KAIST Astrophysics (PH481) - Part 1

Week 2a

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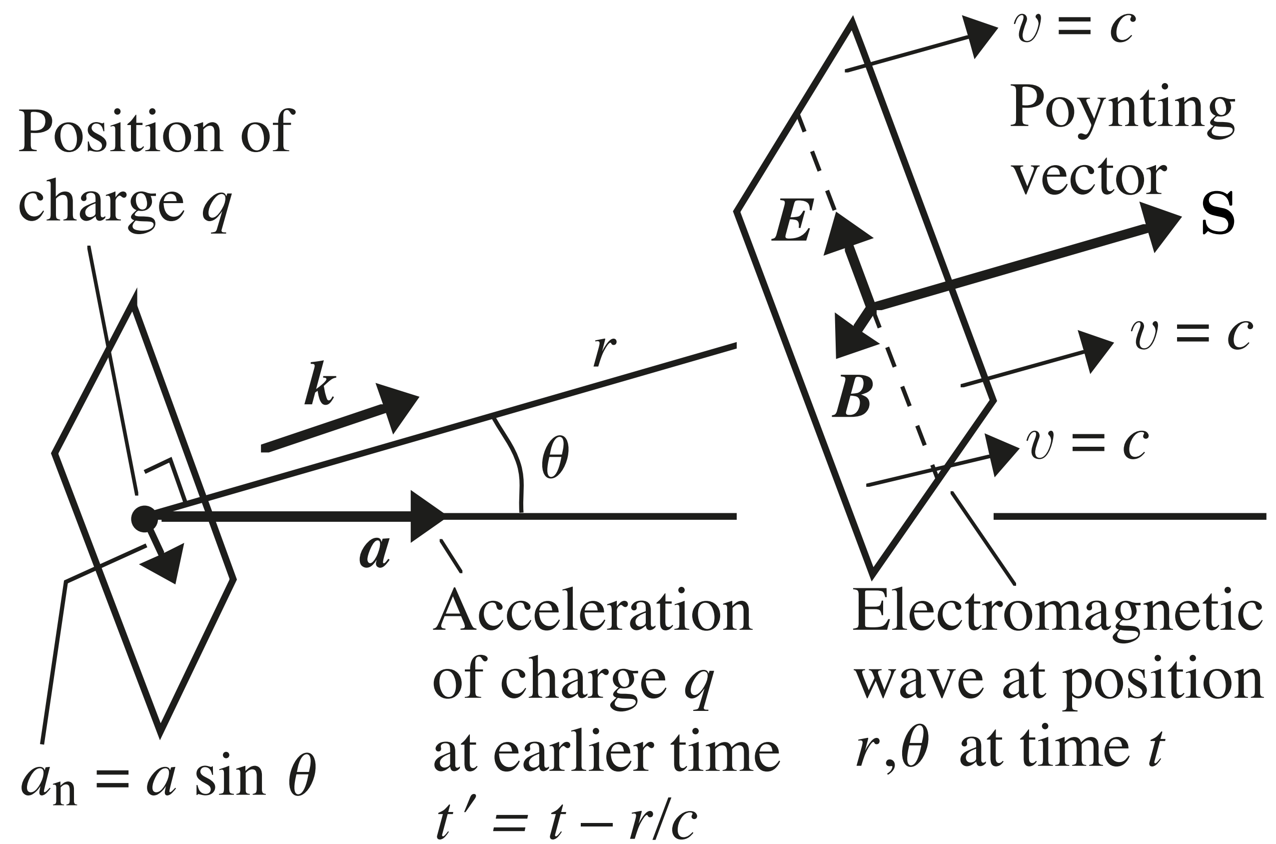
Thermal Bremstrahlung

Radiated electric vector

- If a charged particle undergoes acceleration, electromagnetic wave is created.
  - Electromagnetic waves are transverse. (The electric and magnetic vectors both are perpendicular to the direction of propagation.)
  - The propagating electric vector $\mathbf{E}$ (due to the instantaneous acceleration at the retarded time or earlier time $t' = t - r/c$) lies in the plane defined by the acceleration vector $\boldsymbol{a}$ and the propagation direction $\mathbf{k}$.

$$\mathbf{E}_{\rm rad}(\mathbf{r}, t) \approx \frac{q}{rc^2}\mathbf{k} \times (\mathbf{k} \times \boldsymbol{a})$$

$$\mathbf{B}_{\rm rad}(\mathbf{r}, t) = \mathbf{k} \times \mathbf{E}_{\rm rad}(\mathbf{r}, t)$$



- Poynting vector
  - The energy flux density carried by the electromagnetic wave is described by the Poynting vector:

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}$$

The **retarded time** refers to conditions at the point $\mathbf{r}'$ that existed at a time earlier than t by just the time required for light to travel from $\mathbf{r} = 0$ to $\mathbf{r}$.
The Electric field responds to the changes after the "retarded time" delay.
(For more details, see https://seoncafe.github.io/Teaching_files/2019_astrophysics/lecture2.pdf)

Larmor's Formula

- Larmor Formula (dipole approximation)
  - Energy emitted per unit time into unit solid angle about the propagation vector $\mathbf{k}$:

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \sin^2 \Theta$$

  - Total power emitted into all angles:

$$\boxed{P = \frac{2q^2 a^2}{3c^3}}$$

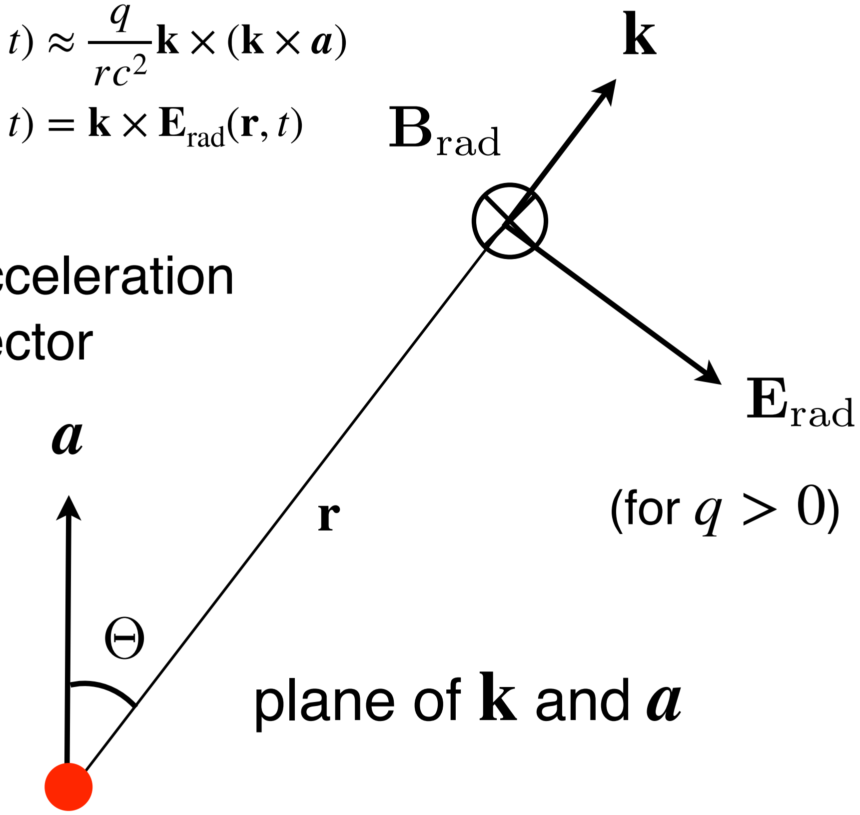
Larmor's formula

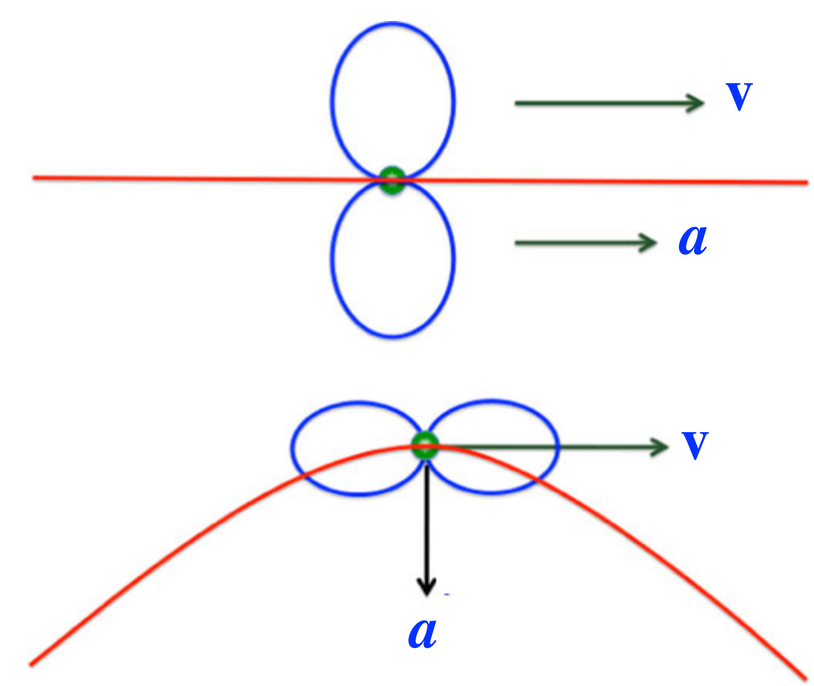
  - The power emitted is proportional to the square of the charge and the square of the acceleration.
  - ***Pattern of the radiation:*** No radiation is emitted along the direction of acceleration, and the maximum is emitted perpendicular to acceleration.
  - The instantaneous direction $\mathbf{E}_{\mathrm{rad}}$ is determined by $\boldsymbol{a}$ and $\mathbf{k}$.
  - The radiation will be linearly polarized in the plane of $\boldsymbol{a}$ and $\mathbf{k}$.

$$\mathbf{E}_{\mathrm{rad}}(\mathbf{r}, t) \approx \frac{q}{rc^2}\mathbf{k} \times (\mathbf{k} \times \boldsymbol{a})$$

$$\mathbf{B}_{\mathrm{rad}}(\mathbf{r}, t) = \mathbf{k} \times \mathbf{E}_{\mathrm{rad}}(\mathbf{r}, t)$$

$\boldsymbol{a}$ = acceleration vector





Bremsstrahlung

- **Bremsstrahlung (in German = “breaking radiation”) (or free-free emission)**
  - Radiation due to the acceleration of a charge in the Coulomb field of another charge.
  - Bremsstrahlung is often called free-free emission because it is produced by free electrons scattering off ions without being captured - the electrons are free before the interaction and remain free afterward.
  - Consider bremsstrahlung radiated from a plasma of temperature T and densities $n_e\,(\mathrm{cm}^{-3})$ electrons with charge $-e$ and $n_i\,(\mathrm{cm}^{-3})$ ions with charge Ze.
  - Then, the ratio between potential to kinetic energies is

$$\frac{\text{Coulomb potential energy}}{\text{thermal kinetic energy}} \approx \frac{Ze^2/\langle r\rangle}{kT} \approx \frac{Ze^2 n_e^{1/3}}{kT} \qquad \leftarrow \quad n_e \approx \frac{1\ \text{electron}}{V} \approx \frac{1\ \text{electron}}{\langle r\rangle^3}$$

$$= 1.67\times10^{-7}\times Z\left(\frac{n_e}{1\ \mathrm{cm}^{-3}}\right)^{1/3}\frac{10^4\ K}{T}$$

$$\ll 1$$

for typical $n_e \ll 1$ and $T \sim 10^4 - 10^8 K$.

  - Therefore, **Coulomb interaction is only a perturbation on the thermal motions of the electrons.**

- **Thermal Bremsstrahlung** is the radiation produced by thermal electrons distributed according to the Maxwell velocity distribution.
  - Thus, it is thermal emission, because it is produced by a source whose emitting particles are in local thermodynamic equilibrium (LTE).

- A full understanding of this process requires a quantum treatment.
  - However, a classical treatment is justified in some regimes, and the formulas so obtained have the correct functional dependence for most of the physical parameters.

- **Bremsstrahlung due to the collision of identical particles (electron-electron, proton-proton) is zero.**
  - Because the accelerations of the two particles are equal in magnitude but opposite in direction. Their radiated electric fields are equal in magnitude but opposite in sign, so that the net radiated electric field approaches zero at distances much larger than the collision impact parameter.

- The bottom line is that only the electron-ion collisions are important, and only the electrons radiate significantly.

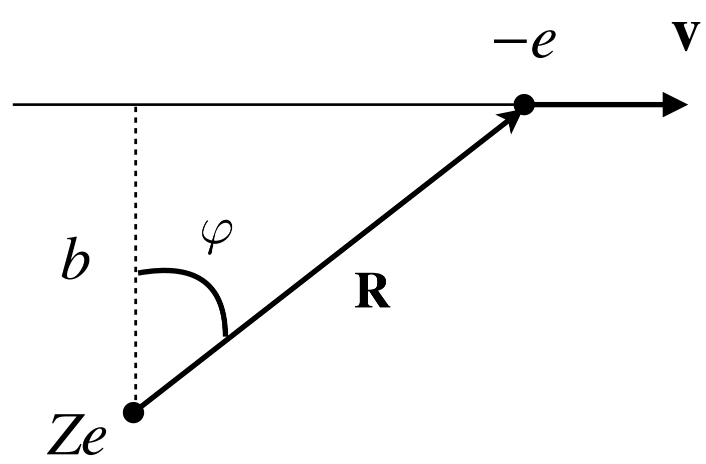
- Assumptions
  - The plasma is assumed to be completely ionized and to consist of elections of charge $-e$ and ions of charge $+Ze$, where Z is the atomic number.
  - In electron-ion bremsstrahlung, we treat the electron as moving in a fixed Coulomb field of the ion, since ***the relative accelerations are inversely proportional to the masses.*** Therefore, radiation from ions can be ignored.

$$\frac{a_i}{a_e} = \frac{m_e}{m_i} \sim (1800)^{-1} < 10^{-3}$$

  - The electrons emit photons with energies $h\nu$ substantially less than their own kinetic energies.
  - The energy loss of an electron, integrated over an entire collision, is only a small part of the electron kinetic energy.

