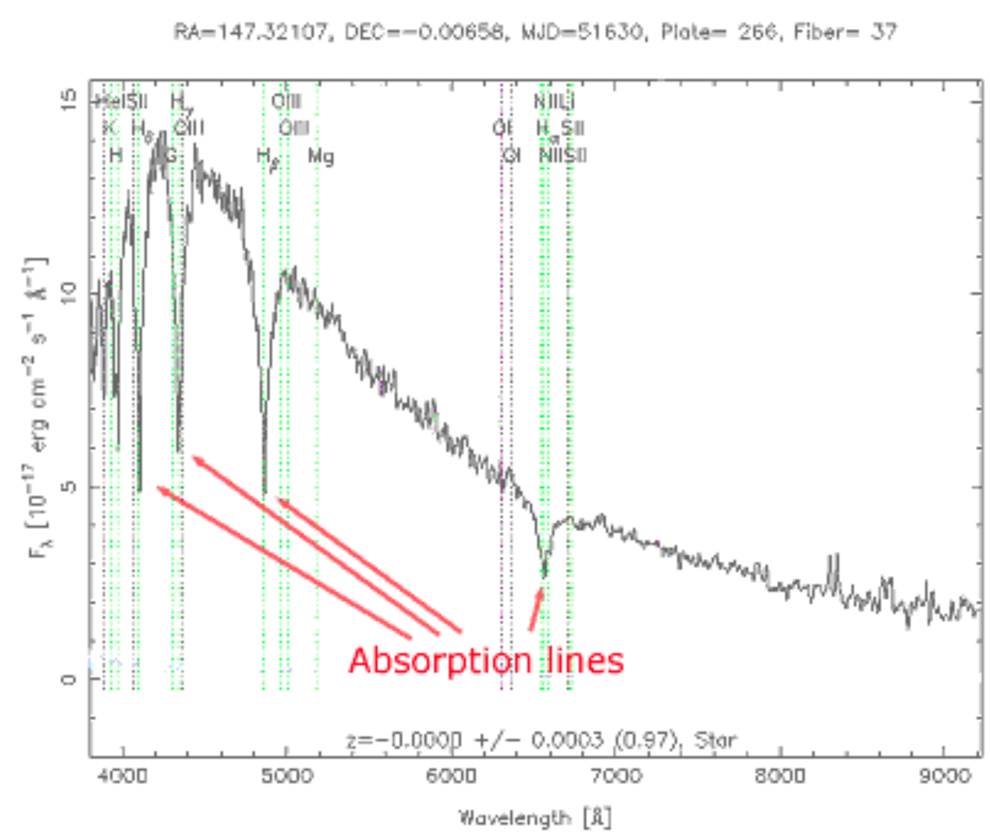
[Homework] Absorption line and emission line spectra

- Temperature of the Solar photosphere is ~ 6000 K. Lots of spectral lines of different elements are observed.



