$(AGN)^2$

5. Comparison of Theory with Observations
6. Internal Dynamics of Gaseous Nebulae

Week 8

April 22 (Monday), 2024

updated on 04/22, 09:51

선광일 (Kwangil Seon)

KASI / UST

5.11 Abundances of the Elements in Nebulae

- Abundances can be derived from measurements of the relative strengths of their emission lines
  - All nebula lines are optically thin. No curve-of-growth effects that complicate stellar atmosphere abundance determinations occur.
  - H, He, N, O, and Ne are observable in the optical spectra.

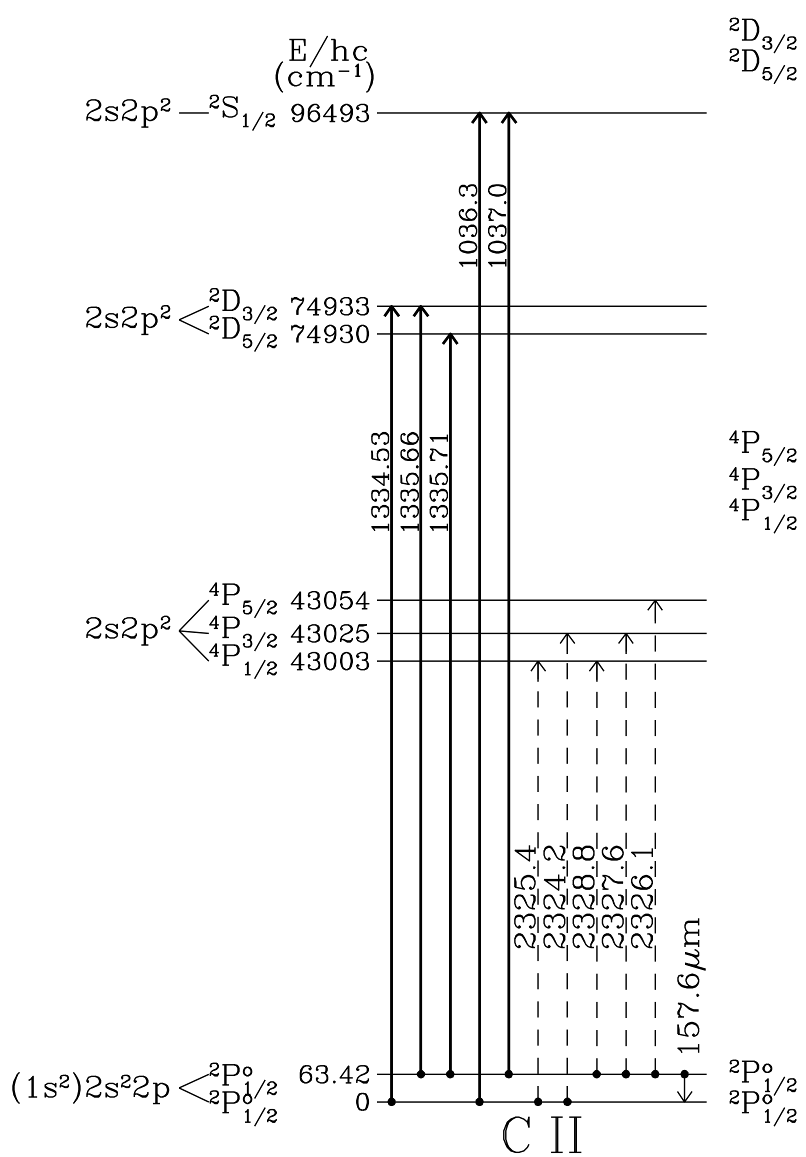
    But, C can be observed in the vacuum UV spectral region.

    Moreover, all stages of ionization of an element are generally not observable in the optical spectral region.
    - For instance, [O II] and [O III] are strong in diffuse nebulae, but O IV and O V are not.
    - However, an [O IV] line is available in the FIR, and O IV] and O V lines are in the vacuum UV.
- Collisionally Excited Lines vs. Recombination Lines
  - **Collisionally excited lines can be quite bright, but their strengths depend strongly on temperature, complicating the determination of relative abundance.** The intensity I_l of an emission line is given by $I_l = \int j_l ds = \int n_i n_e k_l(T) ds$
  - Intensities of recombination lines are given as, for instance, $I_{\mathrm{H}\beta} = \frac{1}{4\pi} \int h\nu_{\mathrm{H}\beta} n_p n_e \alpha^{\mathrm{eff}}_{\mathrm{H}\beta}(\mathrm{H}^0, T) ds$.

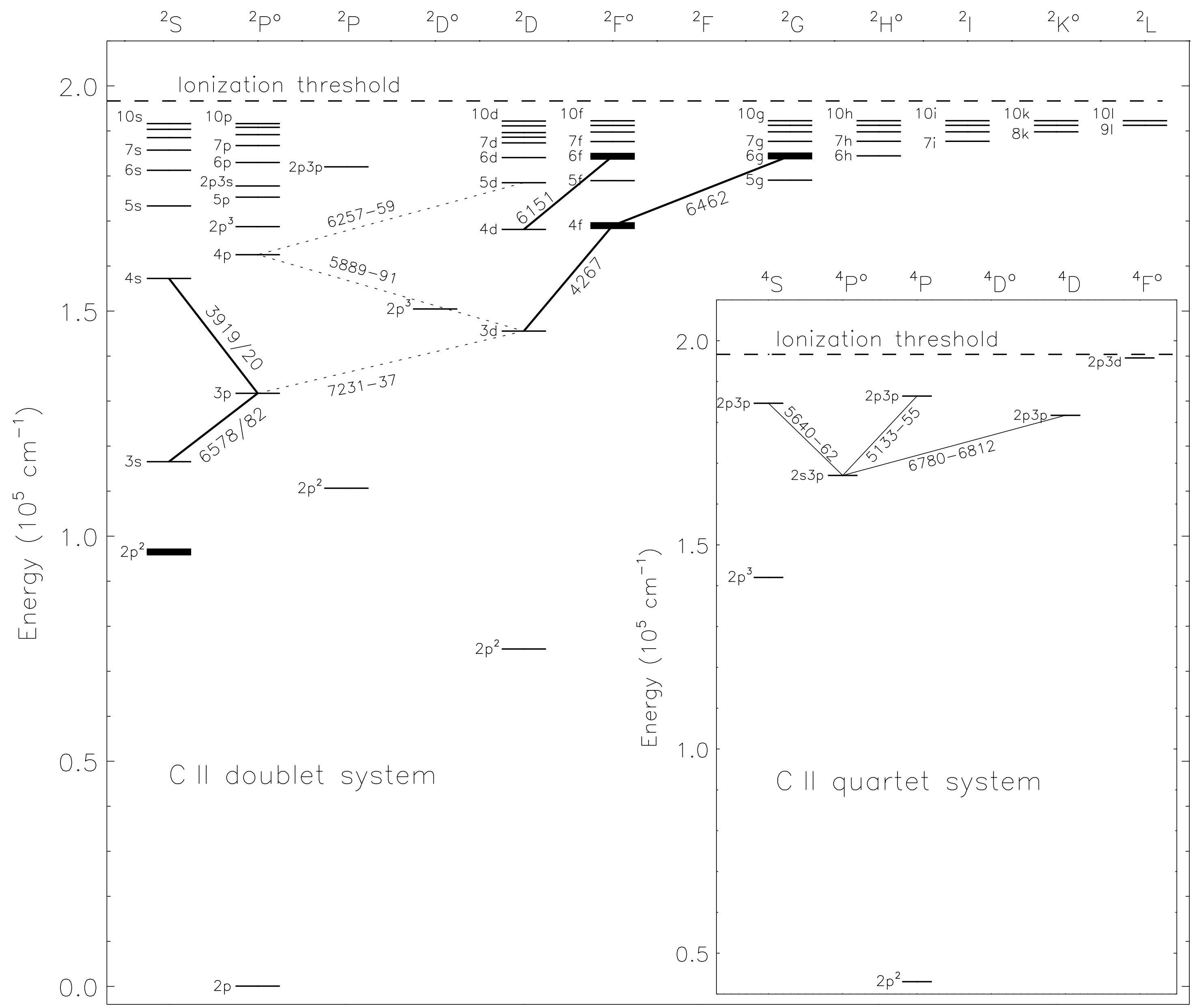
    The recombination rate coefficients are not particularly temperature sensitive and the abundances derived from them do not depend strongly on *T*. However, **recombination lines of elements heavier than He are faint and hard to observe.**

- Recombination lines from heavy elements
  - C II, O IV, and O V have weak permitted emission lines as observed in PNe, which are attributed to resulting from recombination. However, some of these lines may be excited by resonance fluorescence, and their emission coefficients depend not only on temperature and density but also the local radiation field. It is not easy to utilize them.
  - C II $3d\ ^2D - 4f\ ^2F\ \lambda4267$ line cannot be excited by resonance fluorescence, and are suitable for abundance determinations. (see, Fang & Liu, 2011, 2013, MNRAS)

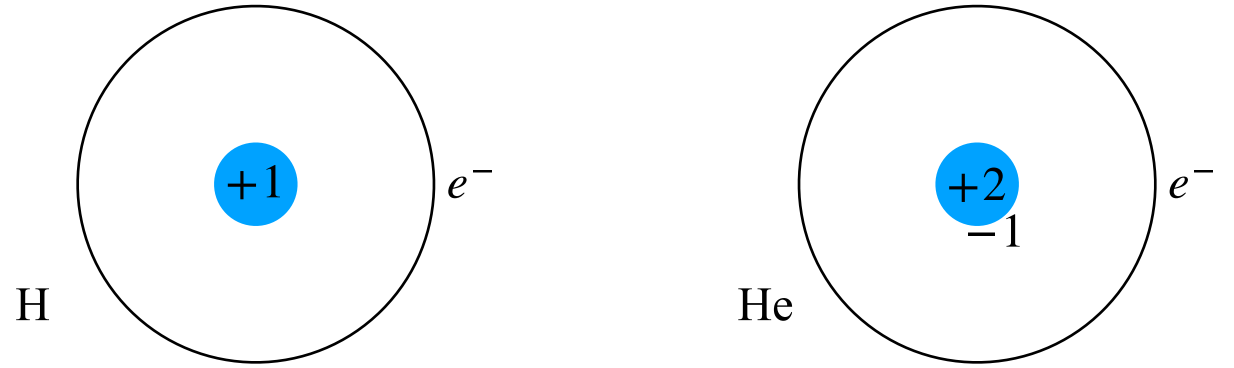
Nieva & Przybilla (2008, A&A, 481, 199)



Draine (2010), Physics of the IS and IGM



- Discrepancy between the temperatures derived from collisionally excited lines and recombination lines has been found in several nebulae.
- He+ abundance in H II regions
  - can be obtained from relative strengths of the radio recombination lines of H I and He I.
  - **At a very high *n* (radio region)**, both H and He are nearly identical one-electron systems except for their masses.



  - Hence, **their relative strengths are directly proportional to their abundances** (as long as they are optically thin, and the nebula is a complete H^+ plus He^+ region, without no H^+ plus He^0 region.
- Abundances of heavy elements
  - For this purpose, strong collisionally excited lines are available. However, the temperature must be determined independently from observational data because the lines are depends sensitively on the temperature.

$$I_\nu = \frac{1}{4\pi}\int h\nu_{12} n_2 n_e q_{12}(T) ds$$ where the collisional excitation rate coefficient

$$q_{12} = \frac{8.63\times 10^{-6}}{T^{1/2}}\frac{\Omega_{21}}{g_1}e^{-\chi/kT}.$$

(Abundance Determination)

- Helium abundance
  - The abundance of He is determined from **comparison of the strengths of radiative recombination lines of H and He** in regions ionized by stars that are sufficiently hot ($T \gtrsim 3.9 \times 10^4$ K) so that He is ionized throughout the H II regions.
- Heavy elements
  - The abundance of heavy elements can be inferred by **comparing the strengths of collisionally excited lines with recombination lines of H**.
  - ***Oxygen:***

$$\left(\lambda_{\mathrm{H}\beta} = 4861.35\text{Å}\right)$$

$$4\pi j([\mathrm{OIII}]\,5008) = n_e n(\mathrm{O}^{+2}) k_{03} \frac{A_{32}}{A_{31} + A_{32}} E_{32}$$

$$4\pi j(\mathrm{H}\beta) = n_e n(\mathrm{H}^+) \alpha_{\mathrm{eff,H}\beta} E_{\mathrm{H}\beta}$$

where

$$\alpha_{\mathrm{eff,H}\beta} \approx 3.03 \times 10^{-14}\, T_4^{-0.874-0.058\ln T_4} \quad \mathrm{cm^3\,s^{-1}}$$

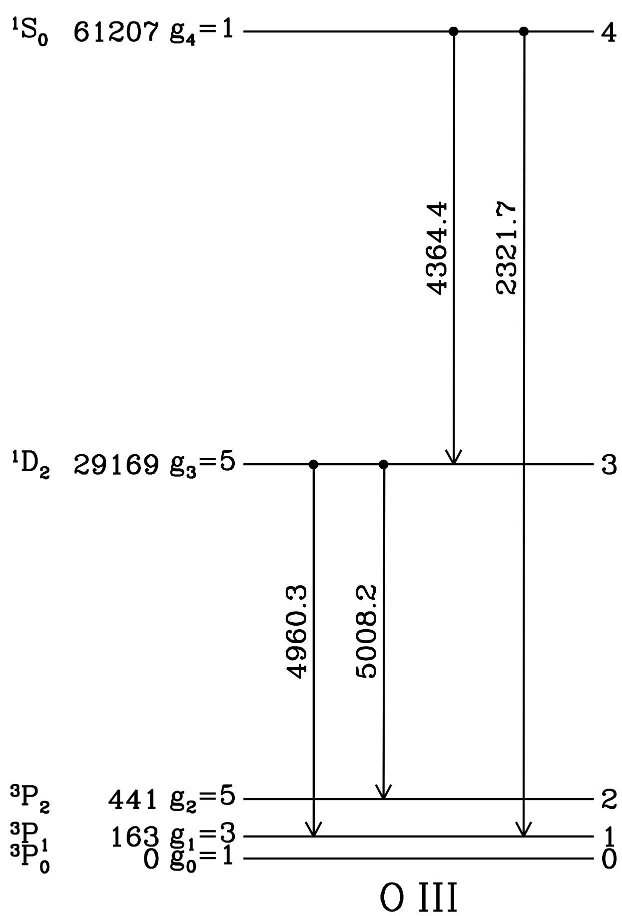
$$k_{03} = 8.62942 \times 10^{-8}\, T_4^{-1/2} \frac{\Omega_{30}}{g_0} e^{-E_{30}/kT}\, \mathrm{cm^3\,s^{-1}} \quad (g_0 = 1)$$

$$E_{32}/k = 29169\,\mathrm{K}, \quad E_{\mathrm{H}\beta}/k = 29588.5\,\mathrm{K}$$

$$\Omega_{30} = 0.243\, T_4^{0.120+0.031\ln T_4}$$

$$A_{32} = 2.0 \times 10^{-2}\ [\mathrm{s^{-1}}]$$

$$A_{31} = 6.8 \times 10^{-3}\ [\mathrm{s^{-1}}]$$



$$\frac{[\mathrm{O\,III}]\,5008}{\mathrm{H}\beta} = 5.091 \times 10^5\, T_4^{0.494+0.089\ln T_4} e^{-2.917/T_4} \frac{n(\mathrm{O}^{+2})}{n(\mathrm{H}^+)}$$