Emission radiated per collision

- **Small-angle scattering** approximation:
  - The electron moves ***rapidly enough*** so that the deviation of its path from a straight line is negligible.



  - **Collision time**: We assume that the interaction between the electron and the ion happens only when the electron passes close to the ion. The characteristic time τ is

$$\tau \approx \frac{b}{v}$$

  - Then, the radiation consists of ***a single pulse*** at an angular frequency of $\omega \approx \dfrac{1}{\tau} = \dfrac{v}{b}$.
    The frequency of the radiation due to a single collision is thus

$$\nu = \frac{\omega}{2\pi} \approx \frac{v}{2\pi b}$$

The smaller the impact parameter, the higher the emitted frequency.

- **Total emitted radiation by a single electron**:
  - During the interaction we assume that the acceleration is constant and equal to

$$a \approx -\frac{Ze^2}{m_e b^2} \quad \leftarrow \quad F = m_e a \approx -\frac{Ze^2}{b^2}$$

  - From the Larmor formula, we get the ***instantaneous power:***

$$P = \frac{2e^2a^2}{3c^3} \approx \frac{2e^2}{3c^3}\frac{Z^2e^4}{m_e^2 b^4} = \frac{2Z^2e^6}{3m_e^2c^3b^4}$$

  - We assume that the acceleration is constant while it is in the vicinity of the ion for time duration τ and to be zero before and after this period.
  - The total energy emitted by the electron is then the sum (integral) for the entire duration of the collision.

$$W(b, \upsilon) \approx P\tau \approx \frac{2Z^2e^6}{3m_e^2c^3b^3\upsilon}$$

  - This is the ***total energy radiated by a single electron of speed υ as it passes an ion of charge Ze with impact parameter b.***

Radiation from Single-speed electron beam

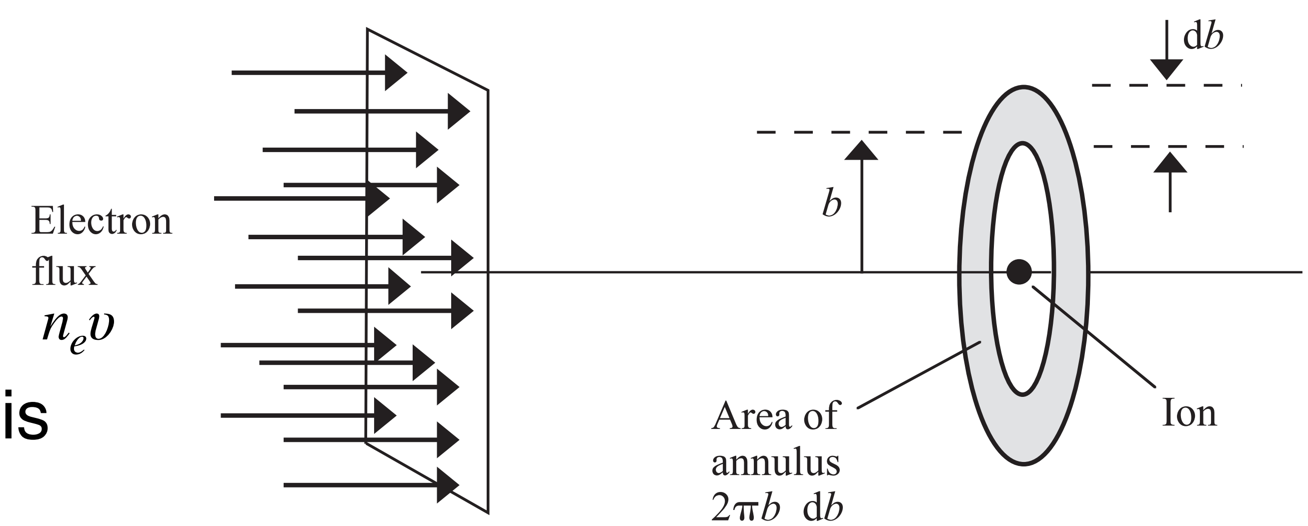
- Consider only the electrons that intersect a narrow annulus of radius b and width db surrounding the ion. One can then calculate the power emitted by those electrons.
- If $n_e =$ the electron density, then the electron flux is $n_e v$. The energy emitted by the annulus is

$$\mathscr{P}_b(b, v)db = W(b, v)n_e v(2\pi b)db$$

- Recall the relation between the impact parameter and frequency:

$$b = \frac{v}{2\pi\nu}$$

$$|db| = \frac{v}{2\pi\nu^2}|d\nu|$$



- The ***power per unit-frequency interval*** is

$$\mathscr{P}_\nu(\nu, v)d\nu = \mathscr{P}_b(b, v)db$$

$$= \frac{8\pi^2}{3}\frac{Z^2e^6}{m_e^2c^3}\frac{n_e}{v}d\nu$$

- Surprisingly, this is independent of frequency.

Radiation from thermal electrons

- Consider the electrons having the Maxwell distribution

$$f(v) = \left(\frac{m_e}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m_e v^2}{2kT}\right)$$

- The volume emissivity is obtained by integrating the power per unit-frequency over the particle velocity and multiplying the number density of ions:

$$j_\nu = n_i \int_{v_{\min}}^{\infty} f(v) \mathscr{P}_\nu(\nu, v) 4\pi v^2 dv$$

  - The lower limit of the integral is ***the smallest velocity that an electron can still emit a photon of energy*** $h\nu$. An electron cannot emit a photon of energy larger than it initially had.

$$h\nu = \frac{1}{2} m_e v_{\min}^2 \;\rightarrow\; v_{\min} = \sqrt{\frac{2h\nu}{m_e}}$$

  - Because $\mathscr{P}_\nu$ is independent of frequency, the frequency dependence comes in only through $v_{\min}$.

$$j_\nu = \frac{32}{3}\left(\frac{2}{3}\frac{\pi^3}{m_e^3 k}\right)^{1/2} \frac{Z^2 e^6}{c^3} n_e n_i \frac{e^{-h\nu/kT}}{T^{1/2}}$$

Spectrum of Thermal Bremsstrahlung

- We have approximately derived the result.
  - There would be errors due to the approximations we have made.
  - However, it turns out to be correct except a minor factor, which is called the Gaunt factor.

$$j_\nu^{\rm ff} = \frac{32}{3}\left(\frac{2}{3}\frac{\pi^3}{m_e^3 k}\right)^{1/2}\frac{Z^2e^6}{c^3}n_e n_i \frac{e^{-h\nu/kT}}{T^{1/2}}\overline{g}_{\rm ff}$$
$$= 6.8\times10^{-38}\overline{g}_{\rm ff}Z^2 n_i n_e T^{-1/2}e^{-h\nu/kT}\quad ({\rm erg\ cm^{-3}\ s^{-1}\ Hz^{-1}})$$

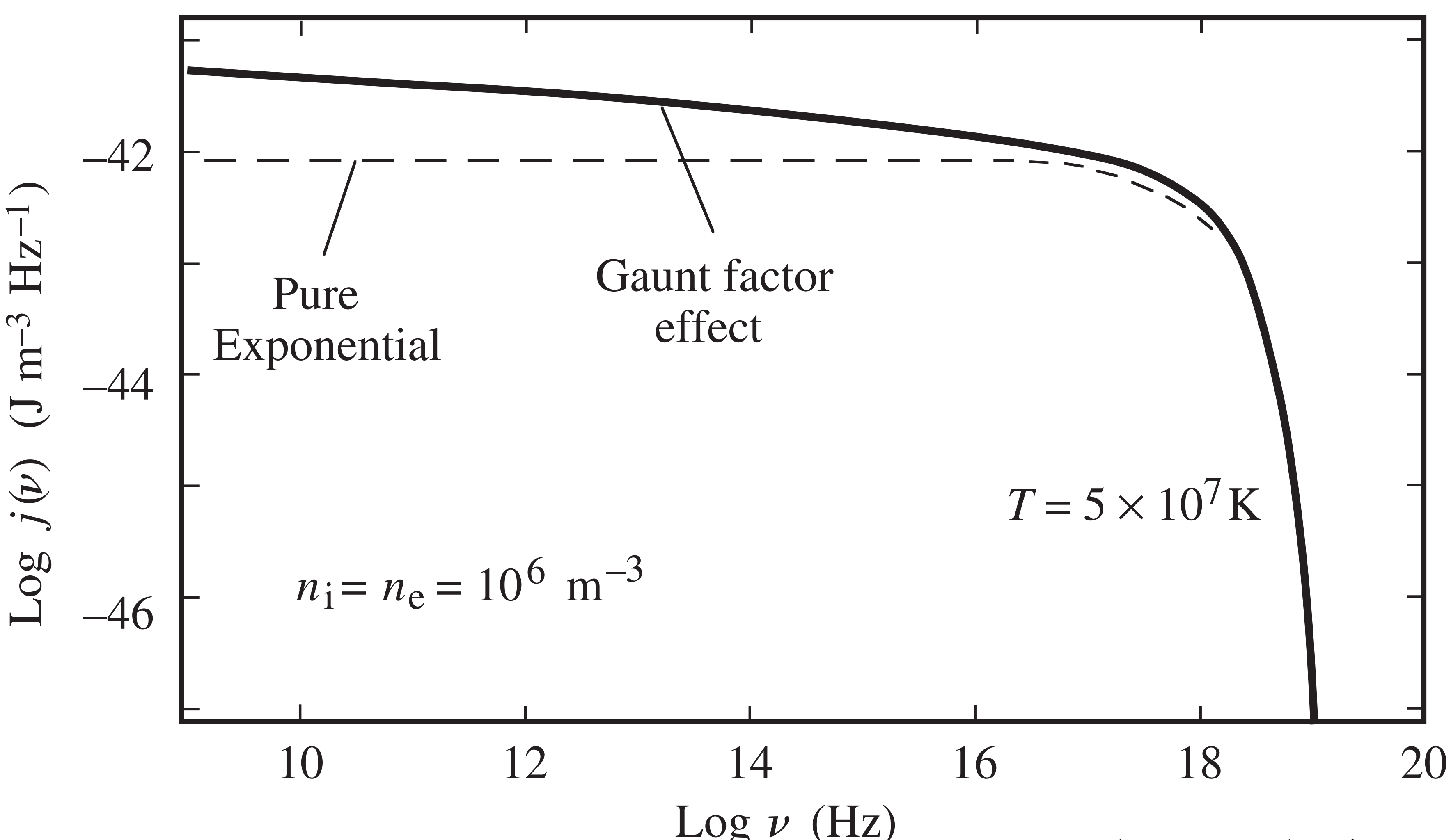
  - where $\overline{g}_{\rm ff}$ is the velocity-averaged free-free Gaunt factor.
  - Summing over all ion species gives the emissivity:

$$j_\nu^{\rm ff} = 6.8\times10^{-38}\sum_i \overline{g}_{\rm ff}Z^2 n_i n_e T^{-1/2}e^{-h\nu/kT}\quad ({\rm erg\ cm^{-3}\ s^{-1}\ Hz^{-1}})$$

- Note that main frequency dependence is $j_\nu^{\mathrm{ff}} \propto e^{-h\nu/kT}$, which shows a "flat spectrum" with a cut off at $\nu \sim hT/h$. The spectrum can be used to determine temperature of hot plasma.

For a hydrogen plasma ($Z = 1$) with $T > 3 \times 10^5$ K at low frequencies $(h\nu \ll kT)$ Gaunt factor is given by

$$\overline{g_{\mathrm{ff}}} = \frac{\sqrt{3}}{\pi} \ln\left(\frac{2.25kT}{h\nu}\right)$$



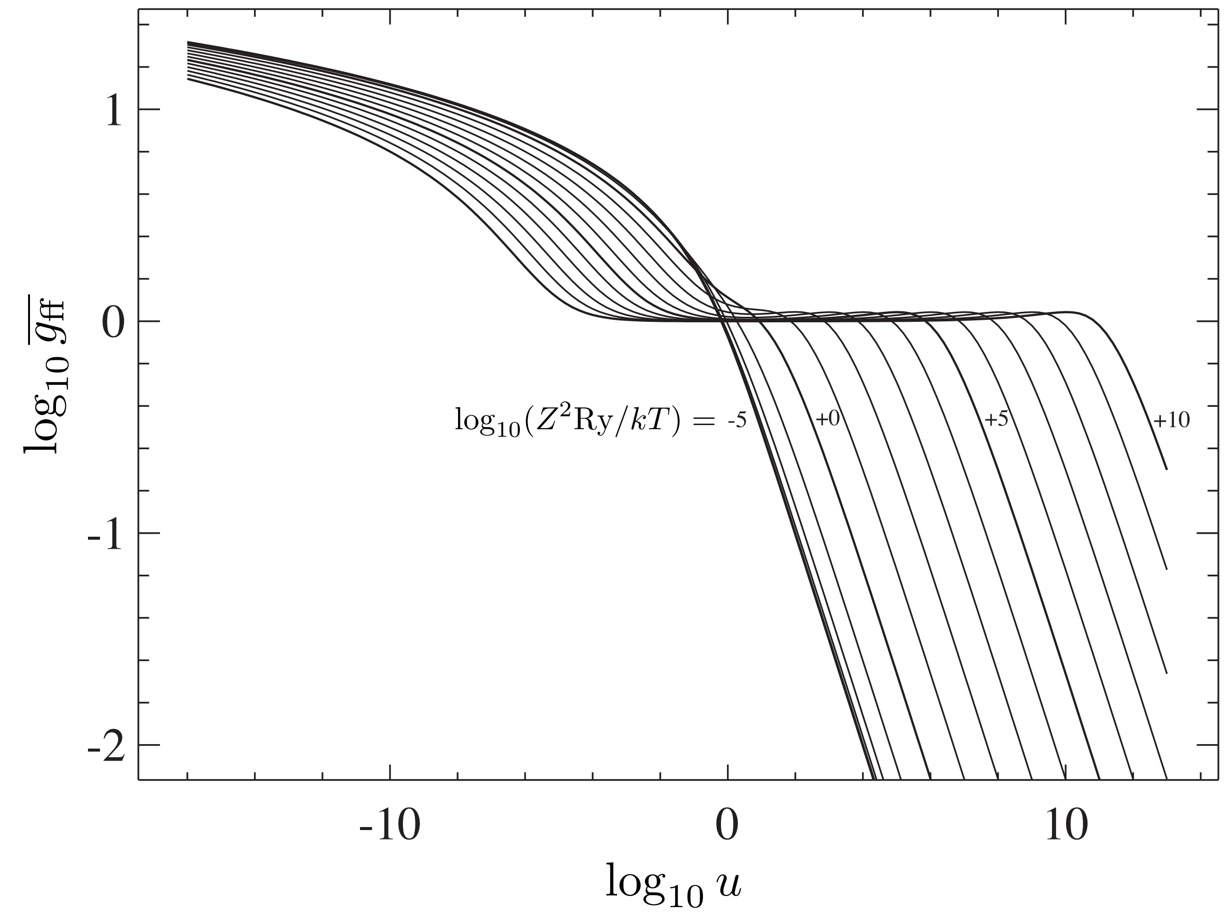
[Bradt, Astrophysics Processes]

