Astrophysics

Lecture 02
September 14 (Wed.), 2022

updated 09/13, 08:38

Kwang-Il Seon
UST / KASI

The Radiative Transfer Equation

- The RT equation without the scattering term,

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

  - ***Absorption coefficient*** is defined as $\alpha_\nu \equiv n\sigma_\nu$ (units: cm^{-1}), meaning the ***total cross-sectional area per unit volume,*** where ρ is the mass density and κ_ν is called the **mass absorption coefficient** or the opacity coefficient.
  - Spontaneous "***emission coefficient" or "emissivity"*** j_ν is the amount of energy emitted per unit time, per unit solid angle, per unit frequency, and per unit volume.

- Optical depth:

$$d\tau_\nu = \alpha_\nu ds, \quad \tau_\nu = \int \alpha_\nu ds$$

- Source function

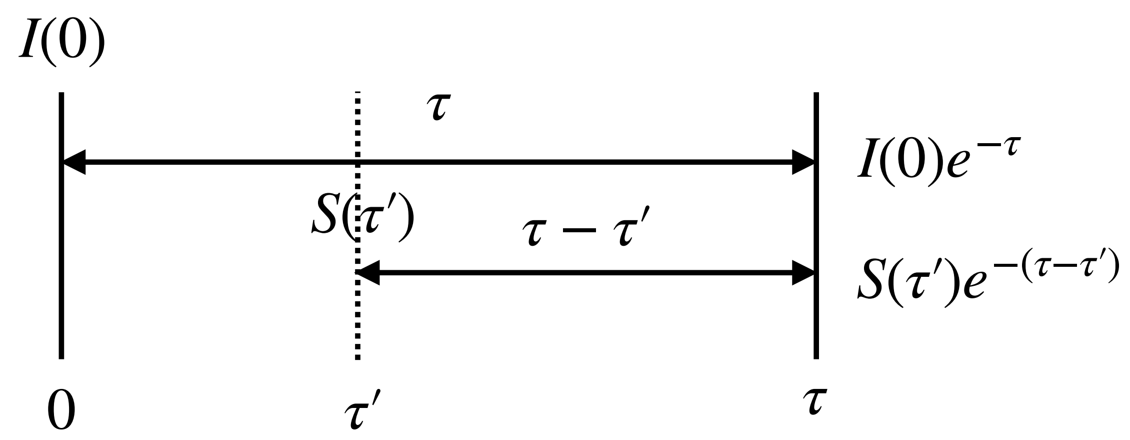
$$S_\nu = \frac{j_\nu}{\alpha_\nu} \quad \longrightarrow \quad \frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

Formal Solution

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$e^{\tau_\nu}\frac{I_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

$$\frac{d}{d\tau_\nu}\left(e^{\tau_\nu} I_\nu\right) = e^{\tau_\nu} S_\nu$$



$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu$$

- The solution is easily interpreted as the sum of two terms:
  - the initial intensity diminished by absorption
  - the integrated source diminished by absorption.

- For a constant source function, the solution becomes

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right) \\ &= S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right) \end{aligned}$$

Mean Free Path

- The ***probability of a photon absorbed*** between optical depths τ_ν and $\tau_\nu + d\tau_\nu$ is

$$P(\tau_\nu)d\tau_\nu = e^{-\tau_\nu}d\tau_\nu$$

  - The mean optical depth traveled is thus equal to unity:

$$\langle\tau_\nu\rangle = \int_0^\infty \tau_\nu P(\tau_\nu)d\tau_\nu = \int_0^\infty \tau_\nu e^{-\tau_\nu}d\tau_\nu = 1$$

  - ***The mean free path is defined as the average distance a photon can travel through an absorbing material until it is absorbed.*** In a homogeneous medium, the mean free path is determined by

$$\boxed{\langle\tau_\nu\rangle = \alpha_\nu \ell_\nu = 1 \quad \rightarrow \quad \ell_\nu = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}}$$

Kirchhoff’s Law in LTE

- TE and LTE
  - In a state of (complete) ***thermodynamic equilibrium (TE)***, no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system. ***In TE, matter and radiation are in equilibrium at the same temperature T.***
  - When the material is (locally) in thermodynamic equilibrium, and only the radiation field is allowed to depart from its TE, we refer to the state of the system as being in ***local thermodynamic equilibrium (LTE)***
  - ***The condition of the validity of the LTE is that the mean free path (of particle collisions) has to be small enough such that the temperature does not vary much over the mean free path. In general, the mean free path of photons is much smaller than particle collisions.***
  - ***In TE, the velocity distribution = Maxwell velocity distribution, Energy level distribution = Boltzmann distribution, ionization = Saha equation, Radiation spectrum = Planck function***
- In the local thermodynamic equilibrium at temperature T, ***the Kirchhoff's law for a system in LTE says that***

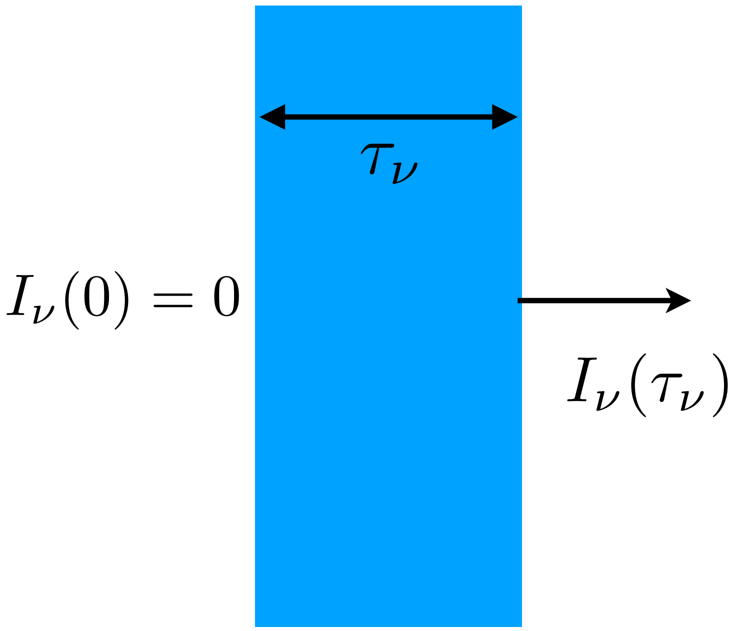
$$\frac{j_\nu}{\alpha_\nu} = B_\nu(T), \quad j_\nu = \alpha_\nu B_\nu(T)$$

- The RT equation in the LTE: $\dfrac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T)$

- ***Blackbody radiation vs. Thermal radiation***
  - ***Blackbody radiation*** means $I_\nu = B_\nu(T)$. An object for which the intensity is the Planck function is emitting blackbody radiation.
  - ***Thermal radiation is defined to be radiation emitted by "matter" in LTE***. Thermal radiation means $S_\nu = B_\nu(T)$. An object for which the source function is the Planck function is emitting thermal radiation.

- ***Thermal radiation becomes blackbody radiation only for optically thick media.***
  - If no light is falling on the back side of the slab, the intensity that is measured on the front side of the slab is

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right) \\ &= B_\nu\left(1 - e^{-\tau_\nu}\right) \end{aligned}$$



  - If the slab is optical thick at frequency $(\tau_\nu \gg 1)$, then

$$I_\nu \approx B_\nu$$

  - If the slab is optically thin $(\tau_\nu \ll 1)$, then

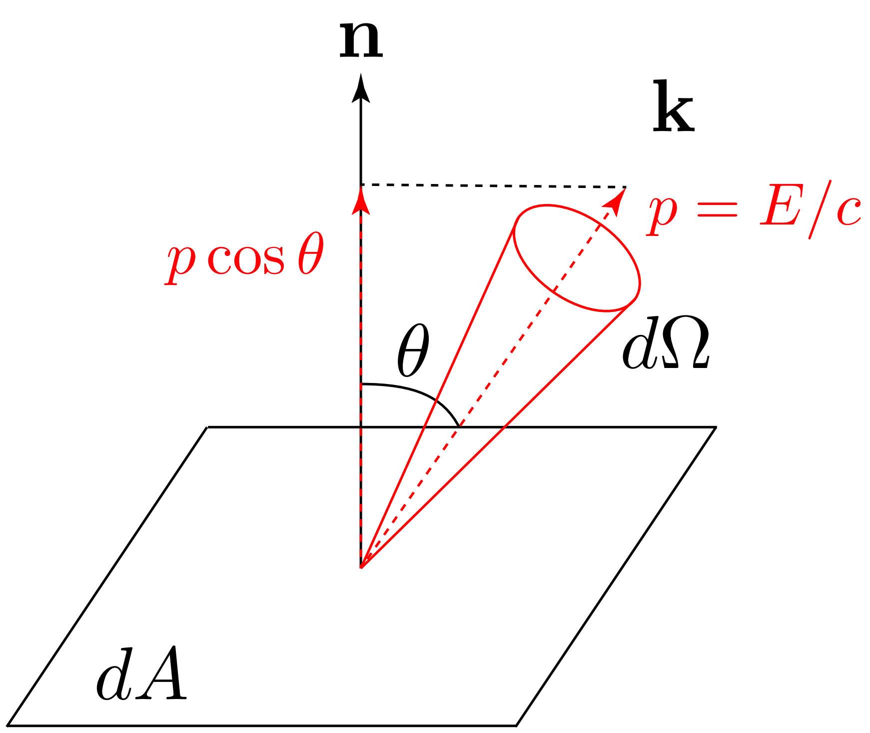
$$I_\nu \approx \tau_\nu B_\nu \ll B_\nu$$

  - This indicates that the radiation, although thermal, will not be blackbody.

Momentum Flux: Radiation Pressure

Radiation pressure due to energy flux propagating along direction $\mathbf{k}$, within solid angle $d\Omega$ and with frequency between $(\nu,\ \nu + d\nu)$, begin transported along $\mathbf{n}$:

- momentum of a photon: $p = E/c$
- force: $F = \dfrac{\Delta p}{\Delta t} = \dfrac{\Delta E}{c \Delta t}$
- radiation pressure = force per unit area



$$p_\nu d\Omega d\nu = \frac{\Delta F_\nu}{\Delta A} = \frac{1}{\Delta A}\frac{\Delta E_\nu / c}{\Delta t}\cos\theta$$
$$= \frac{1}{c} I_\nu \cos^2\theta d\Omega d\nu$$

$$\Longleftarrow \quad \Delta E_\nu = I_\nu \Delta A_k \Delta t \Delta\Omega$$
$$\Delta A_k = \Delta A \cos\theta$$

The first cosine factor is due to the area normal to $\mathbf{k}$ and the second one is due to the projection of the differential flux vector to the normal vector $\mathbf{n}$.

Integrating over solid angle,

$$P\nu = \frac{1}{c}\int I_\nu \cos^2\theta d\Omega = \frac{2\pi}{c}\int_{-1}^{1} I_\nu \mu^2 d\nu$$

$$\boxed{P\nu = \frac{4\pi}{3c} I_\nu = \frac{1}{3} u_\nu}$$

for isotropic radiation field $\left(I_\nu = J_\nu,\ u_\nu = \dfrac{4\pi}{c} J_\nu \right)$

Radiation comes from optically thin, outer parts

- Suppose that S_ν is constant:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)$$

- If $I_\nu(0) = 0$

$$I_\nu(\tau_\nu) = S_\nu\left(1 - e^{-\tau_\nu}\right) = \frac{j_\nu}{\alpha_\nu}\left(1 - e^{-\tau_\nu}\right)$$

- Now, if the dimension of the source is R,

$$I_\nu(\tau_\nu) = \frac{j_\nu R}{\alpha_\nu R}\left(1 - e^{-\tau_\nu}\right) = j_\nu R\left(\frac{1 - e^{-\tau_\nu}}{\tau_\nu}\right)$$

- When the source is optically thin, $I_\nu(\tau_\nu) = j_\nu R \quad (\tau_\nu \ll 1)$
- When the source is optically thick,

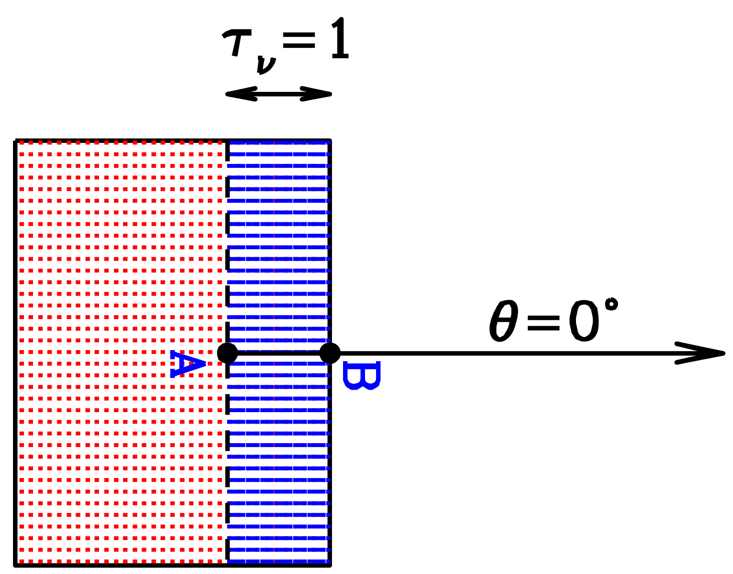
$$I_\nu(\tau_\nu) = \frac{j_\nu}{\alpha_\nu} = j_\nu \ell_{\mathrm{mfp}} \quad (\tau_\nu \gg 1)$$

$$\ell_{\mathrm{mfp}} = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}$$

$$\boxed{I_\nu(\tau_\nu) = j_\nu \cdot \min(R, \ell_{\mathrm{mfp}})}$$

Recall that the mean free path is the distance corresponding to $\tau = 1$.