- ***Nitrogen:***

$$\left(\lambda_{\mathrm{H}\alpha} = 6562.79\text{Å}\right)$$

$$4\pi j([\mathrm{NII}]\,6585) = n_e n(\mathrm{N}^+) k_{03} \frac{A_{32}}{A_{31} + A_{32}} E_{32}$$

$$4\pi j(\mathrm{H}\alpha) = n_e n(\mathrm{H}^+) \alpha_{\mathrm{eff,H}\alpha} E_{\mathrm{H}\alpha}$$

where

$$\alpha_{\mathrm{eff,H}\alpha} \approx 1.17 \times 10^{-13}\, T_4^{-0.942-0.031 \ln T_4} \quad \mathrm{cm}^3\,\mathrm{s}^{-1}$$

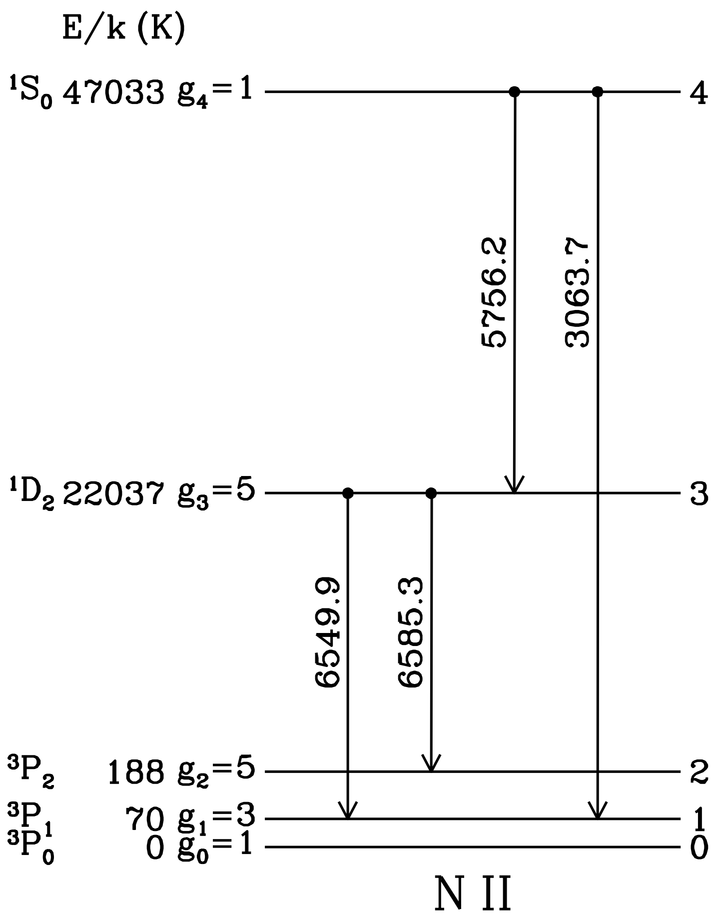
$$k_{03} = 8.62942 \times 10^{-8}\, T_4^{-1/2} \frac{\Omega_{30}}{g_0} e^{-E_{30}/kT}\ \mathrm{cm}^3\,\mathrm{s}^{-1} \qquad (g_0 = 1)$$

$$E_{32}/k = 21849\,\mathrm{K}, \quad E_{\mathrm{H}\alpha}/k = 21916.9\,\mathrm{K}$$

$$\Omega_{30} = 0.303\, T_4^{0.053+0.009 \ln T_4}$$

$$A_{32} = 3.0 \times 10^{-3}\ [\mathrm{s}^{-1}]$$

$$A_{31} = 9.8 \times 10^{-4}\ [\mathrm{s}^{-1}]$$



$$\frac{[\mathrm{N\,II}]\,6585}{\mathrm{H}\alpha} = 1.679 \times 10^5\, T_4^{0.495+0.040 \ln T_4} e^{-2.185/T_4} \frac{n(\mathrm{N}^+)}{n(\mathrm{H}^+)}$$

- Therefore, if the temperature T is known, the relative **abundance of the ion** can be obtained from the measured line ratio.
- The total elemental abundances are then obtained by applying ***ionization correction factors (ICFs)***, which correct for abundances of unobserved ions.

- One-layer model
  - The abundances can be determined on the basis of a model of the structure of the nebula.
  - The simplest model treats the nebula as homogeneous with constant T and n_e, which may be called a one-layer model.
  - Discrepancies in n_e and T determined from different line ratios indicate that this model is too simplified, though **the abundances determined from it are generally correct to within a factor of order two or three.**
- **Discrepancies**
  - Recall that **the temperatures determined from hydrogen recombination lines, the Balmer jump, and free-free brightness temperatures are lower than the temperature determined from collisionally excited (forbidden) line ratios.**
  - The probable explanation of this discrepancy is that t**he temperature is not constant throughout the nebula**, but rather varies from point to point due to variations in the local heating and cooling rates.
  - A complex model approach, as will be discussed in section 5.12, would be required to know the entire temperature structure.
  - However, the discrepancies can easily be understood in a relatively easy way.

- Origin of the discrepancies
  - The emission line ratios determine the temperature of the region where the lines are predominantly originate from.
  - The emission coefficient of collisionally excited lines increases strongly with increasing temperature, and thus the temperature is strongly weighted toward high-temperature regions.

    However, the recombination and free-free emission coefficients decrease with increasing temperature, and thus the resulting temperature is weighted toward low-temperature regions.

    We thus expect that **the collisionally excited lines indicate a higher temperature than do the Balmer jump or radio-frequency measurements.**

- A somewhat more sophisticated scheme to understand the discrepancies
  - The emission rate coefficient is expand in a power series of temperature:

$$\varepsilon_l(T) = \varepsilon_l(T_0) + (T - T_0)\left(\frac{d\varepsilon_l}{dT}\right)_0 + \frac{1}{2}(T - T_0)^2\left(\frac{d^2\varepsilon_l}{dT^2}\right)_0$$

$\varepsilon_l(T) = c_1 T^{-m}$ $(m \approx 1)$ for recombination lines (the recombination rate coefficient)

$\varepsilon_l(T) = \dfrac{c_2 \exp(-\chi/kT)}{T^{1/2}}$ for recombination lines

- The emissivity is given by integrating along the line of sight:

$$\int n_i n_e \varepsilon_l(T)ds = \varepsilon_l(T_0)\int n_i n_e ds + \frac{1}{2}\left(\frac{d^2\varepsilon_l}{dT^2}\right)_0 \int n_i n_e (T-T_0)^2 ds$$

Here, the first order term of $T - T_0$ vanishes if we choose T_0 as the mean temperature,

$$\boxed{T_0 = \frac{\int n_i n_e T ds}{\int n_i n_e ds}} \Rightarrow \text{ mean temperature}$$

and we define $\boxed{t^2 = \frac{\int n_i n_e (T-T_0)^2 ds}{T_0^2 \int n_i n_e ds},}$ representing the variance of temperature.

- If all ions had the same space distribution $n_i(s)$, then both T_0 and t^2 could be determined by combining two line ratios (for instance [O III] ($\lambda 4959 + \lambda 5007)/\lambda 4363$ and [N II] ($\lambda 6543 + \lambda 6583)/\lambda 5755$). Note two unknowns $\varepsilon(T_0)$ and $d^2\varepsilon/dT^2$.

- Then, T_0 and t^2 could be used to determine the abundances of all the ions with measured lines.

  However, all ions do not have the same spatial distribution. For instance, O^{++} is more strongly concentrated to the source of ionizing radiation than N^+.

  The most sophisticated method of all to determine the abundances from the observations is to calculate a complete model of the nebula in an attempt to reproduce all its observed properties, as will be discussed in section 5.12.

- **Notes:**
  - [O III] and [N II] collisional excitation lines trace the abundance of O^{++} and N^+, respectively.
  - The He I recombination line provides the abundance of He^+, and the He II recombination line traces that of He^{++}.
  - Because there is no "observable" He I collisional excitation line, **there is no way to determine the He^0 abundance directly.**
- Observational results
  - He/H abundance ratio
    - $n(\mathrm{He}^+)/n_p = 0.06 - 0.09$ in Orion Nebula, suggesting that the varying amounts of He^0 are present.
    - $n(\mathrm{He}^+)/n_p = 0.009$ in NGC 1982, indicating that the observation was performed in an H^+ plus He^0 zone, where He is predominantly neutral. The exciting star of NGC 1982 is a B1 V star, so the He-ionizing radiation is very weak or absent.
  - **Ionization Correction Factor (ICF)**
    - The above observations indicate that some correction is necessary to estimate the abundance of He, specifically to compensate for the amount of unobserved He^0. An ion of another element can be used its proxy.
    - Empirically, the correction can be based on [S II]$\lambda 6717, 6731$, because S^+ has an ionization potential of 23.4 eV, similar to that of He^0 (24.6 eV). Therefore, to a first approximation

$$\frac{n(\mathrm{He}^0)}{n(\mathrm{He}^+)} = \frac{n(S^+)}{n(S^{++})} \qquad \Rightarrow \qquad n(\mathrm{He}^0) \approx n(\mathrm{He}^+)\frac{n(S^+)}{n(S^{++})}$$

Such ratios are called ICFs, since they correct for unobserved stages of ionization.

- Radio measurements of He^+/H^+ abundance ratios are available for many diffuse nebulae.
  - Radio observations have the advantage that they can be done at large distances in the galactic plane.
  - However, there no known way in which the correction for He^0 can be obtained from radio measurements.
  - NGC 2024 and NGC 1982 are observed to have $n(He^+)/n_p \approx 0$. Radio measurements of H II regions very near the galactic center give quite low $n(He^+)/n_p$ ratios.

    These low ratios are due to their ionizing stars being predominantly cool, rather than suggesting low helium abundance. These indicates corrections are certainly required.
- PNe
  - Many PNe show the presence of both He^{++} and He^+ (He II and He I lines), though some PNe have only He I lines.
  - Nearly all PNe have central stars that are so hot that they have no outer He^+, He^0 zones. Therefore, no correction is necessary for unobserved He^0 in most PNe.
  - Most of PNe have $n(He)/n(H) \approx 0.11$, slightly higher than those of H II regions. Some PNe show twice larger abundances.
  - These differences are real, and the He abundance can help distinguish between different populations of PNe.

- Abundance of the heavy elements in H II regions and PNe
    - They are determined by a combination of collisionally excited lines and hydrogen recombination lines. The line ratio depends mainly on the shape of the stellar continuum from the following relation.

$$\frac{\text{total cooling}}{L(\mathrm{H}\beta)} \approx \frac{\text{cooling by collisional excitation lines}}{L(\mathrm{H}\beta)} \propto \frac{\int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} h(\nu - \nu_0)\sigma_\nu^{\mathrm{p.i.}} d\nu}{\int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} d\nu}$$

    Note also that the strength of collisionally excited lines depends strongly on temperature, which is determined by the shape of the stellar continuum. On the other hand, the recombination lines are not very sensitive to the gas temperature.

    - The gas is hotter in lower-metallicity nebulae.
    - Disadvantages: (1) Collisional excitation lines can be converted into abundances only when the gas temperature is well measured. (2) Additionally, large and rather uncertain corrections may be required for unobserved ion stages.
    - The abundances of a PN are affected by nuclear processing in the central star, where as those in H II regions reflect the composition of the interstellar medium.
- Recombination lines of the heavy elements
    - Large telescopes and CCD detectors have made it possible to detect faint recombination lines of the heavy elements in several H II regions and PNe.
    - Abundances measured with these methods should be far more robust than those inferred from collision excitation lines, as they have similar dependencies on temperature to hydrogen recombination lines.
    - This method would not be affected by temperature fluctuations.
    - However, the lines are faint, and the recombination process is complicated by strong dielectronic recombination.

- Discrepancies in H II regions
  - In H II regions, **the recombination-line abundances tend to be slightly higher than the forbidden-line abundances**, which is attributed to the temperature fluctuations.
  - If this is the case, the correct gas-phase abundances are about 50%-100% higher than those estimated by assuming a constant temperature.
  - $t^2 \approx 0.02 - 0.04$
- Discrepancies in PNe
  - Recent works found differences between collisional and recombination abundances that are larger than a factor of ten, too large to be caused by temperature fluctuations.
  - This suggest that another, presently unknown, process may be affecting one (or both) of the abundance measures.
  - The fact that the discrepancy varies from object to object, indicates that it is not a simple matter of incorrect atomic data. This difference might be related to physical conditions within PNe. If this is the case, then the higher recombination abundances are more likely to be correct.

6.1 Introduction

- The previous chapters have described gaseous nebulae from a static point of view.
- Nebulae certainly have internal motions, and the effects of these motions on their structures cannot be ignored.
  - If an ionized nebula is matter-bounded, it will expand into the surrounding vacuum.
  - On the other hand, if it is ionization-bounded, the hot ionized gas (with $T \approx 10^4$ K) will initially have higher pressure than the surrounding cooler neutral gas ($T \approx 100$ K). Thus, **it will tend to expand until its density is low enough so that the pressures of the two gases are in equilibrium.**
  - [Initial stage] When a hot star forms and the ionizing radiation source is "turned on", ionized volume initially grows in size at a rate of emission of ionizing photons, and an ionizing front separating the ionized and neutral regions propagates into the neutral gas.
- Typical motions in PNe and H II regions
  - Planetary nebulae expand more or less radially with velocities of order 25 km s^{-1}, and the velocity increases radially.
  - Many H II regions are observed to have complex internal velocity distributions that can best be described as turbulent.
- Contents
  - Hydrodynamic equations of motion
  - Ionization fronts and shock fronts that are generated in an expanding, photoionized gas.
  - PNe and H II regions are analyzed
  - Observational material

6.2 Hydrodynamic Equations of Motion

- Hydrodynamics deals with changes and movements in a fluid as forces act on it.
- Two Reference Frames
  - **Lagrangian Description: Lagrangian reference frame - one that moves with the fluid.**
  - **Eulerian Description: Eulerian reference frame - one that is stationary.**
- The time derivative of f, any physical quantity that is a function of position and time, in the Lagrangian reference frame is related to the derivatives in the Eulerian reference frame.

$$\boxed{\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f}$$

  The **Eulerian derivatives** on the right represents (1) changes in f over time and (2) advection, changes due to the flow of upstream material into the region.

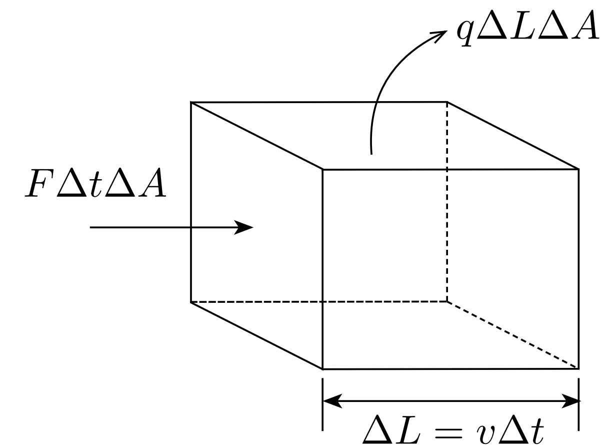
- Hydrodynamic equations
  - Three hydrodynamic equations - conservation of mass, momentum, and energy
  - Two additional equations - an ionization equation and an equation of state.
  - The five equations are needed to fully determine conditions in the gas.

