

Interstellar Medium (ISM)

Week 3

April 02 (Thursday), 2020

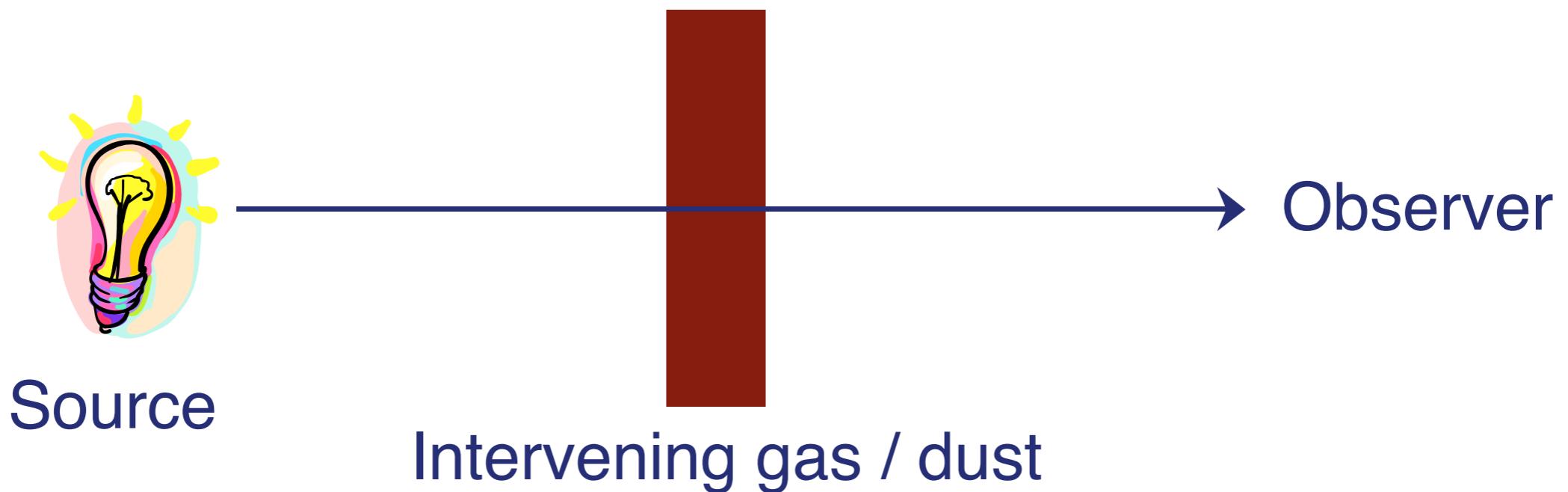
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Radiative Transfer

Radiative Transfer

- How is radiation affected as it propagates through intervening gas and dust media to the observer?



Simplification & Complexity

- Simplification:
 - Astronomical objects are normally much larger than the wavelength of radiation they emit.
 - Diffraction can be neglected.
 - Light rays travel to us along straight lines.
- Complexity:
 - At one point, photons can be traveling in several different directions.
 - For instance, at the center of a star, photons are moving equally in all directions. (However, radiation from a star seen by a distant observer is moving almost exactly radially.)
 - Full specification of radiation needs to say how much radiation is moving in each direction at every point. Therefore, we are dealing with the five- or six-dimensional problem. ($[x, y, z] + [\theta, \phi] + [t]$)



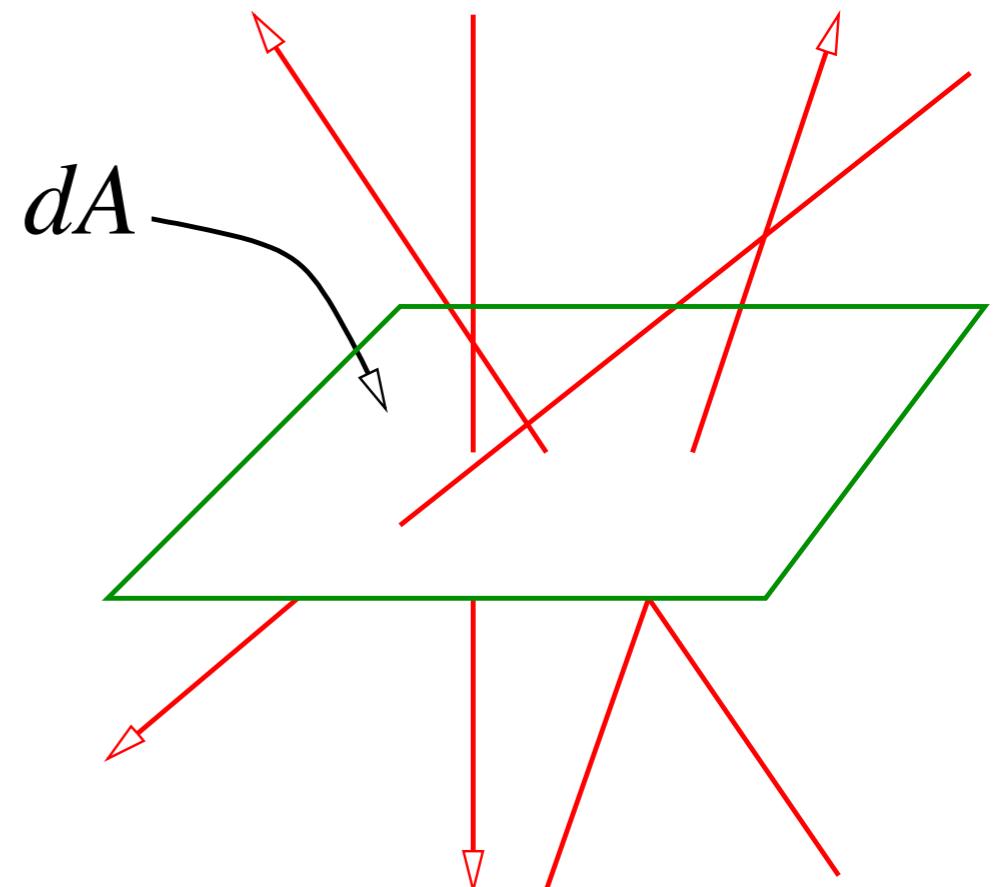
Energy Flux

- Definition
 - Consider a small area dA , exposed to radiation for a time dt .

- Energy flux F is defined as ***the net energy dE passing through the element of area in all directions in the time interval*** so that

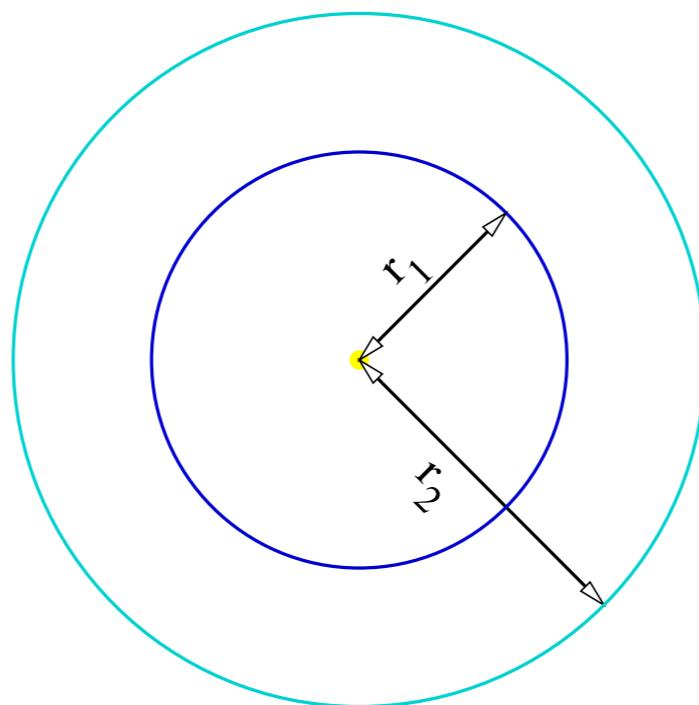
$$dE = F \times dA \times dt$$

- Note that F ***depends on the orientation of the area element dA .***
- Unit: erg cm⁻² s⁻¹



Inverse Square Law

- Flux from an isotropic radiation source, i.e., a source emitting equal amounts of energy in all directions.



- Because of energy conservation, flux through two shells around the source must be the same.

$$4\pi r_1^2 F(r_1) = 4\pi r_2^2 F(r_2)$$

- Therefore, we obtain the inverse square law.

$$F = \frac{\text{const.}}{r^2}$$

Energy Flux Density

- Real detectors are sensitive to a limited range of wavelengths. We need to consider how the incident radiation is distributed over frequency.

Total energy flux: $F = \int F_\nu d\nu$ Integral of F_ν over all frequencies

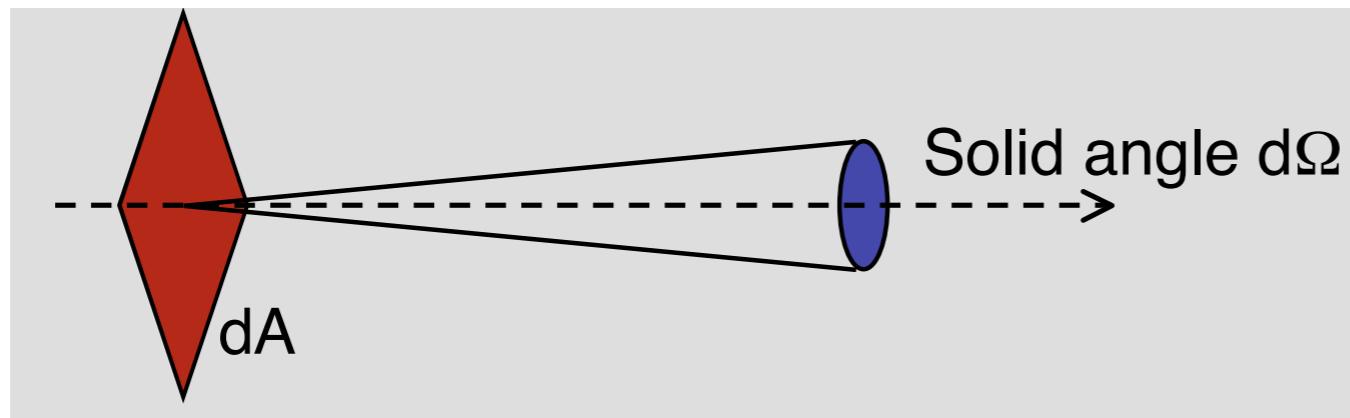
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Units: erg s⁻¹ cm⁻² Hz⁻¹

- F_ν is often called the “flux density.”
- Radio astronomers use a special unit to define the flux density:
1 Jansky (Jy) = 10⁻²³ erg s⁻¹ cm⁻² Hz⁻¹

Specific Intensity or Surface Brightness

- Recall that ***flux is a measure of the energy carried by all rays passing through a given area***
- Intensity is the energy carried along by individual rays.***



- Let dE_ν be the amount of radiant energy which crosses the area dA in a direction \mathbf{k} within solid angle $d\Omega$ about in a time interval dt with photon frequency between ν and $\nu + d\nu$.
- The monochromatic specific intensity I_ν is then defined by the equation.

$$dE_\nu = I_\nu(\mathbf{k}, \mathbf{x}, t) dA_{\mathbf{k}} d\Omega d\nu dt$$

- Unit: $\text{erg s}^{-1} \text{ cm}^{-2} \text{ sr}^{-1} \text{ Hz}^{-1}$
- From the view point of an observer, the specific intensity is called ***surface brightness***.

Relation between the flux and the specific intensity

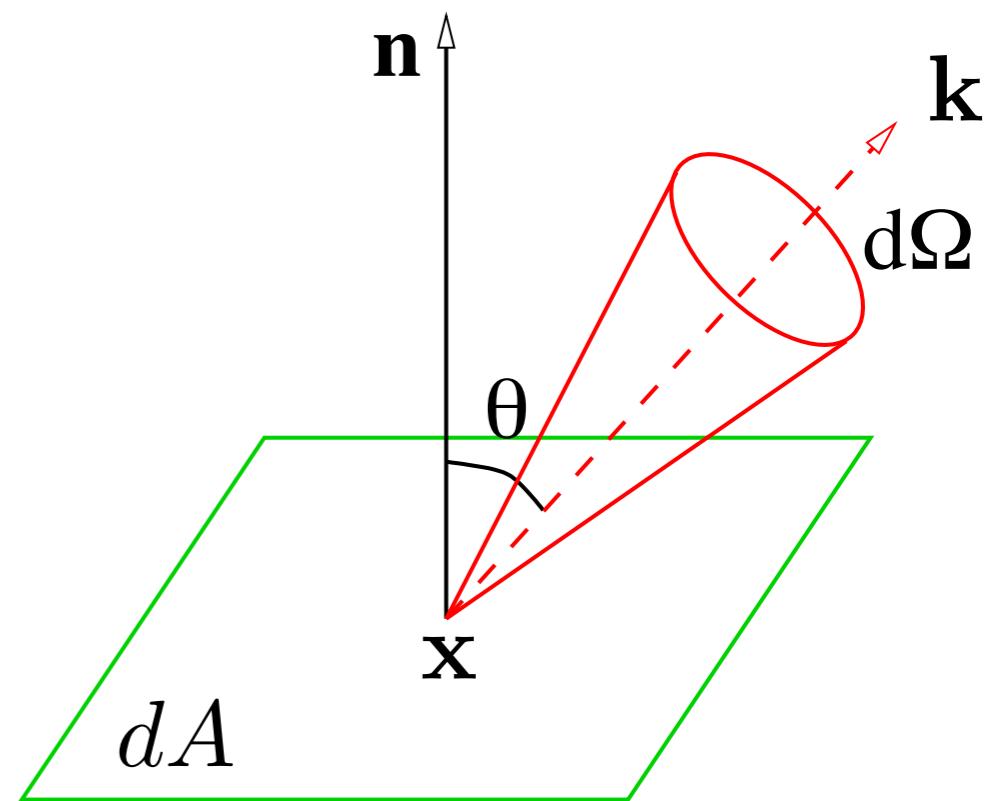
- Let's consider a small area dA , with light rays passing through it at all angles to the normal vector \mathbf{n} of the surface.
- For rays centered about \mathbf{k} , the area normal to \mathbf{k} is

$$dA_{\mathbf{k}} = dA \cos \theta$$

- By the definition,

$$F_{\nu} dAd\nu dt = \int I_{\nu}(\mathbf{k}, \mathbf{x}, t) dA_{\mathbf{k}} d\Omega d\nu dt$$

- Hence, net flux in the direction of \mathbf{n} is given by integrating over all solid angles:



$$F_{\nu} = \int I_{\nu} \cos \theta d\Omega = \int_0^{2\pi} \int_0^{\pi} I_{\nu} \cos \theta \sin \theta d\theta d\phi$$

Note

- Intensity can be defined as per wavelength interval.

$$\begin{aligned} I_\nu |d\nu| &= I_\lambda |d\lambda| \\ \nu I_\nu &= \lambda I_\lambda \end{aligned} \quad \leftarrow \left| \frac{d\nu}{d\lambda} \right| = \frac{c}{\lambda^2} = \frac{\nu}{\lambda}$$

- Integrated intensity is defined as the intensity over all frequencies.

$$I = \int_0^\infty I_\nu d\nu = \int_0^\infty I_\lambda d\lambda$$

- In astrophysics, we plot the spectral energy distribution (SED) as νI_ν versus ν or λI_λ versus λ .

