Astrophysics

Lecture 08
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Bremsstrahlung

[Bremsstrahlung]

- **Bremsstrahlung (= "breaking radiation") (or free-free emission)**: radiation due to the acceleration of a charge in the Coulomb field of another charge.

  Consider bremsstrahlung radiated from a plasma of temperature *T* and densities $n_e\ (\mathrm{cm}^{-3})$ electrons with charge -*e* and $n_i\ (\mathrm{cm}^{-3})$ ions with charge *Ze*.

  We calculate an important ratio:

$$\begin{aligned}\frac{\text{Coulomb potential energy}}{\text{thermal kinetic energy}} \approx \frac{Ze^2/\langle r\rangle}{kT} &\approx \frac{Ze^2/n_e^{1/3}}{kT} \\ &= 1.670\times 10^{-7}\ Z\ \left(\frac{1\ \mathrm{cm}^{-3}}{n_e}\right)^{1/3}\frac{10^4\ K}{T} \\ &\ll 1\end{aligned}$$

  for typical $n_e < 1\ \mathrm{cm}^{-3}$ and $T \sim 10^4 - 10^8$ K.

  Therefore, **Coulomb interaction is only a perturbation on the thermal motions of the electrons.**

A full understanding of this process requires a quantum treatment. However, a classical treatment is justified in some regimes, and the formulas so obtained have the correct functional dependence for most of the physical parameters.

- **Bremsstrahlung due to the collision of identical particles (electron-electron, proton-proton) is zero in the dipole approximation (in absence of external forces)**, because the dipole moment is simply proportional to the center of mass (a constant of motion).

$$\sum e_i \mathbf{r}_i = e \sum \mathbf{r}_i \propto m \sum \mathbf{r}_i = \sum m_i \mathbf{r}_i$$

- Approximations:

(1) In electron-ion bremsstrahlung, we can treat the electron as moving in a fixed Coulomb field of the ion, since the relative accelerations are inversely proportional to the masses.

$$\frac{a_i}{a_e} = \frac{m_e}{m_i} \sim (1800)^{-1} < 10^{-3}$$

(2) A series of small-angle scatterings

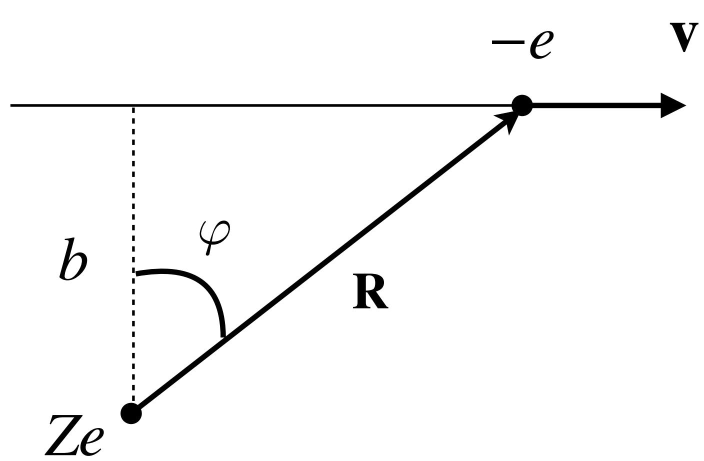
(3) Classical calculation => Quantum correction

(4) Non-relativistic => Relativistic

[Emission from single-speed Electrons]

- **Small-angle scattering** approximation:

  The electron moves rapidly enough so that the deviation of its path from a straight line is negligible.



dipole moment: $\mathbf{d} = -e\mathbf{R}$

$$\ddot{\mathbf{d}} = -e\dot{\mathbf{v}}$$

Take the Fourier transform of the second derivative of the dipole moment.

$$-\omega^2 \bar{\mathbf{d}}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\mathbf{v}} e^{i\omega t} dt \approx -\frac{e}{2\pi}\int_{-\tau}^{\tau} \dot{\mathbf{v}} e^{i\omega t} dt$$

Collision time: the electron would be in close interaction with the ion over a time interval.

$$\tau = \frac{b}{v}$$

For $\omega\tau \gg 1$, the exponential in the integral oscillates rapidly, and the integral is small.

For $\omega\tau \ll 1$, the exponential is essentially unity, so we may write

$$\bar{\mathbf{d}}(\omega) \approx \begin{cases} \dfrac{e}{2\pi\omega^2}\Delta\mathbf{v} & \text{if } \omega\tau \ll 1 \\ 0 & \text{if } \omega\tau \gg 1 \end{cases}$$

where $\Delta\mathbf{v}$ is the change of velocity during the collision.

- **Spectrum of the emitted radiation by a single electron**:

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\left|\bar{d}(\omega)\right|^2 = \begin{cases} \frac{2e^2}{3\pi c^3}|\Delta\mathbf{v}|^2 & \text{if } \omega\tau \ll 1 \\ 0 & \text{if } \omega\tau \gg 1 \end{cases}$$

Let us now estimate $\Delta\mathbf{v}$. Since the path is almost linear, the change in velocity is predominantly normal to the path.

$$F_\perp = F\cos\varphi,\ F = Ze^2/R^2,\ \cos\varphi = b/R,\ R = (b^2+v^2t^2)^{1/2}$$
$$F_\perp = Ze^2/R^2 b/(b^2+v^2t^2)^{1/2}$$

$$\Delta v \approx \Delta v_\perp = \frac{1}{m_e}\int F_\perp dt$$
$$= \frac{Ze^2}{m_e}\int_{-\infty}^{\infty}\frac{b}{(b^2+v^2t^2)^{3/2}}dt = \frac{2Ze^2}{m_e bv} \longleftarrow$$

$$\int_{-\infty}^{\infty}\frac{dx}{(b^2+x^2)^{3/2}} = \left.\frac{x}{b^2(b^2+x^2)^{1/2}}\right|_{-\infty}^{\infty} = \frac{2}{b^2}$$

Thus for small angle scatterings, **the emission from a single collision** is

$$\boxed{\frac{dW(b)}{d\omega} = \begin{cases} \frac{8Z^2e^6}{3\pi c^3 m_e^2 v^2 b^2} & \text{if } b \ll v/\omega \\ 0 & \text{if } b \gg v/\omega \end{cases}}$$

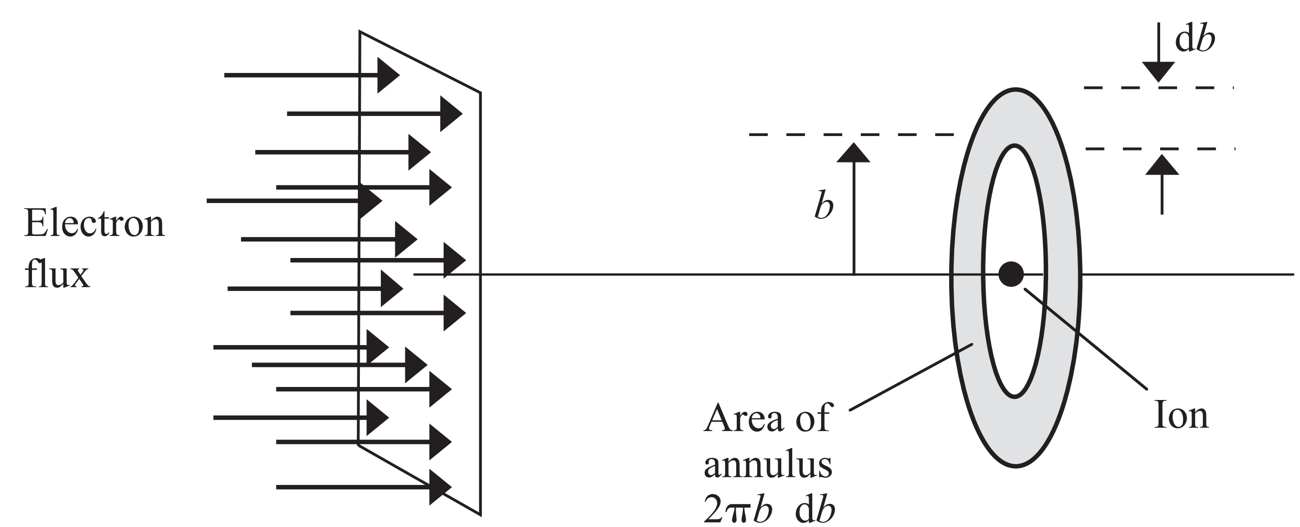
low frequency

high frequency

- **Total spectrum for a medium** with ion density n_i, electron density n_e and for a fixed electron speed.

