$(AGN)^2$

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Introduction

- Temperature
  - The temperature in a static nebula is determined by the balance between **(1) heating by photoionization** and **(2) cooling by recombination and by collisional excitation emission** from the nebula.
- Energy gain and loss
  - **[Gain]** When a photon of energy $h\nu$ is absorbed, an electron (photoelectron) is created, having an energy $\frac{1}{2}mu^2 = h(\nu - \nu_0)$. The electrons produced are rapidly thermalized (see Chap. 2).
  - In ionization equilibrium, the photoionizations are balanced by an equal number of recombinations.
  - **[Loss]** In each recombination, a thermal electron with energy $\frac{1}{2}mu^2$ disappears. An average of this quantity over all recombinations represents the mean energy that "disappears" per recombination.
  - The difference between the mean energy of a newly created electron and the mean energy of a recombining electron represents the net gain in energy per ionization process.
  - **[Loss]** In equilibrium, this net energy gain is balanced by the energy lost by radiation, predominantly by electron collisional excitation of bound levels of abundant ions.
  - **[Loss]** Free-free emission (bremsstrahlung) is another, less important radiative energy-loss mechanism.

Energy Gain by Photoionizaiton

- In a pure H nebula, at any specific location in the nebula, the energy gain (per unit volume per unit time) is

$$G(\mathrm{H}) = n(\mathrm{H}^0)\int_{\nu_0}^{\infty}\frac{4\pi J_\nu}{h\nu}h(\nu-\nu_0)\sigma_\nu^{\mathrm{pi}}(\mathrm{H}^0)d\nu \quad [\mathrm{erg\ cm^{-3}\ s^{-1}}]$$

- In ionization equilibrium,

$$n(\mathrm{H}^0)\int_{\nu_0}^{\infty}\frac{4\pi J_\nu}{h\nu}\sigma_\nu^{\mathrm{pi}}(\mathrm{H}^0)d\nu = n_e n_p \alpha_{\mathrm{A}}(\mathrm{H}^0, T)$$

number of photoionization per unit volume per unit time = number of recombination per unit volume per unit time

- Then, the gain can be expressed as follows:

number of recombination

$$G(\mathrm{H}) = n_e n_p \alpha_{\mathrm{A}}(\mathrm{H}^0, T)\frac{\int_{\nu_0}^{\infty}\frac{4\pi J_\nu}{h\nu}h(\nu-\nu_0)\sigma_\nu^{\mathrm{pi}}(\mathrm{H}^0)d\nu}{\int_{\nu_0}^{\infty}\frac{4\pi J_\nu}{h\nu}\sigma_\nu^{\mathrm{pi}}(\mathrm{H}^0)d\nu} = n_e n_p \alpha_{\mathrm{A}}(\mathrm{H}^0, T)\frac{3}{2}kT_i$$

mean energy gain by electron per photoionization

- The mean energy of a newly created photoelectron depends on the form of the ionizing radiation field, but not on the absolute strength of the radiation.

- For a blackbody spectrum $J_\nu = B_\nu(T_*)$, the initial temperature $T_i \approx T_*$ when $kT_* < h\nu_0$.

Table 3.1

Mean input energy of photoelectrons

Model stellar atmosphere	T_i (K)			
T_* (K)	$\tau_0 = 0$	$\tau_0 = 1$	$\tau_0 = 5$	$\tau_0 = 10$
3.0×10^4	1.58×10^4	1.87×10^4	3.36×10^4	5.02×10^4
3.5×10^4	2.08×10^4	2.48×10^4	4.24×10^4	5.94×10^4
4.0×10^4	2.48×10^4	3.01×10^4	5.48×10^4	8.15×10^4
5.0×10^4	3.33×10^4	4.12×10^4	7.50×10^4	10.60×10^4

- At larger distances from the star, the spectrum of the ionizing radiation is attenuated by absorption in the nebula.
- The higher-energy photons penetrate further into the gas, and the mean energy of the photoelectrons produced at larger optical depths from the star is higher.

Energy Loss by Recombination

- The kinetic energy lost by the electron gas (per unit volume per unit time) in recombination is

$$L_R(\mathrm{H}) = n_e n_p \alpha_{\mathrm{A}}(\mathrm{H}^0, T) kT \quad [\mathrm{erg\ cm^{-3}\ s^{-1}}]$$

mean energy loss by a photon per recombination

$$\alpha_{\mathrm{A}}(\mathrm{H}^0, T) = \sum_{n=1}^{\infty} \alpha_n(\mathrm{H}^0, T) = \sum_{n=1}^{\infty} \sum_{L=0}^{n-1} \alpha_{nL}(\mathrm{H}^0, T) \quad [\mathrm{cm^3\ s^{-1}}]$$

$$\text{where} \quad \alpha_{nL}(\mathrm{H}^0, T) = \frac{1}{kT} \int_0^{\infty} u \sigma_{nL}(\mathrm{H}^0, T) \frac{1}{2} m u^2 f(u) du$$

  - α_{nL} is thus effectively a kinetic energy averaged recombination coefficient.

- Coulomb focusing effect: the cross sections are approximately proportional to u^{-2}.
- Since the recombination cross sections are approximately proportional to u^{-2}, the electrons of lower kinetic energy are preferentially captured, and the mean energy of the captured electrons is somewhat less than $\frac{3}{2} kT_i$.
- In a pure H nebula that had no radiation losses, the thermal equilibrium equation would be $T_{\mathrm{eq}} > T_i$ because of the "heating" due to the preferential capture of the slower electrons.

$$G(\mathrm{H}) = L_R(\mathrm{H}) \quad \rightarrow \quad T_{\mathrm{eq}} > T_i$$

- **On-the-spot approximation:**
  - The radiation field J_ν should include the diffuse radiation as well as the stellar radiation modified by absorption ($e^{-\tau_\nu}$).
  - Every emission of an ionizing photon during a recombination to the level $n = 1$ is assumed to be balanced by absorption of the same photon at a nearby spot in the nebula.
  - Thus production of photons by the diffuse radiation field and recombination to the ground level is simply omitted from the gain and loss rates. Then, the equations are

$$G_{\rm OTS}({\rm H}) = n({\rm H}^0)\int_{\nu_0}^{\infty}\frac{4\pi J_{\nu s}}{h\nu}h(\nu-\nu_0)\sigma_\nu^{\rm pi}({\rm H}^0)d\nu$$

$$= n_e n_p \alpha_{\rm B}({\rm H}^0, T)\frac{\int_{\nu_0}^{\infty}\frac{4\pi J_{\nu s}}{h\nu}h(\nu-\nu_0)\sigma_\nu^{\rm pi}({\rm H}^0)d\nu}{\int_{\nu_0}^{\infty}\frac{4\pi J_{\nu s}}{h\nu}\sigma_\nu^{\rm pi}({\rm H}^0)d\nu}$$

$$L_{\rm OTS}({\rm H}) = n_e n_p \alpha_{\rm B}({\rm H}^0, T)kT \qquad \text{where} \quad \alpha_{\rm B}({\rm H}^0, T) = \sum_{n=2}^{\infty}\alpha_n({\rm H}^0, T)$$

  - The OTS approximation is not as accurate for the equilibrium as it is in the ionization equation, because of the fairly large difference in $h(\nu-\nu_0)$ between the ionizing photons in the stellar and diffuse radiation fields.