How does specific intensity changes along a ray in free space

- Consider a bundle of rays and any two points along the rays. Construct areas dA_1 and dA_2 normal to the rays at these points.
 - Consider the energy carried by the rays passing through both areas. Because energy is conserved,

$$dE_1 = I_1 dA_1 d\Omega_1 d\nu dt = dE_2 = I_2 dA_2 d\Omega_2 d\nu dt$$

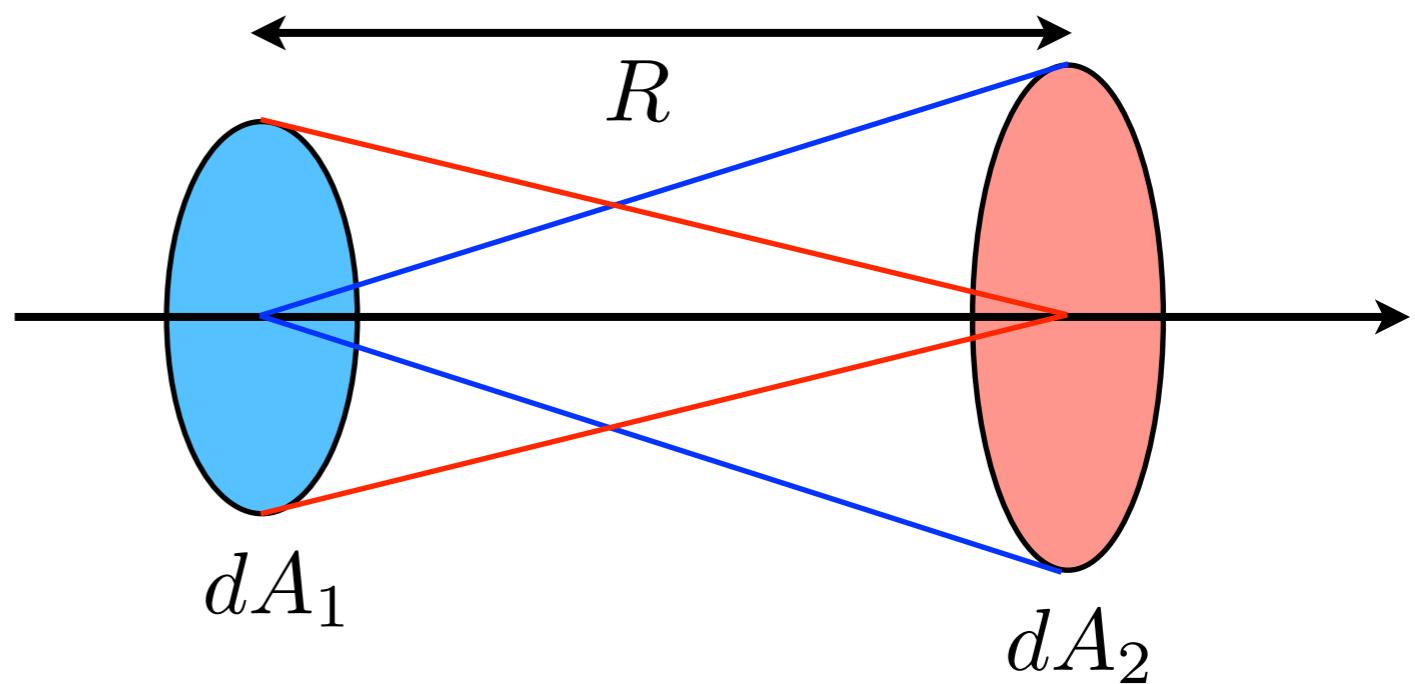
- Here, $d\Omega_1$ is the solid angle subtended by dA_2 at the location 1 and so forth.

$$d\Omega_1 = \frac{dA_2}{R^2}$$

$$d\Omega_2 = \frac{dA_1}{R^2}$$

\rightarrow

$$I_1 = I_2$$



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- Conclusion (the constancy of intensity):
 - the specific intensity remains the same as radiation propagates through free space.

$$I_1 = I_2$$

- If we measure the distance along a ray by variable s , we can express the result equivalently in differential form:

$$\frac{dI}{ds} = 0$$

Specific Energy Density

- Consider a bundle of rays passing through a volume element dV in a direction Ω .
 - Then, the energy density per unit solid angle is defined by

$$dE = u_\nu(\Omega) dV d\Omega d\nu$$

- Since radiation travels at velocity c , the volume element is

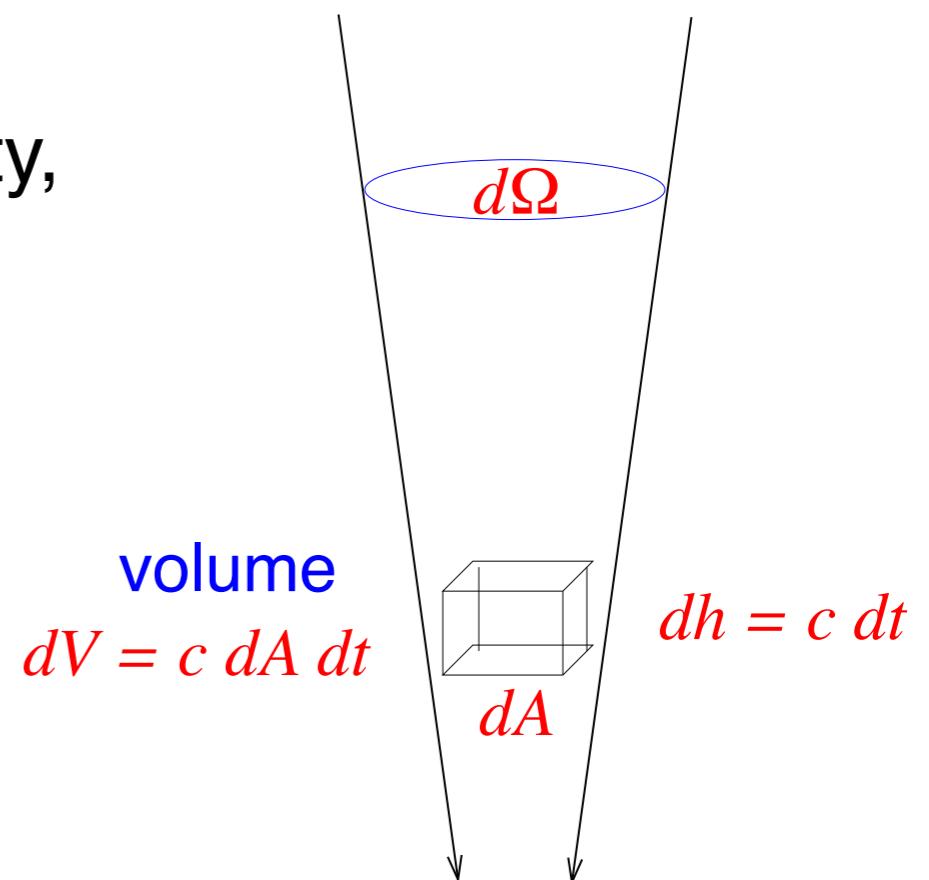
$$dV = dA(cdt)$$

- According to the definition of the intensity,

$$dE = I_\nu dA dt d\Omega d\nu$$

- Then, we have

$$u_\nu(\Omega) = I_\nu(\Omega)/c$$



Energy Density and Mean Intensity

- Integrating over all solid angle, we obtain

$$u_\nu = \int u_\nu(\Omega) d\Omega = \frac{1}{c} \int I_\nu d\Omega$$

- Mean intensity** is defined by

$$J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$$

- Then, the energy density is

$$u_\nu = \frac{4\pi}{c} J_\nu$$

- Total energy density is obtained by integrating over all frequencies.

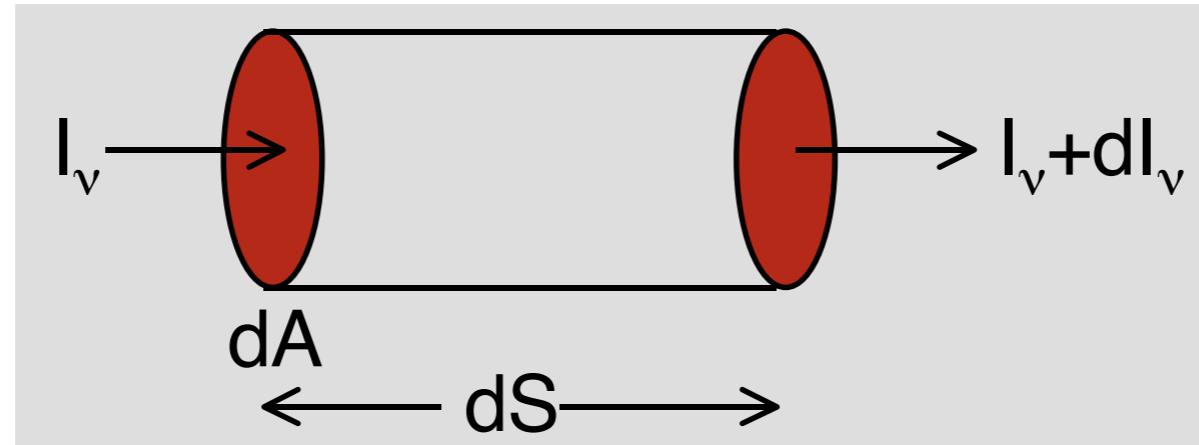
$$u = \int u_\nu d\nu = \frac{4\pi}{c} \int J_\nu d\nu$$

Radiative Transfer Equation

- In reality, as a ray passes through matter, energy may be added, subtracted, or scattered from it by emission, absorption, or scattering.
 - The intensity will not in general remain constant.
 - We need to derive the radiative transfer equation.

Emission

- If the radiation travels through a medium which is itself emitting radiation, that will add to the energy:



- Spontaneous “**emission coefficient**” or “**emissivity**” j_ν is the amount of energy emitted per unit time, per unit solid angle, per unit frequency, and per unit volume:

$$dE = j_\nu dV d\Omega dt d\nu \quad (j_\nu : \text{erg cm}^{-3} \text{ s}^{-1} \text{ sr}^{-1} \text{ Hz}^{-1})$$

- In going a distance ds , a beam of cross section dA travels through a volume $dV = dA ds$. Thus the intensity added to the beam is by ds is

$$dI_\nu = j_\nu ds \qquad \longleftrightarrow \qquad dE = (dI_\nu) dA d\Omega dt d\nu$$

- Therefore, the equation of radiative transfer for pure emission becomes:

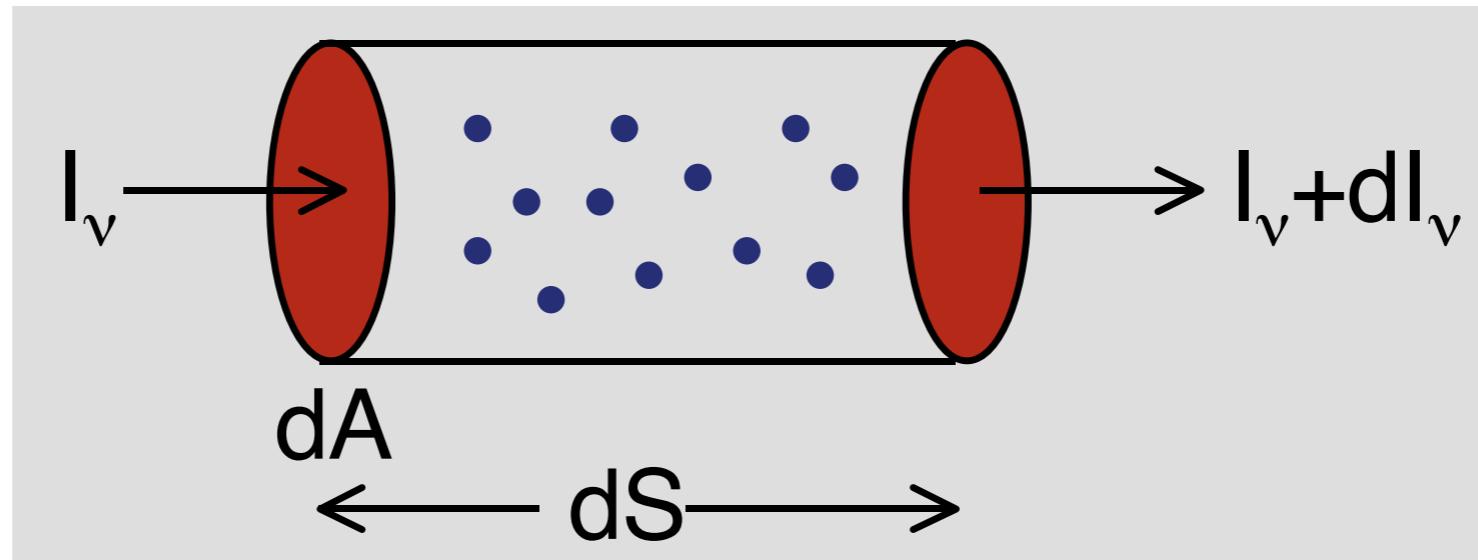
$$\frac{dI_\nu}{ds} = j_\nu$$

- If we know what j_ν is, we can integrate this equation to find the change in specific intensity as radiation propagates through the medium:

$$I_\nu(s) = I_\nu(0) + \int_0^s j_\nu(s')ds'$$

Absorption

- If the radiation travels through a medium which absorbs radiation, the energy in the beam will be reduced:



- Let n denote the number density of absorbers (particles per unit volume).
- Assume that each absorber has a cross-sectional area of σ_ν (in units of cm^2).
- If a beam travels through ds , total area of absorbers is

$$\text{number of absorbers} \times \text{cross-section} = n \times dA \times ds \times \sigma_\nu$$

Fraction of radiation absorbed = fraction of area blocked:

$$\frac{dI_\nu}{I_\nu} = - \frac{ndAds\sigma_\nu}{dA} = -n\sigma_\nu ds$$

$$dI_\nu = -n\sigma_\nu I_\nu ds \equiv -\kappa_\nu I_\nu ds$$

- **Absorption coefficient** is defined as $\kappa_\nu \equiv n\sigma_\nu$ (units: cm^{-1}), meaning the **total cross-sectional area per unit volume**.

