

(AGN)²

- 5. Comparison of Theory with Observations
- 6. Internal Dynamics of Gaseous Nebulae

Week 8

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5.11 Abundances of the Elements in Nebulae

- Abundances can be derived from measurements of the relative strengths of their emission lines
 - All nebula lines are optically thin. No curve-of-growth effects that complicate stellar atmosphere abundance determinations occur.
 - H, He, N, O, and Ne are observable in the optical spectra.
 - But, C can be observed in the vacuum UV spectral region.
 - Moreover, all stages of ionization of an element are generally not observable in the optical spectral region.
 - ▶ For instance, [O II] and [O III] are strong in diffuse nebulae, but O IV and O V are not.
 - ▶ However, an [O IV] line is available in the FIR, and O IV] and O V lines are in the vacuum UV.
- Collisionally Excited Lines vs. Recombination Lines
 - **Collisionally excited lines can be quite bright, but their strengths depend strongly on temperature, complicating the determination of relative abundance.** The intensity I_l of an emission line is given by $I_l = \int j_l ds = \int n_i n_e k_l(T) ds$
 - Intensities of recombination lines are given as, for instance, $I_{H\beta} = \frac{1}{4\pi} \int h\nu_{H\beta} n_p n_e \alpha_{H\beta}^{\text{eff}}(H^0, T) ds$.
The recombination rate coefficients are not particularly temperature sensitive and the abundances derived from them do not depend strongly on T . However, **recombination lines of elements heavier than He are faint and hard to observe.**

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- Recombination lines from heavy elements
 - C II, O IV, and O V have weak permitted emission lines as observed in PNe, which are attributed to resulting from recombination. However, some of these lines may be excited by resonance fluorescence, and their emission coefficients depend not only on temperature and density but also the local radiation field. It is not easy to utilize them.
 - C II $3^2D - 4^2F \lambda 4267$ line cannot be excited by resonance fluorescence, and are suitable for abundance determinations.
 - Discrepancy between the temperatures derived from collisionally excited lines and recombination lines has been found in several nebulae.
 - He⁺ abundance in H II regions
 - can be obtained from relative strengths of the radio recombination lines of H I and He I.
 - **At a very high n (radio region)**, both H and He are nearly identical on-electron systems except for their masses.
 - Hence, **their relative strengths are directly proportional to their abundances** (as long as they are optically thin, and the nebula is a complete H⁺ plus He⁺ region, without no H⁺ plus He⁰ region).
 - Abundances of heavy elements
 - For this purpose, strong collisionally excited lines are available. However, the temperature must be determined from observational data because the lines depends sensitively on the temperature.

$$I_\nu = \frac{1}{4\pi} \int h\nu_{12} n_2 n_e q_{12}(T) ds \quad \text{where the collisional excitation rate coefficient } q_{12} = \frac{8.63 \times 10^{-6}}{T^{1/2}} \frac{\Omega_{21}}{g_1} e^{-\chi/kT}.$$

(Abundance Determination)

- Helium abundance
 - The abundance of He is determined from **comparison of the strengths of radiative recombination lines of H and He** in regions ionized by stars that are sufficiently hot ($T \gtrsim 3.9 \times 10^4$ K) so that He is ionized throughout the H II regions.
- Heavy elements
 - The abundance of heavy elements can be inferred by **comparing the strengths of collisionally excited lines with recombination lines of H.**

Oxygen:

$$\left(\lambda_{\text{H}\beta} = 4861.35 \text{\AA} \right)$$

$$4\pi j([\text{OIII}] 5008) = n_e n(\text{O}^{+2}) k_{03} \frac{A_{32}}{A_{31} + A_{32}} E_{32}$$

$$4\pi j(\text{H}\beta) = n_e n(\text{H}^+) \alpha_{\text{eff}, \text{H}\beta} E_{\text{H}\beta}$$

where

$$\alpha_{\text{eff}, \text{H}\beta} \approx 3.03 \times 10^{-14} T_4^{-0.874 - 0.058 \ln T_4} \text{ cm}^3 \text{ s}^{-1}$$

$$k_{03} = 8.62942 \times 10^{-8} T_4^{-1/2} \frac{\Omega_{30}}{g_0} e^{-E_{30}/kT} \text{ cm}^3 \text{ s}^{-1} \quad (g_0 = 1)$$

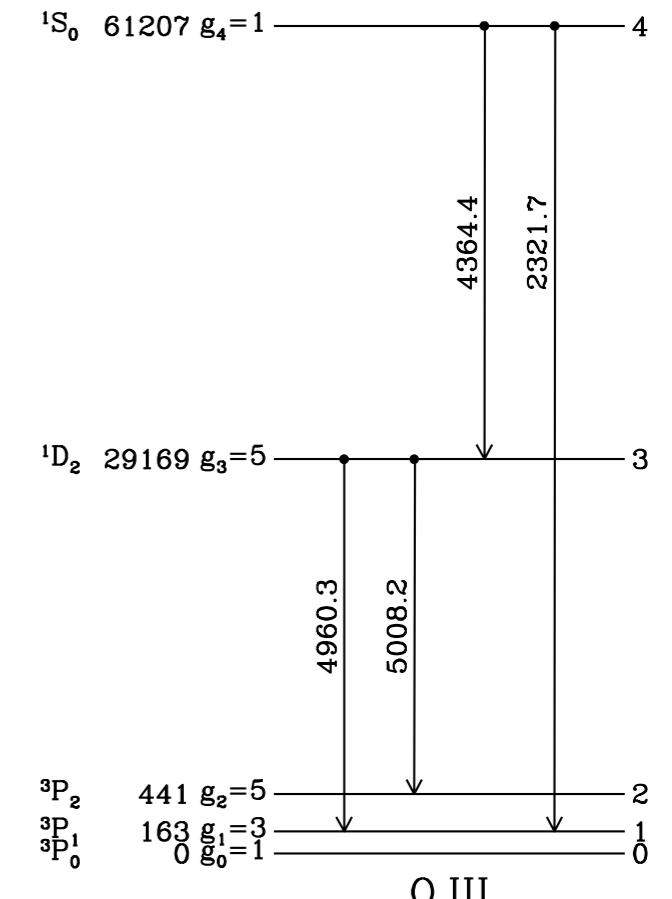
$$E_{32}/k = 29169 \text{ K}, \quad E_{\text{H}\beta}/k = 29588.5 \text{ K}$$

$$\Omega_{30} = 0.243 T_4^{0.120 + 0.031 \ln T_4}$$

$$A_{32} = 2.0 \times 10^{-2} \text{ [s}^{-1}\text{]}$$

$$A_{31} = 6.8 \times 10^{-3} \text{ [s}^{-1}\text{]}$$

$$\frac{[\text{OIII}] 5008}{\text{H}\beta} = 5.091 \times 10^5 T_4^{0.494 + 0.089 \ln T_4} e^{-2.917/T_4} \frac{n(\text{O}^{+2})}{n(\text{H}^+)}$$



- **Nitrogen:**

$$\left(\lambda_{\text{H}\alpha} = 6562.79 \text{\AA} \right)$$

$$4\pi j(\text{[NII]} 6585) = n_e n(\text{N}^+) k_{03} \frac{A_{32}}{A_{31} + A_{32}} E_{32}$$

$$4\pi j(\text{H}\alpha) = n_e n(\text{H}^+) \alpha_{\text{eff}, \text{H}\alpha} E_{\text{H}\alpha}$$

where

$$\alpha_{\text{eff}, \text{H}\alpha} \approx 1.17 \times 10^{-13} T_4^{-0.942 - 0.031 \ln T_4} \text{ cm}^3 \text{ s}^{-1}$$

$$k_{03} = 8.62942 \times 10^{-8} T_4^{-1/2} \frac{\Omega_{30}}{g_0} e^{-E_{30}/kT} \text{ cm}^3 \text{ s}^{-1} \quad (g_0 = 1)$$

