Interstellar Medium (ISM)

Week 10

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Dynamics of H II regions

- A new O star is presumably born within clouds of relatively dense cold gas. The appearance of a source of UV photons will have two effects.
  - First, the gas surrounding the new star will become ionized. Since the mean free path of an UV photon is very short in neutral hydrogen, the photons will be absorbed in a relatively thin surrounding shell of neutral hydrogen, producing new ionization. Thus ***the ionized and neutral gases are separated by an ionization front, which moves rapidly outward*** as more and more atoms become ionized by the stream of photons.
  - Second, the process increases the gas temperature from $\sim 10^2$ K to $\sim 10^4$ K, by a factor of about a hundred. Third, the ionization process itself increases the number of gas particles, by a factor two. As a result, ***the pressure in the ionized gas is ~200 times greater than that in surrounding neutral material.*** This ionized gas cannot be confined and will expand. The ionized and neutral gas are set in motion.
  - Since the expansion velocity is likely to exceed the sound velocity in the surrounding H I region, ***a shock front may be expected to form***, moving out through the neutral gas.
  - The dynamical analysis of H II regions must consider the interactions between the ionization front and the shock front, together with the equations of motion of the gas behind the two fronts.
- This process is not the only way in which ISM is set in motion by means of interaction with stars.
  - There are effects produced by the very high speed continuous mass loss - a ***stellar wind***.
  - Many massive stars terminate their existence in a violent explosive event - a ***supernova***.

- Basic Assumptions:
  - Any disturbances to the cloud structure produced by the formation of a star are neglected. *After a relatively short time (< 10^5 yr), the star reaches a static configuration* in which it can remain for a much longer time (> 3×10^6 yr). The stellar radiant energy output rate and the spectral distribution of the radiation are more or less constant during this phase. The star then produces Lyman continuum (LyC) photons at a constant rate. Since *the star formation time scale is so short, we may take the star to be 'switched on' instantaneously*.
  - The gas around the star will be assumed to be at rest (in the frame of reference of the star).
  - The gas has initially assumed to be uniform in density and temperature.

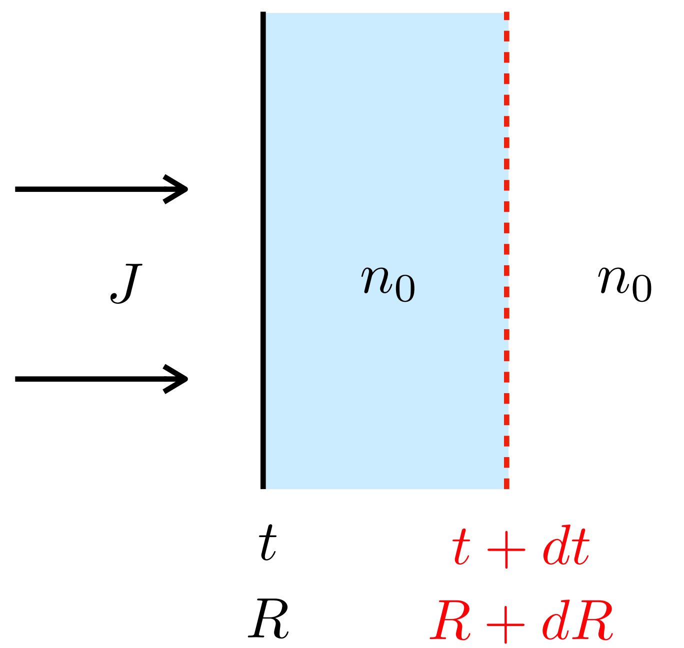
- Ionization front
  - The term "front" describes a more-or-less abrupt boundary between two regions of the ISM with very different properties.
  - An ionized nebula can be approximated as a region of highly ionized gas, separated from the surrounding neutral medium by a thin boundary region, of thickness $\lambda_{\rm mfp} \approx 0.002\,{\rm pc}$. Thus, an H II region is surrounded by an ionization front.

The velocity of the ionization front

- Suppose that at time t the ionization front is located at a distance R from the star and at time $t + dt$ it is at a distance $R + dR$.
  - Let n_0 = number density of the undisturbed neutral hydrogen

    J = number of LyC photons incident normally

    on unit area of the ionization front per unit time.



  - ***Ionization balance at the ionization front***: While the ionization front moves from R to $R + dR$, the photons will ionize all the neutral atoms lying between these two positions $(R,\ R + dR)$.
  - We assume that only one photon is needed to ionize each atom as the front moves the distance dR. In other words, *no recombination occurs within the distance interval* dR. For unit area of the ionization front, the following relation must be satisfied:

$$J \Delta A dt = n_0 \Delta A dR$$

  - Then, the velocity of the ionization front (in a fixed frame of reference) is:

$$\frac{dR}{dt} = \frac{J}{n_0}$$

The initial stage of evolution of an ionized region

- Suppose that the UV source has been suddenly turned on.
  - ***Ionization balance for the ionized region***: We consider two factors:
    - *The radiation field at the ionization front is diluted because of the spherical geometry*.
    - Recombination takes place continuously inside the ionized region, and *some of the UV photons produced by the central source must go to reionize the atoms that have recombined*.

$$\boxed{Q_0 = \left(4\pi R^2\right) J + \left(\frac{4\pi}{3} R^3\right) \alpha_{\mathrm{B}} n_e n_p}$$

  - Inside the ionized sphere, the fractional ionization is near unity. Thus, $n_e = n_p = n_0$. Using this condition, we obtain an equation for the expansion velocity of the ionization front.

$$\frac{dR}{dt} = \frac{J}{n_0} \quad \Rightarrow \quad \frac{dR}{dt} = \frac{Q_0}{4\pi R^2 n_0} - \frac{1}{3} R n_0 \alpha_{\mathrm{B}}$$

    - Let's define the following dimensionless quantities:

$$\rho \equiv R/R_s \quad \text{where } R_s \equiv \left(\frac{3}{4\pi} \frac{Q_0}{\alpha_{\mathrm{B}} n_0^2}\right)^{1/3}$$

$$\tau \equiv t/t_{\mathrm{rec}} \quad \text{where } t_{\mathrm{rec}} \equiv \frac{1}{\alpha_{\mathrm{B}} n_0}$$

Then, the equation in dimensionless form is

$$\frac{d\rho}{d\tau} = \frac{1}{3}\left(\frac{1}{\rho^2} - \rho\right)$$

- The equation can be written:

$$\frac{d\rho}{d\tau} = \frac{1}{3}\left(\frac{1}{\rho^2} - \rho\right) \quad \rightarrow \quad \frac{d\rho^3}{d\tau} = 1 - \rho^3$$

- It's solution is

$$\rho^3 = 1 - e^{-\tau}$$

$$R(t) = R_s \left(1 - e^{-t/t_{\rm rec}}\right)^{1/3}$$

initial condition: $R(t = 0) = 0$

$$\frac{dx}{d\tau} + x = 1$$

$$e^\tau \frac{dx}{d\tau} + e^\tau x = e^\tau$$

$$\frac{d\left(e^\tau x\right)}{d\tau} = e^\tau$$

$$\rightarrow \quad e^\tau x = \int_0^\tau e^{\tau'} d\tau' = e^\tau - 1$$

$$x = 1 - e^{-\tau}$$

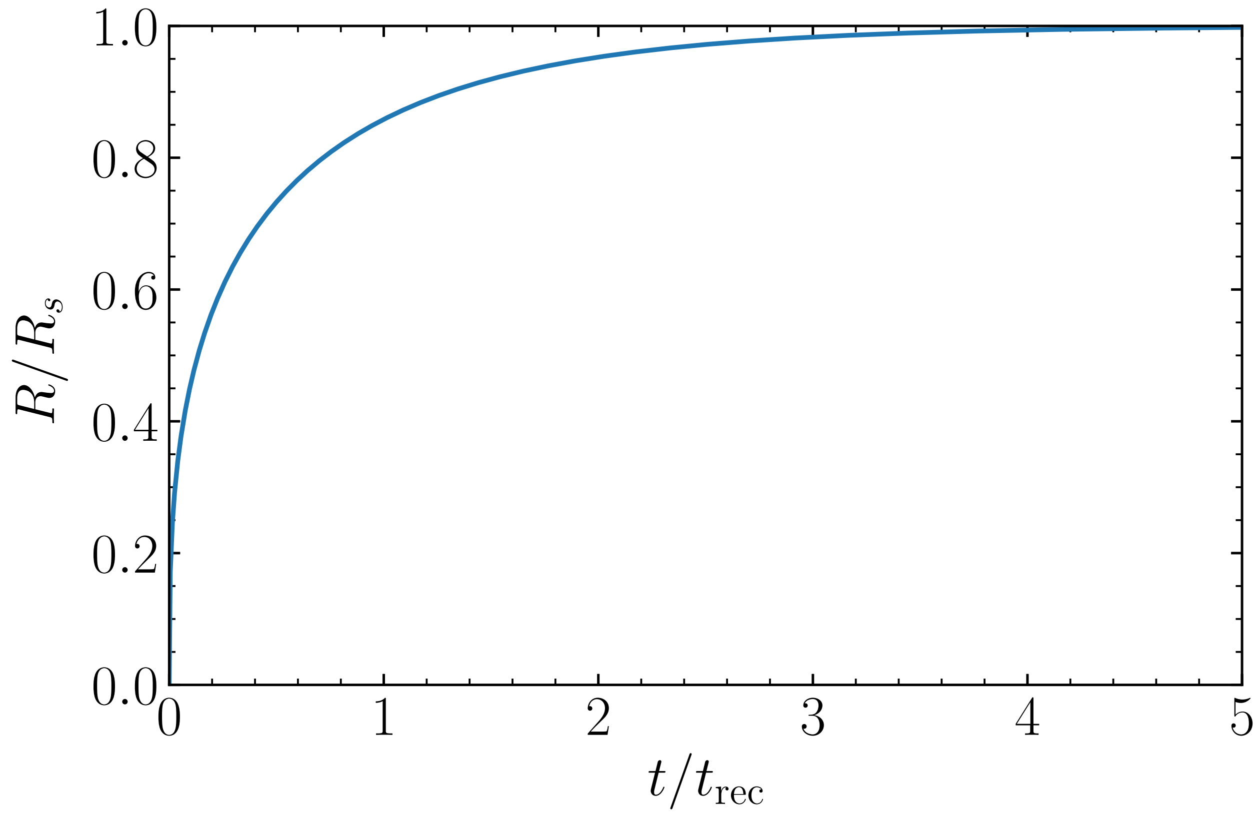
- Scale Parameters:
  - The time scale introduced is the recombination time scale:

$$t_{\rm rec} \equiv \frac{1}{\alpha_{\rm B} n_0} \approx 4000\,{\rm yr}\left(\frac{\alpha_{\rm B}}{2.6\times10^{-13}\,{\rm cm^3\,s^{-1}}}\right)^{-1}\left(\frac{n_0}{30\,{\rm cm^{-3}}}\right)^{-1}$$

the length scale introduced is the Strömgren radius:

$$R_s \equiv \left(\frac{3}{4\pi}\frac{Q_0}{\alpha_{\rm B} n_0^2}\right)^{1/3} \approx 7.2\,{\rm pc}\left(\frac{Q_0}{10^{49}\,{\rm s^{-1}}}\right)^{1/3}\left(\frac{\alpha_{\rm B}}{2.6\times10^{-13}\,{\rm cm^3\,s^{-1}}}\right)^{-1/3}\left(\frac{n_0}{30\,{\rm cm^{-3}}}\right)^{-2/3}$$

- Hence, **the time required to create a Strömgren sphere after turning on a hot star is an order of ~ 4000 yr**. This is also the time it takes the ionized Strömgren sphere to revert to neutral gas after the central UV source has been turned off.



- At times $t \gg t_{\rm rec} \sim 4000\,{\rm yr}$, the gas medium will be fully ionized with radius $R \sim R_s \sim 7\,{\rm pc}$, surrounded by a partially ionized boundary of thickness ~ $\lambda_{\rm mfp} = (n_{\rm H}\sigma_{\rm pi})^{-1} \sim 0.002\,{\rm pc} \ll R_s$.

