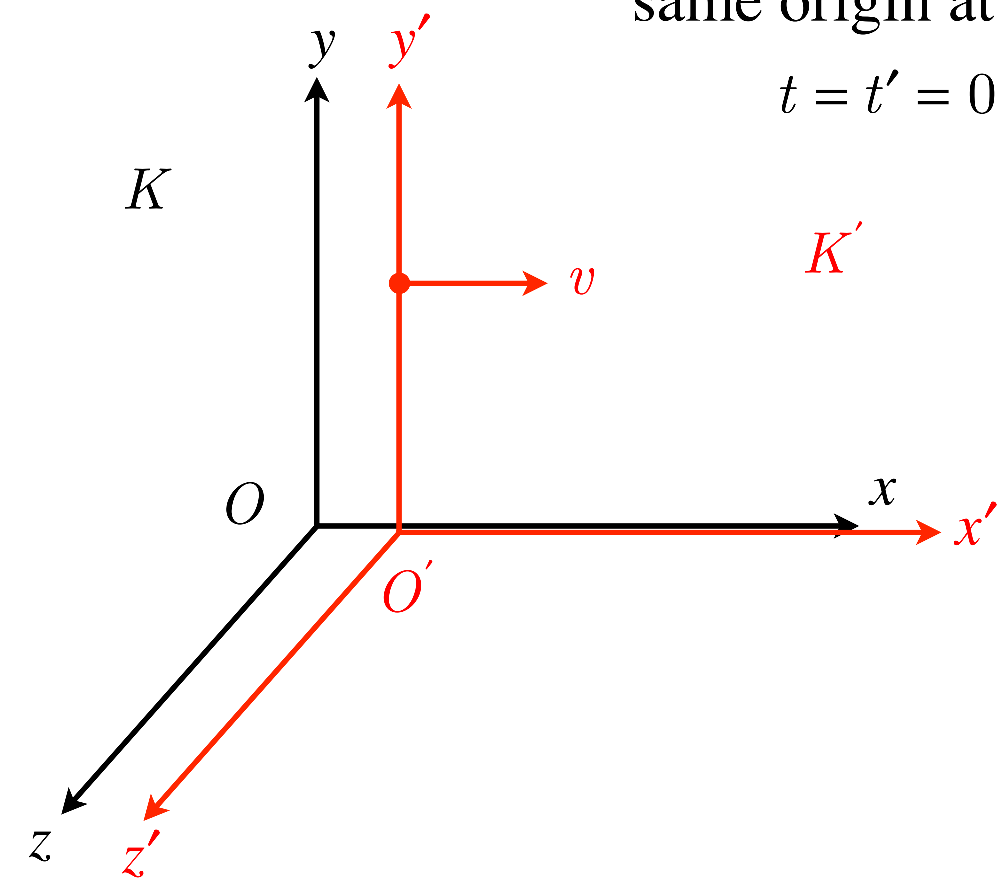
Relativistic Covariance and Kinematics

Galilean Transformation/Relativity

- Galilean transformation is used to transform between the coordinates of two **inertial frames of reference** which differ only by constant relative motion within the constructs of Newtonian physics.

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$



Newton's law is invariant under the Galilean transformation.

However, Maxwell'e equations are not invariant under the Galilean transformation.

- Lorentz transformation is the result of attempts by Lorentz and others to explain how the speed of light was observed to be independent of the reference frame, and to understand the symmetries of the Maxwell's equations.

* Review of Lorentz Transformations *

- **Postulates in the special theory of relativity**

  (1) The laws of nature are the same in two frames of reference in uniform relative motion with no rotation.

  (2) The speed of light is c in all such frames.

- **space-time event**: an event that takes place at a location in space and time.
- **Derivation of Lorentz transforms**:

  If a pulse of light is emitted at the origin at $t = 0$, each observer will see an expanding sphere centered on his own origin. Therefore, we have the equations of the expanding sphere in each frame.

$$x^2 + y^2 + z^2 - c^2t^2 = 0, \quad x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \qquad (1)$$

  Since space is assumed to be homogeneous, the transformation must be linear.

$$x' = a_1 x + a_2 t, \quad y' = y, \quad z' = z, \quad t' = b_1 x + b_2 t$$

  We note that the origin of K' $(x' = 0)$ is a point that moves with speed v as seen in K. Its location in K is given by $x = vt$. Therefore, we have

$$\frac{a_2}{a_1} = -v \qquad \begin{aligned} x' &= a_1(x - vt) \\ y' &= y \\ z' &= z \\ t' &= b_1 x + b_2 t \end{aligned} \qquad (2)$$

Substitute Eqs. (2) into Eq. (1): $x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$

$$a_1{}^2(x - vt)^2 + y^2 + z^2 - c^2(b_1x + b_2t)^2 = x^2 + y^2 + z^2 - c^2t^2$$

$$(a_1{}^2 - c^2b_1{}^2)x^2 - 2(a_1{}^2v + c^2b_1b_2)xt + (a_1{}^2v^2 - c^2b_2{}^2)t^2 + y^2 + z^2 = x^2 + y^2 + z^2 - c^2t^2$$

(Note: we didn't assume that $x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$)

Therefore, the following equations should be satisfied.

$$\begin{aligned} a_1{}^2 - c^2b_1{}^2 &= 1 && (a) \\ (a_1{}^2v + c^2b_1b_2) &= 0 && (b) \\ a_1{}^2v^2 - c^2b_2{}^2 &= -c^2 && (c) \end{aligned} \longrightarrow \begin{aligned} &(a) \quad b_1{}^2 = \frac{a_1{}^2 - 1}{c^2} \\ &(c) \quad b_2{}^2 = 1 + \frac{v^2}{c^2}a_1{}^2 \end{aligned}$$

$$(b) \quad a_1{}^4v^2 = -c^4b_1{}^2b_2^2 = c^2a_1{}^2 + v^2a_1{}^4 - c^2 - v^2a_1{}^2$$

$$\rightarrow \quad a_1 = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \equiv \gamma$$

$$\longrightarrow \quad (a) \quad b_1 = \gamma, \quad (c) \quad b_2 = -\frac{v}{c^2}\gamma$$

Finally, we obtain the Lorentz transformation (and its inverse):

$$\begin{aligned} x' &= \gamma(x - vt) & x &= \gamma(x' + vt') \\ y' &= y & y &= y' \\ z' &= z & z &= z' \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) & t &= \gamma\left(t' + \frac{v}{c^2}x'\right) \end{aligned}$$

where $\gamma \equiv \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \left(1 - \beta^2\right)^{-1/2}$; $\beta \equiv \dfrac{v}{c}$

Lorentz factor $1 \le \gamma < \infty$; $0 \le \beta < 1$

Length Contraction / Time Dilation

- **Length contraction** (Lorentz-Fitzgerald contraction): Suppose a rigid rod of length $L_0 = x_2' - x_1'$ is carried at rest in K'. What is the length as measured in K? The positions of the ends of the rod are marked at the same time in K.

  $$L_0 = x_2' - x_1' = \gamma(x_2 - x_1) = \gamma L$$
  $$L = L_0 / \gamma$$

  Therefore, the rod appears shorter by a factor $1/\gamma$ in K.

  If both carry rods (of the same length when compared at rest) each thinks the other's rod has shrunk!

  It would appear to K' that the two ends of the moving stick were not marked at the same time by the other observer (in K).

- **Time dilation**: Suppose a clock at rest at the origin of K' measures off a time interval $T_0 = t_2' - t_1'$. What is the time interval measured in K? Note that the clock is at rest at the origin of K' so that $x_2' = x_1' = 0$.

  $$T = t_2 - t_1 = \gamma(t_2' - t_1') = \gamma T_0$$
  $$T = \gamma T_0$$

  The time interval has increased by a factor γ, so that the moving clock appears to have slowed down.

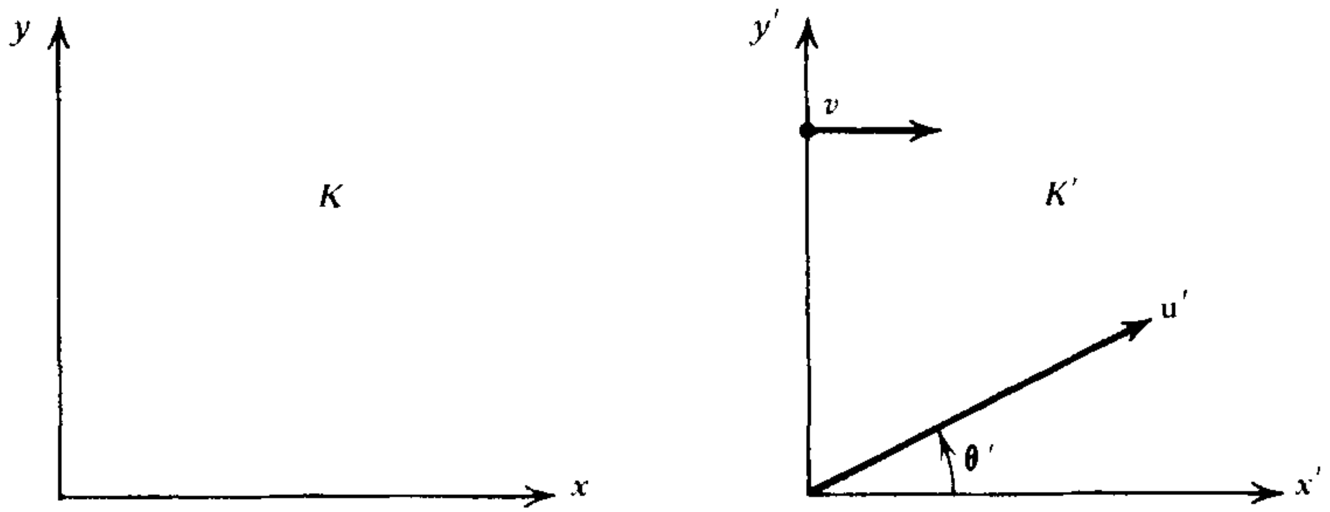
  Time dilation is detected in the increased half-lives of unstable particles moving rapidly in an accelerator or in the cosmic-ray flux.

Transformation of Velocities

- Simultaneity is relative: Simultaneous events at two different spatial points in the primed frame is not simultaneous in the unprimed frame.
- If a point has a velocity $\mathbf{u}'$ in frame K', what is its velocity $\mathbf{u}$ in frame K. Writing Lorentz transformations for differentials

$$dx = \gamma(dx' + v\,dt'), \quad dy = dy', \quad dz = dz'$$

$$dt = \gamma\left(dt' + \frac{v}{c^2}dx'\right)$$



$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma(dt' + v\,dx'/c^2)} = \frac{u_x' + v}{1 + vu_x'/c^2}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + v\,dx'/c^2)} = \frac{u_y'}{\gamma(1 + vu_x'/c^2)}$$

$$u_z = \frac{dz}{dt} = \frac{u_z'}{\gamma(1 + vu_x'/c^2)}$$

or
$$u_{\|} = \frac{u_{\|}' + v}{1 + vu_{\|}'/c^2}$$

$$u_{\perp} = \frac{u_{\perp}'}{\gamma(1 + vu_{\|}'/c^2)}$$

- Aberration formula: the directions of the velocities in the two frames are related by

$$\tan\theta = \frac{u_\perp}{u_\parallel} = \frac{u_\perp{}'}{\gamma(u_\parallel{}' + v)} = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)} \qquad \text{where } u' \equiv |\mathbf{u}'|.$$

- **Aberration of light**

For the case of light: $u' = c$

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + v/c)} = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}$$

$$\cos\theta = \frac{\gamma(\cos\theta' + v/c)}{\sqrt{\gamma^2(\cos\theta' + v/c)^2 + \sin^2\theta'}} = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$\sin\theta = \frac{\sin\theta'}{\sqrt{\gamma^2(\cos\theta' + v/c)^2 + \sin^2\theta'}} = \frac{\sin\theta'}{\gamma\left(1 + \beta\cos\theta'\right)}$$

Using the identity, $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$

The aberration formula can be written as: $\tan\left(\frac{\theta}{2}\right) = \frac{(1/\gamma)\sin\theta'}{1 + \beta\cos\theta' + \cos\theta' + \beta} = \frac{(1/\gamma)\sin\theta'}{(1+\beta)(1+\cos\theta')}$

$$\boxed{\tan\left(\frac{\theta}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \tan\left(\frac{\theta'}{2}\right) \quad \rightarrow \quad \theta < \theta'}$$

- **Beaming ("headlight") effect**:

  If photons are emitted isotropically in K', then half will have $\theta' < \pi/2$ and half $\theta' > \pi/2$.

