Astrophysics

Lecture 09
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Synchrotron Radiation

[Reference]

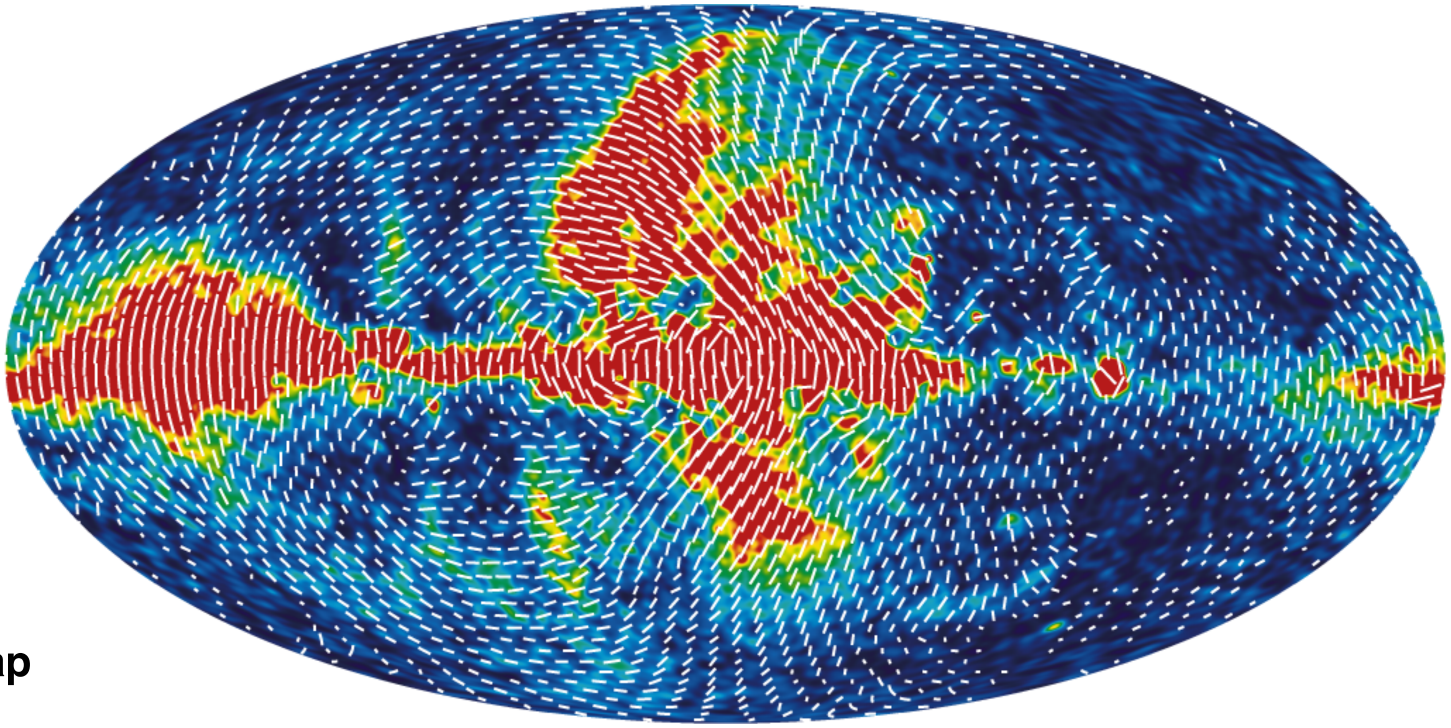
Hale Bradt (2008)
Astrophysics Processes: The Physics of Astronomical Phenomena

[Synchrotron Radiation]

- Particles accelerated by a magnetic field will also radiate. Acceleration by a magnetic field produces ***magnetobremsstrahlung***, the German word for "magnetic braking radiation."
- **Cyclotron radiation**: For non-relativistic velocities, the radiation is called cyclotron radiation. The frequency of emission is simply the frequency of gyration in the magnetic field.
- **Synchrotron radiation**: For extreme relativistic particles, the frequency spectrum is much more complex and can extend to many times the gyration frequency. This radiation is known as synchrotron radiation.
- Synchrotron radiation is ubiquitous in astronomy.

  It accounts for most of the radio emission from active galactic nuclei (AGN), which are thought to be powered by supermassive black holes in galaxies and quasars.

  It dominates the radio continuum emission from star-forming galaxies.



WMAP 23GHz emission and polarization map

Puzzling radiation from the Crab nebula

- A large part of the supernova remnant called the Crab nebula appears as an bluish haze on the sky, and the origin of this light was, for a long time, a big mystery.

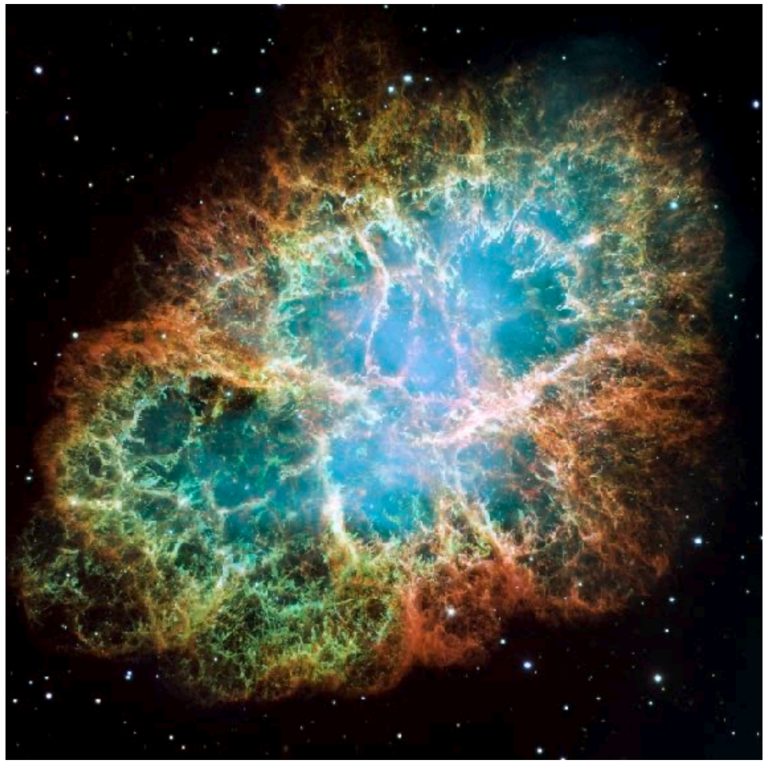
  The bluish luminosity in the optical band could be considered to indicate the presence of a hot, optically thin thermal plasma with a temperature of ~ 50,000 K.

  Such a hot plasma would radiate strong optical emission lines from excited atoms in the plasma. However, no strong spectral lines are observed. The spectrum is a smooth continuum.

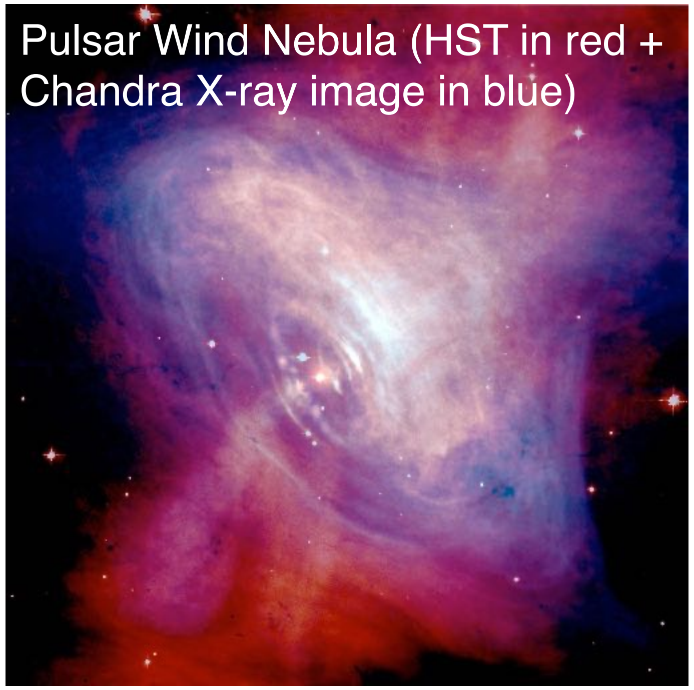
  The puzzle of the source of radiation from the Crab nebula was solved dramatically when a Russian scientist (Iosif Shklovsky) postulated in 1953 that the bluish radiation might be the same as that previously discovered in man-made electron accelerators.

  He therefore suggested that optical astronomers look to see if the radiation from the Crab is polarized. They did (in 1954) and found it to be so.

  Nevertheless, this explanation was quiet a surprise; it required that the radiating electrons have energies in excess of 10^{11} eV. This raised the question of how the electron attained such high energies. Later (1969), it was found that the energy source is a spinning neutron star, the Crab pulsar.



Crab Nebula (HST mosaic image)
The rapidly spinning (30 times a second) neutron star embedded in the center of the nebula is powering the nebula's interior bluish glow. The blue light comes from electrons whirling at nearly the speed of light around magnetic field lines from the neutron star.



Pulsar Wind Nebula (HST in red + Chandra X-ray image in blue)

[Equation of Motion in a uniform magnetic field]

- Consider a particle of mass m and charge q moving in **a uniform magnetic field, with no electric field.**
- Equations of motion:

$$\frac{dE}{dt} = \frac{d(\gamma mc^2)}{dt} = q\mathbf{v}\cdot\mathbf{E} = 0$$
$$\frac{d\mathbf{p}}{dt} = \frac{d(\gamma m\mathbf{v})}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}$$

Recall Four-momentum: $P^\mu = \left(\gamma mc^2,\ \gamma m\mathbf{v}\right)$

Four-force: $F^\mu = \dfrac{dP^\mu}{d\tau} = \gamma\left(\dfrac{1}{c}\dfrac{dE}{dt},\ \dfrac{d\mathbf{p}}{dt}\right)$

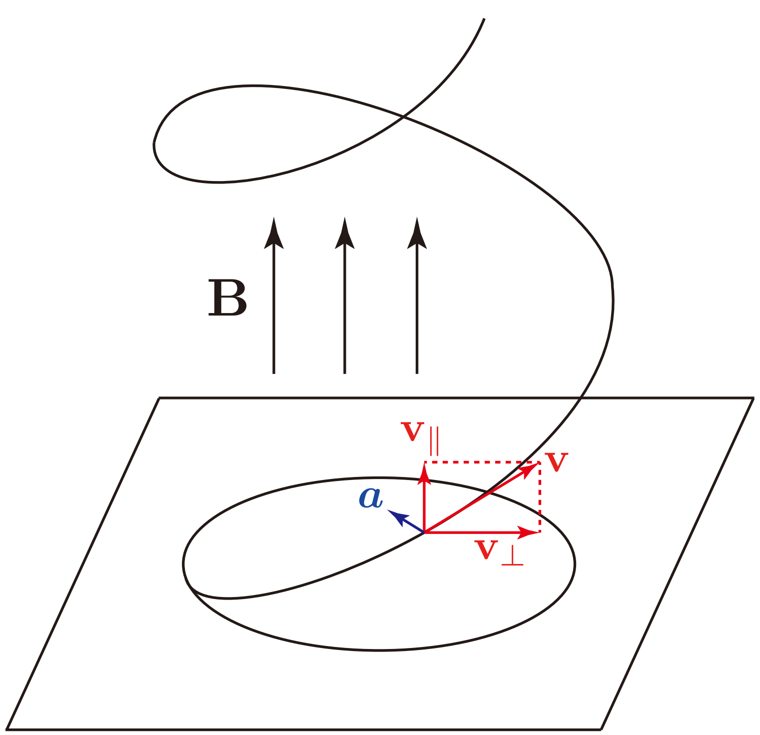
Lorentz Four-force: $F^\mu_{\text{Lorentz}} = \dfrac{q}{c}F^{\mu\nu}U_\nu = \dfrac{q}{c}\gamma\left(\mathbf{E}\cdot\mathbf{v},\ c\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$

The first equation implies that γ = constant (or equivalently $|\mathbf{v}|$ = constant). Therefore, it follows that

$$\boxed{\gamma m\frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}}$$

Decompose the velocity into $\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$, and take dot product with $\mathbf{B}$.

$$\mathbf{B}\cdot\left(\gamma m\frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}\right) \quad\rightarrow\quad \begin{cases} \dfrac{d\mathbf{v}_\parallel}{dt} = 0 \\[2ex] \dfrac{d\mathbf{v}_\perp}{dt} = \dfrac{q}{\gamma mc}\mathbf{v}_\perp\times\mathbf{B} \end{cases}$$



Therefore,

$$\mathbf{v}_\parallel = \text{constant}$$
$$|\mathbf{v}_\perp| = \text{constant}\ \ (\text{since } |\mathbf{v}| = \text{constant})$$

Helical motion: The perpendicular velocity component processes around **B**. Thus, the motion is a combination of the uniform circular motion and the uniform motion along the field.

