

Interstellar Medium (ISM)

Week 5

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선광일 (Kwangil Seon)
KASI / UST

Voigt Line Profile

- The profile can be written using the Voigt function.

$$\phi(\nu) = \frac{1}{\Delta\nu_D \sqrt{\pi}} H(u, a)$$

Here, a is a ratio of the intrinsic broadening to the thermal broadening.

u is a measure of how far you are from the line center, in units of thermal broadening parameter.

In terms of Doppler velocity, u can be expressed as

$$u = \frac{\nu - \nu_0}{\Delta\nu_D} = \frac{\nu - \nu_0}{\nu_0} \frac{c}{v_{\text{th}}}$$

In the velocity term,

$$u = \frac{v}{v_{\text{th}}}, \text{ where } v = \frac{\nu - \nu_0}{\nu_0} c$$

Voigt-Hjerting function:

$$H(u, a) \equiv \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(u - y)^2 + a^2}$$

$$a \equiv \frac{\Gamma}{4\pi\Delta\nu_D}$$

$$u \equiv \frac{\nu - \nu_0}{\Delta\nu_D}$$

$$\Delta\nu_D = \frac{\nu_0}{c} \sqrt{\frac{2kT}{m}}$$

Including the turbulent motion,

$$\Delta\nu_D = \nu_0 \frac{v_{\text{th}}}{c} \rightarrow \Delta\nu_D = \nu_0 \frac{b}{c} = \frac{b}{\lambda_0}$$

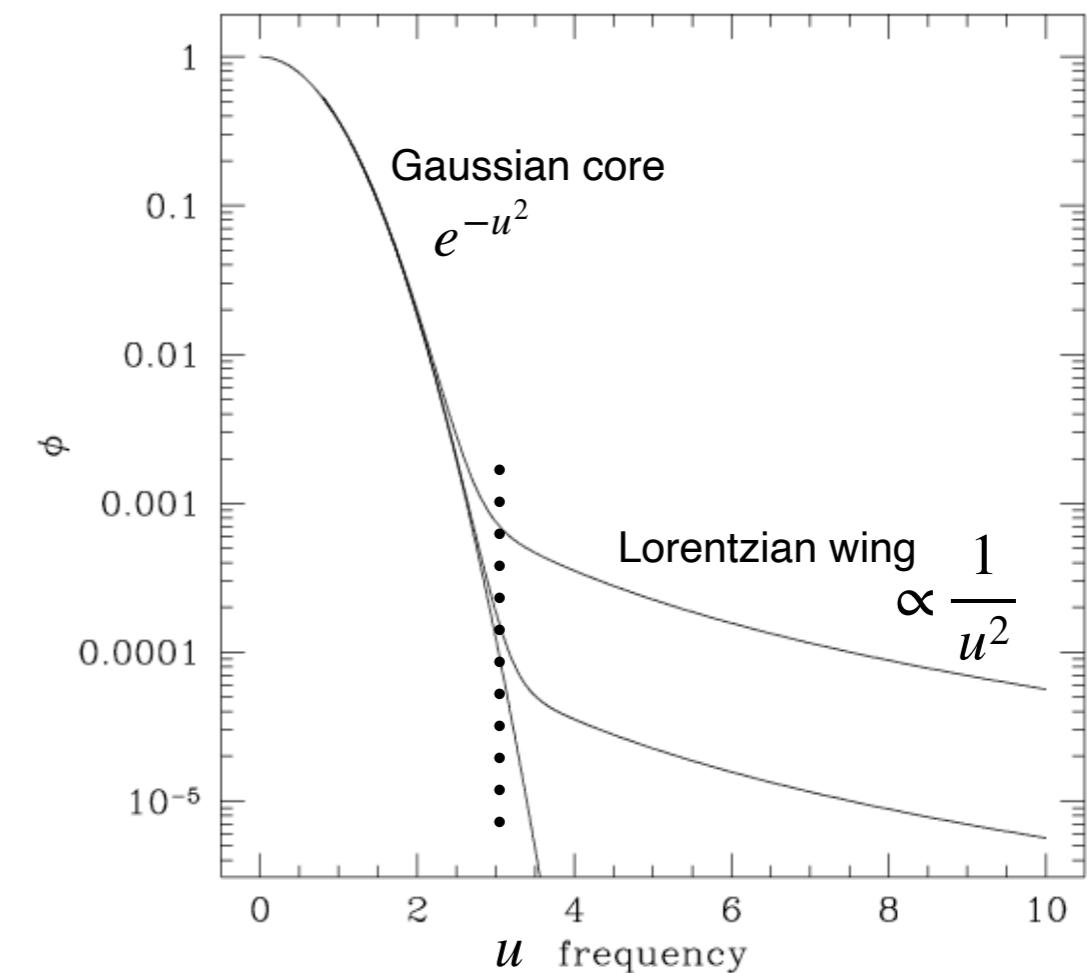
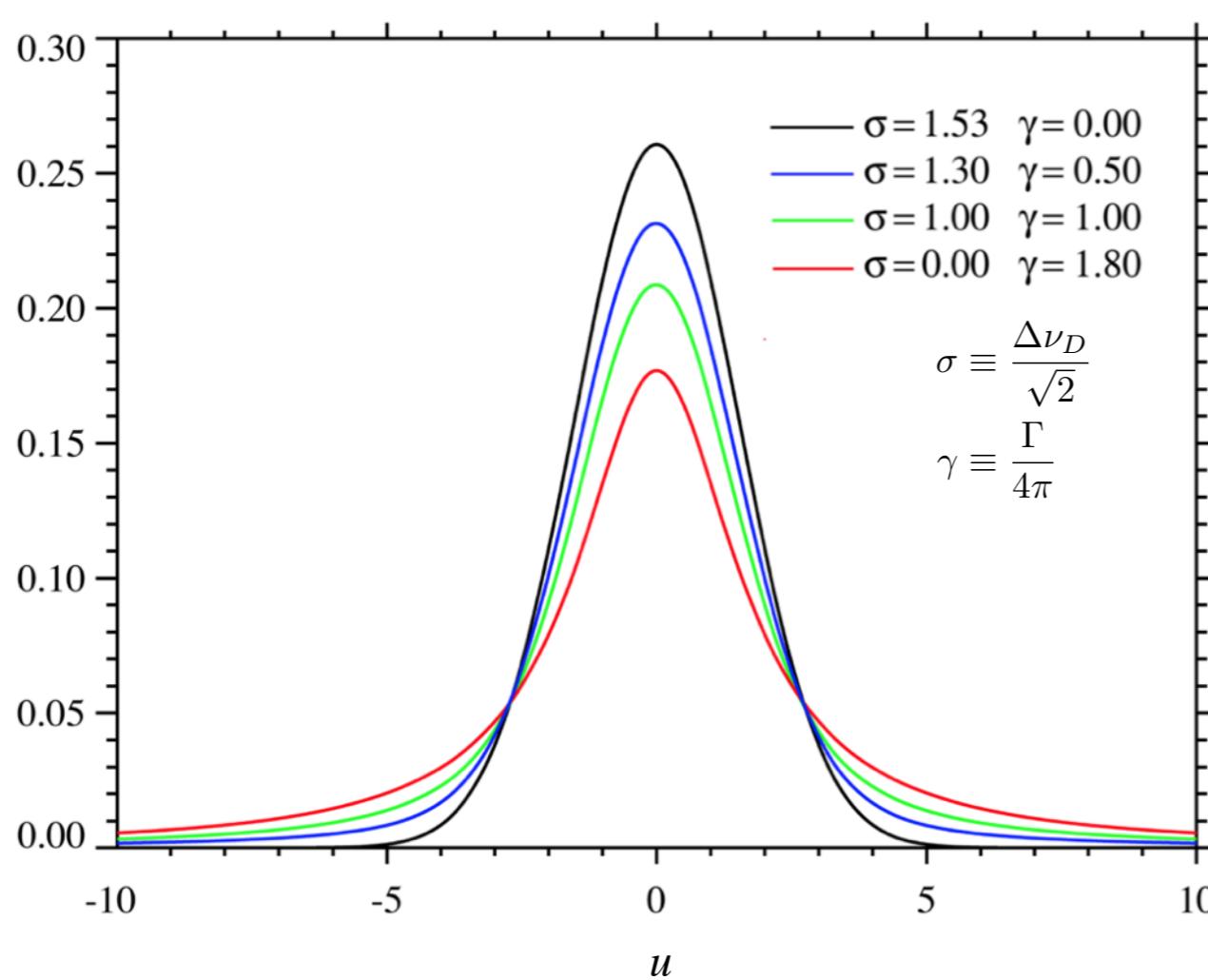
$$\text{where } b = \sqrt{v_{\text{th}}^2 + v_{\text{turb}}^2}, \quad v_{\text{th}} = \sqrt{\frac{2kT}{m}}$$

$$u = \frac{v}{b}$$

Properties of Voigt Function

- For small a , the “core” of the line is dominated by the Gaussian (Doppler) profile, whereas the “wings” are dominated by the Lorentz profile.

$$H(u, a) \equiv \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(u - y)^2 + a^2}$$



- In most case, $a \ll 1$. For Ly α at $T = 100$ K, $a \sim 0.05$.

- Line center:

$$H(0, a) = \exp(a^2) \operatorname{Erfc}(a) \approx 1 - \frac{2}{\sqrt{\pi}}a + a^2 - \mathcal{O}(a^3)$$

- Taylor series expansion of the Voigt function :

$$H(u, a) \equiv \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(u - y)^2 + a^2}$$

- Near the line center ($u \rightarrow 0$), the contribution to the integral is dominated by $y = u$. Therefore,

$$H(u, a) \simeq \frac{a}{\pi} e^{-u^2} \int_{-\infty}^{\infty} \frac{dy}{y^2 + a^2} = e^{-u^2}$$

which is known as the Doppler core.

- In the line wings away from the core ($u \gg 1$), the integral is dominated by $y \sim 0$ because of the rapidly decreasing function in the numerator.

$$H(u, a) \simeq \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{u^2} = \frac{a}{\pi} \frac{\sqrt{\pi}}{u^2} = \frac{a}{\sqrt{\pi} u^2}$$

- In summary, we obtain the Voigt function in a Taylor series expansion around $a = 0$.

$$H(u, a) \approx H(u, 0) + a \left. \frac{dH}{da} \right|_{a=0} \approx e^{-u^2} + a \frac{1}{\sqrt{\pi} u^2}$$

- The first term represents the Gaussian core, provided by the thermal broadening, and the second term represents the Lorentizan damping wing.
- Transition from Doppler core to damping wing can be found by solving:

$$e^{u^2} = \frac{\sqrt{\pi}}{a} u^2 \quad \rightarrow \quad u^2 = \ln \left(\frac{\sqrt{\pi}}{a} \right) + \ln u^2 \quad \text{for hydrogen}$$

$$b = 13 \text{ km s}^{-1} (T/10^4 \text{ K})^{1/2}$$

- The solution for this transcendental equation for Ly α is

$$u^2 \approx 10.31 + \ln \left[\left(\frac{6.265 \times 10^8 \text{ s}^{-1}}{\gamma_{ul}} \right) \left(\frac{1215.67 \text{ \AA}}{\lambda_{ul}} \right) \left(\frac{b}{10 \text{ km s}^{-1}} \right) \right]$$

provided that the quantity in square brackets is not very large or very small. The damping wing for $|u| \gtrsim 3.2$ or velocity shifts $|v| \gtrsim 3.2 (b/10 \text{ km s}^{-1})$.