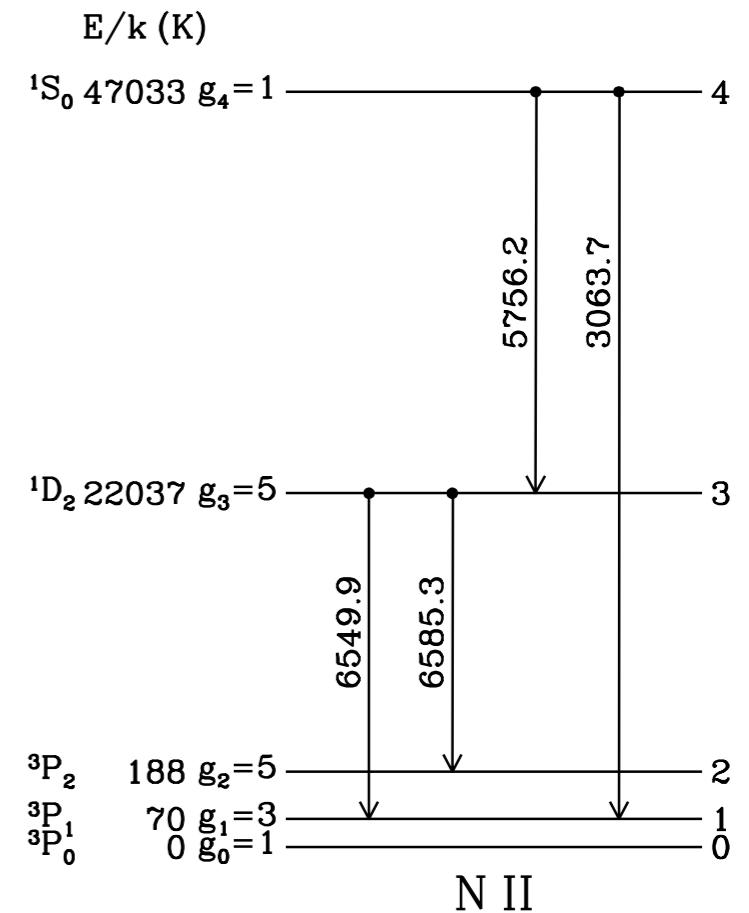
$$E_{32}/k = 21849 \text{ K}, \quad E_{\text{H}\alpha}/k = 21916.9 \text{ K}$$

$$\Omega_{30} = 0.303 T_4^{0.053 + 0.009 \ln T_4}$$

$$A_{32} = 3.0 \times 10^{-3} \text{ [s}^{-1}\text{]}$$

$$A_{31} = 9.8 \times 10^{-4} \text{ [s}^{-1}\text{]}$$

$$\frac{[\text{NII}]}{\text{H}\alpha} = 1.679 \times 10^5 T_4^{0.495 + 0.040 \ln T_4} e^{-2.185/T_4} \frac{n(\text{N}^+)}{n(\text{H}^+)}$$



- Therefore, if the temperature T is known, the relative abundance of the ion can be obtained from the measured line ratio.
- The total elemental abundances are then obtained by applying **ionization correction factors (ICFs)**, which correct for abundances of unobserved ions.

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- One-layer model
 - The abundances can be determined on the basis of a model of the structure of the nebula.
 - The simplest model treats the nebula as homogeneous with constant T and n_e , which may be called a one-layer model.
 - Discrepancies in n_e and T determined from different line ratios indicate that this model is too simplified, though **the abundances determined from it are generally correct to within a factor of order two or three.**
 - Discrepancies
 - Recall that **the temperatures determined from hydrogen recombination lines, the Balmer jump, and free-free brightness temperatures are lower than the temperature determined from collisionally excited (forbidden) line ratios.**
 - The probable explanation of this discrepancy is that the temperature is not constant throughout the nebula, but rather varies from point to point due to variations in the local heating and cooling rates.
 - A complex model approach, as will be discussed in section 5.12, would be required to know the entire temperature structure.
 - However, the discrepancies can easily be understood in a relatively easy way.

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- Origin of the discrepancies
 - The emission line ratios determine the temperature of the region where the lines are predominantly originate from.
 - The emission coefficient of collisionally excited lines increases strongly with increasing temperature, and thus the temperature is strongly weighted toward high-temperature regions.
 - However, the recombination and free-free emission coefficients decrease with increasing temperature, and thus the resulting temperature is weighted toward low-temperature regions.
 - We thus expect that the collisionally excited lines indicate a higher temperature than do the Balmer jump or radio-frequency measurements.
 - A somewhat more sophisticated scheme to understand the discrepancies
 - The emission rate coefficient is expand in a power series of temperature:

$$\varepsilon_l(T) = \varepsilon_l(T_0) + (T - T_0) \left(\frac{d\varepsilon_l}{dT} \right)_0 + \frac{1}{2}(T - T_0)^2 \left(\frac{d^2\varepsilon_l}{dT^2} \right)_0$$

- $\varepsilon_l(T) = c_1 T^{-m}$ ($m \approx 1$) for recombination lines (the recombination rate coefficient)
- $\varepsilon_l(T) = \frac{c_2 \exp(-\chi/kT)}{T^{1/2}}$ for recombination lines

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- The emissivity is given by integrating along the line of sight:

$$\int n_i n_e \epsilon_l(T) ds = \epsilon_l(T_0) \int n_i n_e ds + \frac{1}{2} \left(\frac{d^2 \epsilon_l}{dT^2} \right)_0 \int n_i n_e (T - T_0)^2 ds$$

Here, the first order term of $T - T_0$ vanishes if we choose T_0 as the mean temperature,

$$T_0 = \frac{\int n_i n_e T ds}{\int n_i n_e ds}$$

\Rightarrow mean temperature

and we define

$$t^2 = \frac{\int n_i n_e (T - T_0)^2 ds}{T_0^2 \int n_i n_e ds}$$

- , representing the variance of temperature.
- If all ions had the same space distribution $n_i(s)$, then both T_0 and t^2 could be determined by combining two line ratios (for instance [O III] $(\lambda 4959 + \lambda 5007)/\lambda 4363$ and [N II] $(\lambda 6543 + \lambda 6583)/\lambda 5755$)
 - Then, T_0 and t^2 could be used to determine the abundances of all the ions with measured lines.

However, all ions do not have the same spatial distribution. For instance, O⁺⁺ is more strongly concentrated to the source of ionizing radiation than N⁺.

The most sophisticated method of all to determine the abundances from the observations is to calculate a complete model of the nebula in an attempt to reproduce all its observed properties, as will be discussed in section 5.12.

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- **Notes:**
 - [O III] and [N II] collisional excitation lines trace the abundance of O⁺⁺ and N⁺, respectively.
 - The He I recombination line provides the abundance of He⁺, whereas the He II recombination line traces that of He⁺⁺.
 - Because there is no “observable” He I collisional excitation line, **there is no way to determine the He⁰ abundance directly.**
 - Observational results
 - He/H abundance ratio
 - ▶ $n(\text{He}^+)/n_p = 0.06 - 0.09$ in Orion Nebula, suggesting that the varying amounts of He⁰ are present.
 - ▶ $n(\text{He}^+)/n_p = 0.009$ in NGC 1982, indicating that the observation was performed in an H⁺ plus He⁰ zone, where He is predominantly neutral. The exciting star of NGC 1982 is a B1 V star, so the He-ionizing radiation is very weak or absent.
 - **Ionization Correction Factor (ICF)**
 - ▶ The above observations indicate that some correction is necessary to estimate the abundance of He, specifically to compensate for the amount of unobserved He⁰. An ion of another element can be used its proxy.
 - ▶ Empirically, the correction can be based on [S II] $\lambda\lambda 6717, 6731$, because S⁺ has an ionization potential of 23.4 eV, similar to that of He⁰ (24.6 eV). Therefore, to a first approximation

$$\frac{n(\text{He}^0)}{n(\text{He}^+)} = \frac{n(\text{S}^+)}{n(\text{S}^{++})} \quad \Rightarrow \quad n(\text{He}^0) \approx n(\text{He}^+) \frac{n(\text{S}^+)}{n(\text{S}^{++})}$$

Such ratios are called ICFs, since they correct for unobserved stages of ionization.