- Rearrange the above equations:

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$$

$$\frac{d\mathbf{v}_\parallel}{dt} = 0$$

$$\longrightarrow \quad a_\parallel = 0, \quad a_\perp = \omega_\mathrm{B} v_\perp$$

$$\frac{d\mathbf{v}_\perp}{dt} = \frac{-e}{\gamma m_e c}\mathbf{v}_\perp \times \mathbf{B} = -\frac{\omega_\mathrm{B}}{B}\mathbf{v}_\perp \times \mathbf{B} \quad \left(\text{Here, } \omega_\mathrm{B} \equiv \frac{eB}{\gamma m_e c}\right)$$

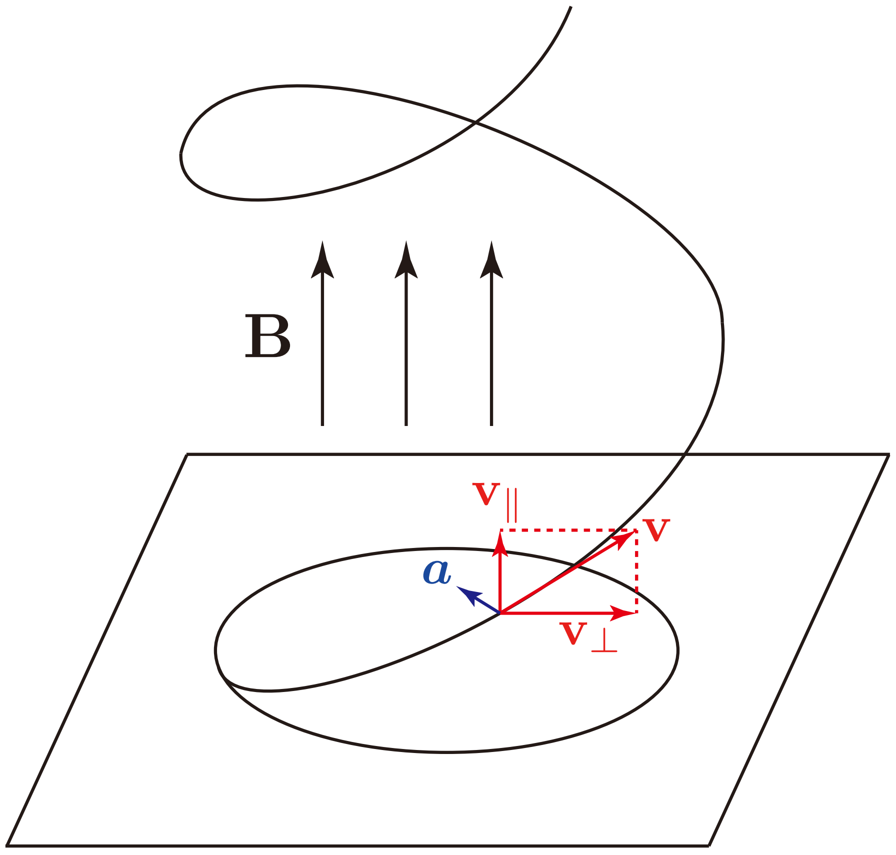
$$\begin{aligned}\frac{d^2\mathbf{v}_\perp}{dt^2} &= -\frac{\omega_\mathrm{B}}{B}\frac{d\mathbf{v}_\perp}{dt} \times \mathbf{B} \\ &= \frac{\omega_\mathrm{B}^2}{B^2}\left(\mathbf{v}_\perp \times \mathbf{B}\right) \times \mathbf{B} \\ &= \frac{\omega_\mathrm{B}^2}{B^2}\left[-\mathbf{v}_\perp\left(\mathbf{B}\cdot\mathbf{B}\right) + \mathbf{B}\left(\mathbf{v}_\perp \cdot \mathbf{B}\right)\right]\end{aligned}$$

$$\frac{d^2\mathbf{v}_\perp}{dt^2} = -\omega_\mathrm{B}^2\mathbf{v}_\perp$$: harmonic oscillator

- Solution:
  - Assuming the initial conditions $\mathbf{v}_\perp \parallel \mathbf{y}$, $\mathbf{r} \parallel \mathbf{x}$, and $z = 0$ at $t = 0$, we obtain the following solution

$$\mathbf{v}(t) = v_\perp \left(-\hat{\mathbf{x}} \sin \omega_{\mathrm{B}} t + \hat{\mathbf{y}} \cos \omega_{\mathrm{B}} t\right) + \hat{\mathbf{z}} v_{\|}$$: harmonic oscillator

$$\mathbf{r}(t) = \frac{v_\perp}{\omega_{\mathrm{B}}} \left(\hat{\mathbf{x}} \cos \omega_{\mathrm{B}} t + \hat{\mathbf{y}} \sin \omega_{\mathrm{B}} t\right) + \hat{\mathbf{z}} v_{\|} t$$



- **Helical motion**: The perpendicular velocity component processes around **B**. Therefore, the motion is ***a combination of the uniform circular motion perpendicular to the magnetic field and the uniform motion along the field***.

- The particle gyrates along the magnetic field lines with the angular velocity ω_B.
- Its trajectory has a helicoidal shape, with gyro radius r_B and pitch angle α.

gyrofrequency: $\omega_\mathrm{B} = \dfrac{eB}{\gamma m_e c}$

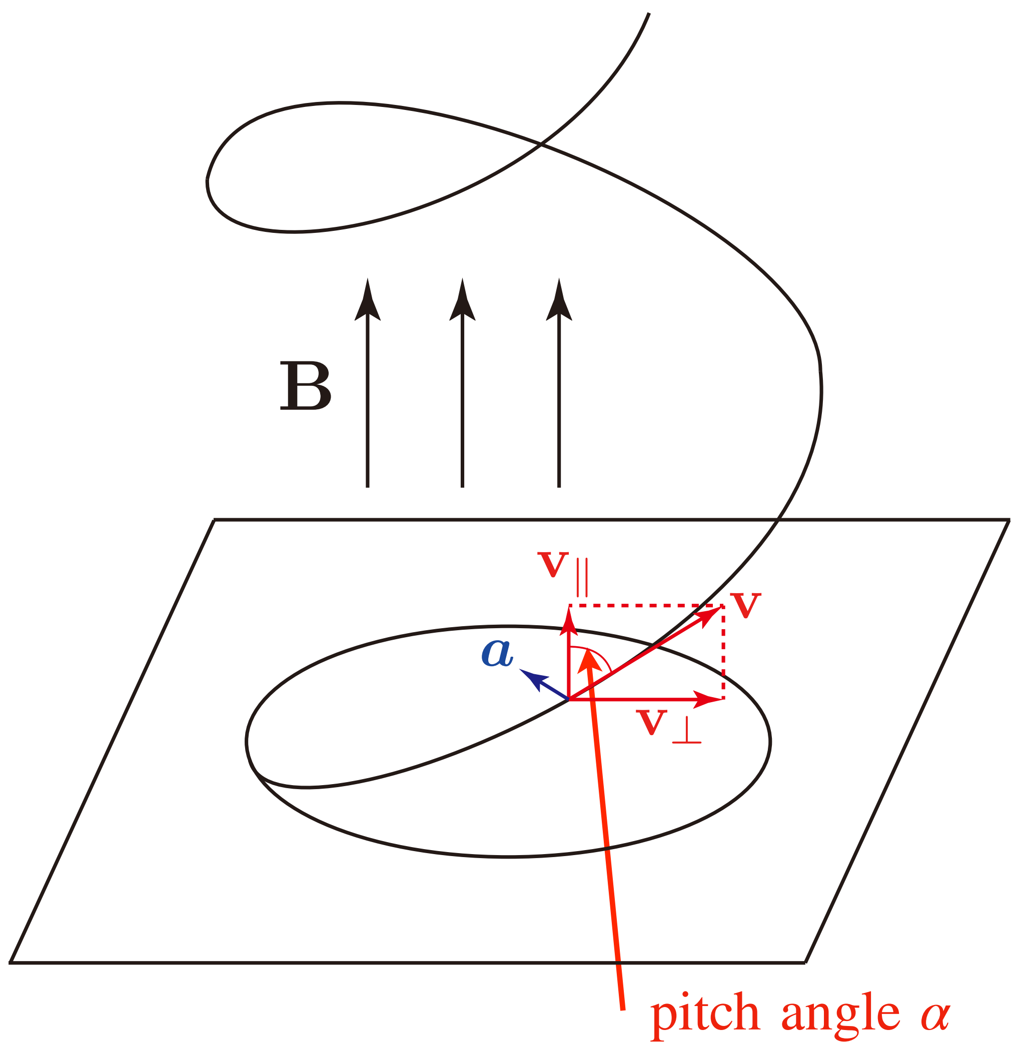
gyroradius: $r_\mathrm{B} = \dfrac{v_\perp}{\omega_\mathrm{B}}$

pitch angle α : $v_\parallel = v\cos\alpha,\ v_\perp = v\sin\alpha$

angle between the magnetic field and velocity
or angle between $\mathbf{v}_\parallel$ and $\mathbf{v}$

$$\omega_\mathrm{B} = \frac{17.6}{\gamma}\frac{B}{\mu\mathrm{G}}\ (\mathrm{Hz})$$

$$r_\mathrm{B} = 1.7\times10^9\ \gamma\beta_\perp\left(\frac{B}{\mu\mathrm{G}}\right)\ (\mathrm{cm})$$



Larmor frequency: $\omega_\mathrm{L} = \dfrac{eB}{m_e c}$ **Larmor radius:** $r_\mathrm{L} = \dfrac{v_\perp}{\omega_\mathrm{L}}$

(It is also called as **Cyclotron frequency** or non-relativistic gyrofrequency.)

[Synchrotron Power Radiated by a Single Electron]

- Total emitted power:

Since $a_\perp = \omega_B v_\perp$ and $a_\parallel = 0$,

$$\begin{aligned} P &= \frac{2q^2}{3c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right) \\ &= \frac{2}{3}\gamma^2\frac{q^4B^2}{m^2c^5}v_\perp^2 = \frac{2}{3}r_e^2 c\beta^2\gamma^2B^2\sin^2\alpha \qquad \leftarrow (v_\perp = v\sin\alpha) \\ &= 2\sigma_T c(\gamma\beta)^2 U_B \sin^2\alpha \end{aligned}$$

where α is the pitch angle, the angle between magnetic field and velocity.

$$\cos\alpha \equiv \frac{\mathbf{v}\cdot\mathbf{B}}{vB},\ \ r_e \equiv \frac{e^2}{mc^2},\ \ \sigma_T = \frac{8\pi}{3}r_e^2,\ \ U_B = \frac{B^2}{8\pi}$$

For an isotropic distribution of velocities, it is necessary to average the formula over all angles.

$$\boxed{\begin{aligned} P &= \frac{4}{3}\sigma_T c(\gamma^2 - 1)U_B \\ &= \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B \end{aligned}} \qquad \leftarrow \qquad \begin{aligned} &\langle\sin^2\alpha\rangle = \frac{1}{4\pi}\int\sin^2\alpha\, d\Omega = \frac{2}{3} \\ &\beta^2 = 1 - \frac{1}{\gamma^2} \end{aligned}$$

Note that $P \propto \begin{cases} m^{-2} & \text{for a fixed velocity } (\gamma) \\ m^{-4} & \text{for a fixed energy } (\gamma mc^2) \end{cases}$ $\longrightarrow$ This indicates that the synchrotron radiation is mostly due to electrons.