  - flux of electrons (per unit area per unit time) incident on one ion = $n_e v$
  - element of area = $2\pi b db$
  - a good approximation is obtained in low-frequency regimes $b \ll v/\omega$:

$$
\begin{aligned}
\frac{dW}{d\omega dV dt} &= n_e n_i v \int_{b_{\min}}^{\infty} \frac{dW(b)}{d\omega} 2\pi b db \\
&= \frac{16e^6}{3c^3 m_e^2 v} n_e n_i Z^2 \int_{b_{\min}}^{b_{\max}} \frac{db}{b} \\
&= \frac{16e^6}{3c^3 m_e^2 v} n_e n_i Z^2 \ln\left(\frac{b_{\max}}{b_{\min}}\right)
\end{aligned}
$$



[Bradt (2008) Astrophysics Processes]

- **Upper limit:** The upper limit is uncertain, but it is of $b_{\max} \sim v/\omega$. Since $b_{\max}$ occurs inside the logarithm, its precise value is not very important.

  The integral is negligible for $b \gg b_{\max} \sim v/\omega$.

- **Lower limits**

  by the small-angle approximation: $\Delta v/v \sim (Ze^2/b)/(m_e v^2/2) < 1 \quad \rightarrow \quad b_{\min} > b_{\min}^{(1)} \equiv Ze^2/m_e v^2$

  by the uncertainty principle: $\Delta x \Delta p \geq \hbar \quad \rightarrow \quad b_{\min} > b_{\min}^{(2)} \equiv \hbar/m_e v$ (de Broglie wavelength)

When $b_{\min}^{(1)} \gg b_{\min}^{(2)}$ or $\frac{1}{2}m_e v^2 \ll Z^2 \mathrm{Ry}$ $\left(\mathrm{Ry} \equiv \frac{m_e c^4}{2\hbar^2} = 13.6 \text{ eV} = \text{Rydberg energy for H atom}\right)$

a classical description of the scattering process is valid. Then, $b_{\min} = b_{\min}^{(1)}$

When $b_{\min}^{(1)} \ll b_{\min}^{(2)}$ or $\frac{1}{2}m_e v^2 \gg Z^2 \mathrm{Ry}$

the uncertainty principle plays an important role. Then, $b_{\min} = b_{\min}^{(2)}$

$$\therefore \; b_{\min} = \max\left(b_{\min}^{(1)}, b_{\min}^{(2)}\right)$$

- For any regime the exact results are conveniently stated in terms of correction factor or **Gaunt factor**. Precise expression of the Gaunt factor comes from QED (Quantum Electrodynamics) computation.

$$4\pi j_\omega(v,\omega) = \frac{dW}{d\omega dV dt} = \frac{16\pi e^6}{3\sqrt{3}c^3 m_e^2 v} n_e n_i Z^2 g_{\rm ff}(v,\omega) \qquad g_{\rm ff}(v,\omega) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{b_{\max}}{b_{\min}}\right)$$

Typically $g_{\rm ff} \approx 1$ to a few.

Tables and plots are available by Bressaard and van de Hulst (1962) and Karzas and Latter (1961).

[Thermal Bremsstrahlung Emission]

- We now average the above single-speed expression over a thermal distribution of electron speeds.

$$f(\mathbf{v})d^3\mathbf{v} = \left(\frac{m_e}{2\pi kT}\right)^{3/2} e^{-m_e v^2/2kT} d^3\mathbf{v} = \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} e^{-m_e v^2/2kT} v^2 dv$$

At frequency ν, the incident velocity must be at least such that $\frac{1}{2}m_e v^2 \geq h\nu$, because otherwise a photon of energy $h\nu$ could not be created.

This cutoff in the lower limit of the integration over electron velocities is called a ***photon discreteness effect***.

$$\frac{dW}{dV\,dt\,d\omega} = \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} \int_{v_{\min}}^{\infty} \frac{dW(v,\omega)}{dV\,dt\,d\omega} v^2 e^{-m_e v^2/2kT} dv \qquad \left(\text{where} \quad v_{\min} \equiv \sqrt{\frac{2h\nu}{m_e}}\right)$$

$$= \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} \frac{16\pi e^6}{3^{3/2} c^3 m_e^2} n_i n_e Z^2 \int_{v_{\min}}^{\infty} \frac{g_{\rm ff}(v,\omega)}{v} v^2 e^{-m_e v^2/2kT} dv$$

$$= \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} \frac{16\pi e^6}{3^{3/2} c^3 m_e^2} n_i n_e Z^2 \int_{v_{\min}}^{\infty} g_{\rm ff}(v,\omega) e^{-m_e v^2/2kT} d(v^2/2)$$

The exponential factor can be written as

$$\exp\left(-\frac{m_e v^2}{2kT}\right) = \exp\left(-\frac{m_e v_{\min}^2}{2kT}\right) \exp\left(-\frac{m_e(v^2 - v_{\min}^2)}{2kT}\right) = \exp\left(-\frac{h\nu}{kT}\right) \exp\left(-\frac{m_e(v^2 - v_{\min}^2)}{2kT}\right)$$

$$\frac{dW}{dV\,dt\,d\omega} = \left(\frac{m_e}{kT}\right)^{3/2}\left(\frac{2}{\pi}\right)^{1/2}\frac{16\pi e^6}{3^{3/2}c^3 m_e^2} n_i n_e Z^2 e^{-h\nu/kT}\left(\frac{m_e}{kT}\right)^{-1}\int_0^\infty g_{\rm ff}(v,\omega)e^{-u}du$$

$$\left(\text{where} \quad u \equiv \frac{m_e(v^2 - v_{\rm min}^2)}{2kT}\right)$$

In terms of $\nu = \omega/(2\pi)$, the volume emissivity is

$$\begin{aligned}
\varepsilon_\nu^{\rm ff} \equiv \frac{dW}{dV\,dt\,d\omega} &= 2\pi\left(\frac{m_e}{kT}\right)^{1/2}\left(\frac{2}{\pi}\right)^{1/2}\frac{16\pi e^6}{3^{3/2}c^3 m_e^2} n_i n_e Z^2 e^{-h\nu/kT}\int_0^\infty g_{\rm ff}(v,\omega)e^{-u}du \\
&= \left(\frac{2}{kT}\right)^{1/2}\frac{32\pi^{3/2}e^6}{3^{3/2}c^3 m_e^{3/2}} n_i n_e Z^2 e^{-h\nu/kT}\int_0^\infty g_{\rm ff}(v,\omega)e^{-u}du \\
&= \frac{2^5\pi e^6}{3m_e c^3}\left(\frac{2\pi}{3km_e}\right)^{1/2} n_i n_e Z^2 T^{-1/2} e^{-h\nu/kT}\overline{g_{\rm ff}}
\end{aligned}$$

$$\boxed{\varepsilon_\nu^{\rm ff} = 6.8\times 10^{-38} n_i n_e Z^2 T^{-1/2} e^{-h\nu/kT}\overline{g_{\rm ff}}}$$

$$\overline{g_{\rm ff}} \equiv \int_0^\infty g_{\rm ff}(v,\omega)e^{-u}du$$

$$g_{\rm ff}(v,\omega) = (\sqrt{3}/\pi)\ln\left(b_{\rm max}/b_{\rm min}\right)$$

where $\overline{g_{\rm ff}}$ is the velocity-averaged free-free Gaunt factor.