Atomic Processes

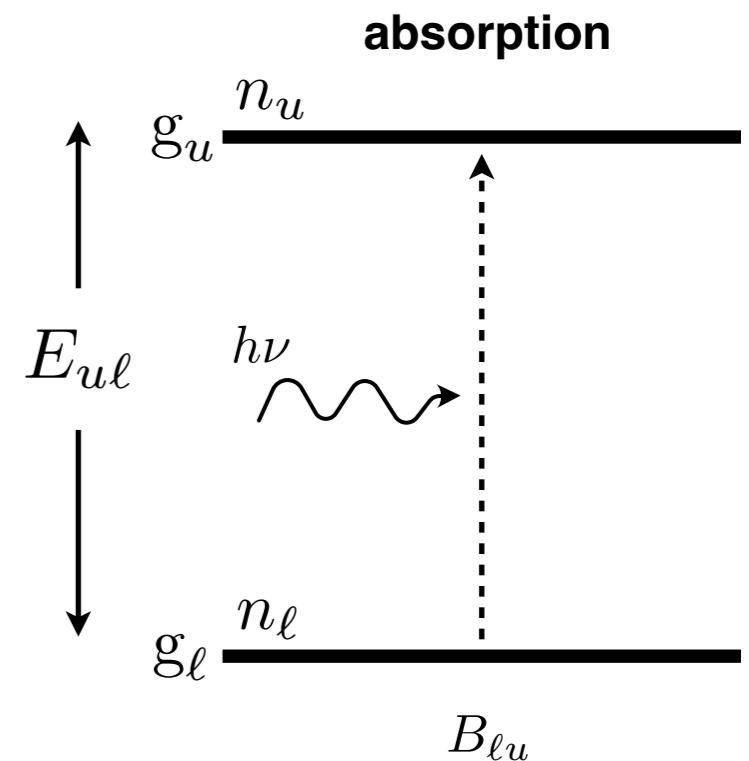
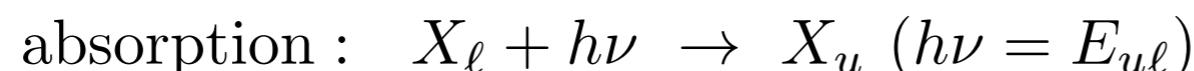
- **Excitation and de-excitation (Transition)**
 - ▶ Radiative excitation (photoexcitation; photoabsorption)
 - ▶ Radiative de-excitation (spontaneous emission and stimulated emission)
 - ▶ Collisional excitation
 - ▶ Collisional de-excitation
- **Emission Line**
 - ▶ Collisionally-excited emission lines
 - ▶ Recombination lines (recombination following photoionization or collisional ionization)
- **Ionization**
 - ▶ Photoionization and Auger-ionization
 - ▶ Collisional Ionization (Direct ionization and Excitation-autoionization)
- **Recombination**
 - ▶ Radiative recombination \Leftrightarrow Photoionization
 - ▶ Dielectronic Recombination (not dielectric!)
 - ▶ Three-body recombination \Leftrightarrow Direct collisional ionization
- **Charge exchange**

Radiative Excitation and De-excitation (Absorption and Emission)

- Three Radiative Transitions and Einstein Coefficients

- **Absorption:**

- If an absorber (atom, ion, molecule, or dust grain) X is in a lower level ℓ and there is radiation present with photons having an energy equal to $E_{u\ell}$. The absorber can absorb a photon and undergo an upward transition.



- The rate per volume at which the absorbers absorb photons will be proportional to both the energy density u_ν of photons of the appropriate energy and the number density n_ℓ of absorbers in the lower level ℓ .

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = - \left(\frac{dn_\ell}{dt} \right)_{\ell \rightarrow u} = n_\ell B_{\ell u} u_\nu$$

- The proportionality constant $B_{\ell u}$ is the **Einstein B coefficient** for the upward transition $\ell \rightarrow u$.

- **Emission:**

- ▶ An absorber X in an excited level u can decay to a lower level ℓ with emission of a photon. There are two ways this can happen:

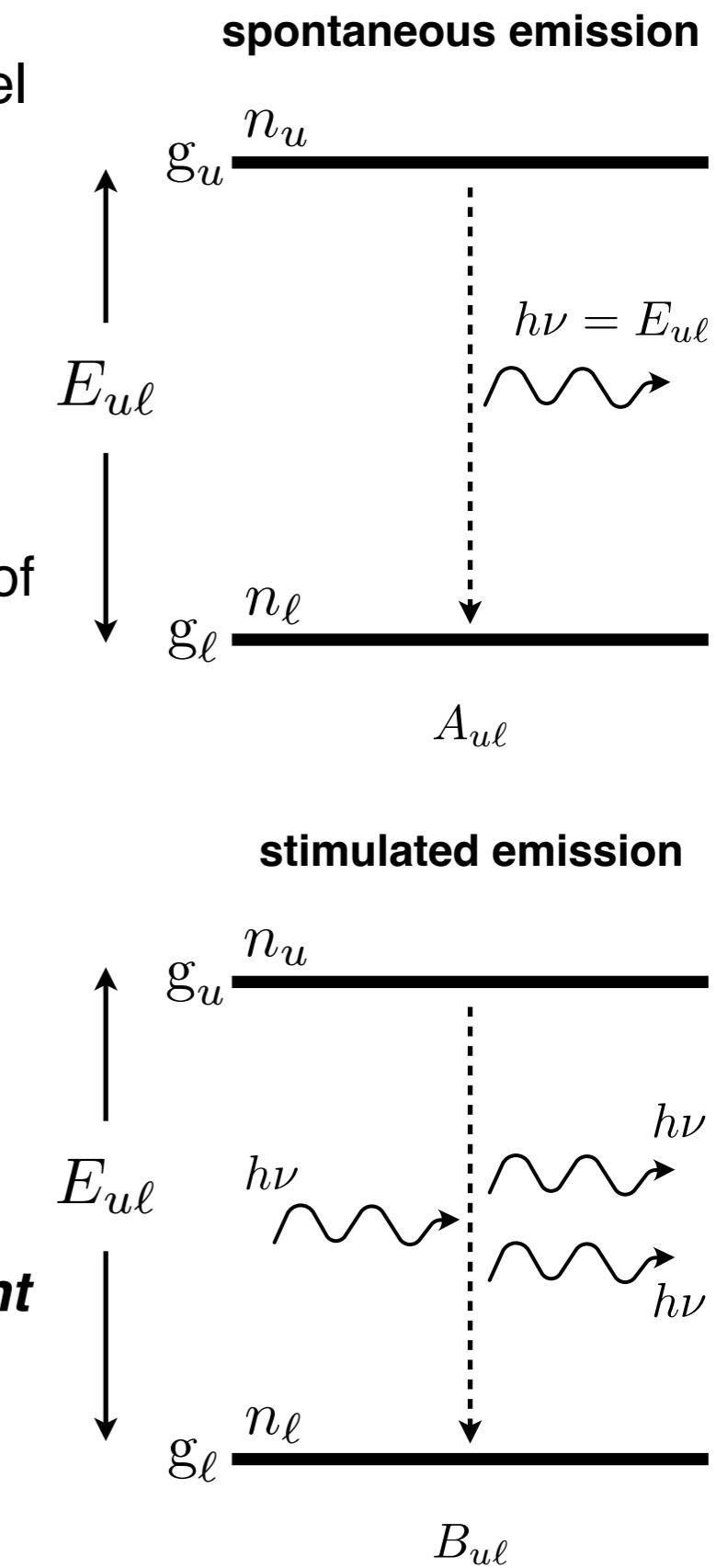
spontaneous emission : $X_u \rightarrow X_\ell + h\nu$ ($h\nu = E_{u\ell}$)

stimulated emission : $X_u + h\nu \rightarrow X_\ell + 2h\nu$ ($h\nu = E_{u\ell}$)

- ▶ **Spontaneous emission** is a random process, independent of the presence of a radiation field.
- ▶ **Stimulated emission** occurs if photons of the identical frequency, polarization, and direction of propagation are already present, and the rate of stimulated emission is proportional to the energy density u_ν of these photons.

$$\left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell} = - \left(\frac{dn_u}{dt} \right)_{u \rightarrow \ell} = n_u (A_{u\ell} + B_{u\ell} u_\nu)$$

- ▶ The probability per unit time $A_{u\ell}$ is the **Einstein A coefficient** for spontaneous transition. The coefficient $B_{u\ell}$ is the **Einstein B coefficient** for the downward transition $u \rightarrow \ell$.



Relations between the Einstein coefficients

- The three Einstein coefficients are not mutually independent.
- ***In thermal equilibrium***, the radiation field becomes the “blackbody” radiation field and the two levels must be populated according to the Boltzmann distribution.

$$(u_\nu)_{\text{TE}} = \frac{4\pi}{c} B_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}$$

$$\left(\frac{n_u}{n_\ell} \right)_{\text{TE}} = \frac{g_u}{g_\ell} e^{-E_{u\ell}/k_B T} \quad \text{Here, } E_{u\ell} = h\nu.$$

- The net rate of change of level u should be equal to zero, in TE.

$$\begin{aligned} \frac{dn_u}{dt} &= \left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} + \left(\frac{dn_u}{dt} \right)_{u \rightarrow \ell} \\ &= n_\ell B_{\ell u} u_\nu - n_u (A_{u\ell} + B_{u\ell} u_\nu) \\ &= 0 \end{aligned}$$

$$n_\ell B_{\ell u} u_\nu - n_u (A_{u\ell} + B_{u\ell} u_\nu) = 0$$

$$(n_\ell B_{\ell u} - n_u B_{u\ell}) u_\nu = n_u A_{u\ell}$$

$$\begin{aligned} u_\nu &= \frac{n_u A_{u\ell}}{n_\ell B_{\ell u} - n_u B_{u\ell}} \\ &= \frac{(n_u A_{u\ell}) / (n_\ell B_{\ell u})}{1 - (n_u B_{u\ell}) / (n_\ell B_{\ell u})} \\ &= \frac{(g_u/g_\ell) e^{-h\nu/kT} (A_{u\ell}/B_{\ell u})}{1 - (g_u/g_\ell) e^{-h\nu/kT} (B_{u\ell}/B_{\ell u})} \quad \leftarrow \quad \frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} e^{-h\nu_{u\ell}/kT_{\text{exc}}} \\ &= \frac{(g_u/g_\ell) (A_{u\ell}/B_{\ell u})}{e^{h\nu/kT} - (g_u/g_\ell) (B_{u\ell}/B_{\ell u})} \end{aligned}$$

Comparing the above eq. with Planck function,

$$u_\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

we can immediately recognize that the following relations should be satisfied.

$$(g_u/g_\ell) (A_{u\ell}/B_{\ell u}) = \frac{8\pi h\nu^3}{c^3}$$

$$(g_u/g_\ell) (B_{u\ell}/B_{\ell u}) = 1$$

Therefore, only one coefficient is independent.

[Note] If there is no stimulated emission ($B_{u\ell} = 0$), the only way to make the left eq. consistent with the Planck function is to assume $h\nu/kT \gg 1$ (Wien's regime). Therefore, the stimulated emission is negligible in the Wien's regime. In other words, the stimulated emission term is required in the Rayleigh-Jean regime.

In summary, we obtained the following relations between the Einstein coefficients.

$$A_{u\ell} = \frac{8\pi h\nu^3}{c^3} B_{u\ell}$$

$$B_{\ell u} = \frac{g_u}{g_\ell} B_{u\ell}$$

$$B_{u\ell} = \frac{c^3}{8\pi h\nu^3} A_{u\ell}$$

$$B_{\ell u} = \frac{g_u}{g_\ell} \frac{c^3}{8\pi h\nu^3} A_{u\ell}$$

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- We note the Einstein coefficients are intrinsic properties of the absorbing material, irrelevant to the assumption of TE. Hence, **the relations between the Einstein coefficients should hold in any condition.**
 - Using the relation, we can rewrite the downward and upward transition rates:

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = n_\ell \frac{g_u}{g_\ell} \frac{c^3}{8\pi h\nu^3} A_{u\ell} u_\nu \quad \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell} = n_u A_{u\ell} \left(1 + \frac{c^3}{8\pi h\nu^3} u_\nu \right)$$

- It is helpful to use a dimensionless quantity, the photon occupation number:

$$n_\gamma \equiv \frac{c^2}{2h\nu^3} I_\nu \quad \xrightarrow{\text{averaging over directions}} \quad \langle n_\gamma \rangle = \frac{c^2}{2h\nu^3} J_\nu = \frac{c^3}{8\pi h\nu^3} u_\nu$$

- Then, the above transition rates are simplified:

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = n_\ell \frac{g_u}{g_\ell} A_{u\ell} \langle n_\gamma \rangle \quad \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell} = n_u A_{u\ell} (1 + \langle n_\gamma \rangle)$$

- The photon occupation number determines the relative importance of stimulated and spontaneous emission: stimulated emission is important only when $\langle n_\gamma \rangle \gg 1$.