(Basics of Hydrodynamics)

- Assumption for hydrodynamics:
  - particle mean free path << size of the region
  - We will describe the equations for conservation of mass, momentum and energy, in 1D space.
- ***Definition***
  - Flux of a hydrodynamic quantity q (for instance, density):

Fluid moves a distance ΔL during a time interval Δt with a velocity v.

$$F \Delta t \Delta A = q \Delta L \Delta A \quad \rightarrow \quad \boxed{F = qv}$$

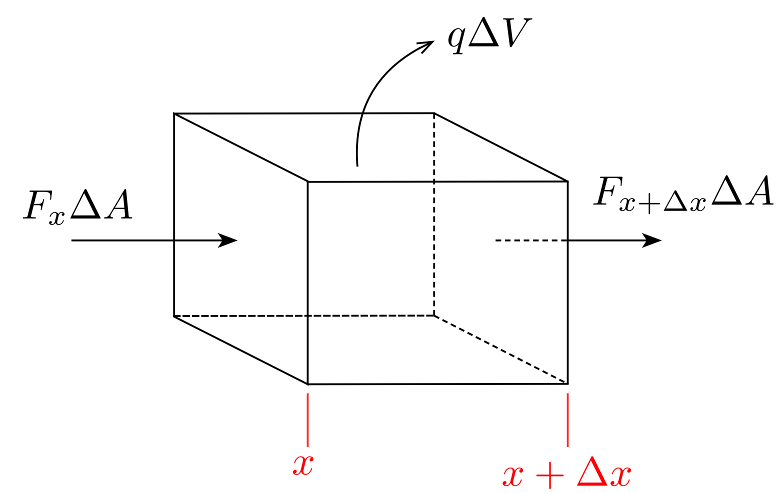


- ***Conservation equation for a quantity*** q
  - Change of the quantity within a volume ΔV for a time interval Δt:

    Here, Δt and Δx are independent.

$$\frac{q\Delta V|_{t+\Delta t} - q\Delta V|_t}{\Delta t} = F\Delta A|_x - F\Delta A|_{x+\Delta x}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial F}{\partial x} \longrightarrow \boxed{\frac{\partial q}{\partial t} = -\frac{\partial (qv)}{\partial x}}$$



  - Here, no sources or sinks of the quantity within ΔV were assumed. If any, the loss and gain terms should be added in the right-had side.

(Mass Conservation)

- Conservation equations
  - *Mass conservation (continuity equation)*
    - mass within a volume $dV = \rho dV$
    - no sources or sinks of material within dV
    - Consider the mass per unit area (dA), contained in the volume

$$\rho dV/dA = \rho dx \longrightarrow \begin{aligned} \frac{\partial}{\partial t}(\rho dx) &= \overbrace{\rho u}^{\text{incoming}} - \overbrace{(\rho + d\rho)(u + du)}^{\text{outgoing}} \\ &= -(\rho du + u d\rho + \cancel{d\rho du}) \end{aligned}$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}}$$

- Mass loss and gain terms should be added in the right-had side, if necessary.

(Momentum Conservation)

- *Momentum conservation (Euler's equation)*

- momentum within dV (per unit area) $=$ $(\rho dV)u/dA = \rho dx u$

  $=$ change of momentum due to fluid flow and gas pressure acting on the surface of dV

$$\frac{\partial}{\partial t}(\rho u dx) = \overbrace{\rho u^2}^{\text{incoming}} - \overbrace{(\rho + d\rho)(u+du)^2}^{\text{outgoing}} + \overbrace{P}^{\text{incoming}} - \overbrace{P + dP}^{\text{outgoing}}$$

$$= \rho u^2 - \left(\rho u^2 + 2\rho u du + \cancel{\rho du^2} + u^2 d\rho + \cancel{2ud\rho du + d\rho du^2}\right) - dP$$

$$\frac{\partial}{\partial t}(\rho u) = -2\rho u \frac{\partial u}{\partial x} - u^2 \frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x}$$

$$\rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} = -\rho u \frac{\partial u}{\partial x} - u\left(\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x}\right) - \frac{\partial P}{\partial x}$$

Using mass conservation, $\frac{\partial u}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$

$$\rho \frac{\partial u}{\partial t} = -\rho u \frac{\partial u}{\partial x} - \frac{\partial P}{\partial x}$$

or

$$\frac{\partial}{\partial t}(\rho u) = -2\rho u \frac{\partial u}{\partial x} - u^2 \frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x}$$

$$= -\frac{\partial}{\partial x}(\rho u^2) - \frac{\partial P}{\partial x}$$

$$\frac{\partial}{\partial t}(\rho u) = -\frac{\partial}{\partial x}\left(\rho u^2 + P\right)$$

- Further terms could be added in the right-hand side, accounting for forces due to gravity, magnetic fields, radiation field, and viscosity.

(Energy Conservation)

- *Energy conservation*

- The first law of thermodynamics states that

  heat added in a system = change in internal energy + work done on surroundings

$$\text{đ}Q = dU + PdV$$

- Internal energy (per particle) for ideal gas is

$$\boxed{\begin{array}{ll} U/N = \dfrac{3}{2}kT & \text{for monatomic gas (translation about 3 axes)} \\ U/N = \dfrac{5}{2}kT & \text{for diatomic gas} \quad \text{(+rotation about 2 axes)} \\ U/N = 3kT & \text{for polyatomic gas (+rotation about 3 axes)} \end{array}}$$

Here, N is the number of particles.

An ideal gas is a theoretical gas composed of many randomly moving point particles whose only interactions are perfectly elastic collisions (no viscosity or heat conduction).

- In general, the internal energy per particle is

$$U/N = \frac{f}{2}kT \quad (f = \text{degree of freedom})$$

At high temperature, molecules have access to an increasing number of vibrational degrees of freedom, as they start to bend and stretch.

- The ideal gas law (the equation of state) for a perfect Maxwellian distribution.

$$\boxed{PV = NkT} \qquad P = \frac{N}{V}kT$$

- ***Specific heat capacity*** is the amount of ***heat energy required to raise the temperature of a material per unit of mass***.

  - specific heat capacity **at constant volume**:

$$c_V \equiv \frac{1}{M}\left(\frac{\partial Q}{\partial T}\right)_V = \frac{1}{M}\left(\frac{\partial U}{\partial T}\right)_V \qquad \boxed{c_V = \frac{f}{2}\frac{k}{m}}$$

$$M = \text{total mass}$$

$$m = M/N = \text{mass per particle}$$

$$m = \mu m_{\rm H}$$

$$(\mu = \text{mean atomic weight per particle})$$

  - specific heat capacity **at constant pressure**:

$$c_P \equiv \frac{1}{M}\left(\frac{\partial Q}{\partial T}\right)_P = \frac{1}{M}\left(\frac{\partial U}{\partial T}\right)_P + \frac{P}{M}\left(\frac{\partial V}{\partial T}\right)_P = \frac{1}{M}\frac{f}{2}Nk + \frac{P}{M}\frac{Nk}{P}$$

$$\boxed{\therefore \quad c_P = \frac{f+2}{2}\frac{k}{m} = c_V + \frac{k}{m}}$$

  - Ratio of specific heat capacities:

$$\gamma \equiv \frac{c_P}{c_V} = \frac{f+2}{f} = \frac{5}{3} \quad \text{for monatomic gas}$$

$$= \frac{7}{5} \quad \text{for diatomic (molecular) gas}$$

$$= \frac{4}{3} \quad \text{for polyatomic (molecular) gas}$$

γ is called the *adiabatic index*.

$c_P > c_V$

This inequality implies that when pressure is held constant, some of the added heat goes into PdV work instead of into internal energy.

- Energy Conservation - limiting cases

  - ***Adiabatic flow*** - negligible heat transport (Internal energy is changed only by work).

$$\text{đ}Q = dU + PdV = Mc_V dT + PdV$$

$$\text{đ}Q = 0$$

$$\rightarrow PdV = -Mc_V dT$$

$$PV = NkT$$

$$\rightarrow VdP + PdV = NkdT$$

We combine two equation and eliminate dT term:

$$\begin{aligned} VdP + PdV &= -\frac{Nk}{Mc_V} PdV \\ &= -\frac{k}{m}\frac{1}{c_V} PdV \end{aligned} \longrightarrow \begin{aligned} VdP &= -\left(1 + \frac{k}{m}\frac{1}{c_V}\right) PdV \\ &= -\frac{1}{c_V}\left(c_V + \frac{k}{m}\right) PdV \\ &= -\gamma PdV \end{aligned}$$

$$\longrightarrow \frac{dP}{P} = -\gamma \frac{dV}{V}$$

We can rewrite this in terms of density:

$$\rho V = M$$

$$\rightarrow \rho dV + V d\rho = 0$$

$$\rightarrow \frac{d\rho}{\rho} = -\frac{dV}{V} \longrightarrow \frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$

In summary,

$$\frac{dP}{P} = -\gamma \frac{dV}{V} \qquad P \propto \rho^{\gamma}$$

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho} \qquad P \propto V^{-\gamma}$$

$$\rightarrow T \propto V^{-(\gamma-1)}$$

adiabatic heating/cooling

- ***Isothermal flow*** - extremely efficient cooling (heat transport).

heat transport timescale << dynamic timescale

This implies the balance between heating and cooling, hence a constant temperature.

From the ideal gas law,

$$P = \frac{N}{V} kT = \rho \frac{kT}{m} \longrightarrow \boxed{\begin{array}{l} P \propto \rho \\ P \propto V^{-1} \end{array}}$$

- **In general**, we have

$$\boxed{\begin{array}{l} P \propto \rho^{\gamma} \\ P \propto V^{-\gamma} \end{array}} \quad (\gamma = 1 \text{ for isothermal gas})$$

A gas that has an equation of state with this power-law form is called a ***polytope***, from the Greek polytropos, meaning “turning many ways” or “versatile.”

(A polystrope should not be confused with a polytrope, which is the n-dimensional generalization of a 2D polygon and 3D polyhedron.)

- ***Specific internal energy*** of the gas (***per unit mass***):

$$\epsilon \equiv U/M$$
$$U/N = \frac{f}{2}kT \longrightarrow \epsilon = \frac{f}{2}\frac{kT}{m} \quad \text{or} \quad \epsilon = \frac{1}{\gamma-1}\frac{kT}{m} = \frac{1}{\gamma-1}\frac{P}{\rho}$$

- ***Total Energy (per unit volume)***:
  - ***Internal energy per unit volume***:
  - ***Kinetic energy due to bulk motion, per unit volume***:
  - ***Work on unit volume***:

$$\mathcal{E}_{\rm int} = \rho\epsilon = \frac{1}{\gamma-1}P$$
$$\mathcal{E}_{\rm kin} = \rho\frac{u^2}{2}$$
$$\mathcal{E}_{\rm mech} = \frac{PdV}{dV} = P$$

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_{\rm int} + \mathcal{E}_{\rm kin} + \mathcal{E}_{\rm mech} \\ &= \rho\left(\frac{u^2}{2} + \epsilon\right) + P\end{aligned} \longrightarrow \mathcal{E} = \rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P$$

- ***Energy conservation***:

$$\frac{\partial\mathcal{E}}{\partial t} = -\frac{\partial(u\mathcal{E})}{\partial x}$$

$$\boxed{\frac{\partial}{\partial t}\left(\rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P\right) = -\frac{\partial}{\partial x}\left[u\left(\rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P\right)\right]}$$

(Sound Wave)

- Suppose that we are surrounded by an ideal gas with a plane parallel symmetry:
  - We consider a region where the gas has initially a uniform density, pressure, and no bulk velocity: $\rho_0,\ P_0,\ u_0 = 0$

    In the uniform gas, we introduce small perturbations of the form:

$$\rho(x,t) = \rho_0 + \rho_1(x,t)$$
$$u(x,t) = u_1(x,t)$$
$$P(x,t) = P_0 + P_1(x,t)$$

$$\longrightarrow \quad P_1 = P - P_0 \propto (\rho_0+\rho_1)^\gamma - \rho_0^\gamma \propto \gamma\rho_0^{\gamma-1}\rho_1 \quad \longrightarrow \quad P_1 = \frac{\gamma P_0}{\rho_0}\rho_1$$

We obtain:

$$\frac{\partial\rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$
$$\rho\frac{\partial u}{\partial t} = -\rho u\frac{\partial u}{\partial x} - \frac{\partial P}{\partial x}$$

$$\longrightarrow \quad \frac{\partial\rho_1}{\partial t} = -\rho_0\frac{\partial u_1}{\partial x}$$
$$\rho_0\frac{\partial u_1}{\partial t} = -\frac{\partial P_1}{\partial x} = -\frac{\gamma P_0}{\rho_0}\frac{\partial \rho_1}{\partial x}$$

$$\longrightarrow \quad \boxed{\frac{\partial^2\rho_1}{\partial t^2} = -\frac{\gamma P_0}{\rho_0}\frac{\partial^2\rho_1}{\partial x^2}}$$

  - The resulting equation represents a **sound wave (acoustic wave)** with a constant sound speed:

$$\boxed{c_s = \left(\frac{\gamma P}{\rho}\right)^{1/2} = \left(\frac{\gamma kT}{m}\right)^{1/2}}$$

$$\boxed{c_s \propto \rho^{(\gamma-1)/2}}$$

For $\gamma > 1$ sound travels more rapidly in a denser gas.

- The sound speed is of the same order as the mean thermal velocity:

$$c_s = 1.2\,\mathrm{km\,s^{-1}} \left(\frac{\gamma}{5/3}\right)^{1/2} \left(\frac{m}{m_p}\right)^{-1/2} \left(\frac{T}{100\,\mathrm{K}}\right)^{1/2}$$

$(m_p = \text{proton mass})$

- ***Sound crossing time***:
  - sound crossing time = time it takes for a signal to cross a region of size L:

$$t_{\mathrm{cross}} = L/c_s$$

  - A small pressure gradient tends to be smoothed out within the sound crossing time. Generally, when a stationary gas is disturbed, the resultant changes in velocity, density, pressure, and temperature are communicated downstream at the sound speed.

    Fast changes occurring on timescales $\ll t_{\mathrm{cross}}$ *will survive, and a shock front forms.*

    Slow changes occurring on timescales $\gg t_{\mathrm{cross}}$ will be damped.

- ***Mach number*** = fluid velocity / sound speed

In a turbulent medium, the root-mean-square Mach number is used to represent the typical turbulent velocity.

$$\boxed{\mathcal{M} \equiv u/c_s}$$

$\mathcal{M} > 1$	supersonic
$\mathcal{M} < 1$	subsonic

$\mathcal{M} \to \infty$	strong shock
$\mathcal{M} \to 1$	weak shock

- The momentum equation
  - describes the motion of a compressible fluid, may be written as

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P - \rho \nabla \phi$$

  - The change in momentum per unit volume (as we follow the fluid) is caused by the force terms, which include the gradient in the pressure P and the force resulting from the gravitational potential ϕ of the stars and of the nebula itself.
  - The gravitational force would be negligible. Viscous dissipation and electromagnetic forces will also be neglected. However, there may be nebulae in which magnetic fields are strong and not negligible.
- The equation of continuity (conservation of mass)

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}$$

- The energy equation is a generalization of the thermal balance equation.

$$\frac{DU}{Dt} = \frac{D}{Dt}\left(\frac{3}{2}\sum_j n_j kT\right) = (G - L) + \frac{P}{\rho}\frac{D\rho}{Dt} - U\nabla \cdot \mathbf{u}$$

$$\boxed{\frac{D\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho}$$

compression + advection

  - Here U is the internal kinetic energy per unit volume, G and L are the energy gain and loss rate per volume per unit time.
  - The second term gives **the heating rate resulting from compression (or expansion) of the gas and the advection of new material** which may have more or less internal energy.
  - The last term gives **the dilation effect**, analogous to the term on the right-hand side of the equation of continuity.