- Inclusion of He and heavy elements
  - Including He in the heating and recombination cooling rates is straightforward:

$$G = G(\mathrm{H}) + G(\mathrm{He})$$

$$G(\mathrm{He}) = n_e n(\mathrm{He}^+)\alpha_\mathrm{A}(\mathrm{He}^0, T)\frac{\int_{\nu_2}^{\infty} \frac{4\pi J_\nu}{h\nu} h(\nu - \nu_2)\sigma_\nu^{\mathrm{pi}}(\mathrm{He}^0)d\nu}{\int_{\nu_2}^{\infty} \frac{4\pi J_\nu}{h\nu}\sigma_\nu^{\mathrm{pi}}(\mathrm{He}^0)d\nu}$$

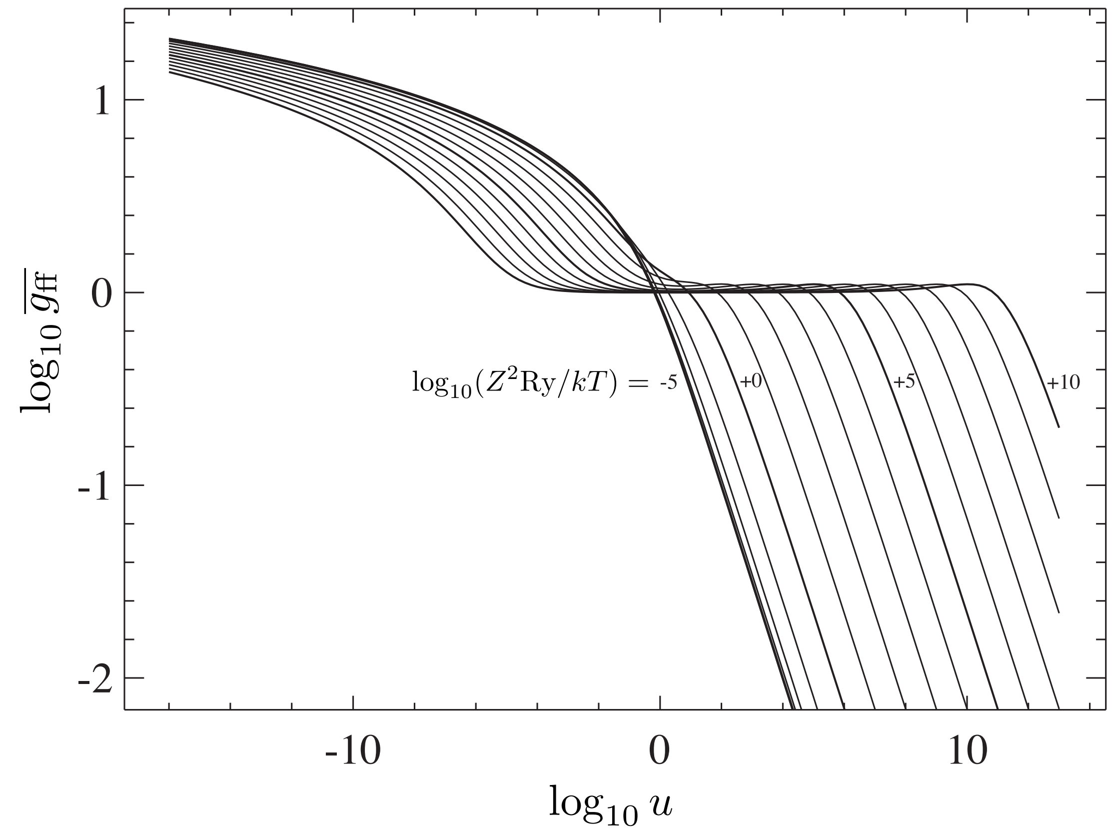
$$L = L_R(\mathrm{H}) + L_R(\mathrm{He})$$

$$L(\mathrm{He}) = n_e n(\mathrm{He}^+)\alpha_\mathrm{A}(\mathrm{He}^0, T)kT$$

  - The heating and recombination cooling rates are proportional to the densities of the ions involved, so the contributions of the heavy elements, which are much less abundant than H and He, can be negligible compared to those of H and He.

Energy Loss by Free-Free Radiation

van Hoof et al. (2014, MNRAS, 444, 420)



$u = h\nu / kT$

- Free-free radiation, in which a continuous spectrum is emitted, is a minor contributor to the cooling rate.
  - The cooling rate by ions of charge Z, integrated over all frequencies is approximately:

$$
\begin{aligned}
L_{FF}(Z) &= 4\pi j_{ff} \\
&= \frac{2^5 \pi e^6 Z^2}{3^{3/2} h m c^3} \left(\frac{2\pi k T}{m} \right) g_{ff} n_e n_+ \quad [\mathrm{erg\ cm^{-3}\ s^{-1}}] \\
&= 1.42 \times 10^{-27} Z^2 T^{1/2} g_{ff} n_e n_+ \quad [\mathrm{erg\ cm^{-3}\ s^{-1}}]
\end{aligned}
$$

$n_+ \approx n_p + n(\mathrm{He}^+)$ the number density of the ions

  - The numerical values of the mean Gaunt factor for free-free emission is a slowly varying function of n_e and T. For nebular conditions in the range $1.0 < g_{ff} < 1.5$.

$$
\boxed{\overline{g}_{\rm ff} \sim \begin{cases} 1 & \text{for } u \sim 1 \\ 1-5 & \text{for } 10^{-4} < u < 1 \end{cases}}
$$

Energy Loss by Collisionally Excited Line Radiation

- Collisional excitation of low-lying energy levels of common ions, such as O^+, O^{++}, and N^+, are the predominant source of radiative cooling.
  - These ions make a significant contribution in spite of their low abundance because they have energy levels with excitation potentials of the order of kT.
  - However, all the levels of H and He have much higher excitation potentials, and therefore are usually not important as collisionally excited coolants.

- Cross section for excitation
  - $\sigma_{12}(u)$ is a function of electron velocity u and is zero below the threshold $\chi = h\nu_{21}$.
  - The main dependence of the excitation cross section is $\sigma \propto u^{-2}$ because of the focusing effect of the Coulomb force.
- Focusing effect of the Coulomb force
  - https://casper.astro.berkeley.edu/astrobaki/index.php/Coulomb_Focusing
  - https://www.youtube.com/watch?v=LXGBGNR5JxI&ab_channel=AaronParsons

Collisional Excitation & De-excitation

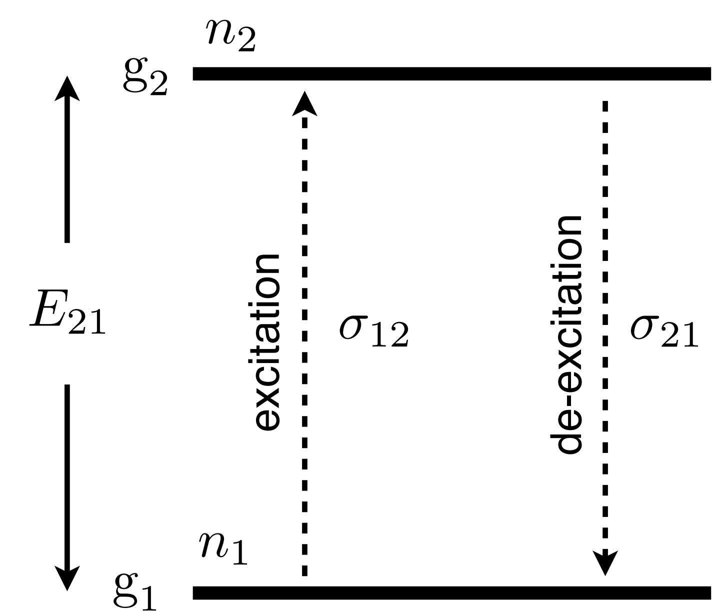
- **Collisional Rate (Two Level Atom)**

  - The cross section σ_{12} for collisional excitation from a lower level 1 to an upper level 2 is, in general, inversely proportional to the impact energy (or v^2) above the energy threshold E_{21} and is zero below.

  - It is convenient to express the collisional cross section in the following form using a dimensionless quantity called the (dimensionless) ***collision strength*** Ω_{12}:

$$\sigma_{12}(v) = (\pi a_0^2)\left(\frac{hR_{\rm H}}{\frac{1}{2}m_e v^2}\right)\frac{\Omega_{12}}{g_1}\ \ {\rm cm}^2 \ \ \text{for} \ \ \frac{1}{2}m_e v^2 > E_{21}$$

$$= \frac{h^2}{4\pi m_e^2 v^2}\frac{\Omega_{12}}{g_1}$$



$$\text{or} \quad \sigma_{12}(E) = \frac{h^2}{8\pi m_e E}\frac{\Omega_{12}}{g_1} \qquad \left(E = \frac{1}{2}m_e v^2\right)$$

$$\text{where,} \quad a_0 = \frac{\hbar^2}{m_e e^2} = 5.12\times 10^{13}\ \ {\rm cm}, \ \text{Bohr radius}$$

$$R_{\rm H} = \frac{m_e e^4}{4\pi\hbar^3} = 109,737\ \ {\rm cm}^{-1}, \ \ \text{Rydberg constant} \qquad \left(\hbar = \frac{h}{2\pi}\right)$$

  - The collision strength Ω_{12} is a function of electron velocity (or energy) but is often approximately constant near the threshold. Here, g_1 and g_2 are the statistical weights of the lower and upper levels, respectively.

- Advantage of using the collision strength is that (1) it removes the primary energy dependence for most atomic transitions and (2) they have the symmetry between the upper and the lower states.