- Gaunt Factor

- Note that the values of Gaunt factor for $u = h\nu/kT \gg 1$ are not important, since the spectrum cuts off for these values.

$$\overline{g_{\rm ff}} \sim \begin{cases} 1 & \text{for } u \sim 1 \\ 1-5 & \text{for } 10^{-4} < u < 1 \end{cases}$$

van Hoof et al. (2014, MNRAS, 444, 420)



$$u = h\nu/kT = 4.8\times10^{11}\nu/T$$

$$\gamma^2 = Z^2{\rm Ry}/kT = 1.58\times10^5 Z^2/T$$

[Thermal Bremsstrahlung (free-free) Absorption]

- Absorption of radiation by free electrons moving in the field of ions:
  - For LTE, Kirchhoff's law says:

$$j_\nu^{\rm ff} = \alpha_\nu^{\rm ff} B_\nu(T) \quad \text{where} \quad B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

  - We can derive

$$\alpha_\nu^{\rm ff} = \frac{4e^6}{3m_e hc}\left(\frac{2\pi}{3km_e}\right)^{1/2} n_i n_e Z^2 T^{-1/2}\nu^{-3}\left(1-e^{-h\nu/kT}\right)\overline{g_{\rm ff}}$$
$$= 3.7\times10^8\ n_i n_e Z^2 T^{-1/2}\nu^{-3}\left(1-e^{-h\nu/kT}\right)\overline{g_{\rm ff}} \quad ({\rm cm}^{-1})$$

  - For $h\nu \gg kT$, $\alpha_\nu^{\rm ff} = 3.7\times10^8 n_i n_e Z^2 T^{-1/2}\nu^{-3}\overline{g_{\rm ff}} \quad ({\rm cm}^{-1}) \longrightarrow \tau_\nu \propto \alpha_\nu^{\rm ff} \propto \nu^{-3}$

  - For $h\nu \ll kT$, $\alpha_\nu^{\rm ff} = \frac{4e^6}{3m_e kc}\left(\frac{2\pi}{3km_e}\right)^{1/2} n_i n_e Z^2 T^{-3/2}\nu^{-2}\overline{g_{\rm ff}} \longrightarrow \tau_\nu \propto \alpha_\nu^{\rm ff} \propto \nu^{-2}$

$$= 0.018 n_i n_e Z^2 T^{-3/2}\nu^{-2}\overline{g_{\rm ff}}$$

  - **Bremsstrahlung self-absorption**: The medium becomes always optically thick at sufficiently small frequency. Therefore, the free-free emission is absorbed inside plasma at small frequencies.

Overall Spectral Shape

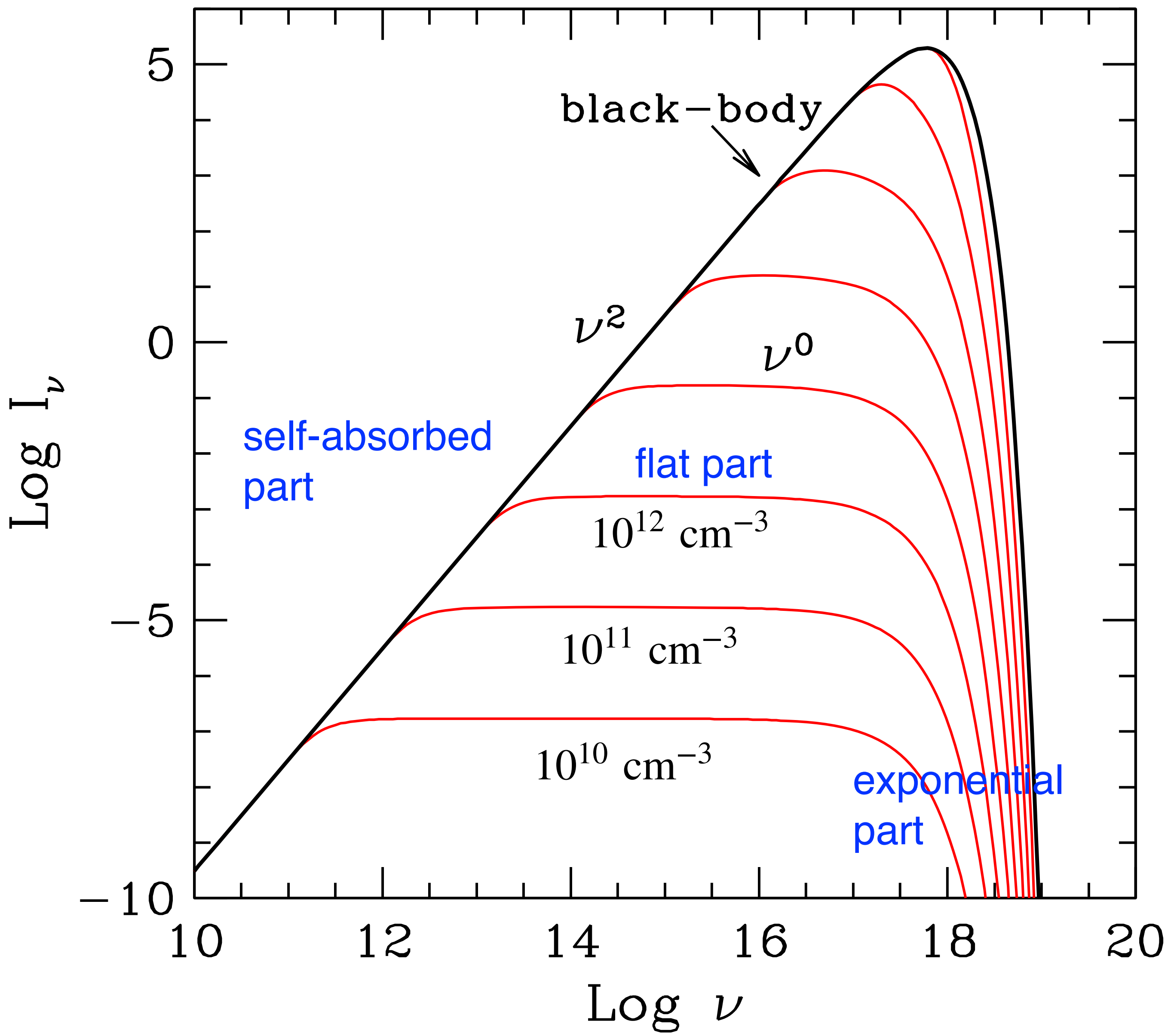
- Spectral shape
  - At low frequencies (optically thick emission),

$$I_\nu = S_\nu = B_\nu \propto \nu^2$$

  - At high frequencies (optically thin emission),

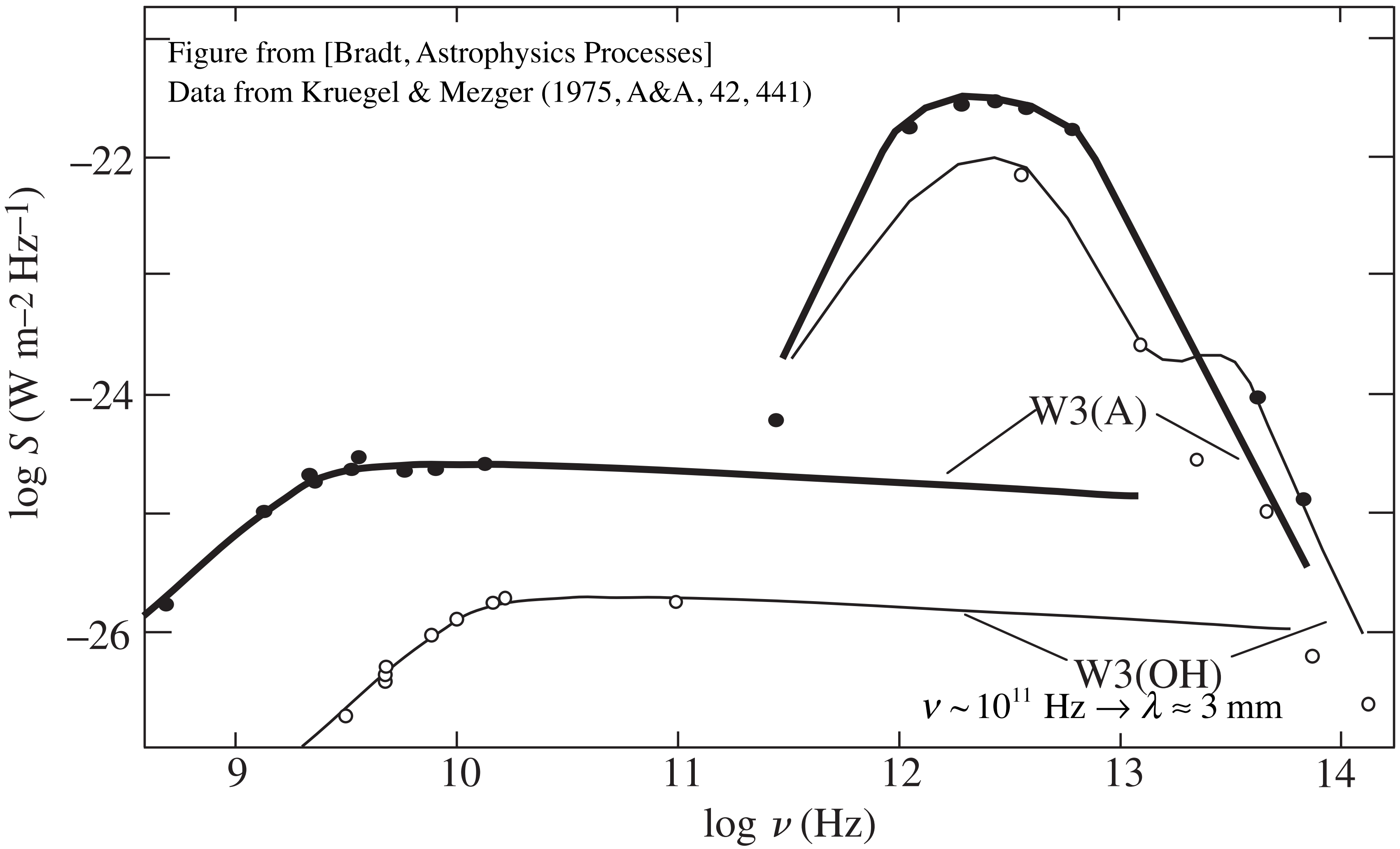
$$I_\nu = \int j_\nu ds \propto e^{-h\nu/kT}$$
$$= \text{constant if } h\nu \ll kT$$

- This spectrum shows the bremsstrahlung intensity from a source of radius $R = 10^{15}$ cm and temperature $T = 10^7$ K.
  - The Gaunt factor is set to unity for simplicity.
  - The density $n_e = n_p$ varies from 10^{10} cm^{-3} to 10^{18} cm^{-3} increasing by a factor 10 for each curve.
  - As the density increases, the optical depth also increases and the spectrum approaches the blackbody one.



Astronomical Examples - H II regions

- The radio spectra of H II regions clearly show the flat spectrum of an optically thin thermal source. The bright stars in the H II regions emit copiously in the UV and thus ionize the hydrogen gas.
- Continuum spectra of two H II regions, W3(A) and W3(OH):
  - Note a flat thermal bremsstrahlung (radio), a low-frequency cutoff (radio, self absorption), and a large peak at high frequency (infrared, $10^{12} - 10^{13}$ Hz, 300-30 μm) due to heated, but still "cold" dust grains in the nebula.