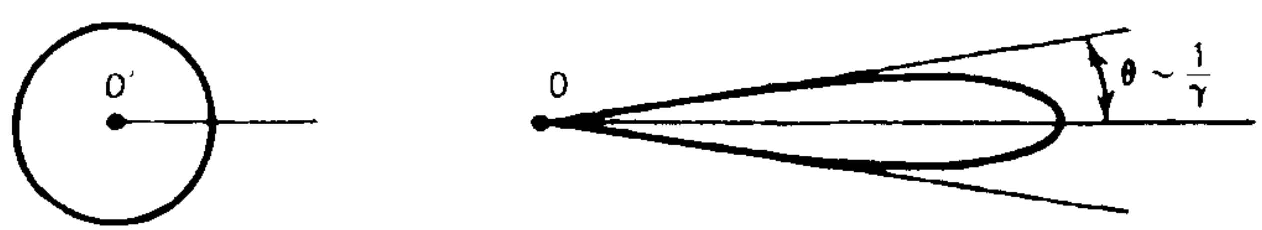
  Consider a photon emitted at right angles to v in K'. Then we have

$$\text{beam half-angle:} \quad \sin\theta_b = \frac{1}{\gamma}, \quad \cos\theta_b = \beta, \quad \text{or} \quad \tan\left(\frac{\theta_b}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}$$

  For highly relativistic speeds, $\gamma \gg 1$, θ_b becomes small:

$$\theta_b \sim \frac{1}{\gamma}$$

  Therefore, in frame K, photons are concentrated in the forward direction, with half of them lying within a cone of half-angle $1/\gamma$. Very few photons will be emitted $\theta \gg 1/\gamma$.



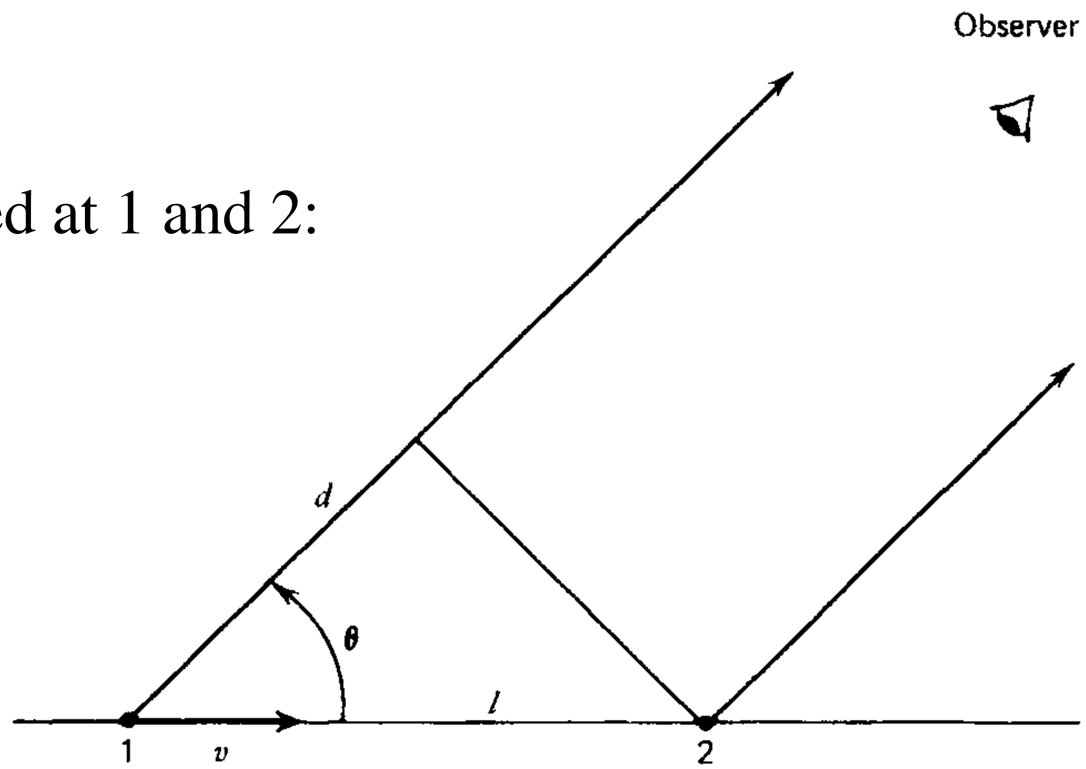
K' K

Doppler Effect

- In the rest frame of the observer K, imagine that the moving source emits one period of radiation as it moves from point 1 to point 2 at velocity v.

  Let frequency of the radiation in the rest frame (K') of the source = ω'. Then the time taken to move from point 1 to point 2 in the observer's frame is given by the time-dilation effect:

$$\Delta t = \Delta t'\gamma = \frac{2\pi}{\omega'}\gamma$$



Difference in arrival times Δt_A of the radiation emitted at 1 and 2:

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t\left(1-\beta\cos\theta\right)$$

Therefore, the observed frequency ω will be

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma(1-\beta\cos\theta)}, \quad \text{or} \quad \boxed{\frac{\omega}{\omega'} = \frac{1}{\gamma(1-\beta\cos\theta)}}$$

Note $1-\beta\cos\theta$ appears even classically. The factor γ^{-1} is purely a relativistic effect.

<u>Transverse (or second-order) Doppler effect :</u>

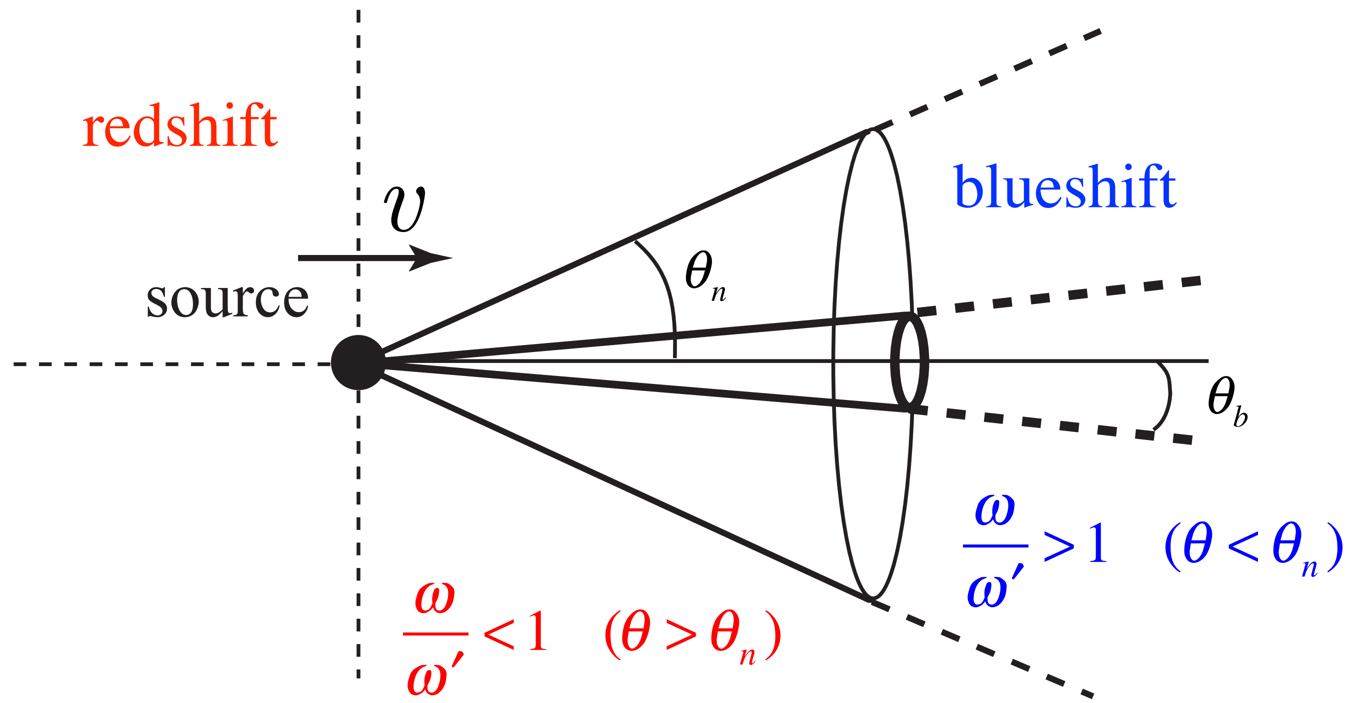
$$\frac{\omega}{\omega'} = \frac{1}{\gamma} \le 1 \quad \text{at} \quad \theta = \pi/2$$

- Beam half-angle: $\sin\theta_b = \gamma^{-1}$
- Angle for null Doppler shift:

$$\frac{\omega}{\omega'} = \frac{1}{\gamma(1-\beta\cos\theta_n)} = 1$$

$$\rightarrow \quad \cos\theta_n = \frac{1-\gamma^{-1}}{\beta} = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{1/2}$$

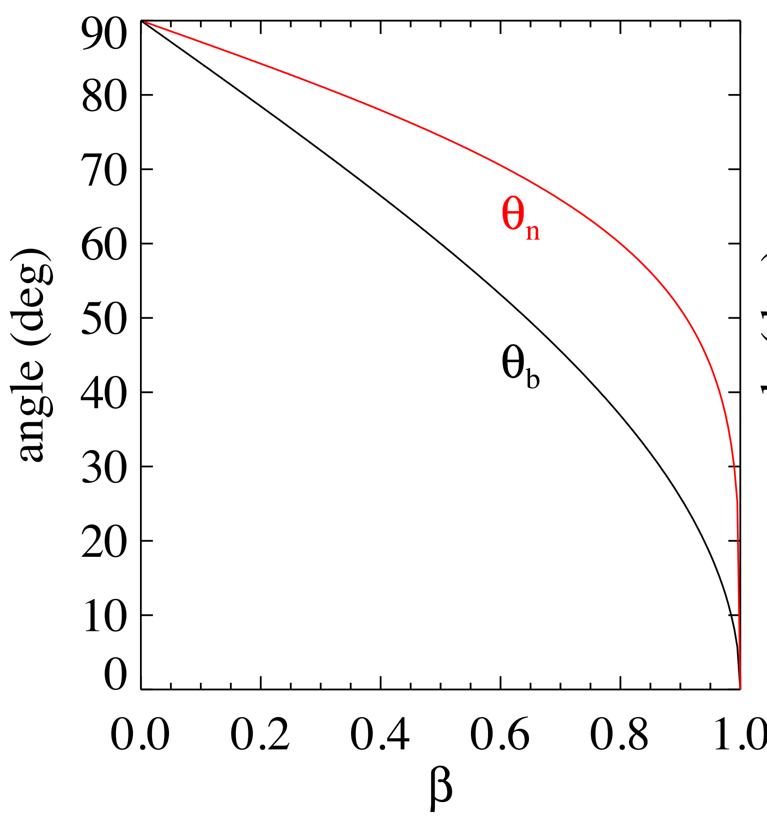
Relativistic Doppler effect can yield redshift even as a source approaches.

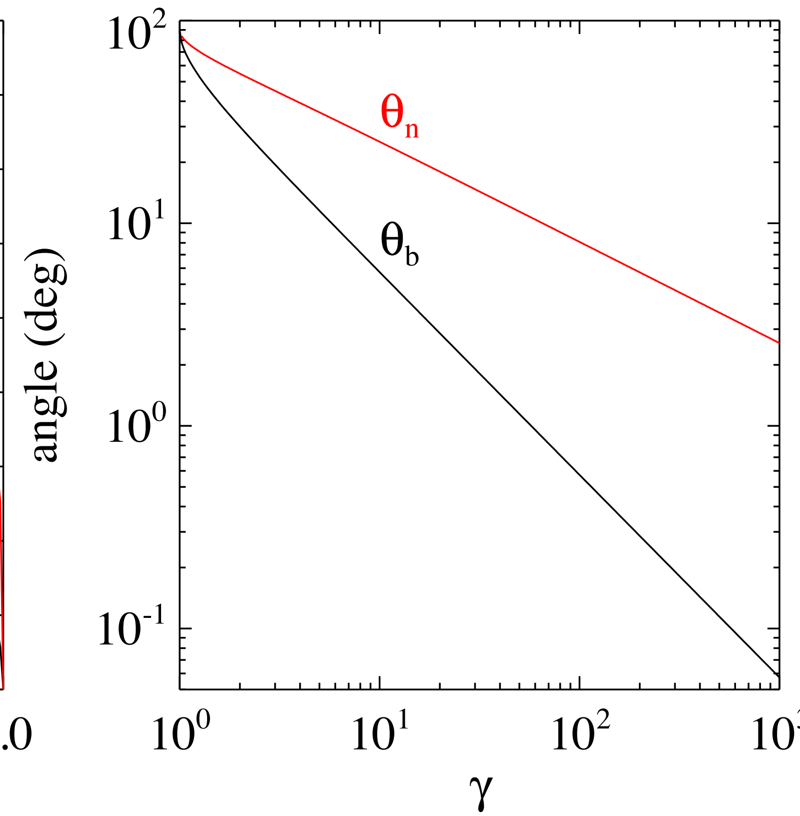


$$\cos\theta_n = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{1/2} \approx 1 - \frac{1}{\gamma} \quad \text{for } \gamma \gg 1$$

$$1 - \frac{\theta_n{}^2}{2} \approx 1 - \frac{1}{\gamma}$$

$$\therefore \quad \theta_n \approx \sqrt{\frac{2}{\gamma}} \approx \sqrt{2\theta_b}$$





- Note $\theta_b \le \theta_n$

Lorentz Invariant

- **Lorentz invariant**: A quantity (scalar) that remains unchanged by a Lorentz transform is said to be a "Lorentz invariant."

$$\begin{aligned} x'^2 + y'^2 + z'^2 - c^2 t'^2 &= \gamma^2 (x - \beta ct)^2 + y^2 + z^2 - \gamma^2 (ct - \beta x)^2 \\ &= \gamma^2 (1 - \beta^2) x^2 + y^2 + z^2 + \gamma^2 (\beta^2 c^2 - c^2) t^2 \\ &= x^2 + y^2 + z^2 - c^2 t^2 \end{aligned}$$

- Proper distance: Since all events are subject to the same transformation, the space-time "interval" between two event is also invariant.

$$ds^2 \equiv dx^2 + dy^2 + dz^2 - c^2 dt^2$$

This is the spatial distance between two events occurring at the same time. This is called the proper distance between the two points.