The principle of detailed balance states that ***in thermodynamic equilibrium each microscopic process is balanced by its inverse.***

$$n_e n_1 v_1 \sigma_{12}(v_1) f(v_1) dv_1 = n_e n_2 v_2 \sigma_{21}(v_2) f(v_2) dv_2$$

Here, v_1 and v_2 are related by $\frac{1}{2}m_e v_1^2 = \frac{1}{2}m_e v_2^2 + E_{21}$, and $f(v)$ is a Maxwell velocity distribution of electrons. Using the Boltzmann equation of thermodynamic equilibrium,

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left(-\frac{E_{21}}{kT}\right)$$

we derive the following relation between the cross-sections for excitation and de-excitation are

$$g_1 v_1^2 \sigma_{12}(v_1) = g_2 v_2^2 \sigma_{21}(v_2) \quad \text{Here, } \frac{1}{2}m_e v_1^2 = \frac{1}{2}m_e v_2^2 + E_{21} \longrightarrow g_1 \cdot (E + E_{21}) \cdot \sigma_{12}(E + E_{21}) = g_2 \cdot E \cdot \sigma_{21}(E)$$

where $E = \frac{1}{2}m_e v_2^2$

and the symmetry of the collision strength between levels.

$$\Omega_{12} = \Omega_{21} \qquad \text{more precisely} \quad \Omega_{12}\left(E + E_{21}\right) = \Omega_{21}\left(E\right)$$

These two relations were derived in the TE condition. However, ***the cross-sections are independent on the assumptions, and thus the above relations should be always satisfied.***

- Collisional excitation and de-excitation rates

The ***collisional de-excitation rate per unit volume per unit time, which is thermally averaged,*** is

$$\left(\frac{dn_1}{dt}\right)_{2\rightarrow 1} = n_e n_2 \int_0^\infty v\sigma_{21}(v) f(v) dv$$
$$= n_e n_2 q_{21} \quad [\mathrm{cm}^{-3}\ \mathrm{s}^{-1}]$$

$$q_{21} \equiv \langle \sigma v \rangle_{2\rightarrow 1}$$

$$\begin{aligned} q_{21} &= \int_0^\infty v\sigma_{21}(v) f(v) dv \\ &= \left(\frac{2\pi\hbar^4}{k_\mathrm{B} m_e^3}\right)^{1/2} T^{-1/2} \frac{\langle \Omega_{21} \rangle}{g_2} \\ &= \frac{8.62942\times 10^{-6}}{T^{1/2}} \frac{\langle \Omega_{21} \rangle}{g_2} \quad [\mathrm{cm}^3\ \mathrm{s}^{-1}], \end{aligned}$$

effective collision strength:

$$\langle \Omega_{21} \rangle \equiv \int_0^\infty \Omega_{21}(E) e^{-E/k_\mathrm{B}T} d(E/k_\mathrm{B}T)$$

and the ***collisional excitation rate per unit volume per unit time*** is

Here, $\frac{1}{2} m_e v_\mathrm{min}^2 = E_{21}$

$$\left(\frac{dn_2}{dt}\right)_{1\rightarrow 2} = n_e n_1 q_{12}$$

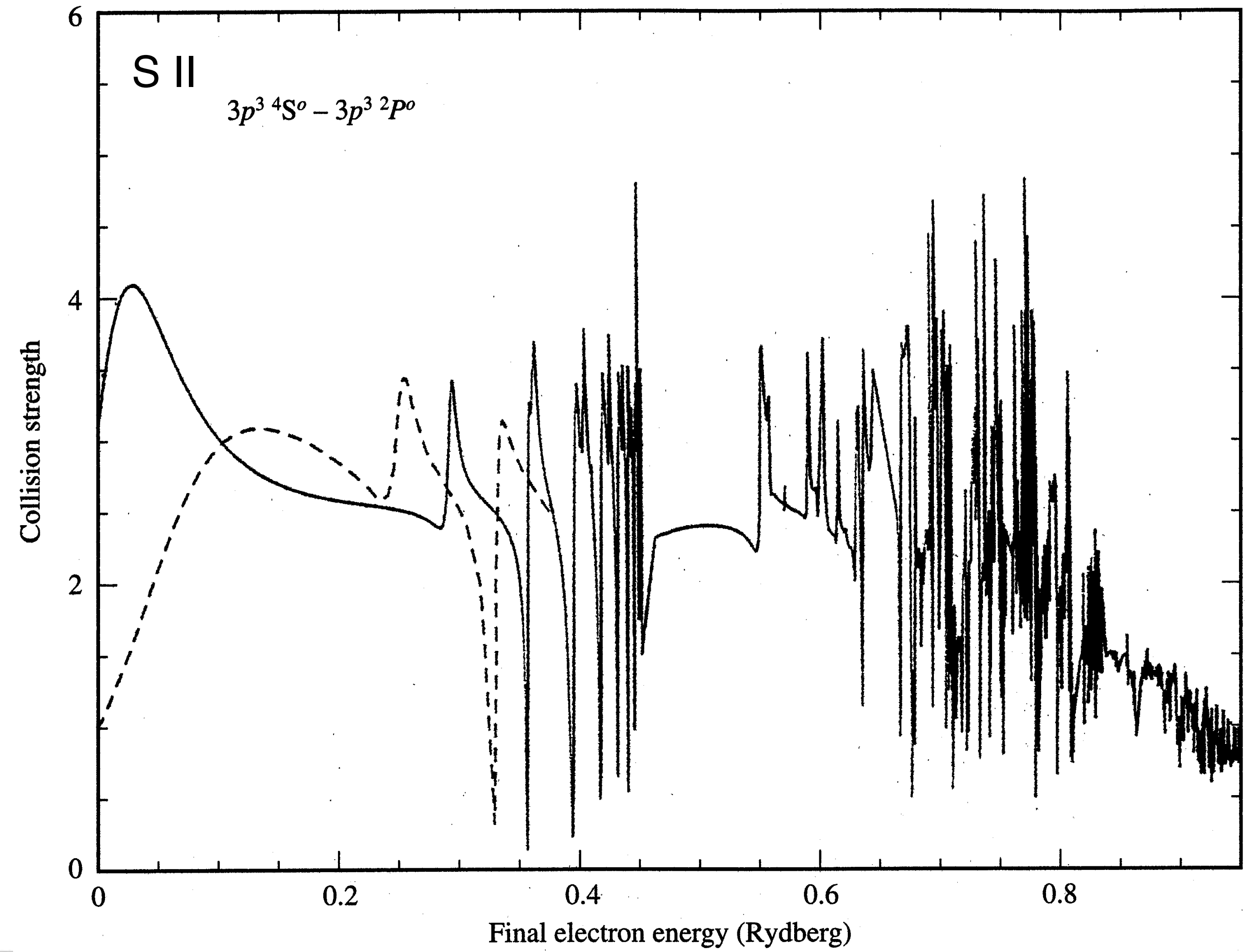
$$q_{12} \equiv \langle \sigma v \rangle_{1\rightarrow 2}$$

$$\begin{aligned} q_{12} &= \int_{v_\mathrm{min}}^\infty v\sigma_{12}(v) f(v) dv \\ &= \left(\frac{2\pi\hbar^4}{k_\mathrm{B} m_e^3}\right)^{1/2} T^{-1/2} \frac{\langle \Omega_{21} \rangle}{g_1} \exp\left(-\frac{E_{21}}{k_\mathrm{B}T}\right) \end{aligned}$$

Here, q_{21} and q_{12} are the collisional rate coefficient for excitation and de-excitation coefficients in units of $\mathrm{cm}^3\ \mathrm{s}^{-1}$, respectively. We also note that ***the rate coefficients for collisional excitation and de-excitation are related by***

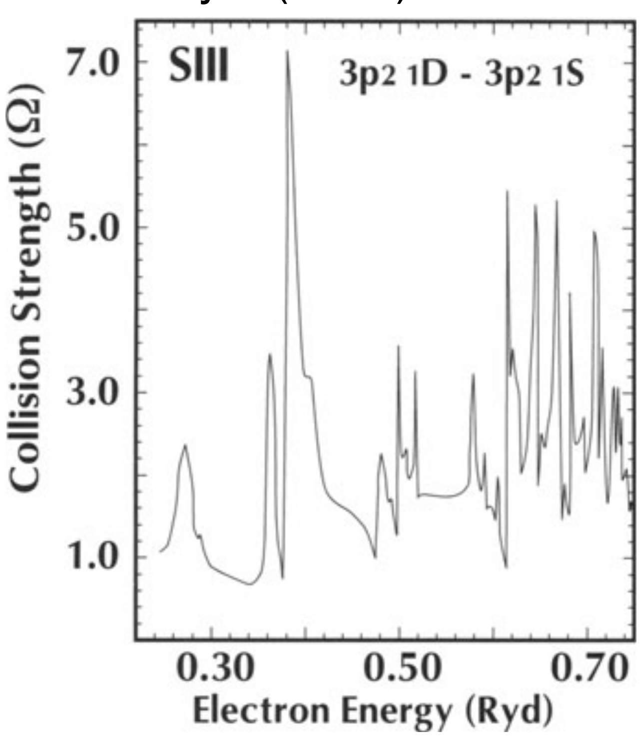
$$q_{12} = \frac{g_2}{g_1} q_{21} \exp\left(-\frac{E_{21}}{k_\mathrm{B}T}\right) \qquad \langle \sigma v \rangle_{1\rightarrow 2} = \frac{g_2}{g_1} \langle \sigma v \rangle_{2\rightarrow 1} \exp\left(-\frac{E_{21}}{k_\mathrm{B}T}\right)$$

- **Collision Strength**
  - Quantum mechanical calculations show that (1) the resonance structure in the collision strengths is important and (2) the collision strength increases with energy for neutral species.
  - The effect of the resonances tends to be averaged out.