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- Radio measurements of He^+/H^+ abundance ratios are available for many diffuse nebulae.
 - ▶ Radio observations have the advantage that they can be done at large distances in the galactic plane.
 - ▶ However, there no known way in which the correction for He^0 can be obtained from radio measurements.
 - ▶ NGC 2024 and NGC 1982 are observed to have $n(\text{He}^+)/n_p \approx 0$. Radio measurements of H II regions very near the galactic center give quite low $n(\text{He}^+)/n_p$ ratios.
 - ▶ These low ratios are due to their ionizing stars being predominantly cool, rather than suggesting low helium abundance. These indicates corrections are certainly required.
 - PNe
 - ▶ Many PNe show the presence of both He^{++} and He^+ , though some PNe have only He I lines.
 - ▶ Nearly all PNe have central stars that are so hot that they have no outer He^+ , He^0 zones. Therefore, no correction is necessary for unobserved He^0 in most PNe.
 - ▶ Most of PNe have $n(\text{He})/n(\text{H}) \approx 0.11$, slightly higher than those of H II regions. Some PNe show twice larger abundances.
 - ▶ These differences are real, and the He abundance can help distinguish between different populations of PNe.

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- Abundance of the heavy elements in H II regions and PNe
 - ▶ They are determined by a combination of collisionally excited lines and hydrogen recombination lines. The line ratio depends mainly on the shape of the stellar continuum from the following relation.

$$\frac{\text{total cooling}}{L(\text{H}\beta)} \approx \frac{\text{cooling by collisional excitation lines}}{L(\text{H}\beta)} \propto \frac{\int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} h(\nu - \nu_0) \sigma_\nu^{\text{p.i.}} d\nu}{\int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} d\nu}$$

Note also that the strength of collisionally excited lines depends strongly on temperature, which is determined by the shape of the stellar continuum. On the other hand, the recombination lines are not very sensitive to the gas temperature.

- ▶ The gas is hotter in lower-metallicity nebulae.
- ▶ Disadvantages: (1) Collisional excitation lines can be converted into abundances only when the gas temperature is well measured. (2) Additionally, large and rather uncertain corrections may be required for unobserved ion stages.
- ▶ The abundances of a PN are affected by nuclear processing in the central star, whereas those in H II regions reflect the composition of the interstellar medium.
- Recombination lines of the heavy elements
 - ▶ Large telescopes and CCD detectors have made it possible to detect faint recombination lines of the heavy elements in several H II regions and PNe.
 - ▶ Abundances measured with these methods should be far more robust than those inferred from collision excitation lines, as they have similar dependencies on temperature to hydrogen recombination lines.
 - ▶ This method would not be affected by temperature fluctuations.
 - ▶ However, the lines are faint, and the recombination process is complicated by strong dielectronic recombination.

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- Discrepancies in H II regions
 - ▶ In H II regions, **the recombination-line abundances tend to be slightly higher than the forbidden-line abundances**, which is attributed to the temperature fluctuations.
 - ▶ If this is the case, the correct gas-phase abundances are about 50%-100% higher than $t^2 \approx 0.02 - 0.04$
 - PNe
 - ▶ Recent works found differences between collisional and recombination abundances that are larger than a factor of ten, too large to be caused by temperature fluctuations.
 - ▶ This suggests that another, presently unknown, process may be affecting one (or both) of the abundance measures.
 - ▶ The fact that the discrepancy varies from object to object, indicates that it is not a simple matter of incorrect atomic data. This difference might be related to physical conditions within PNe. If this is the case, then the higher recombination abundances are more likely to be correct.

6.1 Introduction

- The previous chapters have described gaseous nebulae from a static point of view.
- Nebulae certainly have internal motions, and the effects of these motions on their structures cannot be ignored.
 - If an ionized nebula is matter-bounded, it will expand into the surrounding vacuum.
 - On the other hand, if it is ionization-bounded, the hot ionized gas (with $T \approx 10^4$ K) will initially have higher pressure than the surrounding cooler neutral gas ($T \approx 100$ K) and will therefore tend to expand until its density is low enough so that the pressures of the two gases are in equilibrium.
 - When a hot star forms and the ionizing radiation source is “turned on”, ionized volume initially grows in size at a rate of emission of ionizing photons, and an ionizing front separating the ionized and neutral regions propagates into the neutral gas.
- Typical motions in PNe and H II regions
 - Planetary nebulae expand more or less radially with velocities of order 25 km s^{-1} , and the velocity increases radially.
 - Many H II regions are observed to have complex internal velocity distributions that can best be described as turbulent.
- Contents
 - Hydrodynamic equations of motion
 - Ionization fronts and shock fronts that are generated in an expanding, photoionized gas.
 - PNe and H II regions are analyzed
 - Observational material

6.2 Hydrodynamic Equations of Motion

- Hydrodynamics deals with changes and movements in a fluid as forces act on it.
- Reference Frames
 - **Lagrangian reference frame - one that moves with the fluid.**
 - **Eulerian reference frame - one that is stationary.**
- The time derivative of f , any physical quantity that is a function of position and time, in the Lagrangian reference frame is related to the derivatives in the Eulerian reference frame.

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f$$

The **Eulerian derivatives** on the right represents (1) changes in f over time and (2) advection, changes due to the flow of upstream material into the region.

- Hydrodynamic equations
 - Three hydrodynamic equations - conservation of mass, momentum, and energy
 - Two additional equations - an ionization equation and an equation of state.
 - The five equations are needed to fully determine conditions in the gas.

Basics of Hydrodynamics (from ISM)

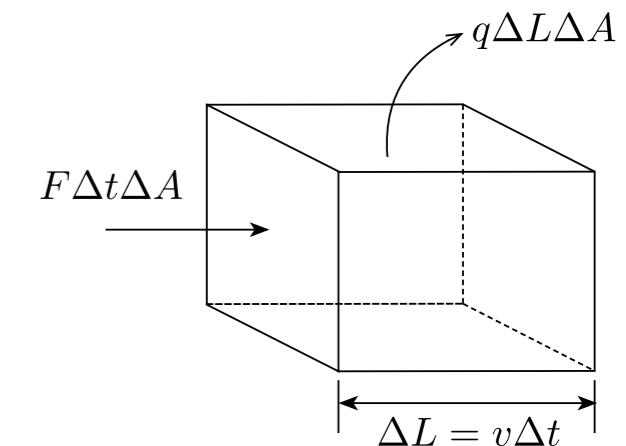
- Assumption for hydrodynamics:
 - particle mean free path \ll size of the region
 - We will describe the equations for conservation of mass, momentum and energy, in 1D space.

- ***Definition***

- Flux of a hydrodynamic quantity q (for instance, density):

Fluid moves a distance ΔL during a time interval Δt with a velocity v .

$$F\Delta t\Delta A = q\Delta L\Delta A \rightarrow F = qv$$



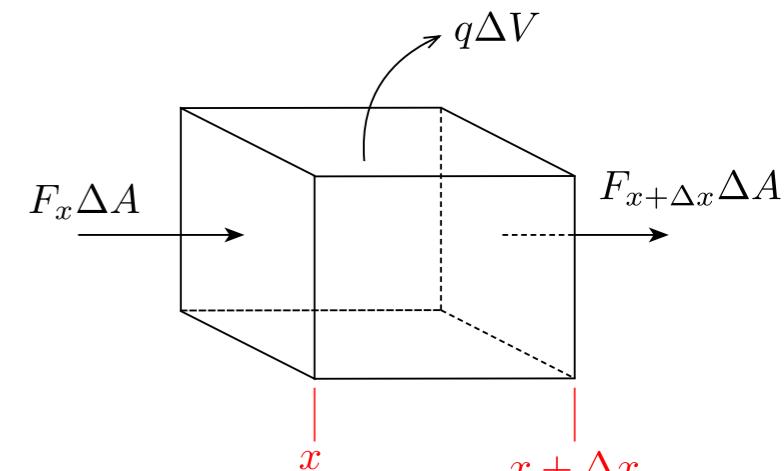
- ***Conservation equation for a quantity*** q

- change of the quantity within a volume ΔV for a time interval Δt :

Here, Δt and Δx are independent.

$$\frac{q\Delta V|_{t+\Delta t} - q\Delta V|_t}{\Delta t} = F\Delta A|_x - F\Delta A|_{x+\Delta x}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial F}{\partial x} \rightarrow \frac{\partial q}{\partial t} = -\frac{\partial(qv)}{\partial x}$$



- Here, no sources or sinks of the quantity within ΔV were assumed. If any, the loss and gain terms should be added in the right-hand side.