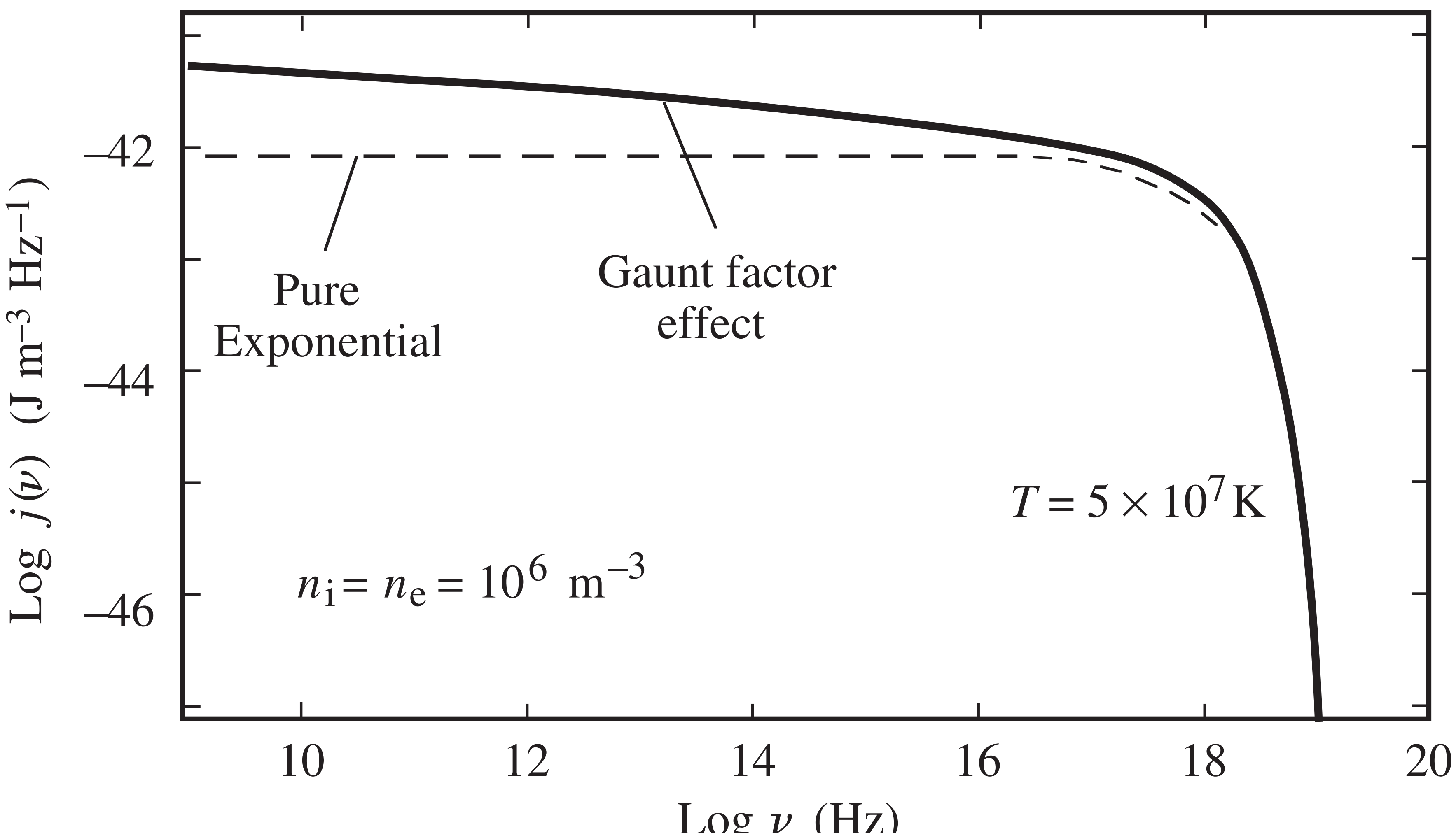
Summing over all ion species gives the emissivity:

$$\boxed{\varepsilon_\nu^{\rm ff} = 6.8\times 10^{-38}\sum_i n_i n_e Z^2 T^{-1/2} e^{-h\nu/kT}\overline{g_{\rm ff}} \quad (\text{erg s}^{-1}\ \text{cm}^{-3}\ \text{Hz}^{-1})}$$

Note that main frequency dependence is $\epsilon_\nu^{\mathrm{ff}} \propto \exp\left(-h\nu/kT\right)$, which shows a "flat spectrum" with a cut off at $\nu \sim kT/h$. The cut-off of the spectrum can be used to determine the temperature of hot plasma.

For a hydrogen plasma ($Z = 1$) with $T > 3 \times 10^5$ K at low frequencies ($h\nu \ll kT$) Gaunt factor is given by

$$\overline{g_{\mathrm{ff}}} = \frac{\sqrt{3}}{\pi} \ln\left(\frac{2.25kT}{h\nu}\right)$$



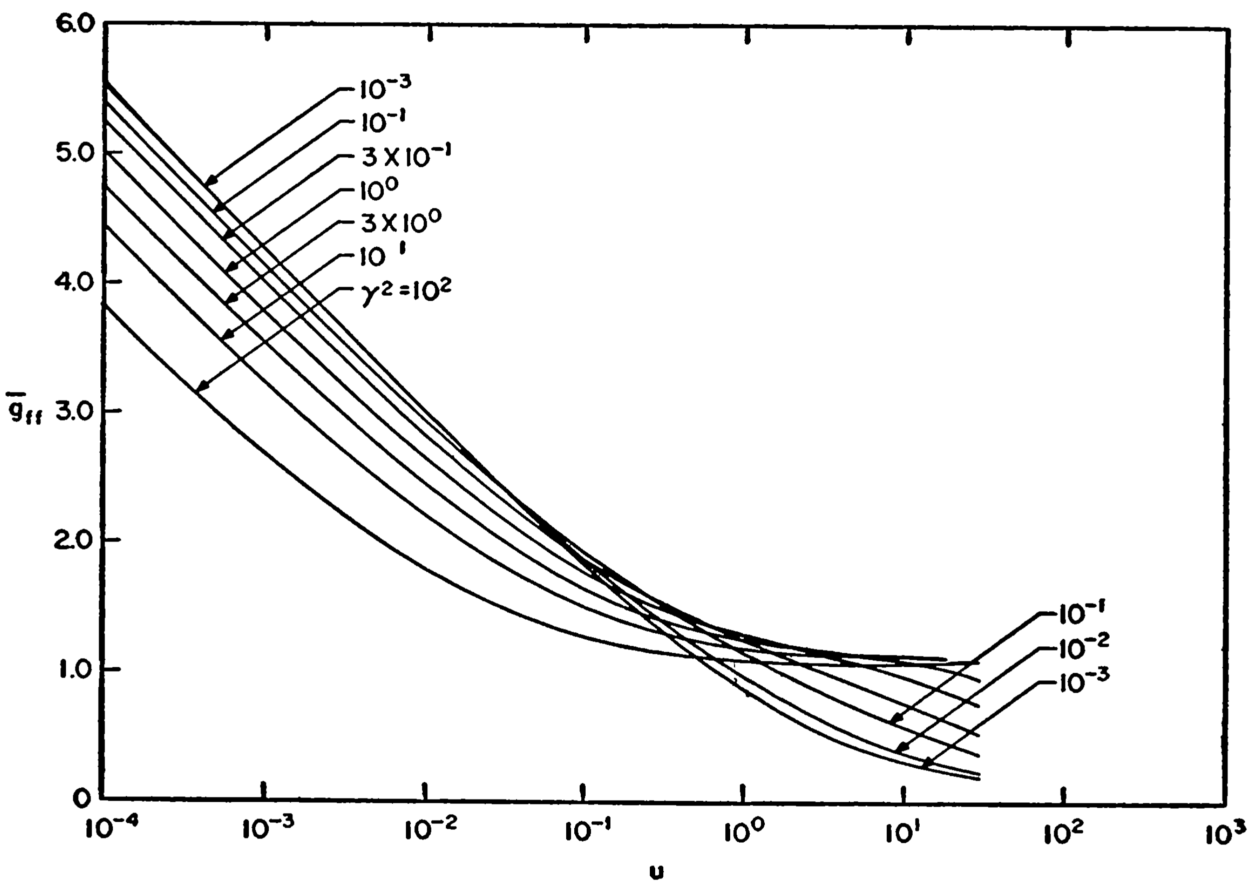
[Bradt, Astrophysics Processes]