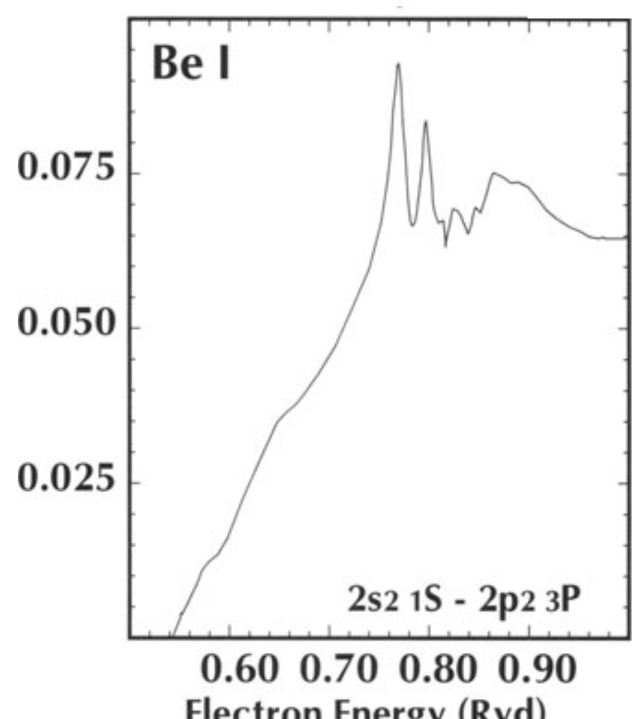


solid - Ramsbottom et al. (1996)
dashed - Cai & Pradhan (1993)

Tayal (1996)



Badnell (1999)



The ***effective collision strength,*** which is thermally averaged, has a value in a range of

$$\langle \Omega_{21} \rangle = \int_0^\infty \Omega_{21}(E) e^{-E/k_{\rm B}T} d(E/k_{\rm B}T)$$

$$10^{-2} < \langle \Omega_{21} \rangle < 10$$

See Table F.1 to F.5 in [Draine]

- Averaged collision strengths

See the CHIANTI atomic database for collision strengths.
https://www.chiantidatabase.org/
https://chianti-atomic.github.io/

Table 3.3
Collision strengths Υ for ^2S–$^2P^o$ transitions in Li-like $2s$ and Na-like $3s$ ions

Ion	$2s\ ^2S,\ 2p\ ^2P^o$	Ion	$3s\ ^2S^o,\ 3p\ ^2P^o$
C^{+3}	8.91	Mg^{+}	16.9
N^{+4}	6.81	Si^{+3}	16.0
O^{+5}	5.21		

Cochrane, D. M., & McWhirter, R. W. P. 1983, PhyS, 28, 25.
McWhirter, R. W. P. 1994, ADNDT, 57, 39.

Table 3.4
Collision strengths Υ for $^2S - {}^3P^o$ transitions in Be-like $2s^2$ and Mg-like $3s^2$ ions

Ion	$^1S,\ ^3P^o$	$^3P^o_0,\ ^3P^o_1$	$^3P^o_0,\ ^3P^o_2$	$^3P^o_1,\ ^3P^o_2$
C^{+2}	1.05	0.96	0.72	2.78
N^{+3}	1.07	1.14	0.83	3.29
O^{+4}	0.82	0.67	0.65	2.32
Al^{+1}	3.35	1.89	1.94	6.72
Si^{+2}	5.56	1.81	3.62	10.4
S^{+4}	1.9	—	—	—

C^{+2} Berrington, K. A., Burke, P. G., Dufton, P. L., & Kingston, A. E. 1985, ADNDT, 33, 195; N^{+3}, O^{+4} Ramsbottom, C. A., Berrington, K. A., Hibbert, A., & Bell, K. L. 1994, Physica Scripta, 50, 246; O^{+4} McKenna, R. C., et al. 1997, ApJ, 486, 571; Al^{+1} Aggarwal, K. M., & Keenan, F. P. 1998, J. Phys B, 31, 4545, and Aggarwal, K. M., & Keenan, F. P. 1994, J. Phys. B, 27, 5321; Si^{+2} Dufton, P. L., & Kingston, A. E. 1994, ADNDT, 57, 273; S^{+4} Dufton, P. L., & Kingston, A. E. 1984, J. Phys. B, 17, 3321 (extrapolated).

Table 3.5
Collision strengths Υ for B-like $2p$, F-like $2p^5$, Al-like $3p$ and Cl-like $3p^5$ ions

Ion	$^1P^o_{1/2},\ ^1P^o_{3/2}$	Ion	$^1P^o_{1/2},\ ^1P^o_{3/2}$
C^{+}	2.15	Si^{+}	5.70
N^{+2}	1.45	S^{+3}	8.54
O^{+3}	2.34	Ar^{+5}	6.33
Ne^{+5}	3.21		
Ne^{+}	0.28	Ar^{+}	2.93
Mg^{+3}	0.36	Ca^{+3}	1.00
Si^{+5}	0.30		

B-like ions from Blum, R. D., & Pradhan, A. K. 1992, ApJS 80, 425; F-like ions from Saraph, H. E. & Tully, J. A. 1994, A&AS, 107, 29; Si^{+} Dufton, P. L., & Kingston, A. E. 1994, ADNDT, 57, 273, S^{+3} Tayal, S. S. 2000, ApJ, 530, 1091; Ar^{+5} Saraph, H. E., & Storey, P. J. 1996, A&AS, 115, 151; Ar^{+}, Ca^{+3} Pelan, J., & Berrington, K. A. 1995, A&AS, 110, 209.

Table 3.6
Collision strengths Υ for C-like $2p^2$, O-like $2p^4$, Si-like $3p^2$ and S-like $3p^4$ ions

Ion	$^3P,\ ^1D$	$^3P,\ ^1S$	$^1D,\ ^1S$	$^3P_0,\ ^3P_1$	$^3P_0,\ ^3P_2$	$^3P_1,\ ^3P_2$	$^3P,\ ^5S^o$
N^{+}	2.64	0.29	0.83	0.41	0.27	1.12	1.27
O^{+2}	2.29	0.29	0.58	0.55	0.27	1.29	0.18
Ne^{+4}	2.09	0.25	0.58	1.41	1.81	5.83	1.51
Ne^{+2}	1.36	0.15	0.27	0.24	0.21	0.77	—
S^{+2}	6.95	1.18	1.38	3.98	1.31	7.87	2.85
Ar^{+4}	3.21	0.56	1.65	2.94	1.84	7.81	—
Ar^{+2}	4.83	0.84	1.22	1.26	0.67	3.09	—

N^{+}, O^{+2}, and Ne^{+4} from Lennon, D. J., & Burke, V. M. 1994, A&AS, 103, 273; Ne^{+2} from Butler, K., & Zeippen, C. J. 1994, A&AS, 108, 1; S^{+2} from Tayal, S. S., and Gupta, G. P. 1999 ApJ 526, 544; Ar^{+2}, Ar^{+4} from Galavis, M. E., Mendoza, C., & Zeippen, C. J. 1995, A&AS, 111, 347.