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- Rearranging the previous equation, we obtain the equation of radiative transfer for pure absorption:

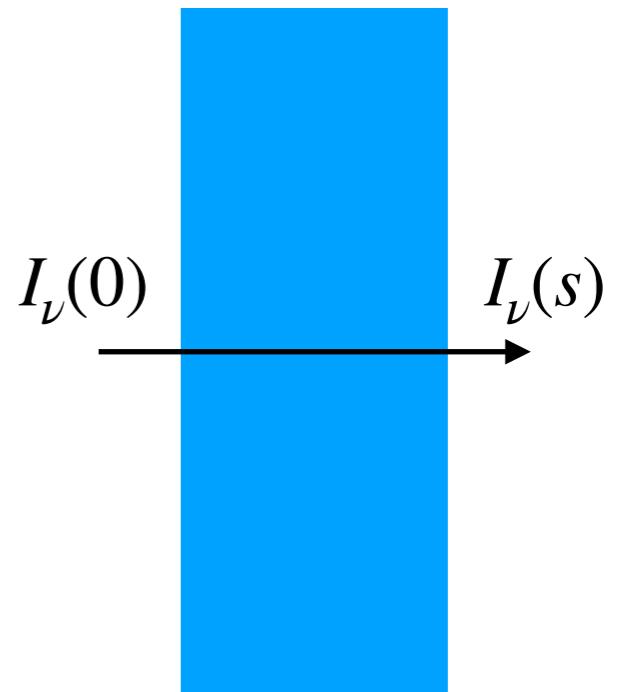
$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu$$

- The amount of reduced energy depends on how much radiation we already have.
- Integrate to find how radiation changes along path:

$$\int_0^s \frac{dI_\nu}{I_\nu} = - \int_0^s \kappa_\nu(s') ds'$$

$$[\ln I_\nu]_0^s = - \int_0^s \kappa_\nu(s') ds'$$

$$I_\nu(s) = I_\nu(0) \exp \left[- \int_0^s \kappa_\nu(s') ds' \right]$$



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- If the absorption coefficient is a constant (example: a uniform density gas of ionized hydrogen), then we obtain

$$I_\nu(s) = I_\nu(0)e^{-\kappa_\nu s}$$

specific intensity
after distance s

initial intensity
at $s = 0$.

radiation exponentially
absorbed with distance

- ***Optical depth:***
 - Imagine radiation traveling into a cloud of absorbing gas, exponential defines a scale over which radiation is attenuated.
 - When $\int_0^s \kappa_\nu(s')ds' = 1$, the intensity will be reduced to $1/e$ of its original value.

- We define the optical depth τ_ν as:

$$\tau_\nu(s) = \int_0^s \kappa_\nu(s')ds' \quad \text{or} \quad d\tau_\nu = \kappa_\nu ds$$

- A medium is said to be ***optically thick*** at a frequency ν if the optical depth for a typical path through the medium satisfies:

$$\tau_\nu(s) > 1$$

- The medium is ***optically thin*** if, instead:

$$\tau_\nu(s) < 1$$

- An optically thin medium is one which a typical photon of frequency ν can pass through without being (significantly) absorbed.

Radiative Transfer Equation

- ***Radiative transfer equation*** with both absorption and emission is

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu$$

absorption emission

- We can rewrite the radiative transfer equation using the optical depth as a measure of ‘distance’ rather than s :

$$\frac{dI_\nu}{\kappa_\nu ds} = -I_\nu + \frac{j_\nu}{\kappa_\nu}$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

- where $S_\nu \equiv j_\nu/\kappa_\nu$ ***is called the source function.*** This is an alternative and sometimes more convenient way to write the equation.

Mean Free Path

- From the exponential absorption law, the probability of a photon absorbed between optical depths τ_ν and $\tau_\nu + d\tau_\nu$ is

$$|dI_\nu| = \left| \frac{dI_\nu}{d\tau_\nu} \right| d\tau_\nu \quad \& \quad |dI_\nu| \propto P(\tau_\nu) d\tau_\nu \quad \rightarrow \quad P(\tau_\nu) = e^{-\tau_\nu}$$

- The mean optical depth traveled is thus equal to unity:

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu P(\tau_\nu) d\tau_\nu = \int_0^\infty \tau_n u e^{-\tau_\nu} d\tau_\nu = 1$$

- The mean free path is defined as the average distance a photon can travel through an absorbing material until it is absorbed.*** In a homogeneous medium, the mean free path is determined by

$$\langle \tau_\nu \rangle = \kappa_\nu \ell_\nu = 1 \quad \rightarrow \quad \ell_\nu = \frac{1}{\kappa_\nu} = \frac{1}{n\sigma_\nu}$$

- A local mean path at a point in an inhomogeneous material can be also defined.

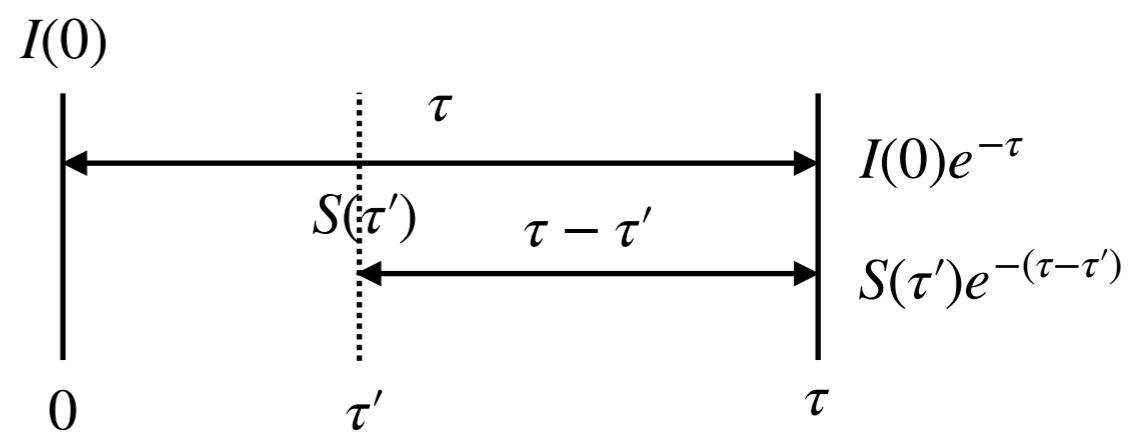
Formal Solution of the RT equation

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$e^{\tau_\nu} \frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

$$\frac{d}{d\tau_\nu} (e^{\tau_\nu} I_\nu) = e^\tau S_\nu$$

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu$$



- The solution is easily interpreted as the sum of two terms:
 - the initial intensity diminished by absorption
 - the integrated source diminished by absorption.
- For a constant source function, the solution becomes

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu} (I_\nu(0) - S_\nu) \end{aligned}$$

Relaxation

- “Relaxation”

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$I_\nu > S_\nu \rightarrow \frac{dI_\nu}{d\tau_\nu} < 0$, then I_ν tends to decrease along the ray

$I_\nu < S_\nu \rightarrow \frac{dI_\nu}{d\tau_\nu} > 0$, then I_ν tends to increase along the ray

- ***The source function is the quantity that the specific intensity tries to approach,*** and does approach if given sufficient optical depth.

As $\tau_\nu \rightarrow \infty$, $I_\nu \rightarrow S_\nu$

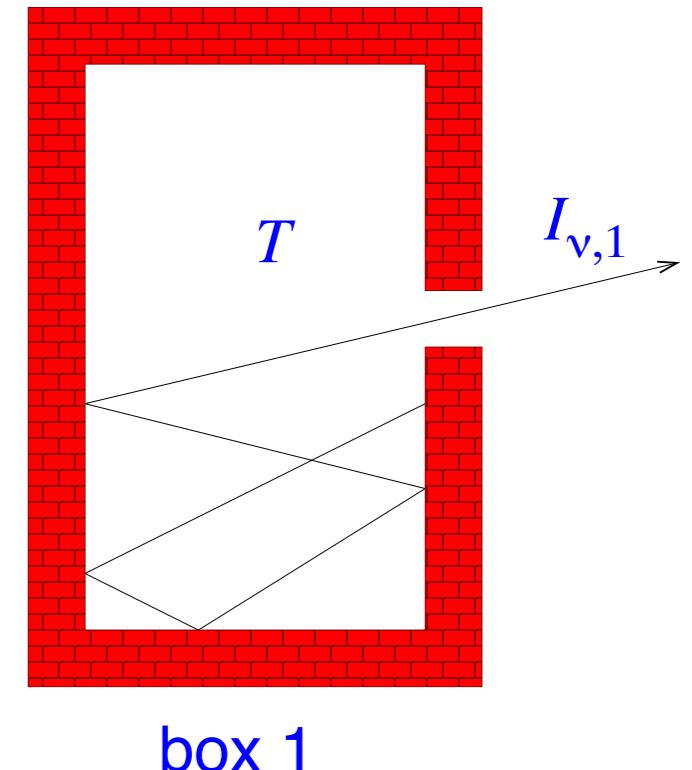
Thermal equilibrium

- In general, equilibrium means a state of balance.
 - ***Thermal equilibrium refers to steady states of temperature, which defines the average energy of material or photons.***
 - In a state of (complete) ***thermodynamic equilibrium (TE)***, no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system. ***In TE, matter and radiation are in equilibrium at the same temperature T.***
 - If the material is (locally) in thermodynamic equilibrium at a well-defined temperature T , ***it is said to be in local thermodynamic equilibrium (LTE) even if it is not in equilibrium with the radiation field.***

Blackbody

- Imagine a container bounded by opaque walls with a very small hole.

- ***Photons will be scattered and absorbed many times, (and eventually trapped and completely absorbed in the box).*** Under such conditions, the particles and photons continually share their kinetic energies. In perfect thermal equilibrium, ***the average particle kinetic energy will equal to the average photon energy, and a unique temperature T can be defined.***

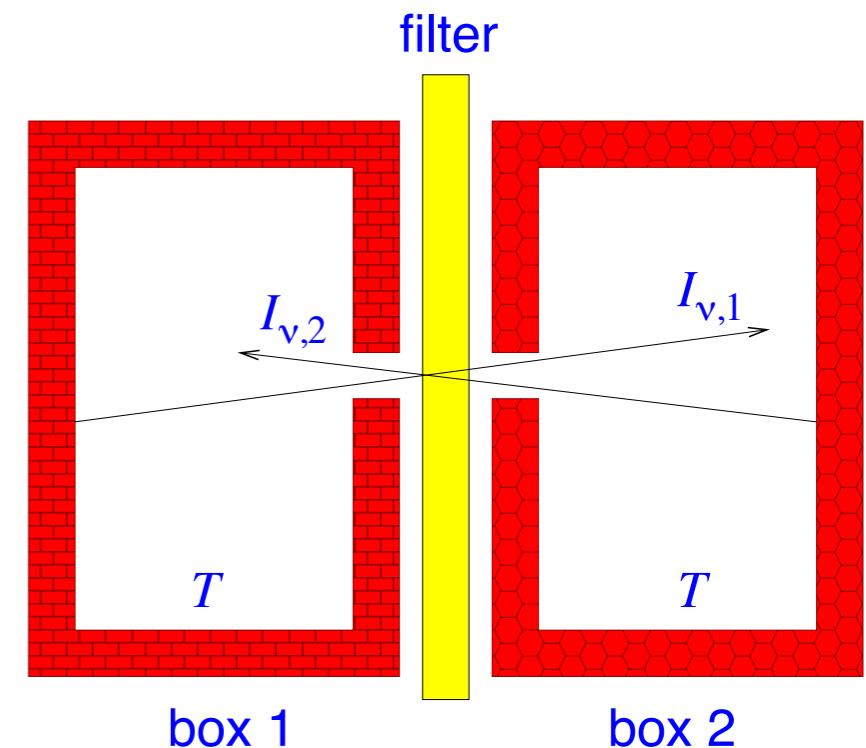


- A **blackbody** is an idealized physical body that absorbs all incident radiation regardless of frequency or angle of incidence (i.e., perfect absorber). The above cavity can be regarded to be a blackbody.
- Radiation from a blackbody in thermal equilibrium is called the **blackbody radiation**.

Blackbody radiation if the universal function.

- Now, consider another cavity (box 2), also at the same temperature, but made of different material or shape and connect two cavities with a filter transparent only in the narrow frequency range ν and $\nu + d\nu$.

- In equilibrium at T , radiation should transfer no net energy from one cavity to the other. Otherwise, one cavity will cool down and the other heats up; this violates the second law of thermodynamics.
- Therefore, the intensity or spectrum that passes through the holes should be a universal function of T and should be isotropic.
- The intensity and spectrum of the radiation emerging from the hole should be independent of the wall material (e.g., wood, copper, or concrete, etc) and any absorbing material that may be inside the cavity.
- The universal function is called the Planck function $B_\nu(T)$.
- This is the blackbody radiation.