- Cooling Time

- The energy balance equation becomes:

$$m_e c^2 \frac{d\gamma}{dt} = -P \longleftarrow P = \frac{2}{3}\gamma^2 \frac{e^4 B^2}{m_e^2 c^3}\beta^2 \left\langle \sin^2\alpha \right\rangle$$

$$\frac{d\gamma}{dt} \simeq -A\gamma^2 \quad \text{where } A = \frac{4e^4B^2}{9m_e^3c^5} \qquad \simeq \frac{4}{9}\gamma^2 \frac{e^4}{m_e^2 c^3} B^2 \quad \leftarrow \quad \beta \simeq 1, \text{ and } \left\langle \sin^2\alpha \right\rangle = 2/3$$

$$\int_{\gamma_0}^{\gamma} \frac{d\gamma'}{\gamma'^2} = -\int_0^t A dt' \quad \rightarrow \quad \boxed{\gamma(t) = \frac{\gamma_0}{1 + A\gamma_0 t} \qquad \text{Here, } \gamma_0 = \gamma(t=0) \text{ and } A \equiv \frac{4e^4B^2}{9m_e^3c^5}.}$$

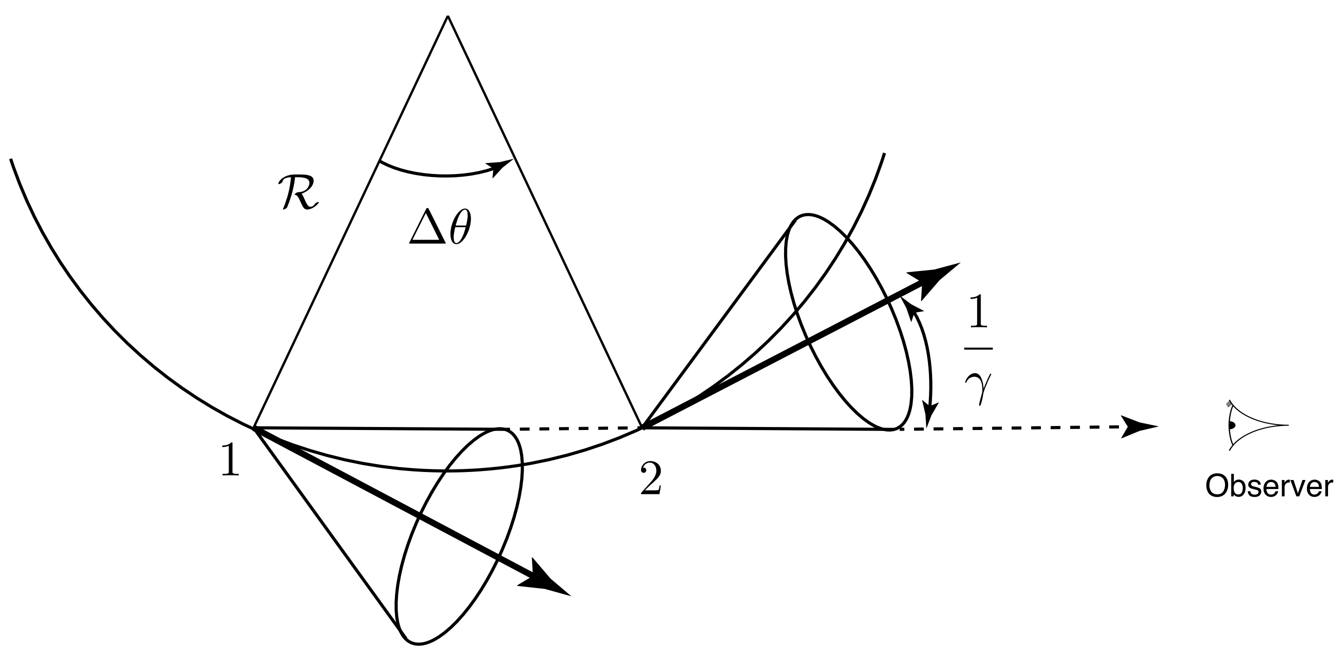
- **Cooling time**: the typical timescale for the electron to loose about of its energy is approximately

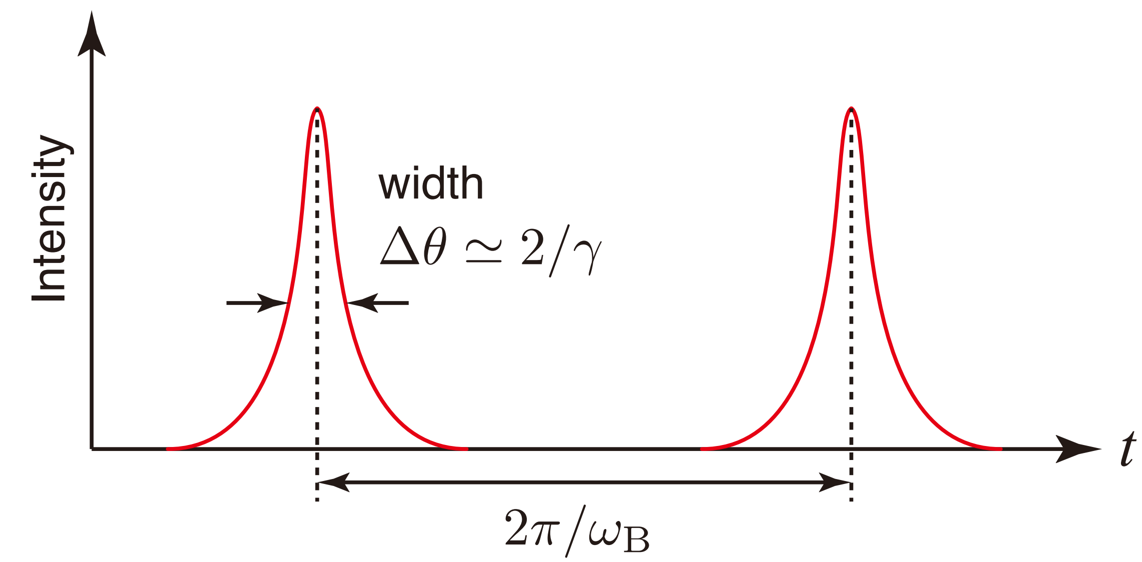
$$t_{\rm cool} = t_{1/2} = \frac{1}{A\gamma_0} = \frac{7.8\times 10^8}{\gamma_0 B^2}\ \text{s}$$

In the vicinity of a supermassive AGN black hole, $B \approx 10^3$ Gauss and for $\gamma \approx 10^3$.

Location	Typical B	$t_{\rm cool}$	cooling length $\approx ct_{\rm cool}$	size of object
Interstellar medium	10^{-6} G	10^{10} years	10^{28} cm	10^{22} cm
Stellar atmosphere	1 G	5 days	10^{15} cm	10^{11} cm
Supermassive black hole	10^4 G	10^{-3} sec	3 10^7 cm	10^{14} cm
White dwarf	10^8 G	10^{-11} sec	3 mm	1000 km
Neutron star	10^{12} G	10^{-19} sec	3 10^{-9} cm	10 km

[Spectrum of Synchrotron Radiation: A Qualitative Discussion]





- Because of beaming effects the emitted radiation fields appear to be concentrated in a narrow set of directions about the particle's velocity.

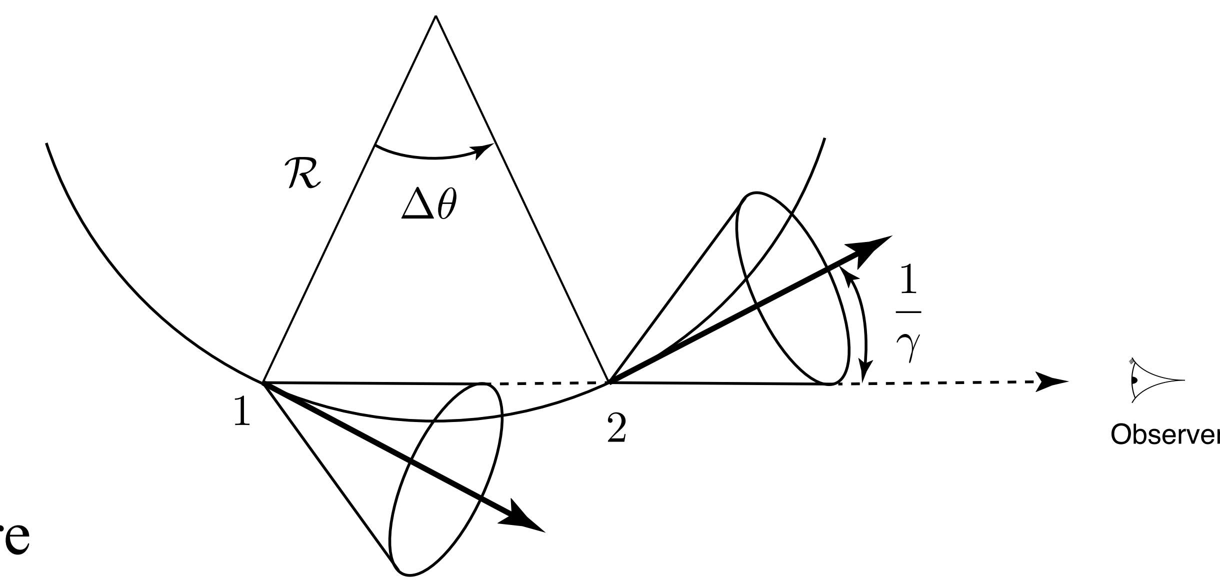
  The observer will see a pulses of radiation confined to a time interval much smaller than the gyration period. The spectrum will thus be spread over a much broader frequency range than one of order $\omega_{\rm B}$.

  The cone of emission has an angular width $\sim 1/\gamma$. Therefore, the observer will see emission over the angular range of $\Delta\theta \simeq 2/\gamma$.

  The radiation appears beamed toward the direction of the observer in **a series of pulses spaced in time (period)** $2\pi/\omega_{\rm B}$ **apart**, but with **each pulse lasting only** $\Delta\theta \simeq 2/\gamma$.

- To Fourier analyze the pulse shape, we need to **calculate the interval of the arrival times of the pulse corresponding to $\Delta\theta \sim 2/\gamma$.**
    - Let's consider an instantaneous rest frame of the electron.

$\Delta s = \mathcal{R}\Delta\theta$ = the path length from point 1 to 2
$\mathcal{R}$ = the radius of curvature of the path
$\Delta t = \Delta s / v$ = time interval from point 1 to 2
$|\Delta \mathbf{v}| = v\Delta\theta$ = velocity change



- From the equation of motion, we find the curvature radius:

$$\gamma m_e \frac{\Delta \mathbf{v}}{\Delta t} = \frac{e}{c}\mathbf{v}\times\mathbf{B}$$

$$\gamma m_e \frac{v\Delta\theta}{\Delta s/v} = \frac{e}{c} vB\sin\alpha \rightarrow \boxed{\mathcal{R} = \frac{\Delta s}{\Delta\theta} = \frac{v}{\omega_B \sin\alpha}}$$

- Therefore the path length is given by

$$\boxed{\Delta s = \mathcal{R}(2/\gamma) = \frac{2v}{\gamma\omega_B\sin\alpha} = \frac{2v}{\omega_L \sin\alpha}}$$

Note that the radius of curvature is different from the gyroradius, which is the projected radius of the curvature radius.

$$\mathcal{R} = \frac{r_B}{\sin^2\alpha} > r_B$$

$$v_\perp = v\sin\alpha$$
$$r_B = \frac{v_\perp}{\omega_B}$$

- Time interval that the particle passes from point 1 to 2:

$$\Delta t = t_2 - t_1 = \frac{\Delta s}{v} \simeq \frac{2}{\omega_\mathrm{L} \sin\alpha}$$

- However, this is the time interval for the particle to travel from point 1 to 2. **We need to calculate the interval of the arrival times of the pulse measured in the observer frame.**

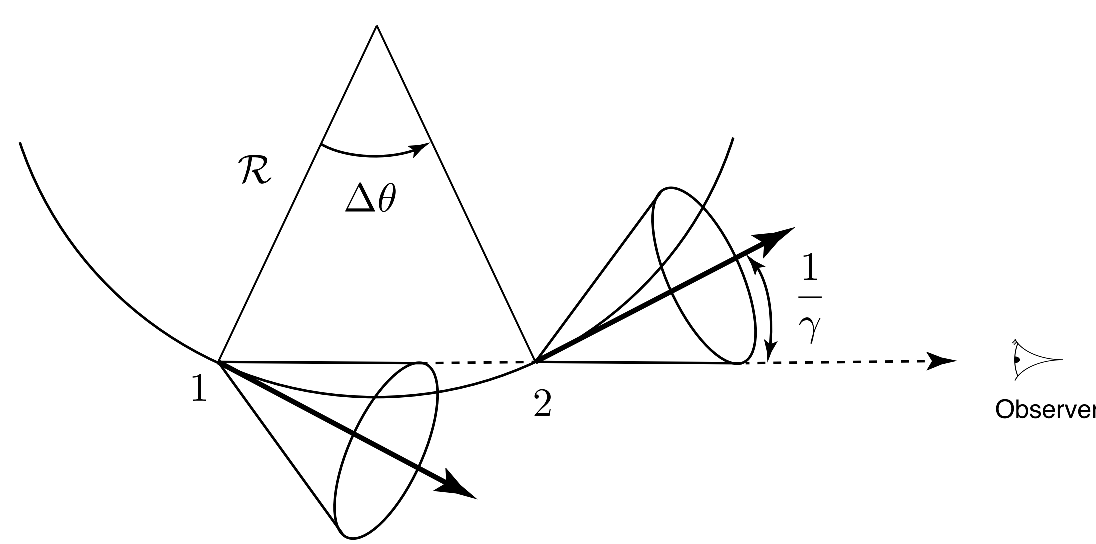
- Note that point 2 is closer than point 1 by $\Delta s/c$. Therefore, the difference of the arrival times of the pulse is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx \frac{1}{\sqrt{(1-\beta)2}} \leftarrow \beta \approx 1$$