Table 3.7
Collision strengths Υ for N-like $2p^3$ and P-like $3p^3$ ions

Ion	$^4S^o,\ ^2D^o$	$^4S^o,\ ^2P^o$	$^2D^o_{3/2},\ ^2D^o_{5/2}$	$^2D^o_{3/2},\ ^2P^o_{1/2}$
O^{+}	1.34	0.40	1.17	0.28
Ne^{+3}	1.40	0.47	1.36	0.34
S^{+}	6.90	3.53	7.47	1.79
Ar^{+3}	1.90	1.18	7.06	1.51

Ion	$^2D^o_{3/2},\ ^2P^o_{3/2}$	$^2D^o_{5/2},\ ^2P^o_{1/2}$	$^2D^o_{3/2},\ ^2D^o_{5/2}$	$^2P^o_{1/2},\ ^2P^o_{3/2}$
O^{+}	0.82	0.33	1.23	0.157
Ne^{+3}	0.51	0.37	0.90	0.34
S^{+}	3.00	2.20	4.99	2.71
Ar^{+3}	2.14	1.53	7.06	2.07

O^{+} Pradhan, A. K. 1976, MNRAS, 177, 31, 1998, and J Phys B, 31, 4317; Ne^{+3}, Giles, K. 1981, MNRAS, 195, 63, and Ramsbottom, C. A., Bell, K. L., & Keenan, F. P. 1998, MNRAS, 293, 233; S^{+} Ramsbottom, C. A., Bell, K. L., Stafford, R. P. 1996, ADNDT, 63, 57; Ar^{+3} Ramsbottom, C. A., & Bell, K. L. 1997, ADNDT, 66, 65.

It is convenient to remember that, for an electron with the mean energy at a typical nebular temperature $T \approx 7{,}500$ K, the cross sections for excitation and deexcitation are $\sigma \approx 10^{-5}\,\langle\Omega\rangle/g$ cm^2.

Sum rule for collision strengths

- Quantum mechanical sum rule for collision strengths for the case where one term consists of a singlet ($S = 0$ or $L = 0$) and the second consists of a multiplet: the collision strength of each fine structure level J is related to the total collision strength of the multiplet by

$$\Omega_{(SLJ,\ S'L'J')} = \frac{(2J'+1)}{(2S'+1)(2L'+1)}\Omega_{(SL,\ S'L')}$$

  Here, $(2J'+1)$ is the statistical weight of an individual **level** in the multiplet, and $(2S'+1)(2L'+1)$ is the statistical weight of the multiplet **term**.

  We can regard the collision strength as "shared" amongst these levels in proportion to the statistical weights of the individual levels ($g_J = 2J+1$).

- ***The flux ratio between the lines in a multiplet is proportional to the ratio of their collision strengths, in a low density medium.*** Then, the flux ratio is determined by the ratio of their statistical weights.

  ✦ C-like ions ($1s^2 2s^2 2p^2 \rightarrow 1s^2 2s^2 2p^2$, such as O^{++}) forbidden or inter combination transitions.

    ground states (triplet) - ${}^3P_0$: ${}^3P_1$: ${}^3P_2$ = 1 : 3 : 5

    excited states (singlets) - ${}^1D_2$, ${}^1S_1$

  ✦ Li-like ions ($1s^2 2s^1 \rightarrow 1s^2 2p^1$, such as C^{+3}) resonance transitions

    ground state (singlet) - ${}^2S_{1/2}$

    excited states (doublet) - ${}^2P_{3/2}$: ${}^2P_{1/2}$ = 2 : 1

Collisional Excitation Lines : mostly Forbidden Lines

- For all the low-lying levels of the ions in H II regions, the excited levels arise from the same electron configurations as the ground level. Radiative transitions between the excited levels and the ground level are forbidden by the electric-dipole selection rules. They occur by magnetic-dipole or electric-quadrupole transitions.

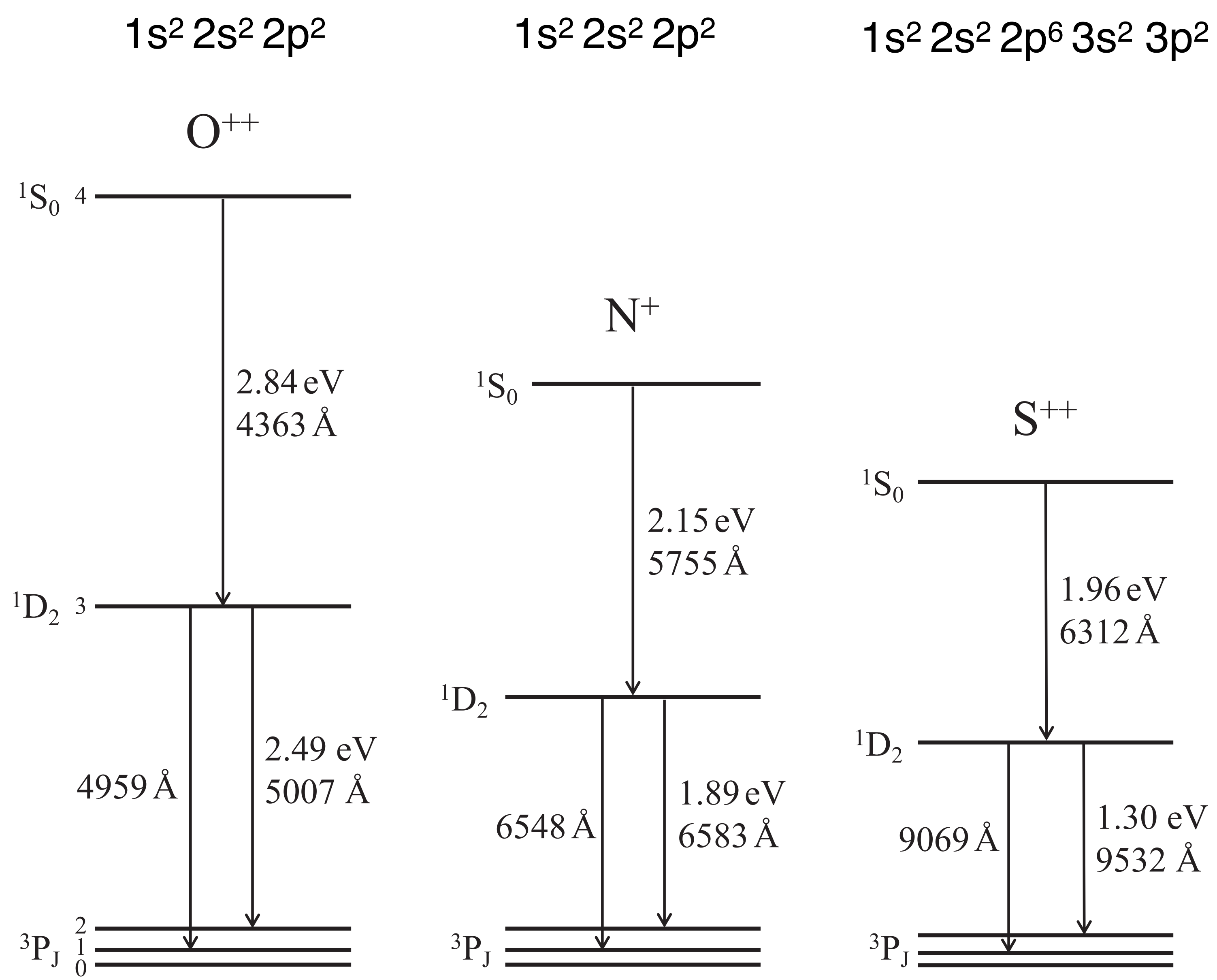


Figure 4.6 [Ryden, Interstellar and Intergalactic Medium]

Collisionally-Excited Emission Line

- Emission line flux
  - **In the low density limit**, the collisional rate between atoms and electrons is much slower than the (spontaneous) radiative de-excitation rate of of the excited level. Thus, we can balance the collisional feeding into level 2 by the rate of radiative transition back down to level 1. The level population is determined by

$$n_e n_1 q_{12} = A_{21} n_2$$

$$\begin{aligned}\frac{n_2}{n_1} &= \frac{n_e q_{21}}{A_{21}} \\ &= \frac{n_e}{A_{21}} \beta \frac{\langle \Omega_{21} \rangle}{g_1} T^{-1/2} \exp\left(-\frac{E_{21}}{kT}\right)\end{aligned}$$

where A_{21} is the Einstein coefficient for spontaneous emission. The line emissivity is given by

$$\begin{aligned}4\pi j_{21} &= E_{21} A_{21} n_2 = E_{21} n_e n_1 q_{12} \\ &= n_e n_1 E_{21} \frac{8.62942 \times 10^{-6}}{T^{1/2}} \frac{\langle \Omega_{21} \rangle}{g_1} \exp\left(-\frac{E_{21}}{kT}\right) \quad [\text{erg cm}^{-3} \text{ s}^{-1}] \\ &\simeq \beta \chi n_e^2 E_{21} T^{-1/2} \frac{\langle \Omega_{21} \rangle}{g_1} \exp\left(-\frac{E_{21}}{kT}\right)\end{aligned}$$

Here, $\beta = \left(\frac{2\pi\hbar^4}{k m_e^2}\right)^{1/2} = 8.62942 \times 10^{-6}$

$\chi = n_1 / n_e$

For low temperature, the exponential term dominates because few electrons have energy above the threshold for collisional excitation, so that the line rapidly fades with decreasing temperature.