Kirchhoff's Law in TE

- In (full) thermodynamic equilibrium at temperature T , by definition, we know that

$$\frac{dI_\nu}{ds} = 0 \quad \text{and} \quad I_\nu = B_\nu(T)$$

- We also note that

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu$$

- Then, we can obtain ***the Kirchhoff's law for a system in TE:***

$$\frac{j_\nu(T)}{\kappa_\nu(T)} = B_\nu(T)$$

- This is remarkable because it connects the properties $j_\nu(T)$ and $\kappa_\nu(T)$ of any kind of matter to the single universal spectrum $B_\nu(T)$.

Kirchhoff's Law in LTE

- Recall that Kirchhoff's law was derived for a system in thermodynamic equilibrium.
- ***Kirchhoff's law applies not only in TE but also in LTE:***
 - Recall that $B_\nu(T)$ is independent of the properties of the radiating /absorbing material.
 - In contrast, both $j_\nu(T)$ and $\alpha_\nu(T)$ depend only on the materials in the cavity and on the temperature of that material; they do not depend on the ambient radiation field or its spectrum.
 - Therefore, the Kirchhoff's law should be true even for the case of LTE.
- ***In LTE, the ratio of the emission coefficient to the absorption coefficient is a function of temperature and frequency only.***
- This generalized version of Kirchhoff's law is an exceptionally valuable tool for calculating the emission coefficient from the absorption coefficient or vice versa.

Implications of Kirchhoff's Law

- A good absorber is a good emitter, and a poor absorber is a poor emitter. (In other words, a good reflector must be a poor absorber, and thus a poor emitter.)

$$j_\nu = \kappa_\nu B_\nu(T) \rightarrow j_\nu \text{ increases as } \kappa_\nu \text{ increases}$$

- It is not possible to thermally radiate more energy than a blackbody, at equilibrium.

$$j_\nu < B_\nu(T) \text{ because } \kappa_\nu < 1$$

- The radiative transfer equation in LTE can be rewritten:

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T)$$

- **Remark:**

- **Blackbody radiation** means $I_\nu = B_\nu(T)$. An object for which the intensity is the Planck function is emitting blackbody radiation.
- **Thermal radiation is defined to be radiation emitted by “matter” in LTE.** Thermal radiation means $S_\nu = B_\nu(T)$. An object for which the source function is the Planck function is emitting thermal radiation.
- **Thermal radiation becomes blackbody radiation only for optically thick media.**

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- To see the difference between thermal and blackbody radiation,
 - consider a slab of material with optical depth τ_ν that is producing thermal radiation.
 - If no light is falling on the back side of the slab, the intensity that is measured on the front side of the slab is

$$\begin{aligned}I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu}) \\&= B_\nu (1 - e^{-\tau_\nu})\end{aligned}$$

- If the slab is optical thick at frequency ν ($\tau_\nu \gg 1$), then

$$I_\nu \approx B_\nu$$

- If the slab is optically thin ($\tau_\nu \ll 1$), then

$$I_\nu \approx \tau_\nu B_\nu \ll B_\nu$$

This indicates that the radiation, although thermal, will not be blackbody.

Thermal radiation becomes blackbody radiation only for optical thick media.

The Planck Spectrum (Quantum Mechanics)

- There is no perfect blackbody.
 - However, the cosmic microwave background comes quite close; stars can sometimes be usefully approximated as blackbodies.
 - By the end of the 19th century, the blackbody spectrum was fairly well known empirically, from laboratory studies. In 1900, Max Planck, using his idea of quantized energies, derived the blackbody spectrum.
- How to calculate the blackbody spectrum?
 - Intensity spectrum is related to the energy density:

$$J_\nu = \frac{c}{4\pi} u_\nu$$

- Energy density =
Number density of photon states x Average energy of each state
- Number density of photon states =
number of states per solid angle per volume per frequency

- (1) Number density of photon state:
 - Consider a photon propagating in direction \mathbf{n} inside a box with dimensions L_x, L_y, L_z in x, y, z directions.

- wave vector: $\mathbf{k} = \frac{2\pi}{\lambda} \mathbf{n} = \frac{2\pi\nu}{c} \mathbf{n}$

- If each dimension of the box is much longer than a wavelength, the photon can be represented by standing wave in the box.
- number of nodes in each direction: $n_x = k_x L_x / 2\pi$
- number of node changes in a wave number interval (if $n_i \gg 1$):

$$\Delta n_x = \frac{L_x \Delta k_x}{2\pi}$$

- number of states in 3D wave vector element $\Delta k_x \Delta k_y \Delta k_z = d^3 k$:

$$\Delta N = \Delta n_x \Delta n_y \Delta n_z = 2 \frac{L_x L_y L_z d^3 k}{(2\pi)^3} = 2 \frac{V d^3 k}{(2\pi)^3}$$

two independent polarizations

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- the density of states (**number of states per solid angle per volume per frequency**):

$$d^3k = k^2 dk d\Omega = \frac{(2\pi)^3 \nu^2 d\nu d\Omega}{c^3}$$

$$\rightarrow \rho_s = \frac{dN}{V d\nu d\Omega} = \frac{2\nu^2}{c^3}$$

- (2) Average energy of each state:

- Each state may contain n photons of energy $h\nu$.
- The energy of the state is $E_n = nh\nu$.
- The probability of a state of energy E_n is proportional to $e^{-\beta E_n}$, where $\beta = (k_B T)^{-1}$ and k_B is the Boltzmann's constant. (from statistical mechanics)
- Therefore, the average energy is:

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = -\frac{\partial}{\partial \beta} \ln \left(\sum_{n=0}^{\infty} e^{-\beta E_n} \right)$$

$$\sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} (e^{-\beta h\nu})^n = (1 - e^{-\beta h\nu})^{-1}$$

$$\langle E \rangle = \frac{h\nu e^{-\beta h\nu}}{1 - e^{-\beta h\nu}} = \frac{h\nu}{\exp(h\nu/k_B T) - 1}$$

- Energy density, integrated over all solid angle:

$$u_\nu = 4\pi\rho_s \langle E \rangle = \frac{8\pi h\nu^3/c^3}{\exp(h\nu/k_B T) - 1}$$

- Planck Law:

$$B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/k_B T) - 1} \text{ or } B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda k_B T) - 1}$$

Spectrum of blackbody radiation

- The frequency dependence of blackbody radiation is given by the ***Planck function***:

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/k_B T) - 1} \quad \text{or} \quad B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda k_B T) - 1}$$

$h = 6.63 \times 10^{-27}$ erg s (Planck's constant)

$k_B = 1.38 \times 10^{-16}$ erg K⁻¹ (Boltzmann's constant)

See "Fundamentals of Statistical and Thermal Physics" (Federick Reif) or "Astrophysical Concepts" (Harwit) for the derivation.

- Energy density:***

$$u_\nu(T) = \frac{4\pi}{c} B_\nu(T) = \frac{8\pi h\nu^3/c^3}{\exp(h\nu/k_B T) - 1}$$

Note that the textbook "Interstellar and Intergalactic Medium" use $\varepsilon_\nu(T)$ to denote the energy density.

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- Photon occupation number:
 - The photon occupation number is dimensionless, and is simply the average number of photons per mode per polarization.

$$n_\gamma(\nu) = \frac{c^2}{2h\nu^3} I_\nu(\nu)$$

- If the radiation field is a blackbody, the photon occupation number is given by

$$n_\gamma(\nu; T) = \frac{1}{\exp(h\nu/k_B T) - 1}$$

Bose-Einstein statistics

Stefan-Boltzmann Law

- Emergent flux is proportional to T^4 .

$$F = \pi \int B_\nu(T) d\nu = \pi B(T)$$

←

$$B(T) = \int B_\nu(T) d\nu = \frac{ac}{4\pi} T^4 = \frac{\sigma}{\pi} T^4$$

$$F = \sigma T^4$$

Stephan – Boltzmann constant : $\sigma = \frac{2\pi^5 k_B^4}{15c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^2 \text{ s}^{-1} \text{ K}^{-4} \text{ sr}^{-1}$

- Total energy density (*another form of the Stefan-Boltzmann law*)

$$u = \frac{4\pi}{c} \int B_\nu(T) d\nu = \frac{4\pi}{c} B(T)$$

$$u(T) = \left(\frac{T}{3400 \text{ K}} \right)^4 \text{ erg cm}^{-3}$$

$$u = aT^4$$

radiation constant : $a \equiv \frac{4\sigma}{c} = 7.57 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$

Rayleigh-Jeans Law & Wien Law

Rayleigh-Jeans Law (low-energy limit)

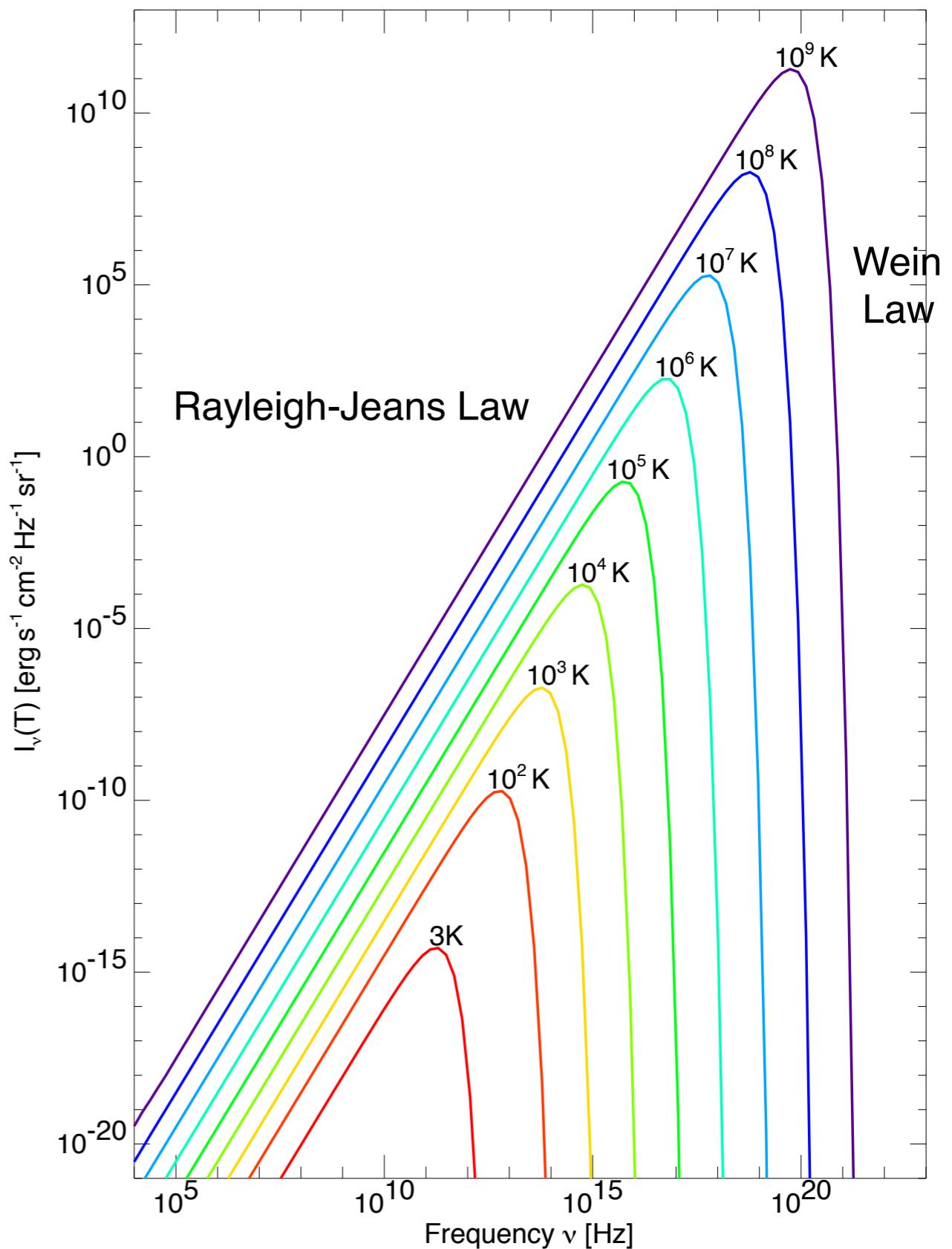
$$h\nu \ll k_B T \quad (\nu \ll 2 \times 10^{10} (T/1\text{ K}) \text{ Hz})$$

$$I_\nu^{RJ}(T) = \frac{2\nu^2}{c^2} k_B T$$

Wien Law (high-energy limit)

$$h\nu \gg k_B T$$

$$I_\nu^W(T) = \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{k_B T}\right)$$



Characteristic Temperatures

- **Brightness Temperature:**

- The brightness temperature is defined to be the temperature such that a blackbody at that temperature would have specific intensity:

$$I_\nu = B_\nu(T_b) \rightarrow T_b(\nu) = \frac{h\nu/k_B}{\ln [1 + 2h\nu^3/(c^2 I_\nu)]}$$

- **Antenna Temperature:**

- Radio astronomers are used to working at very low frequencies. They define the antenna temperature as being the brightness temperature in the Rayleigh-Jeans tail.

$$I_\nu = \frac{2\nu^2}{c^2} k_B T_b \rightarrow T_A \equiv \frac{c^2}{2k_B \nu^2} I_\nu$$

- Radiative transfer equation in the RJ limit:

$$\frac{dT_A}{d\tau_\nu} = -T_A + T \quad (T = \text{temperature of the material in LTE})$$

$$T_A = T_A(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu}) \quad \text{if } T \text{ is constant.}$$

-
- **Color Temperature:**
 - By fitting the spectrum to a blackbody curve without regarding to vertical scale (absolute intensity scale), a color temperature T_c is obtained.
 - The color temperature correctly gives the temperature of a blackbody source of unknown absolute scale.
 - **Effective Temperature:**
 - The effective temperature of a source is obtained by equating the actual flux F to the flux of a blackbody at temperature T_{eff} .

$$F = \int \int I_\nu \cos \theta d\nu d\Omega = \sigma T_{\text{eff}}^4$$

- **Excitation Temperature:**
 - The excitation temperature of level u relative to level ℓ is defined by

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{E_{u\ell}}{k_B T_{\text{exc}}}\right) \rightarrow T_{\text{exc}} \equiv \frac{E_{u\ell}/k_B}{\ln\left(\frac{n_\ell/g_\ell}{n_u/g_u}\right)} \quad (E_{u\ell} \equiv E_u - E_\ell)$$

- Radio astronomers studying the 21 cm line sometimes use the term “**spin temperature**” T_{spin} .