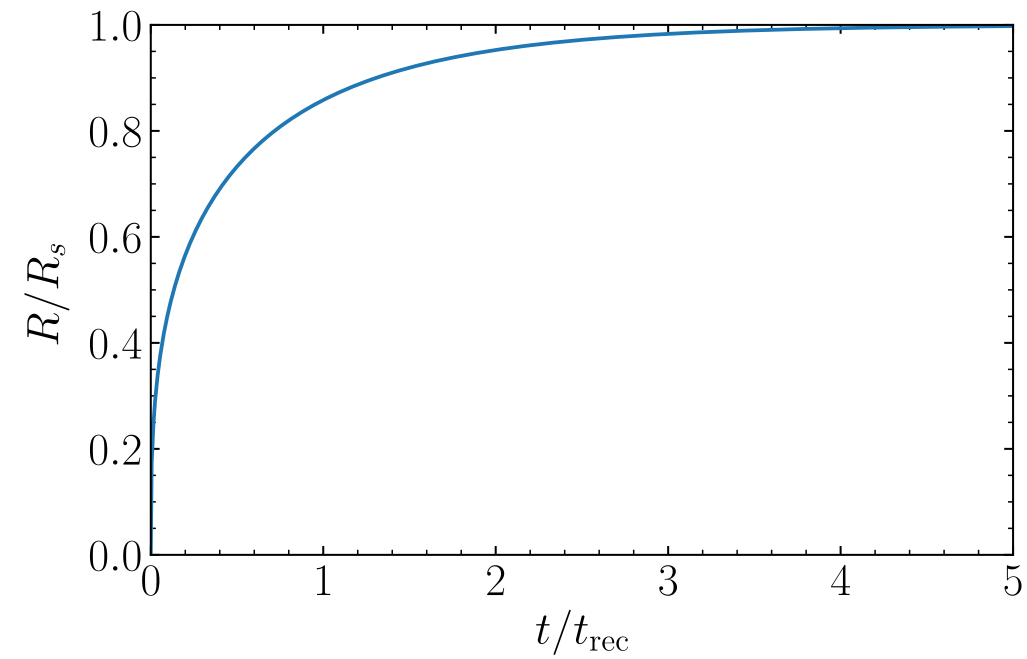
- We can compute the ***rate of expansion of the ionization front***:

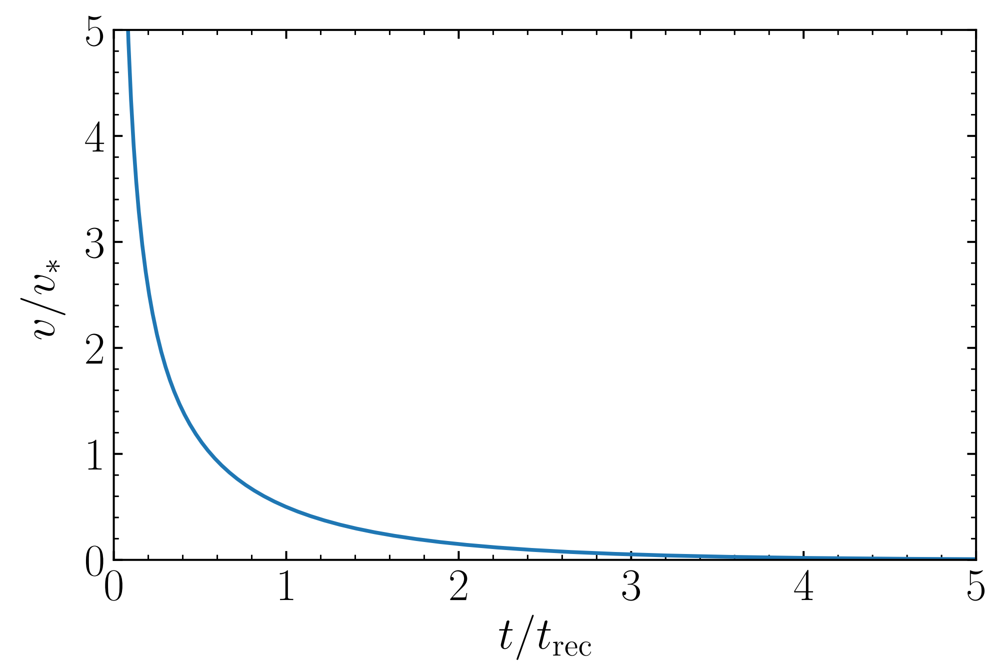
$$\frac{dR}{dt} = \frac{R_s}{3t_{\rm rec}} \frac{e^{-t/t_{\rm rec}}}{(1-e^{-t/t_{\rm rec}})^{2/3}}$$

where the characteristic expansion velocity is

$$v_* \equiv \frac{R_s}{3t_{\rm rec}}$$
$$\simeq 560\,{\rm km\,s^{-1}} \left(\frac{Q_0}{10^{49}\,{\rm s^{-1}}}\right)^{1/3} \left(\frac{n_0}{30\,{\rm cm^{-3}}}\right)^{1/3}$$

This is much larger than the sonic speed $c_s \approx 1\ {\rm km\,s^{-1}}$ in the neutral medium as well as $c_s \approx 10\ {\rm km\,s^{-1}}$ in the ionized medium.





- The expansion speed of the ionization front at two limits:

$$\frac{dR}{dt} \approx \frac{R_s}{3t_{\rm rec}} \left(\frac{t}{t_{\rm rec}}\right)^{-2/3} \qquad \text{for } t \ll t_{\rm rec}$$
$$\frac{dR}{dt} \approx \frac{R_s}{3t_{\rm rec}} e^{-t/t_{\rm rec}} \qquad \text{for } t \gg t_{\rm rec}$$

Note that the expansion speed diverges at *t = 0*.

- The ionization front will initially expand supersonically.

  When will the ionization front expand at subsonic speeds?

$$\frac{dR}{dt} = \frac{R_s}{3t_{\rm rec}} e^{-t/t_{\rm rec}} \lesssim c_{\rm i}$$

$c_{\rm i} \approx 13\,{\rm km\,s^{-1}}$ sound speed in the ionized medium

$$\boxed{t \lesssim t_{\rm sonic} \equiv t_{\rm rec} \ln\left(\frac{R_s}{3t_{\rm rec}}\frac{1}{c_{\rm n}}\right) \approx 3.8 t_{\rm rec} \simeq 15,000\,{\rm yr}}$$

- At this time, the ionization front will have a size of:

$$\boxed{R(t = t_{\rm sonic}) = R_s(1 - e^{-3.8})^{1/3} = 0.9925 R_s}$$

- The ionization front will expand at a supersonic velocity until $t \approx t_{\rm sonic}$ (~ 15,000 yr). By that time, the ionized sphere has reached a radius $R \sim 0.99 R_{\rm S}$ and then it starts to expand at subsonic speed.

- ***However, our analysis has ignored the pressure imbalance between the hot ionized gas inside the Stromgren sphere and the cold neutral gas outside.***
- ***After the sound crossing time*** $t = R_{\rm S}/c_s \sim 0.5\,{\rm Myr}$***, the gas starts to flow outward as a result of the pressure gradient that has build up.***

The final stage of evolution of an ionized region

- Although the ionized sphere approaches ionization equilibrium at $t \gtrsim t_{\rm rec}$, it would be still far from pressure equilibrium.
  - Outside the ionized zone, it will be embedded in the cold neutral medium with a temperature $T \sim 100$ K.
  - Inside the sphere, the heating and cooling processes yield a temperature of $T \sim 10{,}000$ K.
  - Also, the density of particles inside the ionized sphere will doubles when the hydrogen is ionized.
  - Thus, *the pressure inside the sphere will be ~ 200 times higher than the pressure outside, meaning that the ionized gas will begin to expand*.
  - The ionized gas expands as long as it has a higher pressure than its surroundings. This expansion produce a shock and will cease when the hot ionized gas reaches pressure equilibrium with the surrounding cold neutral gas.
- ***The condition of final pressure equilibrium*** can be written in the form:

$$\boxed{2n_{\rm f}kT_{\rm i} = n_0 kT_{\rm n}}$$

$n_{\rm f}$ = number density of the ionized hydrogen.

$T_{\rm i}$ and $T_{\rm n}$ = temperatures of the ionized and neutral gas, typically $T_{\rm i} = 10^4\,{\rm K},\ T_{\rm n} = 10^2\,{\rm K}$.

- The ionized gas sphere must still absorb all the stellar UV photons. Thus,

$$Q_0 = \frac{4}{3}\pi R_{\rm f}^3 n_{\rm f}^2 \alpha_{\rm B}$$

Here, $R_{\rm f}$ is the final radius of the ionized gas sphere. From the pressure equilibrium condition, we obtain the final size:

$$n_{\rm f} = (T_{\rm n}/2T_{\rm i})n_0 \approx 0.005 n_0 \quad \rightarrow \quad R_{\rm f} = (2T_{\rm i}/T_{\rm n})^{2/3} R_{s0} \approx 34 R_{s0}$$

- The ratio of the mass of gas finally ionized to that contained within the initial Strömgren sphere is:

$$\boxed{\frac{M_{\rm f}}{M_{\rm s}} = \frac{R_{\rm f}^3 n_{\rm f}}{R_{s0}^3 n_0} = \frac{2T_{\rm i}}{T_{\rm n}} \approx 200}$$

- This indicates that *the initial Strömgren sphere contains only a very small fraction of the material which, in principle, a star could ultimately ionize.*

The intermediate stage of evolution of an ionized region

- Before the pressure equilibrium is established, the gas density and temperature will be

$$n_\mathrm{i} \approx 2n_0 > n_\mathrm{f} \quad \text{and} \quad T_\mathrm{i} = 10^4\ \mathrm{K}$$

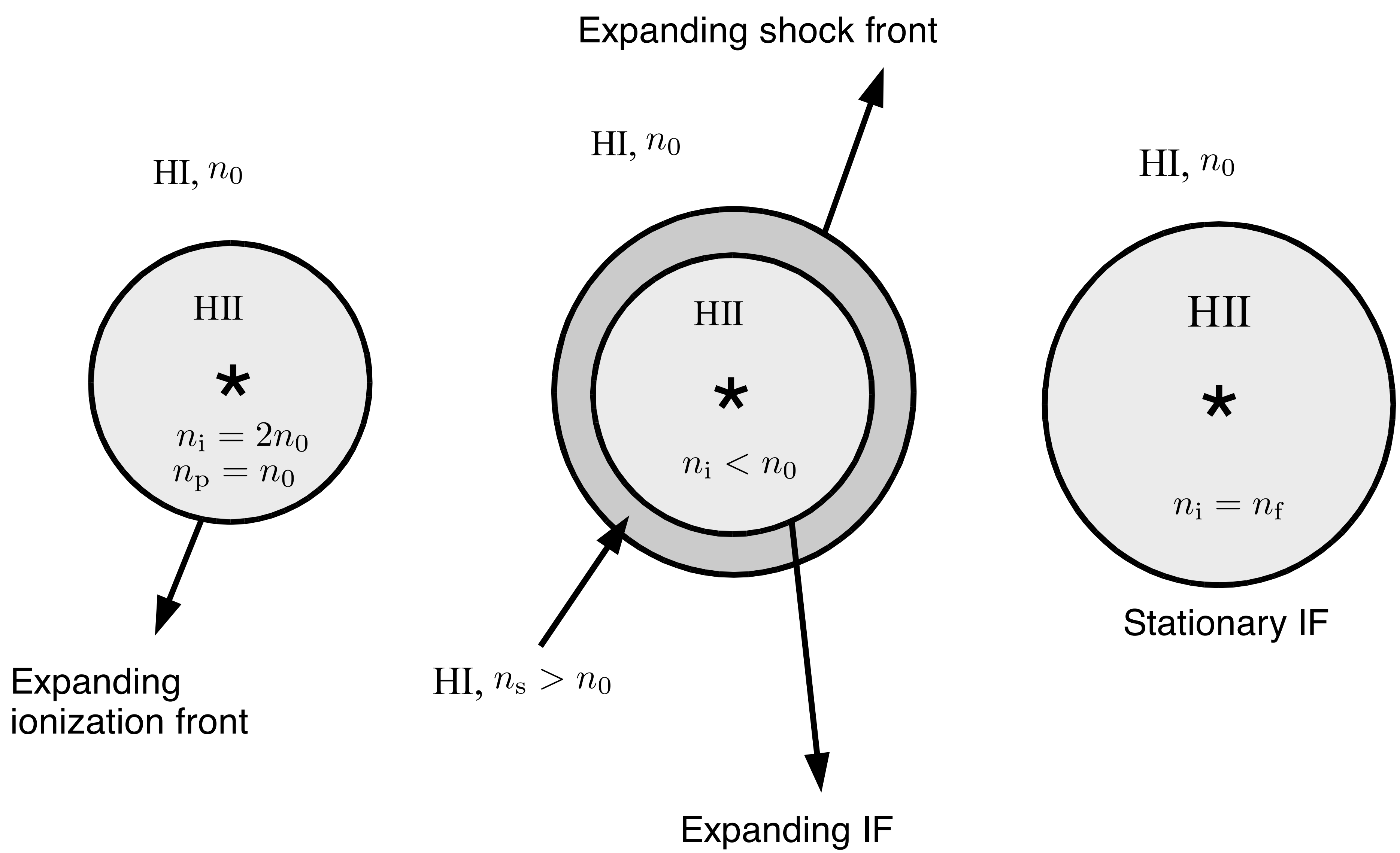
  - Then the isothermal sound speeds of the ionized gas and neutral gas are, respectively:

$$c_\mathrm{i}^2 = \frac{P_\mathrm{i}}{\rho_\mathrm{i}} \approx \frac{2n_0kT_\mathrm{i}}{n_0m_\mathrm{H}} \qquad c_\mathrm{n}^2 = \frac{P_\mathrm{n}}{\rho_\mathrm{n}} = \frac{n_0kT_\mathrm{n}}{n_0m_\mathrm{H}}$$

$$\frac{c_\mathrm{i}}{c_\mathrm{n}} = \left(\frac{n_\mathrm{i}T_\mathrm{i}}{n_0T_\mathrm{n}}\right)^{1/2} \approx \sqrt{200} = 14.14$$

  - The sound speed of the ionized gas is much larger than that of the neutral gas.
  - The ionized gas has a higher pressure and thus plays the role of a piston and pushes a shock wave into the neutral gas. *The expansion speed of the ionized gas is originally equal to about c_i, which is highly supersonic with respect to the sound speed in the neutral gas.*
  - Note also that, at $t \gtrsim t_\mathrm{sonic} \approx 3.8t_\mathrm{rec}$, the expansion speed ($c_\mathrm{i}$) of ionized gas is larger than that of the ionization front.

$\frac{dR}{dt} > c_\mathrm{i}$ at $t \lesssim t_\mathrm{sonic}$	$\longrightarrow$	$\frac{dR}{dt} \approx c_\mathrm{i}$ at $t \approx t_\mathrm{sonic}$	$\longrightarrow$	$\frac{dR}{dt} < c_\mathrm{i}$ at $t \gtrsim t_\mathrm{sonic}$
initial stage		intermediate stage		



Evolutionary scheme of an expanding H II region. (a) The initial stage, (b) expansion with a shock in the neutral gas, (c) the final equilibrium state.