At high temperature, the $T^{-1/2}$ term controls the cooling rate, so the line fades slowly with increasing temperature.

- **In high-density limit**, the level population are set by the Boltzmann equilibrium, and the line emissivity is

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left(-\frac{E_{21}}{kT}\right)$$

$$\begin{aligned} 4\pi j_{21} &= E_{21} A_{21} n_2 \\ &= n_1 E_{21} A_{21} \frac{g_2}{g_1} \exp\left(-\frac{E_{21}}{kT}\right) \\ &\simeq \chi n_e E_{21} A_{21} \frac{g_2}{g_1} \exp\left(-\frac{E_{21}}{kT}\right) \end{aligned}$$

Here, **the line flux scales as n_e rather than n_e^2, but the line flux tends to a constant value at high temperature.**

- **In general**, the equilibrium equation for the balance between the excitation and deexcitation rates of the excited level is

$$n_e n_2 q_{12} = n_e n_2 q_{21} + A_{21} n_2$$

$$\frac{n_2}{n_1} = \frac{n_e q_{12}}{A_{21}} \left[\frac{1}{1 + \frac{n_e q_{21}}{A_{21}}} \right]$$

- Then, the cooling rate is

$$L_C = n_2 A_{21} h\nu_{21} = n_e n_1 q_{12} h\nu_{21} \left[\frac{1}{1 + \frac{n_e q_{21}}{A_{21}}} \right]$$

As $n_e \rightarrow 0$, we obtain $L_C = n_e n_1 q_{12} h\nu_{21}$

As $n_e \rightarrow \infty$, $L_C \rightarrow n_1 \frac{q_{12}}{q_{21}} A_{21} h\nu_{21}$

$$= n_1 \frac{g_2}{g_1} \exp\left(-E_{21}/kT\right) A_{21} h\nu_{21}$$

This is the thermodynamic-equilibrium cooling rate.

- Most ions have more levels, and all ions with ground configurations p^2, p^3, or p^4 have five low-lying levels.

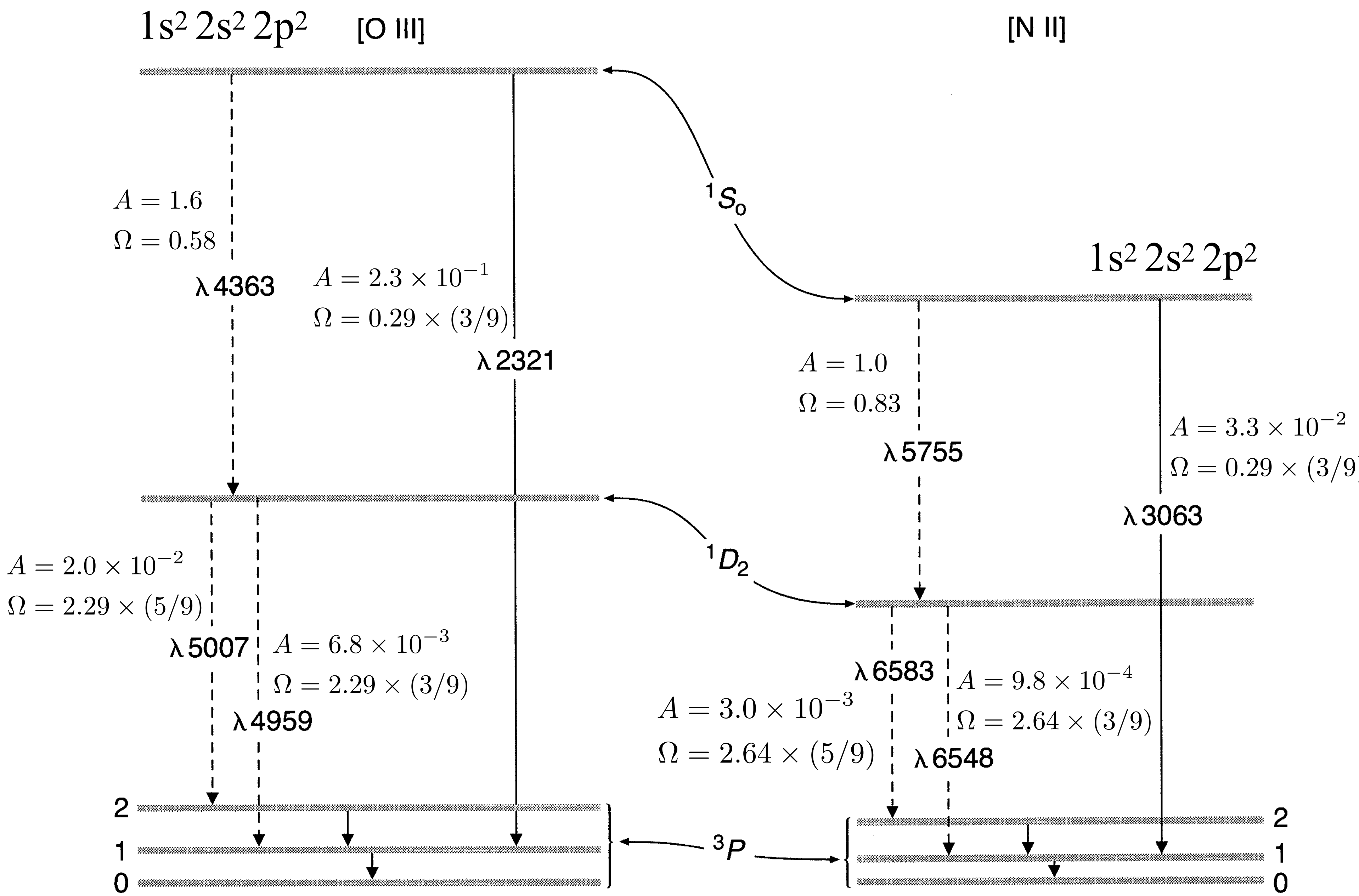


Figure 3.1 [AGN^2]

See Table 3.12 for A and
Table 3.6 for Collision Strength

Ion	$^3P,\ ^1D$	$^3P,\ ^1S$	$^1D,\ ^1S$
N^+	2.64	0.29	0.83
O^{+2}	2.29	0.29	0.58

Table 3.6 Collision Strength

- The equilibrium equations for each of the levels *i* are given by

$$\sum_{j\neq i} n_j n_e k_{ji} + \sum_{j>i} n_j A_{ji} = \sum_{j\neq i} n_i n_e k_{ij} + \sum_{j<i} n_i A_{ij}$$

which, together with the total number of ions

$$\sum_j n_j = n$$

can be solved for the relative population in each level. Then, the collisionally excited radiative cooling rate is

$$L_C = \sum_i n_i \sum_{j<i} A_{ij} h\nu_{ij} \quad [\mathrm{erg\ cm^{-3}\ s^{-1}}]$$

- Critical density
  - For any i, j, collisional deexcitation is not negligible if

$$n_i n_e k_{ij} > n_i \sum_{k<i} A_{ik} \quad \rightarrow \quad n_e k_{ij} > \sum_{k<i} A_{ik}$$

  - For any level i, **critical density** is defined as **the density where the radiative depopulation rate matches the collisional de-excitation for the excited state.**

$$n_c(i) = \sum_{j<i} A_{ij} \Big/ \sum_{j \neq i} k_{ij}$$

  - For $n_e < n_c(i)$, collisional deexcitation of level i is negligible, but for $n_e > n_c(i)$, it is important.

  - For a two level atomic system, the critical density is given by

$$\begin{aligned} A_{21} n_2 &= n_e n_2 q_{21} \\ n_{\rm crit} &= \frac{A_{21}}{q_{21}} \end{aligned} \qquad \rightarrow \qquad \begin{aligned} n_{\rm crit} &= A_{21} \frac{g_2}{\beta \langle \Omega_{21} \rangle} T^{1/2} \\ &= 1.2 \times 10^3 \frac{A_{21}}{10^{-4}\,{\rm s}^{-1}} \frac{g_2}{\langle \Omega_{21} \rangle} \left(\frac{T}{10^4\,{\rm K}} \right)^{1/2} \quad [{\rm cm}^{-3}] \end{aligned}$$

- As can be seen in Tables and the formula, collisional de-excitation is negligible for resonance and most forbidden lines in the ISM.