Atomic Processes

Atomic Processes

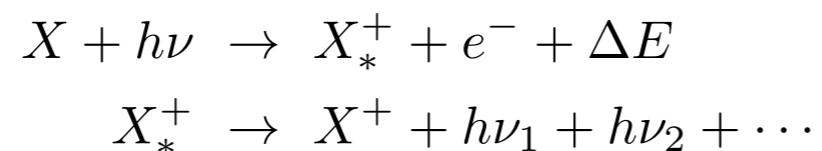
- **Ionization**
 - ▶ Photoionization and Auger-ionization
 - ▶ Collisional Ionization (Direct ionization and Excitation-autoionization)
- **Recombination**
 - ▶ Radiative recombination \Leftrightarrow Photoionization
 - ▶ Dielectronic Recombination (not dielectric!)
 - ▶ Three-body recombination \Leftrightarrow Direct collisional ionization
- **Charge exchange**
- **Excitation and de-excitation (Transition)**
 - ▶ Collisional excitation
 - ▶ Collisional de-excitation
 - ▶ Radiative excitation (photoexcitation; photoabsorption)
 - ▶ Radiative de-excitation (spontaneous emission and stimulated emission)
- **Emission Line**
 - ▶ Collisionally-excited emission lines
 - ▶ Recombination lines

Ionization - [Photoionization]

- Interstellar medium (ISM) is transparent to $h\nu < 13.6 \text{ eV}$ photons, but is very opaque to ionizing photons. In fact, the ISM does not become transparent until $h\nu \sim 1 \text{ keV}$.
 - Sources of ionizing photons include massive, hot young stars, hot white dwarfs, and supernova remnant shocks.
- ***From the Outer Shells***
 - ▶ Photoionization is the inverse process to radiative recombination.



- ▶ If the incoming photon has sufficient energy, it may leave the ionized species in an excited state.

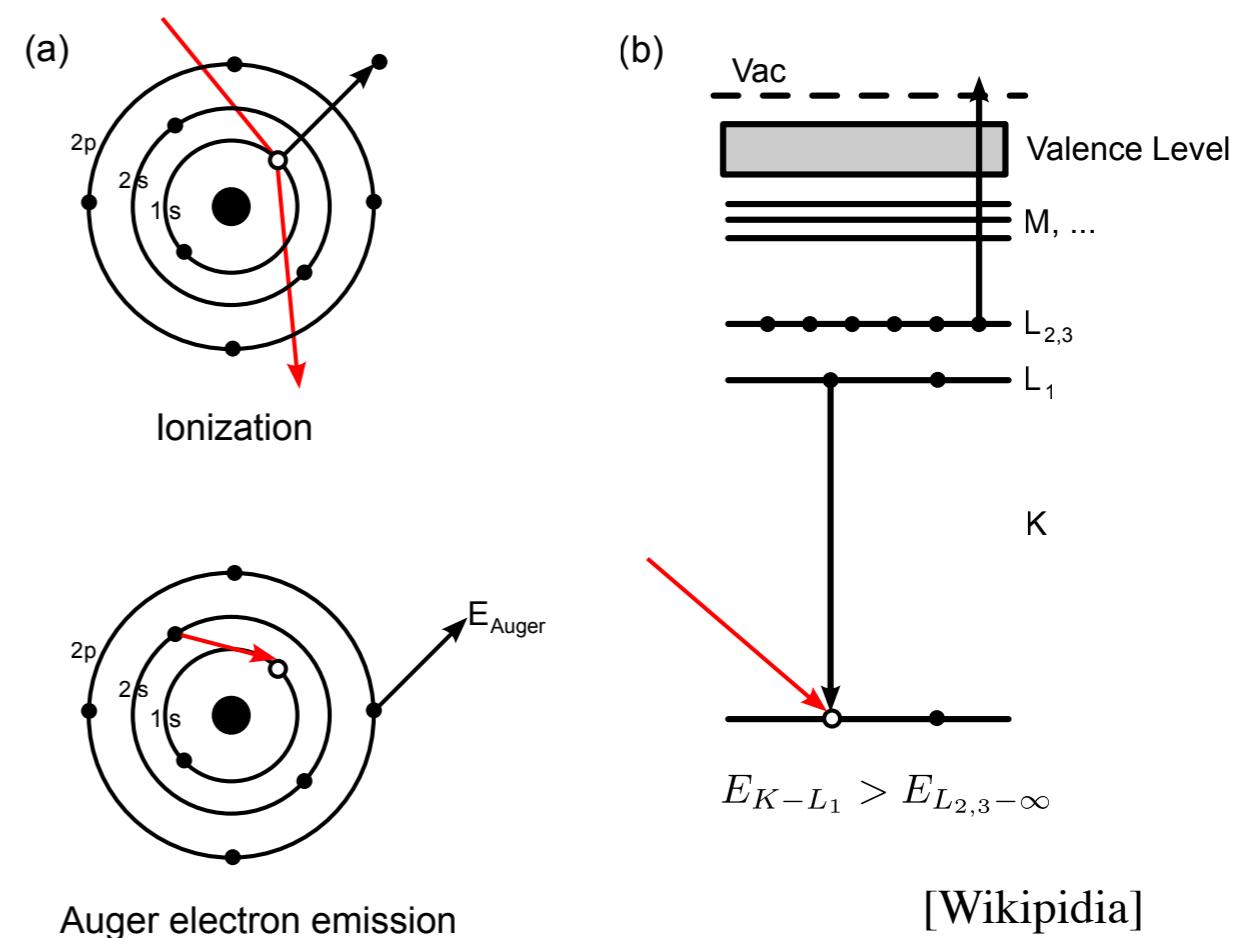
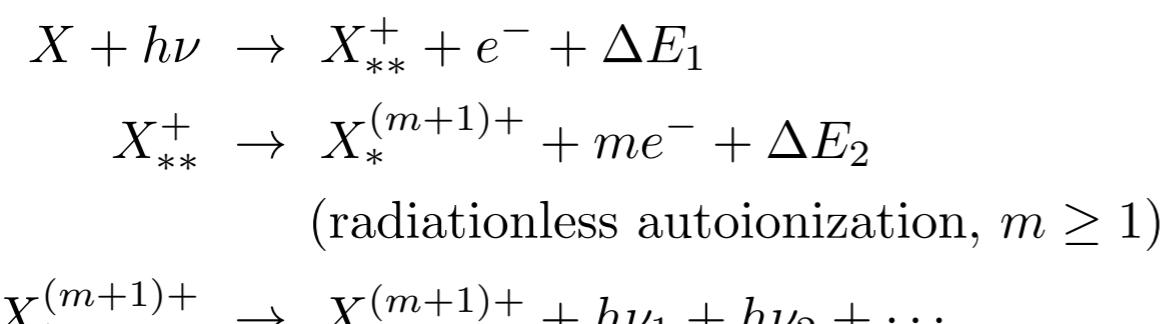


Here, X denotes an atom, molecule, or ion and subscript * indicates an excited states.

ΔE denotes energy carried by the electron.

• Inner Shell Photoionization

- ▶ If the energy of the incoming photon is still higher, it becomes possible to remove one of the inner shell electrons which also results in a change in the electron configuration in the excited species. This may be followed by a radiative readjustment back to the ground state.
- ▶ However, in this case, **Auger ionization** becomes more probable. High frequency photons may eject an inner shell electron from an ion or atom, and the resulting ion may then fill the gap in its inner shell with an outer electron, ejecting another outer electron to remove the energy in a radiationless transition called an Auger transition. Such a process will produce very energetic electrons which will lose their energy in heating up the gas.

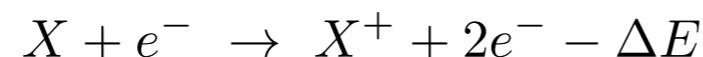


Auger electron emission

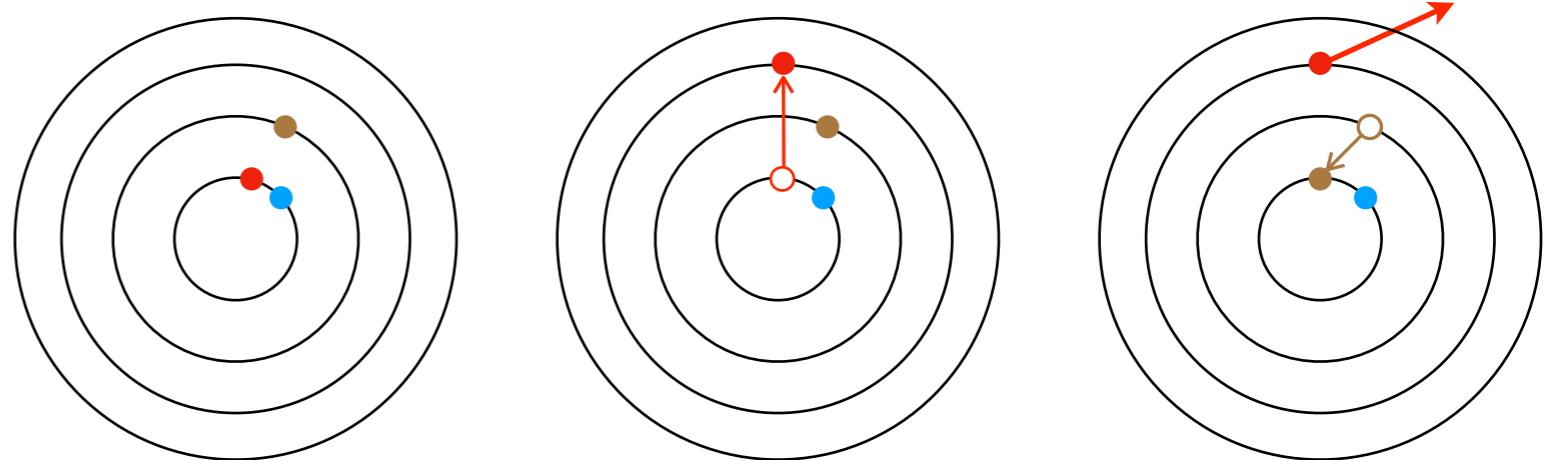
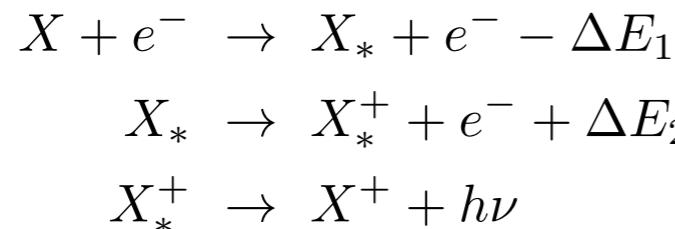
[Wikipedia]

Ionization - [Collisional ionization]

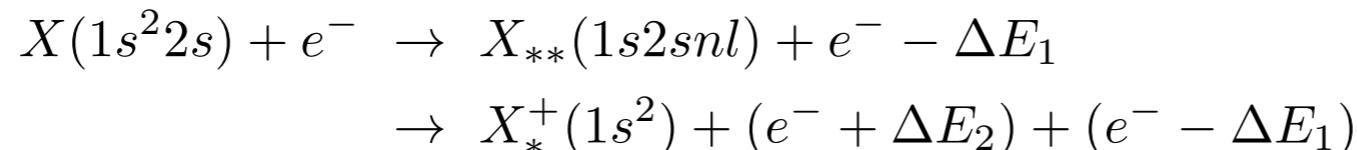
- ***Direct collisional ionization:*** The process whereby an electron strikes an atom or ion X, with sufficient energy to strip out a bound electron:



- ***Excitation-autoionization:*** At sufficiently high electron impact energies, more than one electron of the target may be excited, leaving the atom in an unstable state, which is stabilized by the radiationless ejection of one of outer electrons, possibly followed by a radiative decay of the ionized atom back to its ground state. This process is favored in heavy elements which have a large number of inner shell electrons and only one or two electrons in the outer shell.

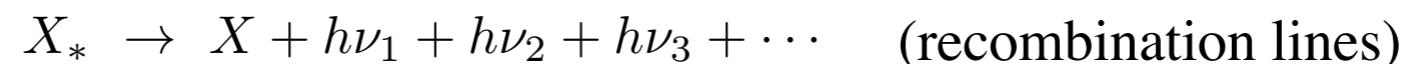
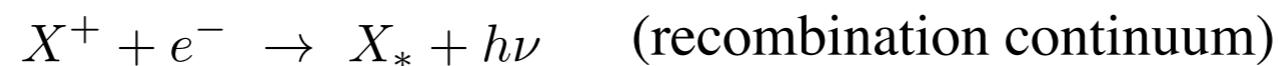


- For example, in collisions of Li-like ions, excitation and autoionization can occur via excitation of the 1s-electron into states with principal quantum numbers $n \geq 2$. After the decay of a doubly excited state, one has an additional electron in the final channel.