- Note that ionization energy is not included, but the kinetic energy of all particles is. It is a reasonably good approximation because all the ionized species are in temperature equilibrium. (The Coulomb-scattering cross sections are so large, and the relaxation times are correspondingly short.)
- The energy equation can be rewritten in a form that includes the internal kinetic energy per unit mass $E = U/\rho$:

$$\frac{DE}{Dt} = \frac{D}{Dt}\left(\frac{U}{\rho}\right) = \frac{1}{\rho}(G - L) - P\frac{D}{Dt}\left(\frac{1}{\rho}\right).$$

$$\begin{aligned}\frac{D(U/\rho)}{Dt} &= \frac{1}{\rho}\frac{DU}{Dt} - \frac{U}{\rho^2}\frac{D\rho}{Dt} = \frac{1}{\rho}(G-L) + \frac{P}{\rho^2}\frac{D\rho}{Dt} - \frac{U}{\rho}\nabla\cdot\mathbf{u} - \frac{U}{\rho^2}\frac{D\rho}{Dt} \\ &= \frac{1}{\rho^2}(G-L) - P\frac{D}{dt}\left(\frac{1}{\rho}\right) - \frac{U}{\rho^2}\underbrace{\left[\rho\nabla\cdot\mathbf{u} + \frac{D\rho}{Dt}\right]}_{=0}\end{aligned}$$

- The ionization equation is

$$\begin{aligned}\frac{Dn(X^{+i})}{Dt} = \frac{\partial n(X^{+})}{\partial t} + \mathbf{u}\cdot\nabla n(X^{+i}) = &- n(X^{+i})\int_{\nu_i}^{\infty}\frac{4\pi J_\nu}{h\nu}\sigma_\nu^{\mathrm{pi}}(X^{+i})d\nu \\ &+ n(X^{+i+1})n_e\alpha_{\mathrm{A}}(X^{+i},T) - n(X^{+i})n_e\alpha_{\mathrm{A}}(X^{+i-1},T) \\ &+ n(X^{+i-1})\int_{\nu_{i-1}}^{\infty}\frac{4\pi J_\nu}{h\nu}\sigma_\nu^{\mathrm{pi}}(X^{+i-1})d\nu - n(X^{+i})\nabla\cdot\mathbf{u}\end{aligned}$$

- The equation of state, the relation between the states of the gas (pressure, density, and temperature):

$$P = \frac{\rho k T}{\mu m_{\rm H}} = n_{\rm tot} k T$$

  - In most situations, the radiation pressure can be neglected, the density of radiation is so low.
  - **Isothermal case** - the balance between heating and cooling processes determine the temperature, and the gas tends to be nearly isothermal.
  - **Adiabatic case** - radiative losses do not occur, and the thermal energy of the gas is preserved.

$$P = K\rho^\gamma, \qquad \gamma = 5/3 \text{ for a monatomic gas (as in H II regions)}$$
$$\gamma = 7/5 \text{ for a diatomic gas}$$

- The time-dependent equations are non-linear integro-differential equations.
  - **The time scale for photoionization and recombination is generally shorter than the dynamical time scale, except in the vicinity of the nebula's edge.** Thus, it is reasonable to assume a static nebula everywhere except in that region.
  - However, note that this assumption might not be the case in the diffuse ionized gas.

- **Advection of neutral material into an expanding H^+.**
  - The advection causes the total luminosity of recombination lines such as Hβ to be smaller than expected.
  - The balance equation for the flux of ionizing photons is

$$\boxed{\underbrace{\Phi(\mathrm{H}^0)}_{\text{\# of ionizing photons}} = \int \left[n_e n_p \alpha_\mathrm{B} + \mathbf{u} \cdot \nabla n \right] dr \approx \underbrace{\int n_e n_p \alpha_\mathrm{B} dr}_{\text{(a) recombination}} + \underbrace{n(\mathrm{H}^0) u}_{\text{(b) advection}}}$$

  Here, the number of ionizing photons entering the region is balanced by (a) the creation of neutral material by recombination and (b) advection.

  - **The flux of ionizing photons is only partially used in sustaining ionization within a volume element; the remainder ionize neutral material that flows into that element.**
  - For a typical H II region, $\Phi(\mathrm{H}^0) = 2 \times 10^9\,\mathrm{cm^{-2}\,s^{-1}}$, $n = 10\,\mathrm{cm^{-3}}$, $u = 20\,\mathrm{km\,s^{-1}}$. The number of ionizing photons used that produce recombinations will be reduced by $n(\mathrm{H}^0)u/\Phi(\mathrm{H}^0) \approx 0.01$, the fraction of the ionizing photons used in the initial ionization of hydrogen rather than ionization following recombination. This effect can be much larger in more rapid flows.

- **In a subsonic flow**, the gas temperature and flow speed are related to one another.
  - From the continuity equation, the mass flux in the time-steady limit ($\partial\rho/\partial t = 0$) is given by

$$\Phi = \rho u \quad [\mathrm{g\,cm^{-2}\,s^{-1}}]$$

    From the momentum equation, the momentum flux is

$$\Pi = P + \rho u^2 = \rho c_s^2 + \rho u^2 \quad \text{for an isothermal gas}$$

$$(\gamma = 1, \text{ the sound speed } c_s^2 = \gamma P/\rho = kT/\mu m_{\mathrm{H}})$$

    In a subsonic flow ($u \ll c_s$),

    the momentum flux is given by $\Pi = \rho c_s^2 \quad \Rightarrow \quad \therefore \quad \Pi u = \Phi c_s^2$

    Since both Π and Φ are conserved, we obtain

$$u = c_s^2 = kT/\mu m_{\mathrm{H}}$$

  - **For a subsonic flow, the gas temperature is proportional to the fluid velocity**, because the velocity is determined by the gas pressure, which is in turn set by the temperature.

6.3 Free Expansions into a Vacuum

- A cloud freely expanding into a vacuum
  - This situation occurs in explosive environments (i.e., envelopes of novae, probably in AGNs).
  - **Riemann problem:** This problem was first treated by Riemann.
  - Initial conditions (at $t = 0$) of an adiabatic gas:

$$\begin{aligned} \rho &= \rho_0 \text{ (constant)} \quad \mathrm{r} \le \mathrm{r}_0 \\ &= 0 \qquad\qquad\qquad \mathrm{r} > \mathrm{r}_0 \\ u &= 0 \quad \text{at all positions} \end{aligned}$$

  - The edge of the gas cloud moves outward with a velocity given by

$$u_e = \frac{2}{\gamma - 1} c_s = 3 c_s$$

    For a monatomic gas ($\gamma = 5/3$), a rarefaction wave moves inward into the undisturbed gas.

    Thus, at later time t, the rarefaction wave has reaches a radius $r_i = r_0 - c_s t$,

    while the outer edge has reached a radius $r_e = r_0 + u_e t$.

  - All the gas between these two radii is moving outward with velocity increasing from 0 at r_i to u_e at r_e. In a spherical nebula, the inward-running rarefaction wave ultimately reaches the center and is reflected, and the gas near the center is then further accelerated outward.

For detailed solutions, see Copson (1950, MNRAS, 110, 238)

6.4 Shocks

- There are cases where the physical variables are nearly discontinuous.
  - shock fronts
  - ionization fronts
- Shock front - across which ρ, u, and P change discontinuously, but the ionization does not change.
  - A real shock front is not an infinitely sharp discontinuity, but **the mean-free path for atomic collisions (which gives the relaxation length) is so short** in comparison with the dimensions of the flow that ρ, u, and P are nearly discontinuous.
  - For this analysis, it is convenient to use **a reference system moving with the shock front**.
  - Assume a plane, steady shock and denote the physical parameters ahead of and behind the shock by subscripts 1 and 2. (0 and 1 in the text book)
- Conservation conditions - Rankine-Hugoniot conditions on the discontinuities at a shock front
  - Momentum and mass conservation conditions across the front

$$P_0 + \rho_0 u_0^2 = P_1 + \rho_1 u_1^2$$
$$\rho_0 u_0 = \rho u_1$$

Here, the velocity components are perpendicular to the front.

  - For many conditions, **the net radiate heating and cooling terms $G - L \approx 0$, and the gas is compressed/expanded adiabatically. An adiabatic process transfers energy to the surroundings only as work.**
  - In this case, the energy conservation condition, combined with the mass conservation, gives

$$\frac{1}{2}u_0^2 + \frac{\gamma}{\gamma - 1}\frac{P_0}{\rho_0} = \frac{1}{2}u_1^2 + \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1}$$

For $\gamma = 5/3$ (monatomic gas), $\frac{1}{2}u_0^2 + \frac{5}{2}\frac{P_0}{\rho_0} = \frac{1}{2}u_1^2 + \frac{5}{2}\frac{P_1}{\rho_1}$

- The first term represents the flow kinetic energy per unit mass.
- The second term may be broken up into two contributions:

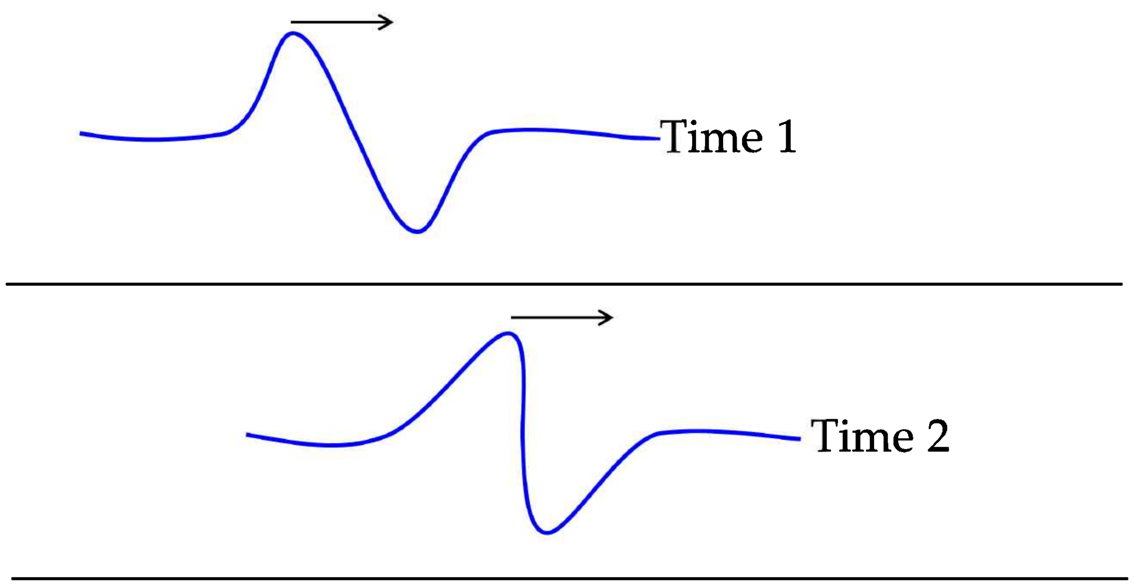
$$\frac{5}{2}P/\rho = \frac{3}{2}P/\rho + P/\rho = \frac{3}{2}kT/\mu \mathrm{m_H} + P/\rho$$

$\Rightarrow$ thermal kinetic energy per unit mass + compressional energy per unit mass

- The Rankine-Hugoniot conditions can be solved to give any three of the quantities in terms of any other three.

(Shock)

- Shock
  - A low-amplitude sound wave traveling through a medium will be adiabatic; that is it will not increase the entropy of the gas through which it passes.
  - For an adiabatic process, the equation of state for the gas is

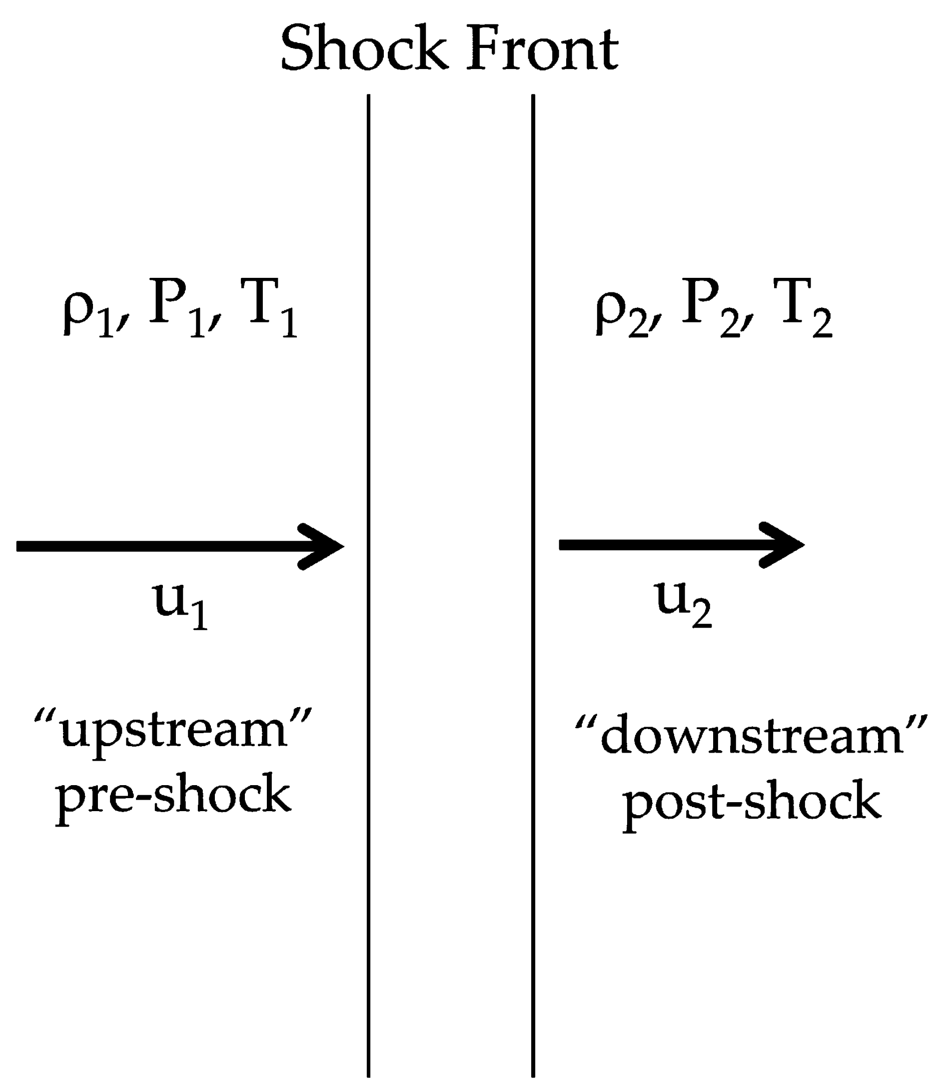


$$c_s \propto \rho^{(\gamma-1)/2}$$

  - Thus, for $\gamma > 1$, sound travels more rapidly in a denser gas.
  - ***For a supersonic gas, the motion itself is faster than the speed of communication, and instead of a smooth transition, the physical quantities (density, pressure, and temperature) undergo a sudden change in values over a small distance.*** This phenomenon is referred to as a ***shock***.
  - We define the **shock front** as the region over which the velocity, density, and pressure of the gas undergo sudden changes. The shock front is a layer whose thickness is comparable to the mean free path between particle collisions.
  - The ordinary sound that we hear every day will not, in practice, steepen into shocks.
  - However, high amplitude pressure fluctuations will rapidly steepen into shocks.

(Shock Front)

- Jump condition (***Rankine-Hugoniot conditions***)
  - Let
    $$\rho = \text{mass density}, \; T = \text{temperature},$$
    $$m = \text{mean molecular mass}$$
  - If a patch is small compared to the shock front's radius of curvature, then we can treat the shock front as if it has ***plane parallel*** symmetry.
  - ***It is convenient to use a frame of reference in which the shock front is stationary.***
  - Let us consider a shock propagating with velocity V_s into a gas that is previously at rest. In the frame of reference of the shock, the gas in the pre-shock region is approaching at a velocity of $-V_s$.
  - In this frame, the bulk velocity $u_1 = -V_s$ of the pre-shock (upstream) gas toward the shock front. The bulk velocity u_2 of the post-shock (downstream) gas points away from the shock front.