[Figure 7.2 Dyson]

- Sound crossing time
  - The ionized region will likely be overpressured relative to its surroundings, in which case it will expand on the sound crossing time.
  - The isothermal sound speed in fully ionized hydrogen is

$$c_s = (2kT/m_{\rm H})^{1/2} = 13\,(T/10^4\,{\rm K})^{1/2}\,{\rm km\,s^{-1}} \qquad p = (n_{\rm HI} + n_e)kT = 2n_{\rm H}kT$$

  - The time for a pressure wave to propagate a distance equal to Strömgren radius is

$$\boxed{t_{\rm sound} = \frac{R_s}{c_s} \approx 2.39 \times 10^5\,\frac{Q_0/10^{49}\,{\rm s^{-1}}}{(n/10^2\,{\rm cm^{-3}})^{2/3}} \quad [{\rm yr}]}$$

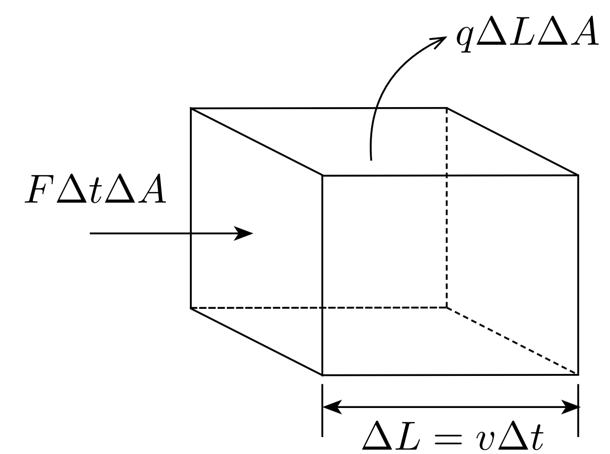
  - This is about a hundred times longer than the recombination time (timescale of the expanding ionization front).

Introduction to Gas Dynamics

- Assumption for hydrodynamics:
  - particle mean free path $\ll$ size of the region
  - We will derive the equations for conservation of mass, momentum and energy, in 1D space.
- ***Definition***
  - Flux of a hydrodynamic quantity q (for instance, density):

    Fluid moves a distance ΔL during a time interval Δt with a velocity v.

$$F \Delta t \Delta A = q \Delta L \Delta A \quad \rightarrow \quad \boxed{F = qv}$$

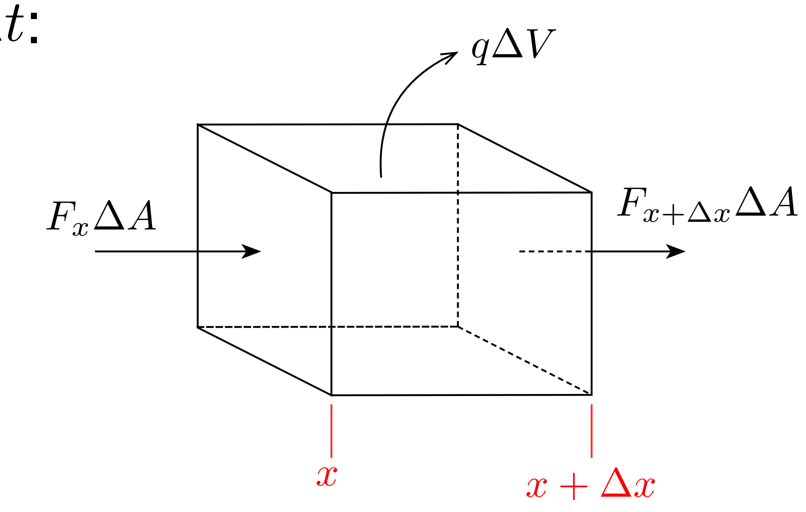


- ***Conservation equation for a quantity*** q
  - change of the quantity within a volume ΔV for a time interval Δt:

    Here, Δt and Δx are independent.

$$\frac{q\Delta V|_{t+\Delta t} - q\Delta V|_t}{\Delta t} = F\Delta A|_x - F\Delta A|_{x+\Delta x}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial F}{\partial x} \longrightarrow \boxed{\frac{\partial q}{\partial t} = -\frac{\partial (qv)}{\partial x}}$$



  - Here, no sources or sinks of the quantity within ΔV were assumed. If any, the loss and gain terms should be added in the right-had side.

Mass Conservation

- Conservation equations
  - ***Mass conservation (continuity equation)***
    - mass within a volume dV = ρdV
    - no sources or sinks of material within dV
    - Consider the mass per unit area (dA), contained in the volume

$$\rho dV/dA = \rho dx \longrightarrow \frac{\partial}{\partial t}(\rho dx) = \overbrace{\rho u}^{\text{incoming}} - \overbrace{(\rho + d\rho)(u + du)}^{\text{outgoing}}$$

$$= -(\rho du + u d\rho + \cancel{d\rho du})$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}}$$

- Mass loss and gain terms should be added in the right-had side, if necessary.

Momentum Conservation

- *Momentum conservation (Euler's equation)*

- momentum within dV (per unit area) = $(\rho dV)u/dA = \rho u dx$

  = change of momentum due to fluid flow and gas pressure acting on the surface of dV

$$\frac{\partial}{\partial t}(\rho u dx) = \overbrace{\rho u^2}^{\text{incoming}} - \overbrace{(\rho + d\rho)(u+du)^2}^{\text{outgoing}} + \overbrace{P}^{\text{incoming}} - \overbrace{P + dP}^{\text{outgoing}}$$

$$= \rho u^2 - \left(\rho u^2 + 2\rho u du + \cancel{\rho du^2} + u^2 d\rho + \cancel{2ud\rho du + d\rho du^2}\right) - dP$$

$$\frac{\partial}{\partial t}(\rho u) = -2\rho u\frac{\partial u}{\partial x} - u^2\frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x}$$
$$= -\frac{\partial}{\partial x}(\rho u^2) - \frac{\partial P}{\partial x}$$

$$\boxed{\frac{\partial}{\partial t}(\rho u) = -\frac{\partial}{\partial x}\left(\rho u^2 + P\right)}$$

or

$$\frac{\partial}{\partial t}(\rho u) = -2\rho u\frac{\partial u}{\partial x} - u^2\frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x}$$
$$\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} = -\rho u\frac{\partial u}{\partial x} - u\left(\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x}\right) - \frac{\partial P}{\partial x}$$

Using mass conservation, $\dfrac{\partial u}{\partial t} = -\dfrac{\partial(\rho u)}{\partial x}$

$$\boxed{\rho\frac{\partial u}{\partial t} = -\rho u\frac{\partial u}{\partial x} - \frac{\partial P}{\partial x}}$$

- Further terms could be added in the right-hand side, accounting for forces due to gravity, magnetic fields, radiation field, and viscosity.

- The following quantity is sometimes known as **Bernoulli's constant**.

$$\boxed{\rho u^2 + P}$$

One may use it to understand why, for example, fast winds engulfing a house causes it to **explode**, rather than implode, because the pressure external to the house becomes lower than its value inside it.

- Viscous force is due to "internal friction" in the fluid (resistivity of the fluid to the flow), as two adjacent fluid parcels move relative to each other.

$$\text{viscous force} \propto \frac{\partial^2 u}{\partial x^2}$$

The viscous force is usually much smaller than force due to gas pressure, but important in high-speed flows with large velocity gradients, as in accretion disks.

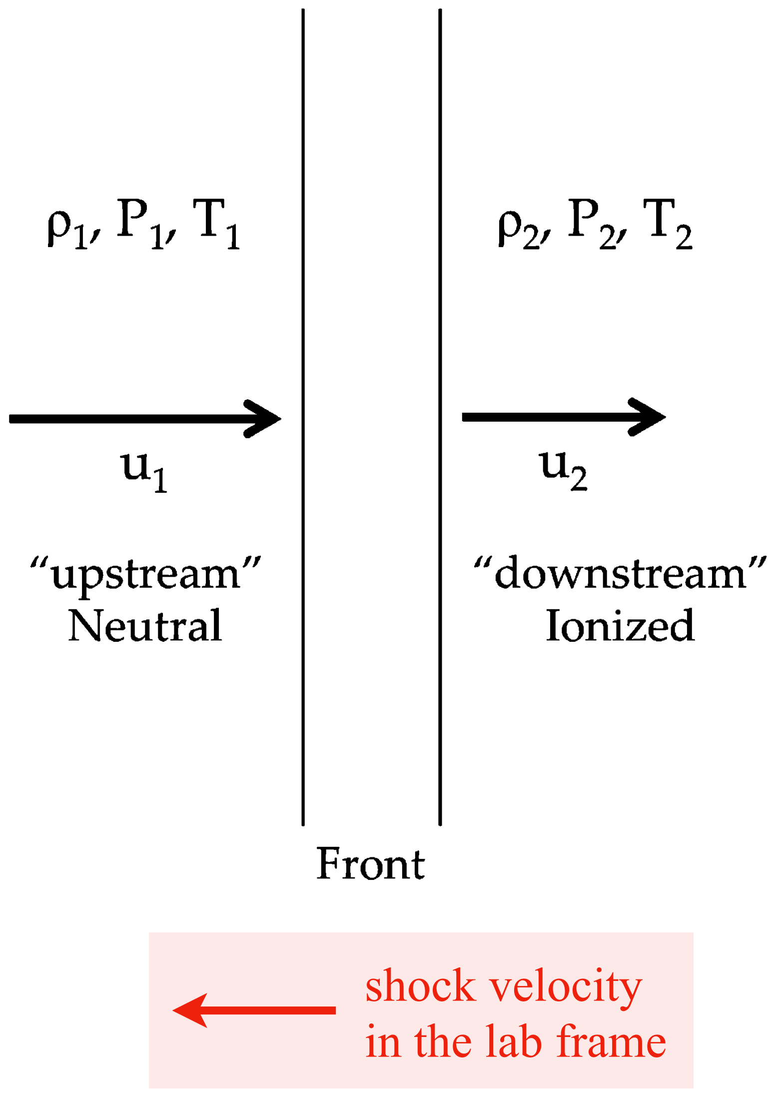
Ionization Front: Jump Condition

- Low density gas, like that of the ISM, can be treated as an ideal gas, with no viscosity with a pressure given by the ideal gas law:

$$P = \frac{\rho k T}{m} \qquad \rho = \text{mass density},\ T = \text{temperature},\ m = \text{mean molecular mass}$$

$\rho,\ P,\ T,\ \mathbf{u}$ = density, pressure, temperature, and bulk velocity

- Let's consider a small patch of the ionization front between the interior of an H II region and its exterior.
  - If the patch is small compared to the ionization front's radius of curvature, then we can treat the ionization front as if it has ***plane parallel*** symmetry.
  - It is convenient to use ***a frame of reference in which the ionization front is stationary***; in this frame, the bulk velocity $\mathbf{u}_1$ of the neutral gas points toward the ionization front. The bulk velocity $\mathbf{u}_2$ of the ionized gas points away from from ionization front.



- Let's consider a steady state solution.
  - We have seen that the speed of the ionization front surrounding a Strömgren sphere changes with time. However, the steady state solution gives us some intuition about the behavior of ionization fronts in general.
  - Then, the mass conservation and momentum conservation equation becomes:

$$\frac{d}{dx}(\rho u) = 0 \qquad \frac{d}{dx}\left(\rho u^2 + P\right) = 0$$

  - Let subscript **1 denote fluid variables in the neutral gas ahead of the I-front**, and subscript **2 denotes fluid variables in the ionizing gas behind the I-front**. Integrating these equation across the ionization front, we obtain:

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

- *The number of H atoms flowing through the ionization front per unit area per second must equal to J, the corresponding number of ionizing photons reaching the front.* Hence, the equation becomes

$$\rho_1 u_1 = \rho_2 u_2 = m_i J$$

Here, $u_1 = \frac{dR}{dt} = \frac{J}{n_0}, \quad \rho_1 = m_i n_0$

where m_i is the mean mass of the gas per newly created positive ion ($m_i = m_\mathrm{H}$ in a pure hydrogen gas). We may also write the equation of momentum conservation using the isothermal sound speeds:

$$\rho_1 \left(u_1^2 + c_1^2\right) = \rho_2 \left(u_2^2 + c_2^2\right)$$

$$\boxed{\begin{aligned} &c_s^2 = \frac{P}{\rho} \text{ for isothermal gas} \\ &P = nkT \end{aligned}}$$

- We will consider a hydrogen gas.

$$c_1 = \left(\frac{kT_1}{m_\mathrm{H}}\right)^{1/2} = 0.91\,\mathrm{km\,s^{-1}} \left(\frac{T_1}{100\,\mathrm{K}}\right)^{1/2}$$ neutral hydrogen gas

$$c_2 = \left(\frac{2kT_2}{m_\mathrm{H}}\right)^{1/2} = 12.9\,\mathrm{km\,s^{-1}} \left(\frac{T_2}{10^4\,\mathrm{K}}\right)^{1/2}$$ fully ionized gas

Here, the number density of particles is $2n_\mathrm{H}$ in a fully-ionized hydrogen gas (downstream) and thus the factor 2 in c_2.