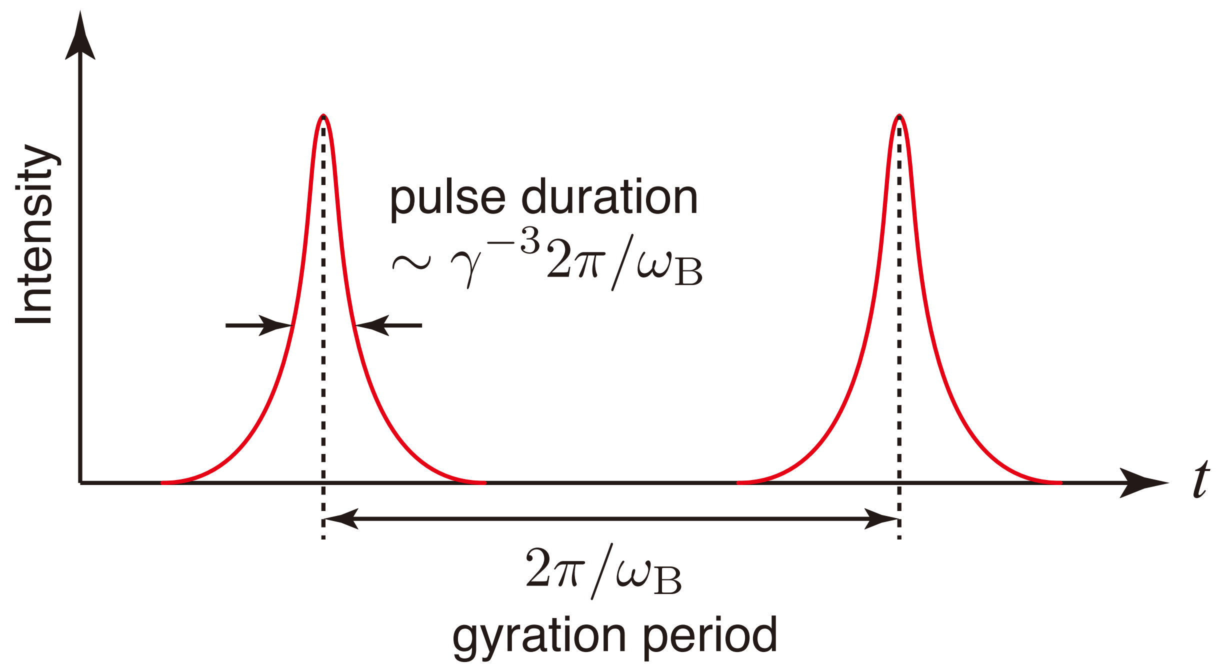
$$\Delta t^A = t_2^A - t_1^A = \Delta t - \frac{\Delta s}{c} = \Delta t\left(1 - \frac{v}{c}\right) \approx \frac{1}{\gamma^2 \omega_\mathrm{L} \sin\alpha} \qquad \leftarrow \quad 1 - \frac{v}{c} \approx \frac{1}{2\gamma^2}$$

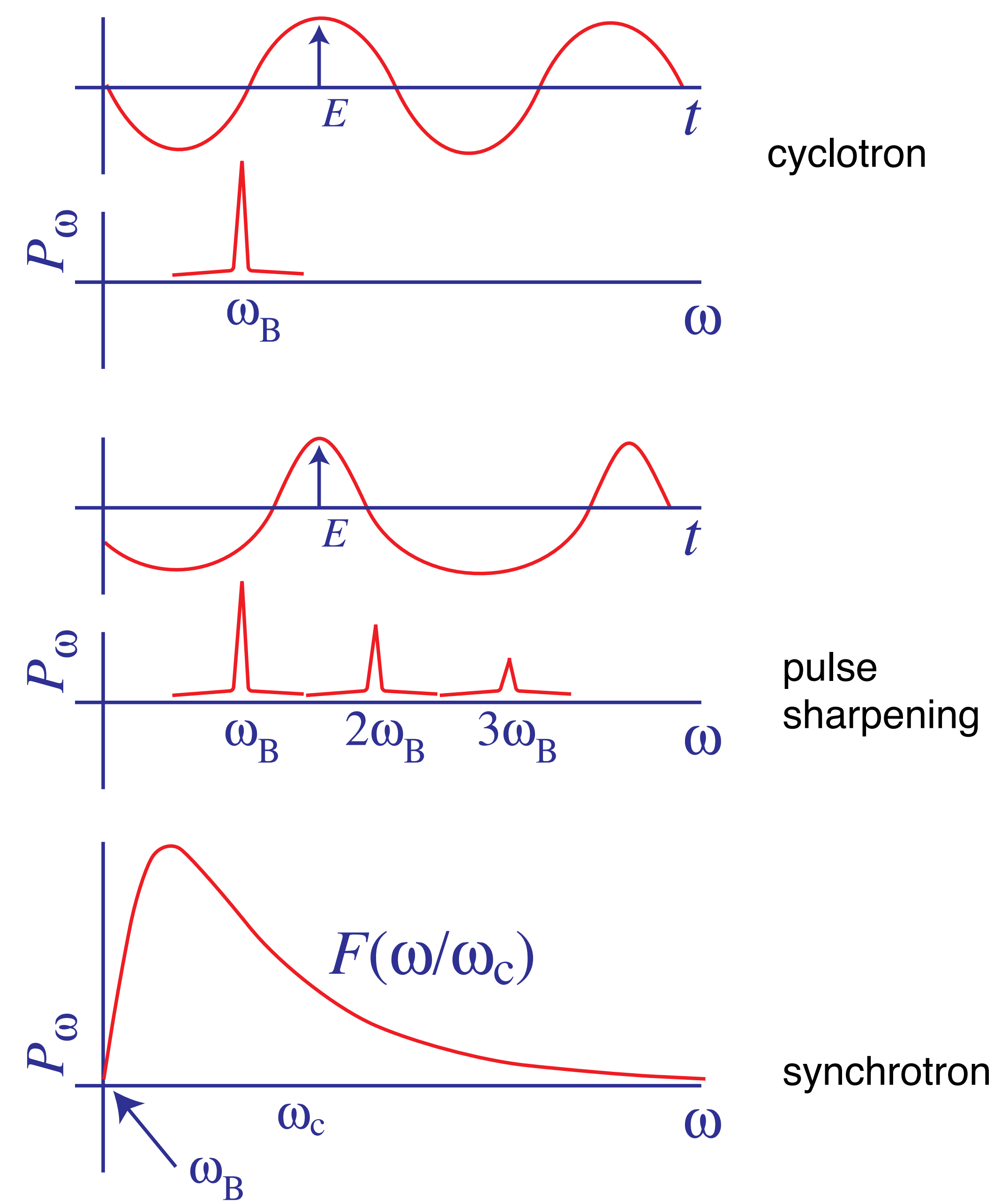
$$\boxed{\Delta t^A = t_2^A - t_1^A \approx \frac{1}{\gamma^2 \omega_\mathrm{L} \sin\alpha} = \frac{1}{\gamma^3 \omega_\mathrm{B} \sin\alpha}}$$

The width of the observed pulses is smaller than the gyration by a factor γ^2.



- Temporal pattern of received pulses:





- We define a critical frequency:

$$\boxed{\omega_c \equiv \frac{3}{2}\gamma^2 \omega_{\rm L} \sin\alpha = \frac{3}{2}\gamma^3 \omega_{\rm B} \sin\alpha}$$

The factor 3/2 is from the accurate calculation. (See Pacholczyk 1970, Radio Astrophysics. Nonthermal Processes in Galactic and Extragalactic Sources)

- From the properties of Fourier transformation, we expect that ***the spectrum will be fairly broad, within the frequency range of*** $\omega_{\rm B} \lesssim \omega \lesssim \omega_c$.

- We can derive an important property of the spectrum for the synchrotron radiation.

Remember that the electric field is a function of $\gamma\theta$, where θ is a polar angle about the direction of motion, because of the beaming effect. Then we can write

$$E(t) \propto F(\gamma\theta)$$

Let time $t = 0$ and the path length $s = 0$ when the pulse is centered on the observer. Then, we find

$$\theta \approx \frac{s}{\mathcal{R}} \quad \text{and} \quad t \approx \frac{s}{v}\left(1 - \frac{v}{c}\right) \approx \frac{s}{v}\frac{1}{2\gamma^2} \qquad \leftarrow \quad 1 - \frac{v}{c} \approx \frac{1}{2\gamma^2}$$

Then we have

$$\omega_c = \frac{3}{2}\gamma^3\omega_{\rm B}\sin\alpha$$

$$\gamma\theta \approx \gamma\frac{s}{\mathcal{R}} = \gamma\left(\frac{s}{v}\omega_{\rm B}\sin\alpha\right) = \gamma\left(2\gamma^2 t\right)\left(\omega_{\rm B}\sin\alpha\right) \propto \omega_c t$$

$$\mathcal{R} = \frac{v}{\omega_{\rm B}\sin\alpha}$$

Hence, the time dependence of the electric field can be written as $E(t) \propto g(\omega_c t)$.

The Fourier transform of the electric field is

$$\begin{aligned} \bar{E}(\omega) &\propto \int_{-\infty}^{\infty} g(\omega_c t) e^{i\omega t} dt \qquad \leftarrow \quad \xi \equiv \omega_c t \\ &= \int_{-\infty}^{\infty} g(\xi) e^{i(\omega/\omega_c)\xi} d\xi \end{aligned}$$

Therefore, **the power per unit frequency is a function of** ω/ω_c :

$$\boxed{P(\omega) \propto \left|\bar{E}(\omega)\right|^2 = C_1 F\left(\frac{\omega}{\omega_c}\right)}$$

[Spectral Index for Power-law Electron Distribution]

- Often the number density of particles with energies between E and $E + dE$ can be approximately expressed in the form ("power law"):

$$N(\gamma)d\gamma = C\gamma^{-p}d\gamma \ (\gamma_1 < \gamma < \gamma_2)$$
$$N(E)dE = CE^{-p}dE \ (E_1 < E < E_2)$$

- The total power radiated per unit volume per unit frequency by such a distribution is given by

$$\begin{aligned} P_{\mathrm{tot}}(\omega) &= \int_{\gamma_1}^{\gamma_2} N(\gamma)P(\omega)d\gamma \\ &\propto \int_{\gamma_1}^{\gamma_2} \gamma^{-p} BF\left(\frac{\omega}{\omega_c}\right)d\gamma \\ &\propto \omega^{-(p-1)/2}B^{(p+1)/2}\int_{x_1}^{x_2} x^{(p-3)/2}F(x)dx \end{aligned}$$

Recall $\omega_c \equiv \frac{3}{2}\gamma^2\omega_{\mathrm{L}}\sin\alpha \ \propto \gamma^2 B$

$\leftarrow$ set $x \equiv \frac{\omega}{\omega_c} \propto \gamma^{-2}B^{-1}\omega$

$dx \propto \gamma^{-3}B^{-1}\omega d\gamma \propto x^{3/2}B^{-1}\omega d\gamma$

$d\gamma \propto x^{-3/2}B^{1/2}\omega^{-1/2}dx$

The extra factor B comes from the more detailed formula.

- Then, the spectrum is also a power law and the power-law spectral index s is related to the particle distribution index p by

$$\boxed{\begin{aligned} P_{\mathrm{tot}}(\omega) &\propto \omega^{-s}B^{s+1} \\ &\propto \omega^{-(p-1)/2}B^{(p+1)/2} \quad \longrightarrow \quad s = \frac{p-1}{2} \end{aligned}}$$

[Spectrum of Synchrotron Radiation: A Detailed Discussion]

- We will use the formula derived in Chapter 3.