- Proper time (interval):

$$c^2 d\tau^2 \equiv -ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

This measures time intervals between events occurring at the same spatial location $(dx = dy = dz = 0)$

If the coordinate differentials refer to the position of the origin of another reference frame traveling with velocity v, then $d\tau = dt\left(1 - \beta^2\right)^{1/2} = dt / \gamma$

This is the time dilation formula in which $d\tau$ is the time interval measured by the frame in motion.

* Four-Vectors *

- **Four-vector**: Invariant in 3D rotations: $dx^2 + dy^2 + dz^2$

  By analogy, the invariance of the space-time interval suggests to define a vector in 4D space (4 dimensional space-time vector or four-vector). The quantities x^μ $(\mu = 0,1,2,3)$ define coordinates of an event in space-time.

$$\vec{x} \equiv x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}$$

Contravariant components

- Minkowski space: Space-time is not a Euclidean space; it is called Minkowski space.

  Minkowski metric:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that this metric is symmetric:

$$\eta_{\mu\nu} = \eta_{\nu\mu}$$

- **Summation convention**:

  The invariant can now be written in terms of the Minkowski metric: $s^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\mu x^\nu$

  In any single term containing **a Greek index repeated twice** (between contravariant and covariant indices), a summation is implied over that index (originated by Einstein). This index is often called a dummy index.

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu$$

Note that $\eta_{\mu\mu} x^\mu$ is regarded as meaningless.

- **Contravariant/Covariant components**

contravariant components: (superscripted)
$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

covariant components: (subscripted)
$$x_\mu = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -ct \\ x \\ y \\ z \end{pmatrix}$$

They are related by

$$x_\mu = \eta_{\mu\nu} x^\nu, \quad x^\mu = \eta^{\mu\nu} x_\nu$$
$$s^2 = x^\mu x_\mu$$

The metric can be used to raise or lower indices.

The components of a position (velocity etc.) vector *contra-vary* with a change of basis vectors to compensate. Transformation rules between the following two vector components are inverse. This is the basic idea of "contravariant" and "covariant."

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu, \qquad \frac{\partial A}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial A}{\partial x^\nu}$$

- **Lorentz transform** (corresponding to a boot along the *x* axis)

transformation matrix:
$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lorentz transformation:
$$\boxed{\begin{aligned} x'^\mu &= \Lambda^\mu{}_\nu x^\nu \\ \Lambda^\mu{}_\nu &= \frac{\partial x'^\mu}{\partial x^\nu} \end{aligned}}$$

Any arbitrary Lorentz transformation can be written in the above form, since the spatial 3D rotation necessary to align the *x* axes before and after the boost are also of linear form.

- **Conditions for the Lorentz transformation**:

From the invariance of s^2, we must have

$$\eta_{\mu\nu}x^\mu x^\nu = \eta_{\sigma\tau}x'^\sigma x'^\tau = \eta_{\sigma\tau}\Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu x^\mu x^\nu$$

This can be true for arbitrary x^μ only if

$$\eta_{\mu\nu} = \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \eta_{\sigma\tau} \quad \text{or equivalently} \quad \eta = \Lambda^T \eta \Lambda \quad \text{in matrix form}$$

Taking determinants yields

$$\det \Lambda = \pm 1$$

Proper Lorentz transformations (to keep the right-handness), which rules out reflections.

$$\det \Lambda = 1$$

Isochronous Lorentz transformations (to ensure that the sense of flow of time is the same in frames)

$$\Lambda^0{}_0 \geq 1$$

- **Lorentz transformation** of the covariant component

$$x'_\mu = \eta_{\mu\tau}x'^\tau = \eta_{\mu\tau}\Lambda^\tau{}_\sigma x^\sigma = \eta_{\mu\tau}\Lambda^\tau{}_\sigma \eta^{\sigma\nu} x_\nu$$

$$\boxed{\therefore\ x'_\mu = \widetilde{\Lambda}_\mu{}^\nu x_\nu \quad \text{where} \quad \widetilde{\Lambda}_\mu{}^\nu \equiv \eta_{\mu\tau}\Lambda^\tau{}_\sigma \eta^{\sigma\nu}, \quad \widetilde{\Lambda}_\mu{}^\nu = \frac{\partial x'_\mu}{\partial x_\nu}}$$

- From the invariance of $s^2 = x^\mu x_\mu$:

$$x'^\sigma x'_\sigma = \Lambda^\sigma{}_\nu x^\nu \widetilde{\Lambda}_\sigma{}^\mu x_\mu = \Lambda^\sigma{}_\nu \widetilde{\Lambda}_\sigma{}^\mu x^\nu x_\mu$$

$$\therefore \Lambda^\sigma{}_\nu \widetilde{\Lambda}_\sigma{}^\mu = \delta^\mu{}_\nu$$

$$\therefore \widetilde{\Lambda}_\sigma{}^\mu = \left(\Lambda^{-1}\right)^\mu{}_\sigma$$

where we have introduced the Kronecker delta

$$\delta^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ Identity matrix

- For any arbitrary contravariant components,

$$Q^\mu = \delta^\mu{}_\nu Q^\nu$$

- Note that

$$\eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu{}_\nu$$

- Inverse transform

$$\widetilde{\Lambda}_\sigma{}^\mu \times \left(x'^\sigma = \Lambda^\sigma{}_\nu x^\nu\right) \quad \rightarrow \quad x^\mu = \widetilde{\Lambda}_\sigma{}^\mu x'^\sigma$$

note: $\widetilde{\Lambda}_\sigma{}^\mu = \left(\Lambda^{-1}\right)^\mu{}_\sigma$

Other Four-vectors

- <u>Four-vector:</u>

	contravariant	covariant
$\vec{A} \quad \rightarrow$	$A^\mu = \eta^{\mu\nu} A_\nu$	$A_\mu = \eta_{\mu\nu} A^\nu$
	$A'^\mu = \Lambda^\mu{}_\nu A^\nu$	$A'_\mu = \widetilde{\Lambda}_\mu{}^\nu A_\nu$

- Consider two four-vectors $\vec{A}$ and $\vec{B}$

$$A'^\mu B'_\mu = \Lambda^\mu{}_\nu \widetilde{\Lambda}_\mu{}^\sigma A^\nu B_\sigma = \delta^\sigma{}_\nu A^\nu B_\sigma = A^\nu B_\nu \qquad \rightarrow \qquad \boxed{\vec{A} \cdot \vec{B} = A^\mu B_\mu = A'^\mu B'_\mu}$$

Therefore, the scalar product of any two four-vectors is a Lorentz invariant or scalar. In particular, the "square" of a four vector is an invariant. Thus, our starting point, the invariance of $s^2 = x^\mu x_\mu$, is seen to be a general property of four-vectors.

- Note

$$\begin{aligned} \vec{A} \cdot \vec{A} &> 0 \quad \rightarrow \quad \text{spacelike four-vector} \\ &= 0 \quad \rightarrow \quad \text{light-like (or null) four-vector} \\ &< 0 \quad \rightarrow \quad \text{time-like four-vector} \end{aligned}$$

$A^0 \quad \rightarrow \quad$ time component

$A^i \quad \rightarrow \quad$ space-components (ordinary three-vector)

$$\vec{A} \cdot \vec{B} = -A^0 B^0 + \mathbf{A} \cdot \mathbf{B} = -A^0 B^0 + A^i B_i \quad (i = 1,2,3)$$

Four-velocity

The (infinitesimally small) difference between the coordinates of two events is also a four-vector. Dividing by the proper time yields a four-vector, the four-velocity:

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \longrightarrow \begin{aligned} U^0 &= \frac{cdt}{d\tau} = c\gamma_u \\ U^i &= \frac{dx^i}{d\tau} = \gamma_u u^i \end{aligned}$$

or $\boxed{\vec{U} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}}$ where $\gamma_u \equiv \left(1 - u^2/c^2\right)^{-1/2}, \quad u \equiv \left|\frac{d\mathbf{x}}{dt}\right|$

length of the four-velocity : $\boxed{\vec{U} \cdot \vec{U} = U^\mu U_\mu = -\left(\gamma_u c\right)^2 + \left(\gamma_u \mathbf{u}\right)^2 = -c^2}$

Transformation of the four-velocity:

$$\begin{aligned} U'^0 &= \gamma(U^0 - \beta U^1) \\ U'^1 &= \gamma(-\beta U^0 + U^1) \\ U'^2 &= U^2 \\ U'^3 &= U^3 \end{aligned} \longrightarrow \begin{aligned} \gamma_{u'} c &= \gamma(c\gamma_u - \beta\gamma_u u^1) \\ \gamma_{u'} u'^1 &= \gamma(-\beta c\gamma_u + \gamma_u u^1) \\ \gamma_{u'} u'^2 &= \gamma_u u^2 \\ \gamma_{u'} u'^3 &= \gamma_u u^3 \end{aligned} \longrightarrow \begin{aligned} \gamma_{u'} &= \gamma\gamma_u (1 - vu^1/c^2) \\ \gamma_{u'} u'^1 &= \gamma\gamma_u (u^1 - v) \end{aligned}$$

$\longrightarrow$ velocity component: $u'^1 = \dfrac{u^1 - v}{1 - vu^1/c^2}$ This is the previously derived formula.

speed: $\gamma_{u'} = \gamma\gamma_u \left(1 - \dfrac{vu^1}{c^2}\right)$

Momentum and Energy

- Four-momentum of a particle with a mass m_0 is defined by

$$P^\mu \equiv m_0 U^\mu \qquad P^0 = m_0 c \gamma_v$$
$$P^i = \gamma_v m_0 \mathbf{v}$$

- In the nonrelativistic limit,

$$P^0 c = m_0 c^2 \gamma = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = m_0 c^2 + \frac{1}{2} m_0 v^2 + \cdots$$

Therefore, we interpret $E \equiv P^0 c = \gamma_v m_0 c^2$ as the total energy of the particle.

The quantity $m_0 c^2$ is interpreted as the rest energy of the particle.

Then,

$$\mathbf{p} \equiv \gamma_v m_0 \mathbf{v}, \quad P^\mu = (E/c, \mathbf{p})$$

Since $\vec{U}^2 = -c^2$, we obtain $\vec{P}^2 = -m_0^2 c^2 = -\dfrac{E^2}{c^2} + |\mathbf{p}|^2$

$$E^2 = m_0^2 c^4 + c^2 |\mathbf{p}|^2$$

- Photons are massless, but we can still define

$$P^\mu = (E/c, \mathbf{p}), \quad E = |\mathbf{p}| c \quad \rightarrow \quad \vec{P}^2 = 0$$

Wavenumber vector and frequency

- Quantum relations:

$$E = h\nu = \hbar\omega \qquad \left(\omega = 2\pi\nu \right)$$
$$p = E / c = \hbar k \qquad \left(k = 2\pi / \lambda \right)$$

We can define four wavenumber vector:

$$\vec{k} \equiv \frac{1}{\hbar}\vec{P} = \left(\frac{\omega}{c}, \mathbf{k} \right)$$

Note that it's a null vector:

$$\vec{k} \cdot \vec{k} = |\mathbf{k}|^2 - \omega^2 / c^2 = 0$$

Then, we obtain an invariant:

$$\vec{k} \cdot \vec{x} = k_\mu x^\mu = \mathbf{k} \cdot \mathbf{x} - \omega t$$

Therefore, the phase of the plane wave is an invariant.