Absorption and Emission Coefficients in terms of Einstein coefficients

- The Einstein coefficients are useful means of analyzing absorption and emission processes. However, we often find it even more useful to use cross section because **the cross section has a natural geometric meaning.**
- (pure) Absorption cross section:**

- The number density of photons per unit frequency interval is $u_\nu/h\nu$. Let $\sigma_{\ell u}(\nu)$ be the cross section for absorption of photons for the transition $\ell \rightarrow u$. Then, the absorption rate is

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = n_\ell \int d\nu \sigma_{\ell u}(\nu) c \frac{u_\nu}{h\nu} \approx n_\ell u_\nu \frac{c}{h\nu_{ul}} \int d\nu \sigma_{\ell u}(\nu)$$

- Here, we assumed that $u_\nu/h\nu$ do not vary appreciably over the profile of the cross section. Therefore, we derive a simple relation between the absorption cross section and the Einstein B coefficient:

$$\int d\nu \sigma_{\ell u}(\nu) = \frac{h\nu_{ul}}{c} B_{\ell u} = \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{ul}^2} A_{ul}$$

- If the cross section has a normalized profile of ϕ_ν , we can write the absorption cross section as follows:

$$\sigma_{\ell u}(\nu) = \frac{h\nu_{ul}}{c} B_{\ell u} \phi_\nu = \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{ul}^2} A_{ul} \phi_\nu \quad \text{with} \quad \int \phi_\nu d\nu = 1$$

- **(effective) Absorption Coefficient**

- We note that the stimulated emission is proportional to the energy density of ambient radiation field. In the radiative transfer equation, it is convenient to include the stimulated emission term in the absorption coefficient as a negative absorption.

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} - \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell}^{\text{stimulated}} = n_\ell B_{\ell u} u_\nu - n_u B_{u \ell} u_\nu$$

$$= n_\ell B_{\ell u} u_\nu - n_u \left(\frac{g_\ell}{g_u} B_{\ell u} \right) u_\nu$$

- Therefore, we may define the cross section for stimulated emission and the net (effective) absorption coefficient as follows:

$$\sigma_{u\ell} = \frac{g_\ell}{g_u} \sigma_{\ell u}$$

$$\kappa_\nu = n_\ell \sigma_{\ell u} - n_u \sigma_{u\ell}$$

$$= n_\ell \sigma_{\ell u} \left(1 - \frac{n_u/n_\ell}{g_u/g_\ell} \right)$$

pure absorption coefficient

- Using the definition of the excitation temperature, we can rewrite them:

$$\kappa_\nu = n_\ell \sigma_{\ell u} \left[1 - \exp \left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}} \right) \right] \quad \text{or} \quad \sigma_\nu^{\text{eff}} = \sigma_{\ell u} \left[1 - \exp \left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}} \right) \right]$$

- **Emission coefficient (Emissivity)**

- The emissivity is defined as the power radiated per unit frequency per unit solid angle per unit volume.
- The line emissivity can be expressed in terms of the spontaneous downward transition rate:

$$4\pi \int d\nu j_\nu = h\nu_{u\ell} \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell}^{\text{spontaneous}}$$

- Comparing with the definition of the Einstein A coefficient, we obtain:
- $$\int d\nu j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell}$$
- If the emission line has a normalized profile of ϕ_ν , we can write the emissivity as follows:

$$j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell} \phi_\nu \quad \text{with} \quad \int d\nu \phi_\nu = 1$$

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- The correction factor for the stimulated emission in absorption coefficient:

- For Ly α line,

$$h\nu_{u\ell} = 10.2 \text{ eV} \rightarrow 1 - \exp\left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}}\right) = 1 - \exp\left(-\frac{1.1837 \times 10^5 \text{ K}}{T_{\text{exc}}}\right)$$

$$\simeq 1 \quad \text{for } T_{\text{exc}} \approx T_{\text{gas}} < 1 \times 10^5 \text{ K}$$

- ▶ The stimulated emission is negligible.

- For 21 cm line,

$$h\nu_{u\ell} = 6 \mu\text{eV} \rightarrow 1 - \exp\left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}}\right) = 1 - \exp\left(-\frac{0.068 \text{ K}}{T_{\text{exc}}}\right)$$

$$\simeq \frac{0.068 \text{ K}}{T_{\text{exc}}} \ll 1 \quad \text{for } T_{\text{exc}} \approx T_{\text{gas}} \sim 100 \text{ K}$$

- ▶ The correction for stimulated emission is very important. **We, therefore, need to take into account the stimulated emission in dealing with the 21 cm line.**

- Two limiting cases:

- At radio and sub-mm frequencies, the upper levels are often appreciably populated, and it is important to include both spontaneous and stimulated emission.
- When we consider propagation of optical, UV, or X-ray radiation in cold ISM, the upper levels of atoms and ions usually have negligible populations, and stimulated emission can be neglected.

- **Source Function:**

$$\begin{aligned}
 S_\nu &= \frac{j_\nu}{\kappa_\nu} \\
 &= \frac{n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell} \phi_\nu^{\text{emiss}}}{n_\ell \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell} \phi_\nu^{\text{abs}} [1 - \exp(-h\nu_{u\ell}/k_B T_{\text{exc}})]} && \leftarrow \frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp(-h\nu_{u\ell}/k_B T_{\text{exc}}) \\
 &= \frac{2h\nu_{u\ell}^3}{c^2} \frac{1}{\exp(h\nu_{u\ell}/k_B T_{\text{exc}}) - 1} && \leftarrow \phi_\nu^{\text{emiss}} = \phi_\nu^{\text{abs}}
 \end{aligned}$$

- This is called the **generalized Kirchhoff's law**.
- **The intrinsic profiles for absorption and emission are the same.**
 - ▶ The source function should approach the Planck function in LTE ($T_{\text{exc}} = T_{\text{kinetic}}$). For this to be true, the intrinsic profile of emission line should be the same as that of absorption line.
 - ▶ We can show that the intrinsic emission and absorption profiles are, indeed, the same, using a semi-classical model for an atom.

Oscillator Strength

- In the previous slides, we characterized the absorption cross section by the Einstein A coefficient. Equivalently, we can express the cross section in terms of the oscillator strength for the absorption transition $\ell \rightarrow u$, defined by the relation:

$$\int \sigma_{\ell u}(\nu) d\nu = \frac{\pi e^2}{m_e c} f_{\ell u} \quad \rightarrow \quad \boxed{\sigma_{\ell u}(\nu) = \frac{\pi e^2}{m_e c} f_{\ell u} \phi_\nu}$$

- Here, the factor $\frac{\pi e^2}{m_e c} = 0.02654 \text{ cm}^2 \text{ Hz}$ is the cross-section, integrated over the line profile, for a classical oscillator model.
- The oscillator strength is the factor which corrects the classical result. The quantum mechanical process can be interpreted as being due to a (fractional) number f of equivalent classical electron oscillators of the same frequency.
- The Einstein A coefficient is related to the absorption oscillator strength of the upward transition by

$$A_{u\ell} = \frac{8\pi^2 e^2 \nu_{u\ell}^2}{m_e c^3} \frac{g_\ell}{g_u} f_{\ell u} = \left(\frac{0.8167 \text{ cm}}{\lambda_{u\ell}} \right)^2 \frac{g_\ell}{g_u} f_{\ell u} [\text{s}^{-1}]$$

- For 21.1 cm line, $g_u = 3$, $g_\ell = 1$ ($g_F = 2F + 1$)

$$A_{u\ell} = 2.88 \times 10^{-15} [\text{s}^{-1}] = (11 \text{ Myr})^{-1} \quad f_{\ell u} = 5.75 \times 10^{-12}$$

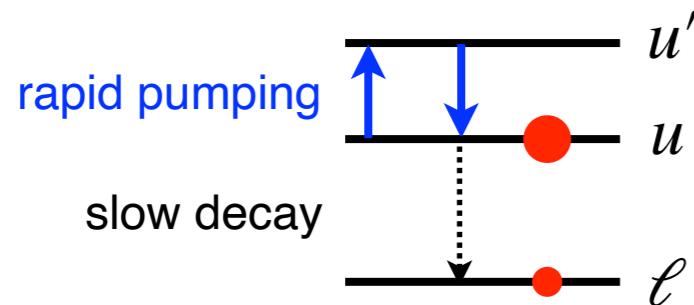
- For Ly α (1215.67 Å) line, $g_u = 3$, $g_\ell = 1$ ($g_L = 2L + 1$)

$$A_{u\ell} = 6.265 \times 10^8 [\text{s}^{-1}] \quad f_{\ell u} = 0.4164 \text{ for } 1^2S_{1/2} \rightarrow 2^2P$$

$f_{\ell u} = 0.27760$ for ${}^2S_{1/2} \rightarrow {}^2P_{3/2}$
 $= 0.13881$ for ${}^2S_{1/2} \rightarrow {}^2P_{1/2}$

Maser Lines

- Population inversion
 - Under some conditions, a process may act to “pump” an excited state u by either collisional or radiative excitation of a higher level u' that then decays to populate level u . If this pumping process is rapid enough (relative to the processes that depopulate u), it may be possible for the relative level populations between u and ℓ to satisfy the inequality (also to have a negative excitation temperature).



$$n_u > \frac{g_u}{g_\ell} n_\ell \quad \rightarrow \quad T_{\text{exc},u\ell} < 0.$$

- When this population inversion occurs, stimulated emission is stronger than pure absorption, and ***the radiation is amplified as it propagates***. Then, the effective absorption coefficient, optical depth, and attenuation factor are

$$\kappa_\nu = \sigma_{\ell u} \left(1 - \frac{n_u/g_u}{n_\ell/g_\ell} \right) < 0, \quad \tau_\nu = \int \kappa_\nu ds < 0, \quad e^{-\tau_\nu} > 1$$

- Maser
 - Such population inversion have been observed for microwave transitions of H I, OH, H₂O, and SiO, and hence we speak of ***maser (microwave amplification by stimulated emission of radiation)*** emission.

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- Observational properties
 - If $|k_B T_{\text{exc}, u\ell}| \gg h\nu$, the RT equation becomes
$$\begin{aligned}T_A &= T_A(0)e^{-\tau_\nu} + T_{\text{exc}}(1 - e^{-\tau_\nu}) \\&= (T_A(0) + |T_{\text{exc}}|)e^{|\tau_\nu|} - |T_{\text{exc}}|\end{aligned}$$
 - The factor $e^{|\tau_\nu|}$ is in some cases very large - some OH and H₂O masers have been observed to have $T_A > 10^{11}$ K.
 - We note that
 - ▶ $e^{|\tau_\nu|}$ is more strongly peaked on the sky than $|\tau_\nu|$ - the angular size of the maser is less than the actual transverse dimension of the maser region.
 - ▶ $e^{|\tau_\nu|}$ is more strongly in ν than $|\tau_\nu|$ - the maser line is narrower than the actual velocity distribution of the maser species.
 - Some maser can be very bright, allowing the use of interferometry, as well as observations of sources at large distances.
 - ▶ This has enabled measurements of proper motion of maser spots in star-forming regions of the Milky Way, as well as in material orbiting a supermassive black hole in the spiral galaxy NGC 4258 (Hernstein et al. 1999).