Ion	ℓ	u	E_ℓ/k (K)	E_u/k (K)	$\lambda_{u\ell}$ (μm)	$n_{\rm H,crit}(u)$ $T = 100\,$K ($\rm cm^{-3}$)	$n_{\rm H,crit}(u)$ $T = 5000\,$K ($\rm cm^{-3}$)
C II	$^2P^o_{1/2}$	$^2P^o_{3/2}$	0	91.21	157.74	2.0×10^3	1.5×10^3
C I	$^3P_0$	$^3P_1$	0	23.60	609.7	620	160
	$^3P_1$	$^3P_2$	23.60	62.44	370.37	720	150
O I	$^3P_2$	$^3P_1$	0	227.71	63.185	2.5×10^5	4.9×10^4
	$^3P_1$	$^3P_0$	227.71	326.57	145.53	2.3×10^4	8.4×10^3
Si II	$^2P^o_{1/2}$	$^2P^o_{3/2}$	0	413.28	34.814	1.0×10^5	1.1×10^4
Si I	$^3P_0$	$^3P_1$	0	110.95	129.68	4.8×10^4	2.7×10^4
	$^3P_1$	$^3P_2$	110.95	321.07	68.473	9.9×10^4	3.5×10^4

Table 17.1 in [Draine]

- However, it is not true for the 21 cm hyperfine structure line of hydrogen.
  - The critical density for 21cm line is

$$n_{\rm crit} \sim 10^{-3}\,(T/100\,{\rm K})^{-1/2}\ \ [{\rm cm^{-3}}]$$

$$A_{21} = 2.88 \times 10^{-15}\ [{\rm s^{-1}}]$$

  - The hyperfine levels are thus essentially in collisional equilibrium in the CNM.

The collisional strengths and other atomic data are available in the CHIANTI atomic database (https://www.chiantidatabase.org/).

Collision strengths at T = 10^4 K

Table 4.1 in The Interstellar Medium [Lequeux]

Ion	Transition l–u	λ µm	A_{ul} $\rm s^{-1}$	Ω_{ul}	n_{crit} $\rm cm^{-3}$
C I	$^3P_0-^3P_1$	609.1354	7.93×10^{-8}	–	(500)
	$^3P_1-^3P_2$	370.4151	2.65×10^{-7}	–	(3000)
C II	$^2P_{1/2}-^2P_{3/2}$	157.741	2.4×10^{-6}	1.80	47 (3000)
N II	$^3P_0-^3P_1$	205.3	2.07×10^{-6}	0.41	41
	$^3P_1-^3P_2$	121.889	7.46×10^{-6}	1.38	256
	$^3P_2-^1D_2$	0.65834	2.73×10^{-3}	2.99	7700
	$^3P_1-^1D_2$	0.65481	9.20×10^{-4}	2.99	7700
N III	$^2P_{1/2}-^2P_{3/2}$	57.317	4.8×10^{-5}	1.2	1880
O I	$^3P_2-^3P_1$	63.184	8.95×10^{-5}	–	2.3×10^4 (5×10^5)
	$^3P_1-^3P_0$	145.525	1.7×10^{-5}	–	3400 (1×10^5)
	$^3P_2-^1D_2$	0.63003	6.3×10^{-3}	–	1.8×10^6
O II	$^4S_{3/2}-^2D_{5/2}$	0.37288	3.6×10^{-5}	0.88	1160
	$^4S_{3/2}-^2D_{3/2}$	0.37260	1.8×10^{-4}	0.59	3890
O III	$^3P_0-^3P_1$	88.356	2.62×10^{-5}	0.39	461
	$^3P_1-^3P_2$	51.815	9.76×10^{-5}	0.95	3250
	$^3P_2-^1D_2$	0.50069	1.81×10^{-2}	2.50	6.4×10^5
	$^3P_1-^1D_2$	0.49589	6.21×10^{-3}	2.50	6.4×10^5
	$^1D_2-^1S_0$	0.43632	1.70	0.40	2.4×10^7
Ne II	$^2P_{1/2}-^2P_{3/2}$	12.8136	8.6×10^{-3}	0.37	5.9×10^5
Ne III	$^3P_2-^3P_1$	15.5551	3.1×10^{-2}	0.60	1.27×10^5
	$^3P_1-^3P_0$	36.0135	5.2×10^{-3}	0.21	1.82×10^4
Si II	$^2P_{1/2}-^2P_{3/2}$	34.8152	2.17×10^{-4}	7.7	(3.4×10^5)
S II	$^4S_{3/2}-^2D_{5/2}$	0.67164	2.60×10^{-4}	4.7	1240
	$^4S_{3/2}-^2D_{3/2}$	0.67308	8.82×10^{-4}	3.1	3270
S III	$^3P_0-^3P_1$	33.4810	4.72×10^{-4}	4.0	1780
	$^3P_1-^3P_2$	18.7130	2.07×10^{-3}	7.9	1.4×10^4
S IV	$^2P_{1/2}-^2P_{3/2}$	10.5105	7.1×10^{-3}	8.5	5.0×10^4
Ar II	$^2P_{1/2}-^2P_{3/2}$	6.9853	5.3×10^{-2}	2.9	1.72×10^6
Ar III	$^3P_2-^3P_1$	8.9914	3.08×10^{-2}	3.1	2.75×10^5
	$^3P_1-^3P_0$	21.8293	5.17×10^{-3}	1.3	3.0×10^4
Fe II	$^6D_{7/2}-^6D_{5/2}$	35.3491	1.57×10^{-3}	–	(3.3×10^6)
	$^6D_{9/2}-^6D_{7/2}$	25.9882	2.13×10^{-3}	–	(2.2×10^6)

Energy Loss by Collisionally Excited Line Radiation of H

- H^+ has no bound levels and thus no lines.
- H^0 may affect the radiative cooling in a nebula.
  - The most important excitation processes from the ground $1\,^2S$ term are

    (1) $1\,^2S \rightarrow 2\,^2P^o$, followed by emission of a Lyα photon with $h\nu = 10.2\,\mathrm{eV}$

    (2) $1\,^2S \rightarrow 2\,^2S$, followed by two photon emission with $h\nu' + h\nu'' = 10.2\,\mathrm{eV}$ and transition probability $A(2\,^2S \rightarrow 1\,^2S) = 8.23\,\mathrm{s}^{-1}$.
  - Cross sections for excitation of neutral atoms by electrons do not vary as u^{-2}, but rise from zero at the threshold, peak at energies several times the threshold, and then decline at high energies, often with superimposed resonances.
  - The mean collision strengths, integrated over the Maxwellian velocity distribution of the electrons vary fairly slows, as Table 3.16 shows.

Table 3.16

Effective collision strengths for H I

T (K)	$1\,^2S, 2\,^2S$	$1\,^2S, 2\,^2P^o$	$1\,^2S, 3\,^2S$	$1\,^2S, 3\,^2P^o$	$1\,^2S, 3\,^2D$
10,000	0.29	0.51	0.066	0.12	0.063
15,000	0.32	0.60	0.071	0.13	0.068
20,000	0.35	0.69	0.077	0.14	0.073

Anderson, H., Balance, C. P., Badnell, N. R., & Summers, H. P. 2000, J.Phys.B, 33, 1255.