- In summary, the equations are

$$\rho_1 u_1 = \rho_2 u_2 = m_i J$$
$$\rho_1 \left(u_1^2 + c_1^2\right) = \rho_2 \left(u_2^2 + c_2^2\right)$$

- We assume that ρ_1 and u_1 are known, and we seek to solve for the unknown ρ_2 and u_2. We obtain a simple quadratic equation for $x \equiv \rho_1/\rho_2 = u_2/u_1$.

$$\frac{\rho_1}{\rho_2}\left(u_1^2 + c_1^2\right) = \left(\frac{\rho_1}{\rho_2}\right)^2 u_1^2 + c_2^2$$
$$u_1^2 x^2 - \left(u_1^2 + c_1^2\right) x + c_2^2 = 0 \longrightarrow x = \frac{1}{2u_1^2}\left[\left(u_1^2 + c_1^2\right) \pm \sqrt{\left(u_1^2 + c_1^2\right)^2 - 4u_1^2 c_2^2}\right]$$

Then, the ratios between densities and velocities are:

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{1}{2u_1^2}\left[\left(u_1^2 + c_1^2\right) \pm \sqrt{\left(u_1^2 + c_1^2\right)^2 - 4u_1^2 c_2^2}\right]$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{1}{2c_2^2}\left[\left(u_1^2 + c_1^2\right) \mp \sqrt{\left(u_1^2 + c_1^2\right)^2 - 4u_1^2 c_2^2}\right]$$

- The roots are real if and only if

$$
\begin{aligned}
f(u_1) &\equiv \left(u_1^2 + c_1^2\right)^2 - 4u_1^2 c_2^2 \\
&= \left(u_1^2 + c_1^2 + 2u_1 c_2\right)\left(u_1^2 + c_1^2 - 2u_1 c_2\right) \geq 0
\end{aligned}
$$

This requires:

$$u_1^2 + c_1^2 - 2u_1 c_2 \geq 0$$

$$\left[u_1 - \left(c_2 + \sqrt{c_2^2 - c_1^2}\right)\right]\left[u_1 - \left(c_2 - \sqrt{c_2^2 - c_1^2}\right)\right] \geq 0$$

Therefore,

$$u_1 \geq u_R \equiv c_2 + \sqrt{c_2^2 - c_1^2} \quad \text{or} \quad u_1 \leq u_D \equiv c_2 - \sqrt{c_2^2 - c_1^2}$$

Note that
$u_R > c_2 > c_1 > u_D$

We also note that

$$u_1^2 + c_1^2 + 2u_1 c_2 = \left[u_1 + \left(c_2 + \sqrt{c_2^2 - c_1^2}\right)\right]\left[u_1 + \left(c_2 - \sqrt{c_2^2 - c_1^2}\right)\right]$$

$$\rightarrow \quad f(u_1) = (u_1^2 - u_\mathrm{R}^2)(u_1^2 - u_\mathrm{D}^2)$$

$$
\begin{aligned}
\frac{u_2}{u_1} &= \frac{\rho_1}{\rho_2} = \frac{1}{2u_1^2}\left[\left(u_1^2 + c_1^2\right) \pm \sqrt{\left(u_1^2 - u_R^2\right)\left(u_1^2 - u_D^2\right)}\right] \\
\frac{\rho_2}{\rho_1} &= \frac{u_1}{u_2} = \frac{1}{2c_2^2}\left[\left(u_1^2 + c_1^2\right) \mp \sqrt{\left(u_1^2 - u_R^2\right)\left(u_1^2 - u_D^2\right)}\right]
\end{aligned}
$$

- The rapidly propagating ionization fronts, with $u_1 \geq u_R$ are called ***R-type fronts (R stands for "rarefied" or rapid)***. The dilatory ionization fronts are called ***D-type fronts (D stands for "dense" or dilatory)***.
  - An R-type front has $u_1 \geq u_\mathrm{R} > c_2 > c_1$, and is supersonic with respect to the neutral medium.
  - A D-type front has $u_1 \leq u_\mathrm{D} < c_1 < c_2$, and is subsonic with respect to the neutral medium.

- For a given front propagation speed u_1, there are two possible values of the density ratio ρ_2/ρ_1 across the ionization front as a function of the propagation speed u_1.
  - The front that has the ***larger density contrast*** is called a ***strong*** front.
  - The front that has the ***smaller density contrast*** is called a ***weak*** front.
  - Thus, there are four types of ionization front: weak R, strong R, weak D, strong D.

$$\frac{\rho_2}{\rho_1} = \frac{1}{2c_2^2}\left[\left(u_1^2 + c_1^2\right) \pm \sqrt{\left(u_1^2 - u_R^2\right)\left(u_1^2 - u_D^2\right)}\right]$$

R-front: $u_1 \geq u_R$ weak $-$; strong $+$
D-front: $u_1 \leq u_D$ weak $+$; strong $-$

  - The solutions for $u_1 = u_R$ and $u_1 = u_D$ are called ***"R-critical"*** and ***"D-critical"***, respectively.

- Since c_2 exceeds c_1 by about one or two order of magnitude in an interstellar ionization front ($c_2 \gg c_1$),

$$u_R = c_2 + \sqrt{c_2^2 - c_1^2} \approx c_2 + c_2\left(1 - \frac{1}{2}\frac{c_1^2}{c_2^2} - \frac{1}{8}\frac{c_1^4}{c_2^4}\right)$$
$$u_D = c_2 - \sqrt{c_2^2 - c_1^2} \approx c_2 - c_2\left(1 - \frac{1}{2}\frac{c_1^2}{c_2^2} - \frac{1}{8}\frac{c_1^4}{c_2^4}\right)$$

$$u_R \approx 2c_2\left(1 - \frac{1}{4}\frac{c_1^2}{c_2^2}\right) > c_2 > c_1 > u_D$$
$$u_D \approx \frac{1}{2}\frac{c_1^2}{c_2}\left(1 + \frac{1}{4}\frac{c_1^2}{c_2^2}\right) < c_1 < c_2 < u_R$$

- Approximate solutions:

R-critical	$\frac{\rho_2}{\rho_1} \approx 2\left(1 - \frac{1}{4}\frac{c_1^2}{c_2^2}\right)$	for $u_1 = u_R$	$\left[\frac{\rho_2}{\rho_1}\right]_{\mathrm{R-critical}} \approx 2$
D-critical	$\frac{\rho_2}{\rho_1} \approx \frac{1}{2}\frac{c_1^2}{c_2^2}\left(1 + \frac{1}{4}\frac{c_1^2}{c_2^2}\right)$	for $u_1 = u_D$	$\left[\frac{\rho_2}{\rho_1}\right]_{\mathrm{D-critical}} \ll 1$
weak R-front	$\frac{\rho_2}{\rho_1} \approx 1 + \frac{c_2^2}{u_1^2}$	for $u_1 \gg u_R$	$1 \approx \left[\frac{\rho_2}{\rho_1}\right]_{\mathrm{weak\ R}} < \left[\frac{\rho_2}{\rho_1}\right]_{\mathrm{strong\ R}}$
strong R-front	$\frac{\rho_2}{\rho_1} \approx \frac{u_1^2}{c_2^2} - 1$		
weak D-front	$\frac{\rho_2}{\rho_1} \approx \frac{c_1^2}{c_2^2} - \frac{u_1^2}{c_1^2}$	for $u_1 \ll u_D$	$\left[\frac{\rho_2}{\rho_1}\right]_{\mathrm{strong\ D}} < \left[\frac{\rho_2}{\rho_1}\right]_{\mathrm{weak\ D}} \ll 1$
strong D-front	$\frac{\rho_2}{\rho_1} \approx \frac{u_1^2}{c_1^2}\left(1 + \frac{c_2^2}{c_1^4}u_1^2\right)$		

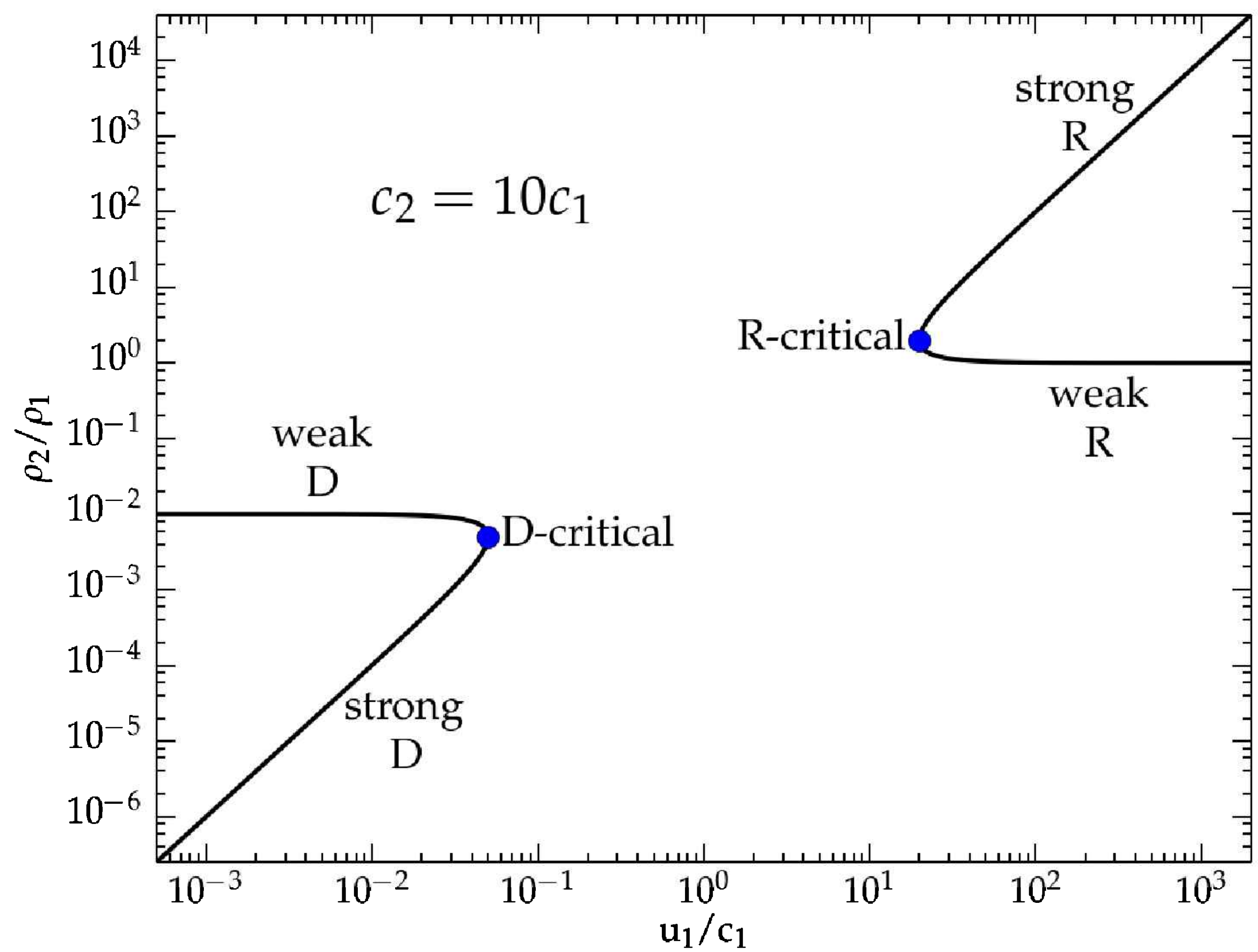


Figure 4.11 [Ryden]

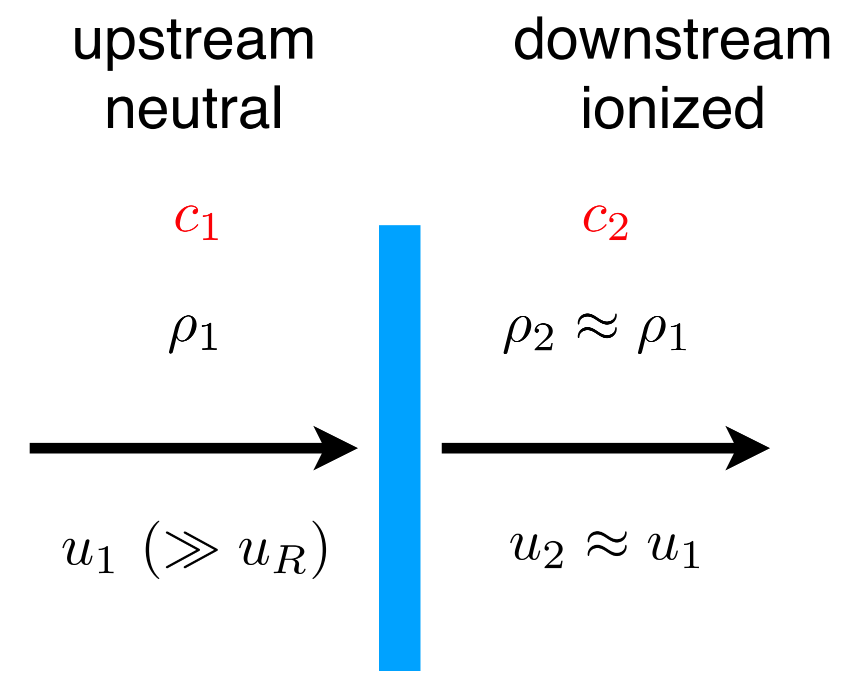
$$u_R \approx 2c_2, \quad u_D \approx \frac{1}{2}\frac{c_1^2}{c_2}$$

R-critical	$\dfrac{\rho_2}{\rho_1} \approx 2$	for $u_1 = u_R$
D-critical	$\dfrac{\rho_2}{\rho_1} \approx \dfrac{1}{2}\dfrac{c_1^2}{c_2^2}$	for $u_1 = u_D$
strong R-front	$\dfrac{\rho_2}{\rho_1} \approx \dfrac{u_1^2}{c_2^2}$	for $u_1 \gg u_R$
weak R-front	$\dfrac{\rho_2}{\rho_1} \approx 1$	
weak D-front	$\dfrac{\rho_2}{\rho_1} \approx \dfrac{c_1^2}{c_2^2} \Rightarrow \rho_1 c_1^2 \approx \rho_2 c_2^2$	for $u_1 \ll u_D$
strong D-front	$\dfrac{\rho_2}{\rho_1} \approx \dfrac{u_1^2}{c_1^2}$	

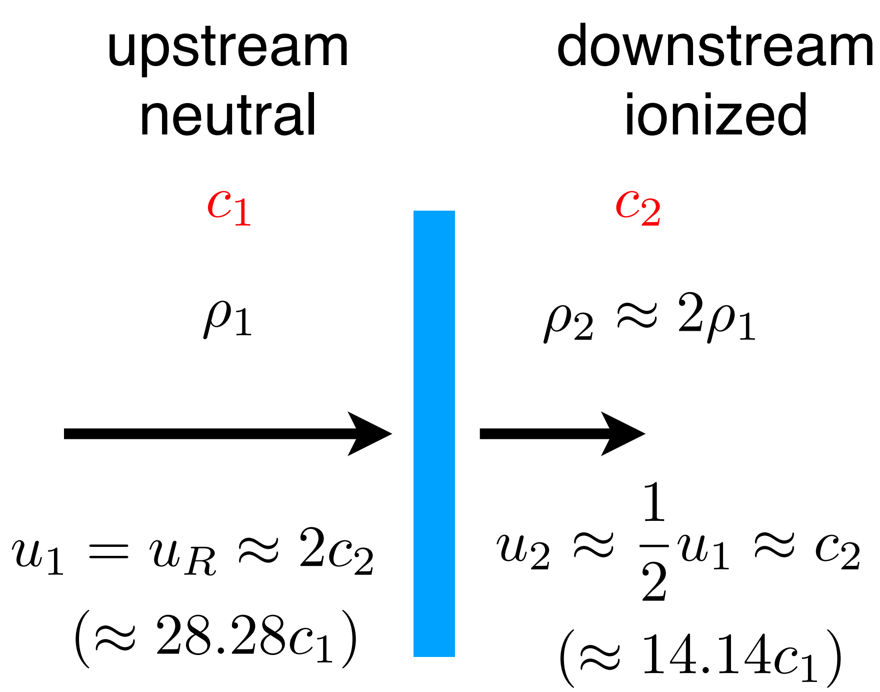
- We note that the four types are not all relevant to H II regions.
  - The strong R type means a lower density in the upstream (neutral gas). The strong R-type fronts are in fact unstable (Rayleigh-Taylor instability). In H II regions, the neutral gas has a higher density than the ionized gas. (or the same density at the initial stage).
  - The strong D type implies that the density in neutral gas increases forever when the ionization front slows down ($\rho_1 \rightarrow \infty$ as $u_1 \rightarrow 0$).
- ***The fronts relevant to the H II regions are weak R-front and weak D-front.***