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\exp\left[i\omega\left(t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c}\right)\right]dt'\right|^2$$

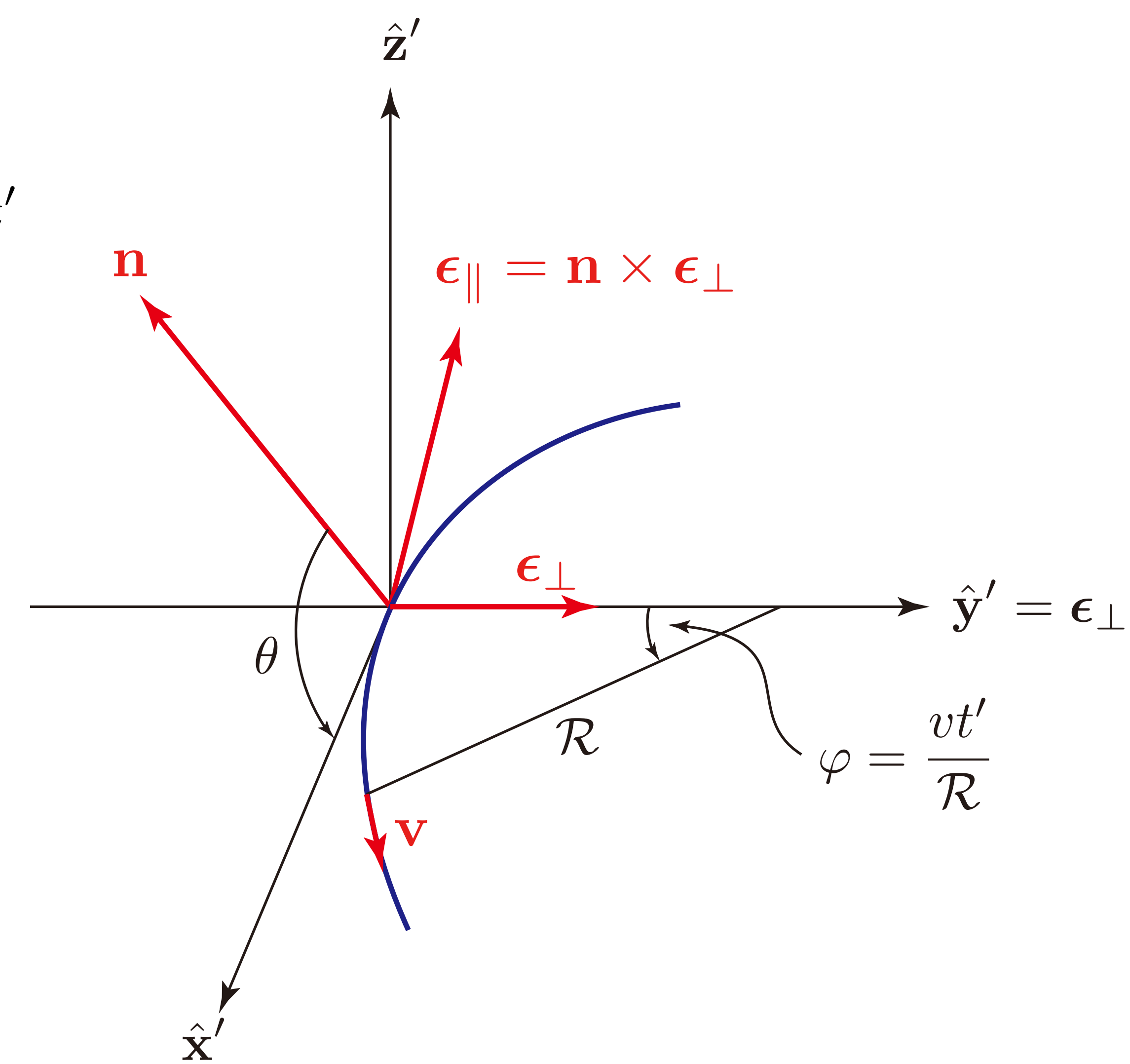
- The coordinate system is chosen so that the particle has velocity $\mathbf{v}$ along the x' axis at time $t'=0$. $\boldsymbol{\epsilon}_\perp$ is a unit vector along the y' axis in the orbital $(x'-y')$ plane. Let θ represent the angle between the observing direction ($\mathbf{n}$) and the velocity vector $\mathbf{v}$ at $t'=0$. Then, an equivalent circular orbit at t' is given by

$$\mathbf{v}(t') = v(\hat{\mathbf{x}}'\cos\varphi + \hat{\mathbf{y}}'\sin\varphi), \quad \text{where } \varphi \equiv \frac{vt'}{\mathcal{R}} = (\omega_B \sin\alpha)t'$$

$$\mathbf{r}(t') = \mathcal{R}(\hat{\mathbf{x}}'\sin\varphi - \hat{\mathbf{y}}'\cos\varphi)$$

Note that (1) $\mathbf{n}\times\hat{\mathbf{x}}' = \sin\theta\,\hat{\mathbf{y}}'$, (2) $\mathbf{n}\times\hat{\mathbf{y}}' = \mathbf{n}\times\boldsymbol{\epsilon}_\perp = \boldsymbol{\epsilon}_\parallel$
(3) $\mathbf{n}\cdot\hat{\mathbf{y}}' = 0$, (4) $\mathbf{n}\cdot\hat{\mathbf{x}}' = \cos\theta$

$$\begin{aligned}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) &= \beta\left[\mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{x}}')\cos\varphi + \mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{y}}')\sin\varphi\right] \\ &= \beta\mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{x}}')\cos\varphi + \beta\left(\mathbf{n}(\mathbf{n}\cdot\hat{\mathbf{y}}') - \hat{\mathbf{y}}'\right)\sin\varphi \\ &= \boldsymbol{\epsilon}_\parallel\beta\sin\theta\cos\varphi - \boldsymbol{\epsilon}_\perp\beta\sin\varphi\end{aligned}$$



- We note that

$$
\begin{aligned}
t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c} &= t' - \frac{\mathcal{R}}{c}\cos\theta\sin\varphi && \leftarrow \quad \mathbf{n}\cdot\hat{\mathbf{x}}' = \cos\theta \\
&= t' - \frac{\mathcal{R}}{c}\left(1-\frac{\theta^2}{2}\right)\left(\varphi - \frac{\varphi^3}{6}\right) && \leftarrow \quad \varphi = \frac{vt'}{\mathcal{R}} \\
&= t'\left[1 - \frac{v}{c}\left(1-\frac{\theta^2}{2}\right)\left(1-\frac{(vt')^2}{6\mathcal{R}^2}\right)\right] && \leftarrow \quad 1-\frac{v}{c} \approx \frac{1}{2\gamma^2} \\
&\approx t'\left[1 - \left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right)\left(1-\left(1-\frac{1}{2\gamma^2}\right)^2\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] \\
&\approx \frac{t'}{2\gamma^2}\left[2\gamma^2 - (2\gamma^2-1)\left(1-\frac{\theta^2}{2}\right)\left(1-\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] && \leftarrow \quad ct' \ll \mathcal{R},\ \theta \ll 1 \\
&\approx \frac{t'}{2\gamma^2}\left[2\gamma^2 - (2\gamma^2-1)\left(1-\frac{\theta^2}{2}-\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] \\
&\approx \frac{t'}{2\gamma^2}\left[2\gamma^2 - (2\gamma^2-1) + 2\gamma^2\left(\frac{\theta^2}{2}+\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] \\
&= \frac{1}{2\gamma^2}\left[\left(1+\gamma^2\theta^2\right)t' + \frac{c^2\gamma^2t'^3}{3\mathcal{R}^2}\right]
\end{aligned}
$$

- We also note that

$$\begin{aligned}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) &= \boldsymbol{\epsilon}_{\parallel}\beta\sin\theta\cos\varphi-\boldsymbol{\epsilon}_{\perp}\beta\sin\varphi \qquad \leftarrow \quad \beta\approx 1\\ &\approx \boldsymbol{\epsilon}_{\parallel}\sin\theta\cos\varphi-\boldsymbol{\epsilon}_{\perp}\sin\varphi\\ &\approx \boldsymbol{\epsilon}_{\parallel}\theta-\boldsymbol{\epsilon}_{\perp}\varphi\\ &= \boldsymbol{\epsilon}_{\parallel}\theta-\boldsymbol{\epsilon}_{\perp}\frac{vt'}{\mathcal{R}}\\ &\approx \boldsymbol{\epsilon}_{\parallel}\theta-\boldsymbol{\epsilon}_{\perp}\frac{ct'}{\mathcal{R}}\end{aligned}$$

$$t'-\frac{\mathbf{n}\cdot\mathbf{r}(t')}{c}\approx\frac{1}{2\gamma^2}\left[\left(1+\gamma^2\theta^2\right)t'+\frac{c^2\gamma^2t'^3}{3\mathcal{R}^2}\right]$$

- We can identify the contribution to the received power in the two orthogonal polarized directions.

$$\frac{dW}{d\omega d\Omega}\equiv\frac{dW_{\parallel}}{d\omega d\Omega}+\frac{dW_{\perp}}{d\omega d\Omega} \longleftarrow \frac{dW}{d\omega d\Omega}=\frac{e^2\omega^2}{4\pi^2c}\left|\int\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\exp\left[i\omega\left(t'-\frac{\mathbf{n}\cdot\mathbf{r}(t')}{c}\right)\right]dt'\right|^2$$

$$\frac{dW_{\perp}}{d\omega d\Omega}=\frac{e^2\omega^2}{4\pi^2c}\left|\int\frac{ct'}{\mathcal{R}}\exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2t'+\frac{c^2\gamma^2t'^3}{3\mathcal{R}^2}\right)\right]dt'\right|^2$$

$$\frac{dW_{\parallel}}{d\omega d\Omega}=\frac{e^2\omega^2\theta^2}{4\pi^2c}\left|\int\exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2t'+\frac{c^2\gamma^2t'^3}{3\mathcal{R}^2}\right)\right]dt'\right|^2 \longleftarrow \theta_\gamma^2\equiv 1+\gamma^2\theta^2$$

Now, define the following variables

$$y\equiv\gamma\frac{ct'}{\mathcal{R}\theta_\gamma}\ \text{and}\ \eta\equiv\frac{\omega\mathcal{R}\theta_\gamma^3}{3c\gamma^3} \longrightarrow$$

$$\frac{dW_{\perp}}{d\omega d\Omega}=\frac{e^2\omega^2}{4\pi^2c}\left(\frac{\mathcal{R}\theta_\gamma^2}{\gamma^2c}\right)^2\left|\int_{-\infty}^{\infty}y\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dy\right|^2$$

$$\frac{dW_{\parallel}}{d\omega d\Omega}=\frac{e^2\omega^2\theta^2}{4\pi^2c}\left(\frac{\mathcal{R}\theta_\gamma}{\gamma c}\right)^2\left|\int_{-\infty}^{\infty}\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dy\right|^2$$

- The integrals are functions only of the parameter η. Since most of the radiation occurs at angle $\theta \approx 0$, η can be written as

$$\eta \approx \eta(\theta = 0) = \frac{\omega \mathcal{R}}{3c\gamma^3} = \frac{\omega v}{3c\gamma^3 \omega_{\mathrm{B}} \sin\alpha} \approx \frac{\omega}{2\omega_c}$$

where $\mathcal{R} = \dfrac{v}{\omega_{\mathrm{B}} \sin\alpha}$

$$\omega_c = \frac{3}{2}\gamma^3 \omega_{\mathrm{B}} \sin\alpha$$

The frequency dependence of the spectrum depends on ω only through ω/ω_c.
The angular dependence uses θ only through the combination $\gamma\theta$.

- The integrals can be expressed in terms of the modified Bessel functions of 1/3 and 2/3 order.

$$\frac{dW_\perp}{d\omega d\Omega} = \frac{e^2\omega^2}{3\pi^2 c}\left(\frac{\mathcal{R}\theta_\gamma^2}{\gamma^2 c}\right)^2 K_{2/3}^2(\eta)$$

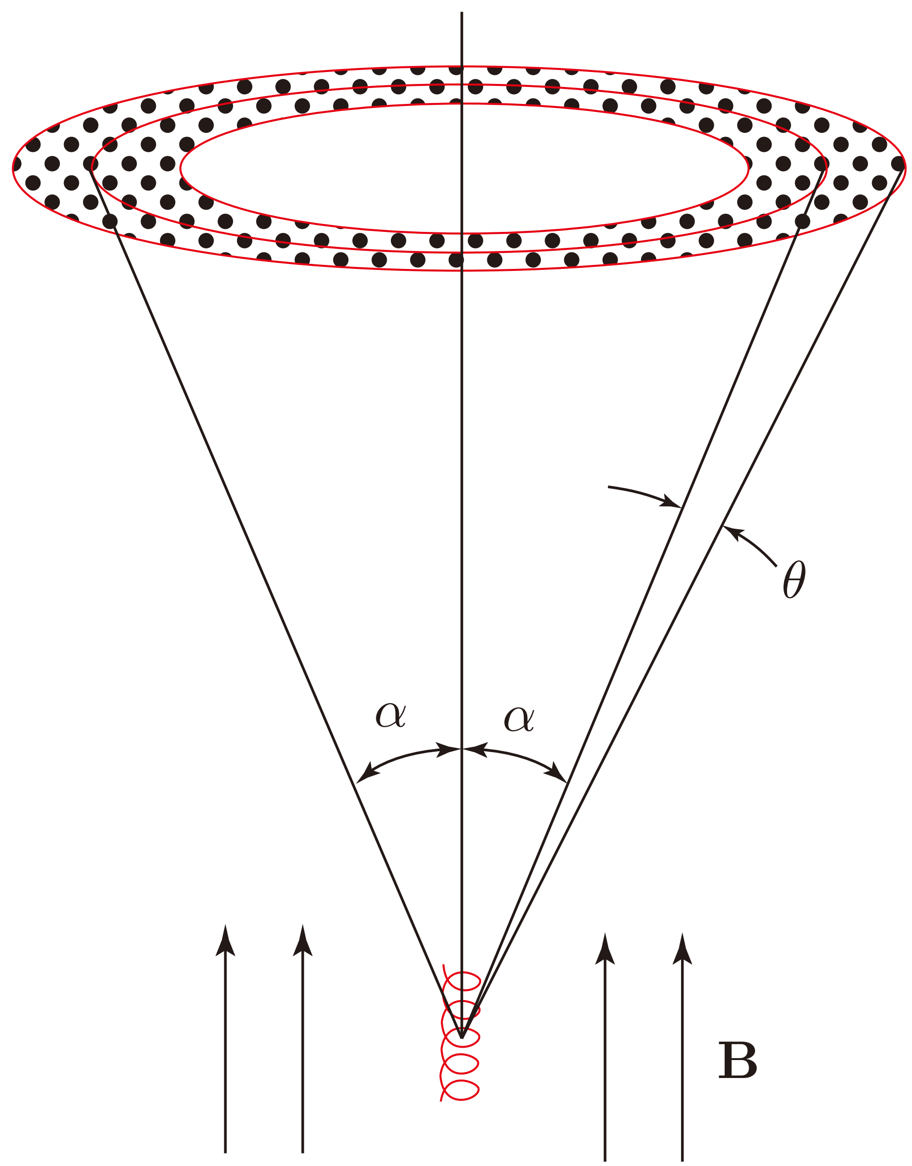
$$\frac{dW_\parallel}{d\omega d\Omega} = \frac{e^2\omega^2\theta^2}{3\pi^2 c}\left(\frac{\mathcal{R}\theta_\gamma}{\gamma c}\right)^2 K_{1/3}^2(\eta)$$

From 10.4.26, 10.4.31, and 10.4.32 of Abramovitz & Stegun (1965)
See Westfold 1959, ApJ, 130, 241

- The energy per frequency range radiated by the particle per complete orbit in the projected normal plane can be obtained by integrating over solid angle.