- The above result indicates that the intensity we see from a thick source comes from a layer of width $R/\tau_\nu = \ell_{\rm mfp}$, i.e., the layer that is optically thin. In other words, we always collect radiation from a layer of the source, down to the depth at which the radiation can escape without being absorbed ($\tau_{\rm layer} = 1$).



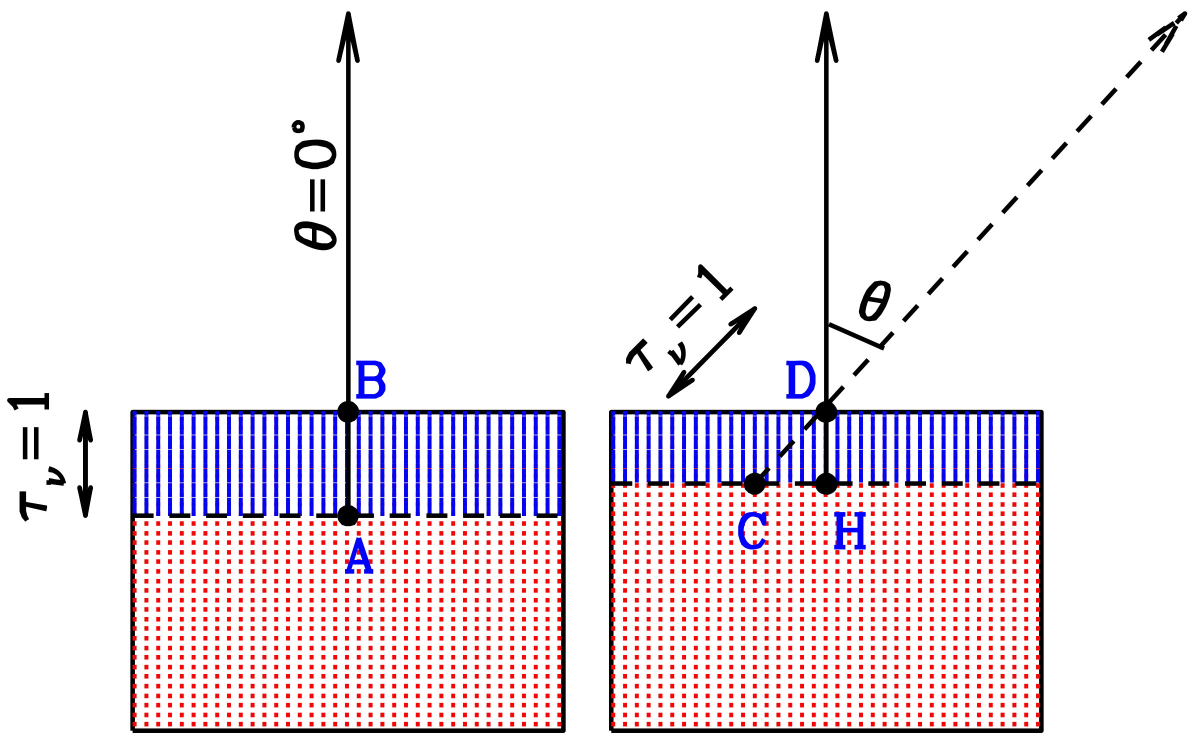
- We also note that the intensity emergent from the medium is related to the unattenuated intensity, which is expected from non-absorbing medium, as follows:

$$I_\nu(\tau_\nu) = j_\nu R\left(\frac{1-e^{-\tau_\nu}}{\tau_\nu}\right) \quad \Rightarrow \quad \boxed{I_\nu(\tau_\nu) = I_\nu^0\left(\frac{1-e^{-\tau_\nu}}{\tau_\nu}\right)}$$

I_ν = attenuated
I_ν^0 = unattenuated

This formula is often used when emission source is uniformly mixed with the absorbing material (e.g., dust).

The $\cos\theta$ Law



emitting volume:

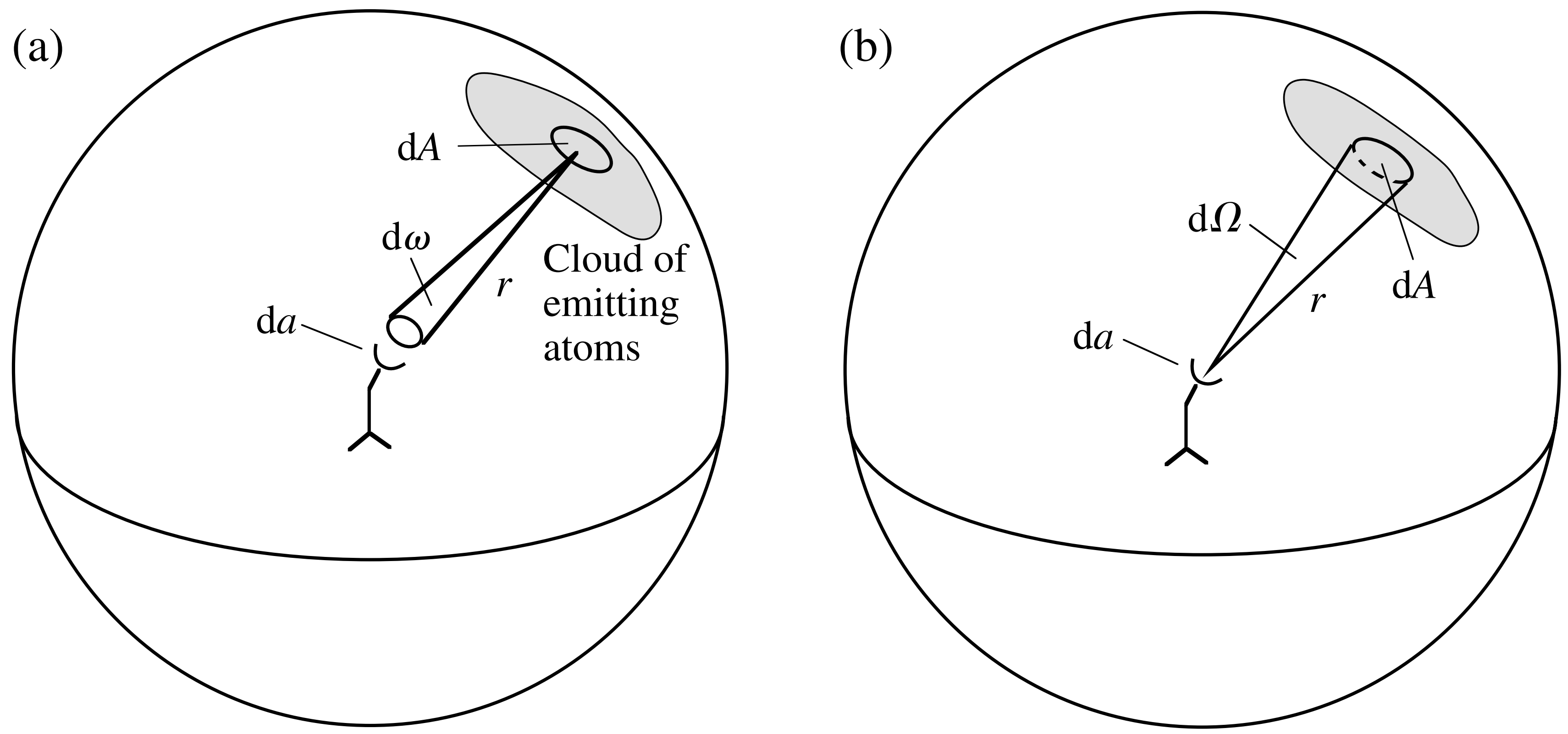
$$V_{\rm emiss} = \Delta A \times \Delta h$$
$$= \Delta A \times \ell_{\rm mfp} \cos\theta$$

- The observer looking at the layer face-on receives the radiation produced in a layer of optical depth $\tau_\nu \sim 1$. The optical depth depends on the viewing angle.
- For a face-on observer, the layer contributing to the emission is maximum, as it is the volume of the layer emitting the unabsorbed radiation.
- For different viewing angles, the radiation reaching the observer makes a path of the same length inside the layer (of unity optical depth), but inclined.
- Consequently, the emitting volume is less, by a $\cos\theta$ factor.

surface brightness = intensity

- The surface brightness describes the power coming from the apparent surface of objects such as the sun or the Crab nebula as a function of angle and frequency.

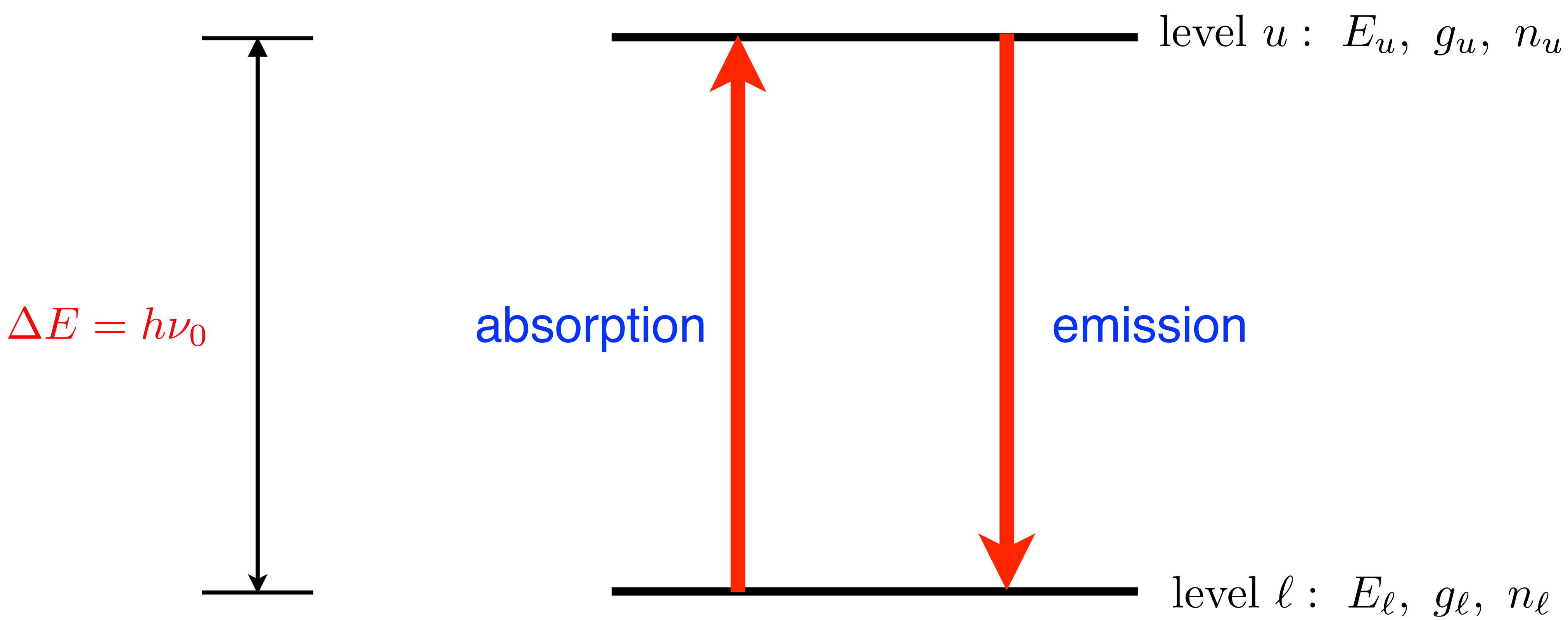
[Bradt, Astronomy Methods]



- (a) Radiation from an atom in the cloud is directed toward the telescope of area da which subtends a small solid angle $d\omega$.
- (b) The beam of the telescope, of solid angle $d\Omega$, views a segment dA of the cloud.

Two Level System: The Einstein Coefficients

- Consider a system with two discrete energy levels $(E_\ell,\ E_u)$ and degeneracies $(g_\ell,\ g_u)$. Let $(n_\ell,\ n_u)$ be the number densities of atoms in levels $(\ell,\ u)$.



Here, g is the statistical weight, i.e., degeneracy of the level.