Evolution of Ionization Front

[1] Weak R front



[2] R-critical front



We will assume that

$$c_1 = (kT_1/m_{\rm H})^{1/2}$$
$$c_2 = (2kT_2/m_{\rm H})^{1/2} = (2T_2/T_1)^{1/2} c_1 = \sqrt{200}\, c_1$$
$$\text{for} \ \ T_1 = 10^2 \,\text{K}, \ T_2 = 10^4 \,\text{K}$$

(1) Weak R front:

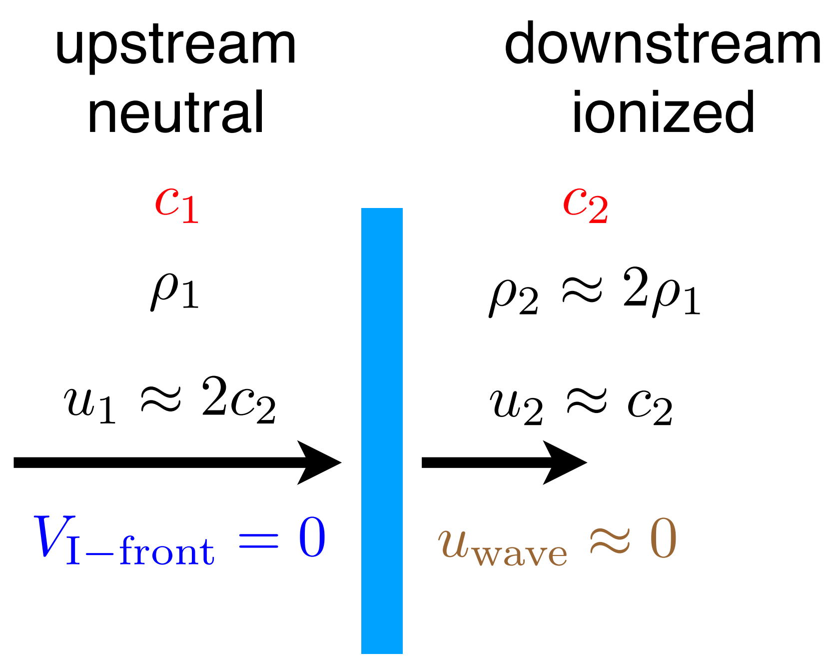
- Initially, the photon flux J is very large. Thus, u_1 is very large, and the ionization front is initially a weak R-type front.The densities of neutral gas and ionized gas are nearly the same: $\rho_2/\rho_1 \approx 1$. (A weak R-type front compresses the gas only slightly.)
- As the ionization front expands, the flux of ionizing photons steadily decreases, and the propagation speed u_1 of the front slows down.

(2) R-critical front:

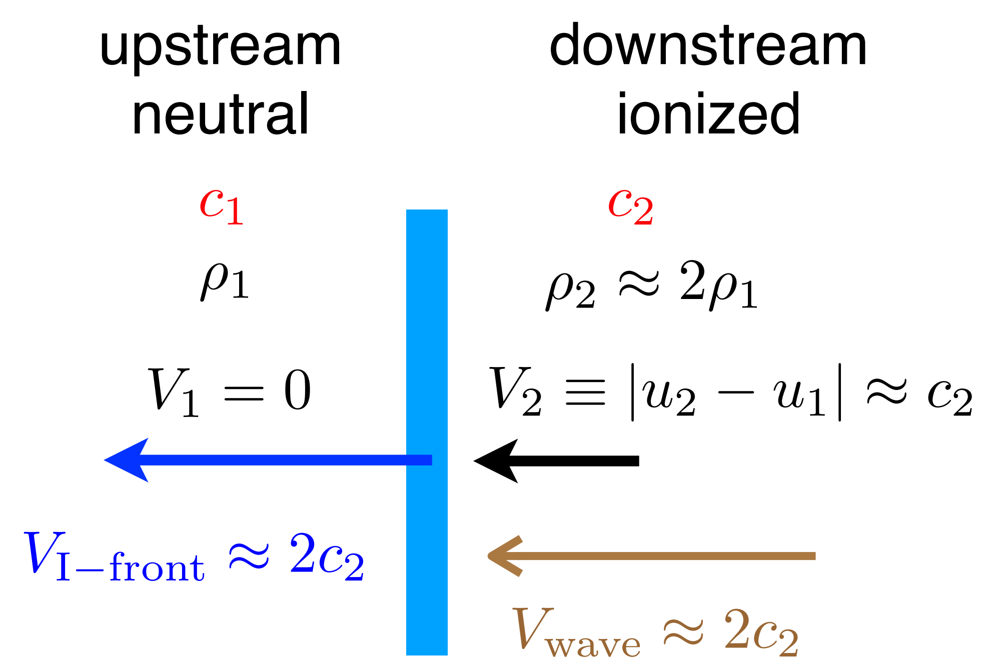
- Eventually, the speed drops to a value $u_1 = u_R \approx 2c_2$.
- At this point, the density ratio has rise to $\rho_2/\rho_1 \approx 2$.
- The speed of the ionized gas is $u_2 \approx (1/2)u_R \approx c_2$ relative to the ionization front, or $u_2 - u_1 \approx -c_2$ in a fixed frame of reference.
- As the ionization front slows down further, the R-type front can no longer exit.

◉ How does the evolution proceed once the ionization front becomes R-critical?

- When the R-critical condition is reached, the gas in the H II region just behind the front is moving at a speed equal to $c_2 \gg c_1$.
- This should derive a shock wave into the pre-ionization front gas. Before this point, the large pressure discrepancy between the H II region and the H I region ahead of it has no chance to act dynamically, because the ionization front races ahead with speed u_1 so much faster than a pressure wave can catch it.
- When the ionization front slows down to a speed $u_1 = u_R \approx 2c_2$, however, the pressure wave (moving at a speed c_2 on top of the speed $u_2 \approx c_2$ that the H II fluid itself moves) can catch up with the ionization front and overtake it.
- In doing so, the pressure wave will steepen into a shock wave, thereby compressing the atomic gas behind it into a denser state that the lagging ionization front then has to eat into.

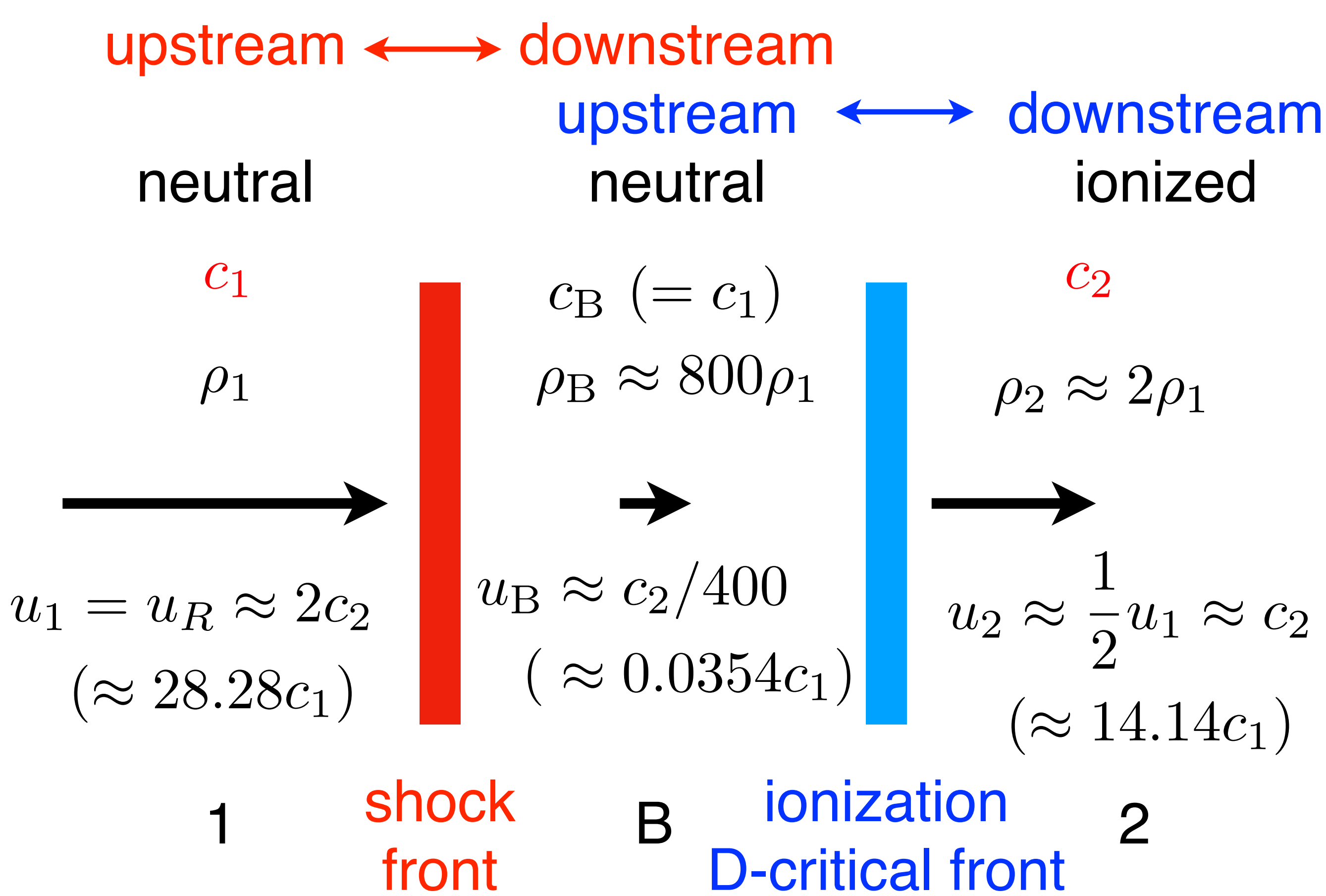


in the rest-frame of ionization front



in the fixed frame

[3] D-critical front



(3) D-critical front:

- As the ionization front slows down further, the R-type front can no longer exit. What happens next is that ***the R-critical ionization front splits into a pair of fronts (shock front + ionization front).***
- ***A leading shock front is followed by a D-critical ionization front.*** The shock front is the boundary between two regions of gas with different density, pressure, and temperature, but no necessarily different ionization states. The shock front propagates with a supersonic speed relative to the gas in the upstream of the shock front.

- We will assume ***an isothermal shock***. Then, the sound speed of the shocked region (B) must be $c_s = c_1$ (from the Rankin-Hugoniot jump condition). Then, using the condition for the D-critical, we obtain the density and speed of the shocked region (B):

$$\frac{\rho_2}{\rho_s} \approx \frac{1}{2}\frac{c_1^2}{c_2^2} = \frac{1}{400}$$

$$\rho_s \approx 400\rho_2 \approx 800\rho_1$$

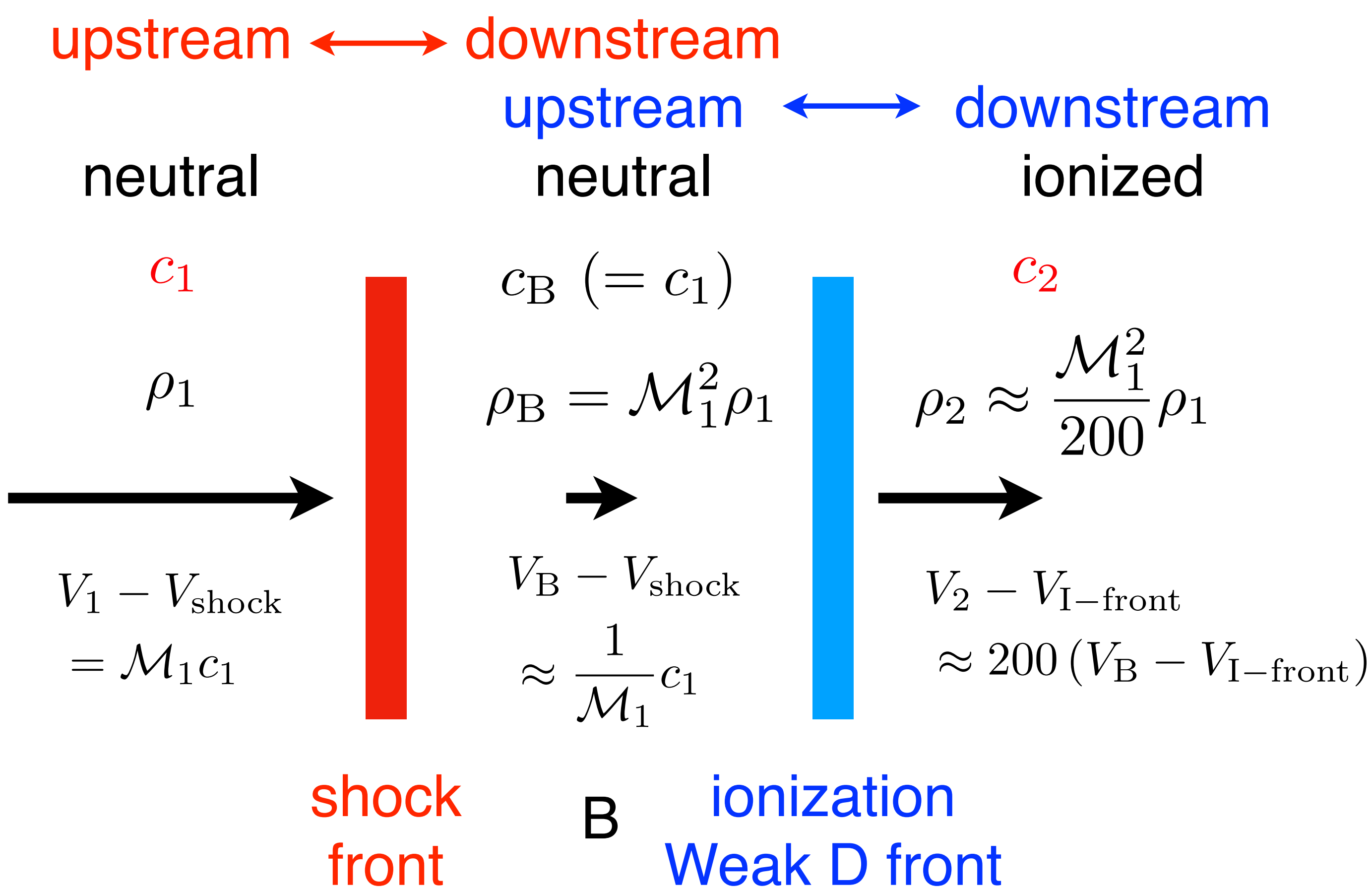
$$\frac{u_s}{u_2} = \frac{\rho_2}{\rho_s}$$