Basics of Hydrodynamics - Mass Conservation (from ISM)

- Conservation equations
 - ***Mass conservation (continuity equation)***

- ▶ mass within a volume $dV = \rho dV$
- ▶ no sources or sinks of material within dV
- ▶ Consider the mass per unit area (dA), contained in the volume

$$\rho dV/dA = \rho dx \quad \longrightarrow \quad \frac{\partial}{\partial t}(\rho dx) = \overbrace{\rho u}^{\text{incoming}} - \overbrace{(\rho + d\rho)(u + du)}^{\text{outgoing}}$$

$$= -(\rho du + ud\rho + \cancel{d\rho du})$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$

- ▶ Mass loss and gain terms should be added in the right-hand side, if necessary.

Basics of Hydrodynamics - Momentum Conservation

- **Momentum conservation (Euler's equation)**

- ▶ momentum within dV (per unit area) = $(\rho dV)u/dA = \rho dx u$
- ▶ = change of momentum due to fluid flow and gas pressure acting on the surface of dV

$$\begin{aligned}\frac{\partial}{\partial t}(\rho u dx) &= \overbrace{\rho u^2}^{\text{incoming}} - \overbrace{(\rho + d\rho)(u + du)^2}^{\text{outgoing}} + \overbrace{P}^{\text{incoming}} - \overbrace{P + dP}^{\text{outgoing}} \\ &= \rho u^2 - \left(\rho u^2 + 2\rho u du + \cancel{\rho du^2} + u^2 d\rho + \cancel{2ud\rho du} + \cancel{d\rho u^2} \right) - dP\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\rho u) &= -2\rho u \frac{\partial u}{\partial x} - u^2 \frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x} \\ \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} &= -\rho u \frac{\partial u}{\partial x} - u \left(\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} \right) - \frac{\partial P}{\partial x}\end{aligned}$$

Using mass conservation, $\frac{\partial u}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$

$$\rho \frac{\partial u}{\partial t} = -\rho u \frac{\partial u}{\partial x} - \frac{\partial P}{\partial x}$$

or

$$\begin{aligned}\frac{\partial}{\partial t}(\rho u) &= -2\rho u \frac{\partial u}{\partial x} - u^2 \frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x} \\ &= -\frac{\partial}{\partial x}(\rho u^2) - \frac{\partial P}{\partial x}\end{aligned}$$

$$\frac{\partial}{\partial t}(\rho u) = -\frac{\partial}{\partial x} (\rho u^2 + P)$$

- ▶ Further terms could be added in the right-hand side, accounting for forces due to gravity, magnetic fields, radiation field, and viscosity.

Basics of Hydrodynamics - Energy Conservation

- *Energy conservation*

- The first law of thermodynamics states that

heat added in a system = change in internal energy + work done on surroundings

$$dQ = dU + PdV$$

- Internal energy (per particle) for ideal gas is

$U/N = \frac{3}{2}kT$ for monatomic gas (translation about 3 axes)

$U/N = \frac{5}{2}kT$ for diatomic gas (+rotation about 2 axes)

$U/N = 3kT$ for polyatomic gas (+rotation about 3 axes)

Here, N is the number of particles.

An ideal gas is a theoretical gas composed of many randomly moving point particles whose only interactions are perfectly elastic collisions (no viscosity or heat conduction).

- In general, the internal energy per particle is

$$U/N = \frac{f}{2}kT \quad (f = \text{degree of freedom})$$

At high temperature, molecules have access to an increasing number of vibrational degrees of freedom, as they start to bend and stretch.

- The ideal gas law (the equation of state) for a perfect Maxwellian distribution.

$$PV = NkT$$

$$P = \frac{N}{V}kT$$

- **Specific heat capacity** is the amount of **heat energy required to raise the temperature of a material per unit of mass**.

- ▶ specific heat capacity **at constant volume**:

$$c_V \equiv \frac{1}{M} \left(\frac{\partial Q}{\partial T} \right)_V = \frac{1}{M} \left(\frac{\partial U}{\partial T} \right)_V$$

$$c_V = \frac{f}{2} \frac{k}{m}$$

M = total mass

$m = M/N$ = mass per particle

$m = \mu m_H$

(μ = mean atomic weight per particle)

- ▶ specific heat capacity **at constant pressure**:

$$c_P \equiv \frac{1}{M} \left(\frac{\partial Q}{\partial T} \right)_P = \frac{1}{M} \left(\frac{\partial U}{\partial T} \right)_P + \frac{P}{M} \left(\frac{\partial V}{\partial T} \right)_P = \frac{1}{M} \frac{f}{2} Nk + \frac{P}{M} \frac{Nk}{P} \quad \boxed{\downarrow}$$

- ▶ Ratio of specific heat capacities:

$$\therefore c_P = \frac{f+2}{2} \frac{k}{m} = c_V + \frac{k}{m}$$

$$\gamma \equiv \frac{c_P}{c_V} = \frac{f+2}{f} = \frac{5}{3} \text{ for monatomic gas}$$

$$= \frac{7}{5} \text{ for diatomic gas}$$

$$= \frac{4}{3} \text{ for polyatomic gas}$$

γ is called the **adiabatic index**.

$$c_P > c_V$$

This inequality implies that when pressure is held constant, some of the added heat goes into PdV work instead of into internal energy.

- Energy Conservation - limiting cases

- **Adiabatic flow** - negligible heat transport (Internal energy is changed only by work).

$$dQ = dU + PdV = Mc_VdT + PdV$$

$$dQ = 0$$

$$\rightarrow PdV = -Mc_VdT$$

$$PV = NkT$$

$$\rightarrow VdP + PdV = NkdT$$

We combine two equation and eliminate dT term:

$$\begin{aligned} VdP + PdV &= -\frac{Nk}{Mc_V} PdV \\ &= -\frac{k}{m} \frac{1}{c_V} PdV \end{aligned}$$

$$\begin{aligned} VdP &= -\left(1 + \frac{k}{m c_V}\right) PdV \\ &= -\frac{1}{c_V} \left(c_V + \frac{k}{m}\right) PdV \\ &= -\gamma PdV \end{aligned}$$



$$\frac{dP}{P} = -\gamma \frac{dV}{V}$$

We can rewrite this in terms of density:

$$\begin{aligned} \rho V &= M \\ \rightarrow \rho dV + Vd\rho &= 0 \\ \rightarrow \frac{d\rho}{\rho} &= -\frac{dV}{V} \end{aligned}$$

$$\longrightarrow \frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$

In summary,

$$\frac{dP}{P} = -\gamma \frac{dV}{V}$$

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$

$$\begin{aligned} P &\propto \rho^\gamma \\ P &\propto V^{-\gamma} \end{aligned}$$

$$\rightarrow T \propto V^{-(\gamma-1)}$$

adiabatic heating/cooling

- **Isothermal flow** - extremely efficient cooling (heat transport).

heat transport timescale << dynamic timescale

This implies the balance between heating and cooling, hence a constant temperature.

From the ideal gas law,

$$P = \frac{N}{V} kT = \rho \frac{kT}{m} \longrightarrow \begin{aligned} P &\propto \rho \\ P &\propto V^{-1} \end{aligned}$$

- **In general**, we have

$$\begin{aligned} P &\propto \rho^\gamma \\ P &\propto V^{-\gamma} \end{aligned}$$

$(\gamma = 1$ for isothermal gas)

A gas that has an equation of state with this power-law form is called a **polytope**, from the Greek polytropos, meaning “turning many ways” or “versatile.”

(A polystrope should not be confused with a polytrope, which is the n-dimensional generalization of a 2D polygon and 3D polyhedron.)