Collisional Excitation & De-excitation

- **Collisional Rate (Two Level Atom)**

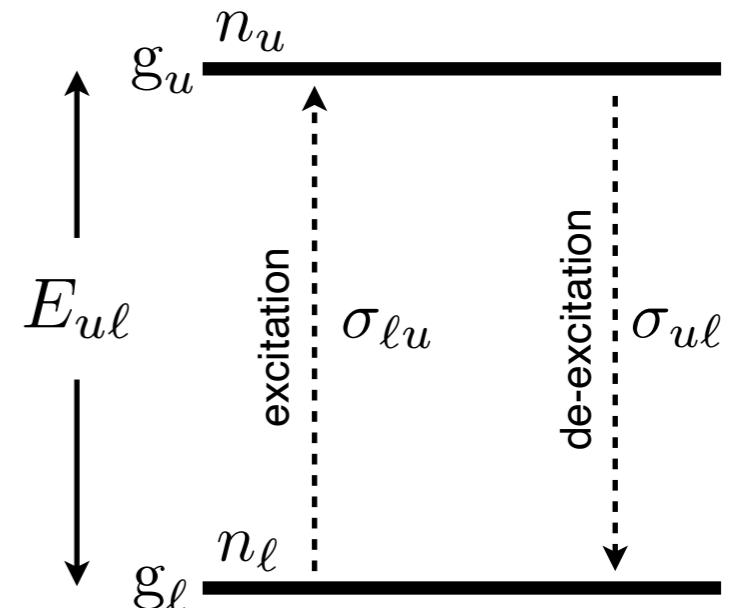
- ▶ The cross section $\sigma_{\ell u}$ for collisional excitation from a lower level ℓ to an upper level u is, in general, inversely proportional to the impact energy (or v^2) above the energy threshold E_{ul} and is zero below.
- ▶ The collisional cross section can be expressed in the following form using a dimensionless quantity called the ***collision strength*** $\Omega_{\ell u}$:

$$\begin{aligned}\sigma_{\ell u}(v) &= (\pi a_0^2) \left(\frac{hR_H}{\frac{1}{2}m_e v^2} \right) \frac{\Omega_{\ell u}}{g_\ell} \text{ cm}^2 \quad \text{for } \frac{1}{2}m_e v^2 > E_{ul} \\ &= \frac{h^2}{4\pi m_e^2 v^2} \frac{\Omega_{\ell u}}{g_\ell}\end{aligned}$$

or $\sigma_{\ell u}(E) = \frac{h^2}{8\pi m_e E} \frac{\Omega_{\ell u}}{g_\ell} \quad \left(E = \frac{1}{2}m_e v^2 \right)$

where, $a_0 = \frac{\hbar^2}{m_e e^2} = 5.12 \times 10^{13}$ cm, Bohr radius

$$R_H = \frac{m_e e^4}{4\pi \hbar^3} = 109,737 \text{ cm}^{-1}, \text{ Rydberg constant} \quad \left(\hbar = \frac{h}{2\pi} \right)$$



- ▶ The collision strength $\Omega_{\ell u}$ is a function of electron velocity (or energy) but is often approximately constant near the threshold. Here, g_ℓ and g_u are the statistical weights of the lower and upper levels, respectively.

- Advantage of using the collision strength is that (1) it removes the primary energy dependence for most atomic transitions and (2) they have the symmetry between the upper and the lower states.

The principle of detailed balance states that ***in thermodynamic equilibrium each microscopic process is balanced by its inverse.***

$$n_e n_\ell v_\ell \sigma_{\ell u}(v_\ell) f(v_\ell) dv_\ell = n_e n_u v_u \sigma_{u\ell}(v_u) f(v_u) dv_u$$

Here, v_ℓ and v_u are related by $\frac{1}{2}m_e v_\ell^2 = \frac{1}{2}m_e v_u^2 + E_{u\ell}$, and $f(v)$ is a Maxwell velocity distribution of electrons. Using the Boltzmann equation of thermodynamic equilibrium,

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right)$$

we derive the following relation between the cross-sections for excitation and de-excitation are

$$g_\ell v_\ell^2 \sigma_{\ell u}(v_\ell) = g_u v_u^2 \sigma_{u\ell}(v_u) \quad \text{Here, } \frac{1}{2}m_e v_\ell^2 = \frac{1}{2}m_e v_u^2 + E_{u\ell} \rightarrow g_\ell \cdot (E + E_{u\ell}) \cdot \sigma_{\ell u}(E + E_{u\ell}) = g_u \cdot E \cdot \sigma_{u\ell}(E)$$

and the symmetry of the collision strength between levels. where $E = \frac{1}{2}m_e v_u^2$

$$\Omega_{\ell u} = \Omega_{u\ell}$$

more precisely $\Omega_{\ell u}(E + E_{u\ell}) = \Omega_{u\ell}(E)$

These two relations were derived in the TE condition. However, ***the cross-sections are independent on the assumptions, and thus the above relations should be always satisfied.***

► Collisional excitation and de-excitation rates

The ***collisional de-excitation rate per unit volume per unit time, which is thermally averaged,*** is

$$\left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell} = n_e n_u \int_0^\infty v \sigma_{u\ell}(v) f(v) dv \\ = n_e n_u k_{u\ell} \quad [\text{cm}^{-3} \text{ s}^{-1}]$$

$$k_{u\ell} = \int_0^\infty v \sigma_{u\ell}(v) f(v) dv \\ = \left(\frac{2\pi\hbar^4}{k_B m_e^3} \right)^{1/2} T^{-1/2} \frac{\langle \Omega_{u\ell} \rangle}{g_u} \\ = \frac{8.62942 \times 10^{-6}}{T^{1/2}} \frac{\langle \Omega_{u\ell} \rangle}{g_u} \quad [\text{cm}^3 \text{ s}^{-1}],$$

$$k_{u\ell} \equiv \langle \sigma v \rangle_{u \rightarrow \ell}$$

effective collision strength:

$$\langle \Omega_{u\ell} \rangle \equiv \int_0^\infty \Omega_{u\ell}(E) e^{-E/k_B T} d(E/k_B T)$$

and the ***collisional excitation rate per unit volume per unit time*** is

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = n_e n_\ell k_{\ell u}$$

$$k_{\ell u} \equiv \langle \sigma v \rangle_{\ell \rightarrow u}$$

$$k_{\ell u} = \int_{v_{\min}}^\infty v \sigma_{\ell u}(v) f(v) dv \quad \text{Here, } \frac{1}{2} m_e v_{\min}^2 = E_{u\ell} \\ = \left(\frac{2\pi\hbar^4}{k_B m_e^3} \right)^{1/2} T^{-1/2} \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} \exp\left(-\frac{E_{u\ell}}{k_B T}\right)$$

Here, $k_{\ell u}$ and $k_{u\ell}$ are the collisional rate coefficient for excitation and de-excitation coefficients in units of $\text{cm}^3 \text{ s}^{-1}$, respectively. We also note that ***the rate coefficients for collisional excitation and de-excitation are related by***

$$k_{\ell u} = \frac{g_u}{g_\ell} k_{u\ell} \exp\left(-\frac{E_{u\ell}}{k_B T}\right) \quad \langle \sigma v \rangle_{\ell \rightarrow u} = \frac{g_u}{g_\ell} \langle \sigma v \rangle_{u \rightarrow \ell} \exp\left(-\frac{E_{u\ell}}{k_B T}\right)$$

Sum rule for collision strengths

- Quantum mechanical sum rule for collision strengths for the case where one term consists of a singlet ($S = 0$ or $L = 0$) and the second consists of a multiplet: the collision strength of each fine structure level J is related to the total collision strength of the multiplet by

$$\Omega_{(SLJ, S'L'J')} = \frac{(2J' + 1)}{(2S' + 1)(2L' + 1)} \Omega_{(SL, S'L')}$$

Here, $(2J' + 1)$ is the statistical weight of an individual level in the multiplet, and $(2S' + 1)(2L' + 1)$ is the statistical weight of the multiplet term.

We can regard the collision strength as “shared” amongst these levels in proportion to the statistical weights of the individual levels ($g_J = 2J + 1$).

- The flux ratio between the lines in a multiplet is proportional to the ratio of their collision strengths, in a low density medium.*** Then, the flux ratio is determined by the ratio of their statistical weights.

- C-like ions ($1s^2 2s^2 2p^2 \rightarrow 1s^2 2s^2 2p^2$) forbidden or inter combination transitions.

ground states (triplet) - ${}^3P_0 : {}^3P_1 : {}^3P_2 = 1 : 3 : 5$

excited states (singlets) - ${}^1D_2, {}^1S_1$

- Li-like ions ($1s^2 2s^1 \rightarrow 1s^2 2p^1$) resonance transitions

ground state (singlet) - ${}^2S_{1/2}$

excited states (doublet) - ${}^2P_{3/2} : {}^2P_{1/2} = 2 : 1$

Collisionally-Excited Emission Line

- Emission line flux

- In the low density limit, the collisional rate between atoms and electrons is much slower than the (spontaneous) radiative de-excitation rate of the excited level. Thus, we can balance the collisional feeding into level u by the rate of radiative transition back down to level ℓ . The level population is determined by

$$n_e n_\ell k_{\ell u} = A_{u\ell} n_u$$

$$\frac{n_u}{n_\ell} = \frac{n_e k_{\ell u}}{A_{u\ell}}$$

$$= \frac{n_e}{A_{u\ell}} \beta \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} T^{-1/2} \exp\left(-\frac{E_{u\ell}}{kT}\right)$$

where $A_{u\ell}$ is the Einstein coefficient for spontaneous emission. The line emissivity is given by

$$4\pi j_{u\ell} = E_{u\ell} A_{u\ell} n_u = E_{u\ell} n_e n_\ell k_{\ell u}$$

$$= n_e n_\ell E_{u\ell} \frac{8.62942 \times 10^{-6}}{T^{1/2}} \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right) \text{ [erg cm}^{-3} \text{ s}^{-1}\text{]}$$

$$\simeq \beta \chi n_e^2 E_{u\ell} T^{-1/2} \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right)$$

Here, $\beta = \left(\frac{2\pi\hbar^4}{km_e^2}\right)^{1/2} = 8.62942 \times 10^{-6}$
 $\chi = n_\ell/n_e$

For low temperature, the exponential term dominates because few electrons have energy above the threshold for collisional excitation, so that the line rapidly fades with decreasing temperature.

At high temperature, the $T^{-1/2}$ term controls the cooling rate, so the line fades slowly with increasing temperature.

-
- ▶ In **high-density limit**, the level population are set by the Boltzmann equilibrium, and the line emissivity is

$$\begin{aligned} \frac{n_u}{n_\ell} &= \frac{g_u}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right) \\ 4\pi j_{u\ell} &= E_{\ell u} A_{u\ell} n_u \\ &= n_\ell E_{\ell u} A_{u\ell} \frac{g_u}{g_\ell} \exp\left(-\frac{E_{\ell u}}{kT}\right) \\ &\simeq \chi n_e E_{\ell u} A_{u\ell} \frac{g_u}{g_\ell} \exp\left(-\frac{E_{\ell u}}{kT}\right) \end{aligned}$$

Here, the line flux scales as n_e rather than n_e^2 , but the line flux tends to a constant value at high temperature.

- ▶ **Critical density** is defined as the density where the radiative depopulation rate matches the collisional de-excitation for the excited state.