$$u_s \approx \frac{1}{400}u_2 \approx \frac{1}{400}c_2 \approx \frac{1}{\sqrt{800}}c_1 = 0.0354c_1$$

$$\longrightarrow \quad \rho_s \approx 800\rho_1, \quad u_s \approx 0.0354c_1, \quad (\mathcal{M}_1 = \sqrt{800})$$

- The shocked region (B) has a very high density, and is almost stationary relative to the ionization front. The velocities u_1, u_s, u_2 are measured in the rest-frame of I-front. The R-critical condition between 1 and 2 is still satisfied.

[4] Weak D front



(4) Weak D front:

- As the H II region expands still further, the leading shock from gradually weakens and the trailing D-critical front develops into a weak D-type front.
- Notice that the weakest of weak-D ionization fronts corresponds to the density discontinuity:

$$\frac{\rho_2}{\rho_B} = \frac{c_1^2}{c_2^2}$$

This is the condition for the static pressure equilibrium in isothermal gas,

$$\rho_2 c_2^2 = \rho_B c_1^2 \qquad (P = \rho c_s^2)$$

the state that we expect for the final Strömgren sphere.

- The condition for the weak D-type front must be satisfied between the regions “B” and “2”.
- In addition to this condition, The shock jump condition should be satisfied between the regions “1” and “B”.
- However, notice that the velocities of the shock front and the ionization front can be different, in general.

$$V_{\mathrm{shock}} \neq V_{\mathrm{I-front}}$$

Intermediate States - expansion phase

- Assumptions:
  - The shocked gas layer is thin.
  - The ionization front follows the shock front and the expansion velocity of ionized sphere is approximately the same as the shock velocity.

$$V_{\mathrm{I-front}} \approx V_{\mathrm{shock}} \qquad \frac{dR}{dt} = V_{\mathrm{s}}$$

- Expansion:
  - The pressure behind a strong "isothermal" shock (high Mach number) is related to the shock velocity:

$$P_{\mathrm{s}} = \rho_0 V_{\mathrm{s}}^2 = n_0 m_{\mathrm{H}} V_s^2$$

  - Now assume that the pressure behind the shock wave is equal to the pressure of the ionized gas (pressure equilibrium).

$$P_{\mathrm{i}} = 2n_{\mathrm{i}}kT = n_{\mathrm{i}} m_{\mathrm{H}} c_{\mathrm{i}}^2 \quad \left(c_{\mathrm{i}}^2 \equiv \frac{2kT}{m_{\mathrm{H}}} \right)$$ for fully-ionized hydrogen gas

  - Then, the shock velocity is given by

$$P_{\mathrm{s}} = P_{\mathrm{i}} \quad \rightarrow \quad V_{\mathrm{s}}^2 = \frac{n_{\mathrm{i}}}{n_0} c_{\mathrm{i}}^2 \quad \rightarrow \quad \frac{V_{\mathrm{s}}^2}{c_{\mathrm{i}}^2} = \frac{n_{\mathrm{i}}}{n_0}$$

- We assume that the amount of fresh neutral gas to be ionized is very small. Then, the ionization balance for the region within R gives

$$Q_0 = \frac{4\pi}{3} R^3 n_\mathrm{i}^2 \alpha_\mathrm{B} \quad \rightarrow \quad R^3 = \frac{3Q_0}{4\pi n_\mathrm{i}^2 \alpha_\mathrm{B}} = R_s^3 \left(\frac{n_0}{n_\mathrm{i}}\right)^2$$

R_s = Strömgren radius for the initial stage.

- Combining with $\dfrac{V_\mathrm{s}^2}{c_\mathrm{i}^2} = \dfrac{n_\mathrm{i}}{n_0}$, the equation for the expansion of the ionization front is

$$R^3 = R_s^3 \left(\frac{c_\mathrm{i}}{V_\mathrm{s}}\right)^4 \quad \longleftarrow \quad \frac{dR}{dt} = V_\mathrm{s}$$

$$\rho \equiv R/R_s, \quad \tau \equiv c_\mathrm{i} t/R_s \quad \longrightarrow \quad \rho^3 \left(\frac{d\rho}{d\tau}\right)^4 = 1 \quad \rightarrow \quad \boxed{\rho^{3/4} \frac{d\rho}{d\tau} = 1}$$

- For a suitable boundary condition, we assume that the initial Strömgren sphere is set up at $\tau_0 = c_\mathrm{i} t_0 / R_s$ (a very small fraction of the lifetime of the H II region):

$$R = R_s \text{ at } \tau = \tau_0$$

Then, the solution of the differential equation is

$$\rho = \left[1 + \frac{7}{4}(\tau - \tau_0)\right]^{4/7} \qquad R = R_s \left(1 + \frac{7}{4} \frac{t - t_0}{R_s/c_\mathrm{i}}\right)^{4/7}$$

- Expanding velocity is

$$\frac{dR}{dt} = c_{\rm i}\left(1 + \frac{7}{4}\frac{t - t_0}{R_s/c_{\rm i}}\right)^{-3/7}$$

- What is the time scale to reach the pressure equilibrium?

$$R(t_{\rm eq}) = R_{\rm f} \approx 34R_s$$

$$R_s\left(1 + \frac{7}{4}\frac{t_{\rm eq}}{R_s/c_{\rm i}}\right)^{4/7} \approx 34R_s$$

$$t_{\rm eq} \approx 273\,(R_s/c_{\rm i})$$

- The expanding velocity at this point is:

$$\boxed{V_{\rm s} = \frac{dR}{dt} = 0.71\,c_{\rm i} \quad \text{at} \quad t_{\rm eq} = 273R_s/c_{\rm i}}$$

Timescales for typical HII region

- Let's examine the case of an O7V star with

$$Q_0 = 10^{49}\,\mathrm{s}^{-1}, \quad n_0 = 10^2\,\mathrm{cm}^{-3}, \quad T = 10^4\,\mathrm{K}$$

- Initial state: recombination time scale $t_{\rm rec} = (n_0 \alpha_{\rm B})^{-1}$

$$R \approx R_s \text{ at } t = t_{\rm rec}$$
$$R_s \approx 3\,\mathrm{pc}\ (\approx 10^{19}\,\mathrm{cm})$$
$$t_{\rm rec} \approx 1000\,\mathrm{yr}$$

$$t \lesssim t_{\rm rec}$$
$$R(t) = R_s \left(1 - e^{-t/t_{\rm rec}}\right)^{1/3}$$
$$\frac{dR}{dt} = \frac{R_s}{3t_{\rm rec}} \frac{e^{-t/t_{\rm rec}}}{(1 - e^{-t/t_{\rm rec}})^{2/3}}$$

- Expansion phase: expansion timescale $t_{\rm exp} = R_s / c_{\rm i}$

$$\text{expansion velocity}: V_{\rm s} \leq 0.65\, c_{\rm i} \text{ at } t \geq t_{\rm exp}$$
$$c_{\rm i} \approx 10\,\mathrm{km\,s}^{-1}$$
$$t_{\rm exp} \approx 3 \times 10^5\,\mathrm{yr} \quad \rightarrow \quad t_{\rm exp} \approx 200\, t_{\rm rec}$$

$$t_{\rm rec} \lesssim t \lesssim t_{\rm eq} \qquad t_0 \approx t_{\rm rec} \ll t_{\rm exp}$$
$$R = R_s \left(1 + \frac{7}{4} \frac{t - t_0}{R_s / c_{\rm i}}\right)^{4/7}$$
$$\frac{dR}{dt} = c_{\rm i} \left(1 + \frac{7}{4} \frac{t - t_0}{R_s / c_{\rm i}}\right)^{-3/7}$$

- Final state: equilibrium timescale $t_{\rm eq} \approx 273 R_s / c_{\rm i}$ (from expansion phase model)

$$R = R_{\rm f} \text{ at } t = t_{\rm eq}$$
$$R_{\rm f} / R_s \approx 34$$
$$t_{\rm eq} \approx 10^8\,\mathrm{yr} \quad \rightarrow \quad t_{\rm eq} \approx 300\, t_{\rm exp}$$

Does the Stromgren sphere reach pressure equilibrium?

- Main-sequence lifetime of an ionizing star

$$t_{\rm MS} \approx 10^{10} \left(\frac{M}{M_\odot}\right)^{-2} {\rm yr} \qquad\qquad t_{\rm MS} \approx 10^{7}\,{\rm yr}\ \ {\rm for}\ M \approx 15 M_\odot$$

- Size
  - During the lifetime of an O star, which is less than 10 Myr, interstellar gas moving at 10 km/s will travel less than 100 pc, which is comparable with the diameter of the large H II regions.
  - Thus, *before an H II region has expanded very far, its central energy source will be extinguished*.
- Time Scale:
  - Main-sequence lifetime of an ionizing star is 10 times smaller than the time scale for the pressure equilibrium:

$$\boxed{t_{\rm MS} \approx 10^{7}\,{\rm yr} \ll t_{\rm eq} \approx 10^{8}\,{\rm yr}}$$

  - ***It is unlikely that the final state (pressure equilibrium) of H II region can be reached during lifetime of star.***

Gas Dynamics

- Gas Dynamics / Shock

Gas Dynamics - Energy Conservation

- ***Energy conservation***
  - The first law of thermodynamics states that

    Heat added in a system = Change in internal energy + Work done on surroundings

$$dQ = dU + PdV$$

  - Internal energy (per particle) for ideal gas is

$$U/N = \frac{3}{2}kT \quad \text{for monatomic gas (translation about 3 axes)}$$
$$U/N = \frac{5}{2}kT \quad \text{for diatomic gas} \quad \text{(+rotation about 2 axes)}$$
$$U/N = 3kT \quad \text{for polyatomic gas (+rotation about 3 axes)}$$

Here, N is the number of particles.

An ideal gas is a theoretical gas composed of many randomly moving point particles whose only interactions are perfectly elastic collisions (no viscosity or heat conduction).

  - In general, the internal energy per particle is

$$U/N = \frac{f}{2}kT \quad (f = \text{degree of freedom})$$

At high temperature, molecules have access to an increasing number of vibrational degrees of freedom, as they start to bend and stretch.

- The ideal gas law (the equation of state) for a perfect Maxwellian distribution.

$$\boxed{PV = NkT} \qquad P = \frac{N}{V}kT$$

- ***Specific heat capacity*** is the amount of ***heat energy required to raise the temperature of a material per unit of mass***.

  - specific heat capacity **at constant volume**:

$$c_V \equiv \frac{1}{M}\left(\frac{\partial Q}{\partial T}\right)_V = \frac{1}{M}\left(\frac{\partial U}{\partial T}\right)_V \qquad \boxed{c_V = \frac{f}{2}\frac{k}{m}}$$

$$M = \text{total mass}$$

$$m = M/N = \text{mass per particle}$$

$$m = \mu m_{\rm H}$$

$$(\mu = \text{mean atomic weight per particle})$$

  - specific heat capacity **at constant pressure**:

$$c_P \equiv \frac{1}{M}\left(\frac{\partial Q}{\partial T}\right)_P = \frac{1}{M}\left(\frac{\partial U}{\partial T}\right)_P + \frac{P}{M}\left(\frac{\partial V}{\partial T}\right)_P = \frac{1}{M}\frac{f}{2}Nk + \frac{P}{M}\frac{Nk}{P}$$

$$\boxed{\therefore \quad c_P = \frac{f+2}{2}\frac{k}{m} = c_V + \frac{k}{m}}$$

  - Ratio of specific heat capacities:

$$\begin{aligned} \gamma \equiv \frac{c_P}{c_V} = \frac{f+2}{f} &= \frac{5}{3} \quad \text{for monatomic gas} \\ &= \frac{7}{5} \quad \text{for diatomic gas} \\ &= \frac{4}{3} \quad \text{for polyatomic gas} \end{aligned}$$

γ is called the *adiabatic index*.

$$c_P > c_V$$

This inequality implies that when pressure is held constant, some of the added heat goes into PdV work instead of into internal energy.