Plane parallel steady-state shock,
in the reference frame of the shock front.

- Let's consider a steady state solution.
  - The gas properties immediately before being shocked ("1") and immediately after being shocked ("2") are obtained from the conservation laws:

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$
$$u_1 \left(\rho_1 \frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} P_1 \right) = u_2 \left(\rho_2 \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} P_2 \right)$$

Dividing the third equation with the first equation:

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}$$

In summary,

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$
$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}$$

Here, *we assume that an adiabatic index is the same on both sides of the shock front.*

- From the three equations, we should be able to derive the changes, ρ_2/ρ_1, u_2/u_1, and P_2/P_1 across the shock.

  It is convenient to use a dimensionless number, the Mach number of the upstream:

$$\mathcal{M}_1 = u_1/c_1,\ c_1^2 = \frac{\gamma P_1}{\rho_1} \quad \rightarrow \quad P_1 = \frac{u_1^2\rho_1}{\gamma \mathcal{M}_1^2}$$

(1) To find the equation for densities:

$$\rho_1 u_1 = \rho_2 u_2 \text{ and } P_1 = \frac{u_1^2\rho_1}{\gamma \mathcal{M}_1^2}$$

$$\begin{aligned} & \rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \\ \rightarrow\ & \rho_1 u_1^2 + u_1^2 \frac{\rho_1}{\gamma \mathcal{M}_1^2} = \frac{(\rho_1 u_1)^2}{\rho_2} + P_2 \\ \rightarrow\ & P_2 = \rho_1 u_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right) \end{aligned}$$

Inserting these relations into the energy conservation equation:

$$\begin{aligned} & \frac{u_1^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2} \\ \rightarrow\ & \frac{u_1^2}{2} + \frac{1}{\gamma-1}\frac{u_1^2}{\mathcal{M}_1^2} = \frac{1}{2}\left(\frac{\rho_1 u_1}{\rho_2}\right)^2 + \frac{\gamma}{\gamma-1}\frac{\rho_1 u_1^2}{\rho_2}\left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right) \\ \rightarrow\ & \frac{1}{2} + \frac{1}{\gamma-1}\frac{1}{\mathcal{M}_1^2} = \frac{1}{2}\left(\frac{\rho_1}{\rho_2}\right)^2 + \frac{\gamma}{\gamma-1}\frac{\rho_1}{\rho_2}\left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right) \end{aligned}$$

$$ax^2 + bx - c = 0$$

where $x = \dfrac{\rho_1}{\rho_2}$

$$a = \frac{1}{2} - \frac{\gamma}{\gamma-1}$$

$$b = \frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_1^2}$$

$$c = \frac{1}{2} + \frac{1}{(\gamma-1)\mathcal{M}_1^2}$$

$$x = \frac{b^2 \pm \sqrt{b^2 + 4ac}}{2a}$$

$$\frac{\rho_1}{\rho_2} = \frac{-\left[\frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_0^2}\right] \pm \frac{\mathcal{M}_1^2-1}{\mathcal{M}_1^2(\gamma-1)}}{1-\frac{2\gamma}{\gamma-1}} = \frac{\gamma\mathcal{M}_1^2 + 1 \mp (\mathcal{M}_1^2 - 1)}{(\gamma+1)\mathcal{M}_1^2} \longrightarrow \frac{\rho_1}{\rho_2} = 1 \quad \text{or} \quad \frac{(\gamma-1)\mathcal{M}_1^2+2}{(\gamma+1)\mathcal{M}_1^2}$$

$$\therefore \quad \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2}$$

(2) Now, we obtain the equation for pressures:

Divide the following equation

$$P_2 = \rho_1 u_1^2 \left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right)$$

with this

$$P_1 = \frac{u_1^2\rho_1}{\gamma\mathcal{M}_1^2}$$

$$\longrightarrow \frac{P_2}{P_1} = \gamma\mathcal{M}_1^2\left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right) = \gamma\mathcal{M}_1^2\left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{(\gamma-1)\mathcal{M}_1^2+2}{(\gamma+1)\mathcal{M}_1^2}\right)$$

$$\therefore \quad \frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1}$$

(3) Using the ideal gas law:

$$P = \frac{\rho k T}{m} \longrightarrow \frac{T_2}{T_1} = \frac{\rho_1}{\rho_2}\frac{P_2}{P_1}$$

Using the equations for densities and pressures:

$$\therefore \quad \frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2+2\right]\left[2\gamma\mathcal{M}_1^2-(\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2}$$

In summary, we obtain the jump conditions:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2} = \frac{u_1}{u_2}$$
$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1}$$
$$\frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2+2\right]\left[2\gamma\mathcal{M}_1^2-(\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2}$$

weak shock

$\mathcal{M} \to 1$ $\quad \frac{\rho_2}{\rho_1} = 1$, $\quad \frac{P_2}{P_1} = 1$, $\quad \frac{T_2}{T_1} = 1$

In the ***lab frame***, let V_s = shock velocity, $v_1,\ v_2$ = gas velocities in upstream (pre-shock) and downstream (post-shock), respectively $(v_1 = 0)$.

Using $u_1 = -V_s$ and $u_2 = v_2 - V_s$, we have

$$\frac{-V_s}{v_2 - V_s} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2}$$

Downstream velocity in the lab frame:

$$v_2 = \frac{2(\mathcal{M}_1^2-1)}{(\gamma+1)\mathcal{M}_1^2}V_s$$

Note a typo in Equation (16.12) of Kwok's book.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2} = \frac{u_1}{u_2}$$
$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1}$$
$$\frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2+2\right]\left[2\gamma\mathcal{M}_1^2-(\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2}$$

monatomic gas: $\gamma = 5/3$

For a strong shock: $\mathcal{M}_1 \gg 1$

$$P_2 \approx \frac{2\gamma\mathcal{M}_1^2}{\gamma+1}P_1 \xrightarrow{P_1=c_1^2\frac{\rho_1}{\gamma}} \frac{2\gamma(u_1/c_1)^2}{\gamma+1} \quad c_1^2\frac{\rho_1}{\gamma}$$

$$T_2 \approx \frac{2(\gamma-1)\gamma}{(\gamma+1)^2}\mathcal{M}_1^2 T_1 = \frac{2(\gamma-1)\gamma}{(\gamma+1)^2}\left(\frac{u_1}{c_1}\right)^2 T_1$$

$$\frac{\rho_2}{\rho_1} \simeq \frac{\gamma+1}{\gamma-1}$$
$$\frac{u_2}{u_1} \simeq \frac{\gamma-1}{\gamma+1}$$
$$P_2 \simeq \frac{2}{\gamma+1}\rho_1 u_1^2$$
$$T_2 \simeq \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{m}{k}u_1^2$$

$$\frac{\rho_2}{\rho_1} \simeq 4$$
$$\frac{u_2}{u_1} \simeq \frac{1}{4}$$
$$P_2 \simeq \frac{3}{4}\rho_1 u_1^2$$
$$T_2 \simeq \frac{3}{16}\frac{m}{k}u_1^2$$

speed of the downstream in the laboratory frame:

$$v_2 \simeq \frac{2}{(\gamma+1)}V_s$$

$$v_2 \simeq \frac{3}{4}V_s$$

For an isothermal shock: $\gamma = 1$

$$\frac{\rho_2}{\rho_1} = \mathcal{M}_1^2 = \frac{u_1}{u_2}$$
$$P_2 = \mathcal{M}_1^2 P_1 = \rho_1 u_1^2 \quad \longrightarrow \quad u_1 u_2 = c_1^2, \quad c_2 = c_1$$
$$T_2 = T_1$$

speed of the downstream in the laboratory frame:

$$v_2 = \left(1 - \frac{1}{\mathcal{M}_1^2}\right)V_s$$

- Consider a strong shock
  - ***No matter how strong the shock is, the gas can only be compressed by a factor of at most 4***:

$$\frac{\rho_2}{\rho_1} \approx 4 \qquad \text{for } \gamma = 5/3$$

(monatomic gas)

$$P_2 \approx \frac{3}{4}\rho_1 u_1^2$$

$$T_2 \approx \frac{3}{16}\frac{m}{k}u_1^2$$

Note that the mean molecular mass (mass per particle) is

$$m = \frac{1.4 m_{\rm H}}{1.1} = 1.273 m_{\rm H} \quad \text{for neutral gas}$$

$$m = \frac{1.4 m_{\rm H}}{2.3} = 0.609 m_{\rm H} \quad \text{for ionized gas}$$

$n \simeq 2.3 n_{\rm H}$ for ionized gas, one electron from an ionized hydrogen two electrons from a doubly-ionized helium.

  - In the lab frame, $V_{\rm s}$ = shock velocity, v_1, v_2 = gas velocities in upstream and downstream, respectively.

$$u_1 = v_1 - V_s = -V_s \ (v_1 = 0)$$
$$u_2 = v_2 - V_s$$

  - Then, the post-shock velocity is

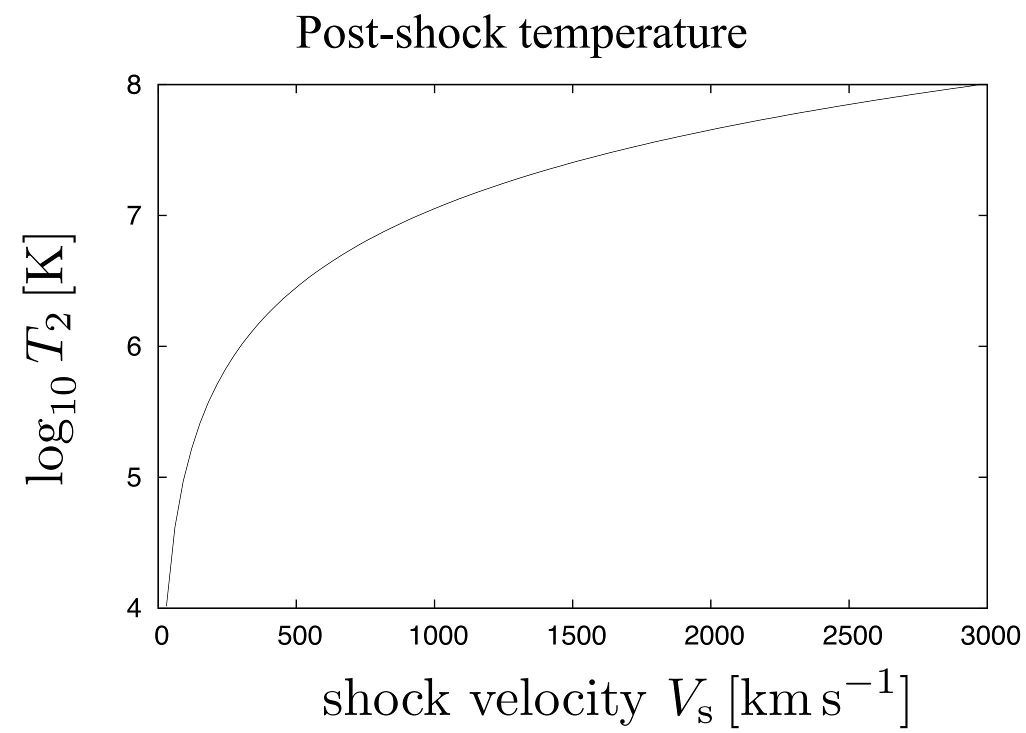
$$\frac{u_2}{u_1} = \frac{v_s - V_{\rm s}}{-V_{\rm s}} = \frac{1}{4} \quad \Rightarrow \quad \boxed{v_2 = \frac{3}{4}V_{\rm s}}$$

  - Hence, ***the post-shock moves in the same direction as the shock front with a velocity of 3/4 of the shock velocity.***

- Then, the post-shock pressure, temperature, specific internal energy, and specific kinetic energy are, respectively,

$$P_2 = \frac{3}{4}\rho_1 V_\mathrm{s}^2$$
$$T_2 = \frac{3m}{16k} V_\mathrm{s}^2$$

for $\gamma = 5/3$



$$\frac{\mathcal{E}_{\mathrm{int},2}}{\rho_2} = \epsilon_2 = \frac{3}{2}\frac{P_2}{\rho_2} = \frac{3}{2}\frac{(3/4)\rho_1 V_\mathrm{s}^2}{4\rho_1}$$
$$\frac{\mathcal{E}_{\mathrm{kin},2}}{\rho_2} = \frac{1}{2}v_2^2$$

$$\frac{\mathcal{E}_{\mathrm{int},2}}{\rho_2} = \frac{9}{32}V_\mathrm{s}^2$$
$$\frac{\mathcal{E}_{\mathrm{kin},2}}{\rho_2} = \frac{9}{32}V_\mathrm{s}^2$$

- A strong shock can produce very high pressures and temperatures. An interstellar shock front with propagation speed $V_s \sim 1000$ km s^{-1} (typical for a supernova shock wave) produces shock heated gas with

$$T_2 \approx 1.38 \times 10^7\,\mathrm{K}\left(\frac{m}{0.609 m_\mathrm{H}}\right)\left(\frac{V_s}{1000\,\mathrm{km\,s^{-1}}}\right)^2$$ or $$T_2 \approx 1.38 \times 10^5\,\mathrm{K}\left(\frac{m}{0.609 m_\mathrm{H}}\right)\left(\frac{V_s}{100\,\mathrm{km\,s^{-1}}}\right)^2$$

assuming the shocks gas is fully ionized hydrogen.

- In general, shock fronts convert supersonic gas into subsonic gas in the shock's frame of reference. Shocks increase density, pressure, and temperature, and decrease bulk velocity relative to the shock front. *Shocks act as entropy generators.*

see also https://www.ita.uni-heidelberg.de/~mattia/teaching/HD_shocks.pdf

- In a gaseous nebula,
  - The heating and cooling rates are order of $G \approx L \approx 10^{-24} n_e n_p \,\mathrm{erg\,cm^{-3}\,s^{-1}}$.

    For a typical nebula with $n_e \approx n_p \approx 10^3\,\mathrm{cm^{-3}}$, intermediate between bright PNe and bright H II regions, $G \approx L \approx 10^{-18}\,\mathrm{erg\,cm^{-3}\,s^{-1}}$.

    At the temperature of $T \approx 10{,}000\,\mathrm{K}$, the internal energy is $U = \dfrac{3}{2} n_e kT \approx 10^{-9}\,\mathrm{erg\,cm^{-3}}$.

    Typical time scales for heating and cooling by radiative processes are $U/G \approx 10^9\,\mathrm{s} \approx 30\,\mathrm{yr}$.
  - Typical velocities in nebulae are of order of a few times the sound speed, at most $30\,\mathrm{km\,s^{-1}} \approx 10^{-12}\,\mathrm{pc\,s^{-1}}$.

    Sized of nebulae are ~ 0.1 pc (PNs) and 10 pc (H II regions). Therefore, the time scale for appreciable expansion or motion is $\sim 10^{11}\,\mathrm{s}$ and $\sim 10^{13}\,\mathrm{s}$, considerably longer than $10^9\,\mathrm{s}$.
  - **The heating/cooling rates due to compression and dilation (expansion) are considerably smaller than the heating/cooling rates due to radiation.**

    Therefore, to a first approximation, the temperature in the nebula is fixed by radiative processes, independently of the hydrodynamic conditions, and a shock front in a nebula may be considered isothermal.