- Transform for $\vec{k}$ (Doppler formula):

$$k'^0 = \gamma(k^0 - \beta k^1) \longrightarrow \omega' = \gamma(\omega - \beta c k^1) = \omega\gamma\left(1 - \frac{v}{c}\cos\theta\right)$$
$$k'^1 = \gamma(-\beta k^0 + k^1)$$
$$k'^2 = k^2$$
$$k'^3 = k^3$$

$$k^1 = (\omega / c)\cos\theta$$

* Tensor Analysis *

- Definition:

  zeroth-rank tensor : Lorentz invariant (scalar) $s' = s$

  first-rank tensor : four-vector $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

  second-rank tensor: $T'^{\mu\nu} = \Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau} T^{\sigma\tau}$

- Covariant components and mixed components:

$$T_{\mu\nu} = \eta_{\mu\sigma}\eta_{\nu\tau}T^{\sigma\tau} \qquad T^{\mu}{}_{\nu} = \eta_{\nu\tau}T^{\mu\tau} \qquad T_{\mu}{}^{\nu} = \eta_{\mu\sigma}T^{\sigma\nu}$$

- Transformation rules:

$$\begin{aligned} T'_{\mu\nu} &= \eta_{\mu\alpha}\eta_{\nu\beta}T'^{\alpha\beta} \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}T^{\gamma\delta} \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}\eta^{\gamma\sigma}\eta^{\delta\tau}T_{\sigma\tau} \\ &= \widetilde{\Lambda}_{\mu}{}^{\sigma}\widetilde{\Lambda}_{\nu}{}^{\tau}T_{\sigma\tau} \end{aligned}$$

$$\begin{aligned} T'^{\mu}{}_{\nu} &= \eta_{\nu\alpha}T'^{\mu\alpha} \\ &= \eta_{\nu\alpha}\Lambda^{\mu}{}_{\sigma}\Lambda^{\alpha}{}_{\delta}T^{\sigma\delta} \\ &= \eta_{\nu\alpha}\Lambda^{\mu}{}_{\sigma}\Lambda^{\alpha}{}_{\delta}\eta^{\delta\tau}T^{\sigma}{}_{\tau} \\ &= \Lambda^{\mu}{}_{\sigma}\widetilde{\Lambda}_{\nu}{}^{\tau}T^{\sigma}{}_{\tau} \end{aligned}$$

$$\begin{aligned} T'^{\nu}_{\mu} &= \eta_{\mu\alpha}T'^{\alpha\nu} \\ &= \eta_{\mu\alpha}\Lambda^{\alpha}{}_{\beta}\Lambda^{\nu}{}_{\tau}T^{\beta\tau} \\ &= \eta_{\mu\alpha}\Lambda^{\alpha}{}_{\beta}\Lambda^{\nu}{}_{\tau}\eta^{\beta\sigma}T_{\sigma}{}^{\tau} \\ &= \widetilde{\Lambda}_{\mu}{}^{\beta}\Lambda^{\nu}{}_{\tau}T_{\sigma}{}^{\tau} \end{aligned}$$

- symmetric tensor = a tensor that is invariant under a permutation of its indices.

$$T^{\mu\nu} = T^{\nu\mu}$$

- antisymmetric tensor : if it alternates sign when any two indices of the subset are interchanged.

$$T^{\mu\nu} = -T^{\nu\mu}$$

- Examples of the second-rank tensors

A product of two vectors: $A^{\mu} B^{\nu}$

$$A'^{\mu} B'^{\nu} = \Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\tau} A^{\sigma} B^{\tau}$$

The Minkowski metric: $\eta^{\mu\nu}$

The Kronecker-delta: $\delta^{\mu}{}_{\nu}$

- Higher-rank tensors
  - Addition: $A^{\mu} + B^{\mu}, \quad F^{\mu\nu} + G^{\mu\nu}$
  - Multiplication: $A^{\mu} B^{\nu}, \quad F^{\mu\nu} G_{\sigma\tau}$
  - Raising and Lowering Indices: The metric can be used to change contravariant indices into covariant ones, and vice versa, by the processes of raising and lowering.
  - Contraction: $A^{\mu} B_{\nu} \quad \rightarrow \quad A^{\mu} B_{\mu}$ scalar

$$T^{\mu\nu}{}_{\sigma} \quad \rightarrow \quad T^{\mu\nu}{}_{\nu} \qquad \text{vector}$$

$$T'^{\mu\nu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \widetilde{\Lambda}_{\nu}{}^{\tau} T^{\alpha\beta}{}_{\tau} = \Lambda^{\mu}{}_{\alpha} \delta^{\tau}{}_{\beta} T^{\alpha\beta}{}_{\tau} = \Lambda^{\mu}{}_{\alpha} T^{\alpha\beta}{}_{\beta}$$

  - Gradients of Tensor Fields: A tensor field is a tensor that is a function of the spacetime coordinates in Cartesian coordinate systems. The <u>gradient operation $\partial / \partial x^{\mu} \equiv \partial_{\mu}$</u> acting on such a field produces a tensor field of on higher rank with <u>μ as a new covariant index</u>.

$$\lambda \quad \rightarrow \quad \frac{\partial \lambda}{\partial x^{\mu}} \equiv \partial_{\mu} \lambda \equiv \lambda,_{\mu} \quad \text{vector (gradient)} \qquad A^{\mu} \quad \rightarrow \quad \frac{\partial A^{\mu}}{\partial x^{\mu}} \equiv \partial_{\mu} A^{\mu} \equiv A^{\mu},_{\mu} \quad \text{scalar (divergence)}$$

  - **Invariance of form or Lorentz covariance or covariance**: A fundamental property of a tensor equation is that if it is true in one Lorentz frame, then it is true in all Lorentz frames. Covariance plays a powerful role in helping decide what the proper equations of physics are.

Mathematical Formulae

- Gamma function

$$\Gamma(x) \equiv \int_0^\infty t^{x-1} e^{-t}\, dt, \quad \Gamma(x) = (x-1)! = (x-1)\Gamma(x-2), \quad \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

- Euler-Mascheroni constant

$$\gamma \equiv \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) = -\int_0^\infty e^{-x}\ln x\, dx = 0.577215664901532$$

- Modified Bessel function of the second kind

$$K_n(x) \equiv \frac{\Gamma(n+1/2)(2x)^n}{\sqrt{\pi}}\int_0^\infty \frac{\cos t}{\left(t^2+x^2\right)^{n+1/2}}\, dt$$

(1) $0 < x < \sqrt{n+1}$

$$K_n(x) \approx \begin{cases} -\ln(x/2) - \gamma & \text{if } n = 0 \\ \dfrac{\Gamma(n)}{2}\left(\dfrac{2}{x}\right)^n & \text{if } n > 0 \end{cases}$$

(2) $x \gg \left|n^2 - 1/4\right|$

$$K_n(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}\left[1 + \frac{(4n^2-1)}{8x}\right]$$

Recurrence formulae

$$K_{n-1}(x) - K_{n+1}(x) = -\frac{2n}{x}K_n(x)$$

$$K_{n-1}(x) + K_{n+1}(x) = -2K_n'(x)$$

Integral formula

$$\int xK_n^2(x)\,dx = \frac{1}{2}x^2\left[K_n^2(x) - K_{n-1}(x)K_{n+1}(x)\right]$$

$$= -xK_{n-1}(x)K_n(x) + \frac{1}{2}x^2\left[K_n^2(x) - K_{n-1}^2(x)\right]$$

[Covariance of Electromagnetic Phenomena]

- Equation of charge conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

$$\boxed{\begin{aligned} \partial_\mu &\equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \\ \partial^\mu &\equiv \frac{\partial}{\partial x_\mu} = \left(-\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \end{aligned}}$$

The above equation can be written as a tensor equation,

$$\frac{\partial j^\mu}{\partial x^\mu} = 0, \quad j^\mu{}_{,\mu} = 0 \quad \text{or} \quad \partial_\mu j^\mu = 0$$

if the **four-current** is defined by

$$j^\mu = \begin{pmatrix} \rho c \\ \mathbf{j} \end{pmatrix} \qquad j_\mu = \begin{pmatrix} -\rho c \\ \mathbf{j} \end{pmatrix}$$

- Note that the Jacobian (determinant) of the transformation from x_μ to x'_μ is simply the determinant of Λ, which is unity. Therefore, the **four-volume element** is an invariant.

$$dx'_0 dx'_1 dx'_2 dx'_3 = \det \Lambda \, dx_0 dx_1 dx_2 dx_3 = dx_0 dx_1 dx_2 dx_3$$

Since ρ is the zeroth component of the four-current, the charge element within a three-volume element is an invariant.

$$de = \rho dx_1 dx_2 dx_3$$

It is also an empirical fact that e is invariant.

- The set of vector and scalar wave equations in the Lorentz gauge is

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{4\pi}{c}(\rho c)$$

If we define the **four-potential**

$$A^\mu = \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix}, \quad A_\mu = \begin{pmatrix} -\phi \\ \mathbf{A} \end{pmatrix}$$

then the wave equations can be written as the tensor equations

$$\frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} = -\frac{4\pi}{c} j^\mu, \qquad \partial_\nu \partial^\nu A^\mu = -\frac{4\pi}{c} j^\mu \quad \text{or} \quad A^{\mu,\nu}_{,\nu} = -\frac{4\pi}{c} j^\mu$$

d'Alembertian operator: $\Box \equiv \dfrac{\partial^2}{\partial x^\nu \partial x_\nu} \;\to\; \Box A^\mu = -\dfrac{4\pi}{c} j^\mu$

- The Lorentz gauge should be preserved under Lorentz transformations.

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \;\to\; \frac{\partial A^\mu}{\partial x^\mu} = 0 \quad \text{or} \quad A^\mu{}_{,\mu} = 0$$

- **Electromagnetic field tensor**:

The fields are expressed in terms of the potentials as

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

The x components of the electric and magnetic fields are explicitly

$$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} = \left(\partial^0 A^1 - \partial^1 A^0\right)$$
$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \left(\partial^2 A^3 - \partial^3 A^2\right)$$

These equations imply that the electric and magnetic fields, six components in all, are the elements of a **second-rank, antisymmetric field-strength tensor**, because a rank two antisymmetric tensor has exactly six independent components.

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \longrightarrow \begin{aligned} F^{0i} &= E_i \\ F^{i0} &= -E_i \\ F^{12} &= -F^{21} = B_3, \cdots \end{aligned}$$

covariant field-strength tensor

$$F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta} \longrightarrow \begin{array}{l} F_{0i} = \eta_{0\alpha}\eta_{i\beta}F^{\alpha\beta} = -F^{0i} \\ F_{i0} = \eta_{i\alpha}\eta_{0\beta}F^{\alpha\beta} = -F^{i0} \\ F_{ij} = \eta_{i\alpha}\eta_{j\beta}F^{\alpha\beta} = F^{ij} \end{array}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \longrightarrow \begin{array}{l} F_{0i} = -E_i \\ F_{i0} = E_i \\ F_{12} = -F_{21} = B_3, \cdots \end{array}$$

- The two Maxwell equations containing sources (inhomogeneous equations):

$$\begin{array}{l} \nabla \cdot \mathbf{E} = 4\pi\rho \\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} \end{array} \longrightarrow \begin{array}{l} \sum_{i=1}^{3} \partial_i E_i = \frac{4\pi}{c} j^0 \\ \partial_2 B_3 - \partial_3 B_2 - \partial_0 E_1 = \frac{4\pi}{c} j^1 \end{array} \longrightarrow \begin{array}{l} -\sum_{i=1}^{3} \partial_i F^{i0} = \frac{4\pi}{c} j^0 \\ -\partial_0 F^{01} - \partial_2 F^{21} - \partial_3 F^{31} = \frac{4\pi}{c} j^1 \end{array}$$

$$\boxed{\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} j^\nu} \quad \text{or} \quad \partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu \qquad \partial^\mu F_{\mu\nu} = -\frac{4\pi}{c} j_\nu \quad \text{or} \quad \partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu$$

- The conservation of charge easily follows from the above equation and the asymmetric property.

$$\partial_\mu \partial_\nu F^{\mu\nu} = -\partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\mu\nu} \;\rightarrow\; \partial_\mu \partial_\nu F^{\mu\nu} = 0$$

$$\partial_\nu j^\nu = -\frac{c}{4\pi} \partial_\nu \partial_\mu F^{\mu\nu} = 0$$

- The "internal" Maxwell equations (homogeneous equations):

$$\begin{aligned} &\nabla \cdot \mathbf{B} = 0 \\ &\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \end{aligned} \longrightarrow \begin{aligned} &\sum_{i=1}^{3} \partial_i B_i = 0 \\ &\partial_2 E_3 - \partial_3 E_2 + \partial_0 B_1 = 0 \end{aligned} \longrightarrow \begin{aligned} &\partial_1 F^{23} + \partial_2 F^{31} + \partial_3 F^{12} = 0 \\ &\partial_2 F^{30} + \partial_3 F^{20} + \partial_0 F^{23} = 0 \end{aligned}$$

$$\boxed{\partial_\mu F^{\nu\sigma} + \partial_\nu F^{\sigma\mu} + \partial_\sigma F^{\mu\nu} = 0} \qquad \text{or} \qquad \partial^\mu F_{\nu\sigma} + \partial^\nu F_{\sigma\mu} + \partial^\sigma F_{\mu\nu} = 0$$

The equation can be written concisely as $F^{[\mu\nu,\sigma]} = 0$ or $F_{[\mu\nu,\sigma]} = 0$, where [ ] around indices denote all permutations of indices, with even permutation contributing with a positive sign and odd permutation with a negative sign, for example,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{[\nu,\mu]}$$

- Transformation of Electromagnetic Fields

- Since $F^{\mu\nu}$ is a second-rank tensor, its components transform in the usual way:

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} F^{\alpha\beta} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

For a pure boost along the x-axis:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow$$

$$E_x{}' = F'^{01} = \Lambda^{0}{}_{0}\Lambda^{1}{}_{1}F^{01} + \Lambda^{0}{}_{1}\Lambda^{1}{}_{0}F^{10} = \gamma^2 E_x - \beta^2\gamma^2 E_x = E_x$$

$$E_y{}' = F'^{02} = \Lambda^{0}{}_{0}\Lambda^{2}{}_{2}F^{02} + \Lambda^{0}{}_{1}\Lambda^{2}{}_{2}F^{12} = \gamma E_y - \beta\gamma B_z$$

$$E_z{}' = F'^{03} = \Lambda^{0}{}_{0}\Lambda^{3}{}_{3}F^{03} + \Lambda^{0}{}_{1}\Lambda^{3}{}_{3}F^{13} = \gamma E_z + \beta\gamma B_y$$

$$B_x{}' = F'^{23} = \Lambda^{2}{}_{2}\Lambda^{3}{}_{3}F^{23} = B_x$$

$$B_y{}' = F'^{31} = \Lambda^{3}{}_{3}\left(\Lambda^{1}{}_{0}F^{30} - \Lambda^{1}{}_{1}F^{31}\right) = \beta\gamma E_z + \gamma B_y$$

$$B_z{}' = F'^{12} = \Lambda^{1}{}_{0}\Lambda^{2}{}_{2}F^{02} + \Lambda^{1}{}_{1}\Lambda^{2}{}_{2}F^{12} = -\beta\gamma E_y + \gamma B_z$$

- In general,

$$\boxed{\begin{array}{ll} \mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} & \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \\ \mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \boldsymbol{\beta}\times\mathbf{B}\right) & \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \boldsymbol{\beta}\times\mathbf{E}\right) \end{array}}$$

The concept of a pure electric or pure magnetic is not Lorentz invariant.