- Energy Conservation - limiting cases
  - ***Adiabatic flow*** - negligible heat transport (Internal energy is changed only by work).

$$dQ = dU + PdV = Mc_V dT + PdV$$
$$dQ = 0$$
$$\rightarrow PdV = -Mc_V dT$$

$$PV = NkT$$
$$\rightarrow VdP + PdV = NkdT$$

We combine two equation and eliminate dT term:

$$\begin{aligned} VdP + PdV &= -\frac{Nk}{Mc_V} PdV \\ &= -\frac{k}{m}\frac{1}{c_V} PdV \end{aligned} \longrightarrow \begin{aligned} VdP &= -\left(1 + \frac{k}{m}\frac{1}{c_V}\right) PdV \\ &= -\frac{1}{c_V}\left(c_V + \frac{k}{m}\right) PdV \\ &= -\gamma PdV \end{aligned}$$

$$\longrightarrow \frac{dP}{P} = -\gamma \frac{dV}{V}$$

We can rewrite this in terms of density:

$$\rho V = M$$
$$\rightarrow \rho dV + V d\rho = 0$$
$$\rightarrow \frac{d\rho}{\rho} = -\frac{dV}{V} \longrightarrow \frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$

In summary,

$$\frac{dP}{P} = -\gamma \frac{dV}{V} \qquad P \propto \rho^\gamma$$
$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho} \qquad P \propto V^{-\gamma}$$
$$\rightarrow T \propto V^{-(\gamma-1)}$$

adiabatic heating/cooling

- ***Isothermal flow*** - extremely efficient cooling (heat transport).

heat transport timescale << dynamic timescale

This implies the balance between heating and cooling, hence a constant temperature.

From the ideal gas law,

$$P = \frac{N}{V} kT = \rho \frac{kT}{m} \longrightarrow \begin{array}{l} P \propto \rho \\ P \propto V^{-1} \end{array}$$

- **In general**, we have

$$\begin{array}{l} P \propto \rho^{\gamma} \\ P \propto V^{-\gamma} \end{array} \quad (\gamma = 1 \text{ for isothermal gas})$$

A gas that has an equation of state with this power-law form is called a ***polytope***, from the Greek polytropos, meaning "turning many ways" or "versatile."

(A polytope should not be confused with a polytrope, which is the n-dimensional generalization of a 2D polygon and 3D polyhedron.)

- ***Specific internal energy*** of the gas (***per unit mass***):

$$\epsilon \equiv U/M$$
$$U/N = \frac{f}{2}kT$$

$$\longrightarrow \quad \epsilon = \frac{f}{2}\frac{kT}{m} \quad \text{or} \quad \epsilon = \frac{1}{\gamma - 1}\frac{kT}{m} = \frac{1}{\gamma-1}\frac{P}{\rho}$$

- ***Total Energy (per unit volume)***:
    - ***Internal energy per unit volume***:
    - ***Kinetic energy due to bulk motion, per unit volume***:
    - ***Work on unit volume***:

$$\mathcal{E}_{\rm int} = \rho\epsilon = \frac{1}{\gamma-1}P$$
$$\mathcal{E}_{\rm kin} = \rho\frac{u^2}{2}$$
$$\mathcal{E}_{\rm mech} = \frac{PdV}{dV} = P$$

$$\boxed{\begin{aligned}\mathcal{E} &= \mathcal{E}_{\rm int} + \mathcal{E}_{\rm kin} + \mathcal{E}_{\rm mech} \\ &= \rho\left(\frac{u^2}{2} + \epsilon\right) + P \end{aligned} \quad \longrightarrow \quad \mathcal{E} = \rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P}$$

- ***Energy conservation***:

$$\frac{\partial \mathcal{E}}{\partial t} = -\frac{\partial (u\mathcal{E})}{\partial x}$$

$$\boxed{\frac{\partial}{\partial t}\left(\rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P\right) = -\frac{\partial}{\partial x}\left[u\left(\rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P\right)\right]}$$

Sound Wave

- Suppose that we are surrounded by an ideal gas with a plane parallel symmetry:
  - We consider a region where the gas has initially a uniform density, pressure, and no bulk velocity: $\rho_0, \ P_0, \ u_0 = 0$

    In the uniform gas, we introduce small perturbations of the form:

$$\rho(x,t) = \rho_0 + \rho_1(x,t)$$
$$u(x,t) = u_1(x,t)$$
$$P(x,t) = P_0 + P_1(x,t)$$

$$\longrightarrow \quad P_1 = P - P_0 \propto (\rho_0 + \rho_1)^\gamma - \rho_0^\gamma \propto \gamma \rho_0^{\gamma-1} \rho_1 \quad \longrightarrow \quad P_1 = \frac{\gamma P_0}{\rho_0} \rho_1$$

We obtain:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$
$$\rho \frac{\partial u}{\partial t} = -\rho u \frac{\partial u}{\partial x} - \frac{\partial P}{\partial x}$$

$$\longrightarrow \quad \frac{\partial \rho_1}{\partial t} = -\rho_0 \frac{\partial u_1}{\partial x}$$
$$\rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial P_1}{\partial x} = -\frac{\gamma P_0}{\rho_0} \frac{\partial \rho_1}{\partial x}$$

$$\longrightarrow \quad \frac{\partial^2 \rho_1}{\partial t^2} = -\frac{\gamma P_0}{\rho_0} \frac{\partial^2 \rho_1}{\partial x^2}$$

  - The resulting equation represents a **sound wave (acoustic wave)** with a constant sound speed:

$$c_s = \left(\frac{\gamma P}{\rho}\right)^{1/2} = \left(\frac{\gamma k T}{m}\right)^{1/2}$$

$$c_s \propto \rho^{(\gamma-1)/2}$$

For $\gamma > 1$ sound travels more rapidly in a denser gas.

- The sound speed is of the same order as the mean thermal velocity:

$$c_s = 1.2\,\mathrm{km\,s^{-1}} \left(\frac{\gamma}{5/3}\right)^{1/2} \left(\frac{m}{m_p}\right)^{-1/2} \left(\frac{T}{100\,\mathrm{K}}\right)^{1/2}$$

$$(m_p = \text{proton mass})$$

- ***Sound crossing time***:
  - sound crossing time = time it takes for a signal to cross a region of size *L*:

$$t_{\rm cross} = L/c_s$$

  - A small pressure gradient tends to be smoothed out within the sound crossing time. Generally, when a stationary gas is disturbed, the resultant changes in velocity, density, pressure, and temperature are communicated downstream at the sound speed.

    Fast changes occurring on timescales $\ll t_{\rm cross}$ *will survive, and a shock front forms.*

    Slow changes occurring on timescales $\gg t_{\rm cross}$ will be damped.

- ***Mach number*** = gas velocity / sound speed

$$\boxed{\mathcal{M} \equiv u/c_s}$$

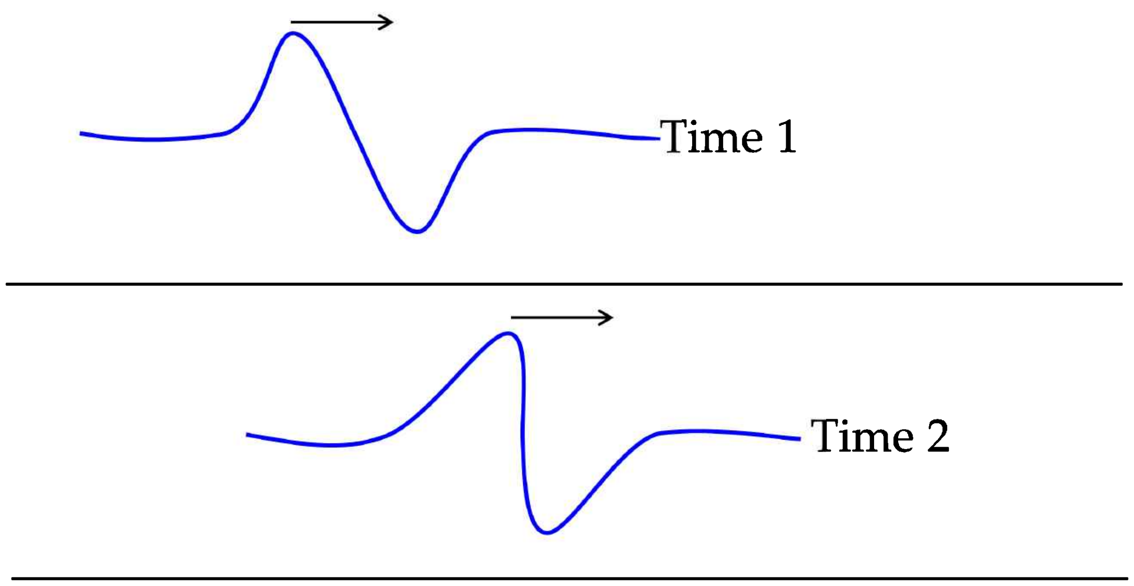
$\mathcal{M} > 1$ supersonic

$\mathcal{M} < 1$ subsonic

Shock

- Shock
  - A low-amplitude sound wave traveling through a medium will be adiabatic; that is it will not increase the entropy of the gas through which it passes.
  - For an adiabatic process, the equation of state for the gas is

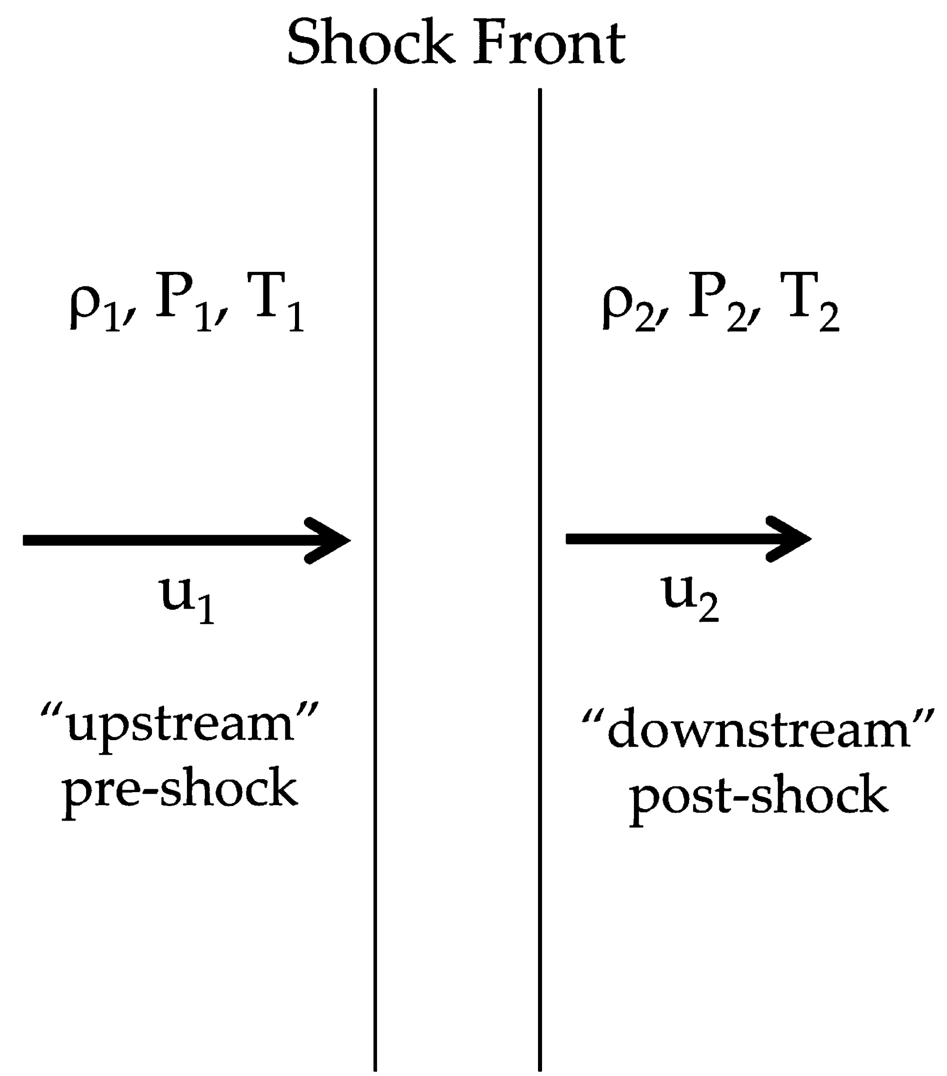
$$c_s \propto \rho^{(\gamma-1)/2}$$



  - Thus, for $\gamma > 1$, sound travels more rapidly in a denser gas.
  - ***For a supersonic gas, the motion itself is faster than the speed of communication, and instead of a smooth transition, the physical quantities (density, pressure, and temperature) undergo a sudden change in values over a small distance.*** This phenomenon is referred to as a ***shock***.
  - We define the shock front as the region over which the velocity, density, and pressure of the gas undergo sudden changes. The shock front is a layer whose thickness is comparable to the mean free path between particle collisions.
  - The ordinary sound that we hear every day will not, in practice, steepen into shocks.
  - However, high amplitude pressure fluctuations will rapidly steepen into shocks.

Shock Front

- Jump condition (***Rankine-Hugoniot conditions***)
  - Let
    $$\rho = \text{mass density},\ T = \text{temperature},$$
    $$m = \text{mean molecular mass}$$
  - If a patch is small compared to the shock front's radius of curvature, then we can treat the shock front as if it has ***plane parallel*** symmetry.
  - ***It is convenient to use a frame of reference in which the shock front is stationary.***
  - Let us consider a shock propagating with velocity V_s into a gas that is previously at rest. In the frame of reference of the shock, the gas in the pre-shock region is approaching at a velocity of $-V_s$.
  - In this frame, the bulk velocity $u_1 = -V_s$ of the pre-shock (upstream) gas toward the shock front. The bulk velocity u_2 of the post-shock (downstream) gas points away from the shock front.



Plane parallel steady-state shock, in the reference frame of the shock front.

- Let's consider a steady state solution.
  - The gas properties immediately before being shocked ("1") and immediately after being shocked ("2") are obtained from the conservation laws:

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$
$$u_1 \left(\rho_1 \frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} P_1 \right) = u_2 \left(\rho_2 \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} P_2 \right)$$

Dividing the third equation with the first equation:

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}$$

In summary,

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$
$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}$$

Here, *we assume that an adiabatic index is the same on both sides of the shock front.*

- From the three equations, we should be able to derive the changes, ρ_2/ρ_1, u_2/u_1, and P_2/P_1 across the shock.