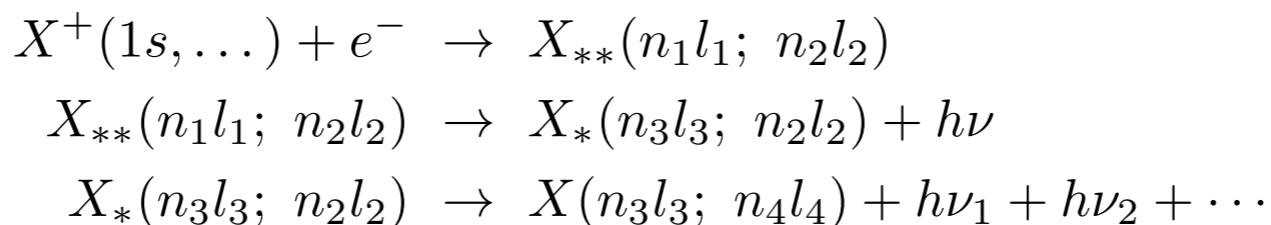
Recombination

- Radiative recombination
 - Radiative recombination is the process of capture of an electron by an ion where the excess energy is radiated away in a photon.
 - The electron is captured into an excited state. The recombined but still excited ion radiates several photons in a radiative cascade, as it returns to the ground state:



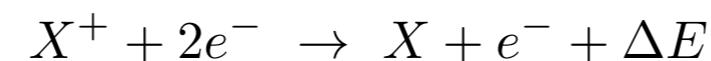
- The photon in the first line represents a **recombination continuum** ($h\nu$) photon. However, photons ($h\nu_1$, $h\nu_2$, $h\nu_3$) represent quantized transitions and are therefore termed **recombination lines**.
- The total effective recombination rate can be written as the sum of the recombination rate to each state.

-
- Dielectronic recombination
 - Dielectronic recombination (DR) is a resonant two-step process in which a few electron is captured by the incident ion into an autoionizing state followed by a stabilizing radiation decay of the resulting ion in the second step. The DR is often more important than radiative recombination.



- The first step is a double-electron process, often called dielectronic capture, through which one free electron is captured and another core electron is simultaneously excited forming a doubly excited state. One of the electron is in an autoionizing state, $n_1 l_1$, and the other is in an excited state, $n_2 l_2$. In the second step, the ion in a doubly excited state emits a photon and decays into a stable state below the ionization limit.
- There is a competitive process for the decay of the doubly excited ion. - the autoionization or Auger decay associated not with the radiative transition but with a change in ion charge.

-
- Three-body Recombination
 - The combination process of an electron with a positive ion in a gas in such a way that the incoming free electron transfers energy and momentum to another free electron in the neighborhood of the ion.



- In most interstellar medium, three-body recombination is unimportant.
- In dense regions with electron densities above 10^4 cm^{-3} , three-body recombination into high levels of the hydrogen atom with principal quantum numbers ($n > 100$) can be important. (Compare Eq. (3.44) and (14.6) in Draine)

Charge Exchange

- During the collision of two ionic species, the charge clouds surrounding each interact, and it is possible that an electron is exchanged between them.



- Since, in virtually all diffuse astrophysical plasmas, hydrogen and helium are overwhelmingly the most abundant species, the charge-exchange reactions which are significant to the ionization balance of the plasma are



Ionization Equilibrium

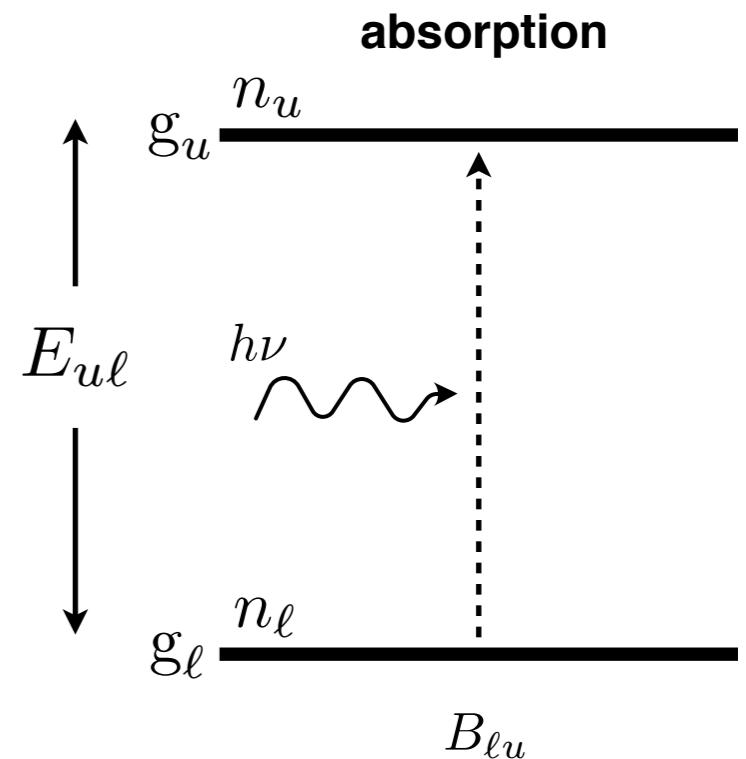
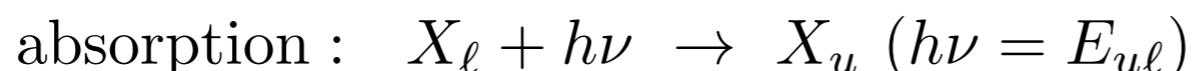
- ***Collisional Ionization Equilibrium (CIE)*** or coronal equilibrium
 - ▶ Balance at a given temperature between collisional ionization from the ground states of the various atoms and ions, and the process of recombination from the higher ionization stages.
 - ▶ In this equilibrium, effectively, all ions are in their ground state.
- ***Photoionization Equilibrium:***
 - ▶ Balance between photo-ionization and the process of recombination.
- Ionization balance under conditions of local thermodynamic equilibrium (LTE)
 - ▶ The ionization equilibrium in LTE is described by the Saha equation.
 - ▶ In LTE, the excited states are all populated according to Boltzmann's law.

Radiative Excitation and De-excitation (Absorption and Emission)

- Three Radiative Transitions and Einstein Coefficients

- **Absorption:**

- If an absorber (atom, ion, molecule, or dust grain) X is in a lower level ℓ and there is radiation present with photons having an energy equal to $E_{u\ell}$. The absorber can absorb a photon and undergo an upward transition.



- The rate per volume at which the absorbers absorb photons will be proportional to both the energy density u_ν of photons of the appropriate energy and the number density n_ℓ of absorbers in the lower level ℓ .

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = - \left(\frac{dn_\ell}{dt} \right)_{\ell \rightarrow u} = n_\ell B_{\ell u} u_\nu$$

- The proportionality constant $B_{\ell u}$ is the **Einstein B coefficient** for the upward transition $\ell \rightarrow u$.

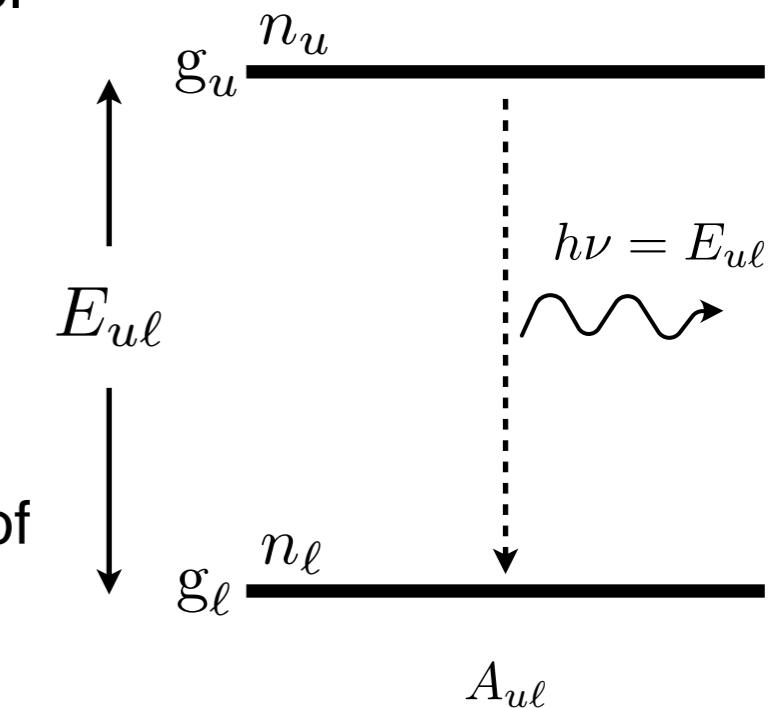
- **Emission:**

- An absorber X in an excited level u can decay to a lower level ℓ with emission of a photon. There are two ways this can happen:

spontaneous emission : $X_u \rightarrow X_\ell + h\nu$ ($h\nu = E_{ul}$)

stimulated emission : $X_u + h\nu \rightarrow X_\ell + 2h\nu$ ($h\nu = E_{ul}$)

spontaneous emission

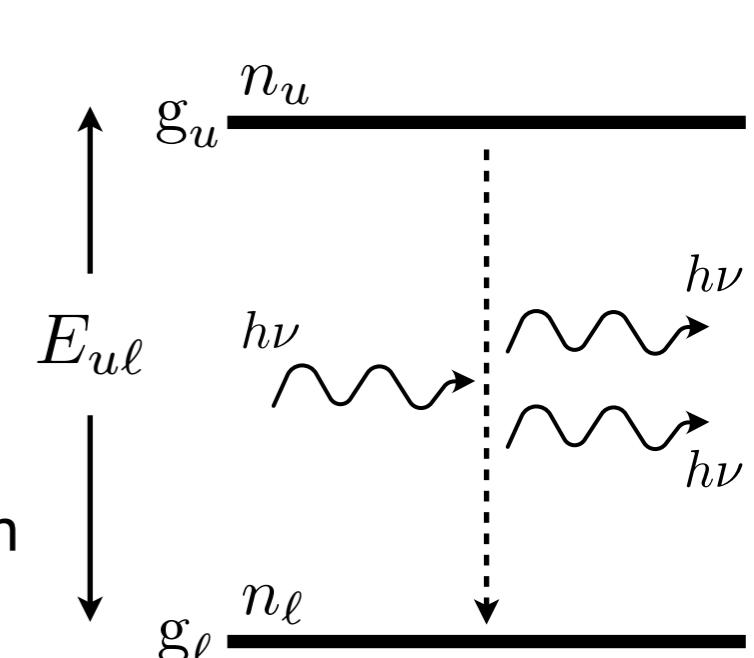


- Spontaneous emission** is a random process, independent of the presence of a radiation field.
- Stimulated emission** occurs if photons of the identical frequency, polarization, and direction of propagation are already present, and the rate of stimulated emission is proportional to the energy density u_ν of these photons.

$$\left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell} = - \left(\frac{dn_u}{dt} \right)_{u \rightarrow \ell} = n_u (A_{ul} + B_{ul}u_\nu)$$

- The probability per unit time A_{ul} is the Einstein A coefficient for spontaneous transition. The coefficient B_{ul} is the Einstein B coefficient for the downward transition $u \rightarrow \ell$.

stimulated emission



B_{ul}

Relations between the Einstein coefficients

- ***In thermal equilibrium***, the radiation field becomes the “blackbody” radiation field and the two levels must be populated according to the Boltzmann distribution.

$$(u_\nu)_{\text{TE}} = \frac{4\pi}{c} B_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}$$

$$\left(\frac{n_u}{n_\ell} \right)_{\text{TE}} = \frac{g_u}{g_\ell} e^{-E_{u\ell}/k_B T} \quad \text{Here, } E_{u\ell} = h\nu.$$

- The net rate of change of level u should be equal to zero, in TE.

$$\begin{aligned} \frac{dn_u}{dt} &= \left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} + \left(\frac{dn_u}{dt} \right)_{u \rightarrow \ell} \\ &= n_\ell B_{\ell u} u_\nu - n_u (A_{u\ell} + B_{u\ell} u_\nu) \\ &= 0 \end{aligned}$$

- We, therefore, obtain the following relations between the Einstein coefficients.

$$A_{u\ell} = \frac{8\pi h\nu^3}{c^3} B_{u\ell}$$

$$B_{\ell u} = \frac{g_u}{g_\ell} B_{u\ell}$$

$$B_{u\ell} = \frac{c^3}{8\pi h\nu^3} A_{u\ell}$$

$$B_{\ell u} = \frac{g_u}{g_\ell} \frac{c^3}{8\pi h\nu^3} A_{u\ell}$$

-
- We note the Einstein coefficients are intrinsic properties of the absorbing material, irrelevant to the assumption of TE. Hence, the relations between the Einstein coefficients should hold in any condition.
 - Using the relation, we can rewrite the downward and upward transition rates:

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = n_\ell \frac{g_u}{g_\ell} \frac{c^3}{8\pi h\nu^3} A_{u\ell} u_\nu \quad \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell} = n_u A_{u\ell} \left(1 + \frac{c^3}{8\pi h\nu^3} u_\nu \right)$$

- It is helpful to use a dimensionless quantity, the photon occupation number:

$$n_\gamma \equiv \frac{c^2}{2h\nu^3} I_\nu \quad \xrightarrow{\text{averaging over directions}} \quad \langle n_\gamma \rangle = \frac{c^2}{2h\nu^3} J_\nu = \frac{c^3}{8\pi h\nu^3} u_\nu$$

- Then, the above transition rates are simplified:

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = n_\ell \frac{g_u}{g_\ell} A_{u\ell} \langle n_\gamma \rangle \quad \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell} = n_u A_{u\ell} (1 + \langle n_\gamma \rangle)$$

- The photon occupation number determines the relative importance of stimulated and spontaneous emission: stimulated emission is important only when $\langle n_\gamma \rangle \gg 1$.