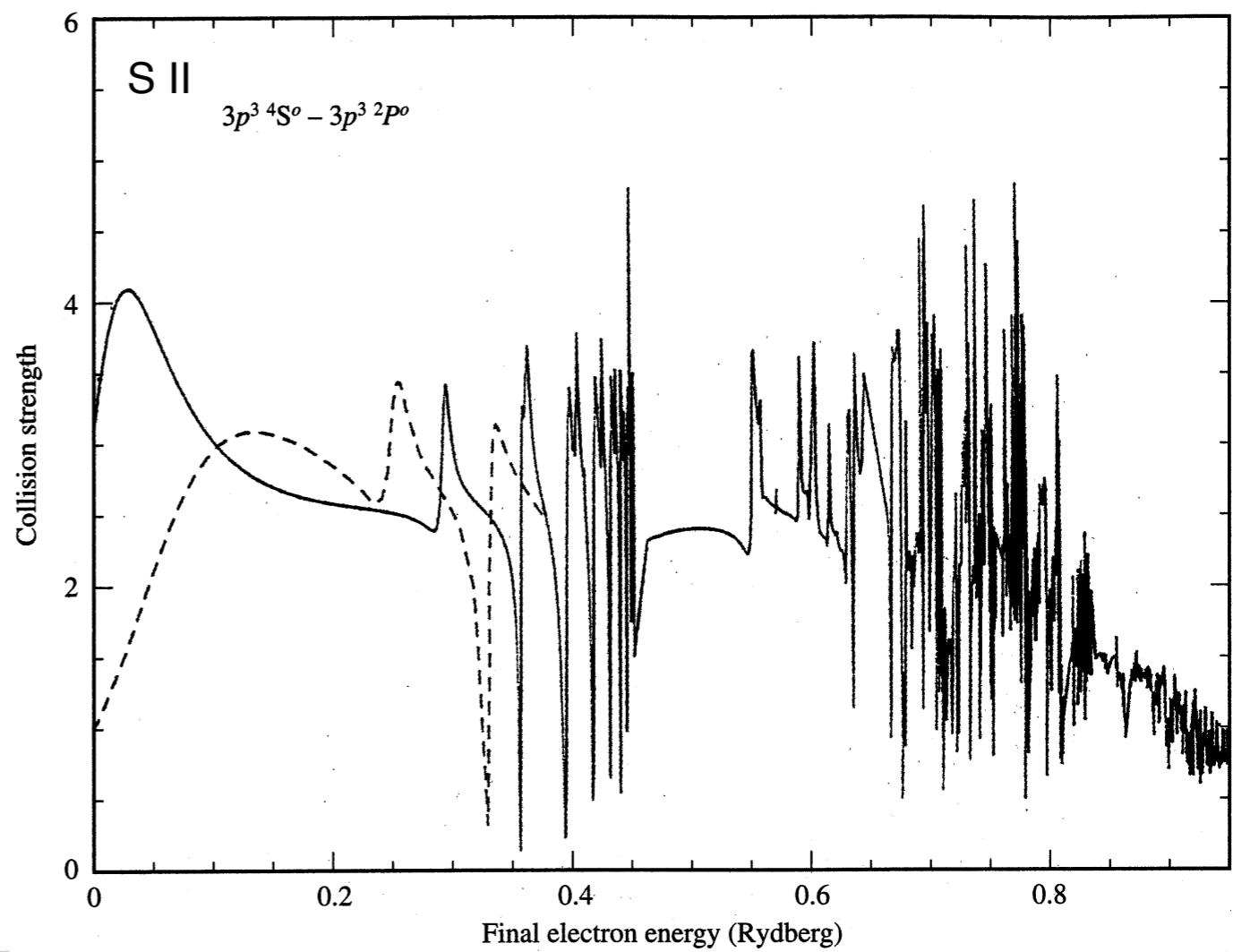
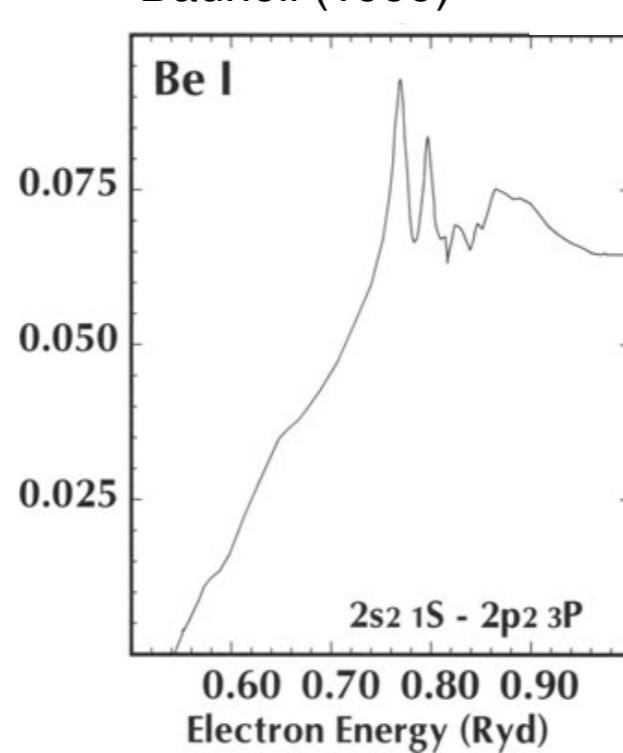
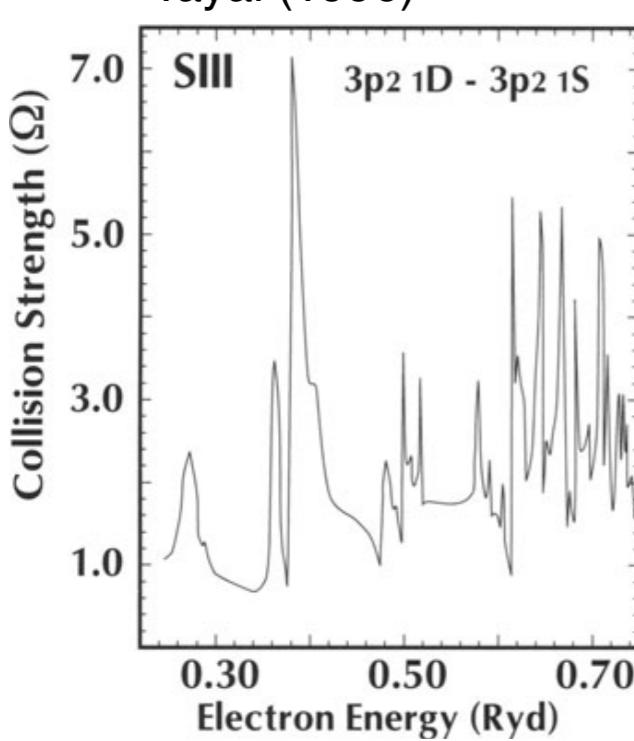
$$\begin{aligned} A_{u\ell} n_u &= n_e n_u k_{u\ell} \\ n_{\text{crit}} &= \frac{A_{u\ell}}{k_{u\ell}} \end{aligned}$$

$$\begin{aligned} \rightarrow n_{\text{crit}} &= A_{u\ell} \frac{g_u}{\beta \langle \Omega_{u\ell} \rangle} T^{1/2} \\ &= 1.2 \times 10^3 \frac{A_{u\ell}}{10^{-4} \text{ s}^{-1}} \frac{g_u}{\langle \Omega_{u\ell} \rangle} \left(\frac{T}{10^4 \text{ K}} \right)^{1/2} [\text{cm}^{-3}] \end{aligned}$$

- ▶ At densities higher than the critical density, collisional de-excitation becomes significant, and the forbidden lines will be weaker as the density increases.

At around the critical density, the “line emissivity vs density” plotted in log-log scale changes slope from +2 to +1.

- Collision Strength
 - Quantum mechanical calculations show that (1) the resonance structure in the collision strengths is important and (2) the collision strength increases with energy for neutral species.



The **effective collision strength**, which is thermally averaged, has a value in a range of

$$\langle \Omega_{ul} \rangle = \int_0^\infty \Omega_{ul}(E) e^{-E/k_B T} d(E/k_B T)$$

$$10^{-2} < \langle \Omega_{ul} \rangle < 10$$

See Table F.1 to F.5 in [Draine]

- As can be seen in Tables and the formula, collisional de-excitation is negligible for resonance and most forbidden lines in the ISM.

Ion	ℓ	u			$n_{H,\text{crit}}(u)$	
			E_ℓ/k (K)	E_u/k (K)	$\lambda_{u\ell}$ (μm)	$T = 100\text{ K}$ (cm^{-3})
C II	$^2\text{P}_{1/2}^o$	$^2\text{P}_{3/2}^o$	0	91.21	157.74	2.0×10^3
CI	$^3\text{P}_0$	$^3\text{P}_1$	0	23.60	609.7	620
	$^3\text{P}_1$	$^3\text{P}_2$	23.60	62.44	370.37	720
O I	$^3\text{P}_2$	$^3\text{P}_1$	0	227.71	63.185	2.5×10^5
	$^3\text{P}_1$	$^3\text{P}_0$	227.71	326.57	145.53	8.4×10^3
Si II	$^2\text{P}_{1/2}^o$	$^2\text{P}_{3/2}^o$	0	413.28	34.814	1.0×10^5
Si I	$^3\text{P}_0$	$^3\text{P}_1$	0	110.95	129.68	4.8×10^4
	$^3\text{P}_1$	$^3\text{P}_2$	110.95	321.07	68.473	9.9×10^4
						1.5×10^3

Table 17.1 in [Draine]

- However, it is not true for the 21 cm hyperfine structure line of hydrogen.
 - The critical density for 21cm line is
- $n_{\text{crit}} \sim 10^{-3} (T/100\text{ K})^{-1/2} [\text{cm}^{-3}]$
- $A_{u\ell} = 2.88 \times 10^{-15} [\text{s}^{-1}]$
- The hyperfine levels are thus essentially in collisional equilibrium in the CNM.

The collisional strengths and other atomic data are available in the CHIANTI atomic database (<https://www.chiantidatabase.org/>).

Ion	Transition l-u	λ μm	A_{ul} s^{-1}	Ω_{ul}	n_{crit} cm^{-3}
C I	$^3\text{P}_0 - ^3\text{P}_1$	609.1354	7.93×10^{-8}	–	(500)
	$^3\text{P}_1 - ^3\text{P}_2$	370.4151	2.65×10^{-7}	–	(3000)
C II	$^2\text{P}_{1/2} - ^2\text{P}_{3/2}$	157.741	2.4×10^{-6}	1.80	47 (3000)
	$^3\text{P}_0 - ^3\text{P}_1$	205.3	2.07×10^{-6}	0.41	41
N II	$^3\text{P}_1 - ^3\text{P}_2$	121.889	7.46×10^{-6}	1.38	256
	$^3\text{P}_2 - ^1\text{D}_2$	0.65834	2.73×10^{-3}	2.99	7700
	$^3\text{P}_1 - ^1\text{D}_2$	0.65481	9.20×10^{-4}	2.99	7700
N III	$^2\text{P}_{1/2} - ^2\text{P}_{3/2}$	57.317	4.8×10^{-5}	1.2	1880
O I	$^3\text{P}_2 - ^3\text{P}_1$	63.184	8.95×10^{-5}	–	$2.3 \times 10^4 (5 \times 10^5)$
	$^3\text{P}_1 - ^3\text{P}_0$	145.525	1.7×10^{-5}	–	$3400 (1 \times 10^5)$
	$^3\text{P}_2 - ^1\text{D}_2$	0.63003	6.3×10^{-3}	–	1.8×10^6
O II	$^4\text{S}_{3/2} - ^2\text{D}_{5/2}$	0.37288	3.6×10^{-5}	0.88	1160
	$^4\text{S}_{3/2} - ^2\text{D}_{3/2}$	0.37260	1.8×10^{-4}	0.59	3890
O III	$^3\text{P}_0 - ^3\text{P}_1$	88.356	2.62×10^{-5}	0.39	461
	$^3\text{P}_1 - ^3\text{P}_2$	51.815	9.76×10^{-5}	0.95	3250
	$^3\text{P}_2 - ^1\text{D}_2$	0.50069	1.81×10^{-2}	2.50	6.4×10^5
	$^3\text{P}_1 - ^1\text{D}_2$	0.49589	6.21×10^{-3}	2.50	6.4×10^5
	$^1\text{D}_2 - ^1\text{S}_0$	0.43632	1.70	0.40	2.4×10^7
	$^2\text{P}_{1/2} - ^2\text{P}_{3/2}$	12.8136	8.6×10^{-3}	0.37	5.9×10^5
Ne II	$^3\text{P}_2 - ^3\text{P}_1$	15.5551	3.1×10^{-2}	0.60	1.27×10^5
	$^3\text{P}_1 - ^3\text{P}_0$	36.0135	5.2×10^{-3}	0.21	1.82×10^4
Si II	$^2\text{P}_{1/2} - ^2\text{P}_{3/2}$	34.8152	2.17×10^{-4}	7.7	(3.4×10^5)
S II	$^4\text{S}_{3/2} - ^2\text{D}_{5/2}$	0.67164	2.60×10^{-4}	4.7	1240
	$^4\text{S}_{3/2} - ^2\text{D}_{3/2}$	0.67308	8.82×10^{-4}	3.1	3270
S III	$^3\text{P}_0 - ^3\text{P}_1$	33.4810	4.72×10^{-4}	4.0	1780
	$^3\text{P}_1 - ^3\text{P}_2$	18.7130	2.07×10^{-3}	7.9	1.4×10^4
S IV	$^2\text{P}_{1/2} - ^2\text{P}_{3/2}$	10.5105	7.1×10^{-3}	8.5	5.0×10^4
Ar II	$^2\text{P}_{1/2} - ^2\text{P}_{3/2}$	6.9853	5.3×10^{-2}	2.9	1.72×10^6
	$^3\text{P}_2 - ^3\text{P}_1$	8.9914	3.08×10^{-2}	3.1	2.75×10^5
Ar III	$^3\text{P}_1 - ^3\text{P}_0$	21.8293	5.17×10^{-3}	1.3	3.0×10^4
	$^6\text{D}_{7/2} - ^6\text{D}_{5/2}$	35.3491	1.57×10^{-3}	–	(3.3×10^6)
Fe II	$^6\text{D}_{9/2} - ^6\text{D}_{7/2}$	25.9882	2.13×10^{-3}	–	(2.2×10^6)