  It is convenient to use a dimensionless number, the Mach number of the upstream:

$$\mathcal{M}_1 = u_1/c_1,\ c_1^2 = \frac{\gamma P_1}{\rho_1} \quad \rightarrow \quad P_1 = \frac{u_1^2 \rho_1}{\gamma \mathcal{M}_1^2}$$

(1) To find the equation for densities:

$$\rho_1 u_1 = \rho_2 u_2 \text{ and } P_1 = \frac{u_1^2\rho_1}{\gamma\mathcal{M}_1^2} \qquad \begin{aligned} & \rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \\ \rightarrow\ & \rho_1 u_1^2 + u_1^2 \frac{\rho_1}{\gamma\mathcal{M}_1^2} = \frac{(\rho_1 u_1)^2}{\rho_2} + P_2 \\ \rightarrow\ & P_2 = \rho_1 u_1^2 \left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right) \end{aligned}$$

Inserting these relations into the energy conservation equation:

$$\begin{aligned} & \frac{u_1^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2} \\ \rightarrow\ & \frac{u_1^2}{2} + \frac{1}{\gamma-1}\frac{u_1^2}{\mathcal{M}_1^2} = \frac{1}{2}\left(\frac{\rho_1 u_1}{\rho_2}\right)^2 + \frac{\gamma}{\gamma-1}\frac{\rho_1 u_1^2}{\rho_2}\left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right) \\ \rightarrow\ & \frac{1}{2} + \frac{1}{\gamma-1}\frac{1}{\mathcal{M}_1^2} = \frac{1}{2}\left(\frac{\rho_1}{\rho_2}\right)^2 + \frac{\gamma}{\gamma-1}\frac{\rho_1}{\rho_2}\left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right) \end{aligned}$$

$$ax^2 + bx - c = 0$$

where $x = \dfrac{\rho_1}{\rho_2}$

$$a = \frac{1}{2} - \frac{\gamma}{\gamma-1}$$

$$b = \frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_1^2}$$

$$c = \frac{1}{2} + \frac{1}{(\gamma-1)\mathcal{M}_1^2}$$

$$x = \frac{b^2 \pm \sqrt{b^2 + 4ac}}{2a}$$

$$\frac{\rho_1}{\rho_2} = \frac{-\left[\frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_0^2}\right] \pm \frac{\mathcal{M}_1^2-1}{\mathcal{M}_1^2(\gamma-1)}}{1-\frac{2\gamma}{\gamma-1}}$$

$$= \frac{\gamma\mathcal{M}_1^2 + 1 \mp (\mathcal{M}_1^2 - 1)}{(\gamma+1)\mathcal{M}_1^2}$$

$$\longrightarrow \quad \frac{\rho_1}{\rho_2} = 1 \quad \text{or} \quad \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2}$$

$$\therefore \quad \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}$$

(2) Now, we obtain the equation for pressures:

Divide the following equation

$$P_2 = \rho_1 u_1^2 \left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right)$$

with this

$$P_1 = \frac{u_1^2 \rho_1}{\gamma\mathcal{M}_1^2}$$

$$\longrightarrow \quad \frac{P_2}{P_1} = \gamma\mathcal{M}_1^2 \left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right)$$

$$= \gamma\mathcal{M}_1^2 \left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2}\right)$$

$$\therefore \quad \frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1}$$

(3) Using the ideal gas law:

$$P = \frac{\rho k T}{m} \quad \longrightarrow \quad \frac{T_2}{T_1} = \frac{\rho_1}{\rho_2}\frac{P_2}{P_1}$$

Using the equations for densities and pressures:

$$\therefore \quad \frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2 + 2\right]\left[2\gamma\mathcal{M}_1^2 - (\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2}$$

In summary, we obtain the jump conditions:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2} = \frac{u_1}{u_2}$$
$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1}$$
$$\frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2+2\right]\left[2\gamma\mathcal{M}_1^2-(\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2}$$

In the ***lab frame***, let V_s = shock velocity, $v_1,\ v_2$ = gas velocities in upstream (pre-shock) and downstream (post-shock), respectively $(v_1 = 0)$.

Using $u_1 = -V_s$ and $u_2 = v_2 - V_s$, we have

$$\frac{-V_s}{v_2 - V_s} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2}$$

Downstream velocity in the lab frame:

$$v_2 = \frac{2(\mathcal{M}_1^2-1)}{(\gamma+1)\mathcal{M}_1^2}V_s$$

Note a typo in Equation (16.12) of Kwok's book.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2} = \frac{u_1}{u_2}$$
$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1}$$
$$\frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2+2\right]\left[2\gamma\mathcal{M}_1^2-(\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2}$$

monatomic gas: $\gamma = 5/3$

For a strong shock: $\mathcal{M}_1 \gg 1$

$$P_2 \approx \frac{2\gamma\mathcal{M}_1^2}{\gamma+1}P_1 \xrightarrow{P_1=c_1^2\frac{\rho_1}{\gamma}} \frac{2\gamma(u_1/c_1)^2}{\gamma+1} \quad c_1^2\frac{\rho_1}{\gamma}$$

$$T_2 \approx \frac{2(\gamma-1)\gamma}{(\gamma+1)^2}\mathcal{M}_1^2 T_1 = \frac{2(\gamma-1)\gamma}{(\gamma+1)^2}\left(\frac{u_1}{c_1}\right)^2 T_1$$

$$\frac{\rho_2}{\rho_1} \simeq \frac{\gamma+1}{\gamma-1}$$
$$\frac{u_2}{u_1} \simeq \frac{\gamma-1}{\gamma+1}$$
$$P_2 \simeq \frac{2}{\gamma+1}\rho_1 u_1^2$$
$$T_2 \simeq \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{m}{k}u_1^2$$

$$\frac{\rho_2}{\rho_1} \simeq 4$$
$$\frac{u_2}{u_1} \simeq \frac{1}{4}$$
$$P_2 \simeq \frac{3}{4}\rho_1 u_1^2$$
$$T_2 \simeq \frac{3}{16}\frac{m}{k}u_1^2$$

speed of the downstream in the laboratory frame:

$$v_2 \simeq \frac{2}{(\gamma+1)}V_s$$

$$v_2 \simeq \frac{3}{4}V_s$$

For an isothermal shock: $\gamma = 1$

$$\frac{\rho_2}{\rho_1} = \mathcal{M}_1^2 = \frac{u_1}{u_2}$$
$$P_2 = \mathcal{M}_1^2 P_1 = \rho_1 u_1^2 \longrightarrow \quad u_1 u_2 = c_1^2, \quad c_2 = c_1$$
$$T_2 = T_1$$

speed of the downstream in the laboratory frame:

$$v_2 = \left(1 - \frac{1}{\mathcal{M}_1^2}\right)V_s$$

- Consider a strong shock
  - ***No matter how strong the shock is, the gas can only be compressed by a factor of at most 4***:

$$\frac{\rho_2}{\rho_1} \approx 4 \qquad \text{for } \gamma = 5/3$$

(monatomic gas)

$$P_2 \approx \frac{3}{4}\rho_1 u_1^2$$

$$T_2 \approx \frac{3}{16}\frac{m}{k}u_1^2$$

Note that the mean molecular mass (mass per particle) is

$$m = \frac{1.4m_{\rm H}}{1.1} = 1.273m_{\rm H} \quad \text{for neutral gas}$$

$$m = \frac{1.4m_{\rm H}}{2.3} = 0.609m_{\rm H} \quad \text{for ionized gas}$$

$n \simeq 2.3n_{\rm H}$ for ionized gas,
one electron from an ionized hydrogen
two electrons from a doubly-ionized helium.

  - In the lab frame, $V_{\rm s}$ = shock velocity, v_1, v_2 = gas velocities in upstream and downstream, respectively.

$$u_1 = v_1 - V_s = -V_s \;\; (v_1 = 0)$$

$$u_2 = v_2 - V_s$$

Then, the post-shock velocity is

$$\frac{u_2}{u_1} = \frac{v_s - V_{\rm s}}{-V_{\rm s}} = \frac{1}{4} \quad \Rightarrow \quad v_2 = \frac{3}{4}V_{\rm s}$$

Hence, ***the post-shock moves in the same direction as the shock front with a velocity of 3/4 of the shock velocity.***

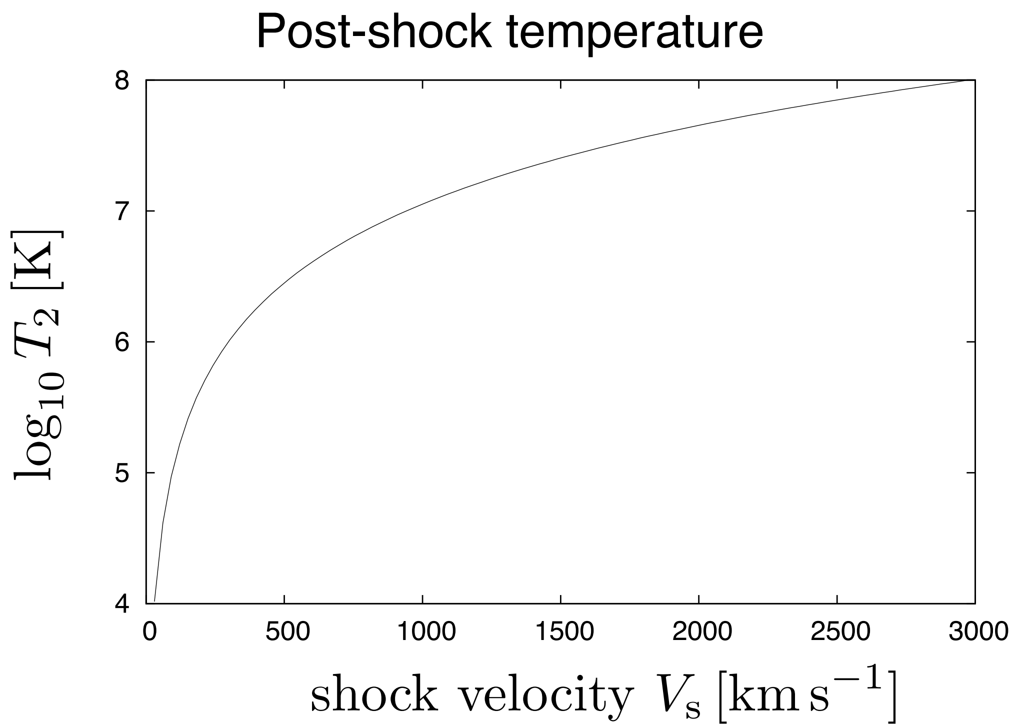
- Then, the post-shock pressure, temperature, specific internal energy, and specific kinetic energy are, respectively,

$$P_2 = \frac{3}{4}\rho_1 V_s^2$$
$$T_2 = \frac{3m}{16k} V_s^2$$

for $\gamma = 5/3$

$$\frac{\mathcal{E}_{\rm int,2}}{\rho_2} = \epsilon_2 = \frac{3}{2}\frac{P_2}{\rho_2} = \frac{3}{2}\frac{(3/4)\rho_1 V_s^2}{4\rho_1}$$
$$\frac{\mathcal{E}_{\rm kin,2}}{\rho_2} = \frac{1}{2}v_2^2$$

$$\frac{\mathcal{E}_{\rm int,2}}{\rho_2} = \frac{9}{32}V_s^2$$
$$\frac{\mathcal{E}_{\rm kin,2}}{\rho_2} = \frac{9}{32}V_s^2$$



- A strong shock can produce very high pressures and temperatures. An interstellar shock front with propagation speed $V_s \sim 1000$ km s^{-1} (typical for a supernova shock wave) produces shock heated gas with

$$T_2 \approx 1.38 \times 10^7\,{\rm K}\left(\frac{m}{0.609 m_{\rm H}}\right)\left(\frac{V_s}{1000\,{\rm km\,s^{-1}}}\right)^2$$ or $$T_2 \approx 1.38 \times 10^5\,{\rm K}\left(\frac{m}{0.609 m_{\rm H}}\right)\left(\frac{V_s}{100\,{\rm km\,s^{-1}}}\right)^2$$

assuming the shocks gas is fully ionized hydrogen.

- In general, shock fronts convert supersonic gas into subsonic gas in the shock's frame of reference. Shocks increase density, pressure, and temperature, and decrease bulk velocity relative to the shock front. *Shocks act as entropy generators.*

Homework (due date: 05/26)

[Q13]

The "cooling time" $\tau_{\rm cool} \equiv |d\ln T/dt|^{-1}$. Suppose the power radiated per unit volume Λ can be approximated by

$$\Lambda \approx A n_{\rm H} n_e \left[T_6^{-0.7} + 0.021 T_6^{1/2} \right]$$

for gas of cosmic abundances, where $A = 1.1\times10^{-22}\,{\rm erg\,cm^3\,s^{-1}}$, and $T_6 \equiv T/10^6\,{\rm K}$. Assume the gas to have $n_{\rm He} = 0.1 n_{\rm H}$, with both H and He fully ionized.

Compute the cooling time (at constant pressure) due to radiative cooling

(a) in a supernova remnant at $T = 10^7\,{\rm K}$, $n_{\rm H} = 10^{-2}\,{\rm cm^{-3}}$.

(b) for intergalactic gas within a dense galaxy cluster (the "intracluster medium") with $T = 10^8\,{\rm K}$, $n_{\rm H} = 10^{-3}\,{\rm cm^{-3}}$.

[Q14]

Consider a strong shock wave propagating into a medium that was initially at rest. Assume the gas to be monatomic ($\gamma = 5/3$). Consider the material just behind the shock front. The gas has an energy density $u_{\rm thermal}$ from random thermal motions, and an energy density $u_{\rm flow}$ from the bulk motion of the shocked gas. If cooling is negligible, calculate the ratio $u_{\rm flow}/u_{\rm thermal}$ in the frame of reference where the shock front is stationary.