Boltzmann distribution & Excitation Temperature

- Boltzmann distribution
  - a probability distribution that gives the probability that a system will be in a certain state as a function of that state's energy and the temperature of the system, when the system is in thermodynamic equilibrium.

$$p_i = \frac{N_i}{N} = \frac{g_i e^{-E_i/k_\mathrm{B}T}}{\sum_i g_i e^{-E_i/k_\mathrm{B}T}} \qquad \rightarrow \qquad \frac{N_j}{N_i} = \frac{g_j}{g_i} e^{-(E_j-E_i)/k_\mathrm{B}T}$$

$$\begin{aligned} g_i &= \text{degeneracy or statistical weight} \\ &= 2J+1 \text{ for a state with an angular momentum } J \end{aligned}$$

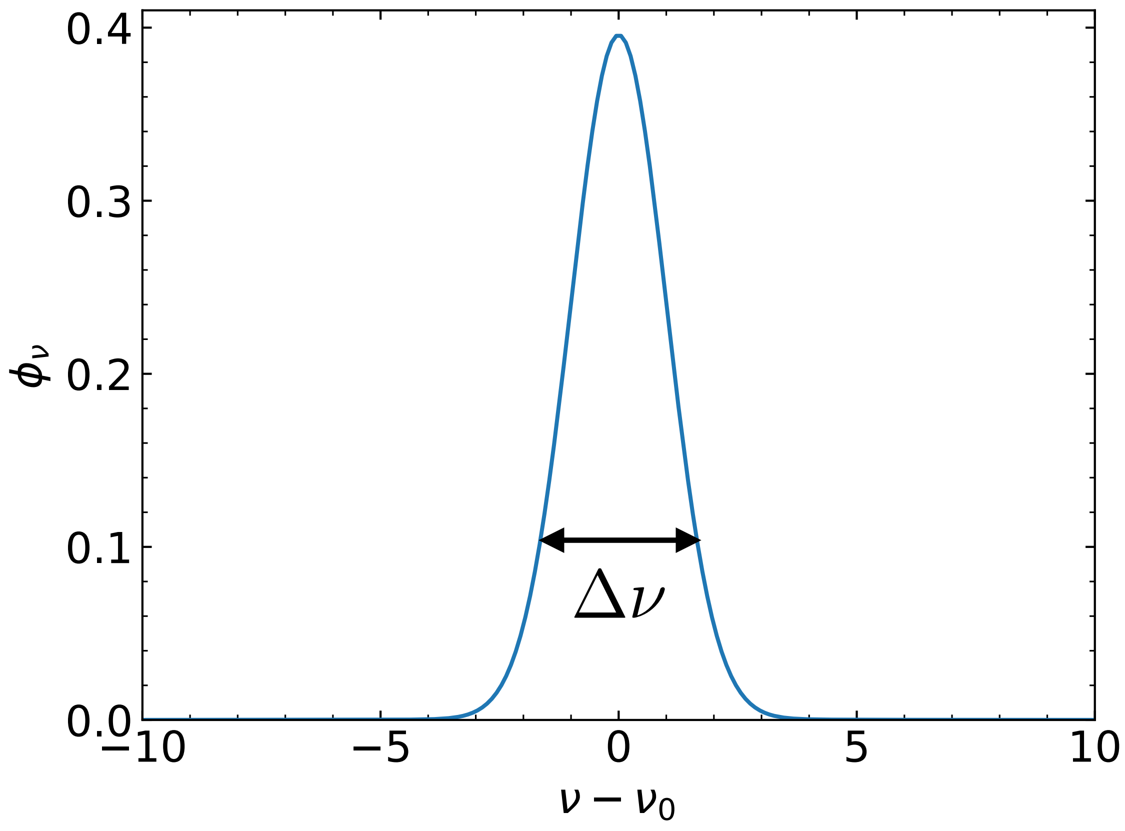
- Excitation temperature
  - Regardless of whether the system is in thermodynamic equilibrium (LTE) or not, the excitation temperature between two levels u and ℓ is defined as:

$$\frac{N_u}{N_\ell} = \frac{g_u}{g_\ell} e^{-(E_u-E_\ell)/k_\mathrm{B}T_\mathrm{exc}}$$

  - In general, the excitation temperatures between three levels, 1, 2, and 3, can all be different.

$$T_{12} \neq T_{13} \neq T_{23}$$

- The absorption probability per unit time is proportional to the mean intensity (or the energy density of photons) at frequency ν_0.
- In general, the energy difference between the two levels have finite width which can be described by a line profile function $\phi(\nu)$.



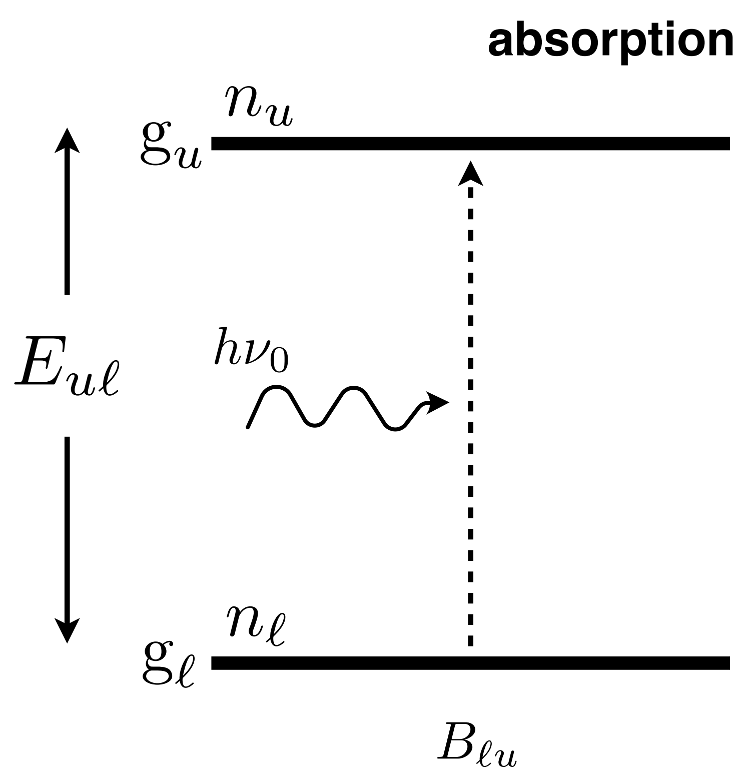
$$\int_0^\infty \phi_\nu d\nu = 1$$

The intensity is described by the mean over the line profile : $\bar{J} = \int J_\nu \phi_\nu d\nu$

Radiative Excitation and De-excitation (Absorption and Emission)

- **Three Radiative Transitions and Einstein Coefficients**
  - ***Absorption***:
    - If an absorber (atom, ion, molecule, or dust grain) X is in a lower level ℓ and there is radiation present with photons having an energy equal to $E_{u\ell}$. The absorber can absorb a photon and undergo an upward transition.

$$\text{absorption}: \quad X_\ell + h\nu_0 \;\rightarrow\; X_u \;(h\nu_0 = E_{u\ell})$$



    - The rate per volume at which the absorbers absorb photons will be proportional to both the (mean) intensity $\bar{J}$ of photons of the appropriate energy and the number density n_ℓ of absorbers in the lower level ℓ.

$$\left(\frac{dn_u}{dt}\right)_{\ell\rightarrow u} = -\left(\frac{dn_\ell}{dt}\right)_{\ell\rightarrow u} = n_\ell B_{\ell u}\bar{J}$$

    - The proportionality constant $B_{\ell u}$ is the ***Einstein B coefficient*** for the upward transition $\ell \rightarrow u$.

- ***Emission***:

  - An absorber X in an excited level u can decay to a lower level ℓ with emission of a photon. There are two ways this can happen:

$$\text{spontaneous emission}: \quad X_u \rightarrow X_\ell + h\nu_0 \quad (h\nu_0 = E_{u\ell})$$

$$\text{stimulated emission}: \quad X_u + h\nu_0 \rightarrow X_\ell + 2h\nu_0 \quad (h\nu_0 = E_{u\ell})$$

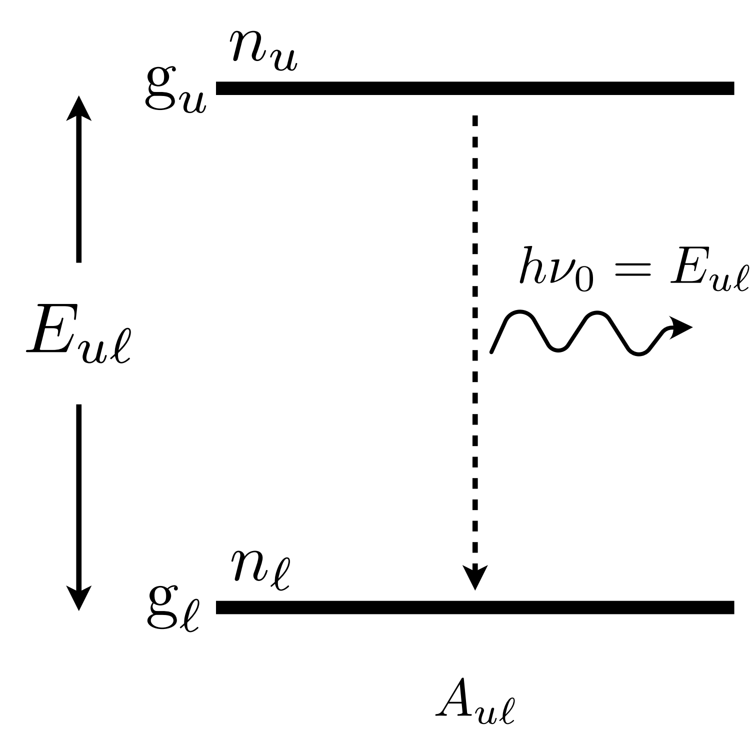
  - ***Spontaneous emission*** is a random process, independent of the presence of a radiation field.

  - ***Stimulated emission*** occurs if photons of the identical frequency, polarization, and direction of propagation are already present, and the rate of stimulated emission is proportional to the intensity $\bar{J}$ of these photons.

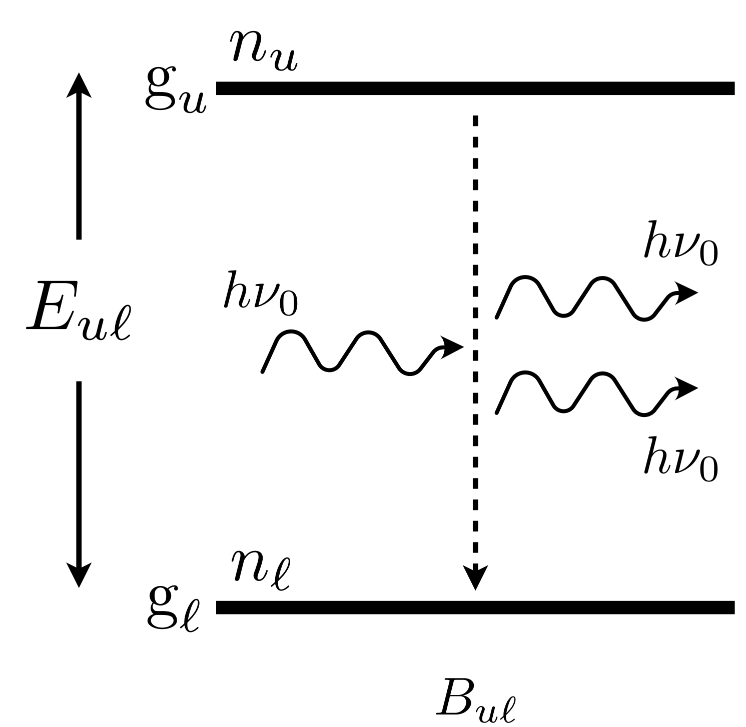
$$\left(\frac{dn_\ell}{dt}\right)_{u\rightarrow\ell} = -\left(\frac{dn_u}{dt}\right)_{u\rightarrow\ell} = n_u\left(A_{u\ell} + B_{u\ell}\bar{J}\right)$$

  - The probability per unit time $A_{u\ell}$ is the ***Einstein A coefficient*** for spontaneous transition. The coefficient $B_{u\ell}$ is the ***Einstein B coefficient*** for the downward transition $u \rightarrow \ell$.

spontaneous emission



stimulated emission



What is stimulated emission?

Stimulated emission from level u to level ℓ:

- The Einstein B-coefficient for stimulated emission is defined by

$B_{u\ell}\bar{J}$ = transition probability per unit time for stimulated emission.

- Einstein found that to derive Planck's law another process was required that was proportional to radiation field and caused emission of a photon.
- **The stimulated emission is precisely coherent (same direction and frequency, etc) with the photon that induced the emission.**

We also note that:

- **The energy density** is often used instead of intensity to define the Einstein B-coefficients, which leads to definitions differing by $c/4\pi$.