- **Specific internal energy** of the gas (*per unit mass*):

$$\begin{aligned}\epsilon &\equiv U/M \\ U/N &= \frac{f}{2}kT\end{aligned}\longrightarrow \epsilon = \frac{f}{2}\frac{kT}{m} \text{ or } \epsilon = \frac{1}{\gamma-1}\frac{kT}{m} = \frac{1}{\gamma-1}\frac{P}{\rho}$$

- **Total Energy (per unit volume):**

► **Internal energy per unit volume:**

$$\mathcal{E}_{\text{int}} = \rho\epsilon = \frac{1}{\gamma-1}P$$

► **Kinetic energy due to bulk motion, per unit volume:**

$$\mathcal{E}_{\text{kin}} = \rho\frac{u^2}{2}$$

► **Work on unit volume:**

$$\mathcal{E}_{\text{mech}} = \frac{PdV}{dV} = P$$

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_{\text{int}} + \mathcal{E}_{\text{kin}} + \mathcal{E}_{\text{mech}} \\ &= \rho\left(\frac{u^2}{2} + \epsilon\right) + P\end{aligned}\longrightarrow \mathcal{E} = \rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P$$

- **Energy conservation:**

$$\frac{\partial \mathcal{E}}{\partial t} = -\frac{\partial(u\mathcal{E})}{\partial x}$$

$$\frac{\partial}{\partial t} \left(\rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P \right) = -\frac{\partial}{\partial x} \left[u \left(\rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P \right) \right]$$

Basics of Hydrodynamics - Sound Wave (from ISM)

- Suppose that we are surrounded by an ideal gas with a plane parallel symmetry:
 - We consider a region where the gas has initially a uniform density, pressure, and no bulk velocity: $\rho_0, P_0, u_0 = 0$

In the uniform gas, we introduce small perturbations of the form:

$$\begin{aligned} \rho(x, t) &= \rho_0 + \rho_1(x, t) & P_1 &= P - P_0 \\ u(x, t) &= u_1(x, t) & \propto (\rho_0 + \rho_1)^\gamma - \rho_0^\gamma \\ P(x, t) &= P_0 + P_1(x, t) & \propto \gamma \rho_0^{\gamma-1} \rho_1 & \longrightarrow P_1 = \frac{\gamma P_0}{\rho_0} \rho_1 \end{aligned}$$

We obtain:

$$\begin{array}{ccc} \frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} & \rightarrow & \frac{\partial \rho_1}{\partial t} = -\rho_0 \frac{\partial u_1}{\partial x} \\ \rho \frac{\partial u}{\partial t} = -\rho u \frac{\partial u}{\partial x} - \frac{\partial P}{\partial x} & & \rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial P_1}{\partial x} = -\frac{\gamma P_0}{\rho_0} \frac{\partial \rho_1}{\partial x} \end{array} \quad \boxed{\frac{\partial^2 \rho_1}{\partial t^2} = -\frac{\gamma P_0}{\rho_0} \frac{\partial^2 \rho_1}{\partial x^2}}$$

- The resulting equation represents a **sound wave (acoustic wave)** with a constant sound speed:

$$c_s = \left(\frac{\gamma P}{\rho} \right)^{1/2} = \left(\frac{\gamma k T}{m} \right)^{1/2}$$

$$c_s \propto \rho^{(\gamma-1)/2}$$

For $\gamma > 1$ sound travels more rapidly in a denser gas.

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- The sound speed is of the same order as the mean thermal velocity:

$$c_s = 1.2 \text{ km s}^{-1} \left(\frac{\gamma}{5/3} \right)^{1/2} \left(\frac{m}{m_p} \right)^{-1/2} \left(\frac{T}{100 \text{ K}} \right)^{1/2}$$

$(m_p = \text{proton mass})$

- **Sound crossing time:**

- ▶ sound crossing time = time it takes for a signal to cross a region of size L :

$$t_{\text{cross}} = L/c_s$$

- ▶ A small pressure gradient tends to be smoothed out within the sound crossing time. Generally, when a stationary gas is disturbed, the resultant changes in velocity, density, pressure, and temperature are communicated downstream at the sound speed.

Fast changes occurring on timescales $\ll t_{\text{cross}}$ will survive, and a shock front forms.

Slow changes occurring on timescales $\gg t_{\text{cross}}$ will be damped.

- **Mach number** = fluid velocity / sound speed

$$\mathcal{M} \equiv u/c_s$$

| | |
|-------------------|------------|
| $\mathcal{M} > 1$ | supersonic |
| $\mathcal{M} < 1$ | subsonic |

- The momentum equation

- describes the motion of a compressible fluid, may be written as

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P - \rho \nabla \phi$$

- The change in momentum per unit volume (as we follow the fluid) is caused by the force terms, which include the gradient in the pressure P and the force resulting from the gravitational potential ϕ of the stars and of the nebula itself.
- The gravitational force would be negligible. Viscous dissipation and electromagnetic forces will also be neglected. However, there may be nebulae in which magnetic fields are strong and not negligible.

- The equation of continuity (conservation of mass)

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}$$

- The energy equation is a generalization of the thermal balance equation.

$$\frac{DU}{Dt} = \frac{D}{Dt} \left(\frac{3}{2} \sum_j n_j k T \right) = (G - L) + \frac{P}{\rho} \frac{D\rho}{Dt} - U \nabla \cdot \mathbf{u}$$

$$\boxed{\frac{D\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho}$$

compression + advection

- Here U is the internal kinetic energy per unit volume, G and L are the energy gain and loss rate per volume per unit time.
- The second term gives **the heating rate resulting from compression (or expansion) of the gas and the advection of new material** which may have more or less internal energy.
- The last term gives **the dilation effect**, analogous to the term on the right-hand side of the equation of continuity.

- Note that ionization energy is not included, but the kinetic energy of all particles is. It is a reasonably good approximation because all the ionized species are in temperature equilibrium. (The Coulomb-scattering cross sections are so large, and the relaxation times are correspondingly short.)
- The energy equation can be rewritten in a form that includes the internal kinetic energy per unit mass $E = U/\rho$:

$$\frac{DE}{Dt} = \frac{D}{Dt} \left(\frac{U}{\rho} \right) = \frac{1}{\rho} (G - L) - P \frac{D}{Dt} \left(\frac{1}{\rho} \right).$$

$$\begin{aligned} \frac{D(U/\rho)}{Dt} &= \frac{1}{\rho} \frac{DU}{Dt} - \frac{U}{\rho^2} \frac{D\rho}{Dt} = \frac{1}{\rho} (G - L) + \frac{P}{\rho^2} \frac{D\rho}{Dt} - \frac{U}{\rho} \nabla \cdot \mathbf{u} - \frac{U}{\rho^2} \frac{D\rho}{Dt} \\ &= \frac{1}{\rho^2} (G - L) - P \frac{D}{dt} \left(\frac{1}{\rho} \right) - \frac{U}{\rho^2} \left[\rho \nabla \cdot \mathbf{u} + \frac{D\rho}{Dt} \right] \\ &\quad \uparrow \qquad \qquad \qquad \uparrow = 0 \end{aligned}$$

- The ionization equation is

$$\begin{aligned} \frac{Dn(X^{+i})}{DT} &= \frac{\partial n(X^+)}{\partial t} + \mathbf{u} \cdot \nabla n(X^{+i}) = - n(X^{+i}) \int_{\nu_i}^{\infty} \frac{4\pi J_\nu}{h\nu} \sigma_\nu^{\text{p.i.}}(X^{+i}) d\nu \\ &\quad + n(X^{+i+1}) n_e \alpha_A(X^{+i}, T) - n(X^{+i}) n_e \alpha_A(X^{+i+1}, T) \\ &\quad + n(X^{+i-1}) \int_{\nu_{i-1}}^{\infty} \frac{4\pi J_\nu}{h\nu} \sigma_\nu^{\text{p.i.}}(X^{+i-1}) d\nu - n(X^{+i}) \nabla \cdot \mathbf{u} \end{aligned}$$

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- The equation of state, the relation between the states of the gas (pressure, density, and temperature):

$$P = \frac{\rho kT}{\mu m_H} = n_{\text{tot}} kT$$

- In most situations, the radiation pressure can be neglected, the density of radiation is so low.
- **Isothermal case** - the balance between heating and cooling processes determine the temperature, and the gas tends to be nearly isothermal.
- **Adiabatic case** - radiative losses do not occur, and the thermal energy of the gas is preserved.

$$P = K\rho^\gamma, \quad \gamma = 5/3 \text{ for a monatomic gas (as in H II regions)}$$

$$\gamma = 7/5 \text{ for a diatomic gas}$$

- The time-dependent equations are non-linear integro-differential equations.
 - **The time scale for photoionization and recombination is generally shorter than the dynamical time scale, except in the vicinity of the nebula's edge.** Thus, it is reasonable to assume a static nebula everywhere except in that region.