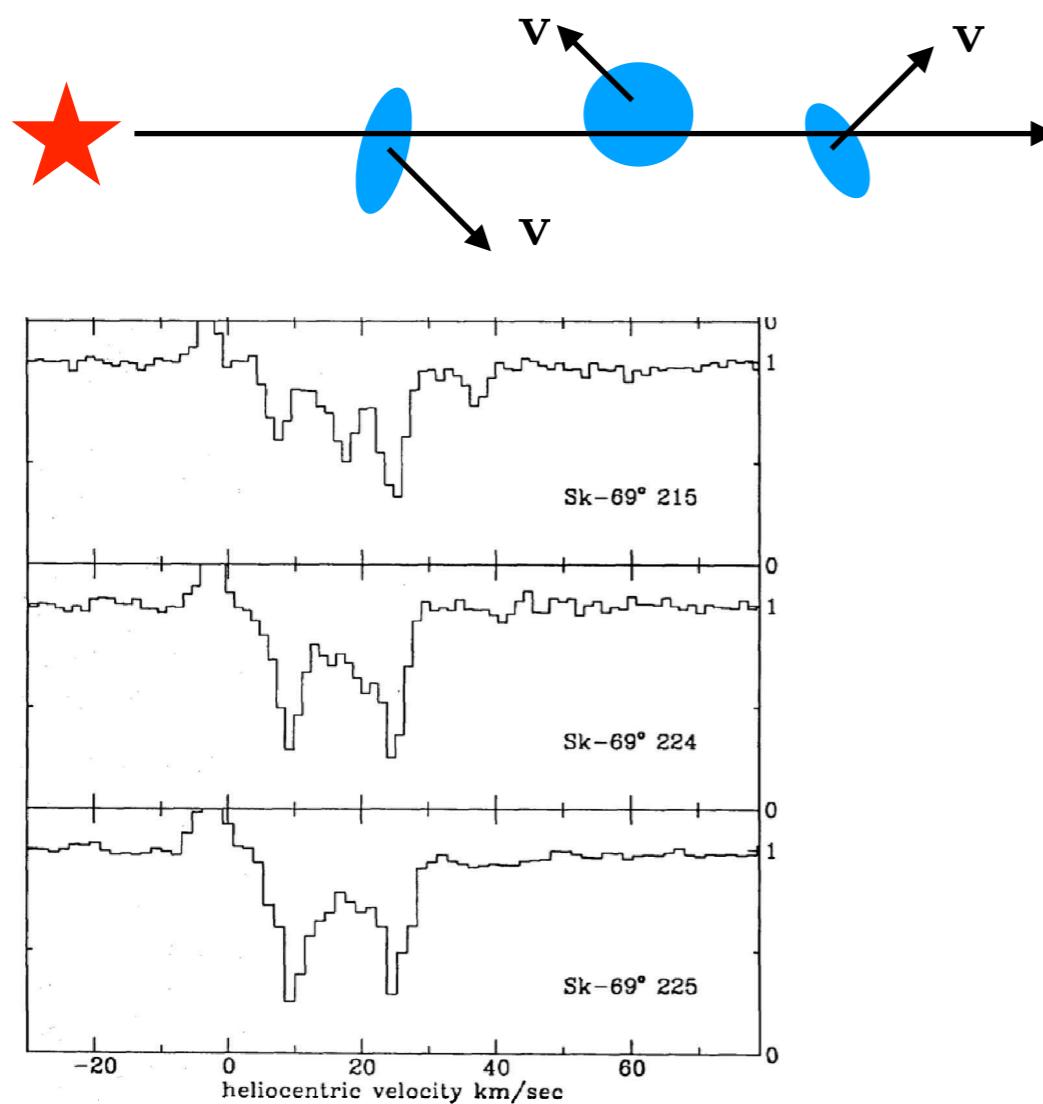
Overall Properties of the Cold Neutral Medium (CNM)

- Overall properties of the CNM
 - Temperature $T \sim 100 \text{ K}$
 - Mean kinetic energy per particle $\langle E \rangle = (3/2)kT \sim 0.013 \text{ eV}$
 - Number density
 - ▶ $n_{\text{atom}} \sim 30 \text{ cm}^{-3}$ for atoms
 - ▶ $n_e \sim 0.04 \text{ cm}^{-3}$ for free electrons
 - Thermal velocity
 - ▶ $v_{\text{th}}(\text{H}) \sim 1.6 \text{ km s}^{-1}$ for hydrogen atoms
 - ▶ $v_{\text{th}}(e) \sim 67 \text{ km s}^{-1}$ for free electrons
 - Mean free path
 - ▶ $\lambda_{\text{mfp}}(\text{HH}) \sim 0.74 \text{ AU}$ for atom-atom collisions
 - ▶ $\lambda_{\text{mfp}}(e\text{H}) \sim 1700 \text{ AU}$ for atom-electron collisions
 - ▶ $\lambda_{\text{mfp}}(ee) \sim 1.9 \times 10^{-3} \text{ AU}$ for electron-electron collisions
 - Collisional time scale
 - ▶ $t_{\text{coll}}(\text{HH}) \sim 2.2 \text{ yr}$ for atom-atom collisions
 - ▶ $t_{\text{coll}}(e\text{H}) \sim 120 \text{ yr}$ for atom-electron collisions
 - ▶ $t_{\text{coll}}(ee) \sim 1.2 \text{ hr}$ for electron-electron collision

See [lecture05 - collisional time scale.pdf](#)
for the detailed calculations of the numerical values

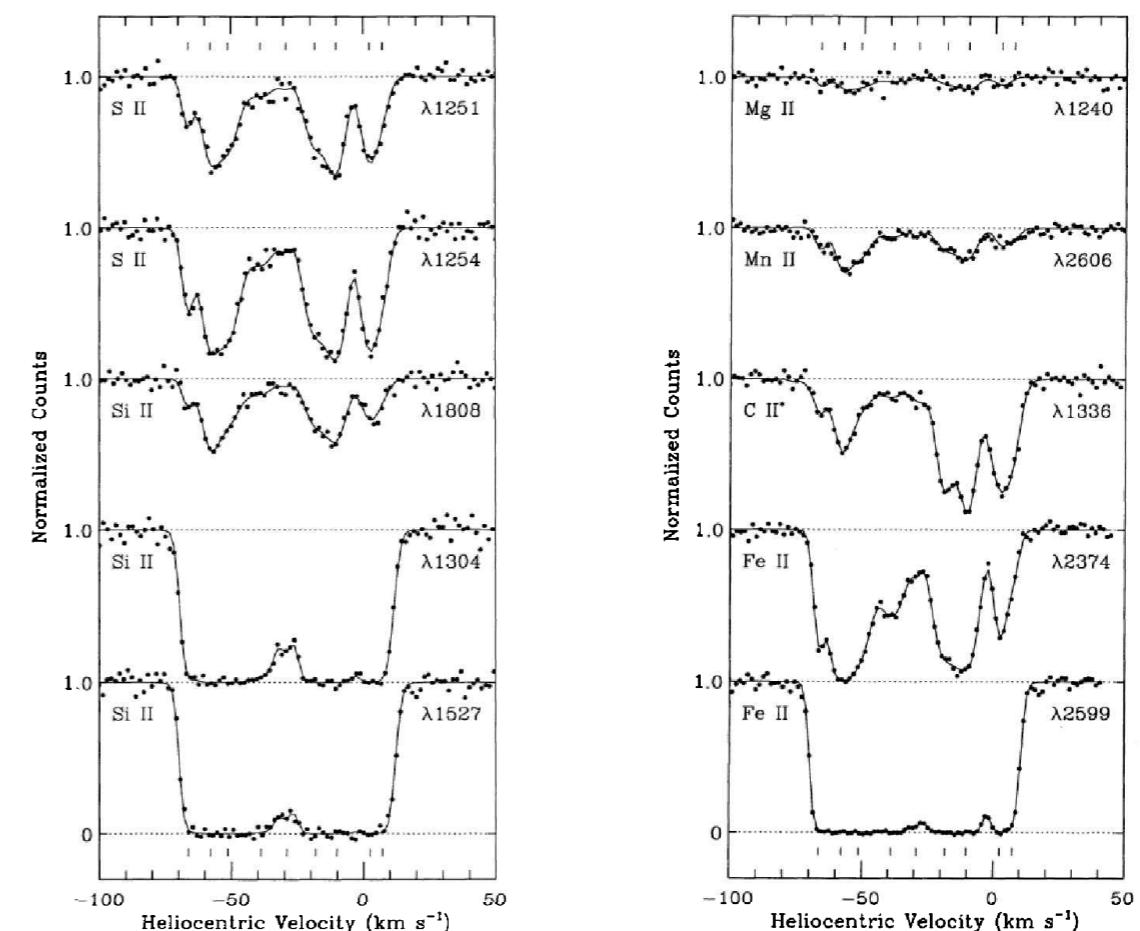
Observations of Absorption Lines Toward the CNM

- The CNM gives rise to a number of absorption features in the spectra of hot background stars (and quasars).
 - The most prominent absorption lines at visible wavelengths are Ca II K and H lines at $\lambda = 3933, 3968 \text{ \AA}$, and Na I D₁ and D₂ doublet lines at $\lambda = 5890, 5896 \text{ \AA}$.



Na I D₂ interstellar absorption line seen along 3 lines of sight to stars in LMC (Molaro et al. 1993)

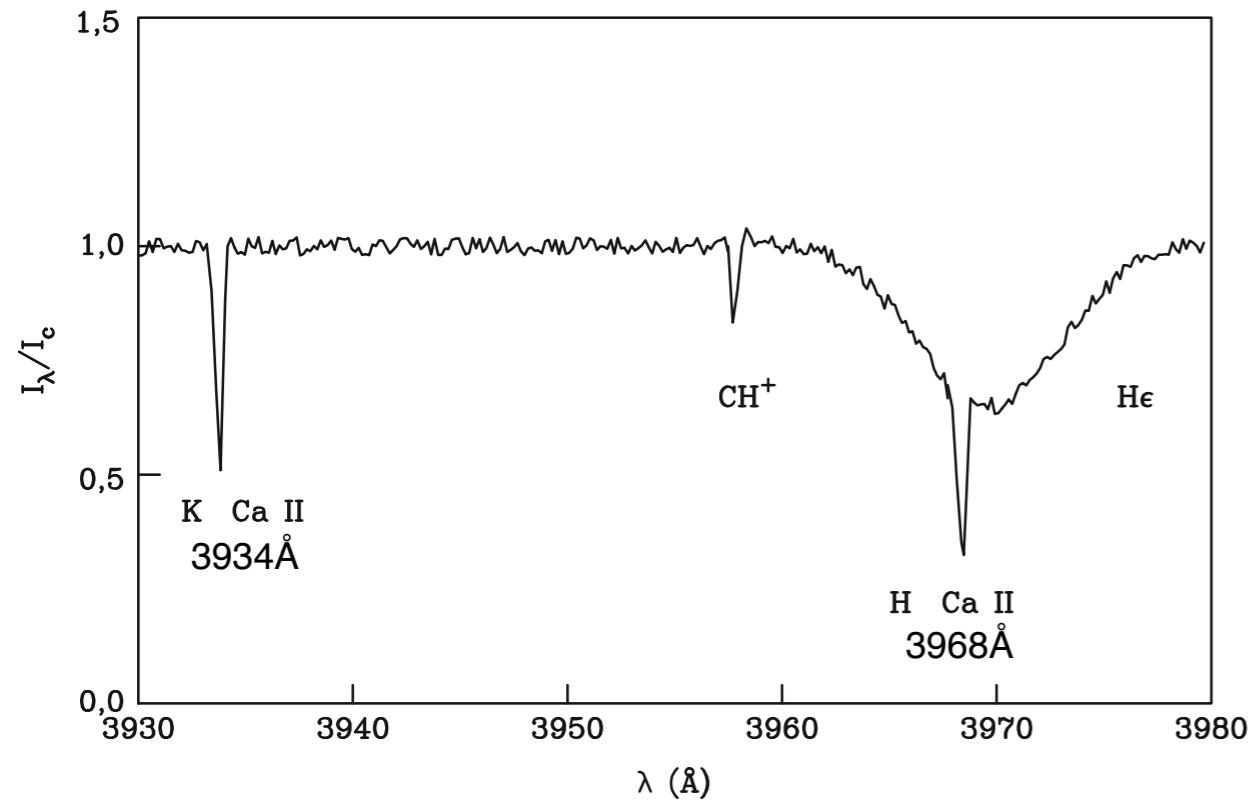
[Note] The cold gas is ~ 100 pc away from Earth, meaning that 5 arcmin corresponds to ~ 0.15 pc.