Relations between the Einstein coefficients

- The three Einstein coefficients are not mutually independent.
- ***In thermal equilibrium***, the radiation field becomes the “blackbody” radiation field and the two levels must be populated according to the Boltzmann distribution.

$$(J_{\nu_0})_{\rm TE} = B_{\nu_0}(T) = \frac{2h\nu_0^3}{c^2}\frac{1}{e^{h\nu_0/k_{\rm B}T}-1}$$

$$\left(\frac{n_u}{n_\ell}\right)_{\rm TE} = \frac{g_u}{g_\ell}e^{-E_{u\ell}/k_{\rm B}T} \quad \text{Here, } E_{u\ell} = h\nu_0.$$

- The net rate of change of level u should be equal to zero, in TE.

$$\frac{dn_u}{dt} = \left(\frac{dn_u}{dt}\right)_{\ell\to u} + \left(\frac{dn_u}{dt}\right)_{u\to\ell}$$

$$\begin{aligned} &= n_\ell B_{\ell u}\bar{J} - n_u\left(A_{u\ell} + B_{u\ell}\bar{J}\right) \\ &= 0 \end{aligned}$$

$$n_\ell B_{\ell u}\bar{J} - n_u\left(A_{u\ell} + B_{u\ell}\bar{J}\right) = 0$$

$$\left(n_\ell B_{\ell u} - n_u B_{u\ell}\right)\bar{J} = n_u A_{u\ell}$$

$$\begin{aligned}
\bar{J} &= \frac{n_u A_{u\ell}}{n_\ell B_{\ell u} - n_u B_{u\ell}} \\
&= \frac{\left(n_u A_{u\ell}\right)/\left(n_\ell B_{\ell u}\right)}{1 - \left(n_u B_{u\ell}\right)/\left(n_\ell B_{\ell u}\right)} \\
&= \frac{\left(\mathrm{g}_u/\mathrm{g}_\ell\right) e^{-h\nu_0/kT}\left(A_{u\ell}/B_{\ell u}\right)}{1 - \left(\mathrm{g}_u/\mathrm{g}_\ell\right) e^{-h\nu_0/kT}\left(B_{u\ell}/B_{\ell u}\right)} \\
&= \frac{\left(\mathrm{g}_u/\mathrm{g}_\ell\right)\left(A_{u\ell}/B_{\ell u}\right)}{e^{h\nu_0/kT} - \left(\mathrm{g}_u/\mathrm{g}_\ell\right)\left(B_{u\ell}/B_{\ell u}\right)}
\end{aligned}$$

Boltzmann distribution

$$\leftarrow \quad \frac{n_u}{n_\ell} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} e^{-h\nu_0/kT_{\mathrm{exc}}}$$

[Note] If there is no stimulated emission $(B_{u\ell} = 0)$, the only way to make the left eq. consistent with the Planck function is to assume $h\nu/kT \gg 1$ (Wien's regime). Therefore, the stimulated emission is negligible in the Wien's regime. In other words, the stimulated emission term is required in the Rayleigh-Jean regime.

Comparing the above eq. with Planck function,

$$J_{\nu_0} = \frac{2h\nu_0^3}{c^2}\frac{1}{e^{h\nu_0/kT} - 1}$$

we can immediately recognize that the following relations should be satisfied.

$$\left(\mathrm{g}_u/\mathrm{g}_\ell\right)\left(A_{u\ell}/B_{\ell u}\right) = \frac{2h\nu_0^3}{c^2}$$

$$\left(\mathrm{g}_u/\mathrm{g}_\ell\right)\left(B_{u\ell}/B_{\ell u}\right) = 1$$

In summary, we obtained the following relations between the Einstein coefficients.

$$A_{u\ell} = \frac{2h\nu^3}{c^2} B_{u\ell} \qquad B_{u\ell} = \frac{c^2}{2h\nu^3} A_{u\ell}$$

$$B_{\ell u} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} B_{u\ell} \qquad B_{\ell u} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell}\frac{c^2}{2h\nu^3} A_{u\ell}$$

- Einstein relations:

$$\boxed{\begin{aligned} g_\ell B_{\ell u} &= g_u B_{u\ell} \\ A_{u\ell} &= \frac{2h\nu_0^3}{c^2} B_{u\ell} \end{aligned}}$$

  - If we can determine any one of the coefficients, these relations allow us to determine the other two.
  - These connect atomic properties $(A_{u\ell}, B_{u\ell}, B_{\ell u})$ and have no reference to the temperature. Thus, the relations must hold whether or not the atoms are in TE.
    - ✦ If the relations were only for TE, the relations would contain the dependence on T.
  - Without stimulated emission, Einstein could not get Planck's law, but only Wien's law.
    - ✦ When $h\nu \gg k_{\rm B}T$ (Wien's limit), level 2 is very sparsely populated relative to level 1. Then, stimulated emission is unimportant compared to absorption.

- We note the Einstein coefficients are intrinsic properties of the absorbing material, irrelevant to the assumption of TE. Hence, ***the relations between the Einstein coefficients should hold in any condition.***
- Using the relation, we can rewrite the downward and upward transition rates:

$$\left(\frac{dn_u}{dt}\right)_{\ell\to u} = n_\ell \frac{g_u}{g_\ell} \frac{c^2}{2h\nu^3} A_{u\ell} J_\nu \qquad \left(\frac{dn_\ell}{dt}\right)_{u\to\ell} = n_u A_{u\ell}\left(1+\frac{c^2}{2h\nu^3}J_\nu\right)$$

- Sometimes, it is helpful to use a dimensionless quantity, the ***photon occupation number***:

$$n_\gamma \equiv \frac{c^2}{2h\nu^3} I_\nu \quad \xrightarrow{\text{averaging over directions}} \quad \langle n_\gamma \rangle = \frac{c^2}{2h\nu^3} J_\nu = \frac{1}{e^{h\nu/k_B T}-1}$$

- Then, the above transition rates are simplified:

$$\left(\frac{dn_u}{dt}\right)_{\ell\to u} = n_\ell \frac{g_u}{g_\ell} A_{u\ell}\langle n_\gamma \rangle \qquad \left(\frac{dn_\ell}{dt}\right)_{u\to\ell} = n_u A_{u\ell}\left(1+\langle n_\gamma\rangle\right)$$

- The photon occupation number determines the relative importance of stimulated and spontaneous emission: stimulated emission is important only when $< n_\gamma > \gg 1$.

Absorption and Emission Coefficients in terms of Einstein coefficients

- The Einstein coefficients are useful means of analyzing absorption and emission processes. However, we often find it even more useful to use cross section because the cross section has a natural geometric meaning.
- ***(pure) Absorption cross section:***
  - The number density of photons per unit frequency interval is $u_\nu/h\nu = (4\pi/c)J_\nu/h\nu$. Let $\sigma_\nu = \sigma_0\phi_\nu$ be the cross section for absorption of photons for the transition $\ell \rightarrow u$. Then, the absorption rate is

$\sigma_\nu = \sigma_0\phi_\nu$

$$\left(\frac{dn_u}{dt}\right)_{\ell\rightarrow u} = n_\ell \int d\nu \sigma_\nu \frac{4\pi J_\nu}{h\nu} \approx n_\ell \frac{4\pi}{h\nu_0}\sigma_0 \int d\nu J_\nu \phi_\nu = n_\ell \frac{4\pi}{h\nu_0}\sigma_0 \bar{J}$$

  - Here, we assumed that J_ν do not vary appreciably over the line profile of the cross section. Therefore, we derive a simple relation between the absorption cross section and the Einstein B coefficient:

$$\frac{4\pi}{h\nu_0}\sigma_0 = B_{\ell u} \quad \rightarrow \quad \sigma_0 = \frac{h\nu_0}{4\pi}B_{\ell u}$$

  - If the cross section has a normalized profile of ϕ_ν, we can write the absorption cross section as follows:

$$\sigma_\nu = \frac{h\nu_0}{4\pi}B_{\ell u}\phi_\nu \quad \text{with} \quad \int \phi_\nu d\nu = 1$$

- ***(effective) Absorption Coefficient***
  - We note that the stimulated emission is proportional to the intensity of ambient radiation field. In the radiative transfer equation, it is convenient to include the stimulated emission term in the absorption coefficient as a negative absorption.

$$\left(\frac{dn_u}{dt}\right)_{\ell\to u} - \left(\frac{dn_\ell}{dt}\right)_{u\to\ell}^{\rm stimulated} = n_\ell B_{\ell u}\bar{J} - n_u B_{u\ell}\bar{J}$$
$$= n_\ell B_{\ell u}\bar{J} - n_u\left(\frac{\mathrm{g}_\ell}{\mathrm{g}_u}B_{\ell u}\right)\bar{J}$$

  - Therefore, we may define the cross section for stimulated emission and the net (effective) absorption coefficient as follows:

$$\sigma_{u\ell} = \frac{\mathrm{g}_\ell}{\mathrm{g}_u}\sigma_{\ell u}$$

$$\kappa_\nu = \underbrace{n_\ell\sigma_{\ell u}}_{\text{pure absorption coefficient}} - n_u\sigma_{u\ell}$$
$$= n_\ell\sigma_{\ell u}\left(1 - \frac{n_u/n_\ell}{\mathrm{g}_u/\mathrm{g}_\ell}\right)$$

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell}e^{-h\nu/k_{\rm B}T_{\rm exc}}$$

  - Using the definition of the excitation temperature, we can rewrite them:

$$\kappa_\nu^{\rm eff} = n_\ell\sigma_{\ell u}\left[1 - \exp\left(-\frac{h\nu_{u\ell}}{k_{\rm B}T_{\rm exc}}\right)\right] \quad \text{or} \quad \sigma_\nu^{\rm eff} = \sigma_{\ell u}\left[1 - \exp\left(-\frac{h\nu_{u\ell}}{k_{\rm B}T_{\rm exc}}\right)\right]$$

- ***Emission coefficient (Emissivity)***
  - The emissivity is defined as the power radiated per unit frequency per unit solid angle per unit volume.
  - The line emissivity can be expressed in terms of the spontaneous downward transition rate:

$$4\pi \int d\nu j_\nu = h\nu_0 \left(\frac{dn_\ell}{dt}\right)_{u\to\ell}^{\text{spontaneous}}$$

  - Comparing with the definition of the Einstein A coefficient, we obtain:

$$\int d\nu j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_0$$

  - If the emission line has a normalized profile of ϕ_ν, we can write the emissivity as follows:

$$j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_0 \phi_\nu \qquad \text{with} \quad \int d\nu \phi_\nu = 1$$

- Source function:

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_u A_{u\ell}}{n_\ell B_{\ell u} - n_u B_{u\ell}}$$

- Using the Einstein relations, the absorption coefficient and source function can be written

$$\alpha_\nu = \frac{h\nu}{4\pi} n_\ell B_{\ell u} \left(1 - \frac{g_\ell n_u}{g_u n_\ell}\right)$$

$$\boxed{S_\nu = \frac{2h\nu^3}{c^2} \left(\frac{n_\ell / n_u}{g_\ell / g_u} - 1\right)^{-1} \quad \rightarrow \quad \text{generalized Kirchhoff's law}}$$

Thermal Emission (LTE)

- If the matter is in TE with itself (but not necessarily with the radiation), we have the Boltzmann distribution. The matter is said to be in LTE.

$$\frac{n_\ell}{n_u} = \frac{g_\ell}{g_2} \exp\left(h\nu/k_\mathrm{B}T\right)$$

- In LTE, we obtain the absorption coefficient and the Kirchhoff's law:

$$\alpha_\nu = \frac{h\nu}{4\pi} n_\ell B_{\ell u} \left[1 - \exp\left(-\frac{h\nu}{k_\mathrm{B}T}\right)\right] \phi_\nu$$

$$\boxed{S_\nu = B_\nu(T) \ \rightarrow \ \text{Kirchhoff's law in LTE}}$$

- The Kirchhoff's law holds even in LTE condition.

Normal & Inverted Populations

- Normal populations:
  - In LTE,

$$\frac{n_u g_\ell}{n_\ell g_u} = \exp\left(-\frac{h\nu}{k_\mathrm{B}T}\right) < 1 \quad \rightarrow \quad \frac{n_\ell}{g_\ell} > \frac{n_u}{g_u}$$

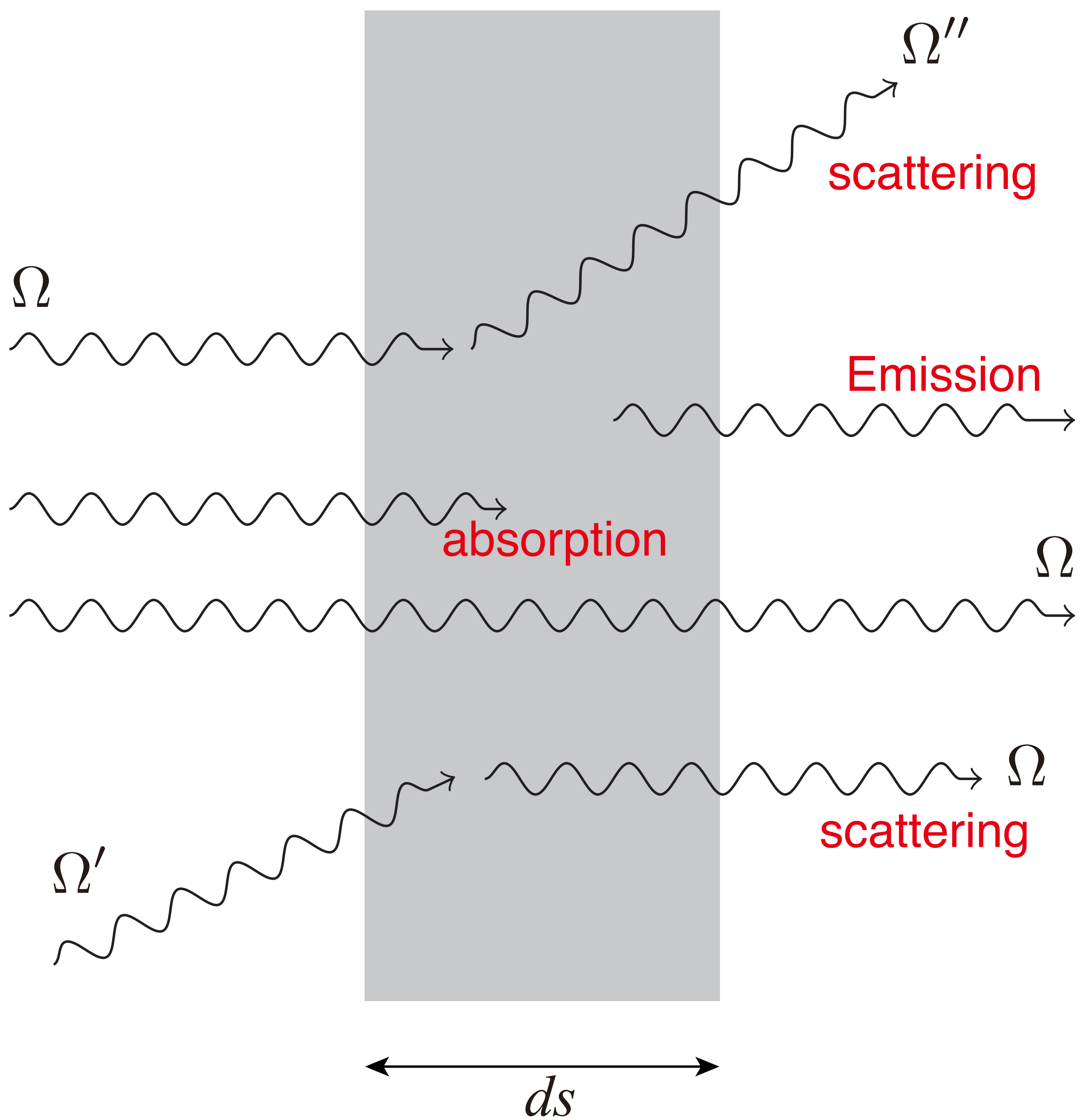
  - The normal populations is usually satisfied even when the material is out of thermal equilibrium.
- Inverted populations:

$$\frac{n_\ell}{g_\ell} < \frac{n_u}{g_u}$$

  - In this case, the absorption coefficient is negative and the intensity increases along a ray.
  - Such a system is said to be a **maser** (**m**icrowave **a**mplification by **s**timulated **e**mission of **r**adiation; also **laser** for **l**ight...).
  - The amplification can be very large. A negative optical depth of -100 leads to an amplification by a factor of $e^{100} = 10^{43}$.