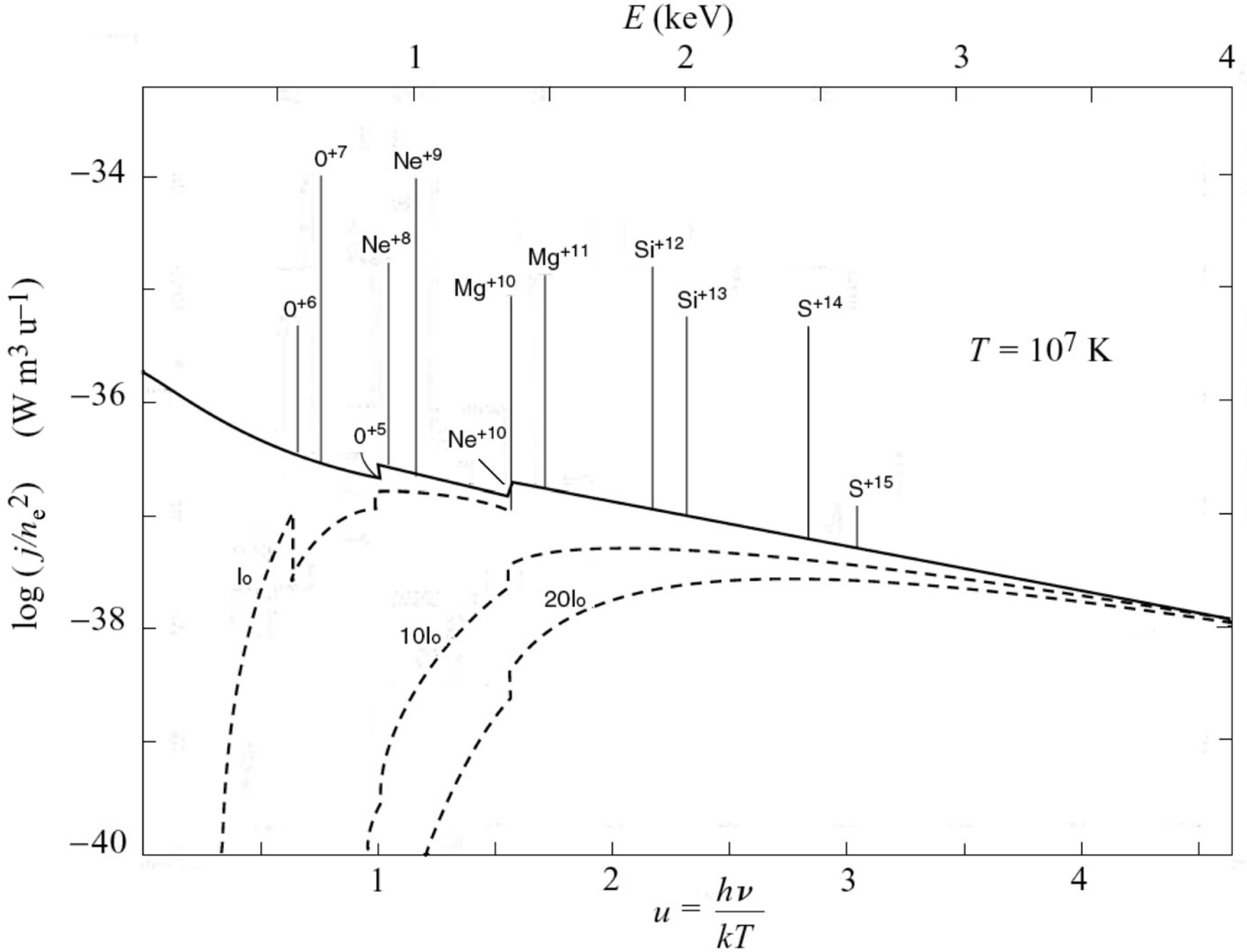
The term H II is pronounced "H two" by astronomers. "H" indicates hydrogen, and "II" is the Roman numeral for 2.

Astronomers use "I" for neutral atoms, "II" for singly-ionized, "III" for doubly-ionized, etc.

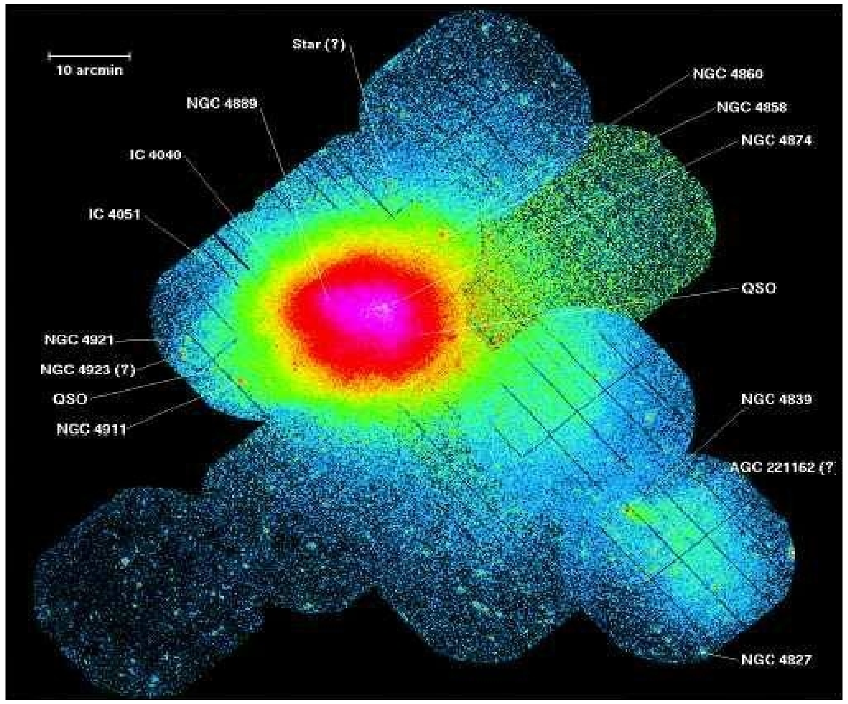
An H II region in the Large Magellanic Cloud (observed with MUSE, VLT)

Astronomical Examples - X-ray emission from galaxy clusters

- Theoretical spectrum for a plasma of temperature 10^7 K that takes into account quantum effects.
  - Comparison with real spectra from clusters of galaxies allows one to deduce the actual amounts of different elements and ionized species in the plasma as well as its temperature.
  - It is only in the present millennium that X-ray spectra taken from satellites (e.g., Chandra and the XMM Newton satellite) have had sufficient resolution to distinguish these narrow lines. The dashed lines show the effect of X-ray absorption by interstellar gas [Bradt, Astrophysics Processes].

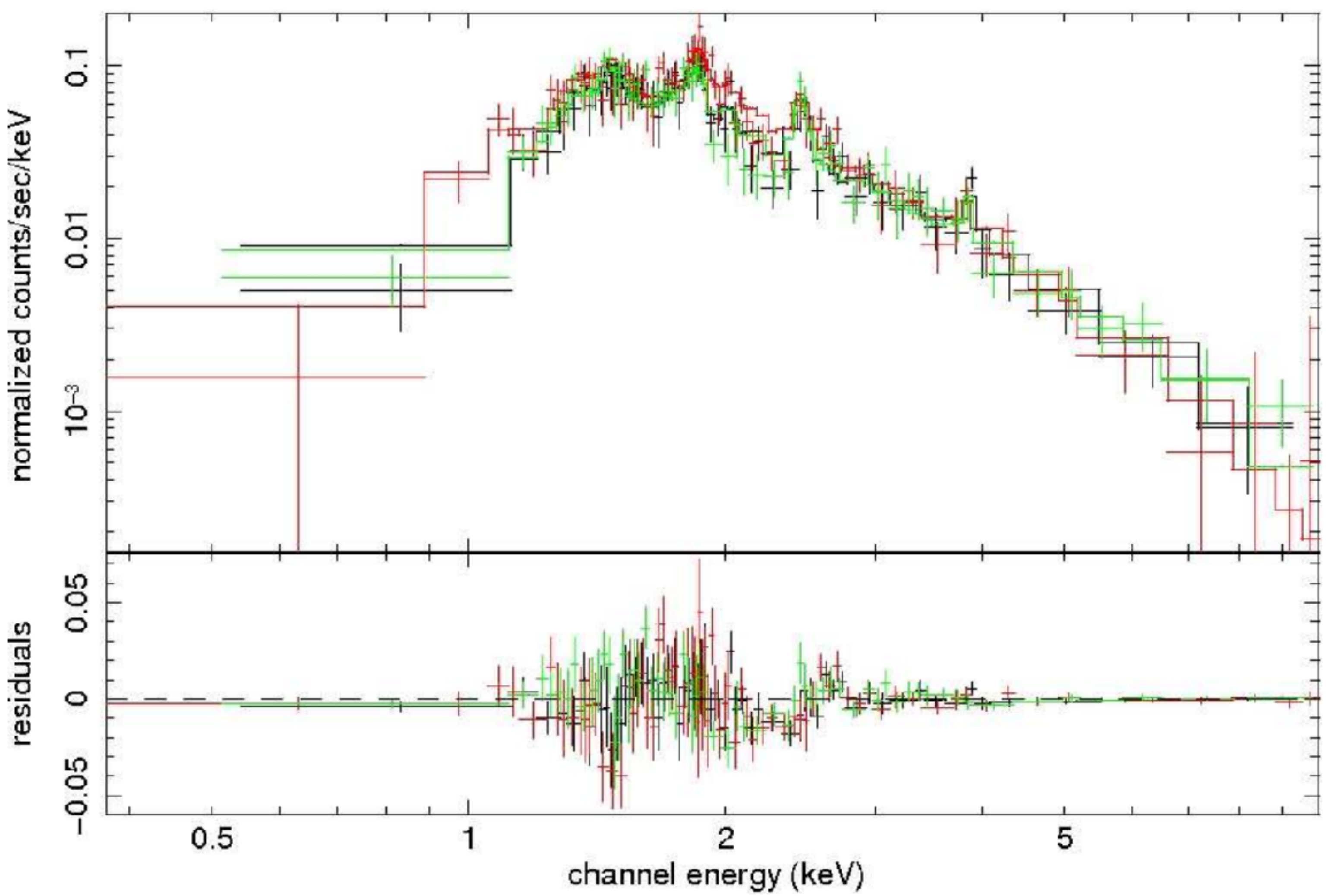


Coma cluster (z = 0.0232), size ~ 1 Mpc



Astronomical Examples - Supernova Remnants

- SNR G346.6-0.2
  - X-ray spectra of the SNR from three of the four telescopes on-board Suzaku (represented by green, red and black).
  - The underlying continuum is thermal bremsstrahlung, while the spectral features are due to elements such as Mg, S, Si, Ca and Fe.
  - The roll over in the spectrum at low and high energies is due to a fall in the detector response, which is forward-modeled together with the spectrum.



Sezer et al. (2011, MNRAS, 415, 301)

Synchrotron Radiation

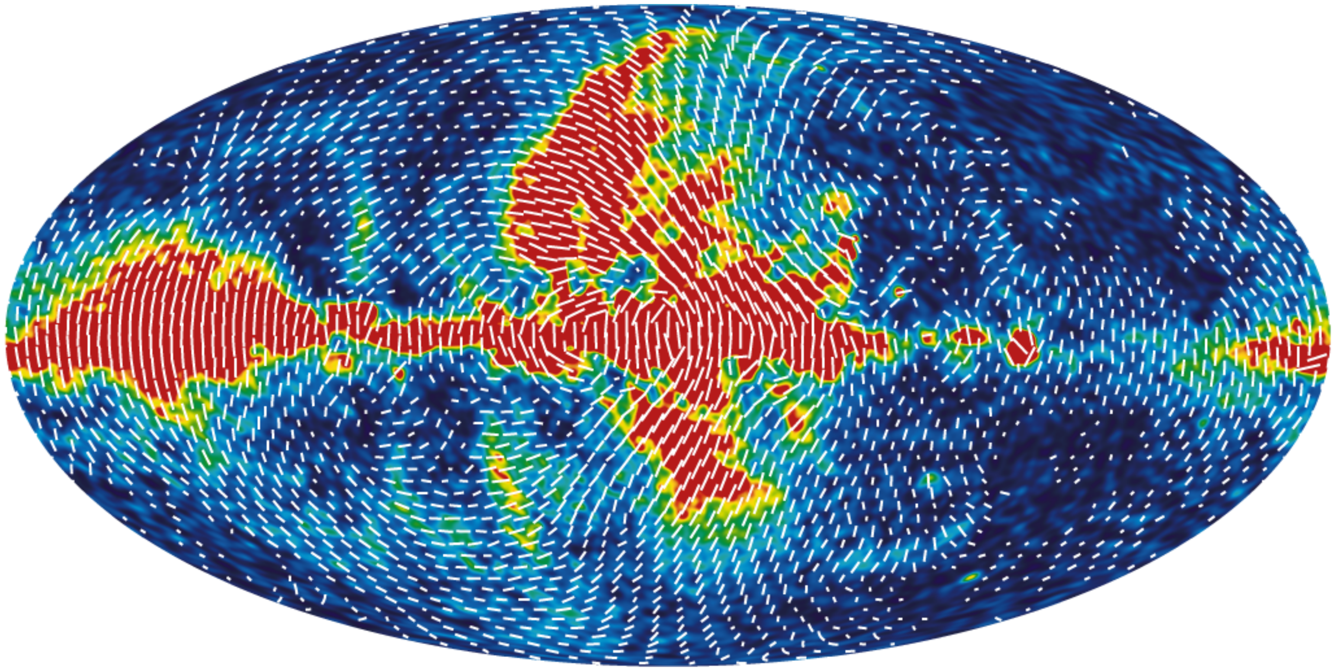
Synchrotron Radiation

- Particles accelerated by a magnetic field will also radiate. Acceleration by a magnetic field produces ***magnetobremsstrahlung***, the German word for “magnetic braking radiation.”

- **Cyclotron radiation**: For non-relativistic velocities, the radiation is called cyclotron radiation. The frequency of emission is simply the frequency of gyration in the magnetic field.
- **Synchrotron radiation**: For extreme relativistic particles, the frequency spectrum is much more complex and can extend to many times the gyration frequency. This radiation is known as synchrotron radiation.

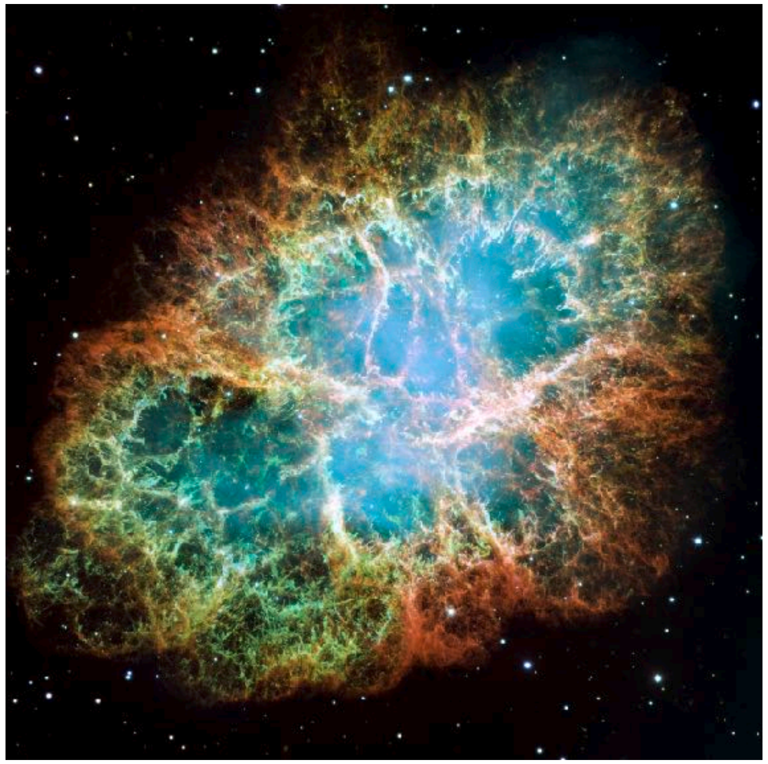
- Synchrotron radiation is ubiquitous in astronomy.
  - It accounts for most of the radio emission from active galactic nuclei (AGN), which are thought to be powered by supermassive black holes in galaxies and quasars.
  - It dominates the radio continuum emission from star-forming galaxies.