- **Advection of neutral material into an expanding H⁺.**
 - The advection causes the total luminosity of recombination lines such as H β to be smaller than expected.
 - The balance equation for the flux of ionizing photons is

$$\Phi(\text{H}^0) = \int [n_e n_p \alpha_B + \mathbf{u} \cdot \nabla n] dr \approx \int n_e n_p \alpha_B dr + n(\text{H}^0) u$$

of ionizing photons

(a) recombination (b) advection

Here, the number of ionizing photons entering the region is balanced by (a) the creation of neutral material by recombination and (b) advection.

- **The flux of ionizing photons is only partially used in sustaining ionization within a volume element; the remainder ionize neutral material that flows into that element.**
- For a typical H II region, $\Phi(\text{H}^0) = 2 \times 10^9 \text{ cm}^{-2} \text{ s}^{-1}$, $n = 10 \text{ cm}^{-3}$, $u = 20 \text{ km s}^{-1}$. The number of ionizing photons used that produce recombinations will be reduced by $n(\text{H}^0)u/\Phi(\text{H}^0) \approx 0.01$, the fraction of the ionizing photons used in the initial ionization of hydrogen rather than ionization following recombination. This effect can be much larger in more rapid flows.

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- The gas temperature and flow speed are related to one another for a subsonic flow.
 - From the continuity equation, the mass flux in the time-steady limit ($\partial\rho/\partial t = 0$) is given by

$$\Phi = \rho u \quad [\text{g cm}^{-2} \text{s}^{-1}]$$

From the momentum equation, the momentum flux is

$$\Pi = P + \rho u^2 = \rho c_s^2 + \rho u^2 \quad \text{for an isothermal gas}$$

$$(\gamma = 1, \text{ the sound speed } c_s^2 = \gamma P / \rho = \sqrt{kT / \mu m_H})$$

In a subsonic flow ($u \ll c_s$),

$$\text{the momentum flux is given by } \Pi = \rho c_s^2 \quad \Rightarrow \quad \therefore \quad \Pi u = \Phi c_s^2$$

Since both Π and Φ are conserved, we obtain

$$u = c_s^2 = kT / \mu m_H$$

- **For a subsonic flow, the gas temperature is proportional to the fluid velocity**, because the velocity is determined by the gas pressure, which is in turn set by the temperature.

6.3 Free Expansions into a Vacuum

- A cloud freely expanding into a vacuum
 - This situation occurs in explosive environments (i.e., envelopes of novae, probably in AGNs).
 - **Riemann problem:** This problem was first treated by Riemann.
 - Initial conditions (at $t = 0$) of an adiabatic gas:

$$\begin{aligned}\rho &= \rho_0 \text{ (constant) } r \leq r_0 \\ &= 0 \quad \quad \quad r > r_0 \\ u &= 0 \quad \text{at all positions}\end{aligned}$$
 - The edge of the gas cloud moves outward with a velocity given by

$$u_e = \frac{2}{\gamma - 1} c_s = 3c_s$$

For a monatomic gas ($\gamma = 5/3$), a rarefaction wave moves inward into the undisturbed gas.

Thus, at later time t , the rarefaction wave has reaches a radius $r_i = r_0 - c_s t$,

while the outer edge has reached a radius $r_e = r_0 + u_e t$.

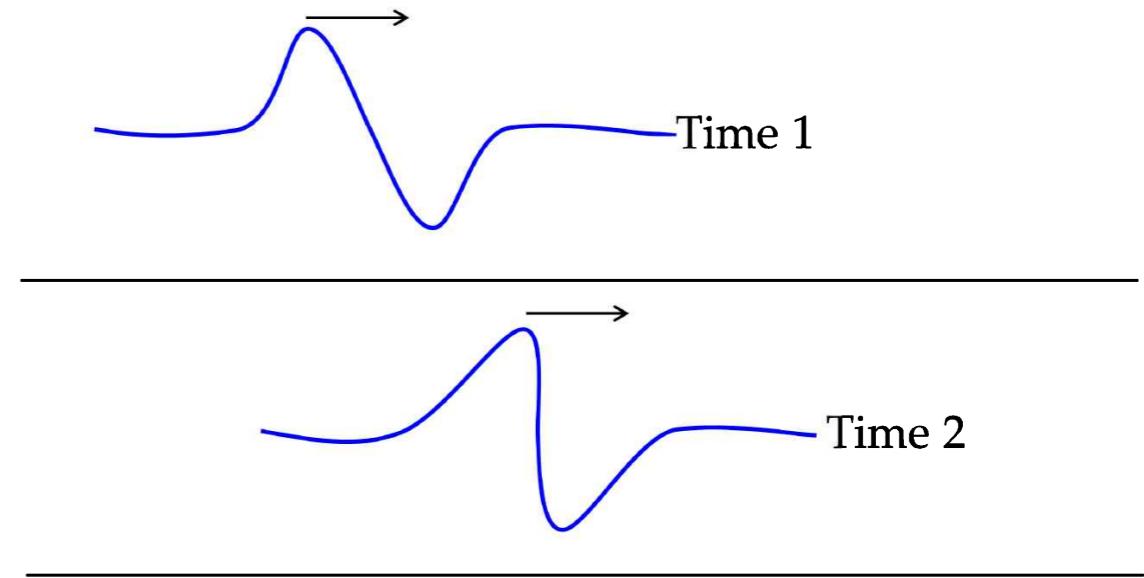
- All the gas between these two radii is moving outward with velocity increasing from 0 at r_i to u_e at r_e . In a spherical nebula, the inward-running rarefaction wave ultimately reaches the center and is reflected, and the gas near the center is then further accelerated outward.

Shock (from ISM)

- Shock

- A low-amplitude sound wave traveling through a medium will be adiabatic; that is it will not increase the entropy of the gas through which it passes.
- For an adiabatic process, the equation of state for the gas is

$$c_s \propto \rho^{(\gamma-1)/2}$$



- Thus, for $\gamma > 1$, sound travels more rapidly in a denser gas.
- *For a supersonic gas, the motion itself is faster than the speed of communication, and instead of a smooth transition, the physical quantities (density, pressure, and temperature) undergo a sudden change in values over a small distance.* This phenomenon is referred to as a **shock**.
- We define the shock front as the region over which the velocity, density, and pressure of the gas undergo sudden changes. The shock front is a layer whose thickness is comparable to the mean free path between particle collisions.
- The ordinary sound that we hear every day will not, in practice, steepen into shocks.
- However, high amplitude pressure fluctuations will rapidly steepen into shocks.

Shock Front

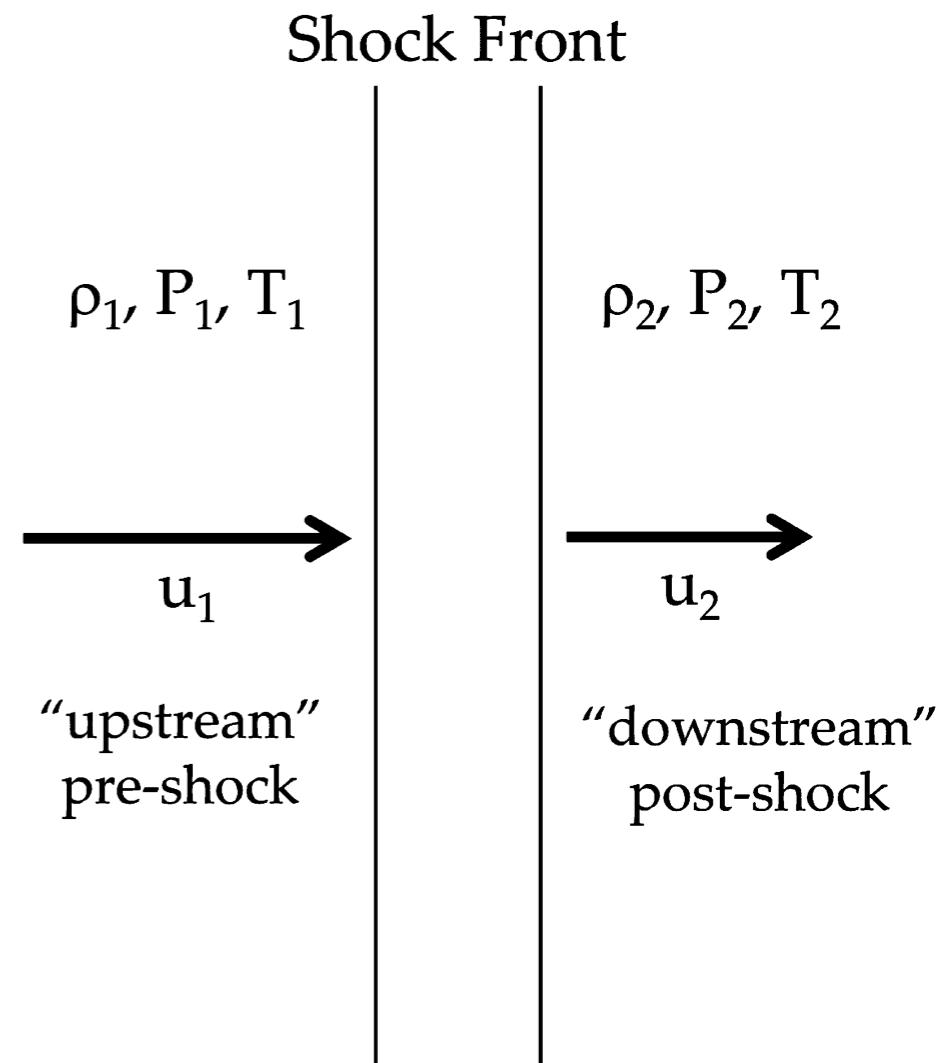
- Jump condition (*Rankine-Hugoniot conditions*)