- Lorenz invariants:

  dot product of F with itself or "square" of F:

$$F^{\mu\nu}F_{\mu\nu} = \sum_{i=1}^{3} F^{0i}F_{0i} + \sum_{i=1}^{3} F^{i0}F_{i0} + \sum_{i\neq j} F^{ij}F_{ij} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right)$$

  determinant of F:

$$\det F = \left(\mathbf{E}\cdot\mathbf{B}\right)^2$$

[Relativistic Mechanics and the Lorentz Four-Force]

- We can define a **four-acceleration** a^{μ} in exactly the same way as we obtained the four-velocity.

$$a^{\mu} \equiv \frac{dU^{\mu}}{d\tau}$$

Note that the four-acceleration and four-velocity are orthogonal:

$$\vec{a} \cdot \vec{U} \equiv \frac{dU^{\mu}}{d\tau} U_{\mu} = \frac{1}{2}\frac{d}{d\tau}\left(U^{\mu}U_{\mu}\right) = \frac{1}{2}\frac{d}{d\tau}\left(-c^{2}\right) = 0$$

- We can also define the four-force F^{μ} from the Lorentz force, so as to obtain a relativistic form of Newton's equation.

$$\boxed{F^{\mu} \equiv m_0 a^{\mu} = \frac{dP^{\mu}}{d\tau}} \qquad \vec{F} = \frac{d\vec{P}}{d\tau} = \gamma\frac{d\vec{P}}{dt} = \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\mathbf{p}}{dt}\right)$$

Since $\mathbf{F}_{\text{Lorentz}} = q\left[\mathbf{E} + \frac{1}{c}\left(\mathbf{v}\times\mathbf{B}\right)\right]$, the **Lorentz four-force** should involve (1) the electromagnetic field tensor and (2) the four-velocity and should also be (3) a four-vector and (4) proportional to the charge of the particle. Therefore, the simplest possibility is

$$\boxed{F^{\mu}{}_{\text{Lorentz}} = \frac{q}{c} F^{\mu\nu} U_{\nu}}$$

- Let's check to see if it is indeed what we want.

$$F^0{}_{\text{Lorentz}} = \frac{q}{c} F^{0\nu} U_\nu = \frac{q}{c} \sum_{i=1}^{3} E_i \gamma v_i = \frac{q}{c} \gamma (\mathbf{E} \cdot \mathbf{v}) \longrightarrow \frac{dE}{dt} = q\mathbf{E} \cdot \mathbf{v} \quad \text{: conservation of energy}$$

The rate of change of particle energy is the mechanical work done on the particle by the field.

$$F^1{}_{\text{Lorentz}} = \frac{q}{c} F^{1\nu} U_\nu = \frac{q}{c} \left(F^{10}(-\gamma c) + F^{12} \gamma v_2 + F^{13} \gamma v_3 \right)$$
$$= \frac{q}{c} \gamma \left(E_1 c + B_3 v_2 - B_2 v_3 \right) \longrightarrow \frac{d\mathbf{p}}{dt} = q \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right)$$

Therefore, we obtained the desired expression for the four-Lorentz force.

- Note that **the four-force is always orthogonal to the four-velocity**:

$$\vec{F} \cdot \vec{U} = m_0 (\vec{a} \cdot \vec{U}) = 0$$

It implies that **every four-force must have some velocity dependence**.

For the Lorentz four-force, in particular, we find

$$\vec{F}_{\text{Lorentz}} \cdot \vec{U} = \frac{q}{c} F^{\mu\nu} U_\mu U_\nu = 0,$$

because $F^{\mu\nu}$ is antisymmetric and $U_\mu U_\nu$ is symmetric.

[Fields of a Uniformly Moving Charge]

- Let's find the fields of a charge moving with constant velocity v along the x axis. In the rest frame of the particle the fields are

$$\mathbf{E}' = \left(E'_x, E'_y, E'_z\right) = \frac{q}{r'^3}\left(x', y', z'\right) \qquad \text{where} \qquad r' = \left(x'^2 + y'^2 + z'^2\right)^{1/2}$$

$$\mathbf{B}' = (0,0,0)$$

inverse transformation of the previous one:

$$\mathbf{E}_\parallel = \mathbf{E}'_\parallel \qquad \mathbf{B}_\parallel = \mathbf{B}'_\parallel$$

$$\mathbf{E}_\perp = \gamma\left(\mathbf{E}'_\perp - \boldsymbol{\beta}\times\mathbf{B}'\right) \qquad \mathbf{B}_\perp = \gamma\left(\mathbf{B}'_\perp + \boldsymbol{\beta}\times\mathbf{E}'\right) \longrightarrow$$

$$E_x = \frac{qx'}{r'^3} \qquad B_x = 0$$

$$E_y = \gamma\frac{qy'}{r'^3} \qquad B_y = -\gamma\beta\frac{qz'}{r'^3}$$

$$E_z = \gamma\frac{qz'}{r'^3} \qquad B_z = \gamma\beta\frac{qy'}{r'^3}$$

Since $x' = \gamma(x - vt),\ y' = y,\ z' = z$, we obtain

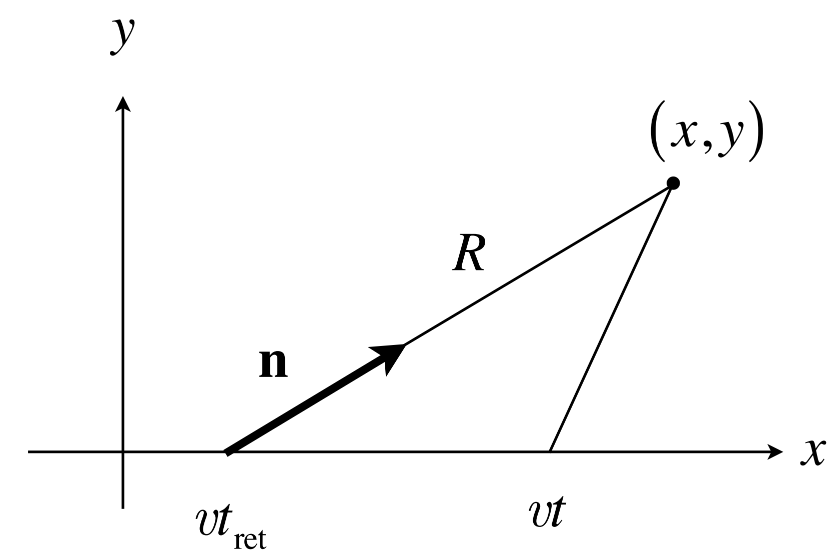
$$\boxed{\begin{array}{ll} E_x = \gamma\dfrac{q(x - vt)}{r^3} & B_x = 0 \\ E_y = \gamma\dfrac{qy}{r^3} & B_y = -\gamma\beta\dfrac{qz}{r^3} \\ E_z = \gamma\dfrac{qz}{r^3} & B_z = \gamma\beta\dfrac{qy}{r^3} \end{array}}$$

where $\quad r = \left[(x - vt)^2 + y^2 + z^2\right]^{1/2}$

Is this equivalent to the fields given by the Lienard-Wiechert potentials?

- Velocity field from the retarded potential

- For simplicity, assume $z = 0$.



$$\mathbf{E} = \left(E_x, E_y, E_z\right) = \gamma \frac{q}{r^3}(x - vt, y, z)$$

$$= \gamma \frac{q}{\left(\gamma^2 \bar{x}^2 + y^2\right)^{3/2}} (\bar{x}, y, 0) \quad \text{where} \quad \bar{x} \equiv x - vt$$

Let us first find where the retarded position of the particle is.

$$t_{\text{ret}} \equiv t - R / c$$

$$R^2 = \left(x - vt_{\text{ret}}\right)^2 + y^2 = \left(\bar{x} + \beta R\right)^2 + y^2$$

$$\mathbf{n} = \frac{\left(\bar{x} + \beta R\right)}{R}\hat{\mathbf{x}} + \frac{y}{R}\hat{\mathbf{y}} = \left(\frac{\bar{x}}{R} + \beta\right)\hat{\mathbf{x}} + \frac{y}{R}\hat{\mathbf{y}}$$

$$R \longrightarrow \quad (1-\beta^2)^2 R^2 - 2\bar{x}\beta R - \bar{x}^2 - y^2 = 0$$

$$R^2 - 2\bar{x}\gamma^2 \beta R - \gamma^2\left(\bar{x}^2 + y^2\right) = 0$$

$$R = \gamma^2 \beta \bar{x} \pm \left[\gamma^4 \beta^2 \bar{x}^2 + \gamma^2\left(\bar{x}^2 + y^2\right)\right]^{1/2}$$

$$= \gamma^2 \beta \bar{x} \pm \gamma\left[\gamma^2 \beta^2 \bar{x}^2 + \left(\bar{x}^2 + y^2\right)\right]^{1/2}$$

$$= \gamma^2 \beta \bar{x} \pm \gamma\left(\gamma^2 \bar{x}^2 + y^2\right)^{1/2}$$

positive solution $\rightarrow$ $R = \gamma^2 \beta \bar{x} + \gamma\left(\gamma^2 \bar{x}^2 + y^2\right)^{1/2}$

$$(1) \quad \mathbf{n} - \boldsymbol{\beta} = \frac{\bar{x}}{R}\hat{\mathbf{x}} + \frac{y}{R}\hat{\mathbf{y}}$$

$$\mathbf{E} = \gamma \frac{q}{\left(\gamma^2 \bar{x}^2 + y^2\right)^{3/2}}(\mathbf{n} - \boldsymbol{\beta})$$

$$(2) \quad \left(\gamma^2 \bar{x}^2 + y^2\right)^{1/2} = \frac{R - \gamma^2 \beta \bar{x}}{\gamma} = R\gamma\left(\frac{1}{\gamma^2} - \frac{\beta \bar{x}}{R}\right)$$

$$= R\gamma\left(1 - \beta^2 - \frac{\beta \bar{x}}{R}\right)$$

$$= R\gamma\left[1 - \beta\left(\frac{\bar{x}}{R} + \beta\right)\right]$$

$$= R\gamma\left(1 - \mathbf{n} \cdot \boldsymbol{\beta}\right) = R\gamma\kappa$$

$$\therefore \ \mathbf{E} = q\frac{\left(\mathbf{n} - \boldsymbol{\beta}\right)}{\gamma^2 \kappa^3 R^2} = q\frac{\left(\mathbf{n} - \boldsymbol{\beta}\right)\left(1 - \beta^2\right)}{\kappa^3 R^2} \quad \text{: velocity field}$$

- Time-dependence of the electric field at a point

- Let us choose the field point to be at $(0,b,0)$.