- Gaunt Factor

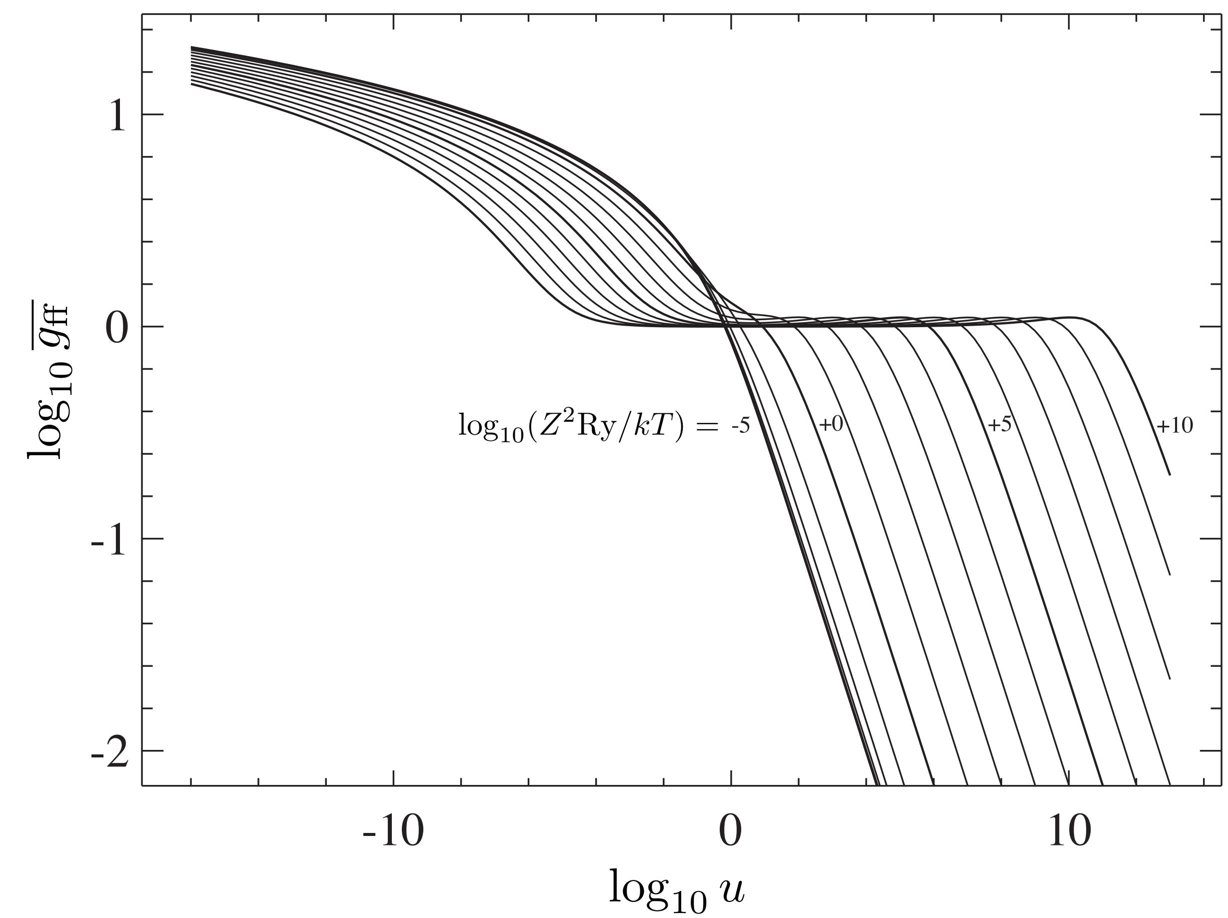
- Note that the values of Gaunt factor for $u = h\nu/kT \gg 1$ are not important, since the spectrum cuts off for these values.

$$\overline{g_{\rm ff}} \sim \begin{cases} 1 & \text{for } u \sim 1 \\ 1-5 & \text{for } 10^{-4} < u < 1 \end{cases}$$

Karzas & Latter (1961, ApJS, 6, 167)



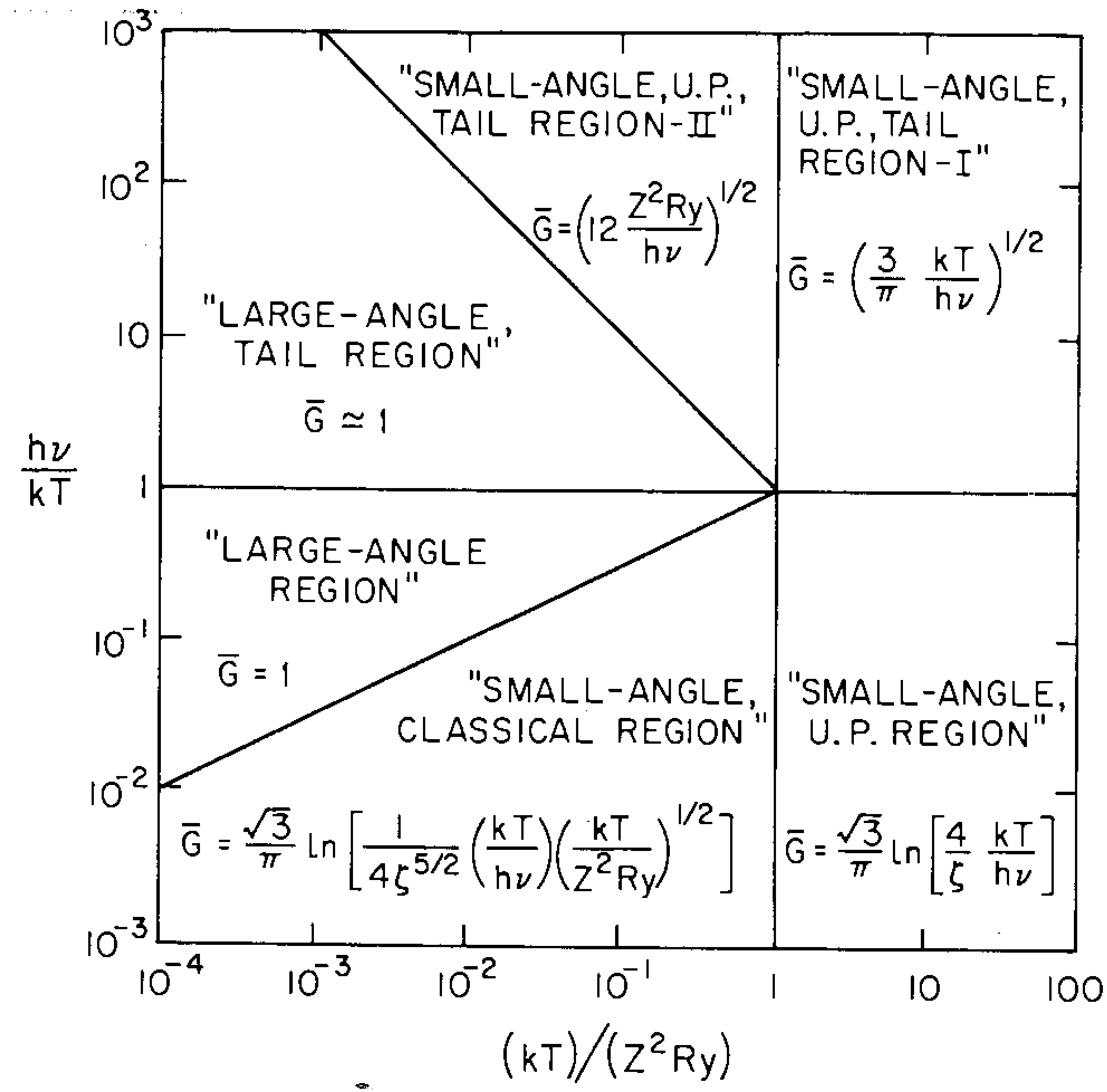
van Hoof et al. (2014, MNRAS, 444, 420)



$$u = h\nu/kT = 4.8 \times 10^{11} \nu/T$$

$$\gamma^2 = Z^2{\rm Ry}/kT = 1.58 \times 10^5 Z^2/T$$

- Novikov & Thorne (1973, in Black Holes, Les Houches)