RT equation including the Scattering Effect

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\mathrm{ext}} I_\nu + j_\nu + \underbrace{\alpha_\nu^{\mathrm{scatt}} \int \Phi_\nu(\Omega' \to \Omega) I_\nu(\Omega') d\Omega'}_{\Rightarrow\ j_\nu^{\mathrm{scatt}}}$$



- extinction cross section

$$\sigma_\nu^{\mathrm{ext}} = \sigma_\nu^{\mathrm{abs}} + \sigma_\nu^{\mathrm{scatt}}$$

- extinction coefficient

$$\alpha_\nu^{\mathrm{ext}} = \alpha_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{scatt}}$$
$$= n\sigma_\nu^{\mathrm{abs}} + n\sigma_\nu^{\mathrm{scatt}}$$

- scattering phase function

$$\Phi_\nu(\Omega' \to \Omega)$$

$$\int \Phi_\nu(\Omega' \to \Omega) d\Omega = 1$$

Scattering Effects: Pure Scattering

- Assumptions
  - ***isotropic scattering:*** scattered equally into equal solid angles
  - ***coherent scattering*** (elastic or monochromatic scattering): the total amount of radiation scattered per unit frequency is equal to the total amount absorbed in the same frequency range.
  - Thompson scattering (scattering from non-relativistic electrons) is nearly coherent.
- **scattering coefficient**

$$j_\nu^{\rm sca} = \alpha_\nu^{\rm sca} \int \Phi_\nu(\Omega' \rightarrow \Omega) I_\nu(\Omega') d\Omega$$

$$= \alpha_\nu^{\rm sca} \frac{1}{4\pi} \int I_\nu d\Omega' = \alpha_\nu^{\rm sca} J_\nu$$

In the textbook,
the scattering coefficient is denoted by σ_ν.

- **source function**

$$S_\nu = J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$$

- **radiative transfer equation**

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\rm sca} (I_\nu - J_\nu)$$

  - This is an integro-differential equation, and cannot be solved by the formal solution.
  - $\rightarrow$ Random walks, Rosseland approximation, or Eddington approximation

Random Walks in an "infinite" medium

- Random walks: let's consider a single photon rather than a beam of photons (i.e., ray).
- <u>In an infinite, homogeneous medium</u>, net displacement of the photon after *N* free paths is zero, because the average displacement, being a vector, must be zero.

$$\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \cdots + \mathbf{r}_N \quad \rightarrow \quad \langle \mathbf{R} \rangle = 0$$

- root mean square net displacement:

$$\begin{aligned} l_*^2 \equiv \langle \mathbf{R}^2 \rangle &= \langle \mathbf{r}_1^2 \rangle + \langle \mathbf{r}_2^2 \rangle + \langle \mathbf{r}_3^2 \rangle + \cdots + \langle \mathbf{r}_N^2 \rangle \\ &\quad + 2 \langle \mathbf{r}_1 \cdot \mathbf{r}_2 \rangle + 2 \langle \mathbf{r}_1 \cdot \mathbf{r}_3 \rangle + \cdots \\ &\approx N l^2 \ \leftarrow \ \text{Note} \ \ \langle \mathbf{r}_i^2 \rangle \approx l^2, \quad \langle \mathbf{r}_i \cdot \mathbf{r}_j \rangle = 0 \ \ (i \neq j) \end{aligned}$$

$$\boxed{\therefore \ l_* = \sqrt{N} l}$$

$$\langle \mathbf{r}_i \cdot \mathbf{r}_j \rangle = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \mu d\mu d\phi = 0$$

- The cross terms involve averaging the cosine of the angle between the directions before and after scattering, and this vanishes for isotropic scattering and for any scattering with front-back symmetry (Thompson or Rayleigh scattering).

Random Walks in “finite” medium

- In a finite medium, a photon generated somewhere within the medium will scatter until it escapes completely.
- For regions of large optical depth, the mean number of scatterings to escape is roughly determined by $l_* \approx L$ (displacement = the typical size of the medium).

$$l_* = \sqrt{N} l \approx L \ \rightarrow \ N \approx L^2/l^2 = L^2 (n\sigma_\nu^{\mathrm{sca}})^2$$
$$\therefore \ N \approx \tau^2 \quad (\tau \gg 1)$$

- For regions of small optical depth, the probability of scatterings within τ is

$$\int_0^\tau P(\tau')d\tau' = 1 - e^{-\tau} \approx \tau \qquad \therefore \ N \approx \tau \qquad (\tau \ll 1)$$

- In summary, for any optical thickness, the mean number of scatterings is

$$N \approx \tau^2 + \tau \quad \text{or} \quad N \approx \max(\tau, \ \tau^2)$$

Combined Scattering and Absorption

- The RT equation including the scattering and absorption:

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\mathrm{ext}} I_\nu + j_\nu + \alpha_\nu^{\mathrm{scatt}} \int \Phi_\nu(\Omega' \to \Omega) I_\nu(\Omega') d\Omega'$$

- Assuming ***the LTE condition and an isotropic scattering***, we obtain

$$j_\nu = \alpha_\nu^{\mathrm{abs}} B_\nu$$

$$\begin{aligned} j_\nu^{\mathrm{sca}} &= \alpha_\nu^{\mathrm{sca}} \int \Phi_\nu(\Omega' \to \Omega) I_\nu(\Omega') d\Omega \\ &= \alpha_\nu^{\mathrm{sca}} J_\nu \end{aligned} \qquad \Rightarrow \qquad \frac{j_\nu^{\mathrm{sca}}}{\alpha_\nu^{\mathrm{sca}}} = J_\nu$$

  - This indicates that ***the mean intensity plays a role of source function for scattering***.

- Then, the RT equation becomes

$$\frac{dI_\nu}{ds} = -(\alpha_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{sca}}) I_\nu + \alpha_\nu^{\mathrm{abs}} B_\nu + \alpha_\nu^{\mathrm{sca}} J_\nu$$

- The RT equation to the case of combined absorption and scattering.

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\rm abs}(I_\nu - B_\nu) - \alpha_\nu^{\rm sca}(I_\nu - J_\nu)$$
$$= -(\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca})(I_\nu - S_\nu) = -\alpha_\nu^{\rm ext}(I_\nu - S_\nu)$$
$$\text{where} \quad S_\nu \equiv \frac{\alpha_\nu^{\rm abs} B_\nu + \alpha_\nu^{\rm sca} J_\nu}{\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}} \quad \text{and} \quad \alpha_\nu^{\rm ext} \equiv \alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}$$

- ***The source function is a weighted average of the two source functions (for absorption and scattering).***
- extinction coefficient: $\alpha_\nu^{\rm ext} \equiv \alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}$
- optical depth: $d\tau_\nu \equiv \alpha_\nu^{\rm ext} ds$
- If a matter element is deep inside a medium (i.e., in TE),

$$J_\nu = B_\nu \;\rightarrow\; S_\nu = B_\nu$$

- If the element is isolate in free space, $J_\nu = 0 \;\rightarrow\; S_\nu = \alpha_\nu^{\rm abs} B_\nu / \alpha_\nu^{\rm ext}$

- generalized mean free path:

$$l_\nu = (\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca})^{-1}$$

- probability of a (random walk) step ending in absorption:

$$\epsilon_\nu = \alpha_\nu^{\rm abs}/\alpha_\nu^{\rm ext}$$

- probability for scattering (known as the ***single-scattering albedo***)

$$a_\nu = 1 - \epsilon_\nu = \alpha_\nu^{\rm sca}/\alpha_\nu^{\rm ext}$$

- source function:

$$\begin{aligned} S_\nu &= \epsilon_\nu B_\nu + (1 - \epsilon_\nu) J_\nu \\ &= (1 - a_\nu) B_\nu + a_\nu J_\nu \end{aligned}$$

Random Walks with Scattering and Absorption

- In an infinite medium, every photon is eventually absorbed.
  - Since a random walk can be terminated with probability $\epsilon = \alpha^{\rm abs}/\alpha^{\rm ext}$ at the end of each free path, the mean number of free paths is given by

mean number of free paths x probability of termination = 1

$$N\epsilon = 1 \;\rightarrow\; N = 1/\epsilon$$

  - diffusion length (thermalization length, effective mean path, or effective free path): a measure of the net displacement between the points of creation and destruction of a typical photon.

$$\begin{aligned} l_* &\approx \sqrt{N} l = l/\sqrt{\epsilon} \\ &\approx (\alpha_\nu^{\rm ext})^{-1}\sqrt{\alpha_\nu^{\rm ext}/\alpha_\nu^{\rm abs}} \\ &\approx (\alpha_\nu^{\rm abs}\alpha_\nu^{\rm ext})^{-1/2} \end{aligned}$$

- In a finite medium:
  - The behavior depends on whether its size L is larger or smaller than the effective free path ℓ_*.
  - Effective optical thickness: $\tau_* = L/l_* \approx \sqrt{\tau_{\rm abs}(\tau_{\rm abs} + \tau_{\rm sca})} = \sqrt{\tau_{\rm abs}\tau_{\rm ext}}$

$$\text{where } \tau_{\rm abs} \equiv \alpha_\nu^{\rm abs} L, \;\; \tau_{\rm sca} \equiv \alpha_\nu^{\rm sca} L, \;\; \tau_{\rm ext} \equiv \alpha_\nu^{\rm ext}$$

    - If effectively thin or translucent $(\tau_* \ll 1, L \ll \ell_*)$, most photons will escape the medium before being destroyed.
  - Luminosity of thermal source with volume V is

$$L_\nu = 4\pi j_\nu V = 4\pi \alpha_\nu B_\nu V \;\; (\tau_* \ll 1)$$

    - If effectively thick, we expect $I_\nu \rightarrow B_\nu$, $S_\nu \rightarrow B_\nu$, and only the photons emitted within an effective path length of the boundary will have a reasonable chance of escaping before being absorbed.

$$L_\nu = \pi \alpha_\nu^{\rm abs} B_\nu A l_* = \pi\sqrt{\epsilon_\nu} B_\nu A \qquad (F = \pi B \text{ at surface of the source})$$

Approximate Solutions

- How to solve the radiative transfer equation:

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\mathrm{ext}} \left(I_\nu - S_\nu\right)$$

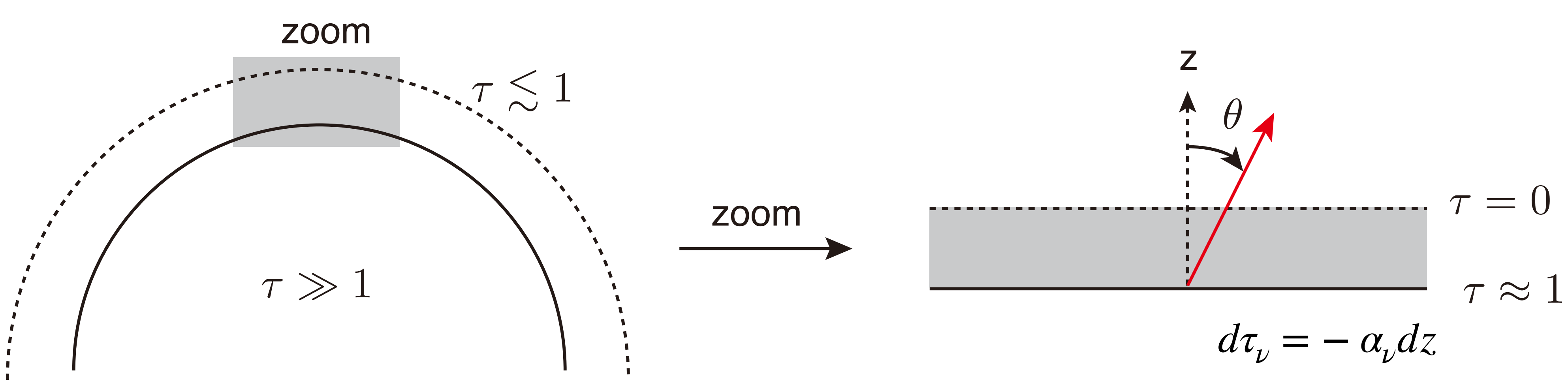
$$S_\nu = \left(1 - \epsilon_\nu\right) J_\nu + \epsilon_\nu B_\nu \quad \text{and} \quad \epsilon_\nu = \alpha_\nu^{\mathrm{abs}} / \alpha_\nu^{\mathrm{ext}}$$

- We will learn two approximations to solve the equation in a plane-parallel medium, primarily developed for the stellar atmosphere.