  This involves no loss in generality. Then,

$$E_x = -\frac{q\gamma vt}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = -\frac{q}{b^2}\frac{\gamma vt / b}{(\gamma^2 v^2 t^2 / b^2 + 1)^{3/2}}$$

$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = \frac{q\gamma}{b^2}\frac{1}{(\gamma^2 v^2 t^2 / b^2 + 1)^{3/2}}$$

$$E_z = 0$$

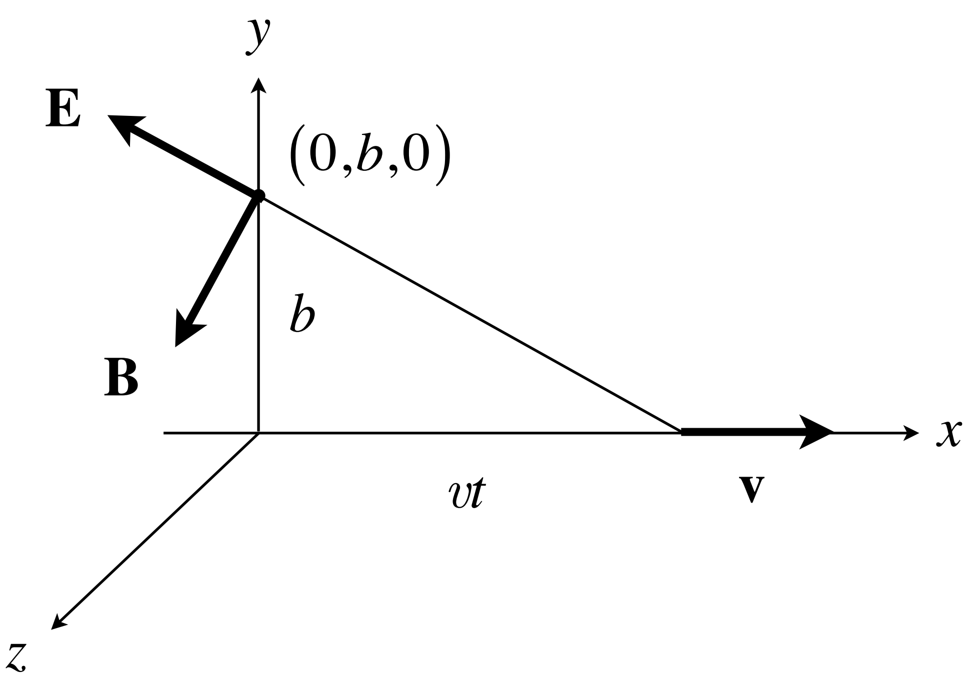
$$B_x = 0$$

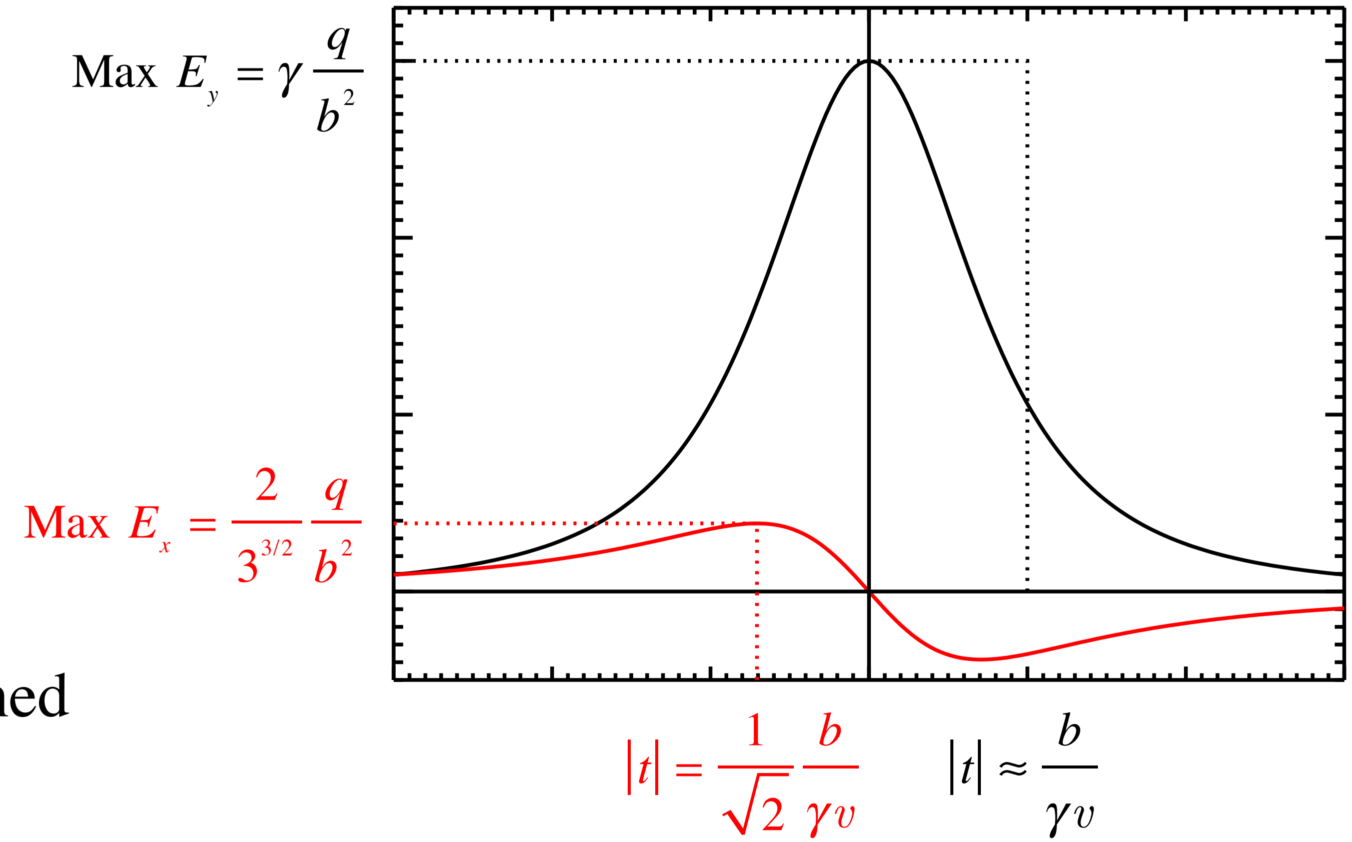
$$B_y = 0$$

$$B_z = \beta E_y$$

As $\gamma \gg 1 \rightarrow |E_x| \ll E_y$

The field of a highly relativistic charge appears to be a pulse of radiation traveling in the same direction as the charge and confined to the tranverse plane.





- Spectrum of the pulse

- Spectrum of this pulse of virtual radiation.

$$\hat{E}(\omega) = \frac{1}{2\pi}\int E_y(t)e^{i\omega t}\,dt$$

$$= \frac{q\gamma b}{2\pi}\int_{-\infty}^{\infty}\left(\gamma^2 v^2 t^2 + b^2\right)^{-3/2} e^{i\omega t}\,dt = \frac{q\gamma b}{2\pi}\int_{0}^{\infty}\left(\gamma^2 v^2 t^2 + b^2\right)^{-3/2}\left(e^{i\omega t} + e^{-i\omega t}\right)dt$$

$$= \frac{q\gamma b}{\pi}\int_{0}^{\infty}\left(\gamma^2 v^2 t^2 + b^2\right)^{-3/2} \cos\omega t\,dt$$

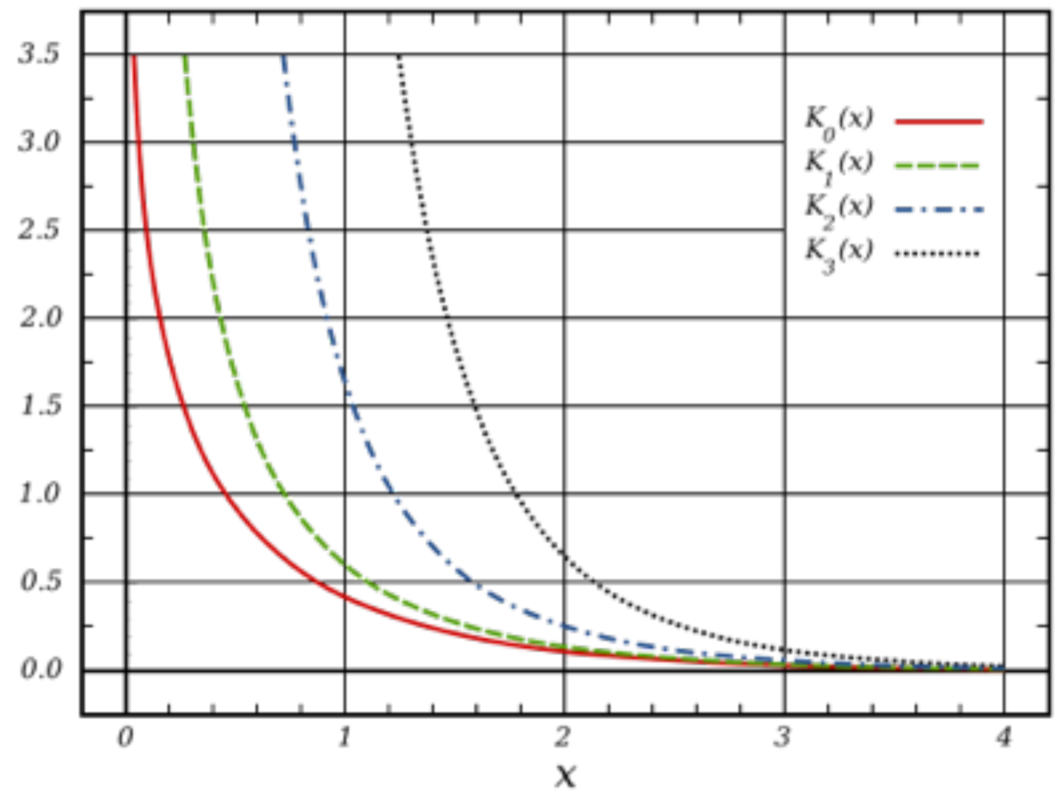
This integral can be done in terms of the modified Bessel function:

$$K_n(x) \equiv \frac{\Gamma(n+1/2)(2x)^n}{\sqrt{\pi}}\int_0^\infty \frac{\cos t}{\left(t^2+x^2\right)^{n+1/2}}\,dt$$

Gamma function: $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$

$$\hat{E}(\omega) = \frac{q\gamma b}{\pi}\left(\frac{\gamma^2 v^2}{\omega^2}\right)^{-3/2}\frac{1}{\omega}\int_0^\infty\left(\omega^2 t^2 + \frac{b^2\omega^2}{\gamma^2 v^2}\right)^{-3/2}\cos\omega t\,d\omega t$$

$$= \frac{q}{\pi b v}\frac{b\omega}{\gamma v}K_1\left(\frac{b\omega}{\gamma v}\right)$$

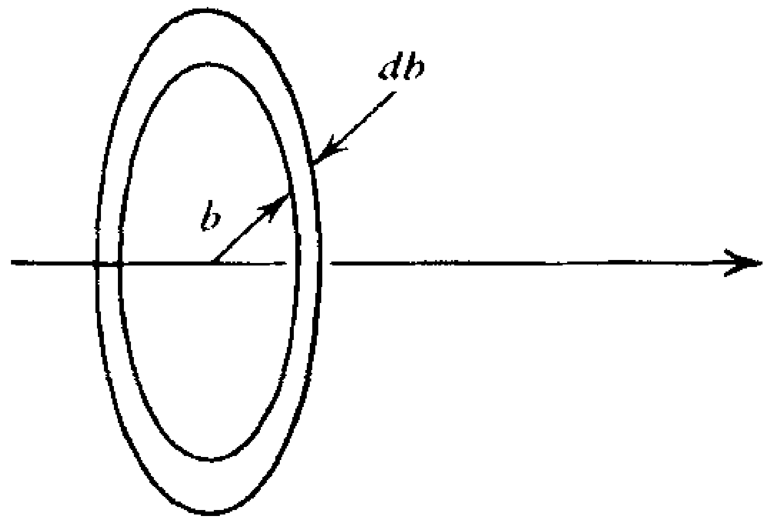


Thus the spectrum is

$$\frac{dW}{dA\,d\omega} = c\left|\hat{E}(\omega)\right|^2 = \frac{q^2}{\pi^2 b^2 v^2}\left(\frac{b\omega}{\gamma v}\right)^2 K_1^{\,2}\left(\frac{b\omega}{\gamma v}\right)$$

The spectrum starts cut off for $\omega > \gamma v / b$.

$$\Delta\omega \sim \frac{1}{\Delta t} \sim \gamma v / b$$



- Total energy per unit frequency range is obtained by

$$\frac{dW}{d\omega} = 2\pi \int_{b_{\min}}^{b_{\max}} \frac{dW}{dA d\omega} b\, db$$

The lower limit has been chosen as some minimum distance, such that the approximation of the field by means of classical electrodynamics and a point charge is valid.

$$\frac{dW}{d\omega} = \frac{2q^2 c}{\pi v^2} \int_x^{\infty} y K_1{}^2(y) dy \qquad \text{where} \quad y \equiv \frac{\omega b}{\gamma v}, \text{ and} \quad x \equiv \frac{\omega b_{\min}}{\gamma v}$$

$$= \frac{2q^2 c}{\pi v^2} \left[x K_0(x) K_1(x) - \frac{1}{2} x^2 \left(K_1{}^2(x) - K_0{}^2(x) \right) \right]$$

- Two limiting cases:

(1) $\omega \ll \dfrac{\gamma v}{b_{\min}}$ $(x \ll 1)$,

$$xK_0(x)K_1(x) - \frac{1}{2}x^2\left(K_1{}^2(x) - K_0{}^2(x)\right)$$
$$\approx x\left(-\ln(x/2) - \gamma\right)\frac{1}{x} - \frac{x^2}{2}\left[\frac{1}{x^2} - \left(\ln(x/2) + \gamma\right)^2\right] \longrightarrow \frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2}\ln\left(0.68\frac{\gamma v}{\omega b_{\min}}\right)$$
$$\approx \ln\left[\frac{2}{x}e^{-(\gamma+1/2)}\right]$$
$$= \ln\left(\frac{0.68}{x}\right)$$

(2) $\omega \gg \dfrac{\gamma v}{b_{\min}}$ $(x \gg 1)$,

$$xK_0(x)K_1(x) - \frac{1}{2}x^2\left(K_1{}^2(x) - K_0{}^2(x)\right) \longrightarrow \frac{dW}{d\omega} = \frac{q^2c}{2v^2}\exp\left(-\frac{2\omega b_{\min}}{\gamma v}\right)$$
$$\approx x\frac{\pi}{2x}e^{-2x} - \frac{1}{2}x^2\frac{\pi}{2x}e^{-2x}\left[\left(\frac{3}{8x}\right)^2 - \left(\frac{1}{8x}\right)^2\right]$$
$$= \frac{\pi}{4}e^{-2x}$$

[Emission from Relativistic Particles]

- Total emitted power:

Imagine **an instantaneous rest frame** *K'*, such that the particle has zero velocity at a certain time. We can then calculate the radiation emitted by use of the dipole (Larmor) formula.