- We note that the emitted radiation is almost completely confined to the solid angle shown shaded in the following figure, which lies within an angle $1/\gamma$ of a cone of half-angle α. Therefore, the integral over the solid angle can be approximated by

$$\frac{dW_\parallel}{d\omega} = \int_0^\pi \frac{dW_\parallel}{d\omega d\Omega} 2\pi \sin\theta d\theta \approx \int_{-\infty}^{\infty} \frac{dW_\parallel}{d\omega d\Omega} 2\pi \sin\alpha d\theta$$



- Therefore,

$$\frac{dW_\perp}{d\omega} = \frac{2e^2\omega^2\mathcal{R}^2 \sin\alpha}{3\pi c^3\gamma^4} \int_{-\infty}^{\infty} \theta_\gamma^4 K_{2/3}^2(\eta) d\theta$$
$$\frac{dW_\parallel}{d\omega} = \frac{2e^2\omega^2\mathcal{R}^2 \sin\alpha}{3\pi c^3\gamma^4} \int_{-\infty}^{\infty} \theta_\gamma^2 K_{1/3}^2(\eta) d\theta$$

The infinite integral limits on the integral are convenient and permissible, because the integrand is concentrated to small values of $\Delta\theta$ about α, of order $1/\gamma$.

- The emitted power per frequency is obtained by dividing the orbital period of the charge $T = 2\pi/\omega_{\mathrm{B}}$:

$$P_\parallel(\omega) \equiv \frac{1}{T}\frac{dW_\parallel}{d\omega}$$
$$P_\perp(\omega) \equiv \frac{1}{T}\frac{dW_\perp}{d\omega}$$

- Emitted power:

$$P_\perp(\omega) \equiv \frac{\sqrt{3}e^3 B \sin\alpha}{4\pi m_e c^2}\left[F(x) + G(x)\right]$$

$$P_\parallel(\omega) \equiv \frac{\sqrt{3}e^3 B \sin\alpha}{4\pi m_e c^2}\left[F(x) - G(x)\right]$$

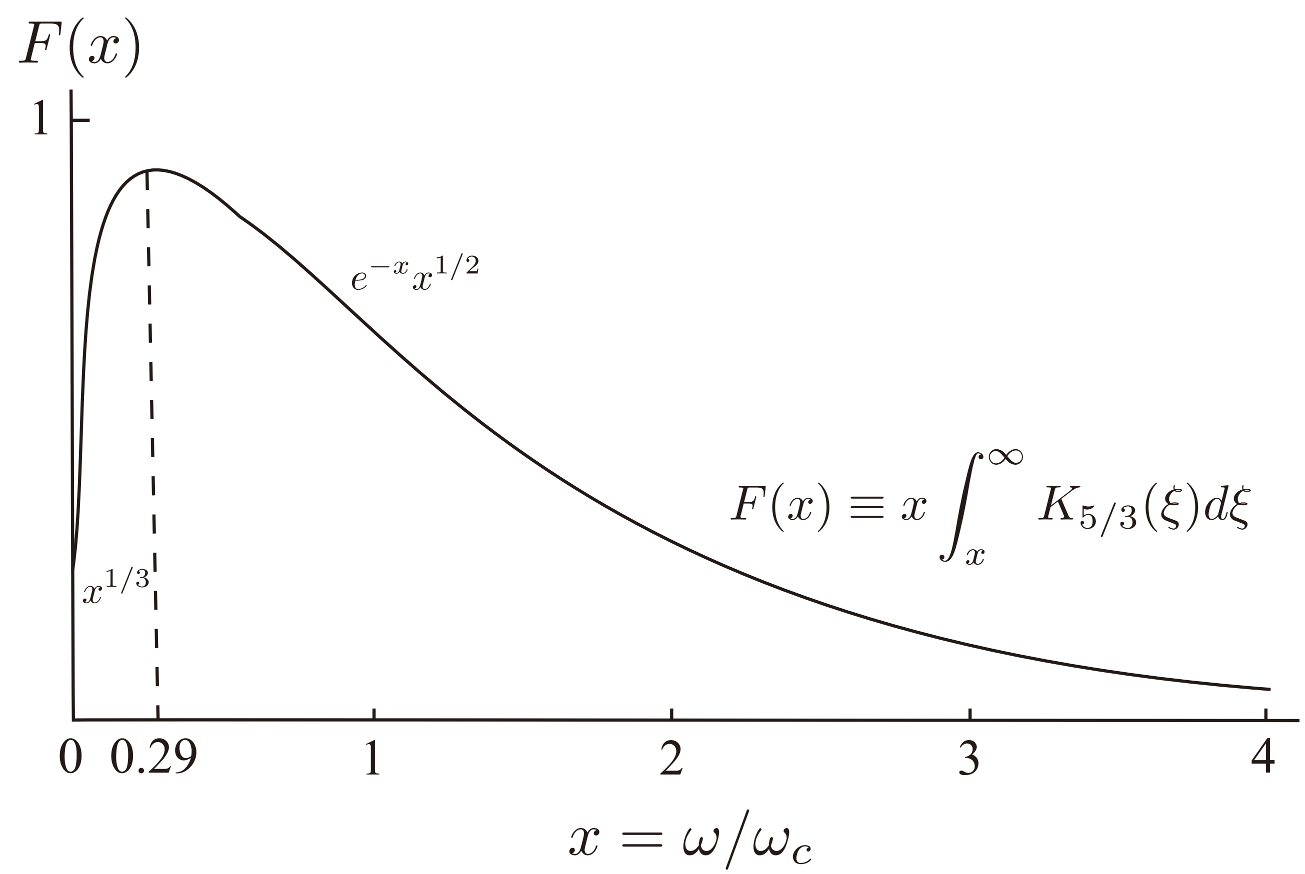
where
$$F(x) \equiv x\int_x^\infty K_{5/3}(\xi)d\xi$$
$$G(x) \equiv xK_{2/3}(x)$$
$$x \equiv \omega/\omega_c$$

- Total emitted power per frequency:

$$P(\omega) \equiv P_\parallel(\omega) + P_\perp(\omega) = \frac{\sqrt{3}e^3 B \sin\alpha}{2\pi m_e c^2}F(x)$$

$$F(x) \sim \frac{4\pi}{\sqrt{3}\Gamma(1/3)}\left(\frac{x}{2}\right)^{1/3} \quad \text{if} \quad x \ll 1$$

$$F(x) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-x}x^{1/2} \quad \text{if} \quad x \gg 1$$



- For a power-law distribution of electrons $N(\gamma)d\gamma = C\gamma^{-p}d\gamma \quad (\gamma_1 < \gamma < \gamma_2)$, we obtain the total power per unit volume per unit frequency:

$$
\begin{aligned}
P_{\rm tot}(\omega) &= \int N(\gamma)P(\omega)d\gamma \\
&\equiv \frac{\sqrt{3}e^3 CB\sin\alpha}{2\pi m_e c^2(p+1)}\Gamma\left(\frac{p}{4}+\frac{19}{12}\right)\Gamma\left(\frac{p}{4}-\frac{1}{12}\right)\left(\frac{m_e c\omega}{3eB\sin\alpha}\right)^{-(p-1)/2}
\end{aligned}
$$

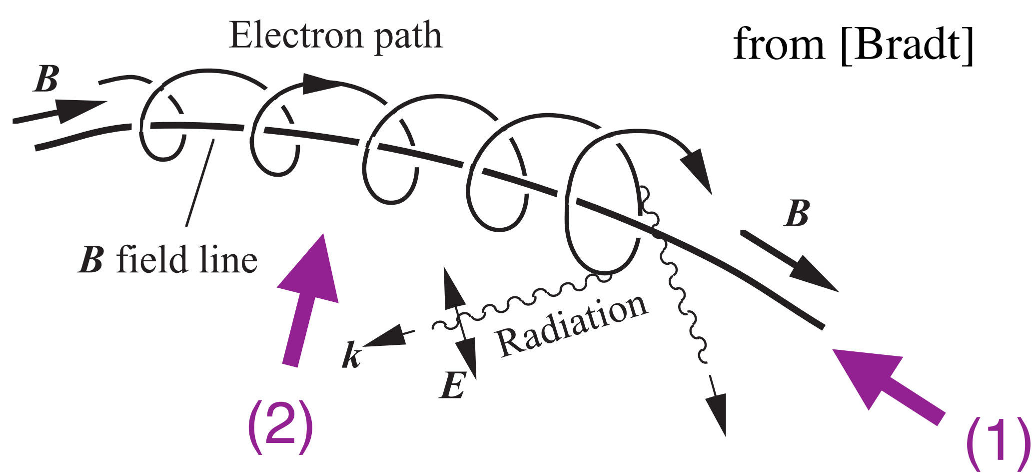
$$\boxed{P_{\rm tot}(\omega) \propto B^{(p+1)/2}\omega^{-(p-1)/2}}$$

- For the complete derivation of the formula, see Westfold (1959).

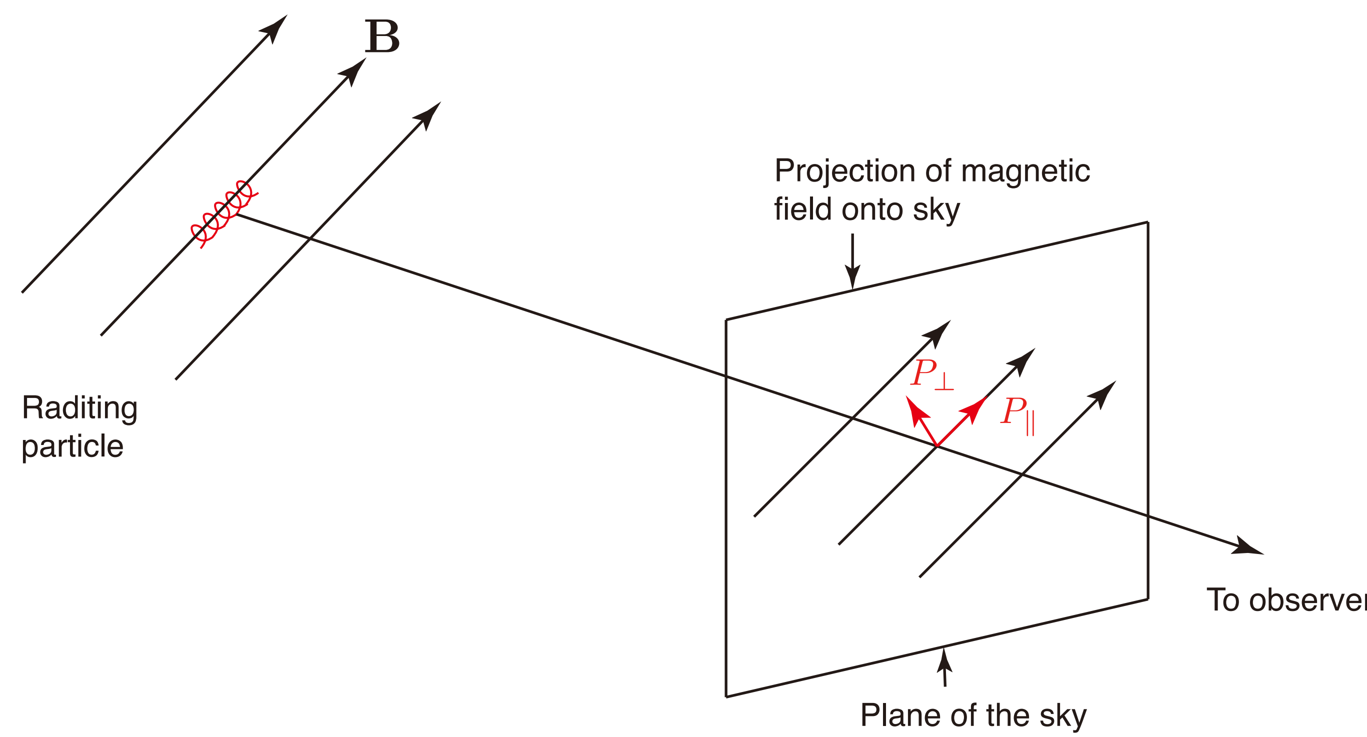
[Polarization of Synchrotron Radiation]