U.P. = Uncertainty principle

- To obtain the formulas for non-thermal bremsstrahlung, one needs to know the actual distributions of velocities, and the formula for emission from a single-speed electron must be averaged over that distribution. One also must have the appropriate Gaunt factors.
- Integrated Bremsstrahlung emission per unit volume:

$$\begin{aligned}\varepsilon^{\mathrm{ff}} \equiv \int \varepsilon^{\mathrm{ff}}(\nu)d\nu &= \frac{2^5\pi e^6}{3m_e c^3}\left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-1/2} n_i n_e Z^2 \int e^{-h\nu/kT}\overline{g_{\mathrm{ff}}}d\nu \\ &= \frac{2^5\pi e^6}{3m_e c^3}\left(\frac{2\pi}{3km_e}\right)^{1/2}\left(\frac{kT^{1/2}}{h}\right) n_i n_e Z^2 \int_0^\infty e^{-u}\overline{g_{\mathrm{ff}}}du \\ &= \left(\frac{2\pi kT}{3m_e}\right)^{1/2} \frac{2^5\pi e^6}{3hm_e c^3} n_i n_e Z^2 \overline{g_{\mathrm{B}}}\end{aligned}$$

$$\boxed{\varepsilon^{\mathrm{ff}}\left(\equiv \frac{dW}{dtdV}\right) = 1.42\times 10^{-27} n_i n_e Z^2 T^{1/2}\overline{g_{\mathrm{B}}}\ \mathrm{erg\ cm^{-3}\ s^{-1}} \longrightarrow \varepsilon_{\mathrm{ff}} \propto T^{1/2}}$$

where frequency average of the velocity averaged Gaunt factor:

$$\begin{aligned}\overline{g_{\mathrm{B}}} &= \int_0^\infty e^{-u}\overline{g_{\mathrm{ff}}}du \quad (u = h\nu/kT) \\ &= 1.3 \pm 0.2\end{aligned}$$

$$\overline{g_{\mathrm{B}}} \approx 1 + \frac{0.44}{1+0.058\left[\ln(T/10^{5.4}Z^2K)\right]^2}$$

$$\text{for } 10^{4.2}\ \mathrm{K} \le T/Z^2 \le 10^{8.2}\ \mathrm{K},\ \text{Draine (2011)}$$

[Thermal Bremsstrahlung (free-free) Absorption]

- Absorption of radiation by free electrons moving in the field of ions:

  For thermal system, Kirchoff's law says:

$$\frac{1}{4\pi}\frac{dW}{dV\,dt\,d\nu} = j_\nu^{\rm ff} = \alpha_\nu^{\rm ff} B_\nu(T) \qquad B_\nu(T) = \left(2h\nu^3/c^2\right)\left[\exp(h\nu/kT) - 1\right]^{-1}$$

We have then

$$\boxed{\begin{aligned}\alpha_\nu^{\rm ff} &= \frac{4e^6}{3m_e hc}\left(\frac{2\pi}{3km_e}\right)^{1/2} n_i n_e Z^2 T^{-1/2}\nu^{-3}\left(1 - e^{-h\nu/kT}\right)\overline{g_{\rm ff}} \\ &= 3.7\times10^8\ n_i n_e Z^2 T^{-1/2}\nu^{-3}\left(1 - e^{-h\nu/kT}\right)\overline{g_{\rm ff}}\quad ({\rm cm}^{-1})\end{aligned}}$$

For $h\nu \gg kT$, $\alpha_\nu^{\rm ff} = 3.7\times10^8 n_i n_e Z^2 T^{-1/2}\nu^{-3}\overline{g_{\rm ff}}\ \ ({\rm cm}^{-1}) \longrightarrow$

For $h\nu \ll kT$, $\alpha_\nu^{\rm ff} = \frac{4e^6}{3m_e kc}\left(\frac{2\pi}{3km_e}\right)^{1/2} n_i n_e Z^2 T^{-3/2}\nu^{-2}\overline{g_{\rm ff}} \longrightarrow$

$$= 0.018 n_i n_e Z^2 T^{-3/2}\nu^{-2}\overline{g_{\rm ff}}$$

$$\boxed{1 - e^{-h\nu/kT} \approx 1 - (1 - h\nu/kT) = h\nu/kT}$$

$$\boxed{\begin{aligned}&\tau_\nu \propto \alpha_\nu^{\rm ff} \propto \nu^{-3}\ \text{for}\ h\nu \gg kT \\ &\tau_\nu \propto \alpha_\nu^{\rm ff} \propto \nu^{-2}\ \text{for}\ h\nu \ll kT\end{aligned}}$$

Bremsstrahlung self-absorption: **The medium becomes always optically thick at sufficiently small frequency.** Therefore, the free-free emission is absorbed inside plasma.

[Overall Spectral Shape]

- An approximate formula for the free-free Gaunt factor is given by Draine (2011).

$$\overline{g_{\rm ff}} \approx 6.155(Z\nu_9)^{-0.118} T_4^{0.177} \quad (0.14 < Z\nu_9/T_4^{3/2} < 250) \quad \text{where } \nu_9 = \nu/10^9 \text{ Hz}, \quad T_4 = T/10^4 \text{ K}$$

- Emission and absorption coefficients:

$$j_\nu = \frac{1}{4\pi}\varepsilon_\nu \approx 3.35 \times 10^{-40} n_i n_e Z^{1.882} T_4^{-0.323} \nu_9^{-0.118} \text{ erg cm}^{-3} \text{ s}^{-1} \text{ Hz}^{-1} \text{ sr}^{-1}$$

$$\alpha_\nu = \frac{j_\nu}{B_\nu} \approx 3.37 \times 10^{-7} n_i n_e Z^{1.882} T_4^{-1.323} \nu_9^{-2.118} \text{ pc}^{-1}$$

- Optical depth:

$$\tau_\nu = \int \alpha_\nu ds \approx 3.37 \times 10^{-7} Z^{1.882} T_4^{-1.323} \nu_9^{-2.118} \left(\frac{n_i}{n_p}\right) \left[\frac{\text{EM}}{\text{cm}^{-6}\text{pc}}\right] \quad \text{where EM } \equiv \int n_e n_p ds$$

- SED (Spectral Energy Density) from a uniform sphere

$$\text{for } \tau_\nu \gg 1, \ h\nu \ll kT \longrightarrow I_\nu = S_\nu = B_\nu \qquad F_\nu = \pi B_\nu \left(\frac{R}{d}\right)^2 \propto \nu^2 \ \ \text{(Rayleigh-Jeans Law)}$$

$$\text{for } \tau_\nu \ll 1, \ h\nu \ll kT \longrightarrow I_\nu = \int j_\nu ds \qquad F_\nu = 4\pi j_\nu \left(\frac{4\pi R^2}{3}\right) \frac{1}{4\pi d^2} \propto \nu^{-0.1}$$