  - Rosseland approximation
  - Eddington approximation

Plane-parallel Atmosphere

- Stellar Atmosphere
  - The atmosphere of a star is defined as its outer regions that determine the properties of the radiative flux that emanates from its surface.
  - The calculations of the structure of a stellar atmosphere necessitate the knowledge of the detailed monochromatic opacity and can only be undertaken with powerful numerical capabilities such as those provided by modern computers.
  - However, by making appropriate approximations, certain interesting results can be obtained analytically.
- ***Plane-parallel atmosphere***
  - We ignore the curvature of the underlying body. In a plane-parallel medium with a frequency independent opacity. This simplification is especially suitable for models of bodies where the atmosphere is a relatively thin layer around the body, i.e., "locally virtually flat."
  - The optical depth is defined as a function of the vertical distance *z* rather than the distance long the ray path *s*. It increases as we go deeper down. The normal convention is to take $\tau = 0$ at the top of the stellar atmosphere

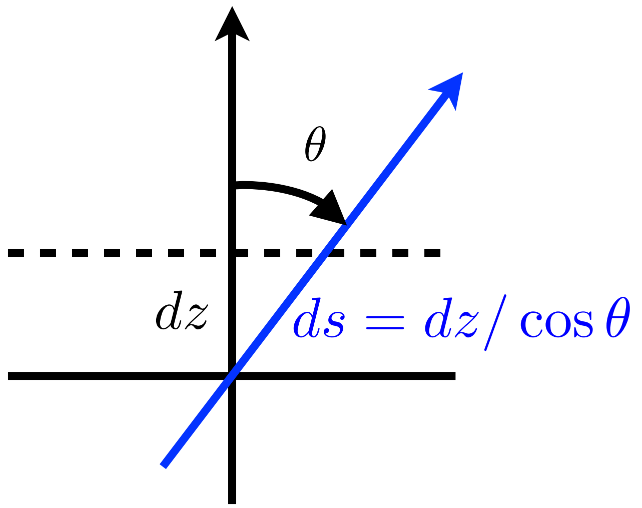


(1) Radiative Diffusion: Rosseland Approximation

- Imagine a plane-parallel medium (in which ρ, T depend only on depth *z*).

$$ds = \frac{dz}{\cos\theta} = \frac{dz}{\mu} \;\rightarrow\; \mu \frac{\partial I_\nu(z,\mu)}{\partial z} = -\alpha_\nu^{\rm ext}(I_\nu - S_\nu)$$

$$I_\nu(z,\mu) = S_\nu - \frac{\mu}{\alpha_\nu^{\rm ext}} \frac{\partial I_\nu}{\partial z}$$



- "zeroth" approximation: when the point in question is deep in the material, all quantities changes slowly on the scale of a mean free path $\ell = 1/\alpha^{\rm ext}$ and the derivative term above is very small.

$$I_\nu^{(0)}(z,\mu) \approx S_\nu^{(0)}(T) \qquad \therefore\; J_\nu^{(0)} = S_\nu^{(0)} \text{ and } I_\nu^{(0)} = S_\nu^{(0)} = B_\nu$$

LTE condition

This is independent of the angle.

- "first" approximation:

$$I_\nu^{(1)}(z,\mu) \approx S_\nu^{(0)} - \frac{\mu}{\alpha_\nu^{\rm ext}} \frac{\partial I_\nu^{(0)}}{\partial z} = B_\nu(T) - \frac{\mu}{\alpha_\nu^{\rm ext}} \frac{\partial B_\nu(T)}{\partial z} \rightarrow \text{linear in } \mu$$

- Net specific flux along *z*: the angle-independent part of the intensity does not contribute to the flux.

$$F_\nu(z) = \int I_\nu^{(1)}(z,\mu)\cos\theta d\Omega = 2\pi \int_{-1}^{+1} I_\nu^{(1)}(z,\mu)\mu d\mu$$

$$= -\frac{2\pi}{\alpha_\nu^{\rm ext}}\frac{\partial B_\nu}{\partial z}\int_{-1}^{+1}\mu^2 d\mu$$

$$= -\frac{4\pi}{3\alpha_\nu^{\rm ext}}\frac{\partial B_\nu(T)}{\partial T}\frac{\partial T}{\partial z}$$

- Total integrated flux:

$$F(z) = \int_0^\infty F_\nu(z)d\nu = -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty \frac{1}{\alpha_\nu^{\rm ext}}\frac{\partial B_\nu}{\partial T}d\nu$$

let’s define the Rosseland mean absorption coefficient

$$\frac{1}{\alpha_R} \equiv \frac{\int_0^\infty \frac{1}{\alpha_\nu^{\rm ext}}\frac{\partial B_\nu}{\partial T}d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu}$$

use

$$\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \frac{\partial}{\partial T} \int_0^\infty B_\nu d\nu = \frac{\partial \left(\sigma T^4 / \pi \right)}{\partial T} = \frac{4\sigma T^3}{\pi}$$

Then, we obtain the Rosseland approximation to radiative flux

$$F(z) = -\frac{16\sigma T^3}{3\alpha_R} \frac{\partial T}{\partial z} \to -\chi \nabla T$$

which is also called ***the equation of radiative diffusion***.

- The flux equation can be interpreted as a heat conduction with an "effective heat conductivity," $\chi = 16\sigma T^3 / 3\alpha_R$.
- At which frequencies the Rosseland mean becomes important?
  - The mean involves a weighted average of $1/\alpha^{\text{ext}}$ so that frequencies at which **the extinction coefficient is small (transparent) tend to dominate.**
  - The weighting function $\partial B_\nu / \partial T$ has a shape similar to that of the Planck function, but it peaks at $h\nu_{\max} = 3.8 k_{\text{B}} T$, instead of $h\nu_{\max} = 2.8 k_{\text{B}} T$.

(2) Eddington Approximation

- In Eddington approximation, the intensities are assumed to approach isotropy, and not necessarily their thermal values.

  In the Rosseland approximation, the intensities approach the Planck function at large effective depths.

- Near isotropy can be introduced by assuming that the intensity is linear in μ. (frequency is suppressed for convenience).

$$I(\tau, \mu) = a(\tau) + b(\tau)\mu$$

- Let us take the first three moments.

mean intensity: $$J \equiv \frac{1}{2}\int_{-1}^{+1} I d\mu = a$$

flux: $$H \equiv \frac{1}{2}\int_{-1}^{+1} \mu I d\mu = \frac{b}{3} \quad \rightarrow \quad \boxed{K = \frac{1}{3}J}$$

This relation is called the Eddington approximation

radiation pressure: $$K \equiv \frac{1}{2}\int_{-1}^{+1} \mu^2 I d\mu = \frac{a}{3}$$

  - Compare with the following equations for the isotropic radiation.

$$p = \frac{1}{3}u \quad \left(p \equiv \frac{1}{c}\int I\cos^2\theta d\Omega, \ u(\mathbf{\Omega}) = \frac{1}{c}I\right)$$

- optical depth and the transfer equation:

$$d\tau(z) \equiv -\alpha^{\mathrm{ext}} dz, \quad \mu \frac{\partial I}{\partial \tau} = I - S$$

- Note: source function is independent to μ (because $S = (1-\epsilon)J + \epsilon B$).
- Integrate the above equation and obtain the following equations.

$$\frac{1}{2}\int_{-1}^{+1} d\mu \left(\mu \frac{\partial I}{\partial \tau} = I - S\right) \rightarrow \frac{\partial H}{\partial \tau} = J - S$$

$$\frac{1}{2}\int_{-1}^{+1} d\mu\mu \left(\mu \frac{\partial I}{\partial \tau} = I - S\right) \rightarrow \frac{\partial K}{\partial \tau} = H \rightarrow \frac{1}{3}\frac{\partial J}{\partial \tau} = H$$

- The two equations can be combined to yield:

$$\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = J - S \quad \rightarrow \boxed{\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = \epsilon(J - B)}$$

- Let us define a new optical depth $\tau_* \equiv \sqrt{3\epsilon}\tau = \sqrt{3\tau_{\mathrm{abs}}(\tau_{\mathrm{abs}} + \tau_{\mathrm{sca}})}$
- The radiative equation is then $\boxed{\dfrac{\partial^2 J}{\partial \tau_*^2} = J - B}$
- This equation is sometimes called the ***radiative diffusion equation***. Given the temperature structure of the medium, $B(\tau)$, the equation can be solved for J.

Grey Atmosphere

- Grey Atmosphere
  - For example, by supposing that the opacity if independent of frequency, some illustrative and insightful results can be found. A stellar atmosphere while assuming such an opacity spectrum is called a grey atmosphere.
  - Nevertheless, the radiation field present in such an atmosphere will posses a frequency distribution since matter emits a radiative spectrum associated to its temperature.
  - We will find an expression for the temperature profile (with respect to depth) and the flux emanating from a grey atmosphere.
  - It is more convenient to use a convention where τ increases inward. Then, the RT equation is given by

$$\boxed{\mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu}$$

Here, τ is independent of frequency.

  - (1) By integrating this equation over all frequencies, the RT equation becomes

$$\boxed{\mu \frac{dI}{d\tau} = I - S}$$

where $I = \int_0^\infty I_\nu d\nu, \quad S = \int_0^\infty S_\nu d\nu$

- Radiative equilibrium
  - If there exists no energy sources or sinks in a stellar atmosphere, the amount of energy absorbed by a given layer is equal to the energy that it emits.
  - In the plane-parallel approximation, this assumption implies that the energy generated in the stellar interior passes out in the form of a constant energy flux through the outer layers of the atmosphere since the energy emitted is distributed over the same area independently of depth.

$$\int\int \alpha_\nu I_\nu d\nu d\Omega = \int\int j_\nu d\nu d\Omega$$

total energy absorbed (or scattered by matter) = energy that it radiates

  - In the grey atmosphere, using the definition of the source function, we obtain

$$\int\int \alpha_\nu I_\nu d\nu d\Omega = \int\int \alpha_\nu S_\nu d\nu d\Omega \quad \rightarrow \quad \alpha \int J_\nu d\nu = \alpha \int S_\nu d\nu$$

$$\boxed{J = S}$$

- (2) Average the first equation over the solid angle:

$$\frac{1}{2}\int_{-1}^{1} d\mu\left(\mu\frac{dI_\nu}{d\tau}\right) = \frac{1}{2}\int_{-1}^{1} d\mu I_\nu - \frac{1}{2}\int_{-1}^{1} d\mu S_\nu$$

$$\boxed{\frac{dH}{d\tau} = J - S \quad \rightarrow \quad \frac{dH}{d\tau} = 0 \quad \text{from the radiative equilibrium } (J = S) \qquad H = \text{constant}}$$

- (3) Multiply the first equation by μ and average over the solid angle:

$$\frac{1}{2}\int_{-1}^{1} d\mu\left(\mu^2\frac{dI}{d\tau}\right) = \frac{1}{2}\int_{-1}^{1} d\mu\,(\mu I) - \frac{1}{2}\int_{-1}^{1} d\mu\,(\mu S)$$

$$\boxed{\frac{dK}{d\tau} = H}$$

if the source function is isotropic.

- Integrating over the above equation, we obtain the pressure as a function optical depth:

$$\boxed{K(\tau) = H\,(\tau + c)}$$

Here, Hc is the constant of integration.