- Let

ρ = mass density, T = temperature,

m = mean molecular mass

- If a patch is small compared to the shock front's radius of curvature, then we can treat the shock front as if it has *plane parallel* symmetry.
- ***It is convenient to use a frame of reference in which the shock front is stationary.***
- Let us consider a shock propagating with velocity V_s into a gas that is previously at rest. In the frame of reference of the shock, the gas in the pre-shock region is approaching at a velocity of $-V_s$.
- In this frame, the bulk velocity $u_1 = -V_s$ of the pre-shock (upstream) gas toward the shock front. The bulk velocity u_2 of the post-shock (downstream) gas points away from the shock front.



Plane parallel steady-state shock,
in the reference frame of the shock front.

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- Let's consider a steady state solution.
 - The gas properties immediately before being shocked (“1”) and immediately after being shocked (“2”) are obtained from the conservation laws:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

$$u_1 \left(\rho_1 \frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} P_1 \right) = u_2 \left(\rho_2 \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} P_2 \right)$$

Dividing the third equation with the first equation:

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}$$

In summary,

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2 \\ \rho_1 u_1^2 + P_1 &= \rho_2 u_2^2 + P_2 \\ \frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} &= \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2} \end{aligned}$$

Here, we assume that an adiabatic index is the same on both sides of the shock front.

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- From the three equations, we should be able to derive the changes, ρ_2/ρ_1 , u_2/u_1 , and P_2/P_1 across the shock.

It is convenient to use a dimensionless number, the Mach number of the upstream:

$$\mathcal{M}_1 = u_1/c_1, \quad c_1^2 = \frac{\gamma P_1}{\rho_1} \quad \rightarrow \quad P_1 = \frac{u_1^2 \rho_1}{\gamma \mathcal{M}_1^2}$$

(1) To find the equation for densities:

$$\begin{aligned} \rho_1 u_1^2 + P_1 &= \rho_2 u_2^2 + P_2 \\ \rho_1 u_1 = \rho_2 u_2 \text{ and } P_1 = \frac{u_1^2 \rho_1}{\gamma \mathcal{M}_1^2} &\rightarrow \rho_1 u_1^2 + u_1^2 \frac{\rho_1}{\gamma \mathcal{M}_1^2} = \frac{(\rho_1 u_1)^2}{\rho_2} + P_2 \\ &\rightarrow P_2 = \rho_1 u_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right) \end{aligned}$$

Inserting these relations into the energy conservation equation:

$$\begin{aligned} \frac{u_1^2}{2} + \frac{\gamma}{\gamma-1} \frac{P_1}{\rho_1} &= \frac{u_2^2}{2} + \frac{\gamma}{\gamma-1} \frac{P_2}{\rho_2} \\ \rightarrow \frac{u_1^2}{2} + \frac{1}{\gamma-1} \frac{u_1^2}{\mathcal{M}_1^2} &= \frac{1}{2} \left(\frac{\rho_1 u_1}{\rho_2} \right)^2 + \frac{\gamma}{\gamma-1} \frac{\rho_1 u_1^2}{\rho_2} \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right) \\ \rightarrow \frac{1}{2} + \frac{1}{\gamma-1} \frac{1}{\mathcal{M}_1^2} &= \frac{1}{2} \left(\frac{\rho_1}{\rho_2} \right)^2 + \frac{\gamma}{\gamma-1} \frac{\rho_1}{\rho_2} \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right) \end{aligned}$$



$$ax^2 + bx - c = 0$$

where $x = \frac{\rho_1}{\rho_2}$

$$a = \frac{1}{2} - \frac{\gamma}{\gamma-1}$$

$$b = \frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_1^2}$$

$$c = \frac{1}{2} + \frac{1}{(\gamma-1)\mathcal{M}_1^2}$$

$$x = \frac{b^2 \pm \sqrt{b^2 + 4ac}}{2a}$$

$$\frac{\rho_1}{\rho_2} = \frac{-\left[\frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_0^2}\right] \pm \frac{\mathcal{M}_1^2 - 1}{\mathcal{M}_1^2(\gamma-1)}}{1 - \frac{2\gamma}{\gamma-1}}$$

→

$$\frac{\rho_1}{\rho_2} = 1 \quad \text{or} \quad \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2}$$

$$\therefore \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}$$

(2) Now, we obtain the equation for pressures:

Divide the following equation

$$P_2 = \rho_1 u_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right)$$

with this

$$P_1 = \frac{u_1^2 \rho_1}{\gamma \mathcal{M}_1^2}$$



$$\frac{P_2}{P_1} = \gamma \mathcal{M}_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right)$$

$$= \gamma \mathcal{M}_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2} \right)$$

$$\therefore \frac{P_2}{P_1} = \frac{2\gamma \mathcal{M}_1^2 - (\gamma-1)}{\gamma+1}$$

(3) Using the ideal gas law:

$$P = \frac{\rho k T}{m} \quad \rightarrow \quad \frac{T_2}{T_1} = \frac{\rho_1}{\rho_2} \frac{P_2}{P_1}$$

Using the equations for densities and pressures:

$$\therefore \frac{T_2}{T_1} = \frac{[(\gamma-1)\mathcal{M}_1^2 + 2][2\gamma \mathcal{M}_1^2 - (\gamma-1)]}{(\gamma+1)^2 \mathcal{M}_1^2}$$

In summary, we obtain the jump conditions:

$$\begin{aligned}\frac{\rho_2}{\rho_1} &= \frac{(\gamma + 1)\mathcal{M}_1^2}{(\gamma - 1)\mathcal{M}_1^2 + 2} = \frac{u_1}{u_2} \\ \frac{P_2}{P_1} &= \frac{2\gamma\mathcal{M}_1^2 - (\gamma - 1)}{\gamma + 1} \\ \frac{T_2}{T_1} &= \frac{[(\gamma - 1)\mathcal{M}_1^2 + 2][2\gamma\mathcal{M}_1^2 - (\gamma - 1)]}{(\gamma + 1)^2\mathcal{M}_1^2}\end{aligned}$$

In the ***lab frame***, let V_s = shock velocity, v_1, v_2 = gas velocities in upstream (pre-shock) and downstream (post-shock), respectively ($v_1 = 0$).