Suppose that the particle emits a total amount of energy dW' in this frame in time dt'. The momentum of this radiation is zero, $d\mathbf{p}' = 0$, because the emission is symmetrical in the frame.

The energy in a frame *K* moving with velocity $-\mathbf{v}$ w.r.t. the particle is:

$$dW = \gamma dW' \longleftarrow dE = cdP^0 = c\tilde{\Lambda}^0_\mu dP'^\mu = c\tilde{\Lambda}^0_0 dP'^0 = \gamma dE'$$

The time interval dt is simply

$$dt = \gamma dt'$$

The total power emitted in frames *K* and *K'* are given by

$$P = \frac{dW}{dt}, \qquad P' = \frac{dW'}{dt'}$$

Thus **the total emitted power is a Lorentz invariant** for any emitter that emits with front-back symmetry in its instantaneous rest frame.

$$P = P'$$

- **the Larmor formula in covariant form:**

  Recall that $\vec{a}\cdot\vec{U}=0$, and because $\vec{U}=(c,\mathbf{0})$ in the instantaneous rest frame of the particle, we have

$$a_0'=0 \quad\to\quad |\mathbf{a}'|^2 = a_k' a'^k = a_\mu' a'^\mu = \vec{a}\cdot\vec{a}$$

  Therefore,

$$P' = \frac{2q^2}{3c^3}|\mathbf{a}'|^2 \quad\longrightarrow\quad P = \frac{2q^2}{3c^3}\vec{a}\cdot\vec{a}$$

- Expression of P in terms of the three-vector acceleration

  Recall

$$dt = \gamma\left(dt' + \frac{v}{c^2}dx'_{\|}\right)$$

$$u_{\|} = \frac{u_{\|}'+v}{1+vu_{\|}'/c^2}$$

$$u_{\perp} = \frac{u_{\perp}'}{\gamma(1+vu_{\|}'/c^2)}$$

$$\longrightarrow$$

$$\sigma \equiv (1+vu_{\|}'/c^2)$$

$$dt = \gamma dt'\sigma$$

$$u_{\|} = \frac{u_{\|}'+v}{\sigma}$$

$$u_{\perp} = \frac{u_{\perp}'}{\gamma\sigma}$$

$$\longrightarrow$$

$$dt = \gamma dt'\sigma$$

$$du_{\|} = \frac{du_{\|}'}{\sigma} - \frac{u_{\|}'+v}{\sigma^2}\frac{v}{c^2}du_{\|}' = \frac{du_{\|}'}{\sigma^2}\left(1-\frac{v^2}{c^2}\right) = \frac{du_{\|}'}{\gamma^2\sigma^2}$$

$$du_{\perp} = \frac{du_{\perp}'}{\gamma\sigma} - \frac{u_{\perp}'}{\gamma\sigma^2}\frac{v}{c^2}du_{\|}' = \frac{1}{\gamma\sigma^2}\left(\sigma du_{\perp}' - \frac{vu_{\perp}'}{c^2}du_{\|}'\right)$$

Hence,

$$a_\parallel = \frac{du_\parallel}{dt} = \frac{1}{\gamma^3\sigma^3}\frac{du'_\parallel}{dt'}$$

$$a_\perp = \frac{du_\perp}{dt} = \frac{1}{\gamma^2\sigma^3}\left(\sigma\frac{du'_\perp}{dt'} - \frac{vu'_\perp}{c^2}\frac{du'_\parallel}{dt'}\right)$$

$\longrightarrow$

Transformation of three-vector acceleration:

$$\boxed{\begin{aligned} a_\parallel &= \frac{1}{\gamma^3\sigma^3}a'_\parallel \\ a_\perp &= \frac{1}{\gamma^2\sigma^3}\left(\sigma a'_\perp - \frac{vu'_\perp}{c^2}a'_\parallel\right) \\ &\text{where} \quad \sigma \equiv \left(1 + \frac{vu'_\parallel}{c^2}\right) \end{aligned}}$$

In an instantaneous rest frame of a particle,

$$u'_\parallel = u'_\perp = 0,\ \ \sigma = 1$$

$$\boxed{\begin{aligned} a'_\parallel &= \gamma^3 a_\parallel \\ a'_\perp &= \gamma^2 a_\perp \end{aligned}}$$

Note $\tan\theta'_a \equiv \frac{a'_\perp}{a'_\parallel} = \frac{1}{\gamma}\frac{a_\perp}{a_\parallel} = \frac{1}{\gamma}\tan\theta_a$

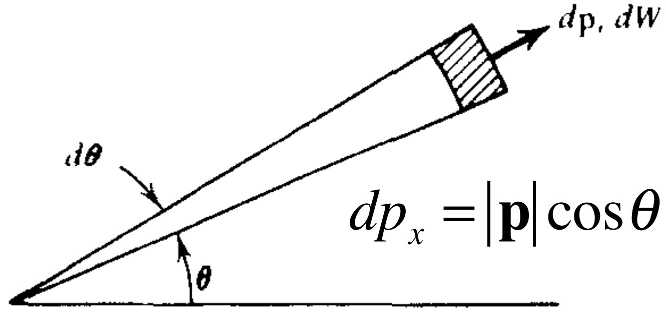
Thus we can write

$$P = \frac{2q^2}{3c^3}|\mathbf{a}'|^2 = \frac{2q^2}{3c^3}\left(a'^2_\parallel + a'^2_\perp\right)$$

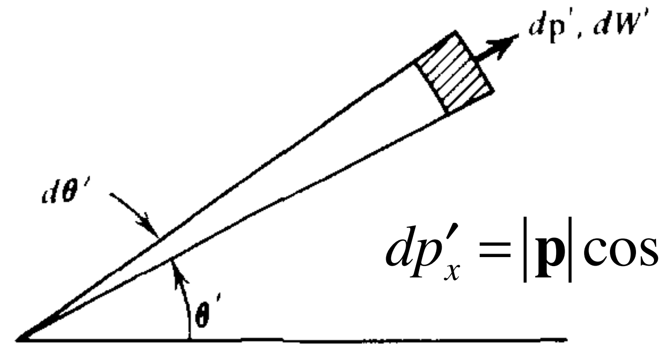
$\longrightarrow$

$$\boxed{P = \frac{2q^2}{3c^3}\gamma^4\left(\gamma^2 a_\parallel^2 + a_\perp^2\right)}$$

- <u>Angular Distribution of Emitted and Received Power</u>



$$dp_x = |\mathbf{p}|\cos\theta$$



$$dp'_x = |\mathbf{p}|\cos\theta'$$

Note:

$$d\phi' = d\phi$$

In the instantaneous rest frame of the particle, let us consider an amount of energy *dW'* that is emitted into the solid angle $d\Omega' = \sin\theta' d\theta' d\phi'$ (see the above figure).

$$\mu \equiv \cos\theta \rightarrow d\Omega = d\mu d\phi \qquad \mu' \equiv \cos\theta' \rightarrow d\Omega' = d\mu' d\phi'$$

Recall $\cos\theta = \dfrac{\cos\theta' + \beta}{1+\beta\cos\theta'} \rightarrow \mu = \dfrac{\mu' + \beta}{1+\beta\mu'}$, or inverse $\mu' = \dfrac{\mu - \beta}{1-\beta\mu}$

$$d\mu = \frac{d\mu'}{1+\beta\mu'} - \frac{\mu'+\beta}{(1+\beta\mu')^2}\beta d\mu' \longrightarrow \boxed{\begin{array}{l} d\mu = \dfrac{d\mu'}{\gamma^2(1+\beta\mu')^2}, \ d\mu = \gamma^2(1-\beta\mu)^2 d\mu' \\ d\Omega = \dfrac{d\Omega'}{\gamma^2(1+\beta\mu')^2}, \ d\Omega = \gamma^2(1-\beta\mu)^2 d\Omega' \end{array}}$$

- Power

Recall that energy and momentum form a four-vector

$$P^{\mu} = \left(\frac{E}{c}, \mathbf{p}\right), \text{ and } |\mathbf{p}| = \frac{E}{c} \quad \longrightarrow \quad dW = \gamma\left(dW' + v dp'_x\right) = \gamma\left(1+\beta\mu'\right)dW'$$

$$\therefore\ dW = \gamma\left(1+\beta\mu'\right)dW',\ dW = \gamma^{-1}\left(1-\beta\mu\right)^{-1} dW'$$

$$\frac{dW}{d\Omega} = \gamma^{3}\left(1+\beta\mu'\right)^{3}\frac{dW'}{d\Omega'},\ \frac{dW}{d\Omega} = \gamma^{-3}\left(1-\beta\mu\right)^{-3}\frac{dW'}{d\Omega'}$$

In the rest frame, the power emitted in a unit time interval is $\frac{dP'}{d\Omega'} \equiv \frac{dW'}{dt'd\Omega'}$

However, in the observer's frame, there are two possible choices for the time interval to calculate the power.

(1) $dt = \gamma dt'$:

This is the time interval during which the emission occurs. With this choice we obtain **the emitted power**.

(2) $dt_A = \gamma(1-\beta\mu)dt'$, or $dt_A = \gamma^{-1}(1+\beta\mu')^{-1}dt'$:

This is the time interval of the radiation as received by a stationary observer in K. With this choice we obtain **the received power**.

- Thus we obtain the two results:

$$\frac{dP_e}{d\Omega} = \gamma^2 (1+\beta\mu')^3 \frac{dP'}{d\Omega'} = \gamma^{-4} (1-\beta\mu)^{-3} \frac{dP'}{d\Omega'}$$
$$\frac{dP_r}{d\Omega} = \gamma^4 (1+\beta\mu')^4 \frac{dP'}{d\Omega'} = \gamma^{-4} (1-\beta\mu)^{-4} \frac{dP'}{d\Omega'}$$

P_r is the power actually measured by an observer. It has the expected symmetry property of yielding the inver transformation by interchanging primed and unprimed variables, along with a change of sign of β.

P_e is used in the discussion of emission coefficient.

In practice, the distinction between emitted and received power is often not important, since they are equal in an average sense for stationary distributions of particles.

- Beaming effect:

If the radiation if isotropic in the particle's frame, then the angular distribution in the observer's frame will be highly peaked in the forward direction for highly relativistic velocities.

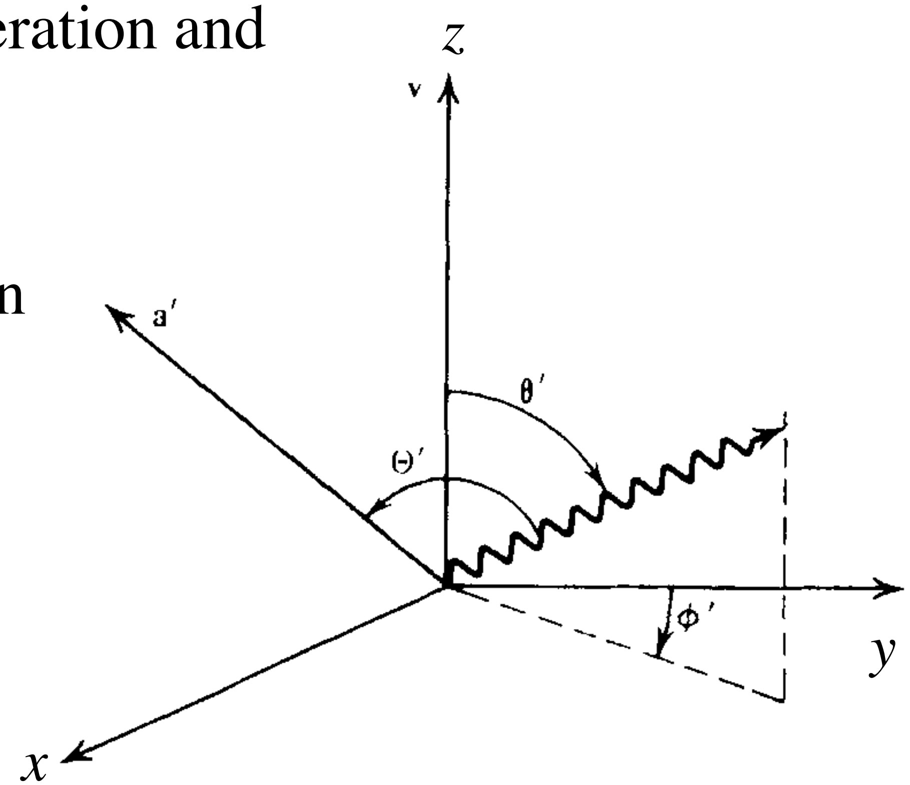
The factor $\gamma^{-4}(1-\beta\mu)^{-4}$ is sharply peaked near $\theta \approx 0$ with an angular scale of order $1/\gamma$.

$$\gamma^{-4}(1-\beta\mu)^{-4} \approx \gamma^{-4}\left[1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right)\right]^{-4} = \gamma^{-4}\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^{-4} = \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^4$$

- Dipole emission from a slowly moving particle

$$\frac{dP'}{d\Omega'} = \frac{q^2 a'^2}{4\pi c^3}\sin^2\Theta'$$

$\Theta' =$ the angle between the acceleration and the direction of emission.