Resulting Thermal Equilibrium

- The temperature at each point in a static nebula is determined by the equilibrium between heating and cooling rates:

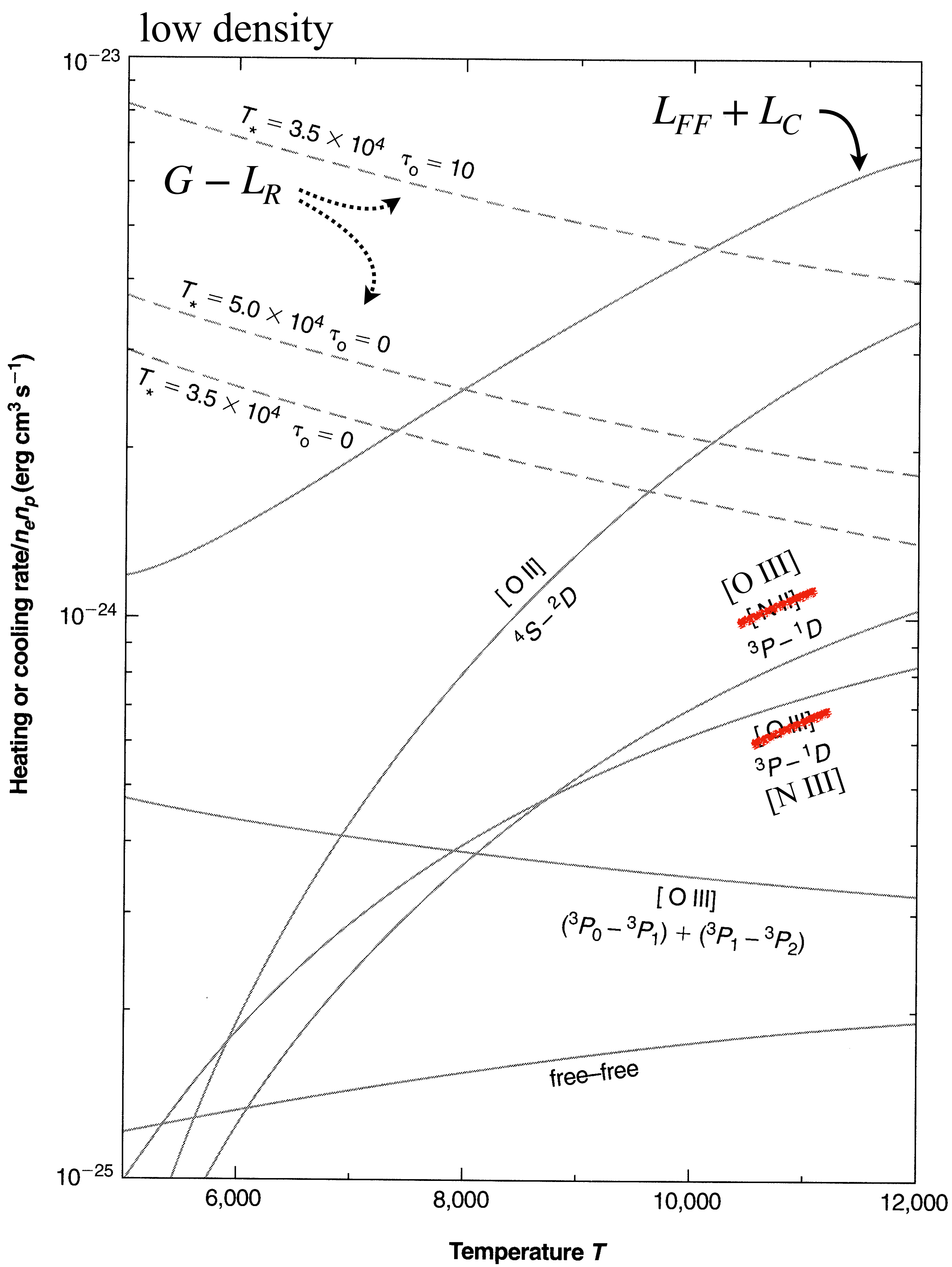
$$G = L_R + L_{FF} + L_C$$

  - In the low-density limit, all the terms in G, L_R, L_{FF}, and L_C are proportional to n_e and to the density of some ion. Therefore, the resulting temperature is independent of the total density, but do depend on the relative abundances of the various ions.
  - When collisional deexcitation begins to be important, the cooling rate at a given temperature is decreased, and the equilibrium temperature for a given stellar radiation field is therefore somewhat increased.

- It is convenient to rewrite the above balance equation in the form

$$G - L_R = L_{FF} + L_C$$

  - $G - L_R$ is the "effective heating rate", representing the net energy gained in photoionization processes.

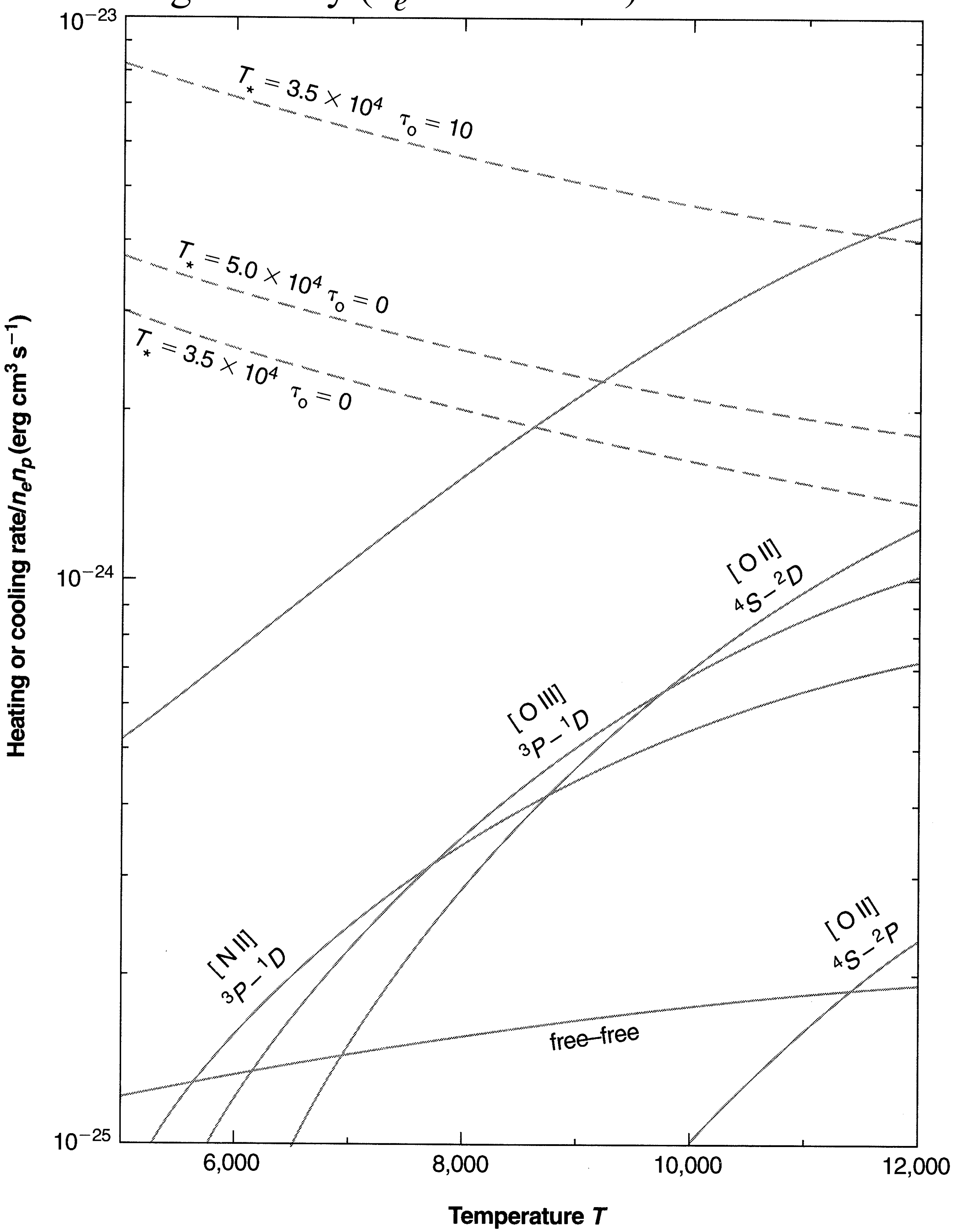


- Dashed lines - net effective heating rates $(G - L_R)$ for various stellar spectra.
- Top solid black curve - total radiative cooling rate $(L_{FF} + L_C)$.
- Other solid curves - individual contributions to radiative cooling.

- For each level, the cooling rate is small if $kT \ll \chi$, then increases rapidly and peaks at $kT \approx \chi$, and then decreases slowly for $kT > \chi$.

- The equilibrium temperature is given by the intersection of a dashed curve and the total radiative cooling rate curve.
- The increased optical depth τ_0 (or increased stellar temperature T_*) increases T by increasing G.
- Typical nebular temperatures are $T \approx 7000 - 8000\,\mathrm{K}$.

high density ($n_e \approx 10^4\,\mathrm{cm}^{-3}$)



- At high electron densities, collisional deexcitation can modify the radiative cooling rate and thus the nebular temperature.
- Higher temperatures occur at high densities.
- Similarly, lower abundances of the heavy elements tend to decrease the cooling rate and thus to increase the equilibrium temperature
- Under conditions of very high ionization, as in the central part of a planetary nebula,
  - the ionization is high enough that there is very little H^0, O^+, or O^{++}, and the main coolants are Ne^{+4} and C^{+3}.
  - The nebular temperature can be $T \lesssim 2 \times 10^4$ K.