Absorption and Emission Coefficients in terms of Einstein coefficients

- The Einstein coefficients are useful means of analyzing absorption and emission processes. However, we often find it even more useful to use cross section because the cross section has a natural geometric meaning.
- Absorption cross section:**
 - The number density of photons per unit frequency interval is $u_\nu/h\nu$. Let $\sigma_{\ell u}(\nu)$ be the cross section for absorption of photons for the transition $\ell \rightarrow u$. Then, the absorption rate is

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = n_\ell \int d\nu \sigma_{\ell u}(\nu) c \frac{u_\nu}{h\nu} \approx n_\ell u_\nu \frac{c}{h\nu} \int d\nu \sigma_{\ell u}(\nu)$$

- Here, we assumed that $u_\nu/h\nu$ do not vary appreciably over the line profile of the cross section. Therefore, we derive a simple relation between the absorption cross section and the Einstein B coefficient:

$$\int d\nu \sigma_{\ell u}(\nu) = \frac{h\nu_{u\ell}}{c} B_{\ell u} = \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell}$$

- If the cross section has a normalized profile of ϕ_ν , we can write the absorption cross section as follows:

$$\sigma_{\ell u}(\nu) = \frac{h\nu_{u\ell}}{c} B_{\ell u} \phi_\nu = \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell} \phi_\nu \quad \text{with} \quad \int \phi_\nu d\nu = 1$$

- **Absorption Coefficient**

- We note that the stimulated emission is proportional to the energy density of ambient radiation field. In the radiative transfer equation, it is convenient to include the stimulated emission term in the absorption coefficient as a negative absorption.

$$\begin{aligned} \left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} - \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell}^{\text{stimulated}} &= n_\ell B_{\ell u} u_\nu - n_u B_{u \ell} u_\nu \\ &= n_\ell B_{\ell u} u_\nu - n_u \left(\frac{g_\ell}{g_u} B_{\ell u} \right) u_\nu \end{aligned}$$

- Therefore, we may define the cross section for stimulated emission and the net (effective) absorption coefficient as follows:

$$\begin{aligned} \sigma_{u\ell} &= \frac{g_\ell}{g_u} \sigma_{\ell u} \\ \kappa_\nu &= n_\ell \sigma_{\ell u} - n_u \sigma_{u\ell} \\ &= n_\ell \sigma_{\ell u} \left(1 - \frac{n_u/n_\ell}{g_u/g_\ell} \right) \end{aligned}$$

- Using the definition of the excitation temperature, we can rewrite them:

$$\kappa_\nu = n_\ell \sigma_{\ell u} \left[1 - \exp \left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}} \right) \right] \quad \text{or} \quad \sigma_\nu^{\text{eff}} = \sigma_{\ell u} \left[1 - \exp \left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}} \right) \right]$$

- ***Emission coefficient (Emissivity)***

- The emissivity is defined as the power radiated per unit frequency per unit solid angle per unit volume.
- The line emissivity can be expressed in terms of the spontaneous downward transition rate:

$$4\pi \int d\nu j_\nu = h\nu_{u\ell} \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell}^{\text{spontaneous}}$$

- Comparing with the definition of the Einstein A coefficient, we obtain:

$$\int d\nu j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell}$$

- If the emission line has a normalized profile of ϕ_ν , we can write the emissivity as follows:

$$j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell} \phi_\nu \quad \text{with} \quad \int d\nu \phi_\nu = 1$$

-
- The correction factor for the stimulated emission in absorption coefficient:
 - For Ly α line,

$$h\nu_{u\ell} = 10.2 \text{ eV} \rightarrow 1 - \exp\left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}}\right) = 1 - \exp\left(-\frac{1.1837 \times 10^5 \text{ K}}{T_{\text{exc}}}\right)$$

$$\simeq 1 \quad \text{for } T_{\text{exc}} \approx T_{\text{gas}} < 1 \times 10^5 \text{ K}$$

- ▶ The stimulated emission is negligible.
 - For 21 cm line,
- $$h\nu_{u\ell} = 6 \mu\text{eV} \rightarrow 1 - \exp\left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}}\right) = 1 - \exp\left(-\frac{0.068 \text{ K}}{T_{\text{exc}}}\right)$$
- $$\simeq \frac{0.068 \text{ K}}{T_{\text{exc}}} \ll 1 \quad \text{for } T_{\text{exc}} \approx T_{\text{gas}} \sim 100 \text{ K}$$
- ▶ The correction for stimulated emission is very important. We, therefore, need to take into account the stimulation emission in dealing with the 21 cm line.

- **Source Function:**

$$\begin{aligned}
 S_\nu &= \frac{j_\nu}{\kappa_\nu} \\
 &= \frac{n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell} \phi_\nu}{n_\ell \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell} \phi_\nu [1 - \exp(-h\nu_{u\ell}/k_B T_{\text{exc}})]} \quad \leftarrow \frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp(-h\nu_{u\ell}/k_B T_{\text{exc}}) \\
 &= \frac{2h\nu_{u\ell}^3}{c^2} \frac{1}{\exp(h\nu_{u\ell}/k_B T_{\text{exc}}) - 1}
 \end{aligned}$$

- This is called **generalized Kirchhoff's law**.

Oscillator Strength

- In the previous slides, we characterized the absorption cross section by the Einstein A coefficient. Equivalently, we can express the cross section in terms of the oscillator strength for the absorption transition $\ell \rightarrow u$, defined by the relation:

$$\int \sigma_{\ell u}(\nu) d\nu = \frac{\pi e^2}{m_e c} f_{\ell u} \quad \rightarrow \quad \sigma_{\ell u}(\nu) = \frac{\pi e^2}{m_e c} f_{\ell u} \phi_\nu$$

- Here, the factor $\frac{\pi e^2}{m_e c}$ is the cross-section for a classical oscillator model.
- The oscillator strength is the factor which corrects the classical result. The quantum mechanical process can be interpreted as being due to a (fractional) number f of equivalent classical electron oscillators of the same frequency.
- The Einstein A coefficient is related to the absorption oscillator strength of the upward transition by

$$A_{u\ell} = \frac{8\pi^2 e^2 \nu_{u\ell}^2}{m_e c^3} \frac{g_\ell}{g_u} f_{\ell u} = \left(\frac{0.8167 \text{ cm}}{\lambda_{u\ell}} \right)^2 \frac{g_\ell}{g_u} f_{\ell u} [\text{s}^{-1}]$$

- For 21.1 cm line, $g_u = 3$, $g_\ell = 1$ ($g_F = 2F + 1$) $f_{\ell u} = 5.75 \times 10^{-12}$

$$A_{u\ell} = 2.88 \times 10^{-15} [\text{s}^{-1}]$$

- For Ly α (1215.67 Å) line, $g_u = 3$, $g_\ell = 1$ ($g_L = 2L + 1$) $f_{\ell u} = 0.4164$

$$A_{u\ell} = 6.265 \times 10^8 [\text{s}^{-1}]$$

Collisional Excitation & De-excitation

- **Collisional Rate (Two Level Atom)**

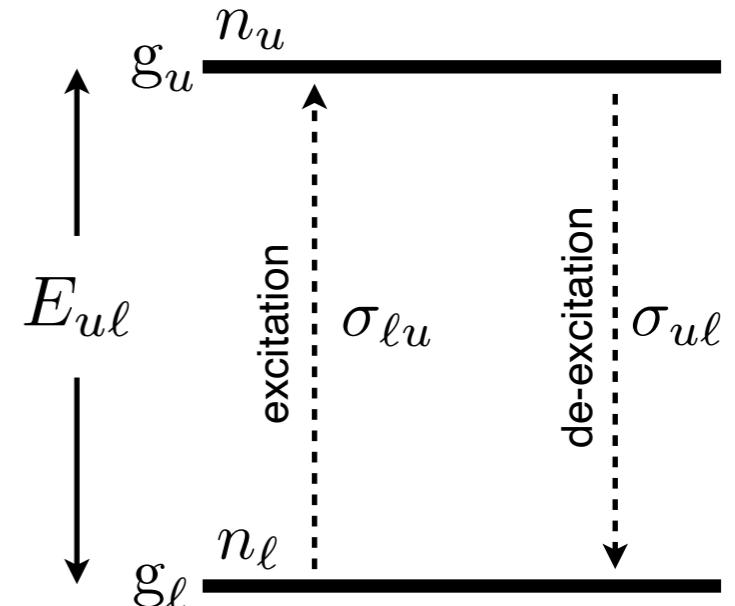
- ▶ The cross section $\sigma_{\ell u}$ for collisional excitation from a lower level ℓ to an upper level u is, in general, inversely proportional to the impact energy (or v^2) above the energy threshold E_{ul} and is zero below.
- ▶ The collisional cross section can be expressed in the following form using a dimensionless quantity called the ***collision strength*** $\Omega_{\ell u}$:

$$\begin{aligned}\sigma_{\ell u}(v) &= (\pi a_0^2) \left(\frac{hR_\infty}{\frac{1}{2}m_e^2 v^2} \right) \frac{\Omega_{\ell u}}{g_\ell} \text{ cm}^2 \quad \text{for } \frac{1}{2}m_e^2 v^2 > E_{ul} \\ &= \frac{h^2}{4\pi m_e^2 v^2} \frac{\Omega_{\ell u}}{g_\ell}\end{aligned}$$

or $\sigma_{\ell u}(E) = \frac{h^2}{8\pi m_e E} \frac{\Omega_{\ell u}}{g_\ell} \quad (E = \frac{1}{2}m_e v^2)$

where, $a_0 = \frac{\hbar^2}{m_e e^2} = 5.12 \times 10^{13}$ cm, Bohr radius

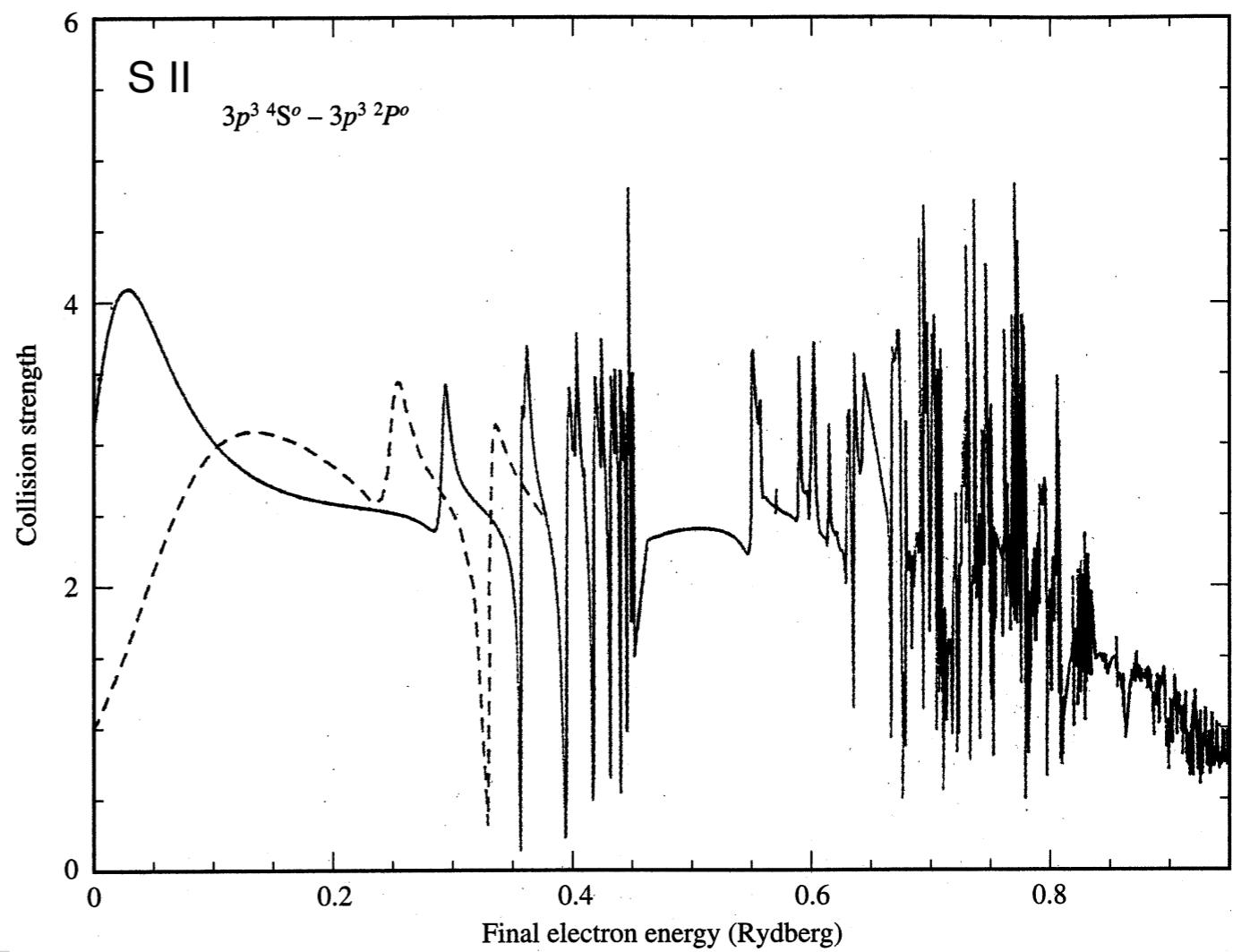
$$R_\infty = \frac{m_e e^4}{4\pi \hbar^3} = 109,737 \text{ cm}^{-1}, \text{ Rydberg constant} \quad \left(\hbar = \frac{h}{2\pi} \right)$$



- ▶ The collision strength $\Omega_{\ell u}$ is a function of electron velocity (or energy) but is often approximately constant near the threshold. Here, g_ℓ and g_u are the statistical weights of the lower and upper levels, respectively.