UV interstellar absorption lines toward an O-type star HD93521. (Spitzer & Fitzpatrick 1993)

[Note] (1) multiple velocity components and (2) line saturation on Si II and Fe II.

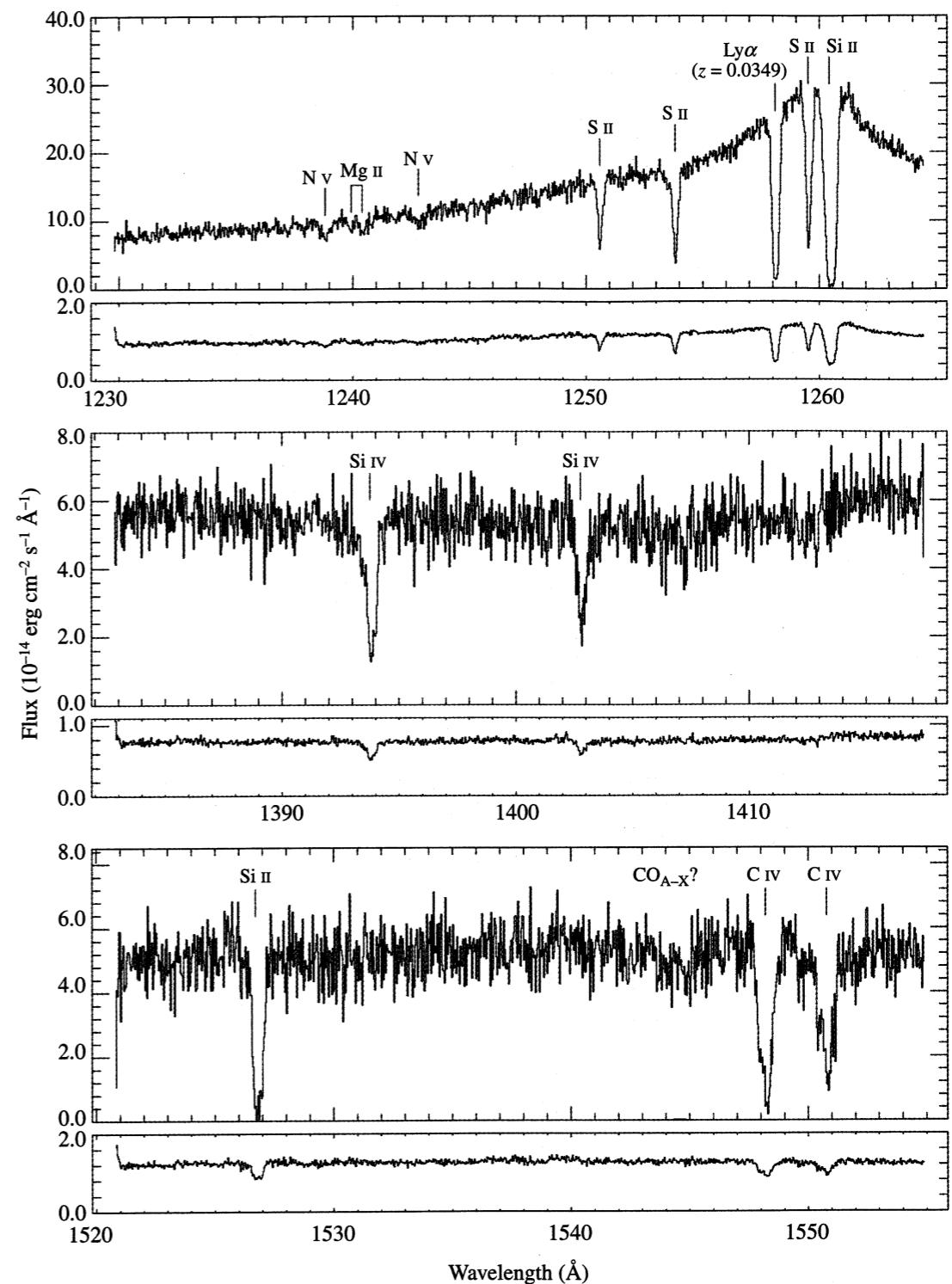
The multiple velocities are due primarily to the differential rotation of our galaxy. (clouds at different distances)



Interstellar absorption lines in the spectrum of ζ Oph (O9.5V).

Note that the Ca II H line occurs inside the H ϵ hydrogen line, which is much broader and of stellar origin.

Figure 4.6 in Astrophysics of the Interstellar Medium [Maciel]



Interstellar absorption lines toward the Seyfert 1 galaxy ESO 141-055.

Figure 5.5 in Physics and Chemistry of the Interstellar Medium [Kwok]

-
- The alkali metals (Li, K, and Na) and alkaline earth metals (Ca) produce absorption lines at visible wavelengths ($4000 \text{ \AA} < \lambda < 7300 \text{ \AA}$, $1.7 \text{ eV} < E < 3.1 \text{ eV}$); these elements have loosely bound outer electrons.
 - Most other elements produce UV absorption lines ($\lambda < 4000 \text{ \AA}$, $E > 3.1 \text{ eV}$).
 - Therefore, the study of the CNM was extensively made by the launch of orbiting UV telescopes (Copernicus, IUE, etc).
 - In particular, Ly α ($\lambda = 1215.67 \text{ \AA}$; $E = 10.2 \text{ eV}$) from hydrogen.
 - Interstellar absorption lines at visible wavelengths were also found from neutral atoms such as Ca I, K I, Li I, ions such as Ti II, and diatomic molecules such as CH, NH, CN, CH $^+$ and C $_2$.
 - [Note] The first discovery of interstellar molecules was made by the detection of CH absorption at $\lambda \sim 4300$ (4315) \AA (Swings & Rosenfeld 1937), not at radio wavelengths.
 - CH, NH, and CN are referred to as “**radicals**”, in chemistry, meaning molecules that contain at least one unpaired electron. They quickly combine with one another, or with single atoms in laboratory. But, in the low density of the ISM, they have long lifetimes.

-
- The composition and excitation of interstellar gas can be studied using absorption lines that appear in the spectra of background stars (or other sources).
 - The interstellar lines are typically narrow compared to spectral features produced by absorption in stellar photospheres, and in practice can be readily distinguished.
 - For instance, consider the Fraunhofer lines in the Sun's spectrum. The Ca H and K lines, with equivalent widths of 14Å and 19Å respectively, are the strongest absorption lines. The Na I D₁ and D₂ lines have equivalent widths of 0.6Å and 0.8Å.
 - For many interstellar lines, the equivalent width is sufficiently small that the mÅ is a convenient unit.
 - It is normally possible to detect absorption only by the ground state (and perhaps the excited fine-structure levels of the ground electronic state) - the populations in the excited electronic states are too small to be detected in absorption.
 - The widths of absorption lines are usually determined by Doppler broadening, with line widths of a few km s⁻¹ (or $\Delta\lambda/\lambda \approx 10^{-5}$) - often observed in cool clouds.
 - Absorption lines (and emission lines) contains a lots of information about number density, temperature, chemical abundances, ionization states, and excitation states.
 - However, interpreting the information requires understanding the ways in which light interacts with baryonic matter, radiative transfer.
 - **We need to know the line profile to analyze absorption lines.**

Optical Depth

- The optical depth in an absorption line can be written

$$\tau_\nu = \frac{\pi e^2}{m_e c} f_{\ell u} \left(1 - \frac{n_u/g_u}{n_\ell/g_\ell} \right) N_\ell \phi_\nu \simeq \frac{\pi e^2}{m_e c} f_{\ell u} N_\ell \phi_\nu$$

Here, $N_\ell \equiv \int n_\ell ds$ is the column density of the absorbers.

The line profile is given by $\phi_\nu = \frac{1}{\Delta\nu_D \sqrt{\pi}} H(u, a)$, and its value at the line center is

$$\begin{aligned} \phi_\nu(\nu = \nu_{\ell u}) &= \frac{1}{\nu_{\ell u}(b/c)\sqrt{\pi}} H(0, a) & u &= \frac{\nu - \nu_{\ell u}}{\Delta\nu_D} = \frac{\nu - \nu_{u\ell}}{\nu_{\ell u}(b/c)} \\ &\approx \frac{1}{\nu_{\ell u}(b/c)\sqrt{\pi}} & &= \frac{v}{b} \quad \left(v = \frac{\nu - \nu_{\ell u}}{\nu_{\ell u}} c, b = \sqrt{2}v_{\text{rms}} = \sqrt{\frac{2k_B T}{m}} \right) \end{aligned}$$

The correction factor for stimulated emission is negligible for the optical lines. Then, dropping the correction factor, the optical depth can be written

$$\tau_\nu = \tau_0 H(u, a)$$

Here, τ_0 is the optical depth at the line center.

$$\tau_0 = \frac{\sqrt{\pi}e^2}{m_e c} f_{\ell u} \frac{\lambda_{\ell u}}{b} N_\ell$$

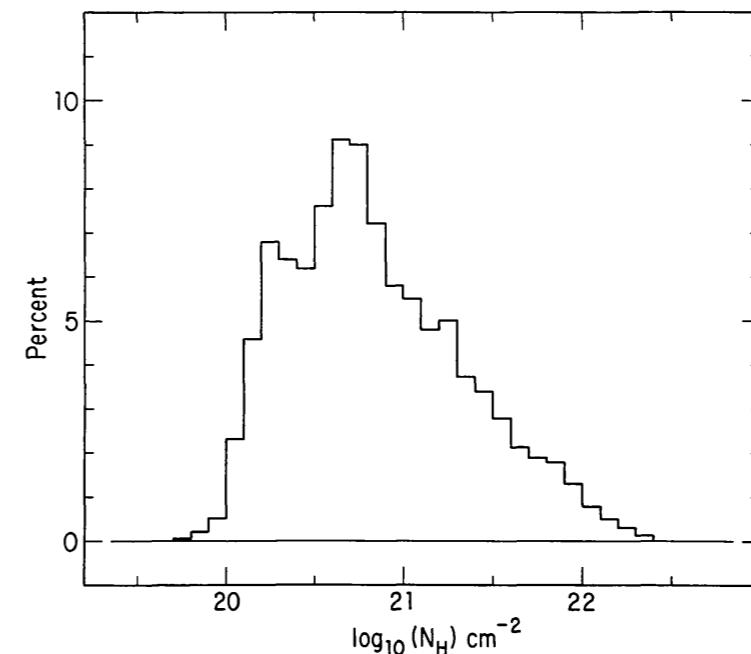
- The central optical depth for Ly α is

$$\tau_0 = 0.7580 \left(\frac{N_\ell}{10^{13} \text{ cm}^{-2}} \right) \left(\frac{f_{\ell u}}{0.4164} \right) \left(\frac{\lambda_{\ell u}}{1215.67 \text{ \AA}} \right) \left(\frac{10 \text{ km s}^{-1}}{b} \right)$$

- In the WNM, Ly α will be optically thin ($\tau_0 < 1$) when $N_\ell < 10^{13} \text{ cm}^{-2}$ and optically thick ($\tau_0 > 1$) when $N_\ell > 10^{13} \text{ cm}^{-2}$.
- In the CNM, Ly α will be optically thin when $N_\ell < 10^{12} \text{ cm}^{-2}$ and optically thick when $N_\ell > 10^{12} \text{ cm}^{-2}$.
- In Milky Way, the total column density of hydrogen atom is $N_\ell \sim 10^{20} - 10^{21} \text{ cm}^{-2}$.

The percentage of the sky covered by H I at a given N_H .

Figure 4 in Dickey & Lockman (1990, ARA&A)



- As a reference, the column density of the Earth's atmosphere, looking upward from sea level, is $N \sim 2 \times 10^{25} \text{ cm}^{-2}$.

Absorption Line Shapes

- Lyman α absorption line profiles for $b = 10 \text{ km s}^{-1}$

$$F_\nu/F_\nu(0) = e^{-\tau_0 H(u,a)}$$

- When $\tau_0 < 1$, $F_\nu/F_\nu(0) \approx 1 - \tau_\nu$ and thus the shape of an absorption line resembles the upside-down Voight function.
- When $\tau_0 \gg 1$, the absorption line saturates at its center and becomes increasingly “box-shaped.”