- ***Using the Eddington approximation***, the mean intensity is obtained to be

$$\boxed{J(\tau) = 3H\,(\tau + c)}$$

$c = \frac{2}{3}$ can be obtained as shown in the next slide

How to derive the constant of integration:

$$\mu \frac{dI}{d\tau} = I - S$$

$$e^{-\tau/\mu} \frac{dI}{d(\tau/\mu)} - e^{-\tau/\mu} I = -e^{-\tau/\mu} S$$

$$\frac{d}{d(\tau/\mu)} \left(e^{-\tau/\mu} I \right) = -e^{-\tau/\mu} S$$

$$\left[e^{-\tau'/\mu} I \right]_{\tau'=\infty}^{\tau} = -\int_{\infty}^{\tau} e^{-\tau'/\mu} S d(\tau'/\mu)$$

$$I(\tau, \mu) = \int_{\tau}^{\infty} e^{-(\tau'-\tau)/\mu} S d(\tau'/\mu) \qquad \text{for } \mu > 0$$

$$I(0, \mu) = \int_0^{\infty} e^{-\tau'/\mu} S d(\tau'/\mu) = 3H \int_0^{\infty} e^{-\tau'/\mu} (\tau' + c) d(\tau'/\mu) \longrightarrow \boxed{I(0, \mu) = 3H(\mu + c)}$$

Now, from the definition of the outward flux H at $\tau = 0$:

$$H = \frac{1}{2} \int_0^1 \mu I(0, \mu) d\mu = \frac{3H}{2} \left(\frac{1}{3} + \frac{c}{2} \right)$$

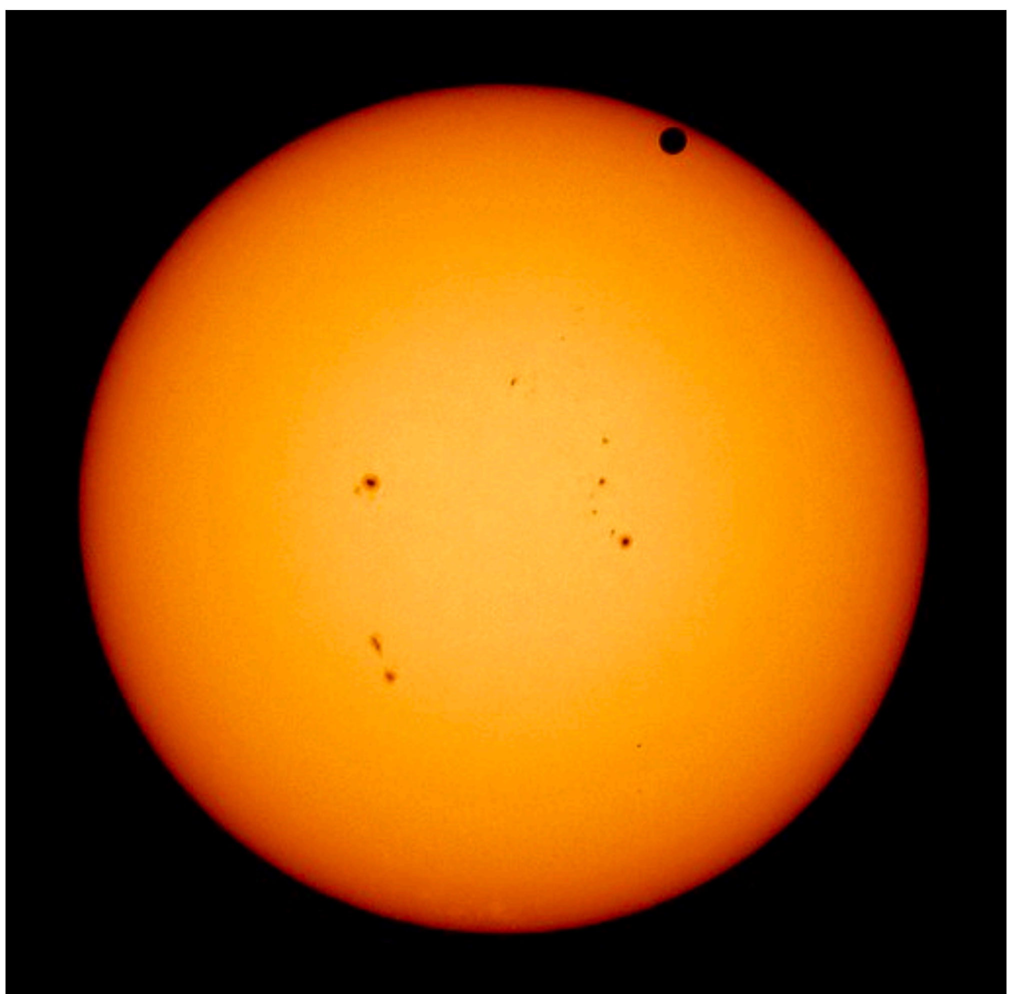
Note that $I(0, \mu) = 0$ for $\mu < 0$.

This gives the value of the constant of integration to be

$$\boxed{c = \frac{2}{3}}$$

Limb Darkening

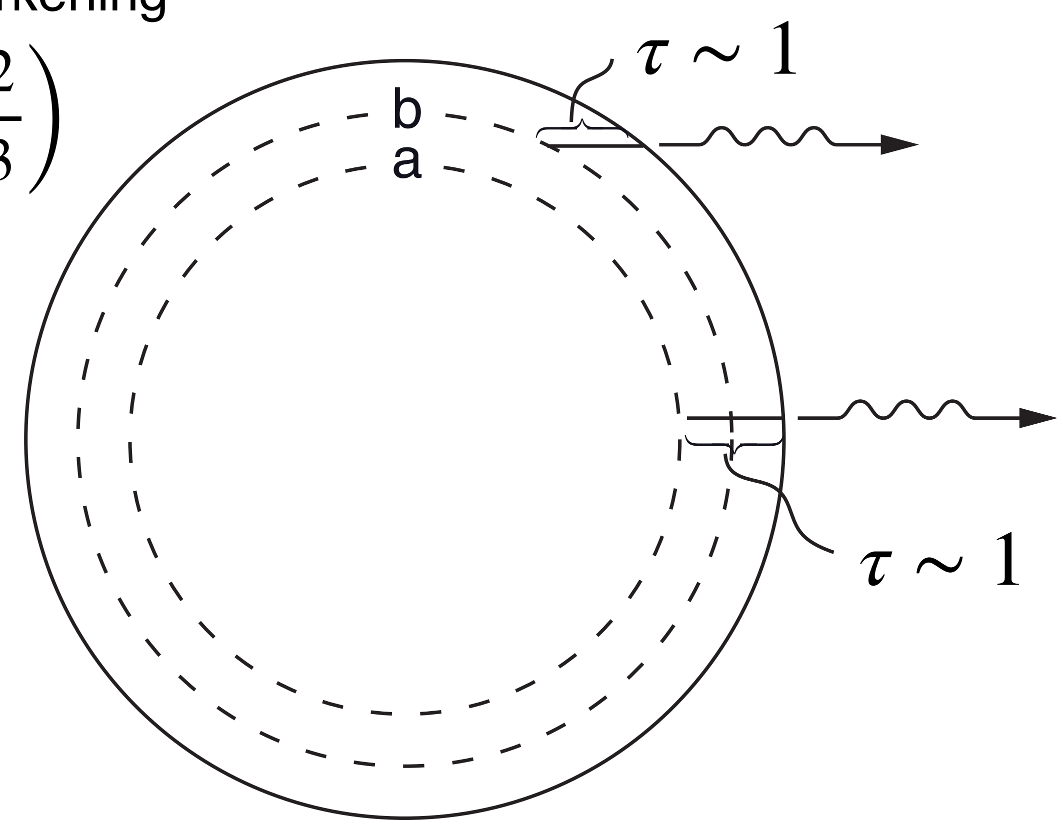
- Observations show that the radiation intensity at the center of the solar disk is larger than the corresponding intensity detected near its limb. This is called the limb darkening effect.
- Since the local temperature in the atmosphere increases with depth and the radiation leaving the star comes from layers with optical depth approximately equal to ~ 1, the layers responsible for this radiation are hotter in the center than near the limb.
- The Eddington limb darkening law can be estimated with the results found for the grey atmosphere.



[credit] Brocken Inaglory

Eddington limb darkening

$$\frac{I_\nu(\mu)}{I_\nu(1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right)$$



$T_a > T_b \Rightarrow S_a > S_b$

- Limb Darkening
  - The intensity at the surface of star is

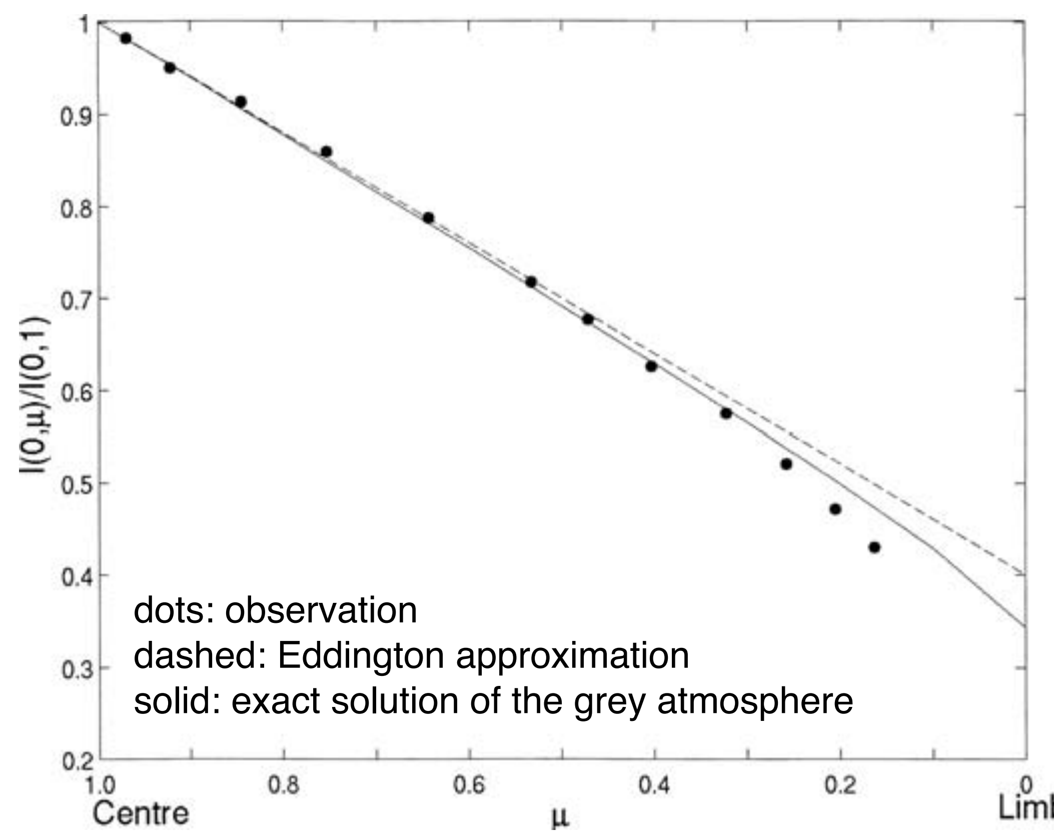
$$I(\tau=0,\mu) = \int_0^\infty S(\tau) e^{-\tau/\mu} \frac{d\tau}{\mu}$$
$$= \int_0^\infty 3H\left(\tau + \frac{2}{3}\right) e^{-\tau/\mu} d(\tau/\mu)$$
$$= 3H\left(\mu + \frac{2}{3}\right)$$

$$S = J = 3H\left(\tau + \frac{2}{3}\right)$$

  - The direction of the intensity coming from the center of the disk defines the angle $\theta = 0$ or $\mu = 1$, while the intensity near the limb of the disk is related to values of $\mu < 1$. The ratio of these intensities is

$$\boxed{\frac{I(0,\mu)}{I(0,1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right)}$$

This is called the Eddington limb darkening law.



for wavelength $\lambda = 5485\text{Å}$ [Pierce et al. 1950}

  - At the limb ($\mu = 0$) this ratio gives 0.4. Therefore, this demonstrates that the intensity for the grey-atmosphere approximation at the limb is 40% of the value of that at the stellar disk's center.

- Temperature Profile
  - From the definition of the effective temperature:

$$H = \frac{1}{4\pi}\int \cos\theta I d\Omega = \frac{1}{4\pi}F = \frac{1}{4\pi}\sigma_{\rm SB}T_{\rm eff}^4$$

  - Relation between the intensity and flux that comes outward:

$$J(\tau) = \frac{1}{\pi}F(\tau) = \frac{1}{\pi}\sigma_{\rm SB}T^4$$

Here, T is the effective temperature at τ.

  - Combining these, we obtain

$$J(\tau) = 3H\left(\tau + \frac{2}{3}\right) \longrightarrow T^4 = \frac{3}{4}T_{\rm eff}^4\left(\tau + \frac{2}{3}\right)$$

$$\boxed{T = T_{\rm eff}\left[\frac{3}{4}\left(\tau + \frac{2}{3}\right)\right]^{1/4}}$$

  - This equation tells us how temperature varies inside a grey atmosphere.