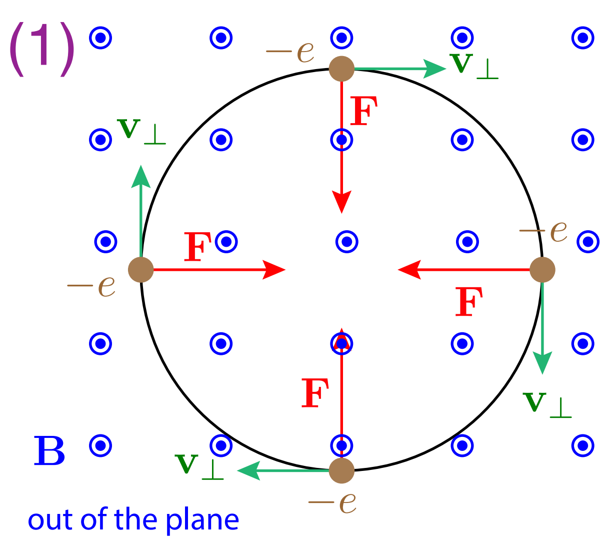
- In general, the radiation from a single charge will be elliptically polarized.

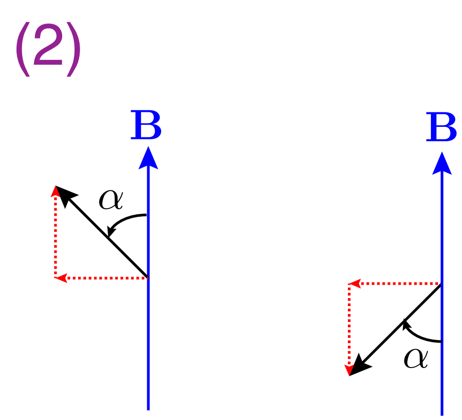
  The electric field is in the same plane as the acceleration vector, which is perpendicular to the magnetic field. For any reasonable distribution of particles that varies smoothly with pitch angle, the elliptical component will cancel out as emission cones will contribute equally from both sides of the line of sight. Thus, on average, **the radiation will be partially linearly polarized perpendicular to the magnetic field.**



from [Bradt]







Recall the electric field is in the plane made by the acceleration vector and line of sight (Lecture 04).

$$\mathbf{E}_{\rm rad} \propto \mathbf{n} \times (\mathbf{n} \times \boldsymbol{a}) = \mathbf{n}\,(\mathbf{n} \cdot \boldsymbol{a}) - \boldsymbol{a}$$

- Degree of linear polarization of **a single energy**:

$$\Pi(\omega) \equiv \frac{P_\perp(\omega) - P_\parallel(\omega)}{P_\perp(\omega) + P_\parallel(\omega)} = \frac{G(x)}{F(x)}$$

- For particles with **a power law distribution of energies**:

$$\begin{aligned}\Pi(\omega) &= \frac{\int G(x)\gamma^{-p}d\gamma}{\int F(x)\gamma^{-p}d\gamma} \qquad \leftarrow \ \gamma \propto x^{-1/2} \\ &= \frac{\int G(x)x^{(p-3)/2}dx}{\int F(x)x^{(p-3)/2}dx} \\ &= \frac{(p+1)/2}{2}\frac{1}{\frac{p-3}{4}+\frac{4}{3}} \\ &= \frac{p+1}{p+\frac{7}{3}}\end{aligned}$$

$$\int_0^\infty x^\mu F(x)dx = \frac{2^{\mu+1}}{\mu+2}\Gamma\left(\frac{\mu}{2}+\frac{7}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right)$$

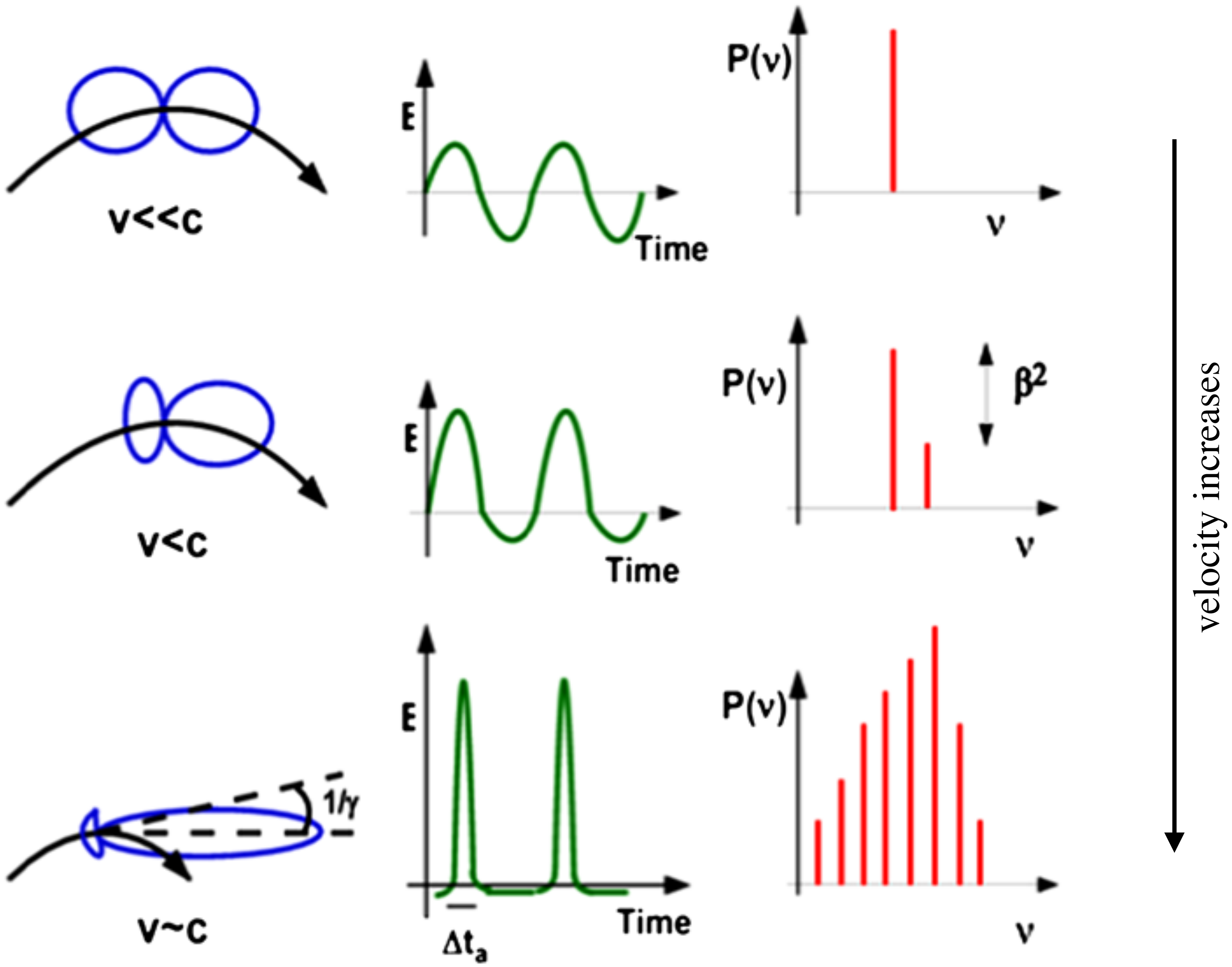
$$\int_0^\infty x^\mu G(x)dx = 2^\mu\Gamma\left(\frac{\mu}{2}+\frac{4}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right)$$

- For particles of **a single energy**, the polarization degree of the frequency integrated radiation is

$$\begin{aligned}\Pi(\omega) &= \frac{\int G(x)dx}{\int F(x)dx} = \frac{p+1}{p+\frac{7}{3}} \qquad \leftarrow \quad p=3 \\ &= \frac{3}{4} \\ &= 75\%\end{aligned}$$

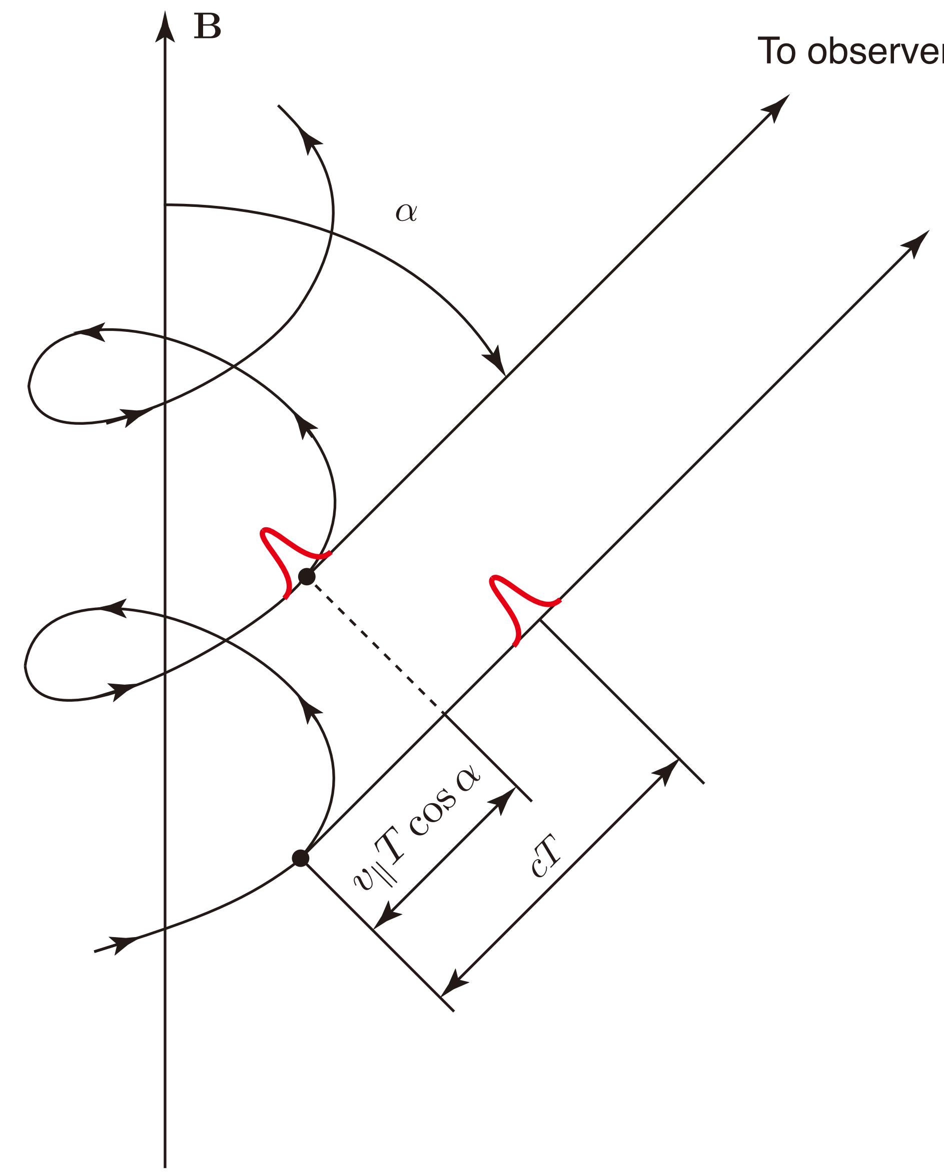
[Transition from Cyclotron to Synchrotron Emission]

- For low energies, the electric field components vary sinusoidally with the same frequency as the gyration in the magnetic field. The spectrum consists of a single line.
- When v/c increases, higher harmonics of the fundamental frequency begin to contribute.
- For very relativistic velocities, the originally sinusoidal form of $E(t)$ has now become a series of sharp pulses, which is repeated at time intervals $2\pi/\omega_B$. The spectrum now involves a great number of harmonics, the envelope of which approaches the form of the function $F(x)$.