    Across the actual shock front, the temperature is higher behind the front than ahead of it.
  - **Isothermal Jump condition**: In the hot region immediately behind the front, the radiation rate is large and the gas is very rapidly cooled, so that relatively close behind the shock the gas is again at the equilibrium temperature, the same temperature as in the gas just ahead of the shock. (In general, the equilibrium temperature is independent of the density.) Then,

$$\gamma = 1 \quad \Rightarrow \quad \frac{P_0}{\rho_0} = \frac{P_1}{\rho_1} = \frac{kT}{\mu m_{\mathrm{H}}}$$

6.5 Ionization Fronts and Expanding H^+ Regions

- Ionization front - across which not only ρ, u, and P, but also the degree of ionization, change discontinuously.
  - This is a good approximation at the edge of an ionization-bounded region, because the ionization decreases very sharply in a distance of the order of the mean free path of an ionizing photon ($\sim 10^{-4}$ pc for the density $n_{\rm H} = 10^3\,{\rm cm}^{-3}$) ($\sigma_0^{\rm pi} = 6.3 \times 10^{-18}\,{\rm cm}^{-2}$)
  - The momentum and mass conservation conditions still apply.
  - However, the energy-conservation condition is different from that for a shock front, because energy is added to the gas crossing the ionization front.
  - Furthermore, the rate of flow of gas through the ionization front is fixed by the flux of ionizing photons arriving at the front, since each ionizing photon produces one electron-ion pair. Thus,

$$\rho_1 u_1 = \rho_2 u_2 = m_i \phi_i$$

where m_i is the mean mass of the ionized gas per newly created electron-ion pair, and $\phi_i = \phi({\rm H}^0)$ is the flux of ionizing photons:

$$\phi({\rm H}^0) = \frac{Q({\rm H}^0)}{4\pi r^2} = \int_{\nu_0}^{\infty} \frac{\pi F_\nu}{h\nu} d\nu$$

  - The speed of the ionization front is given by

$$u = \frac{Q({\rm H}^0)}{4\pi r^2 n} - \frac{\alpha_{\rm B} n r}{3} = \frac{\phi({\rm H}^0)}{n} - \frac{\alpha_{\rm B} n r}{3} \quad [{\rm cm\,s^{-1}}],$$

which can be integrated to find

$$r^3 = \frac{3Q({\rm H}^0)}{\alpha_{\rm B} n^2}\left[1 - \exp(-\alpha_{\rm B} t)\right] \quad [{\rm cm}^3]$$

- Let $q^2/2$ the excess kinetic energy per unit mass transferred to the gas in the ionization processes. Then,

$$\phi_i\left(\frac{1}{2}m_i q^2\right) = \int_{\nu_0}^{\infty} \frac{\pi F_\nu}{h\nu}(h\nu - h\nu_0)d\nu$$

- The energy conservation across the ionization front is expressed in the form:

$$\frac{1}{2}u_0^2 + \frac{5}{2}\frac{P_0}{\rho_0} + \frac{1}{2}q^2 = \frac{1}{2}u_1^2 + \frac{5}{2}\frac{P_1}{\rho_1}$$

- In an ionization front, the ionization conditions and hence the heating/cooling rates are quite different on the two sides of the front.

$$\frac{P_0}{\rho_0} = \frac{kT_0}{\mu_0 m_\mathrm{H}} \quad \text{and} \quad \frac{P_1}{\rho_1} = \frac{kT_1}{\mu_1 m_\mathrm{H}}$$

  - T_0 and T_1 are temperatures determined by the heating and cooling rates ahead of the shock and behind it, respectively. μ_0 and μ_1 are the corresponding mean molecular weights.

    Order of magnitude estimates are $T_0 \approx 100$ K, $T_1 \approx 10{,}000$ K, $\mu_0 \approx 1$, $\mu_1 \approx 1/2$. Therefore, the sound speeds are

$$\boxed{c_1 = \left(\frac{kT_1}{\mu_1 m_\mathrm{H}}\right)^{1/2} \approx 12.9\,\mathrm{km\,s^{-1}} \gg c_0 = \left(\frac{kT_0}{\mu_0 m_\mathrm{H}}\right)^{1/2} \approx 0.9\,\mathrm{km\,s^{-1}}}$$

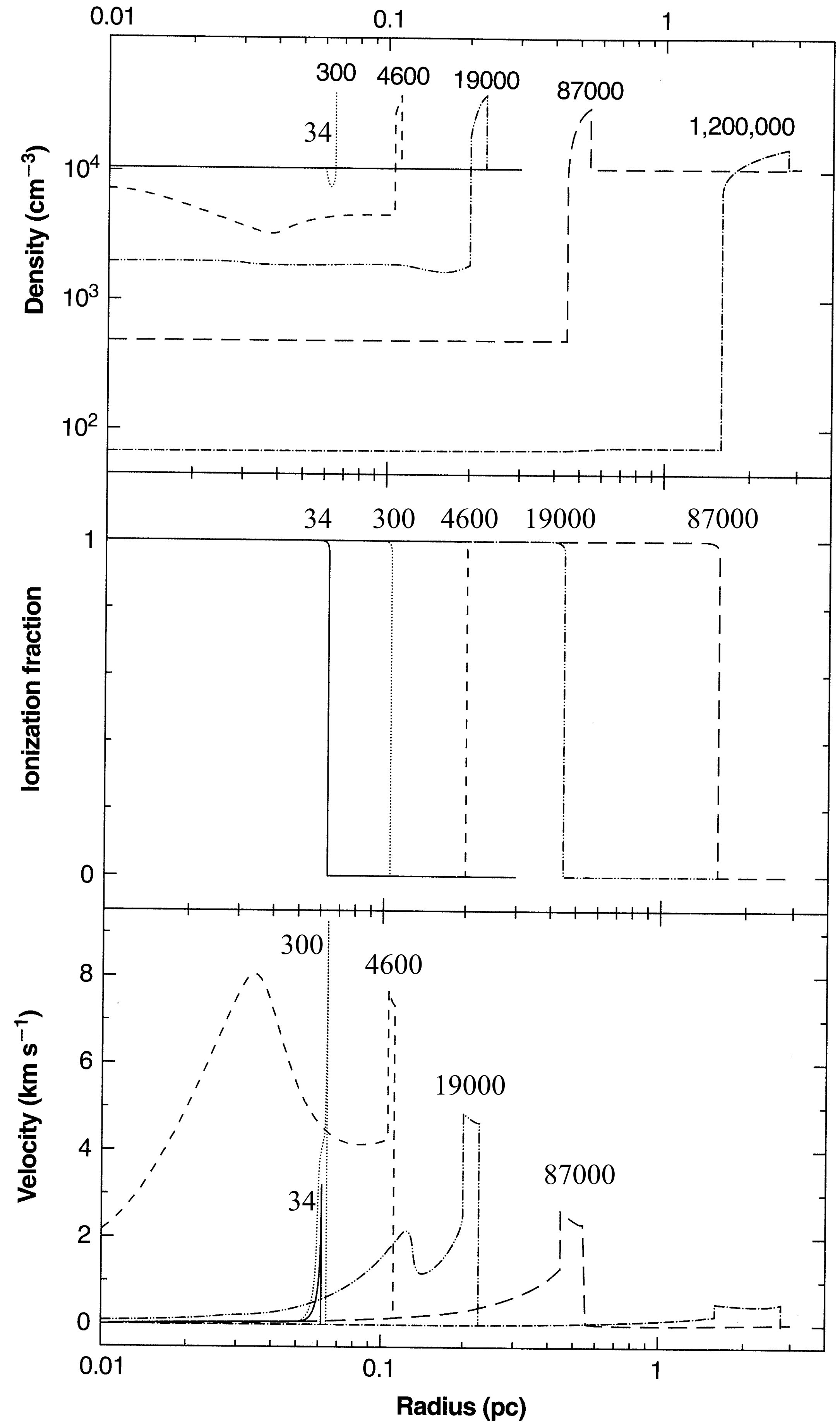
  - The thickness of this "isothermal shock front" is fixed by the radiation rate and is order of 10^{-3} pc.

- [Figure 6.1]

  Simplified model of expanding H II regions around an O star.

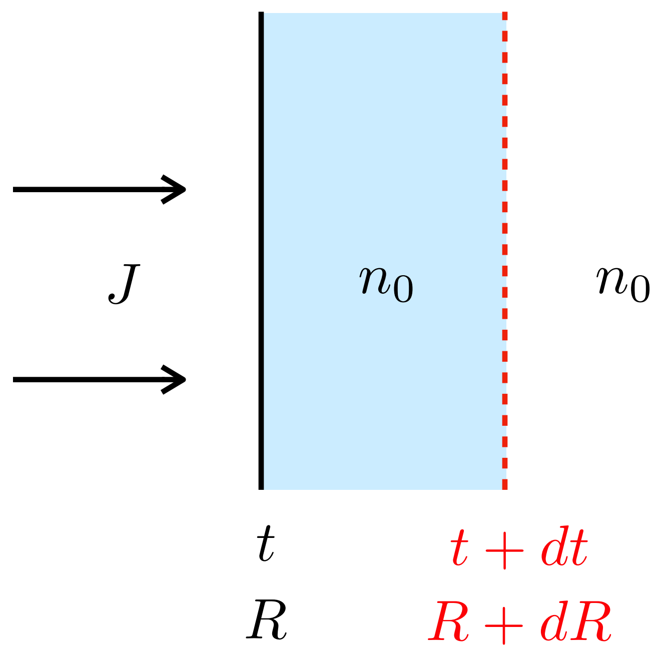
  This is the result of the numerical integration of the system of partial differential equations.

  - The time since the star turned on is indicated by the numbers in the upper panel.
  - The star emits 10^{48} hydrogen-ionizing photons per second and initially surrounded by a homogeneous atomic medium at a density of $10^4\ \mathrm{cm}^{-3}$. The atomic gas has a constant temperature of 100 K, and so the region has constant pressure.
  - The equilibrium radius of the Strömgren sphere is 2.1×10^{17} cm, which is reached only at very long time.



(Velocity of Ionization Front)

- Suppose that at time t the ionization front is located at a distance R from the star and at time $t + dt$ it is at a distance $R + dR$.



- Let n_0 = number density of the undisturbed neutral hydrogen

  J = number of Lyman continuum photons incident normally on unit area of the ionization front per unit time.

- ***Ionization balance at the ionization front***: While the ionization front moves from R to $R + dR$, the photons will ionize all the neutral atoms lying between these two positions $(R,\ R + dR)$.
- We assume that only one photon is needed to ionize each atom as the front moves the distance dR. In other words, no recombination occurs within the distance interval dR. For unit area of the ionization front, the following relation must be satisfied:

$$J \Delta A dt = n_0 \Delta A dR$$

- Then, the velocity of the ionization front (in a fixed frame of reference) is:

$$\frac{dR}{dt} = \frac{J}{n_0}$$

(The initial stage of evolution of an ionized region)

- Suppose that the UV source has been suddenly turned on.
  - ***Ionization balance for the ionized region***: We consider two factors:
    - *The radiation field at the ionization front is diluted because of the spherical geometry*.
    - Recombination takes place continuously inside the ionized region, and *some of the UV photons produced by the central source must go to reionize the atoms that have recombined*.

$$\boxed{Q_0 = \left(4\pi R^2\right) J + \left(\frac{4\pi}{3} R^3\right) \alpha_{\mathrm{B}} n_e n_p}$$

  - Inside the ionized sphere, the fractional ionization is near unity. Thus, $n_e = n_p = n_0$. Using this condition, we obtain an equation for the expansion velocity of the ionization front.

$$\frac{J}{n_0} = \frac{dR}{dt} = \frac{Q_0}{4\pi R^2 n_0} - \frac{1}{3} R n_0 \alpha_{\mathrm{B}}$$

    - Let's define the following dimensionless quantities:

$$\rho \equiv R/R_s \quad \text{where } R_s \equiv \left(\frac{3}{4\pi}\frac{Q_0}{\alpha_{\mathrm{B}} n_0^2}\right)^{1/3}$$

$$\tau \equiv t/t_{\mathrm{rec}} \quad \text{where } t_{\mathrm{rec}} \equiv \frac{1}{\alpha_{\mathrm{B}} n_0}$$

Then, the equation in dimensionless form is

$$\frac{d\rho}{d\tau} = \frac{1}{3}\left(\frac{1}{\rho^2} - \rho\right)$$

- The equation can be written:

$$\frac{d\rho}{d\tau} = \frac{1}{3}\left(\frac{1}{\rho^2} - \rho\right) \quad \rightarrow \quad \frac{d\rho^3}{d\tau} = 1 - \rho^3$$

- It's solution is

$$\rho^3 = 1 - e^{-\tau}$$

$$\boxed{R(t) = R_s\left(1 - e^{-t/t_{\rm rec}}\right)^{1/3}}$$

initial condition: $R(t=0)=0$

$$\frac{dx}{d\tau} + x = 1$$

$$e^{\tau}\frac{dx}{d\tau} + e^{\tau}x = e^{\tau}$$

$$\frac{d\left(e^{\tau}x\right)}{d\tau} = e^{\tau}$$

$$\rightarrow \quad e^{\tau}x = \int_0^{\tau} e^{\tau'}d\tau' = e^{\tau} - 1$$

$$x = 1 - e^{-\tau}$$

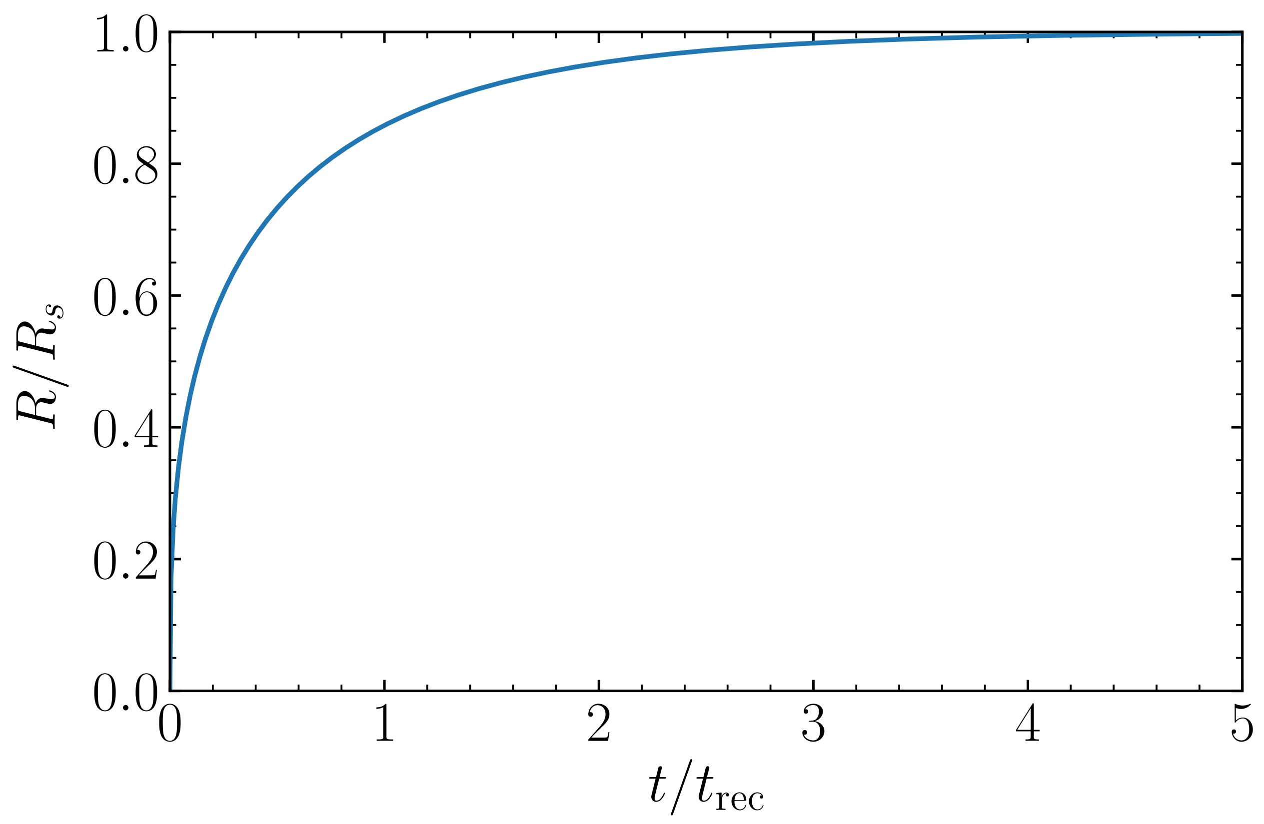
- Scale Parameters:
  - The time scale introduced is the recombination time scale:

$$t_{\rm rec} \equiv \frac{1}{\alpha_{\rm B} n_0} \approx 4000\,{\rm yr}\left(\frac{\alpha_{\rm B}}{2.6\times10^{-3}\,{\rm cm^3\,s^{-1}}}\right)^{-1}\left(\frac{n_0}{30\,{\rm cm^{-3}}}\right)^{-1}$$

the length scale introduced is the Strömgren radius:

$$R_s \equiv \left(\frac{3}{4\pi}\frac{Q_0}{\alpha_{\rm B} n_0^2}\right)^{1/3} \approx 7\,{\rm pc}\left(\frac{Q_0}{10^{49}\,{\rm s^{-1}}}\right)^{1/3}\left(\frac{\alpha_{\rm B}}{2.6\times10^{-3}\,{\rm cm^3\,s^{-1}}}\right)^{-1/3}\left(\frac{n_0}{30\,{\rm cm^{-3}}}\right)^{-2/3}$$

- Hence, **the time required to create a Strömgren sphere after turning on a hot star is an order of ~ 4000 yr**. This is also the time it takes the ionized Strömgren sphere to revert to neutral gas after the central UV source has been turned off.