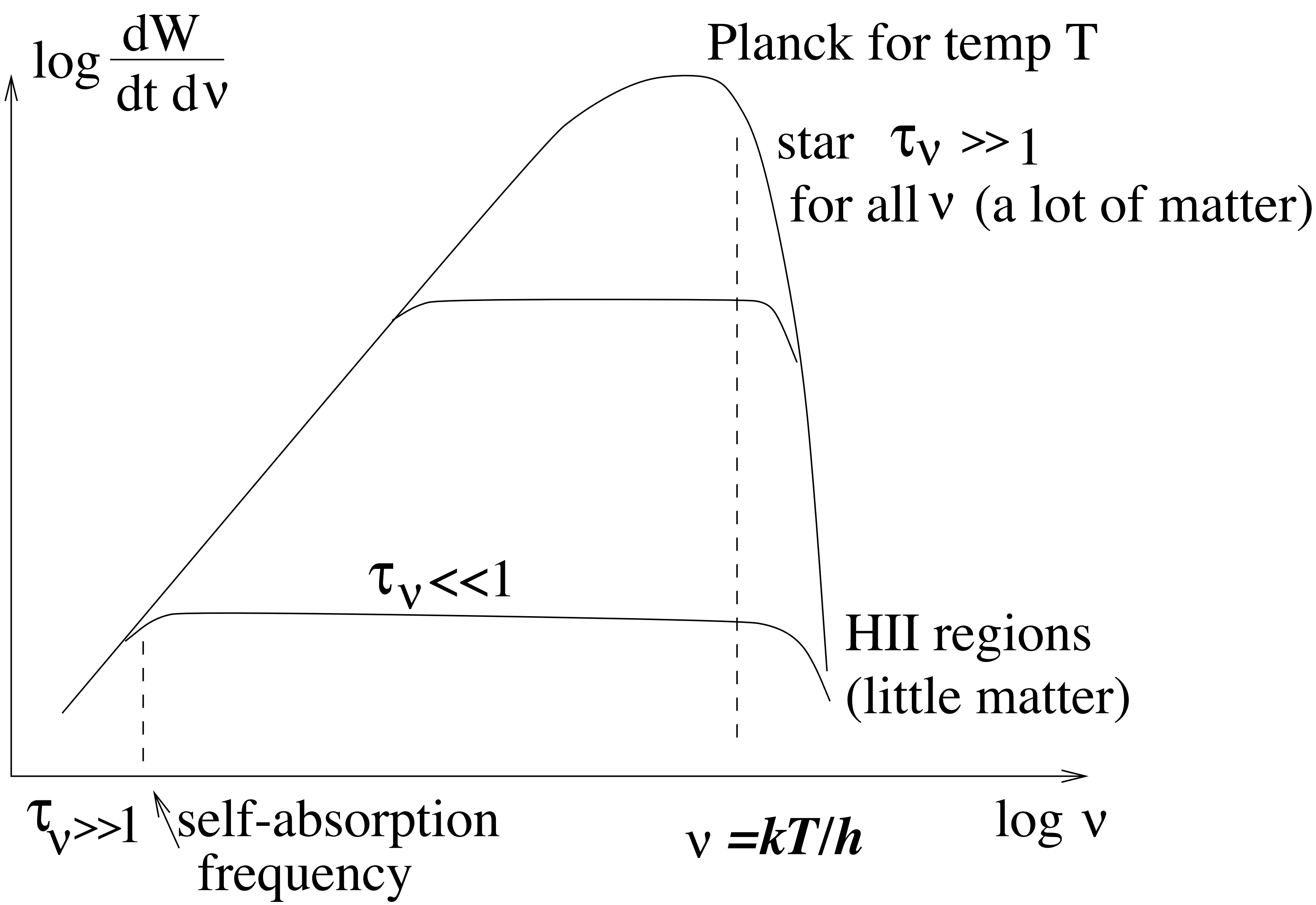


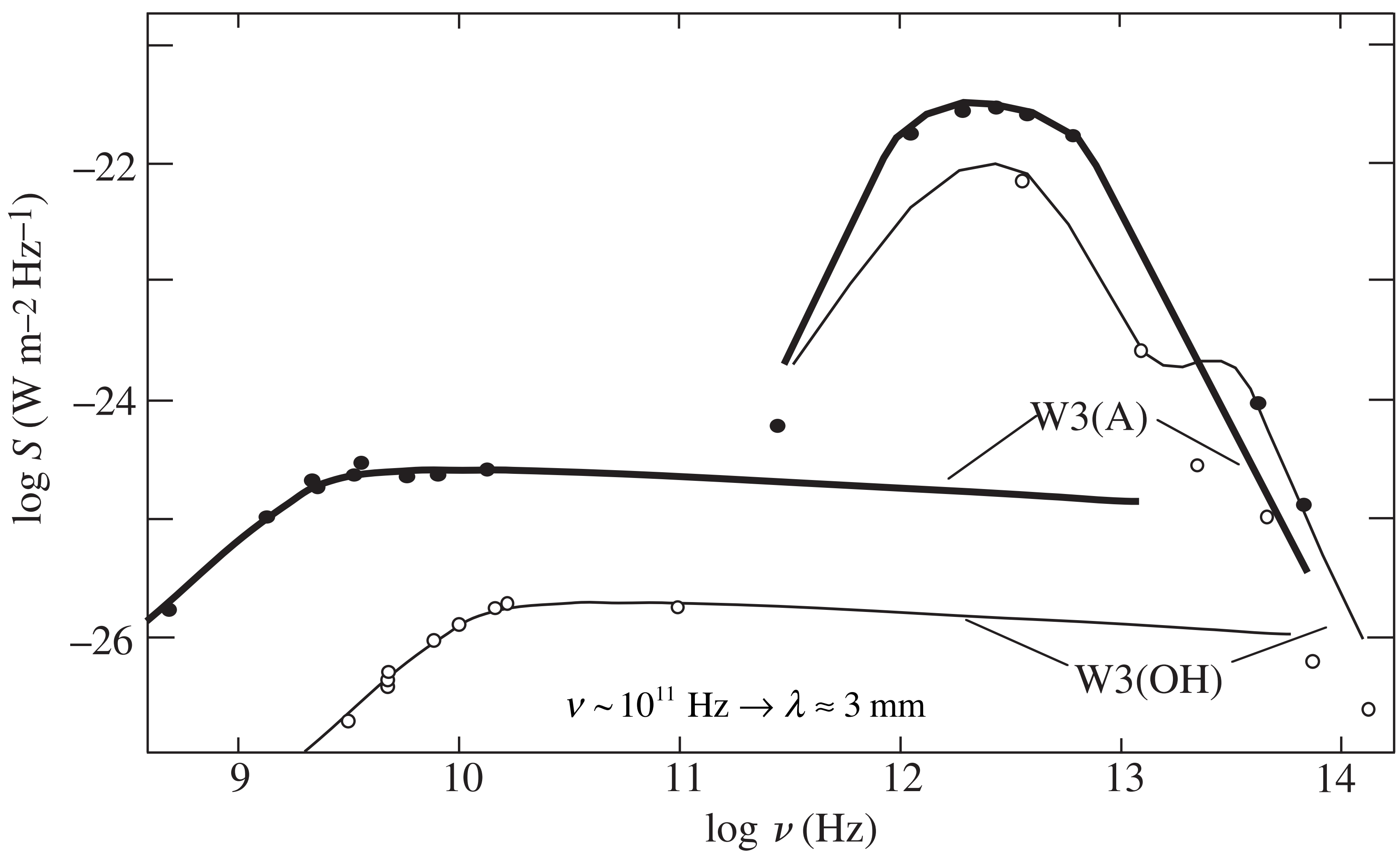
Figure from the Lecture Note of J. Poutanen

Astronomical Examples - H II regions

- The radio spectra of H II regions clearly show the flat spectrum of an optically thin thermal source. The bright stars in the H II regions emit copiously in the UV and thus ionize the hydrogen gas.
- Continuum spectra of two H II regions, W3(A) and W3(OH):