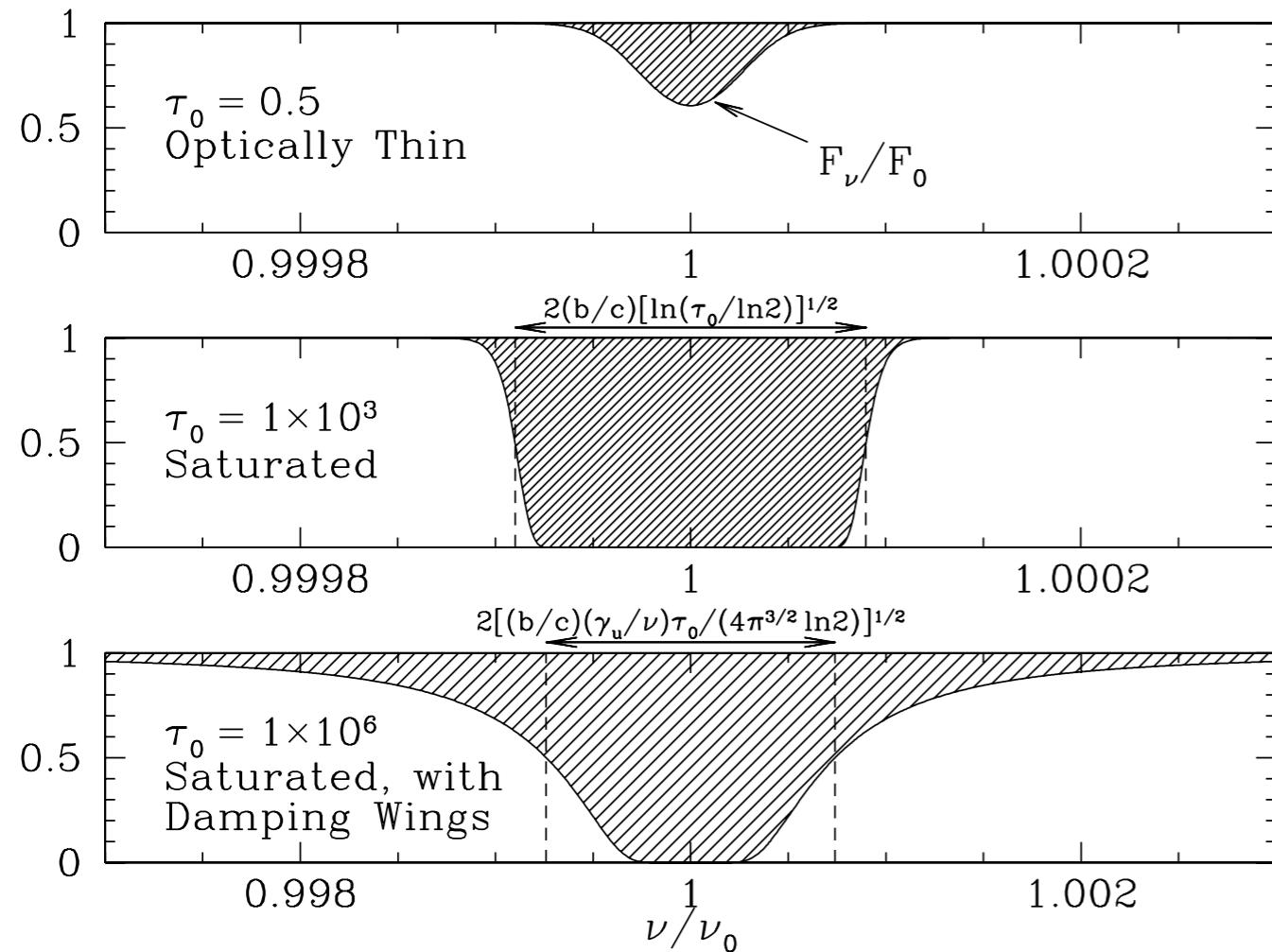
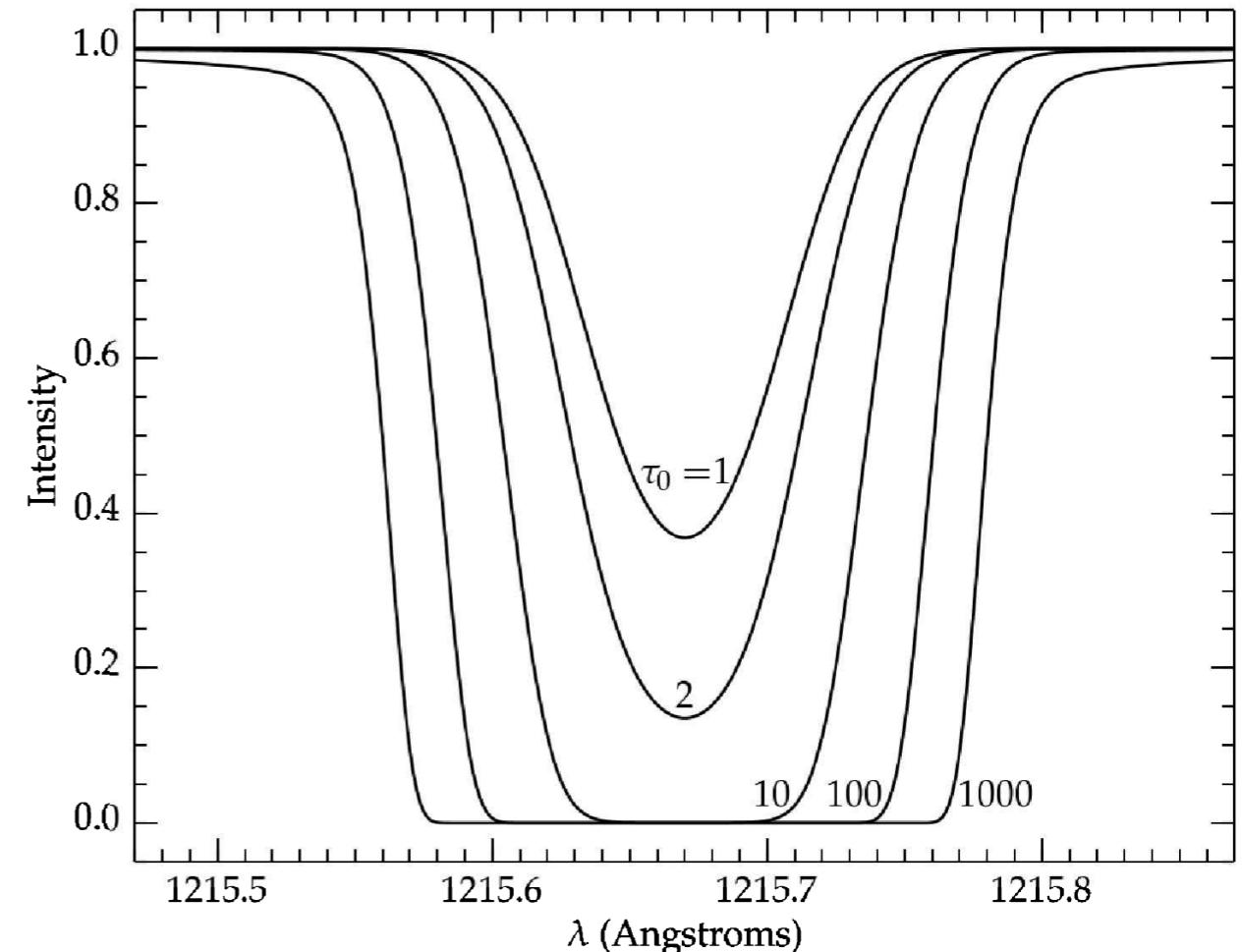


Figure 9.1 in [Draine]

Note the different abscissa in the lowest panel.



Lyman α absorption lines for $b = 10 \text{ km s}^{-1}$.

Figure 2.6 in [Ryden]

Homework (due date: 04/17)

from Draine's problems

- [Q5]** 1 A local HI cloud is interposed between us and the cosmic microwave background with temperature $T_{\text{CMB}} = 2.7255 \text{ K}$. Suppose that the HI in the cloud has a spin temperature $T_{\text{spin}} = 50 \text{ K}$, and that the optical depth at line-center (of the 21 cm line) is $\tau = 0.1$. The cloud is extended. We observe the cloud with a radio telescope.
- What will be the (absolute) brightness temperature T_B at line-center of the 21 cm line? Express your answer in deg K. You may assume that $h\nu \ll kT_B$.
 - What will be the (absolute) intensity at line-center of the 21 cm line? Express your answer in Jy sr^{-1} .
- 2 Consider a photon of frequency $h\nu$ entering a slab of material containing two-level atoms with excitation temperature T_{ul} . At the frequency of the photon, let the optical depth of the slab be τ .
- Let P_{abs} be the probability that the original photon will undergo absorption before exiting from the slab. Give an expression for P_{abs} in terms of τ
- Hint: here, τ is the optical depth for pure absorption.
- Consider a photon that crossed the slab without being absorbed. Let $P_{\text{stim.em.}}$ be the probability that the incident photon will stimulate emission of one or more photons. Give an expression for $P_{\text{stim.em.}}$ in terms of τ and $h\nu/kT_{ul}$.
- 3 Suppose that we have a molecule with three energy levels – denoted 0, 1, 2 – ordered according to increasing energy, $E_0 < E_1 < E_2$. Let g_0, g_1, g_2 be the degeneracies of the levels. Suppose that there is radiation present with $h\nu = E_2 - E_0$, due to an external source plus emission in the $2 \rightarrow 0$ transition.
- Let ζ_{02} be the absorption probability per unit time for a molecule in level 0, with a transition to level 2. Let A_{20}, A_{21} , and A_{10} be the Einstein A coefficients for decays $2 \rightarrow 0$, $2 \rightarrow 1$, and $1 \rightarrow 0$ by spontaneous emission of a photon. Ignore collisional processes.
- Hint: this mean no external radiation field for $2 \rightarrow 1$ and $1 \rightarrow 0$
- Ignoring possible absorption of photons in the $2 \rightarrow 1$ and $1 \rightarrow 0$ transitions, obtain an expression for the ratio n_1/n_0 , where n_i is the number density of molecules in level i .
 - How large must ζ_{02} be for this molecule to act as a maser in the $1 \rightarrow 0$ transition?
 - Is it possible for this system to have maser emission in the $2 \rightarrow 1$ transition? If so, what conditions must be satisfied?
- Hint: find a relation between A_{21} and A_{10} to make the solution self-consistent.

[Q6] Voigt profile

- We want to derive an approximate formula for the transition point from the Gaussian core to the Lorentz wing, which is defined by

$$u^2 = \ln(\sqrt{\pi}/a) + \ln u^2 \quad \text{or} \quad x = \ln(\sqrt{\pi}/a) + \ln x, \text{ where } x \equiv u^2$$

The above equation can be expressed in the form:

$$x = g(x) \text{ where } g(x) = \ln x + \ln(\sqrt{\pi}/a)$$

This equation can be solved using "**Fixed Point Iteration Method.**" Starting from any initial point x_0 , the following recursive process gives an approximate solution of the equation.

$$x_{n+1} = g(x_n)$$

- (1) Find a numerical solution x_* for Ly α line with $b = 10 \text{ km s}^{-1}$, which is appropriate for Ly α in the WNM with $T \sim 10000 \text{ K}$.
- (2) Let's denote the width parameter as a_* for $b = 10 \text{ km s}^{-1}$. This means that

$$x_* = \ln x_* + \ln(\sqrt{\pi}/a_*)$$

Now, for any parameter a which is different from a_* , you may express the constant term in $g(x)$ as follows:

$$\ln(\sqrt{\pi}/a) = \ln(a_*/a) + \ln(\sqrt{\pi}/a_*)$$

To find the solution for $a \neq a_*$ (but, $a \approx a_*$), choose an initial guess to be $x_0 = x_*$. Show that the solution for any a can be expressed as (after only a single iteration):

$$x_1 = x_* + \ln(a_*/a)$$

Insert numerical values into the above equation and compare it with Eq. (2.39) in Ryden's book (our textbook).

- (3) Insert numerical values into the above equation compare it with the results in this lecture note and Eq. (6.42) in Draine's book.