- At times $t \gg t_{\rm rec} \sim 4000\,{\rm yr}$, the gas medium will be fully ionized with radius $R \sim R_s \sim 7\,{\rm pc}$, surrounded by a partially ionized boundary of thickness $\sim \lambda_{\rm mfp} = (n_{\rm H}\sigma_{\rm pi})^{-1} \sim 0.002\,{\rm pc} \ll R_s$.

- Jump conditions across an ionization front

$$\frac{\rho_1}{\rho_0} = \frac{c_0^2 + u_0^2 \pm \left[(c_0^2 + u_0^2)^2 - 4c_1^2 u_0^2\right]^{1/2}}{2c_1^2}$$

- Because ρ_1/ρ_0 must be real, there are two allowed ranges of speed of the ionization front:

$$\left(c_0^2 + u_0^2\right)^2 - 4c_1^2 u_0^2 = \left(c_0^2 + u_0^2 - 2c_1 u_1\right)\left(c_0^2 + u_0^2 + 2c_1 u_1\right) \geq 0$$

$$u_0^2 + c_0^2 - 2u_0 c_1 = \left[u_0 - \left(c_1 + \sqrt{c_1^2 - c_0^2}\right)\right]\left[u_0 - \left(c_1 - \sqrt{c_1^2 - c_0^2}\right)\right] \geq 0$$

$$\boxed{u_0 \geq c_1 + \sqrt{c_1^2 - c_0^2} \equiv u_R \approx 2c_1 \qquad \text{or} \qquad u_0 \leq c_1 - \sqrt{c_1^2 - c_0^2} \equiv u_D \approx \frac{c_0^2}{2c_1}}$$

where the approximations apply for $c_1 \gg c_0$ because, in H II regions, the neutral gas has a higher density than the ionized gas (or the same density at the initial stage).

- Here, R stands for "rare" or "low-density" gas, and D stands for "dense" or "high-density" gas.

R-type fronts have $u_0 \geq u_R \gg c_0$ and, therefore these fronts move supersonically into the undisturbed gas ahead of them.

D-type fronts have $u_0 \leq u_D < c_0$, so they move subsonically with respect to the gas ahead of them.

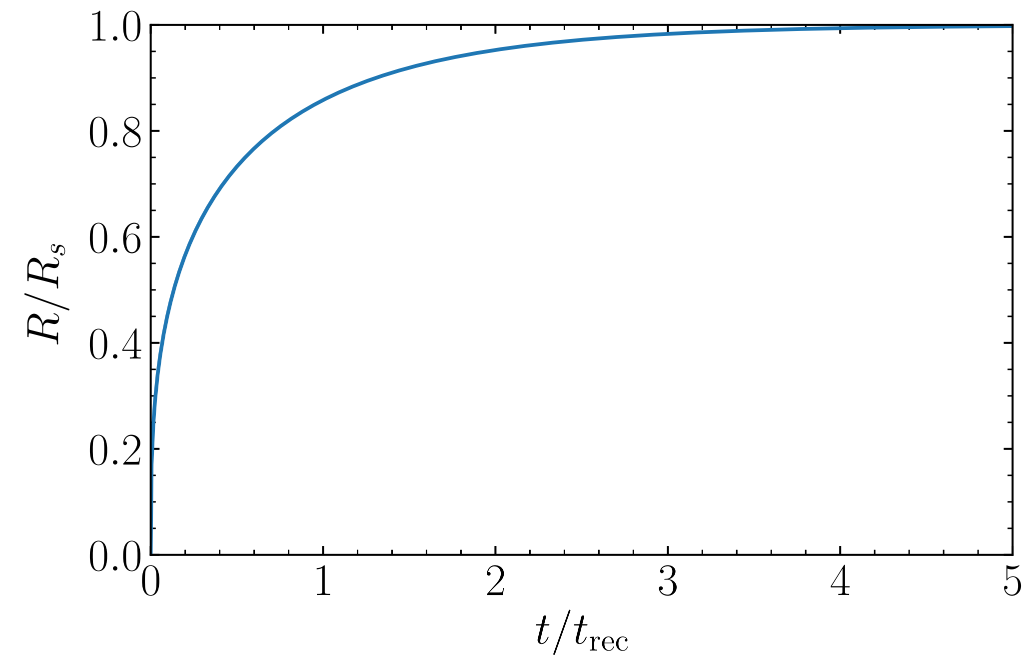
- We can compute the ***rate of expansion of the ionization front***:

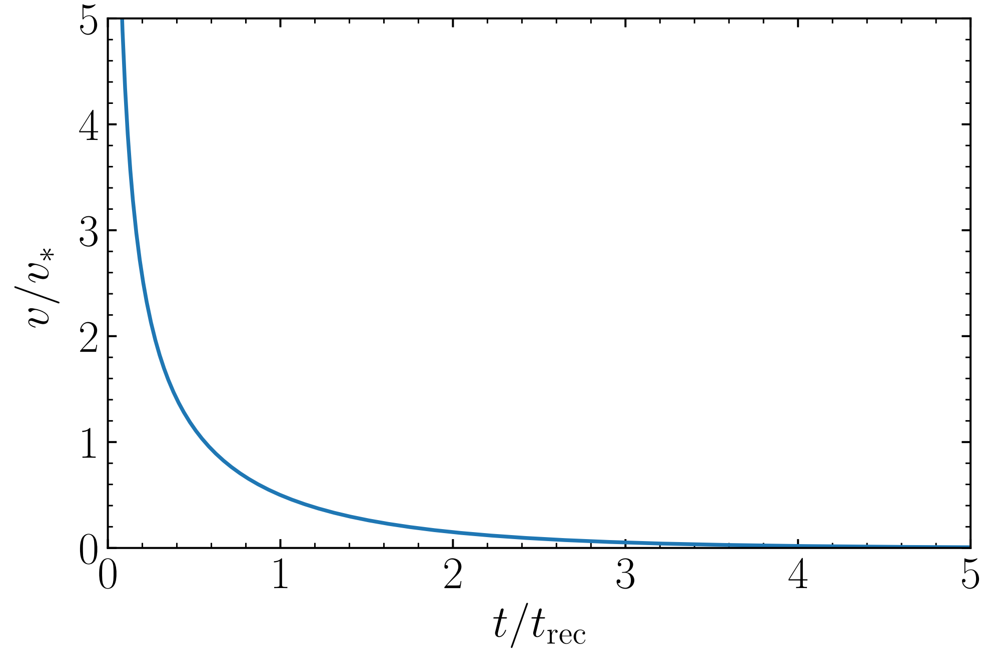
$$\frac{dR}{dt} = \frac{R_s}{3t_{\rm rec}} \frac{e^{-t/t_{\rm rec}}}{(1-e^{-t/t_{\rm rec}})^{2/3}}$$

where the characteristic expansion velocity is

$$v_* \equiv \frac{R_s}{3t_{\rm rec}}$$
$$\simeq 560\,{\rm km\,s^{-1}} \left(\frac{Q_0}{10^{49}\,{\rm s^{-1}}}\right)^{1/3} \left(\frac{n_0}{30\,{\rm cm^{-3}}}\right)^{1/3}$$

This is much larger than the sonic speed $c_s \approx 1\ {\rm km\,s^{-1}}$ in the neutral medium as well as $c_s \approx 10\ {\rm km\,s^{-1}}$ in the ionized medium.





- The expansion speed of the ionization front at two limits:

$$\frac{dR}{dt} \approx \frac{R_s}{3t_{\rm rec}} \left(\frac{t}{t_{\rm rec}}\right)^{-2/3} \qquad \text{for } t \ll t_{\rm rec}$$
$$\frac{dR}{dt} \approx \frac{R_s}{3t_{\rm rec}} e^{-t/t_{\rm rec}} \qquad \text{for } t \gg t_{\rm rec}$$

Note that the expansion speed diverges at t = 0.

- The ionization front will initially expand supersonically. When will the ionization front expand at subsonic speeds?

$$\frac{dR}{dt} = \frac{R_s}{3t_{\rm rec}} e^{-t/t_{\rm rec}} \lesssim c_{\rm i}$$

$c_{\rm i} \approx 13\,{\rm km\,s^{-1}}$ sound speed in the ionized medium

$$\boxed{t \lesssim t_{\rm sonic} \equiv t_{\rm rec} \ln\left(\frac{R_s}{3t_{\rm rec}}\frac{1}{c_{\rm i}}\right) \approx 3.8 t_{\rm rec} \simeq 15,000\,{\rm yr}}$$

- At this time, the ionization front will have a size of:

$$\boxed{R(t = t_{\rm sonic}) = R_s(1 - e^{-3.8})^{1/3} = 0.9925 R_s}$$

- The ionization front will expand at a supersonic velocity until $t \approx t_{\rm sonic}$ ($\sim$ 15,000 yr). By that time, the ionized sphere has reached a radius $R \sim 0.99 R_S$ and then it starts to expand at subsonic speed.

- ***At $t = R_S/c_s \sim 0.5\,{\rm Myr}$, the gas starts to flow outward as a result of the pressure gradient that has build up.***

(The intermediate stage of evolution of an ionized region)

- Before the pressure equilibrium is established, the gas density and temperature will be

$$n_{\rm i} \approx 2n_0 > n_{\rm f} \quad \text{and} \quad T_{\rm i} = 10^4\ {\rm K}$$

  - Then the isothermal sound speeds of the ionized gas and neutral gas are, respectively:

$$c_{\rm i}^2 = \frac{P_{\rm i}}{\rho_{\rm i}} \approx \frac{2n_0 k T_{\rm i}}{n_0 m_{\rm H}} \qquad c_{\rm n}^2 = \frac{P_{\rm n}}{\rho_{\rm n}} = \frac{n_0 k T_{\rm n}}{n_0 m_{\rm H}}$$

$$\frac{c_{\rm i}}{c_{\rm n}} = \left(\frac{n_{\rm i} T_{\rm i}}{n_0 T_{\rm n}}\right)^{1/2} \approx \sqrt{200} = 14.14$$

$$\boxed{P_{\rm i} = 2n_0 k T_i = 200 P_{\rm n}}$$

  - The sound speed of the ionized gas is much larger than that of the neutral gas.
  - The ionized gas has a higher pressure and thus plays the role of a piston and pushes a shock wave into the neutral gas.

    The expansion speed of the ionized gas is originally equal to about $c_{\rm i}$, which is highly supersonic with respect to the sound speed in the neutral gas.

  - Note also that, at $t \gtrsim t_{\rm sonic} \approx 3.8 t_{\rm rec}$, **the expansion speed ($c_{\rm i}$) of ionized gas is larger than that of the ionization front.**

$$\frac{dR}{dt} > c_{\rm i} \text{ at } t \lesssim t_{\rm sonic} \longrightarrow \frac{dR}{dt} \approx c_{\rm i} \text{ at } t \approx t_{\rm sonic} \longrightarrow \frac{dR}{dt} < c_{\rm i} \text{ at } t \gtrsim t_{\rm sonic}$$

initial stage — intermediate stage

(The final stage of evolution of an ionized region)

- Although the ionized sphere approaches ionization equilibrium at $t \gtrsim t_{rec}$, it would be still far from pressure equilibrium.
  - Outside the ionized zone, it will be embedded in the cold neutral medium with a temperature $T \sim 100$ K.
  - Inside the sphere, the heating and cooling processes yield a temperature of $T \sim 10{,}000$ K.
  - Also, the density of particles inside the ionized sphere will doubles when the hydrogen is ionized.
  - Thus, *the pressure inside the sphere will be ~ 200 times higher than the pressure outside, meaning that the ionized gas will begin to expand*.
  - The ionized gas expands as long as it has a higher pressure than its surroundings. This expansion produce a shock and will cease when the hot ionized gas reaches pressure equilibrium with the surrounding cold neutral gas.
- ***The condition of final pressure equilibrium*** can be written in the form:

$$\boxed{2n_{\rm f}kT_{\rm i} = n_0 kT_{\rm n}}$$

$n_{\rm f}$ = number density of the ionized hydrogen.

$T_{\rm i}$ and $T_{\rm n}$ = temperatures of the ionized and neutral gas, typically $T_{\rm i} = 10^4$ K, $T_{\rm n} = 10^2$ K.