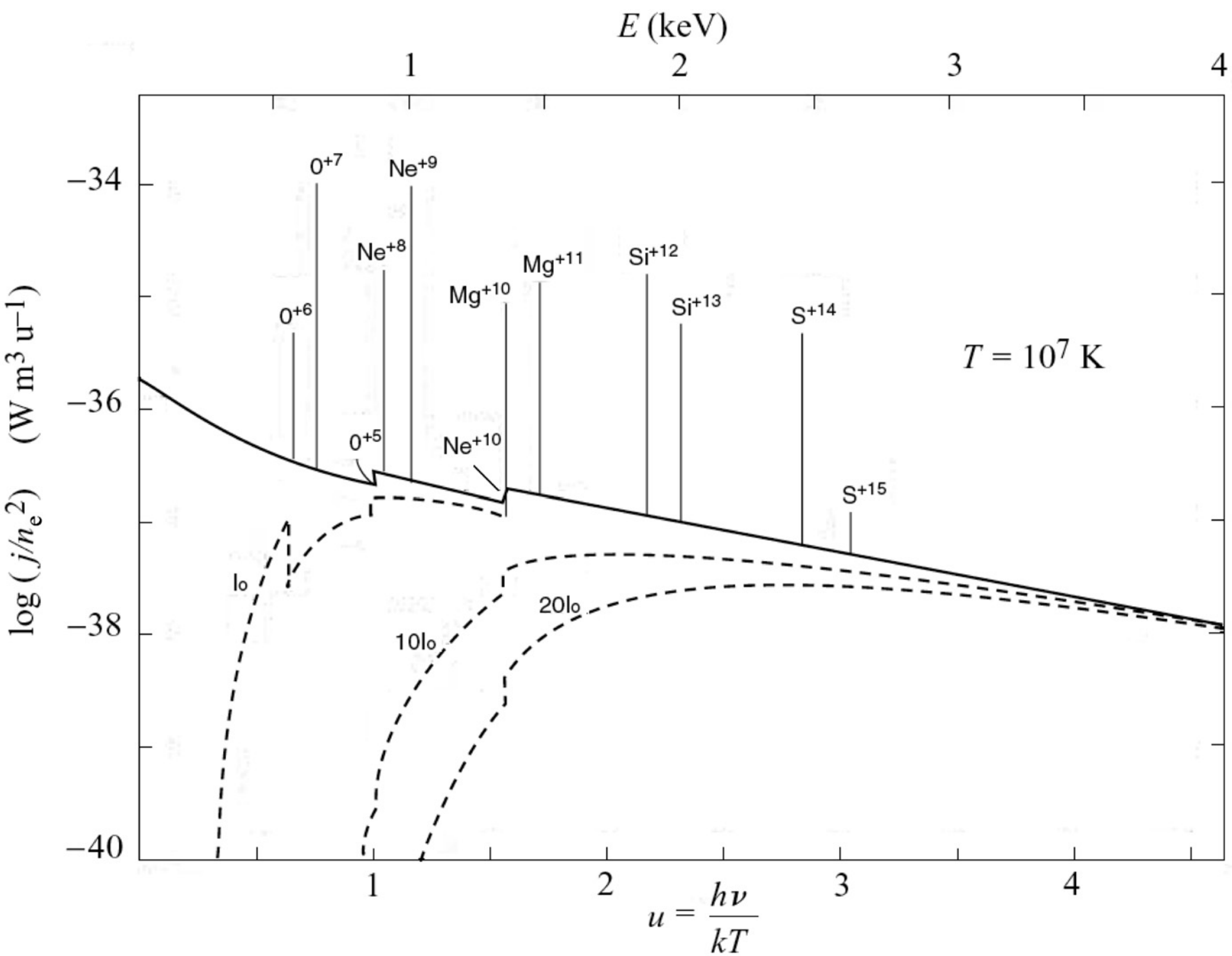
  Note a flat thermal bremsstrahlung (radio), a low-frequency cutoff (radio, self absorption), and a large peak at high frequency (infrared, $10^{12} - 10^{13}$ Hz) due to heated, but still "cold" dust grains in the nebula.

Figure from [Bradt, Astrophysics Processes]
Data from Kruegel & Mezger (1975, A&A, 42, 441)

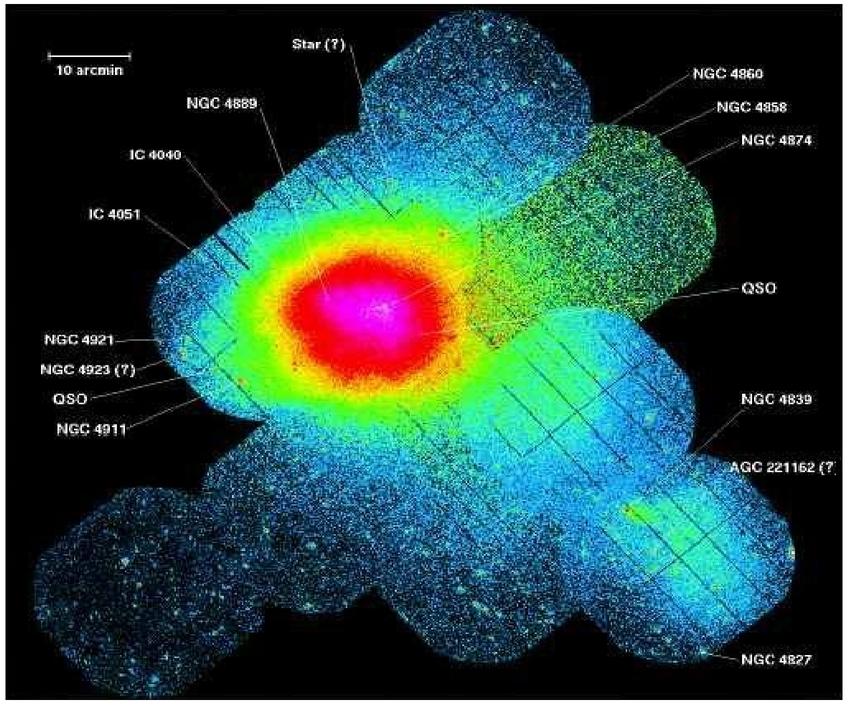


Astronomical Examples - X-ray emission

- Theoretical spectrum for a plasma of temperature 10^7 K that takes into account quantum effects. Comparison with real spectra from clusters of galaxies allows one to deduce the actual amounts of different elements and ionized species in the plasma as well as its temperature. It is only in the present millennium that X-ray spectra taken from satellites (e.g., Chandra and the XMM Newton satellite) have had sufficient resolution to distinguish these narrow lines The dashed lines show the effect of X-ray absorption by interstellar gas [Bradt, Astrophysics Processes].



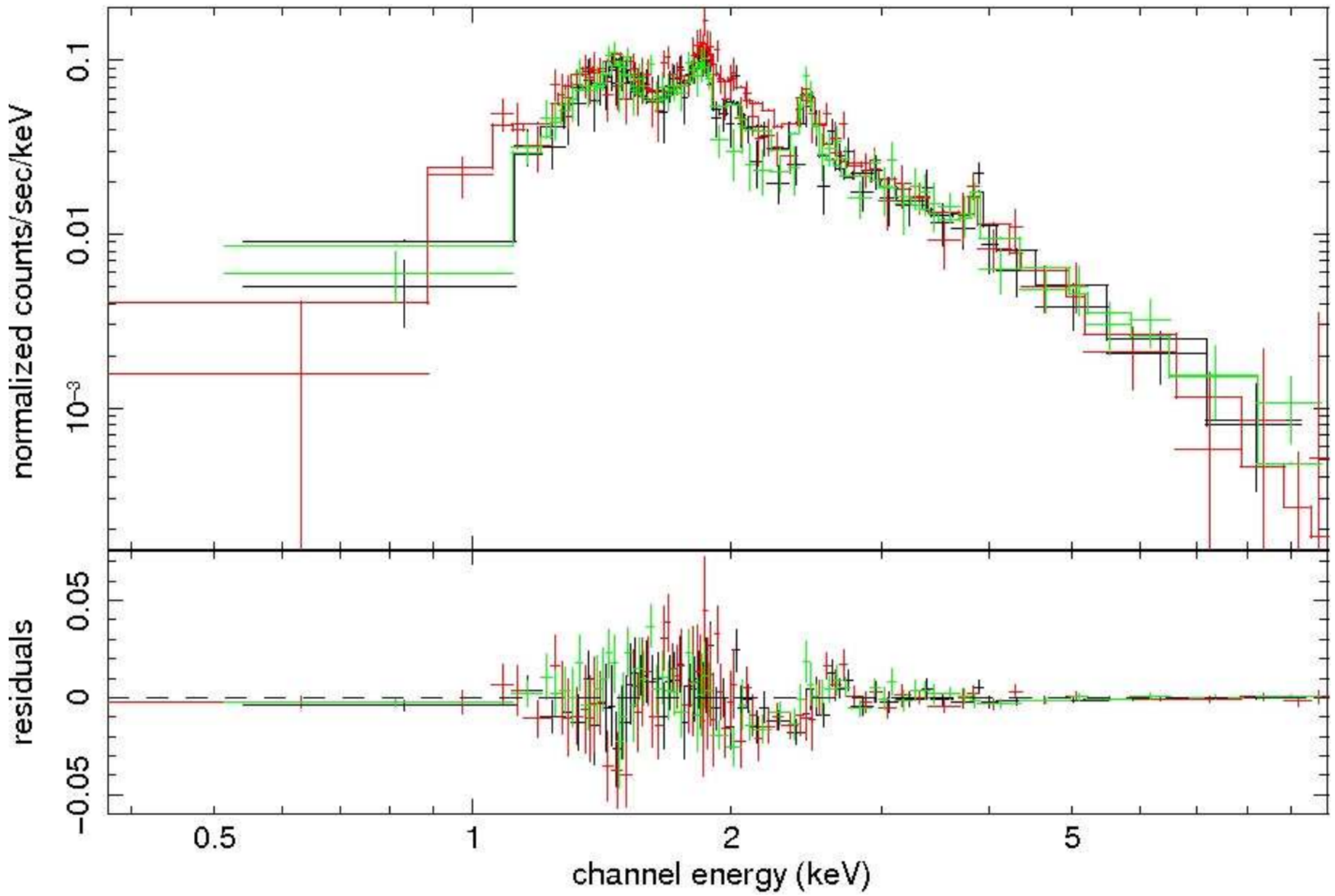
Coma cluster (z = 0.0232), size ~ 1 Mpc



Astronomical Examples - Supernova Remnants

- SNR G346.6-0.2

X-ray spectra of the SNR from three of the four telescopes on-board Suzaku (represented by green, red and black). The underlying continuum is thermal bremsstrahlung, while the spectral features are due to elements such as Mg, S, Si, Ca and Fe. The roll over in the spectrum at low and high energies is due to a fall in the detector response, which is forward-modeled together with the spectrum.