WMAP 23GHz emission and polarization map

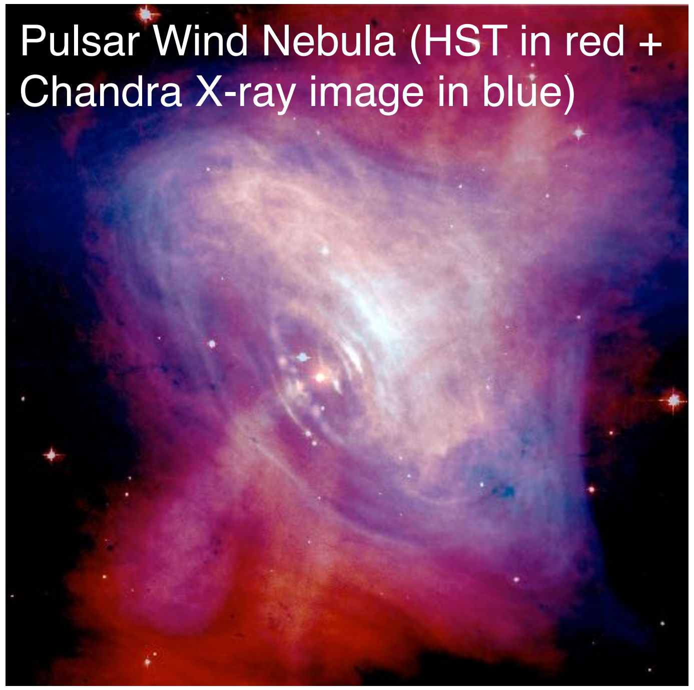


Puzzling radiation from the Crab nebula

- A large part of the supernova remnant called the Crab nebula appears as an bluish haze on the sky, and the origin of this light was, for a long time, a big mystery.
  - The bluish luminosity in the optical band could be considered to indicate the presence of a hot, optically thin thermal plasma with a temperature of ~ 50,000 K.
  - Such a hot plasma would radiate strong optical emission lines from excited atoms in the plasma. However, no strong spectral lines are observed. The spectrum is a smooth continuum.
  - The puzzle of the source of radiation from the Crab nebula was solved dramatically when a Russian scientist (I. S. Shklovsk) postulated in 1953 that the bluish radiation might be the same as that previously discovered in man-made electron accelerators.
  - He therefore suggested that optical astronomers look to see if the radiation from the Crab is polarized. They did (in 1954) and found it to be so.
  - Nevertheless, this explanation was quiet a surprise; it required that the radiating electrons have energies in excess of 10^{11} eV. This raised the question of how the electron attained such high energies. Later (1969), it was found that the energy source is a spinning neutron star, the Crab pulsar.



Crab Nebula (HST mosaic image)
The rapidly spinning (30 times a second) neutron star embedded in the center of the nebula is powering the nebula's interior bluish glow. The blue light comes from electrons whirling at nearly the speed of light around magnetic field lines from the neutron star.



Pulsar Wind Nebula (HST in red + Chandra X-ray image in blue)

Larmor's formula

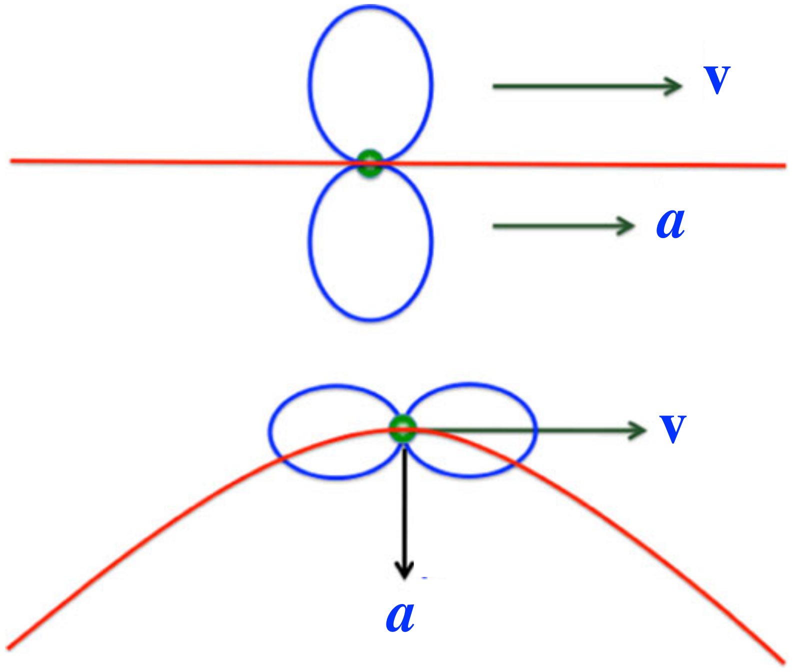
- Relativistic version of the Larmor's formula
  - Recall that

$$P' = \frac{2q^2}{3c^3} \left| \boldsymbol{a}' \right|^2$$

in an instantaneous rest frame of the particle.

  - The power emitted by an accelerated electron is obtained to be

$$\boxed{P = \frac{2q^2}{3c^3} \gamma^4 \left(a_\perp^2 + \gamma^2 a_\parallel^2 \right)}$$



$a_\perp$: acceleration perpendicular to the electron velocity $\mathbf{v}$

$a_\parallel$: acceleration parallel to the electron velocity $\mathbf{v}$

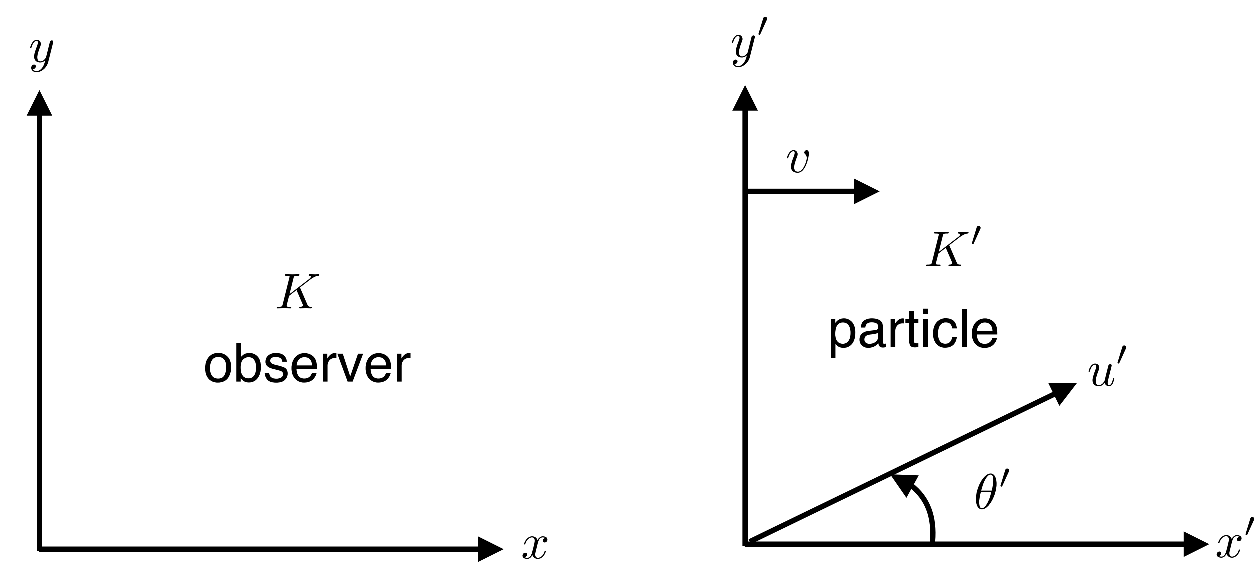
$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad : \quad \text{Lorentz factor}$$

Beaming Effect

- Imaging two frames *K* and *K'*. *K'* moves with a velocity v relative to *K*.
- If a point has a velocity $\mathbf{u}'$ in frame *K'*, what is its velocity $\mathbf{u}$ in frame *K*?

$$\boxed{u_\parallel = \frac{u'_\parallel + v}{1 + v u'_\parallel / c^2} \qquad u_\perp = \frac{u'_\perp}{\gamma\left(1 + v u'_\parallel / c^2\right)}}$$



- Aberration formula: the directions of the velocities in the two frames are related by

$$\tan\theta = \frac{u_\perp}{u_\parallel} = \frac{u'_\perp}{\gamma\left(u'_\parallel + v\right)} = \frac{u' \sin\theta'}{\gamma\left(u' \cos\theta' + v\right)}$$

θ = angle of the velocity measured in K
θ' = angle measured in *K'*

- **Aberration of light** ($u' = c$, $\beta = v/c$)

$$\tan\theta = \frac{\sin\theta'}{\gamma\left(\cos\theta' + \beta\right)}$$
$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$
$$\sin\theta = \frac{\sin\theta'}{\gamma\left(1 + \beta\cos\theta'\right)}$$

$$\longrightarrow \quad \boxed{\tan\left(\frac{\theta}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^2 \tan\left(\frac{\theta'}{2}\right) \quad \rightarrow \quad \theta < \theta'}$$

- **Beaming ("headlight") effect**:
  - Beam half-angle = the half-angle of the cone that includes one-half of the rays.
  - If photons are emitted isotropically in K', then half will have $\theta' < \pi/2$ and half $\theta' > \pi/2$.
  - Consider a photon emitted at right angles ($\theta' = \pi/2$) to v in K'. Then the half-angle corresponds to the following angle in K.

beam half-angle: $\tan\left(\frac{\theta_b}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^2$ $\qquad \sin\theta_b = \frac{1}{\gamma}$ or $\cos\theta_b = \beta$

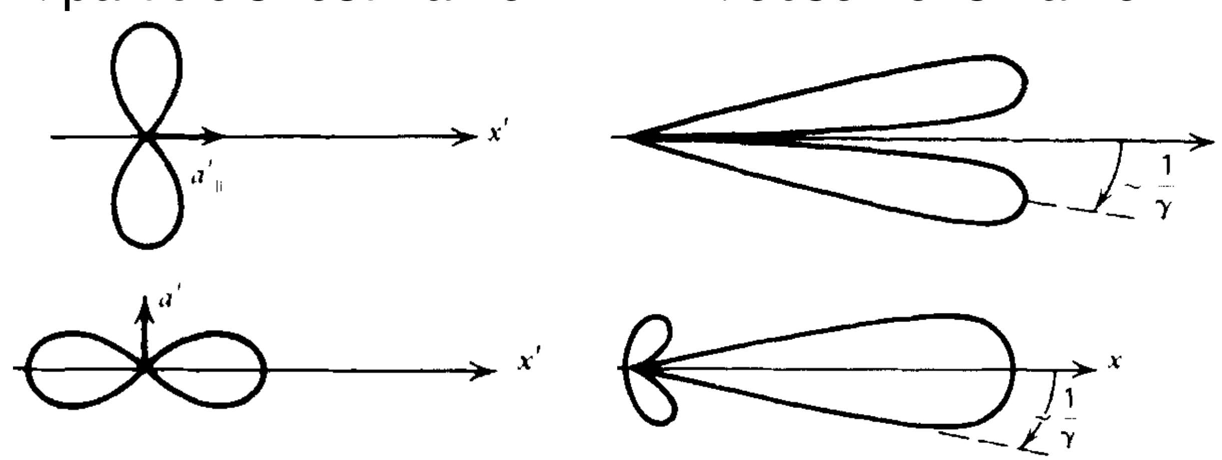
  - For highly relativistic speeds, $\gamma \gg 1$, θ_b becomes small:

$$\boxed{\theta_b \approx \frac{1}{\gamma}}$$

K' : particle's rest frame: $\qquad$ K : observer's frame:

parallel acceleration:

perpendicular acceleration:



  - Therefore, in frame K, ***photons are concentrated in the forward direction, with half of them lying within a cone of half-angle*** $1/\gamma$.