Using $u_1 = -V_s$ and $u_2 = v_2 - V_s$, we have

$$\frac{-V_s}{v_2 - V_s} = \frac{(\gamma + 1)\mathcal{M}_1^2}{(\gamma - 1)\mathcal{M}_1^2 + 2}$$

Downstream velocity in the lab frame:

$$v_2 = \frac{2(\mathcal{M}_1^2 - 1)}{(\gamma + 1)\mathcal{M}_1^2} V_s$$

Note a typo in Equation (16.12) of Kwok's book.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)\mathcal{M}_1^2}{(\gamma - 1)\mathcal{M}_1^2 + 2} = \frac{u_1}{u_2}$$

$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma - 1)}{\gamma + 1}$$

$$\frac{T_2}{T_1} = \frac{[(\gamma - 1)\mathcal{M}_1^2 + 2][2\gamma\mathcal{M}_1^2 - (\gamma - 1)]}{(\gamma + 1)^2\mathcal{M}_1^2}$$

For a strong shock: $\mathcal{M}_1 \gg 1$

$$P_2 \approx \frac{2\gamma\mathcal{M}_1^2}{\gamma + 1} P_1 \xrightarrow{P_1 = c_1^2 \frac{\rho_1}{\gamma}} \frac{2\gamma(u_1/c_1)^2}{\gamma + 1} c_1^2 \frac{\rho_1}{\gamma}$$

$$T_2 \approx \frac{2(\gamma - 1)\gamma}{(\gamma + 1)^2} \mathcal{M}_1^2 T_1 = \frac{2(\gamma - 1)\gamma}{(\gamma + 1)^2} \left(\frac{u_1}{c_1}\right)^2 T_1$$

speed of the downstream in the laboratory frame:

$$\frac{\rho_2}{\rho_1} \simeq \frac{\gamma + 1}{\gamma - 1}$$

$$\frac{u_2}{u_1} \simeq \frac{\gamma - 1}{\gamma + 1}$$

$$P_2 \simeq \frac{2}{\gamma + 1} \rho_1 u_1^2$$

$$T_2 \simeq \frac{2(\gamma - 1)}{(\gamma + 1)^2} \frac{m}{k} u_1^2$$

$$v_2 \simeq \frac{2}{(\gamma + 1)} V_s$$

monatomic gas: $\gamma = 5/3$

$$\frac{\rho_2}{\rho_1} \simeq 4$$

$$\frac{u_2}{u_1} \simeq \frac{1}{4}$$

$$P_2 \simeq \frac{3}{4} \rho_1 u_1^2$$

$$T_2 \simeq \frac{3}{16} \frac{m}{k} u_1^2$$

$$v_2 \simeq \frac{3}{4} V_s$$

For an isothermal shock: $\gamma = 1$

speed of the downstream in the laboratory frame:

$$\frac{\rho_2}{\rho_1} = \mathcal{M}_1^2 = \frac{u_1}{u_2}$$

$$P_2 = \mathcal{M}_1^2 P_1 = \rho_1 u_1^2$$

$$T_2 = T_1$$

$$v_2 = \left(1 - \frac{1}{\mathcal{M}_1^2}\right) V_s$$

$$u_1 u_2 = c_1^2$$

$$c_2 = c_1$$

- Consider a strong shock
 - **No matter how strong the shock is, the gas can only be compressed by a factor of at most 4:**

$$\begin{aligned}\frac{\rho_2}{\rho_1} &\approx 4 & \text{for } \gamma = 5/3 \\ P_2 &\approx \frac{3}{4} \rho_1 u_1^2 \\ T_2 &\approx \frac{3}{16} \frac{m}{k} u_1^2\end{aligned}$$

(monatomic gas)

Note that the mean molecular mass (mass per particle) is

$$m = \frac{1.4m_{\text{H}}}{1.1} = 1.273m_{\text{H}} \quad \text{for neutral gas}$$

$$m = \frac{1.4m_{\text{H}}}{2.3} = 0.609m_{\text{H}} \quad \text{for ionized gas}$$

$n \simeq 2.3n_{\text{H}}$

for ionized gas,
one electron from an ionized hydrogen
two electrons from a doubly-ionized helium.

- In the lab frame, V_s = shock velocity, v_1 , v_2 = gas velocities in upstream and downstream, respectively.

$$u_1 = v_1 - V_s = -V_s \quad (v_1 = 0)$$

$$u_2 = v_2 - V_s$$

- Then, the post-shock velocity is

$$\frac{u_2}{u_1} = \frac{v_s - V_s}{-V_s} = \frac{1}{4} \Rightarrow v_2 = \frac{3}{4} V_s$$

- Hence, **the post-shock moves in the same direction as the shock front with a velocity of 3/4 of the shock velocity.**

- Then, the post-shock pressure, temperature, specific internal energy, and specific kinetic energy are, respectively,

$$P_2 = \frac{3}{4} \rho_1 V_s^2$$

$$T_2 = \frac{3m}{16k} V_s^2$$

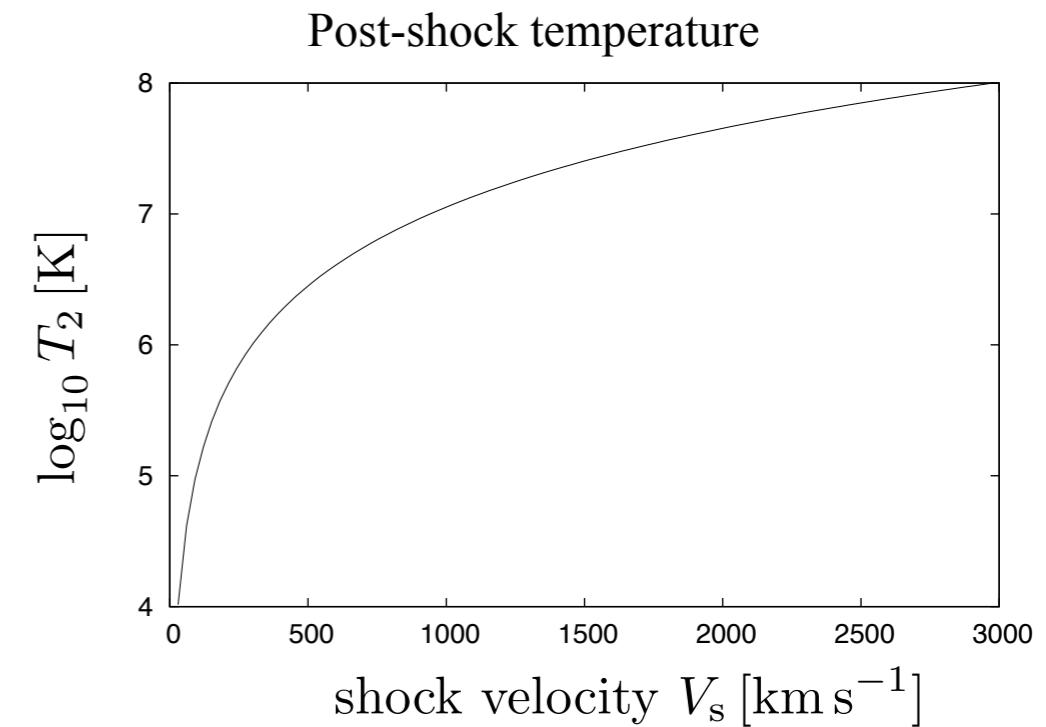
for $\gamma = 5/3$

$$\frac{\mathcal{E}_{\text{int},2}}{\rho_2} = \epsilon_2 = \frac{3}{2} \frac{P_2}{\rho_2} = \frac{3}{2} \frac{(3/4)\rho_1 V_s^2}{4\rho_1}$$

$$\frac{\mathcal{E}_{\text{kin},2}}{\rho_2} = \frac{1}{2} v_2^2$$

$$\frac{\mathcal{E}_{\text{int},2}}{\rho_2} = \frac{9}{32} V_s^2$$

$$\frac{\mathcal{E}_{\text{kin},2}}{\rho_2} = \frac{9}{32} V_s^2$$



- A strong shock can produce very high pressures and temperatures. An interstellar shock front with propagation speed $V_s \sim 1000 \text{ km s}^{-1}$ (typical for a supernova shock wave) produces shock heated gas with

$$T_2 \approx 1.38 \times 10^7 \text{ K} \left(\frac{m}{0.609m_{\text{H}}} \right) \left(\frac{V_s}{1000 \text{ km s}^{-1}} \right)^2$$

or

$$T_2 \approx 1.38 \times 10^5 \text{ K} \left(\frac{m}{0.609m_{\text{H}}} \right) \left(\frac{V_s}{100 \text{ km s}^{-1}} \right)^2$$

assuming the shocks gas is fully ionized hydrogen.

- In general, shock fronts convert supersonic gas into subsonic gas in the shock's frame of reference. Shocks increase density, pressure, and temperature, and decrease bulk velocity relative to the shock front. *Shocks act as entropy generators.*

6.4 Shocks

6.5 Ionization Fronts and Expanding H⁺ Regions

6.6 Magnetic Fields

6.7 Stellar Winds