Sezer et al. (2011, MNRAS, 415, 301)

Cooling function

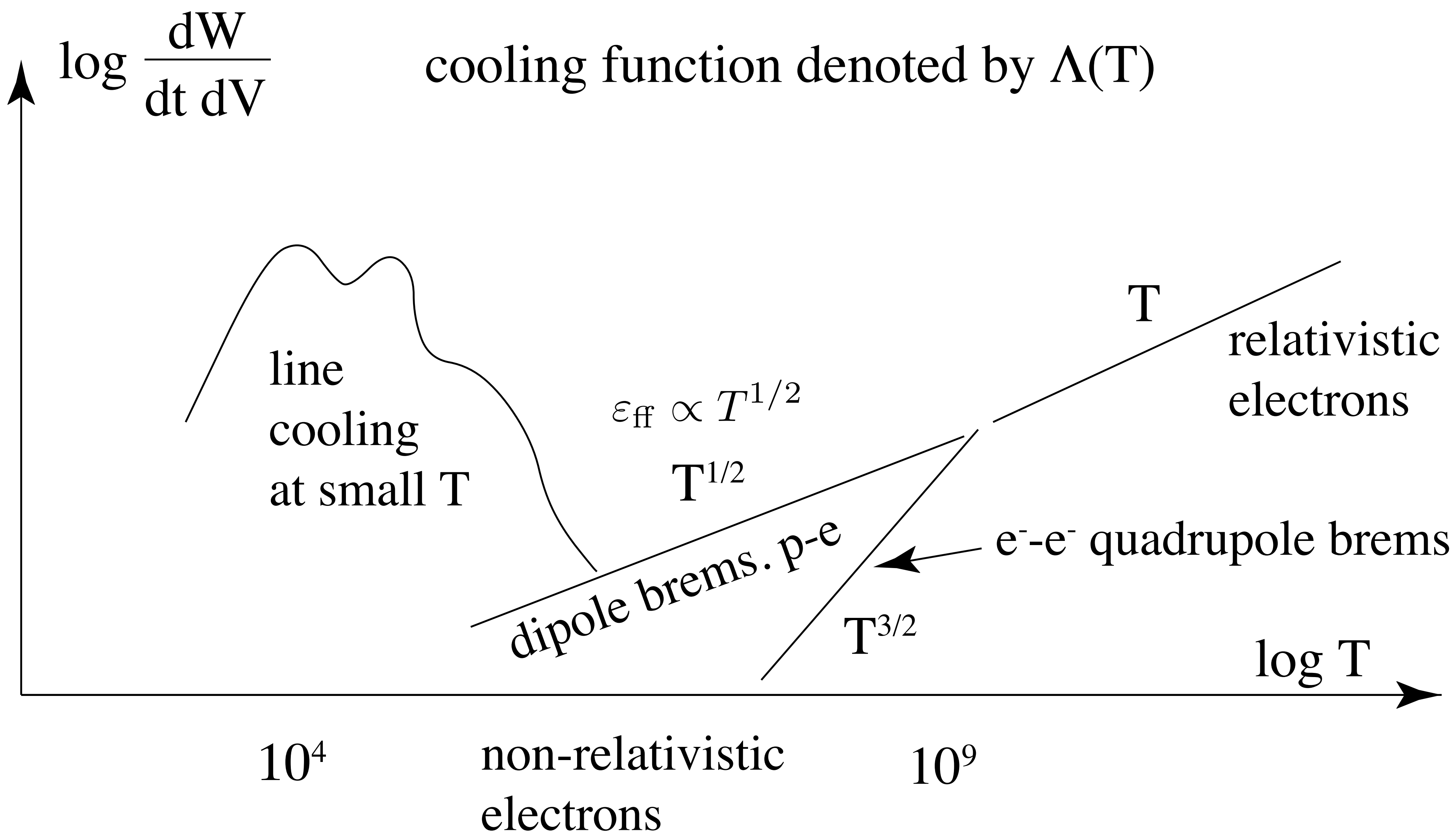


Figure from the Lecture Note of J. Poutanen

[Relativistic Bremsstrahlung]*

- Normally, the ions move rather slowly in comparison to the electrons.

  However, in a frame of reference in which electron is initially at rest, the ion appears to move rapidly toward the electron. **The electrostatic field of the ion appears to the electron to be a pulse of electromagnetic radiation. This radiation then Compton (or Thompson) scatters off the electron to produce emitted radiation.** Transforming back to the rest frame of the ion (or lab frame) we obtain the bremsstrahlung emission of the electron. **Relativistic bremsstrahlung can be regarded as the Compton scattering of the virtual quanta of the ion's electrostatic field as seen in the electron's frame.**

- In the (primed) electron rest frame, the spectrum of the pulse of the virtual quanta:

$$\frac{dW'}{dA'd\omega'} = \frac{q^2}{\pi^2 b'^2 v^2}\left(\frac{b'\omega'}{\gamma v}\right)^2 K_1^2\left(\frac{b'\omega'}{\gamma v}\right) = \frac{(Ze)^2}{\pi^2 b'^2 v^2}\left(\frac{b'\omega'}{\gamma v}\right)^2 K_1^2\left(\frac{b'\omega'}{\gamma c}\right) \quad \leftarrow \; v \approx c$$

(in the ultrarelativistic limit)

In the low-frequency limit, the scattered radiation is

$$\frac{dW}{d\omega} = \sigma_T \frac{dW'}{dA'd\omega'} \qquad \left(\sigma_T = \frac{2\pi}{3}\frac{e^4}{m_e^2 c^4}\right)$$

Transverse lengths are unchanged, $b = b'$, and $\omega = \gamma\omega'(1+\beta\cos\theta')$. The scattering is forward-backward symmetric, we therefore have the averaged relation $\omega = \gamma\omega'$.

- For a plasma with a single-speeds

$$\begin{aligned}
\frac{dW}{dV\,dt\,d\omega} &= n_e n_i v \int_{b_{\min}}^{\infty} \frac{dW(b)}{d\omega} 2\pi b\,db \\
&= \frac{16 Z^2 e^6}{3 c^3 m_e^2} n_e n_i \int_{b_{\min}}^{b_{\max}} \left(\frac{b\omega}{\gamma^2 c}\right) K_1\left(\frac{b\omega}{\gamma^2 c}\right) db \\
&= \frac{16 Z^2 e^6}{3 c^3 m_e^2} n_e n_i \ln\left(\frac{0.68\gamma^2 c}{\omega b_{\min}}\right)
\end{aligned}$$

- For a Maxwell distribution of electrons, a useful approximate expression for the frequency integrated power is given by Novikov & Thorne (1973).

$$\varepsilon_\nu^{\mathrm{ff}} = 1.4 \times 10^{-27} n_i n_e Z^2 T^{1/2} \overline{g_{\mathrm{B}}} \left(1 + 4.4 \times 10^{-10} T\right) \quad (\mathrm{erg\ s^{-1}\ cm^{-3}})$$

See also Itoh et al. (2000, ApJS, 128, 125), Zekovic (2013, arXiv:1310.5639v1)

- At higher frequencies Klein-Nishina corrections must be used.