Doppler Effect

- In the rest frame of the observer K, imagine that the moving source emits one period of radiation as it moves from point 1 to point 2 at velocity v.
  - Let frequency of the radiation in the rest frame K' of the source to be ω'. Then the time taken to move from point 1 to point 2 in the observer's frame is given by the time-dilation effect:

$$\Delta t = \Delta t' \gamma = \frac{2\pi}{\omega'}\gamma$$

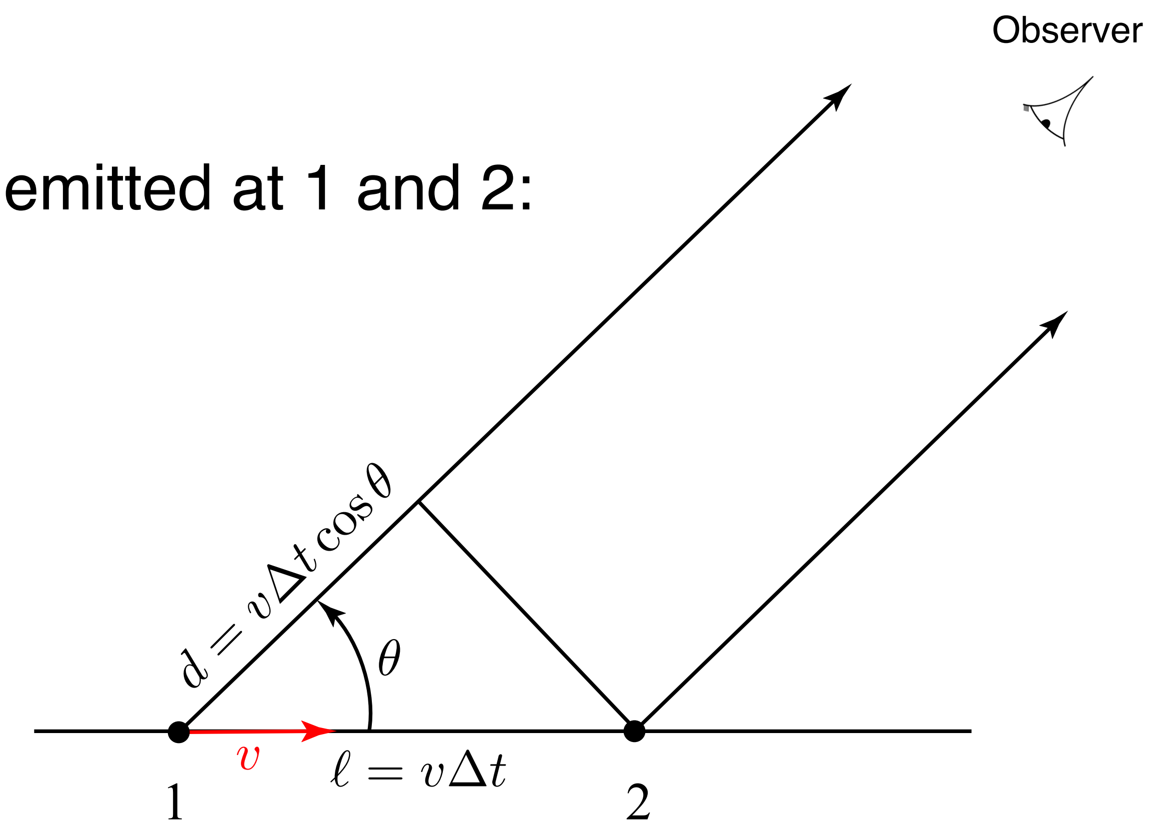
  - Difference in arrival times Δt_A of the radiation emitted at 1 and 2:

$$\Delta t_A = t_A(2) - t_A(1) = \Delta t - \frac{d}{c} = \Delta t \left(1 - \frac{v}{c}\cos\theta\right)$$

  - Therefore, the observed frequency ω will be

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma(1-\beta\cos\theta)} \quad \boxed{\frac{\omega}{\omega'} = \frac{1}{\gamma(1-\beta\cos\theta)}}$$

  - Note that $1 - \beta\cos\theta$ appears even classically.
  - The factor γ^{-1} is purely a relativistic effect.



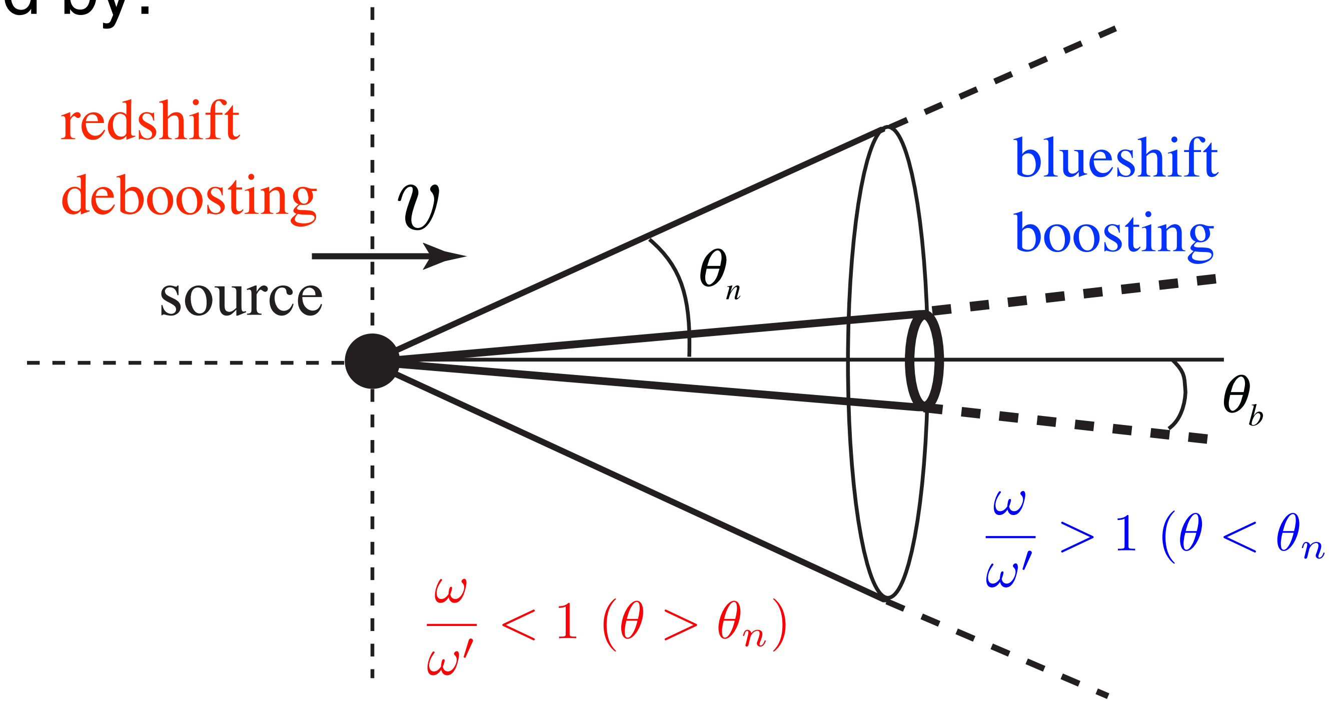
Photons emitted from point 2 will travel a smaller distance than photons from point 1 by the distance d.

Angle for null Doppler shift

- ***Angle for null Doppler shift*** is defined by:

$$\frac{\omega}{\omega'} = \frac{1}{\gamma\,(1-\beta\cos\theta_n)} = 1$$

$$\rightarrow \quad \cos\theta_n = \frac{1-\gamma^{-1}}{\beta} = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{-1/2}$$

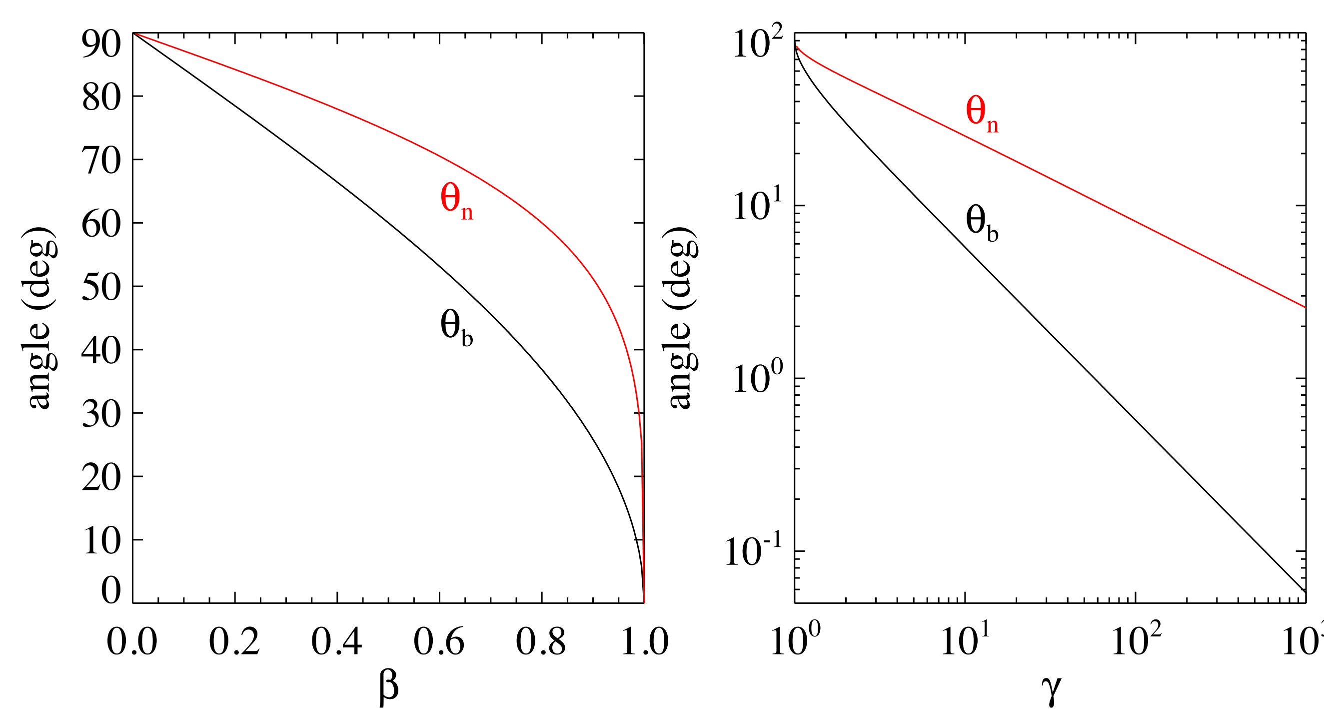


- Relativistic Doppler effect can yield redshift even as a source approaches.

$$\cos\theta_n = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{-1/2} \approx 1 - \frac{1}{\gamma} \quad \text{for} \quad \gamma \gg 1$$

$$1 - \frac{\theta_n^2}{2} \approx 1 - \frac{1}{\gamma}$$

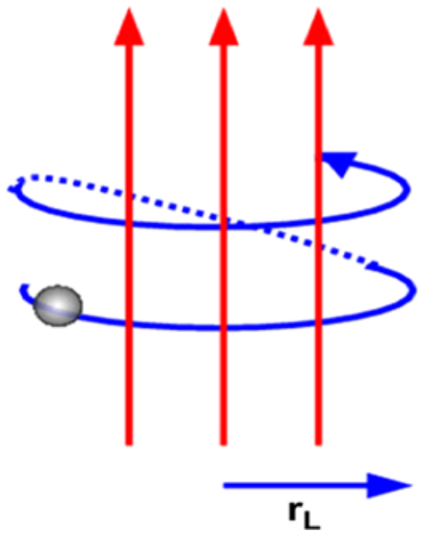
$$\boxed{\theta_n \approx \sqrt{\frac{2}{\gamma}} \approx \sqrt{2\theta_b}}$$



- Note $\theta_b < \theta_n$

Equation of Motion in a uniform magnetic field

- Consider a particle of mass *m* and charge *q* moving in **a uniform magnetic field, with no electric field.**
- Equations of motion:



$$\frac{dE}{dt} = \frac{d(\gamma m c^2)}{dt} = q\mathbf{v}\cdot\mathbf{E} = 0$$
$$\frac{d\mathbf{p}}{dt} = \frac{d(\gamma m \mathbf{v})}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}$$

- The first equation implies that $\underline{\gamma = \text{constant (or equivalently } |\mathbf{v}| = \text{constant)}}$. Therefore, the second equation becomes

$$\gamma m \frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}$$

- Decompose the velocity into $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel$, and take dot product with **B**.

$$\mathbf{B}\cdot\left(\gamma m \frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}\right) \quad \rightarrow \quad \begin{cases} \dfrac{d\mathbf{v}_\parallel}{dt} = 0 \\[2ex] \dfrac{d\mathbf{v}_\perp}{dt} = \dfrac{q}{\gamma m c}\mathbf{v}_\perp \times \mathbf{B} \end{cases}$$

-

- Rearrange the above equations:

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

$$\boxed{\frac{d\mathbf{v}_{\parallel}}{dt} = 0} \longrightarrow \begin{array}{l} a_{\parallel} = 0 \\ a_{\perp} = \omega_{\mathrm{B}} v \end{array}$$

$$\frac{d\mathbf{v}_{\perp}}{dt} = \frac{-e}{\gamma m_e c}\mathbf{v}_{\perp} \times \mathbf{B} = -\frac{\omega_{\mathrm{B}}}{B}\mathbf{v}_{\perp} \times \mathbf{B} \quad \left(\text{Here, } \omega_{\mathrm{B}} \equiv \frac{eB}{\gamma m_e c}\right)$$

$$\boxed{\frac{d^2\mathbf{v}_{\perp}}{dt^2} = -\,\omega_{\mathrm{B}}^2 \mathbf{v}_{\perp}}$$: harmonic oscillator