Boundary conditions: Two-stream approximation

- To solve the second order differential equation, we need two boundary conditions. The boundary conditions can be provided in several ways. One way to do is to use two-stream approximation, in which the entire radiation field is represented by radiation at just two angles, i.e., $\mu = \pm\mu_0$:

$$I(\tau,\mu) = I^+(\tau)\delta(\mu-\mu_0) + I^-(\tau)\delta(\mu+\mu_0)$$

- The two terms denote the outward and inward intensities. Then, the three moments are

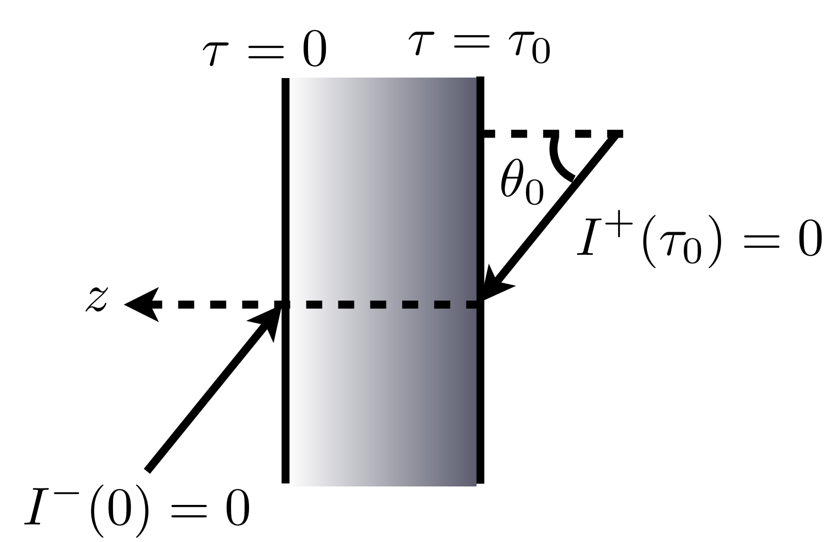
$$\begin{aligned} J &= \frac{1}{2}\left(I^+ + I^-\right) \\ H &= \frac{1}{2}\mu_0\left(I^+ - I^-\right) \\ K &= \frac{1}{2}\mu_0^2\left(I^+ + I^-\right) \end{aligned} \quad \rightarrow \ \text{we obtain } \mu_0 = \frac{1}{\sqrt{3}} \ \text{ in order to satisty } K = \frac{1}{3}J$$

$$(\theta_0 = \cos^{-1}\mu_0 = 54.74^\circ)$$

- Using $H = \dfrac{1}{3}\dfrac{\partial J}{\partial \tau}$, we obtain: $\boxed{I^+ = J + \dfrac{1}{3}\dfrac{\partial J}{\partial \tau}, \quad I^- = J - \dfrac{1}{3}\dfrac{\partial J}{\partial \tau}}$

Suppose the medium extends from $\tau = 0$ to $\tau = \tau_0$ and there is no incident radiation. Then, we obtain two boundary conditions:

$$I^+(\tau_0) = 0 \ \text{ and } \ I^-(0) = 0 \quad \rightarrow \quad \boxed{\begin{aligned} \frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau} &= J \quad \text{at} \ \ \tau = 0 \\ \frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau} &= -J \ \ \text{at} \ \ \tau = \tau_0 \end{aligned}}$$



Iteration Method

- Recall

$$\frac{dI(s)}{ds} = -\alpha^{\rm ext} I(s) + \alpha^{\rm sca} \int \Phi(\Omega, \Omega') I(s, \Omega') d\Omega' + j(s)$$

or

$$\frac{dI(\tau)}{d\tau} = -I(\tau) + a \int \Phi(\Omega, \Omega') I(\tau, \Omega') d\Omega' + \mathsf{S}(\tau) \qquad \left(d\tau \equiv \alpha^{\rm ext} ds, \ \mathsf{S}(\tau) \equiv \frac{j(\tau)}{\alpha^{\rm ext}} \right)$$

- Let I_0 be the intensity of photons that come directly from the source, I_1 the intensity of photons that have been scattered once by dust, and I_n the intensity after *n* scatterings. Then,

$$I(s) = \sum_{n=0}^{\infty} I_n(s)$$

- The intensities I_n satisfy the equations.

$$\begin{aligned} \frac{dI_0(\tau)}{d\tau} &= -I_0(\tau) + \mathsf{S}(\tau) \\ \frac{dI_n(\tau)}{d\tau} &= -I_n(\tau) + a \int \Phi(\Omega, \Omega') I_{n-1}(\tau, \Omega') d\Omega' \\ &\equiv -I_n(\tau) + \mathsf{S}_{n-1}(\tau) \qquad \left(\mathsf{S}_{n-1}(\tau) \equiv a \int \Phi(\Omega, \Omega') I_{n-1}(\tau, \Omega') d\Omega' \right) \end{aligned}$$

- Then, the formal solutions are:

$$\rightarrow \quad \begin{aligned} I_0(\tau) &= e^{-\tau} I_0(0) + \int_0^{\tau} e^{-(\tau - \tau')} \mathsf{S}(\tau') d\tau' \\ I_n(\tau) &= e^{-\tau} I_n(0) + \int_0^{\tau} e^{-(\tau - \tau')} \mathsf{S}_{n-1}(\tau') d\tau' \end{aligned}$$

(1) application to the edge-on galaxies

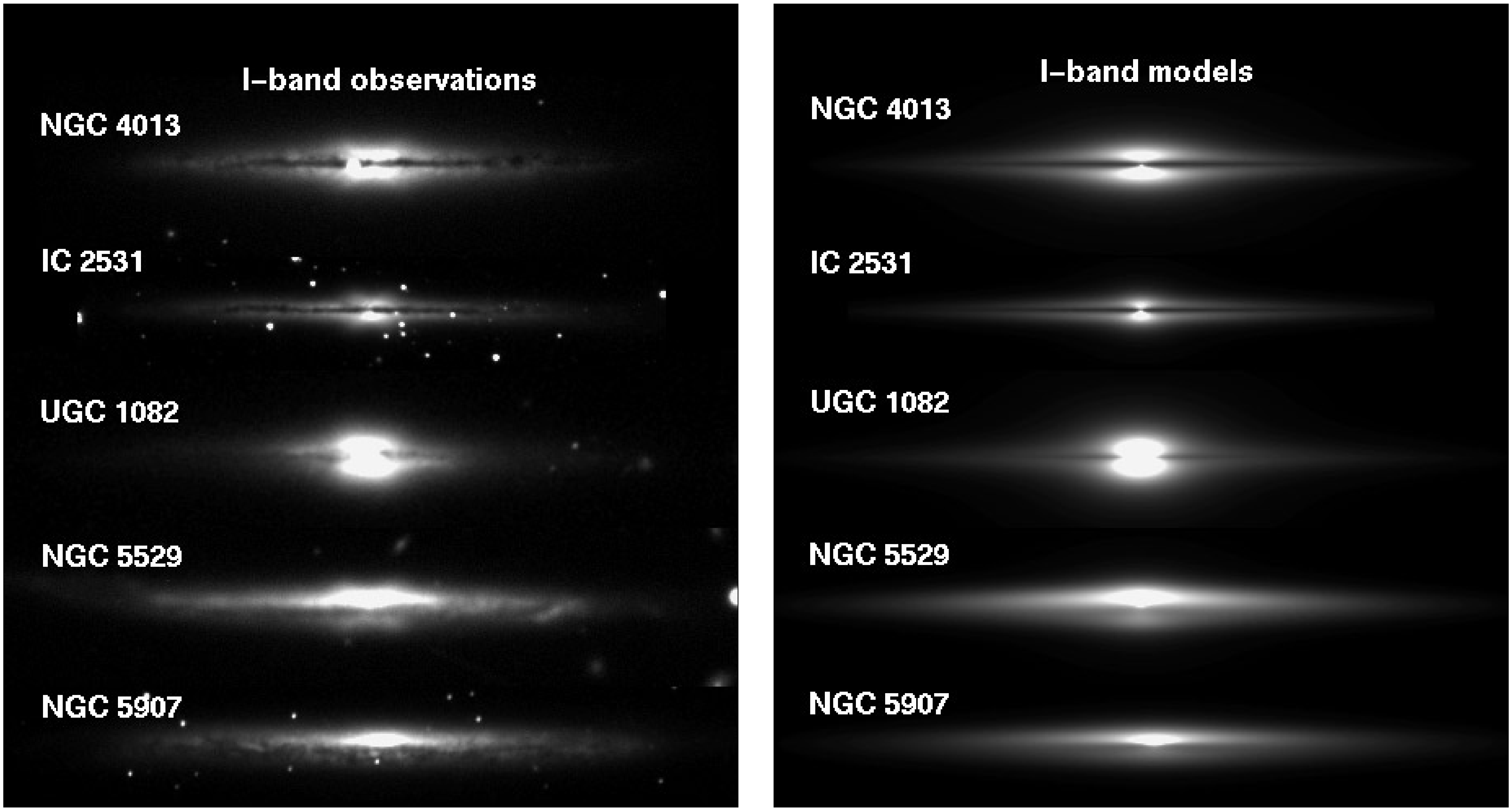
- The solution can be further simplified by assuming that

$$\frac{I_n}{I_{n-1}} \approx \frac{I_1}{I_0} \quad (n \geq 2)$$

- Then, the infinite series becomes

$$I_n \approx I_0 \sum_{n=0}^{\infty} \left(\frac{I_1}{I_0} \right)^n = \frac{I_0}{1 - I_1/I_0}$$

- Kylafis & Bahcall (1987) and Xilouris et al. (1997, 1998, 1999) applied this approximation to model the dust radiative transfer process in the edge-on galaxies.



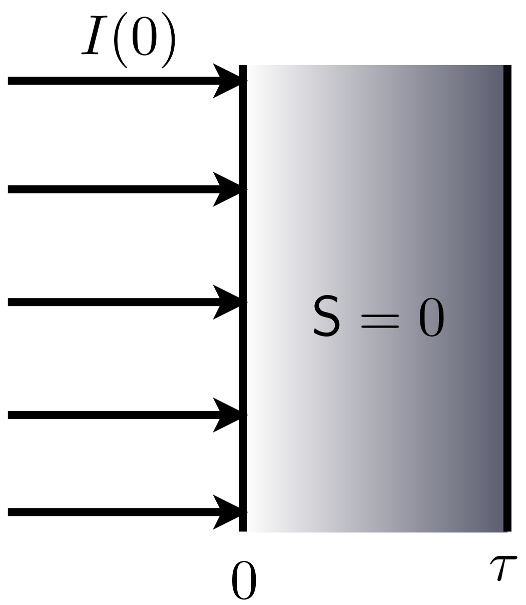
Xilouris et al. (1999)

(2) solution for the perfect forward scattering

- Assume the very strong forward-scattering

$$\Phi(\Omega, \Omega') = \delta(\Omega' - \Omega)$$
$$\rightarrow \quad \mathsf{S}_{n-1}(\tau) = a I_{n-1}(\tau)$$

- The iterative solutions are:



$$I_0(\tau) = e^{-\tau} I_0(0)$$
$$\rightarrow \quad \mathsf{S}_0(\tau) = a I_0(\tau) = a e^{-\tau} I_0(0)$$
$$I_1(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_0(\tau') d\tau' = (a\tau) e^{-\tau} I_0(0)$$
$$\rightarrow \quad \mathsf{S}_1(\tau) = a I_1(\tau) = (a^2 \tau) e^{-\tau} I_0(0)$$
$$I_2(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_1(\tau') d\tau' = \frac{(a\tau)^2}{2} I_0(0)$$
$$\rightarrow \quad \mathsf{S}_2(\tau) = a I_2(\tau) = (a^3 \tau^2) e^{-\tau} I_0(0)$$
$$I_3(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_2(\tau') d\tau' = \frac{(a\tau)^3}{3 \times 2} I_0(0)$$
$$\vdots$$
$$\rightarrow \quad \mathsf{S}_{n-1}(\tau) = a I_{n-1}(\tau) = (a^n \tau^{n-1}) e^{-\tau} I_0(0)$$
$$I_n(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_{n-1}(\tau') d\tau' = \frac{(a\tau)^n}{n!} I_0(0)$$

The final solutions are:

$$I^{\mathrm{direc}}(\tau) = e^{-\tau} I(0)$$
$$\begin{aligned} I^{\mathrm{scatt}}(\tau) &= \sum_{n=1}^{\infty} I_n(\tau) = \sum_{n=1}^{\infty} \frac{(a\tau)^n}{n!} e^{-\tau} I(0) \\ &= (e^{a\tau} - 1) e^{-\tau} I(0) \\ &\approx a\tau e^{-\tau} I(0) \quad \text{if} \quad a\tau \ll 1 \\ &\approx a\tau I(0) \qquad \text{if} \quad \tau \ll 1 \end{aligned}$$

$$\begin{aligned} I^{\mathrm{tot}}(\tau) &= I^{\mathrm{direc}}(\tau) + I^{\mathrm{scatt}}(\tau) \\ &= e^{-(1-a)\tau} I(0) \\ &= e^{-\tau_{\mathrm{abs}}} I(0) \end{aligned}$$

Homeworks (deadline 09/24)

(1) Explain the physical meaning of the result obtained for the perfect forward scattering.

$$I^{\mathrm{tot}}(\tau) = e^{-\tau_{\mathrm{abs}}} I(0)$$

(2) The scattering phase function for dust grains is known to be well described by the following Henyey-Greenstein function.

$$\Phi(\mu) = \frac{1}{2} \frac{1-g^2}{(1+g^2-2g\mu)^{3/2}}$$

Here, $-1 \leq g \leq 1$, $\mu = \cos\theta = \Omega \cdot \Omega'$, and θ is the scattering angle $(0 \leq \theta \leq \pi)$.

(a) Plot the shape of the phase function $\Phi(\mu)$ as a function of μ for the cases of $g = -0.9,\ 0.5,\ 0.0,\ 0.5,\ 0.9$.

(b) Explain the trend of the phase function as g increases.

(c) Calculate the fraction of the photons being scattered within $0 \le \theta \le \pi/4 = 45°$ for $g = -0.9,\ 0.5,\ 0.0,\ 0.5,\ 0.9$. You may can calculate the integral numerically (or analytically).

$$\int_{\cos(0)}^{\cos(\pi/4)} \Phi(\mu)d\mu = \sum_{i=1}^{N} \Phi(\mu_i)\Delta\mu$$

$$\Delta\mu = \frac{\cos(\pi/4) - \cos(0)}{N}$$

$$\mu_i = \cos(0) + \left(i - \frac{1}{2}\right)\Delta\mu \quad (i = 1, \cdots, N) \quad N = \text{a large number}$$

To plot the figures and calculate, we may want to use numpy and matplotlib packages of python.