- The ionized gas sphere must still absorb all the stellar UV photons. Thus,

$$Q_0 = \frac{4}{3}\pi R_\mathrm{f}^3 n_\mathrm{f}^2 \alpha_\mathrm{B}$$

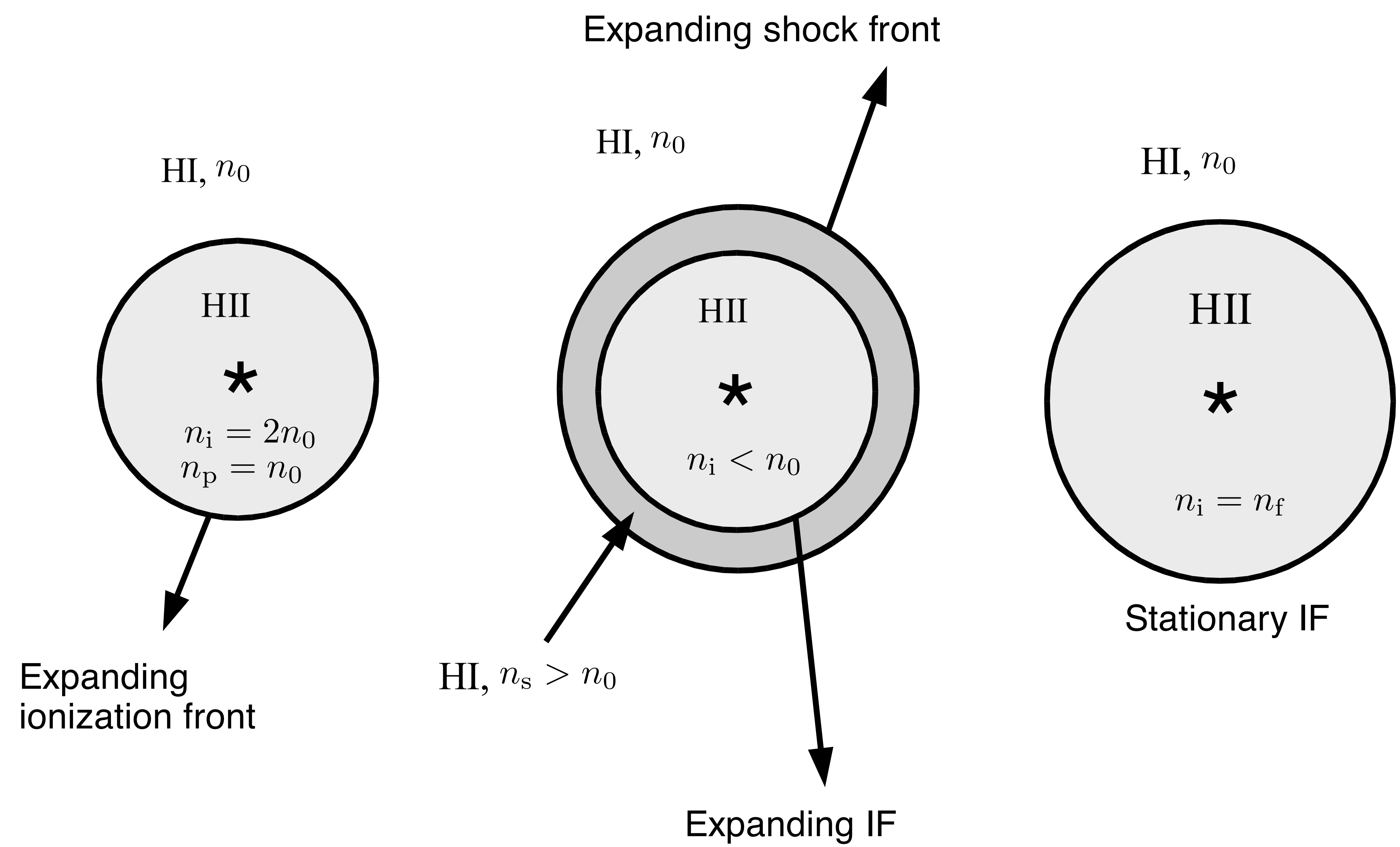
Here, R_f is the final radius of the ionized gas sphere. From the pressure equilibrium condition, we obtain the final size:

$$n_\mathrm{f} = (T_\mathrm{n}/2T_\mathrm{i})n_0 \approx 0.005 n_0 \quad \rightarrow \quad R_\mathrm{f} = (2T_\mathrm{i}/T_\mathrm{n})^{2/3} R_{s0} \approx 34 R_{s0}$$

- The ratio of the mass of gas finally ionized to that contained within the initial Strömgren sphere is:

$$\boxed{\frac{M_\mathrm{f}}{M_\mathrm{s}} = \frac{R_\mathrm{f}^3 n_\mathrm{f}}{R_{s0}^3 n_0} = \frac{2T_\mathrm{i}}{T_\mathrm{n}} \approx 200}$$

- This indicates that *the initial Strömgren sphere contains only a very small fraction of the material which, in principle, a star could ultimately ionize.*



Evolutionary scheme of an expanding H II region. (a) The initial stage, (b) expansion with a shock in the neutral gas, (c) the final equilibrium state.

[Figure 7.2 Dyson]

Note that The jump conditions for the ionization front are discussed in the ISM lecture.

6.6 Magnetic Fields

- Magnetic fields permeate the Galaxy and can affect the gas dynamics.
  - Ions tend to follow magnetic field lines, and the field lines can be compressed or expanded by electromagnetic effects.
- Two simple limits for interactions between an ionized gas and a magnetic field
  - These are determined by the ratio between the thermal energy density $E_{\rm th} = nkT$ and the magnetic energy density $E_{\rm B} = B^2/8\pi$.
  - $E_{\rm th} \gg E_{\rm B}$: the gas is in control and field lines follow the matter as it expands or contracts.
    - For a spherical expansion, the number of field lines per unit area at radius r is proportional to r^{-2} while the gas density is proportional to r^{-3}. Then, $B \propto n^{2/3}$.
  - $E_{\rm th} \ll E_{\rm B}$: the magnetic field is in control and matter will flow along field lines.
- In the presence of magnetic fields and ions,
  - the gas can couple with the field and create a magnetohydrodynamical (MHD) wave (e.g., Alfven waves), which transmits disturbances over appreciable lengths.
  - In many cases, these MHD waves move faster than the sound speed so that a “magnetic precursor” precedes the shock.
  - Gas can be compressed and accelerated by this precursor before the shock front arrives.
  - As a result, the effects of the shock can be spread over a greater distance and the shock jump will tend to be less sharp.

- Two types of magnetic shocks
  - J-type shock occurs when the fluid undergoes a discontinuous jump, as in the case with no magnetic field. This occurs when the shock is fast or B is small.
  - C-type shock is one where the magnetic precursor is strong enough to make all flow variables continuous. This occurs for slower shocks or strong B.
- Various forms of MHD waves can occur.
  - In may cases, the MHD waves are supersonic relative to the gas.
  - In a non-magnetized gas, supersonic turbulence is dissipated into heat quickly by the resulting shocks.
  - In a magnetized ionized gas, if the gas is coupled to the magnetic field, then supersonic MHD wave motions result in coherent gas motions that do not produce heat. Such waves are said to be non-dissipative and can persist for some time. The presence of spectral lines with supersonic broadening is one possible indication of the presence of a magnetic field.
- Equipartition
  - The ISM is in approximate energy equipartition between turbulent and magnetic energies.

$$\frac{B^2}{8\pi} \approx \frac{1}{2}\rho u^2 \approx 10^{-12}\,\mathrm{erg\,cm^{-3}} \quad \text{in the local ISM}$$

  - There are exceptions.
  - The details that establish energy equipartition are not well understood.

6.7 Stellar Winds

- Blue-shifted emission lines from stars
  - Many “early type” stars (O, B, and Wolf-Rayet), older, lower-mass PNe central stars, and some cooler stars (especially supergiants) show both emission and absorption lines.
  - They often show blue-shifted emission lines, relative to the photospheric velocity indicating that it arises in outflowing gas.
  - These stars are losing mass and the outflowing stellar wind affects the surrounding nebula.
- Radiation Pressure - Eddington Luminosity
  - Winds are mainly driven by the force of the outward flow of radiation past the stellar atmosphere.
  - Each photon carries a momentum $h\nu/c$. The total momentum passing a unit area at a distance r from the center of star of luminosity L is $p = L/4\pi r^2 c$.
  - If the gas is very highly ionized then electron scattering is the dominant opacity source. For photon energies smaller than the electron rest mass ($h\nu \ll m_e c^2 \approx 0.5\,\mathrm{MeV}$), the electron scattering cross section is given by the Thomson cross section:

$$\sigma_\mathrm{T} = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 0.67 \times 10^{-24}\ [\mathrm{cm}^2]$$

  - The total momentum an electron receives per unit time, due to scattering are coupled by strong electrostatic forces.
  - In static equilibrium, this outward force must be balanced by the inward pull of gravity on protons. This leads to the Eddington limit for stability against radiation pressure.

$$\frac{L_\mathrm{Edd}}{4\pi r^2 c}\sigma_\mathrm{T} = \frac{GMm_\mathrm{H}}{r^2} \quad \Rightarrow \quad L_\mathrm{Edd} = \frac{4\pi c GMm_\mathrm{H}}{\sigma_\mathrm{T}} \approx 3 \times 10^4 L_\odot \frac{M}{M_\odot}$$

    Stars more luminous than this must drive away their outer layers.
  - However, the gas is not highly ionized, typical photoionization opacities are $\sim 10^6$ larger than σ_T and line-center opacities are larger still. Therefore, theses opacity sources are more efficient at driving a wind and the above equation overestimates the limiting luminosity

- Structure of a radiatively driven stellar wind
  - Dynamical equations + equation of statistical and thermal equilibrium.
  - The calculations show that, as a rule of thumb, the terminal velocity of the wind is within a factor of a few times of the escape velocity from the star.

    The escape velocity is obtained by equating the kinetic energy in the wind with the potential energy at the surface.

$$\frac{1}{2}\rho u^2 = \frac{GM\rho}{r} \quad u \sim (2GM/r)^{1/2} \sim 10^3\,\mathrm{km\,s^{-1}} \text{ for many stars.}$$

- Observations
  - Stellar winds are studied by detailed analysis of the emission-absorption profiles (P-cygni) seen in the stellar lines.
  - Most lines are found in the vacuum UV so space observations are important.
  - Mass loss rate $\dot{m} = 4\pi r^2 \rho u$ can be derived from the density and velocity of the wind, which are determined from spectral analysis.

Table 6.1
Stellar mass loss rates

Star	Type	T_* (K)	$R/R_\odot$	$L_*/L_\odot$	$\dot{m}$ ($M_\odot$ /yr)	u (km s^{-1})
θ^1 Ori C	O7p V	4.5×10^4	8	2.5×10^5	4.0×10^{-7}	1000
9 Sgr	O4 V	4.6×10^4	16	1.0×10^6	5.0×10^{-6}	2750
ζ Pup	O4 I	4.2×10^4	19	1.0×10^6	5.0×10^{-6}	2485
NGC 6210	PN	9.0×10^4	0.29	5.0×10^3	2.2×10^{-9}	2180
NGC 6543	PN	6.0×10^4	0.70	5.7×10^3	4.0×10^{-8}	1900
NGC 7009	PN	8.8×10^4	0.45	1.1×10^4	2.8×10^{-9}	2770

The results assumes spherical symmetry.

In real stars rotation and magnetic fields complicate the issue, as they do in the sun.

- Shocks
  - The rapidly moving stellar wind overtakes the surrounding, slow-moving, nebular material, and creates a shock.
  - In the simplest case the outer regions of the stellar wind is in pressure equilibrium with the nebula. The density of the wind at the shock interface is

$$\rho_s = \frac{\dot{m}}{4\pi r^2 u}$$

  - From the jump conditions, the shock temperature is

$$T_s = \frac{\rho}{n}\frac{u^2}{9k} \approx 0.6\, m_{\mathrm{H}}\frac{u^2}{9k} = 8\times 10^6\left(\frac{u}{1000\,\mathrm{km/s}}\right)^2 \;[\mathrm{K}]$$

  where a solar composition and full ionization of H and He are assumed.

  - The resulting pressure is $P_s = \dfrac{\dot{m} m_{\mathrm{H}} u}{36\pi r^2}$.

$$P_s = nkT_s = nk\frac{\rho_s}{n}\frac{u^2}{9k} = \frac{\dot{m}u}{36\pi r^2}$$

These equations seem to be typos.

  - This pressure is often sufficient to create a swept-up shell of nebular material that creates an outward-moving shock wave, but one that is much slower and cooler than the shock at the outer edge of the wind.

Note that $T_s = \dfrac{\rho}{n}\dfrac{u^2}{9k}$ is obtained when $\gamma = 11/9$.

$$T_2 \approx \frac{3}{16}\frac{m}{k}V_s^2 \qquad v_2 = \frac{3}{4}V_s \quad \Rightarrow \quad T_2 \approx \frac{1}{3}\frac{m}{k}v_2^2$$

- Bubble of hot gas
  - The wind often creates a bubble of hot gas surrounding the star.
  - The temperature of the shock at the wind-nebula interface is so hot that it emits mainly X-ray, with little optical emission.

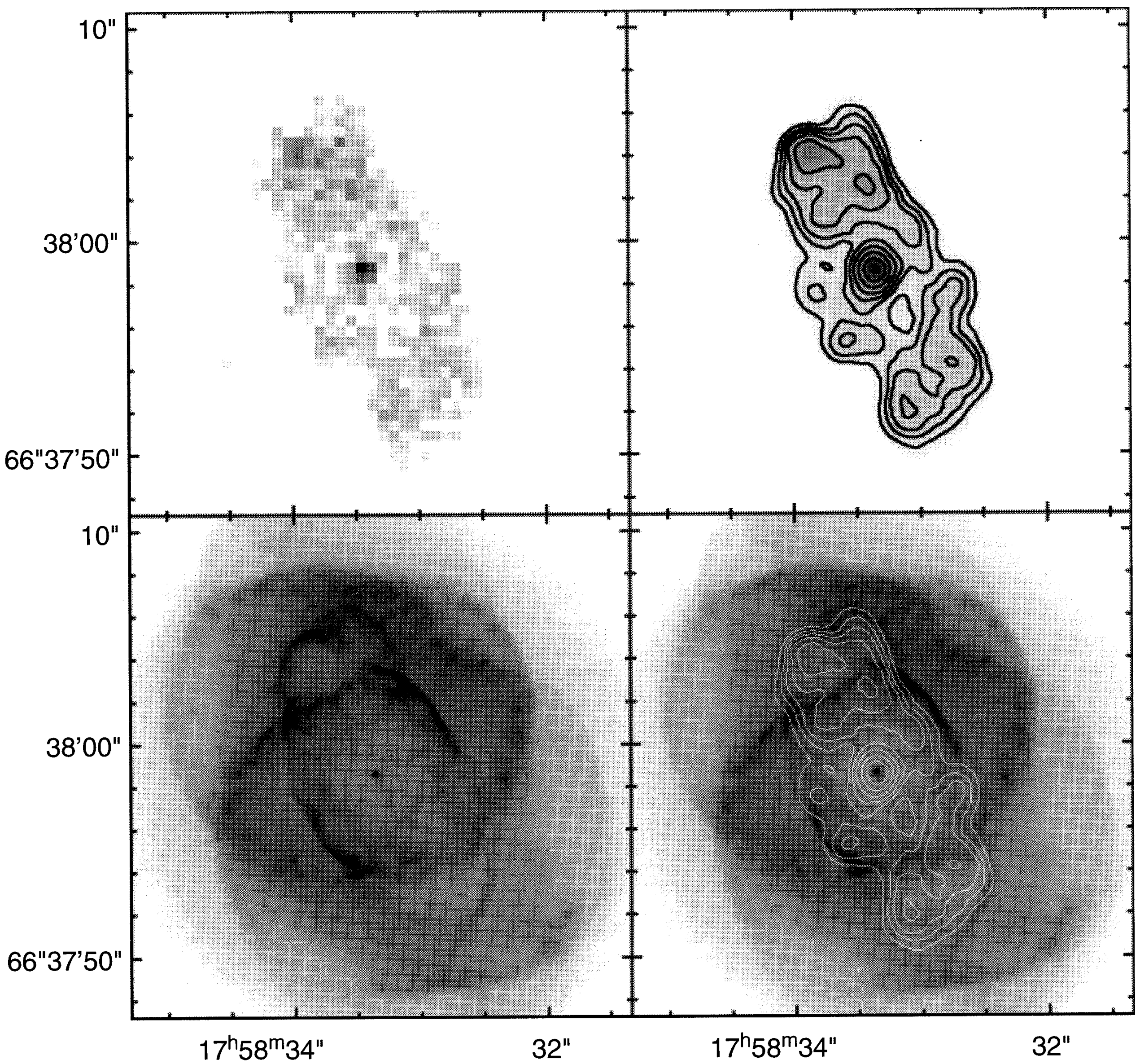


Figure 6.2
A comparison of X-ray and optical images of the planetary nebula NGC 6543. The upper Chandra images are in X-ray light and the lower HST images are in the light of Hα. The hot gas detected in the X-rays fills the central regions of the nebula.

The wind has created a central region filled with hot gas detected in X-rays.

The optical emission comes from surrounding regions of the nebula.

- The stellar wind can also interact with dense ionized flows that are embedded in the nebula.