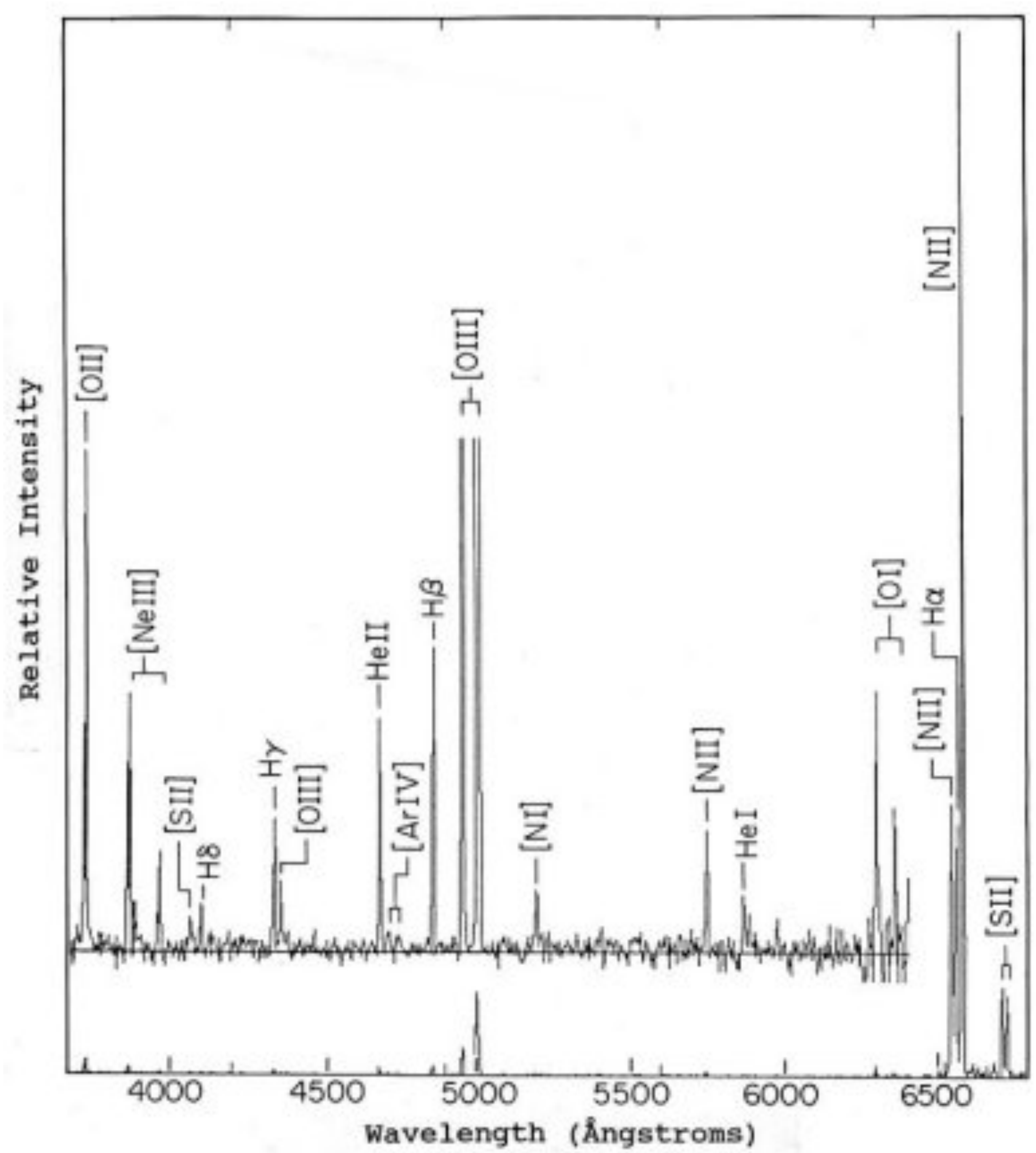
- ***Optical spectrum of stars is an absorption line spectrum*** - see dark absorption lines superimposed on a bright continuum.





Two strong absorption lines are Na I D lines due to sodium.

- However, ***emission nebulae typically show emission line spectra***: (Spectral lines are stronger than the continuum.)



Spectrum from an emission nebula

Homework:
- Explain why this difference happens?
- Deadline: Sept. 11 (Wed.)

Hint for homework

- Recall the solution for RT equation when the source function is constant.

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

- Assume optically thin regions, $\tau_\nu \ll 1$ and show that the above equation becomes

$$I_\nu(D) \approx I_\nu(0) + \tau_\nu(D)\big(S_\nu - I_\nu(0)\big) \quad \text{at } \tau_\nu = D$$

- See the following three figures, and explain why some objects show absorption line spectra, but some show emission line spectra.
- Note that $T(\tau_0 = 0) > T(\tau_0 = D)$ for the case of the stellar atmosphere.