Using $a'_{\parallel} = \gamma^3 a_{\parallel},\ a'_{\perp} = \gamma^2 a_{\perp}$ and $\frac{dP_r}{d\Omega} = \gamma^{-4}(1-\beta\mu)^{-4}\frac{dP'}{d\Omega'}$, we obtain

$$\boxed{\frac{dP_r}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{\left(\gamma^2 a_{\parallel}^2 + a_{\perp}^2\right)}{(1-\beta\mu)^4}\sin^2\Theta'}$$

To use this formula, we must relate Θ' to the angles in K.

(1) Acceleration parallel to velocity: $\Theta' = \theta',\ a_{\perp} = 0$

$$\sin^2\Theta' = 1-\mu'^2 = 1-\left(\frac{\mu-\beta}{1-\beta\mu}\right)^2 = \frac{1-\mu^2}{\gamma^2(1-\beta\mu)^2} \longrightarrow \frac{dP_{\parallel}}{d\Omega} = \frac{q^2 a_{\parallel}^2}{4\pi c^3}\frac{1-\mu^2}{(1-\beta\mu)^6}$$

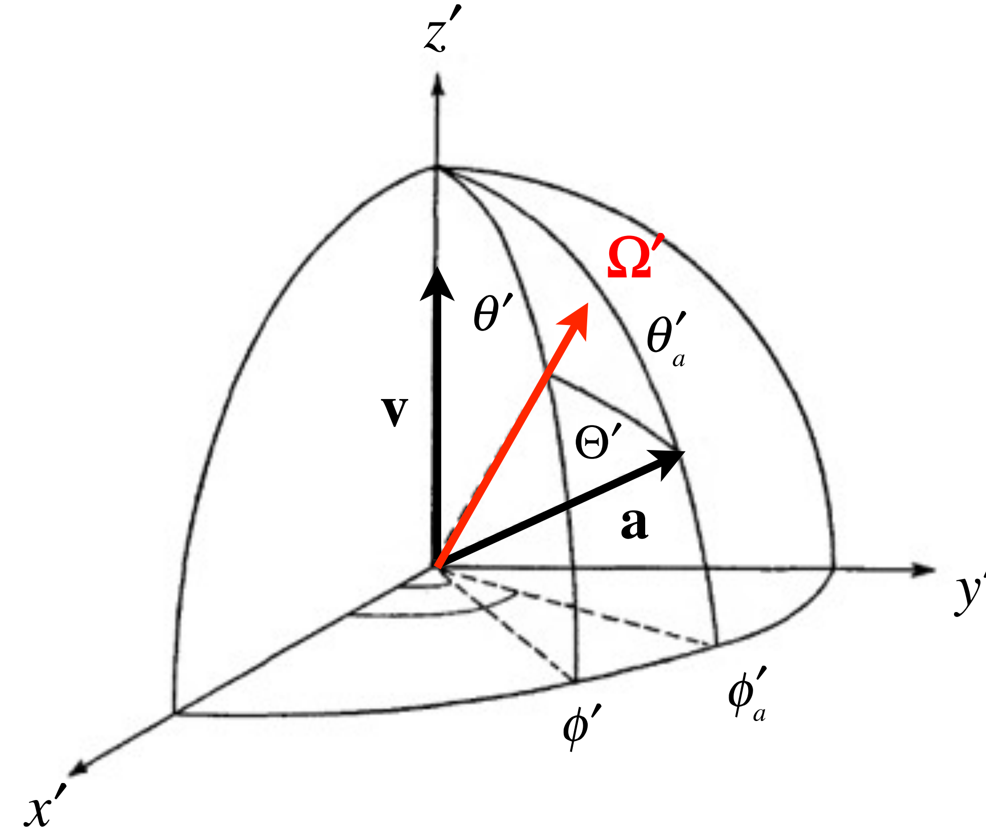
(2) Acceleration perpendicular to velocity: $\cos\Theta' = \sin\theta'\cos\phi',\ a_{\parallel} = 0$ (when a is in y-direction in the above figure)

$$\sin^2\Theta' = 1-\frac{(1-\mu^2)\cos^2\phi}{\gamma^2(1-\beta\mu)^2} \longrightarrow \frac{dP_{\perp}}{d\Omega} = \frac{q^2 a_{\perp}^2}{4\pi c^3}\frac{1}{(1-\beta\mu)^4}\left[1-\frac{(1-\mu^2)\cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right]$$

(3) In general

$$\cos\Theta' = \mu'\mu'_a + \left(1-\mu'^2\right)^{1/2}\left(1-\mu_a'^2\right)^{1/2}\cos\left(\phi'-\phi'_a\right)$$

See Eq. (219) in Chadrasekhar (1960)

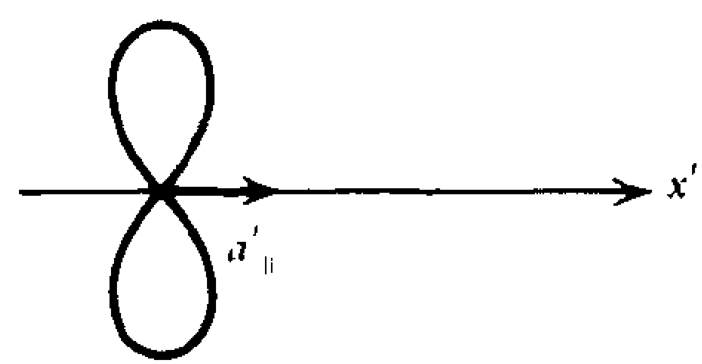


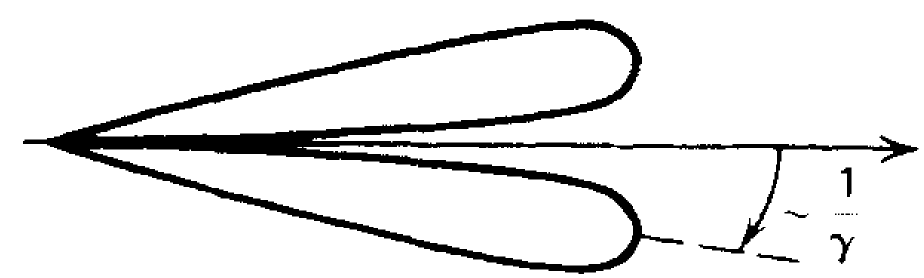
- In the extreme relativistic limit, the radiation becomes strongly peaked in the forward direction.

particle's rest frame:

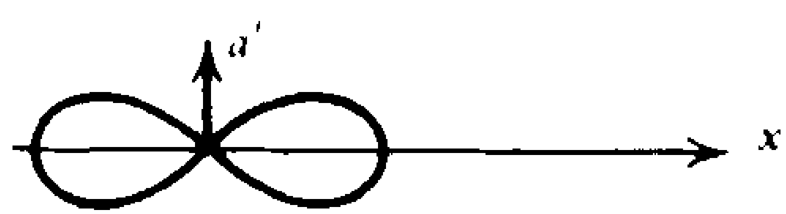
observer's frame:

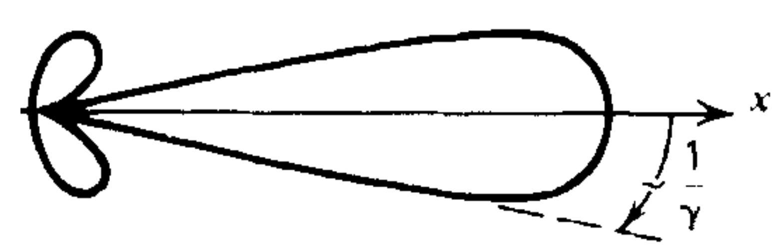
parallel acceleration:





perpendicular acceleration:





[Invariant Phase Volumes and Specific Intensity]

- **Phase volume**

Consider a group of particles that occupy a slight spread in position and in momentum at a particular time. In a rest frame comoving with the particles, they occupy a spatial volume element and a momentum volume element.

$$\begin{aligned} d^3\mathbf{x}' &= dx'dy'dz' \\ d^3\mathbf{p}' &= dp'_x dp'_y dp'_z \end{aligned} \quad \longrightarrow$$

phase volume in the comoving frame:

$$d\mathcal{V}' \equiv d^3\mathbf{x}' d^3\mathbf{p}' = dx'dy'dz'dp'_x dp'_y dp'_z$$

In the observer's frame, $dx = \gamma^{-1}dx', \ dy = dy', \ dz = dz'$

$$dp_x = \gamma\left(dp'_x + \beta dP'_0\right), \ dp_y = dp'_y, \ dp_z = dp'_z$$

We note that $dP'_0 = 0 + \mathcal{O}(dp'^2_x)$ because the velocities are near zero in the comoving frame and the energy is quadratic in velocity. Therefore, we have

$$dp_x = \gamma dp'_x \quad \text{and} \quad \boxed{d\mathcal{V}' \equiv d^3\mathbf{x}' d^3\mathbf{p}' = d^3\mathbf{x}\, d^3\mathbf{p} \equiv d\mathcal{V}} \ \text{: Lorentz invariant}$$

This contains no reference to particle mass, and therefore it has applicability to photons.

The phase space density

$$f \equiv \frac{dN}{d\mathcal{V}}$$

is an invariant, since the number of particles within the phase volume element is a countable quantity and itself invariant.

- **Specific Intensity and Source Function**

Definition of the energy density per unit solid angle per frequency range.

$$h\nu f p^2 dp d\Omega = U_\nu(\Omega) d\Omega d\nu$$

Since $p = h\nu / c$ and $U_\nu(\Omega) = I_\nu / c$ we find that $\boxed{\dfrac{I_\nu}{\nu^3} = \text{Lorentz invariant}}$

Because the source function occurs in the transfer equation as the difference $I_\nu - S_\mu$, the source function must have the same transformation properties as the intensity.

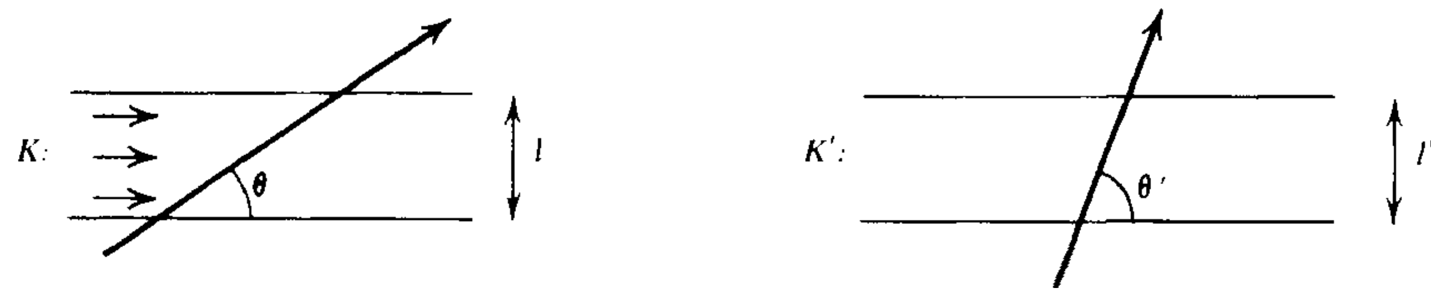
- **Optical Depth, Absorption Coefficient and Emission Coefficient**

The optical depth must be an invariant, since $e^{-\tau}$ gives the fraction of photons passing through the material, and this involves simple counting.

$$\boxed{\tau = \text{Lorentz invariant}}$$

- **Absorption Coefficient and Emission Coefficient**

Consider the optical depth in two frames:



Then, the optical depth is

$$\tau = \frac{l\alpha_\nu}{\sin\theta} = \frac{l}{\nu\sin\theta}\nu\alpha_\nu = \text{Lorentz invariant}$$

Note that $\nu\sin\theta$ is proportional to the y component of the photon four-momentum $\vec{k} = \left(\frac{\omega}{c}, \mathbf{k}\right)$.

Both k_y and l are the same in both frames, being perpendicular to the motion. Therefore, we have

$$\boxed{\nu\alpha_\nu = \text{Lorentz invariant}}$$

Finally, we obtain the transformation of the emission coefficient from the definition of the source function: $S_\nu \equiv j_\nu / \alpha_\nu$

$$\boxed{\frac{j_\nu}{\nu^2} = \text{Lorentz invariant}}$$