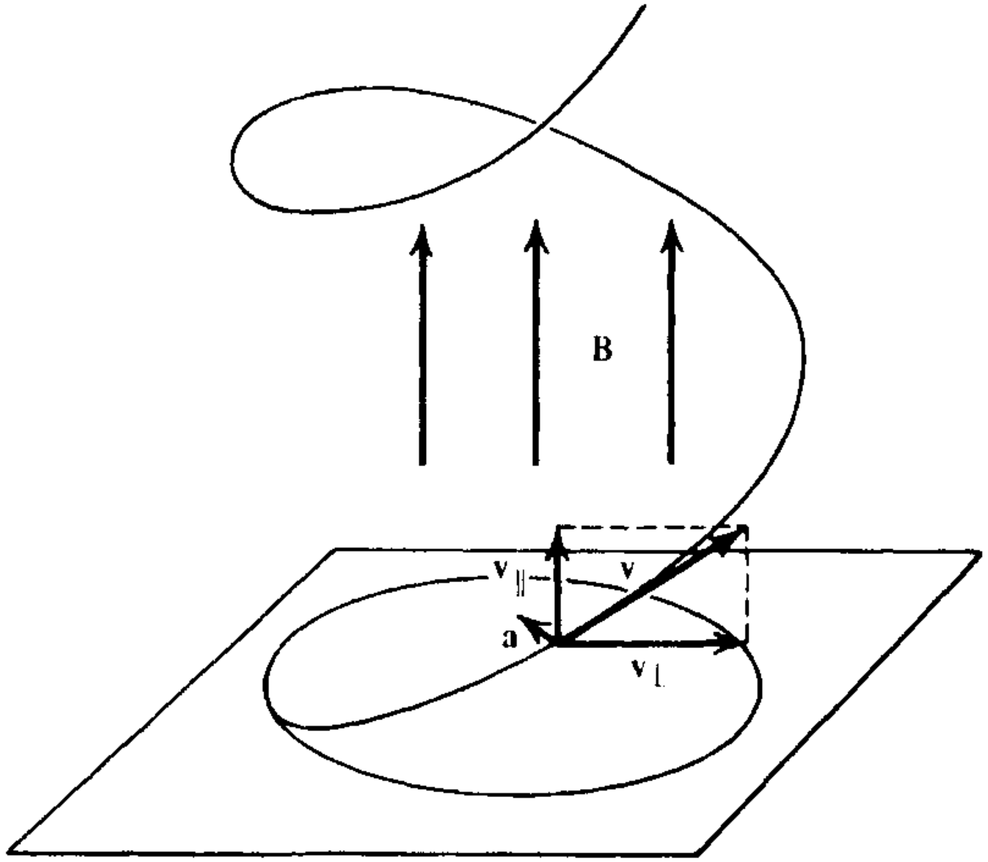
- Solution: $\mathbf{v}(t) = v_{\perp}\left(-\hat{\mathbf{x}}\sin\omega_{\mathrm{B}}t + \hat{\mathbf{y}}\cos\omega_{\mathrm{B}}t\right) + \hat{\mathbf{z}}v_{\parallel}$

$$\mathbf{r}(t) = \frac{v_{\perp}}{\omega_{\mathrm{B}}}\left(\hat{\mathbf{x}}\cos\omega_{\mathrm{B}}t + \hat{\mathbf{y}}\sin\omega_{\mathrm{B}}t\right) + \hat{\mathbf{z}}v_{\parallel}t$$

(initial conditions : $\mathbf{v}_{\perp} \parallel \mathbf{y}$ and $\mathbf{x}_{\perp} \parallel \mathbf{x}$ at $t = 0$)

$$\mathbf{v}_{\parallel} = \text{constant}$$

$$|\mathbf{v}_{\perp}| = \text{constant (since } |\mathbf{v}| = \text{constant)}$$



- **Helical motion**: The perpendicular velocity component processes around **B**. Therefore, the motion is ***a combination of the uniform circular motion perpendicular to the magnetic field and the uniform motion along the field***.

- The particle gyrates along the magnetic field lines with the angular velocity ω_B.
- Its trajectory has a helicoidal shape, with gyroradius radius r_B and pitch angle α.

gyrofrequency: $\omega_B = \dfrac{eB}{\gamma m_e c}$

gyroradius: $r_B = \dfrac{v_\perp}{\omega_B}$

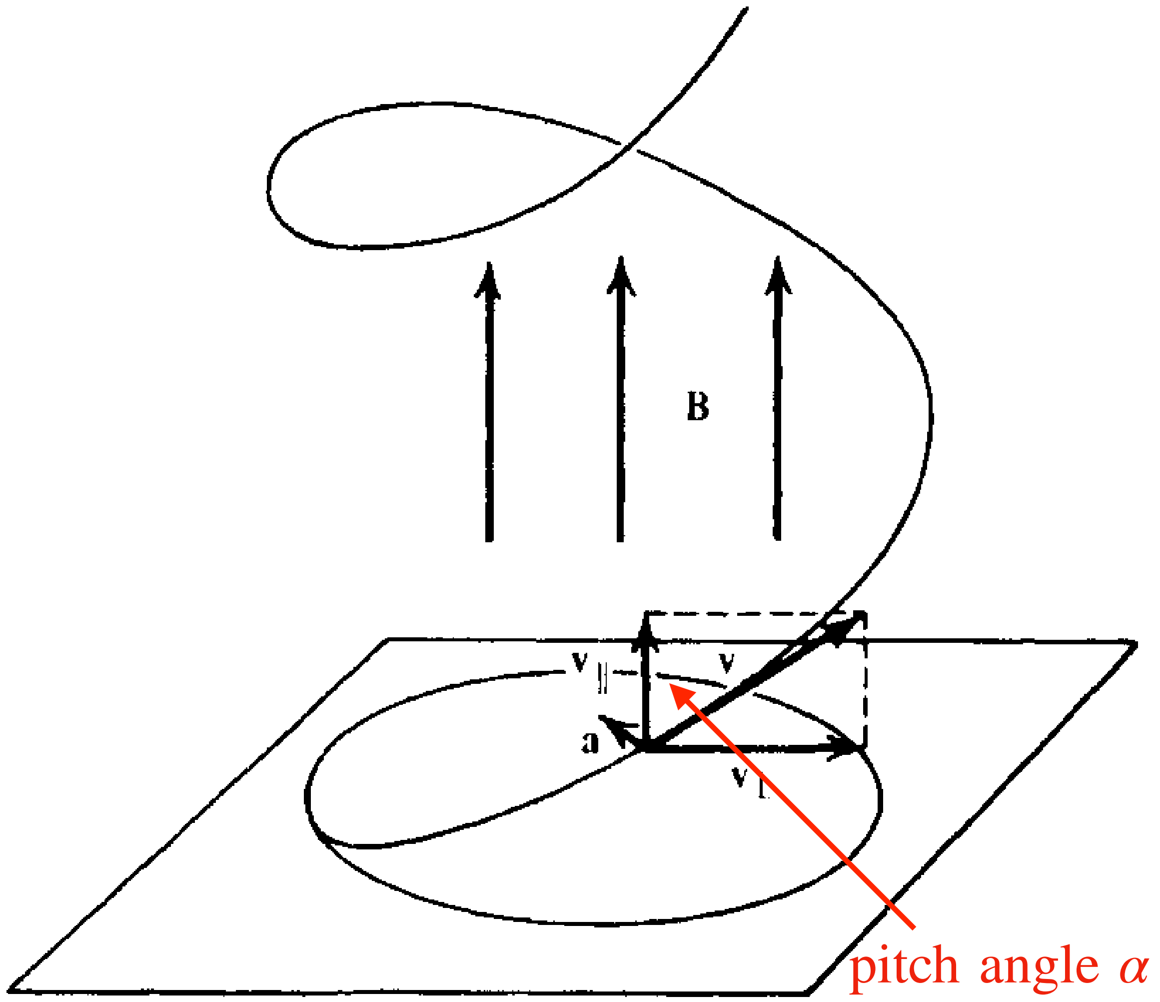
pitch angle α **:** $v_{\|} = v\cos\alpha, v_\perp = v\sin\alpha$

angle between the magnetic field and velocity

angle between $\mathbf{v}_{\|}$ and $\mathbf{v}$

$$\omega_B = \frac{17.6}{\gamma}\frac{B}{\mu\mathrm{G}}\ (\mathrm{Hz})$$

$$r_B = 1.7\times 10^9\ \gamma\beta_\perp\left(\frac{B}{\mu\mathrm{G}}\right)\ (\mathrm{cm})$$



Larmor frequency: $\omega_L = \dfrac{eB}{m_e c}$ **Larmor radius:** $r_L = \dfrac{v_\perp}{\omega_L}$

(Cychrotron frequency,
non-relativistic gyrofrequency)

Synchrotron Power Radiated by a Single Electron

- Relativistic version of the Larmor's formula gives the radiated power:

$$P = \frac{2q^2}{3c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right)$$

- Total emitted power:
  - Since $a_\perp = \omega_B v_\perp$ and $a_\parallel = 0$, we obtain $P = \dfrac{2}{3}\gamma^2\dfrac{q^4B^2}{m^2c^5}v_\perp^2$.
  - Using the definitions:

$$\text{pitch angle}: \ \cos\alpha = \frac{\mathbf{v}\cdot\mathbf{B}}{vB}, \ \text{classical electron radius}: r_e = \frac{e^2}{m_e c}$$

$$\text{Thomson scattering cross}-\text{section}: \sigma_T = \frac{8\pi}{3}r_e^2, \ \text{magnetic field energy}: U_\mathrm{B} = \frac{B^2}{8\pi}$$

$$\boxed{P = 2\sigma_T c(\gamma\beta)^2 U_\mathrm{B}\sin^2\alpha}$$

  - For an isotropic distribution of velocities, it is necessary to average the formula over all pitch angles.

$$\boxed{P = \frac{4}{3}\sigma_T c(\gamma^2 - 1)U_\mathrm{B}} \quad \leftarrow \quad \langle\sin^2\alpha\rangle = \frac{1}{4\pi}\int \sin^2\alpha \, d\Omega = \frac{2}{3}$$

$$\beta^2 = 1 - \frac{1}{\gamma^2}$$

Cooling Time

- The energy balance equation becomes:

$$mc^2\frac{d\gamma}{dt} = -P \quad \leftarrow E = \gamma mc^2$$

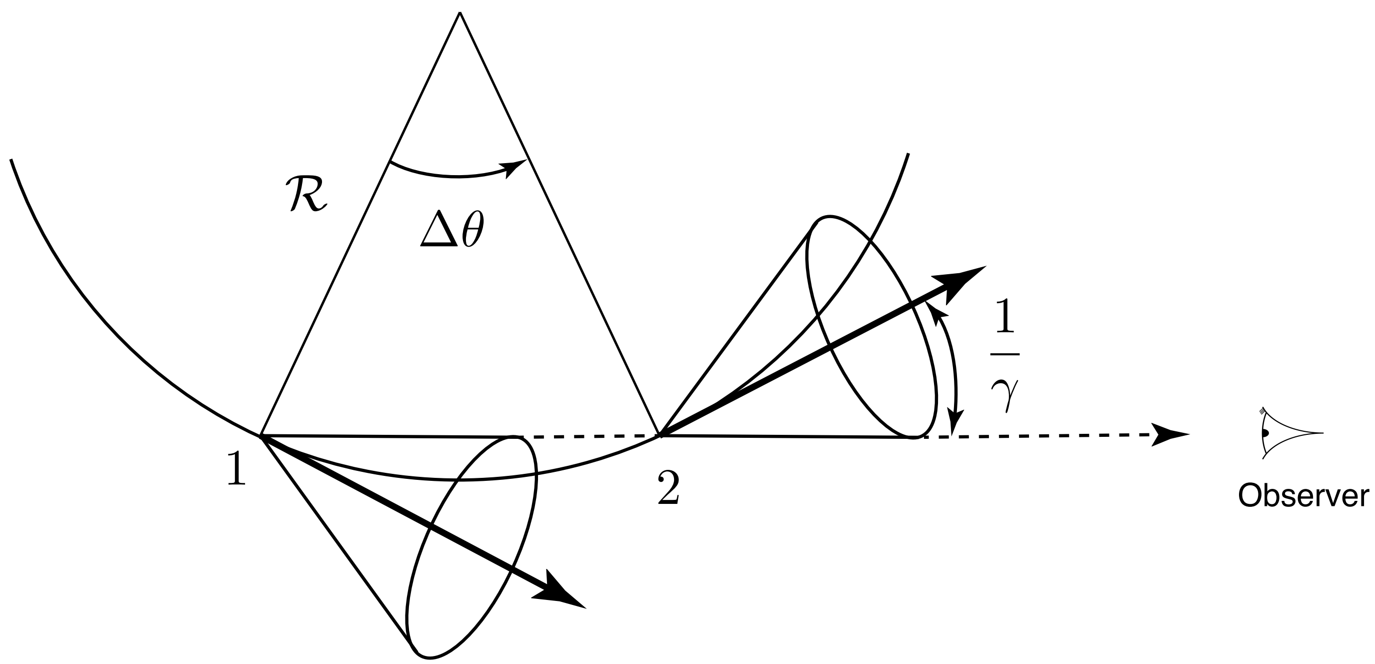
  - **cooling time**: the typical timescale for the electron to loose about of its energy is approximately

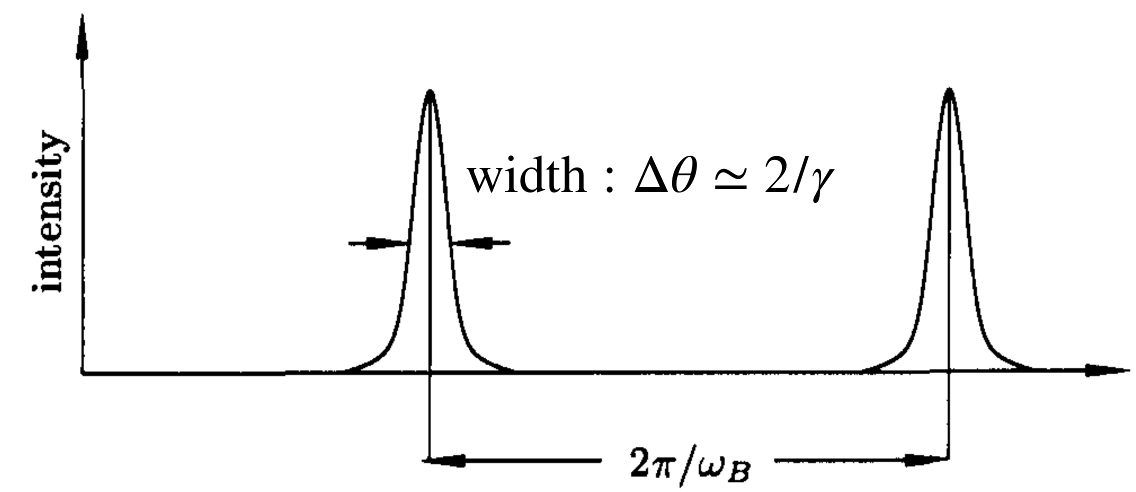
$$t_{\rm cool} = \frac{\text{energy}}{\text{cooling rate}} = \frac{\gamma mc^2}{P} \approx \frac{23 \text{ years}}{\gamma (B/\text{Gauss})^2} \quad (\text{if } \gamma \gg 1)$$