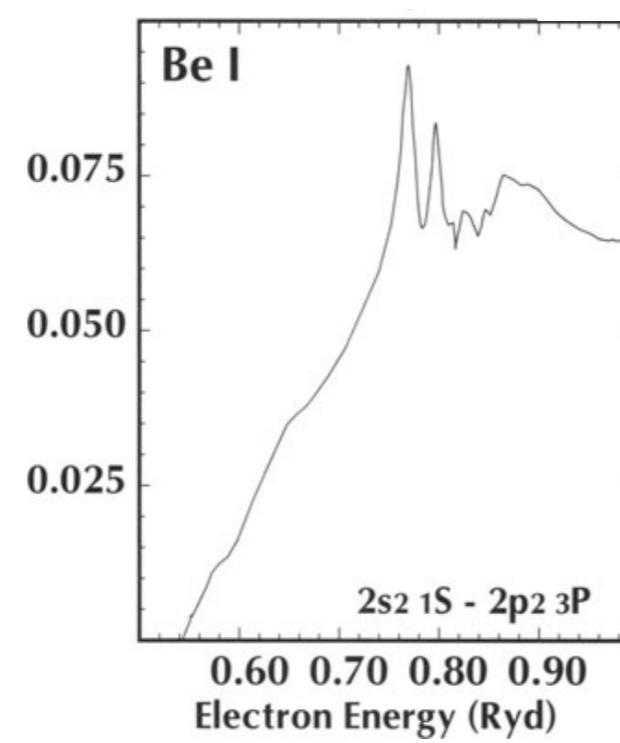
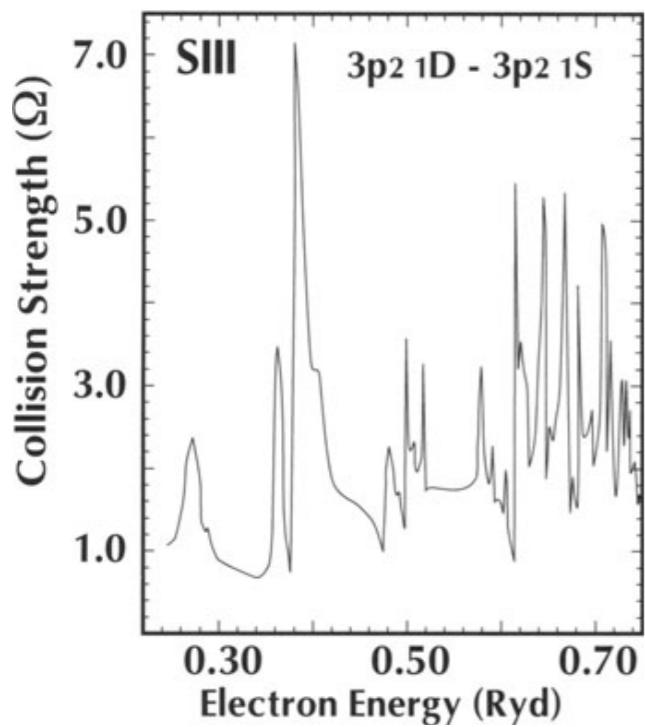
- Collision Strength

- Quantum mechanical calculations show that (1) the resonance structure in the collision strengths is important and (2) the collision strength increases with energy for neutral species.



Tayal (1996)

Badnell (1999)



solid - Ramsbottom et al. (1996)
dashed - Cai & Pradhan (1993)

- Advantage of using the collision strength is that (1) it removes the primary energy dependence for most atomic transitions and (2) they have the symmetry between the upper and the lower states.

The principle of detailed balance states that ***in thermodynamic equilibrium each microscopic process is balanced by its inverse.***

$$n_e n_\ell v_\ell \sigma_{\ell u}(v_\ell) f(v_\ell) dv_\ell = n_e n_u v_u \sigma_{u\ell}(v_u) f(v_u) dv_u$$

Here, v_ℓ and v_u are related by $\frac{1}{2}m_e v_\ell^2 = \frac{1}{2}m_e v_u^2 + E_{u\ell}$, and $f(v)$ is a Maxwell velocity distribution of electrons. Using the Boltzmann equation of thermodynamic equilibrium,

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right)$$

we derive the following relation between the cross-sections for excitation and de-excitation are

$$g_\ell v_\ell^2 \sigma_{\ell u}(v_\ell) = g_u v_u^2 \sigma_{u\ell}(v_u) \quad \text{Here, } \frac{1}{2}m_e v_\ell^2 = \frac{1}{2}m_e v_u^2 + E_{u\ell}$$

or

$$g_\ell(E + E_{u\ell}) \sigma_{\ell u}(E + E_{u\ell}) = g_u E \sigma_{u\ell}(E)$$

and the symmetry of the collision strength between levels.

$$\Omega_{\ell u} = \Omega_{u\ell}$$

more precisely $\Omega_{u\ell}(E + E_{u\ell}) = \Omega_{u\ell}(E)$

These two relations were derived in the TE condition. However, ***the cross-sections are independent on the assumptions, and thus the above relations should be always satisfied.***

► Collisional excitation and de-excitation rates

The ***collisional de-excitation rate per unit volume per unit time, which is thermally averaged,*** is

$$\begin{aligned} \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell} &= n_e n_u \int_0^\infty v \sigma_{u\ell}(v) f(v) dv \\ &= n_e n_u k_{u\ell} \quad [\text{cm}^{-3} \text{ s}^{-1}] \end{aligned}$$

$$k_{u\ell} \equiv \langle \sigma v \rangle_{u \rightarrow \ell}$$

$$\begin{aligned} k_{u\ell} &= \int_0^\infty v \sigma_{u\ell}(v) f(v) dv \\ &= \left(\frac{2\pi\hbar^4}{k_B m_e^3} \right)^{1/2} T^{-1/2} \frac{\langle \Omega_{u\ell} \rangle}{g_u} \\ &= \frac{8.62942 \times 10^{-6}}{T^{1/2}} \frac{\langle \Omega_{u\ell} \rangle}{g_u} \quad [\text{cm}^3 \text{ s}^{-1}], \end{aligned}$$

$$\text{Here, } \langle \Omega_{u\ell} \rangle = \int_0^\infty \Omega_{u\ell}(E) e^{-E/k_B T} d(E/k_B T)$$

and the ***collisional excitation rate per unit volume per unit time*** is

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = n_e n_\ell k_{\ell u}$$

$$k_{\ell u} \equiv \langle \sigma v \rangle_{\ell \rightarrow u}$$

$$\begin{aligned} k_{\ell u} &= \int_{v_{\min}}^\infty v \sigma_{\ell u}(v) f(v) dv \quad \text{Here, } \frac{1}{2} m_e v_{\min}^2 = E_{u\ell} \\ &= \left(\frac{2\pi\hbar^4}{k_B m_e^3} \right)^{1/2} T^{-1/2} \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right) \end{aligned}$$

Here, $k_{\ell u}$ and $k_{u\ell}$ are the collisional rate coefficient for excitation and de-excitation coefficients in units of $\text{cm}^3 \text{ s}^{-1}$, respectively. We also note that ***the rate coefficients for collisional excitation and de-excitation are related by***

$$k_{\ell u} = \frac{g_u}{g_\ell} k_{u\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right)$$

$$\langle \sigma v \rangle_{\ell \rightarrow u} = \frac{g_u}{g_\ell} \langle \sigma v \rangle_{u \rightarrow \ell} \exp\left(-\frac{E_{u\ell}}{kT}\right)$$

Sum rule for collision strengths

- Quantum mechanical sum rule for collision strengths for the case where one term consists of a singlet ($S = 0$ or $L = 0$) and the second consists of a multiplet: the collision strength of each fine structure level J is related to the total collision strength of the multiplet by

$$\Omega_{(SLJ, S'L'J')} = \frac{(2J' + 1)}{(2S' + 1)(2L' + 1)} \Omega_{(SL, S'L')}$$

Here, $(2J'+1)$ is the statistical weight of an individual level in the multiplet, and $(2S'+1)(2L'+1)$ is the statistical weight of the multiplet.

We can regard the collision strength as “shared” amongst these levels in proportion to the statistical weights of the individual levels ($g_J = 2J+1$).

- We will learn that the flux ratio between the fine structure lines in a multiplet is proportional to the ratio of their collision strengths, provided that the energy gap between the fine structure levels are small. Then, the flux ratio is determined by the ratio of their statistical weights.

- C-like ions ($1s^2 2s^2 2p^2 \rightarrow 1s^2 2s^2 2p^2$) forbidden or inter combination transitions.

ground states (triplet) - ${}^3P_0 : {}^3P_1 : {}^3P_2 = 1 : 3 : 5$

excited states (singlets) - ${}^1D_2, {}^1S_1$

- Li-like ions ($1s^2 2s^1 \rightarrow 1s^2 2p^1$) resonance transitions

ground state (singlet) - ${}^2S_{1/2}$

excited states (doublet) - ${}^2P_{3/2} : {}^2P_{1/2} = 2 : 1$

Collisionally-Excited Emission Line

- Emission line flux

- ◆ In the low density limit, the collisional rate between atoms and electrons is much slower than the (spontaneous) radiative de-excitation rate of the excited level. Thus, we can balance the collisional feeding into level u by the rate of radiative transition back down to level ℓ . The level population is determined by

$$\begin{aligned}
 n_e n_\ell k_{\ell u} &= A_{u\ell} n_u \\
 \frac{n_u}{n_\ell} &= \frac{n_e k_{\ell u}}{A_{u\ell}} \\
 &= \frac{n_e}{A_{u\ell}} \beta \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} T^{-1/2} \exp\left(-\frac{E_{u\ell}}{kT}\right)
 \end{aligned}$$

where $A_{u\ell}$ is the Einstein coefficient for spontaneous emission. The line emissivity is given by

$$\begin{aligned}
 4\pi j_{u\ell} &= E_{u\ell} A_{u\ell} n_u = E_{u\ell} n_e n_\ell k_{\ell u} \\
 &= n_e n_\ell E_{u\ell} \frac{8.62942 \times 10^{-6}}{T^{1/2}} \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right) \quad [\text{erg cm}^{-3} \text{ s}^{-1}] \\
 &\simeq \beta \chi n_e^2 E_{u\ell} T^{-1/2} \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right)
 \end{aligned}$$

Here, $\beta = \left(\frac{2\pi\hbar^4}{km_e^2}\right)^{1/2} = 8.62942 \times 10^{-6}$
 $\chi = n_\ell/n_e$

For low temperature, the exponential term dominates because few electrons have energy above the threshold for collisional excitation, so that the line rapidly fades with decreasing temperature.

At high temperature, the $T^{-1/2}$ term controls the cooling rate, so the line fades slowly with increasing temperature.

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- ♦ In **high-density limit**, the level population are set by the Boltzmann equilibrium, and the line emissivity is

$$\begin{aligned} 4\pi j_{ul} &= E_{\ell u} A_{ul} n_u \\ \frac{n_u}{n_\ell} &= \frac{g_u}{g_\ell} \exp\left(-\frac{E_{ul}}{kT}\right) \\ &= n_\ell E_{\ell u} A_{ul} \frac{g_u}{g_\ell} \exp\left(-\frac{E_{\ell u}}{kT}\right) \\ &\simeq \chi n_e E_{\ell u} A_{ul} \frac{g_u}{g_\ell} \exp\left(-\frac{E_{\ell u}}{kT}\right) \end{aligned}$$

- ♦ Here, the line flux scales as n_e rather than n_e^2 , but the line flux tends to a constant value at high temperature.
- ♦ **Critical density** is defined as the density where the radiative depopulation rate matches the collisional de-excitation for the excited state.

$$A_{ul} n_u = n_e n_u k_{ul}$$

$$n_{\text{crit}} = \frac{A_{ul}}{k_{ul}} \quad \rightarrow \quad n_{\text{crit}} = A_{ul} \frac{g_u}{\beta \langle \Omega_{ul} \rangle} T^{1/2} \quad [\text{cm}^{-3}]$$

- ♦ At densities higher than the critical density, collisional de-excitation becomes significant, and the forbidden lines will be weaker as the density increases.

At around this density, the line emissivity vs density plotted in log-log scale changes slope from +2 to +1.

Spontaneous decay rates and critical densities for several forbidden lines

Species	Transition	$\lambda(\mu\text{m})$	$A(\text{s}^{-1})$	$n_c(\text{cm}^{-3})^{\dagger}$
Optical forbidden lines				
N ⁺	${}^1D_2 - {}^3P_2$	0.65834	3.0×10^{-3}	7.0×10^5
O ⁺	${}^2D_{5/2} - {}^4S_{3/2}$	0.37288	3.6×10^{-5}	3.1×10^3
	${}^2D_{3/2} - {}^4S_{3/2}$	0.37260	1.8×10^{-4}	1.6×10^4
O ⁺⁺	${}^1D_2 - {}^3P_2$	0.50069	2.0×10^{-2}	8.6×10^4
Infrared fine-structure lines				
C ⁰	${}^3P_1 - {}^3P_0$	609.135	7.93×10^{-8}	1.7×10^2
	${}^3P_2 - {}^3P_1$	370.415	2.68×10^{-7}	7.1×10^2
C ⁺	${}^2P_{3/2} - {}^2P_{1/2}$	157.741	2.36×10^{-6}	3.0×10^3
O ⁰	${}^3P_1 - {}^3P_2$	63.184	8.95×10^{-5}	9.8×10^5
	${}^3P_0 - {}^3P_1$	145.526	1.70×10^{-5}	1.5×10^5
O ⁺⁺	${}^3P_2 - {}^3P_1$	51.8	9.8×10^{-5}	3.8×10^3
	${}^3P_1 - {}^3P_0$	88.4	2.6×10^{-5}	1.7×10^3
Si ⁺	${}^2P_{3/2} - {}^2P_{1/2}$	34.815	2.13×10^{-4}	3.4×10^5

[†] Calculated for $T_e = 10^4$ K.

Adapted from Osterbrock 1989, *Astrophysics of Gaseous Nebulae and Active Galactic Nuclei*, and Watson 1985, *Physica Scripta*, **T11**, 33.

Dependence on density

- Under the conditions of very low density and weak radiation fields,
 - ▶ The vast majority of the atoms reside in the ground state.
collisional excitation timescale >> radiative decay time scale
This condition will remain true even if the excited state has a radiative lifetime of several second, which is frequently the case for the forbidden transitions observed in ionized astrophysical plasmas.
 - ▶ flux of an emission line \propto flux number of collisions \propto product of the number densities of the two colliding species by the probability that a collision will produce a collisional excitation.
 - ▶ If the energy gap between the ground state and the excited state $E_{\ell u}$ is much larger than the mean energy of the colliding species ($\sim k_B T$), then, because there are few very energetic collisions, relatively few collisional excitations can occur. Therefore, the resulting emission line will be very much weaker than when $E_{ul} < k_B T$.
This gives us ***the possibility of measuring temperature from the relative strengths of lines coming from excited levels at different energies above the ground state.***

- At high enough densities,
 - ▶ The collisional timescales are short.
 - ▶ The population in any upper level is set by the balance between collisional excitation, and the collisional de-excitation out of these levels, and are governed by the Boltzmann equilibrium.
- At intermediate densities,
 - ▶ The collisional rates and the radiative decay rates are compatible.
 - ▶ The intensity of an emission line is determined by both the temperature and the density.
 - ▶ If the temperature is known, the density can be determined from the intensity ratio of two such lines.