[Ghisellini] Radiative Processes in High Energy Astrophysics

[Distinction between Received and Emitted Power]



- If $T = 2\pi/\omega_B$ is the orbital period of the projected motion, then time-delay effects will give a period between the arrival of pulses T_A satisfying

$$T_A = T\left(1 - \frac{v_\parallel}{c}\cos\alpha\right) = T\left(1 - \frac{v}{c}\cos^2\alpha\right)$$
$$\approx T\left(1 - \cos^2\alpha\right) = \frac{2\pi}{\omega_B}\sin^2\alpha$$

  Therefore, **the fundamental observed frequency is $\omega_B/\sin^2\alpha$ rather than ω_B.**

- This leads to two modifications to the preceding results, neither of which is serious, fortunately.

(1) Spacing of the harmonics is $\omega_B/\sin^2\alpha$. For extreme relativistic particles this is not important, because one sees a continuum rather than the harmonic structure. Note that we did take the Doppler effects in deriving the pulse width Δt_A and consequently for the critical frequency ω_c. The continuum radiation is still a function of ω/ω_c.

(2) The emitted power was found by dividing the energy by the period T. But the received power must be obtained by dividing by T_A. Thus we have $P_r = P_e/\sin^2\alpha$.

The average power emitted and received will be the same, because the total number of emitted and received pulses must be the same in the long run. These corrections are not important for most cases of interest.

[Synchrotron Self-Absorption]

- **Opacity**

In order to calculate the opacity for **non-thermal velocity distribution** of electrons. We first need to generalize the Einstein coefficients to **include continuum states.**

For a given energy of a photon $h\nu$ there are many possible transitions, meaning that the absorption coefficient should be obtained by summing over all upper states 2 and lower stats 1:

$$\boxed{\alpha_\nu = \frac{h\nu}{4\pi}\sum_{E_1}\sum_{E_2}\left[n(E_1)B_{12} - n(E_2)B_{21}\right]\phi_{21}(\nu)}$$

The profile function $\phi_{21}(\nu)$ is essentially a Dirac delta-function $\phi_{21}(\nu) = \delta\left(\nu - \left(E_2 - E_1\right)/h\right)$ that restrict the summations to those states differing by an energy $h\nu = E_2 - E_1$.

In terms of the Einstein coefficients, ***the emitted power*** is given by

$$P(\nu, E_2) = h\nu\sum_{E_1} A_{21}\phi_{21}(\nu) = (2h\nu^3/c^2)h\nu\sum_{E_1} B_{21}\phi_{21}(\nu) \longleftarrow A_{21} = \left(2h\nu^3/c^2\right)B_{21}$$

- Using this, the part due to stimulated emission of α_ν can be represented in terms of P:

$$-\frac{h\nu}{4\pi}\sum_{E_1}\sum_{E_2} n(E_2)B_{21}\phi_{21}(\nu) = -\frac{c^2}{8\pi h\nu^3}\sum_{E_2} n(E_2)P(\nu, E_2)$$

- The true absorption coefficient (first part) is:

$$\frac{h\nu}{4\pi}\sum_{E_1}\sum_{E_2} n(E_1)B_{12}\phi_{21}(\nu) = \frac{c^2}{8\pi h\nu^3}\sum_{E_2} n(E_2 - h\nu)P(\nu, E_2) \longleftarrow \begin{array}{l} B_{12} = B_{21} \\ E_1 = E_2 - h\nu \end{array}$$

Therefore, we have
$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\sum_{E_2}\left[n(E_2 - h\nu) - n(E_2)\right]P(\nu, E_2)$$

We need to convert the discrete summation to an integral over continuum energy (or momentum).

(1) Let $f(p)d^3p \equiv$ number of electrons per volume with momenta in d^3p about p.

(2) Number of quantum states per volume in $d^3p = g\dfrac{d^3p}{h^3}$ (g = 2 for spin 1/2 particles)

uncertainty principle

$\Delta x \Delta p = h \quad \rightarrow \quad \dfrac{dx^3 d^3p}{h^3} = 1$

(3) Electron density per quantum state = $\dfrac{f(p)d^3p}{gd^3p/h^3} = \dfrac{h^3}{g}f(p)$

Therefore, we can make the replacements

$$\boxed{\sum_2 \rightarrow \frac{g}{h^3}\int d^3p \quad \text{from (2)}} \qquad \boxed{n(E_2) \rightarrow \frac{h^3}{g}f(p) \quad \text{from (3)}}$$

Then, the absorption coefficient becomes

$$\boxed{\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int d^3p_2\left[f(p_2^*) - f(p_2)\right]P(\nu, E_2)}$$

where p_2^* is the momentum corresponding to energy $E_2 - h\nu$.

- <u>For a thermal distribution of particles, we can derive the **Kirchhoff's Law for continuum states**.</u>

Boltzmann distribution $$f(p) = K \exp\left[-\frac{E(p)}{kT}\right]$$

$$f(p_2^*) - f(p) = K \exp\left(-\frac{E_2 - h\nu}{kT}\right) - K \exp\left(-\frac{E_2}{kT}\right)$$
$$= f(p_2)\left(e^{h\nu/kT} - 1\right)$$

Thus, the absorption coefficient is

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\left(e^{h\nu/kT} - 1\right)\int d^3p_2 f(p_2) P(\nu, E_2)$$
$$= \frac{1}{4\pi}\frac{c^2}{2h\nu^3}\left(e^{h\nu/kT} - 1\right) 4\pi j_\nu$$

This integral is the total power per volume per frequency range. i.e., $4\pi j_\nu$

Therefore, we obtained the Kirchhoff's Law for thermal emission.

$$\boxed{\alpha_\nu = \frac{j_\nu}{B_\nu(T)}}$$

- **<u>For an isotropic, and extremely relativistic electron distribution</u>**, we can use energy instead of momentum:

$$E = \sqrt{(pc)^2 + (mc^2)^2} \approx pc \longrightarrow d^3p = 4\pi p^2 dp = \frac{4\pi}{c^3}E^2 dE$$

$$f(p)4\pi p^2 dp = N(E)dE \longrightarrow f(p) = \frac{N(E)dE}{4\pi p^2 dp} = \frac{N(E)}{(4\pi/c^3)E^2}$$

$$\Rightarrow \begin{cases} d^3p f(p) = dE E^2 \dfrac{N(E)}{E^2} \\ d^3p f(p^*) = dE E^2 \dfrac{N(E^*)}{E^{*2}} \end{cases}$$

where $E^* = E - h\nu$

Then,

$$\boxed{\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int dE P(\nu, E)E^2\left[\frac{N(E-h\nu)}{(E-h\nu)^2} - \frac{N(E)}{E^2}\right]}$$

Assume that $h\nu \ll E$ (in fact, a necessary condition for the application of classical electrodynamics) and expand to first order in $h\nu$.

Taylor expansion:

$$\frac{N(E-h\nu)}{(E-h\nu)^2} \approx \frac{N(E)}{E^2} - h\nu\frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right] + \mathcal{O}\left((h\nu)^2\right)$$

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2}\int dE P(\nu, E)E^2\frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right]$$

- <u>For a power law distribution of particles:</u>

$$N(E) = CE^{-p}$$

$$-E^2\frac{d}{dE}\left[\frac{N(E)}{E^2}\right] = (p+2)CE^{-(p+1)} = \frac{(p+2)N(E)}{E}$$

$$\Rightarrow \alpha_\nu = \frac{(p+2)c^2}{8\pi\nu^2}\int dE P(\nu, E)\frac{N(E)}{E}$$

$$\propto \nu^{-2}\int dE F(x)\frac{E^{-p}}{E} \quad \leftarrow \quad \text{set } x = \frac{\omega}{\omega_c} \propto \nu\gamma^{-2} \propto \nu E^{-2}$$

$$\propto \nu^{-2}\int \nu^{1/2}x^{-3/2}dx F(x)\nu^{-(p+1)/2}x^{(p+1)/2}$$

$$\boxed{\alpha_\nu \propto B^{(p+2)/2}\nu^{-(p+4)/2}}$$

Note $\alpha_\nu \propto \nu^{-(p+4)/2}$ indicates that **the synchrotron emission is optically thick at low frequencies and optically thin at high frequencies.**

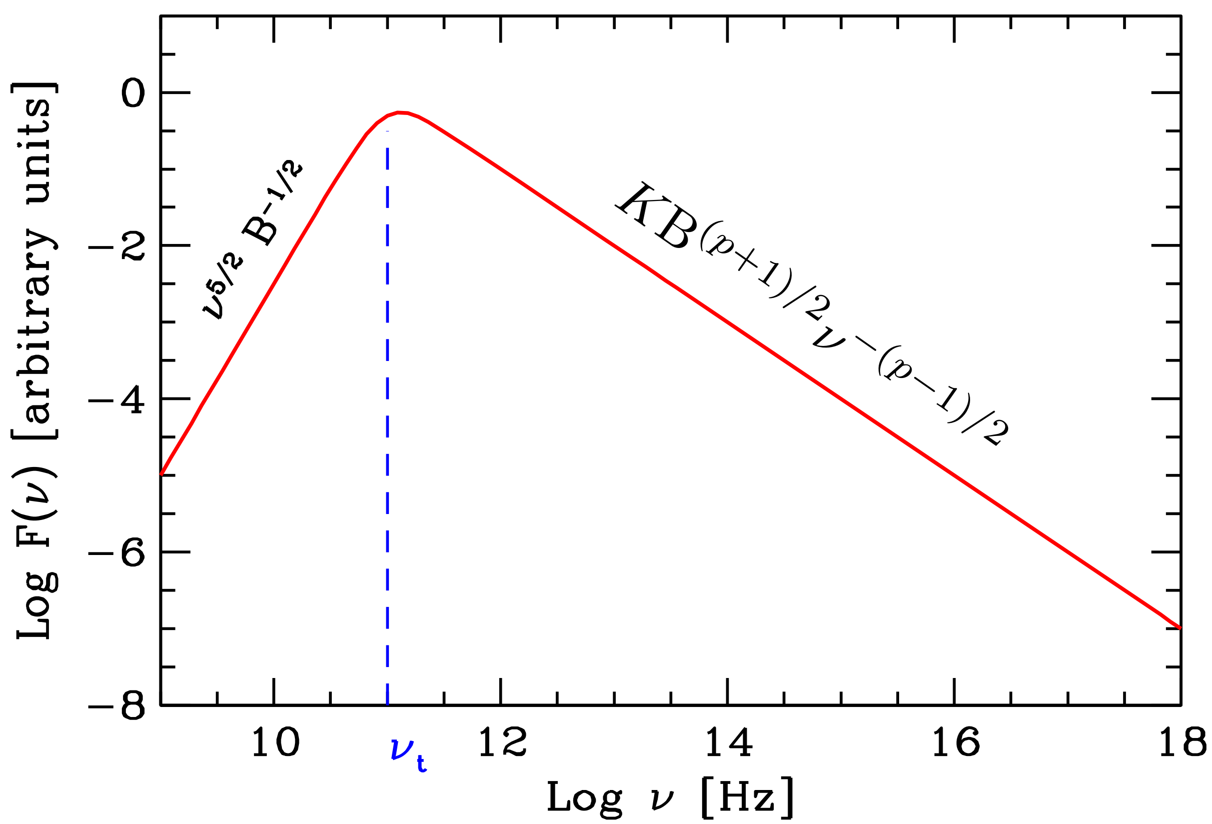
The source function is

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{P(\nu)}{4\pi\alpha_\nu} \propto B^{-1/2}\nu^{5/2} \longleftarrow \begin{aligned} P(\nu) &\propto B^{(p+1)/2}\nu^{-(p-1)/2} \\ \alpha_\nu &\propto B^{(p+2)/2}\nu^{-(p+4)/2} \end{aligned}$$

For optically thin synchrotron emission, $\boxed{I_\nu = \int j_\nu ds \propto B^{(p+1)/2}\nu^{-(p-1)/2}}$ high frequency

For optically thick emission, $\boxed{I_\nu = S_\nu \propto B^{-1/2}\nu^{5/2}}$ low frequency

Therefore, the synchrotron spectrum from a power-law distribution of electrons has a shape like the following figure.



- Observations of the self-absorption part could determine B.
- Observations of the thin part can then determine the proportional constant K and the electron slope p.

Homework (due date: 11/13)

[Q10]

An ultrarelativistic electron emits synchrotron radiation. We derived that its energy decreases with time according to (page 10 in this lecture; problem 6.1 in Rybicki & Lightman):

$$\gamma(t) = \frac{\gamma_0}{1 + A\gamma_0 t} \qquad \text{Here, } \gamma_0 = \gamma(t=0) \text{ and } A \equiv \frac{4e^4B^2}{9m_e^3c^5}.$$

In contrast, we showed γ is constant (page 5 in this lecture; page 168 in Rybicki & Lightman). These two results seem to be inconsistent with each other.

How does one reconcile the decrease of γ with the result of constant γ? What is your idea to reconcile this apparent discrepancy.