  - for $\gamma = 10^3$

Location	Typical B	$t_{\rm cool}$	cooling length $\approx ct_{\rm cool}$	size of object
Interstellar medium	10^{-6} G	10^{10} years	10^{28} cm	10^{22} cm
Stellar atmosphere	1 G	5 days	10^{15} cm	10^{11} cm
Supermassive black hole	10^4 G	10^{-3} sec	$3\ 10^7$ cm	10^{14} cm
White dwarf	10^8 G	10^{-11} sec	3 mm	1000 km
Neutron star	10^{12} G	10^{-19} sec	$3\ 10^{-9}$ cm	10 km

Spectrum of Synchrotron Radiation: A Qualitative Discussion

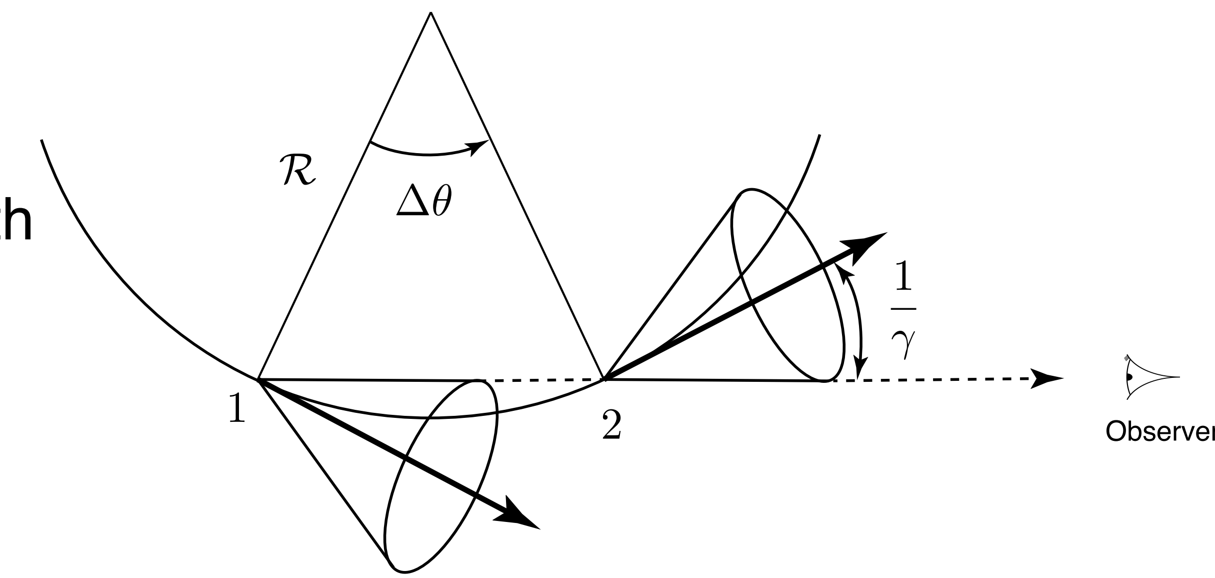




- Because of beaming effects the emitted radiation fields appear to be concentrated in a narrow set of directions about the particle's velocity.
  - The observer will see a pulses of radiation confined to a time interval much smaller than the gyration period.
  - The cone of emission has an angular width $\sim 1/\gamma$. Therefore, the observer will see emission over the angular range of $\Delta\theta \simeq 2/\gamma$.
  - The radiation appears beamed toward the direction of the observer in ***a series of pulses spaced in time (period)*** $2\pi/\omega_{\rm B}$ ***apart***, but with ***each pulse lasting only*** $\Delta\theta \simeq 2/\gamma$.
  - The spectrum will thus be spread over a much broader frequency range than on of order $\omega_{\rm B}$.

- To Fourier analyze the pulse shape, we need to **calculate the interval of the arrival times of the pulse corresponding to $\Delta\theta \sim 2/\gamma$.**
  - Let's consider an instantaneous rest frame of the electron.

$\Delta s = \mathcal{R}\Delta\theta$ = the pathlength from point 1 to 2
$\mathcal{R}$ = the radius of curvature of the path
$\Delta t = \Delta s/v$ = time interval from point 1 to 2
$|\Delta \mathbf{v}| = v\Delta\theta$ = velocity change



  - From the equation of motion, we find the curvature radius:

$$\gamma m_e \frac{\Delta \mathbf{v}}{\Delta t} = \frac{e}{c}\mathbf{v} \times \mathbf{B}$$

$$\gamma m_e \frac{v\Delta\theta}{\Delta s/v} = \frac{e}{c} vB\sin\alpha \rightarrow \boxed{\mathcal{R} = \frac{\Delta s}{\Delta\theta} = \frac{v}{\omega_\mathrm{B}\sin\alpha}}$$

Note that the radius of curvature is different from the gyroradius, which is the projected radius of the curvature radius.

  - Therefore the path length is given by

$$\boxed{\Delta s = \mathcal{R}(2/\gamma) = \frac{2v}{\gamma\omega_\mathrm{B}\sin\alpha} = \frac{2v}{\omega_\mathrm{L}\sin\alpha}}$$

- Time interval that the particle passes from point 1 to 2:

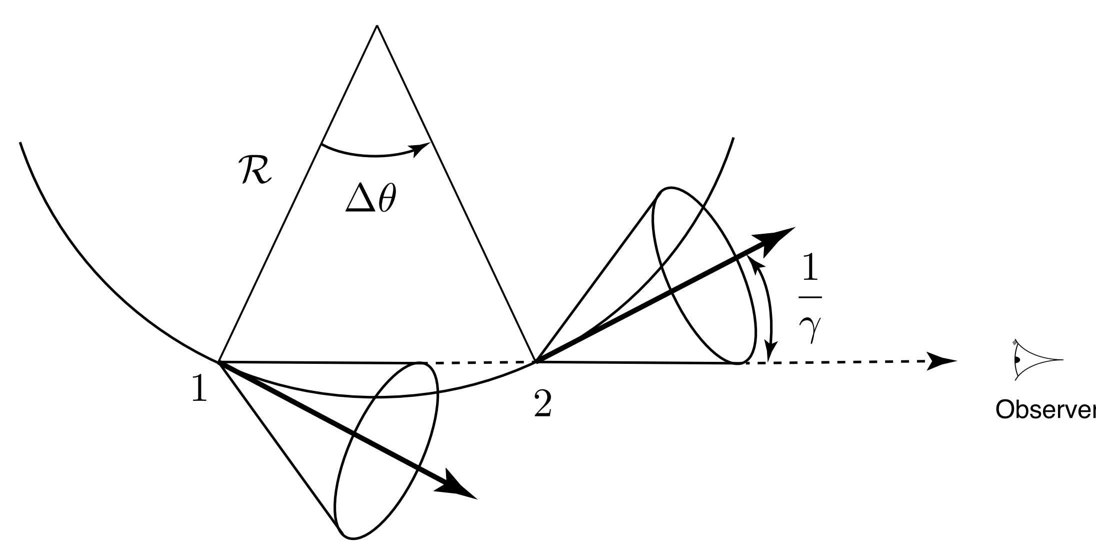
$$\Delta t = t_2 - t_1 = \frac{\Delta s}{v} \simeq \frac{2}{\omega_{\rm L} \sin\alpha}$$

- However, this is the time interval for the particle to travel from point 1 to 2. We need to calculate the interval of the arrival times of the pulse measured in the observer frame.
- Note that point 2 is closer than point 1 by $\Delta s/c$. Therefore, the difference of the arrival times of the pulse is

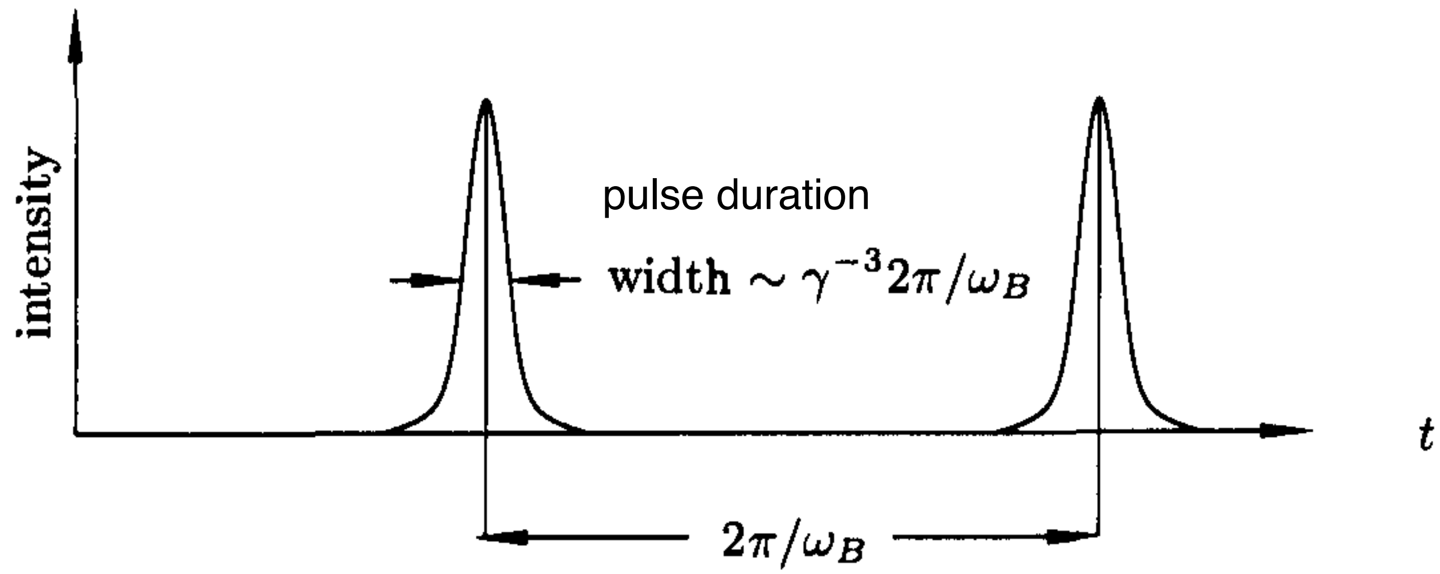
$$\Delta t^A = t_2^A - t_1^A = \Delta t - \frac{\Delta s}{c} = \Delta t\left(1 - \frac{v}{c}\right) \approx \frac{1}{\gamma^2 \omega_{\rm L}\sin\alpha} \qquad \leftarrow \quad 1 - \frac{v}{c} \approx \frac{1}{2\gamma^2}$$

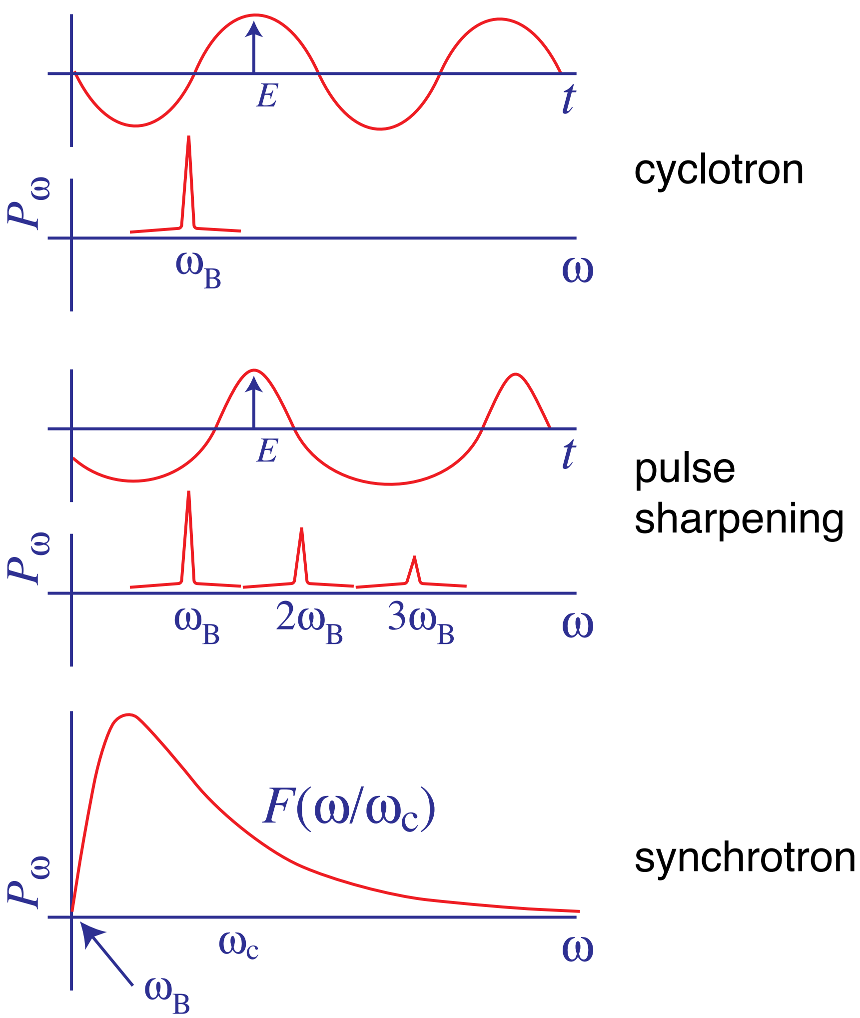
$$\boxed{\Delta t^A = t_2^A - t_1^A \approx \frac{1}{\gamma^2 \omega_{\rm L}\sin\alpha} = \frac{1}{\gamma^3 \omega_{\rm B}\sin\alpha}}$$

The width of the observed pulses is smaller than the gyration by a factor γ^3.



- Temporal pattern of received pulses:





- We define a critical frequency:

$$\omega_c \equiv \frac{3}{2}\gamma^2 \omega_{\rm L} \sin\alpha = \frac{3}{2}\gamma^3 \omega_{\rm B} \sin\alpha$$

The factor 3/2 is from the accurate calculation. (See Pacholczyk 1970, Radio Astrophysics. Nonthermal Processes in Galactic and Extragalactic Sources)

- From the properties of Fourier transformation, we expect that ***the spectrum will be fairly broad, within the frequency range of*** $\omega_{\rm B} \lesssim \omega \lesssim \omega_c$.