

Astrophysics

Lecture 11

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[Inverse Compton Power for Single Scattering]

- **Assumptions:**

- (1) Isotropic distributions of photons and electrons.
- (2) The change in energy of the photon in the electron's rest frame is negligible.

(Thomson scattering is applicable in the electron's rest frame). $\epsilon'_1 \approx \epsilon'$ or $h\nu'_1 = h\nu'$

- **Total Inverse Compton Power:**

We first note that the power is invariant because the energy and time transform in the same way:

$$\frac{dE_1}{dt} = \frac{dE'_1}{dt'} \rightarrow P = P'$$

Therefore, the **power contained in the scattered radiation** in the laboratory frame can be expressed in terms of the quantities in the electron's rest frame.

$$P' = c\sigma_T \int \epsilon'_1 f'_{ph}(\mathbf{p}') d^3\mathbf{p}'$$

Here, $f'_{ph}(\mathbf{p}')$ is the number density of photons with momentum between \mathbf{p}' and $\mathbf{p}' + d\mathbf{p}'$ and $c f'_{ph}(\mathbf{p}') d^3\mathbf{p}'$ is the incident photon flux in the electrons' rest frame.

$$f_{ph} = \frac{dN_{ph}}{d^3\mathbf{x} d^3\mathbf{p}}$$

The number density is invariant because the number of particles and the phase space volume element are Lorentz invariants.

$$f'_{ph}(\mathbf{p}') = f_{ph}(\mathbf{p})$$

Recall that **the momentum volume element transforms in the same way as the energy under Lorentz transforms**. Therefore,

$$d^3\mathbf{p}' = \gamma (1 - \beta \cos \theta) d^3\mathbf{p}$$

The scattered photon energy in the electron rest frame can be expressed in terms of the incident photon energy in the lab frame:

$$\begin{aligned}\epsilon'_1 &= \epsilon' \quad (\text{Thomson scattering}) \\ &= \gamma (1 - \beta \cos \theta) \epsilon \quad (\text{Lorentz transform})\end{aligned}$$

Then the power contained in the scattered radiation is

$$\begin{aligned}P = P' &= c\sigma_T \gamma^2 \int \epsilon (1 - \beta \cos \theta)^2 f_{\text{ph}}(\mathbf{p}) d^3\mathbf{p} \\ &= c\sigma_T \gamma^2 \int \int \epsilon (1 - \beta \cos \theta)^2 f_{\text{ph}}(p) p^2 dp d\Omega \\ &= c\sigma_T \gamma^2 \left[\int \epsilon f_{\text{ph}}(p) p^2 dp (4\pi) \right] \left[\int (1 - \beta \cos \theta)^2 \frac{d\Omega}{4\pi} \right]\end{aligned}$$

For an isotropic distribution of photons, integrating over θ , we have the last term:

$$\langle (1 - \beta \cos \theta)^2 \rangle = 1 + \frac{1}{3} \beta^2 \quad \leftarrow \quad \langle \cos \theta \rangle = 0, \quad \langle \cos^2 \theta \rangle = 1/3$$

Finally, we obtain the power contained in the scattered radiation given by

$$P = c\sigma_T \gamma^2 \left(1 + \frac{1}{3}\beta^2\right) U_{\text{ph}}$$

where

$$U_{\text{ph}} = \int \epsilon f_{\text{ph}}(p) p^2 dp / (4\pi) = \int \epsilon f_{\text{ph}}(\mathbf{p}) d^3\mathbf{p}$$

is the initial photon energy density.

- **Mean photon energy after the inverse Compton scattering:**

Now, we consider the ***monochromatic*** radiation field with energy $\epsilon = h\nu = \text{constant}$. Then, the initial photon energy density is

$$U_{\text{ph}} = \epsilon \int f_{\text{ph}}(\mathbf{p}) d^3\mathbf{p} = \epsilon n_{\text{ph}} \quad \longleftarrow \quad n_{\text{ph}} \equiv \int f_{\text{ph}} d^3\mathbf{p} = \int \frac{dN_{\text{ph}}}{d^3\mathbf{x} d^3\mathbf{p}} d^3\mathbf{p}$$

The scattering power can be expressed by the multiplication of the number scattering events per unit time and the mean photon energy after scattering.

$$P = \langle \epsilon_1 \rangle \times \frac{dN_{\text{scatt}}}{dt}$$

The **scattering rate** in the lab frame is related to that in the electron's rest frame.

$$\frac{dN_{\text{scatt}}}{dt} = \frac{1}{\gamma} \frac{dN'_{\text{scatt}}}{dt'} \quad \text{because the number of event is invariant: } dN_{\text{scatt}} = dN'_{\text{scatt}}$$

The scattering rate in the electron's rest frame is calculated, as done for the scattering power:

$$\begin{aligned} \frac{dN'_{\text{scatt}}}{dt'} &= c\sigma_T \int f'_{\text{ph}}(\mathbf{p}') d^3\mathbf{p}' = c\sigma_T \gamma \int (1 - \beta \cos \theta) f_{\text{ph}}(\mathbf{p}) d^3\mathbf{p} \\ &= c\sigma_T \gamma n_{\text{ph}} \end{aligned}$$

Then, we obtain the scattering rate:

$$\boxed{\frac{dN_{\text{scatt}}}{dt} = c\sigma_T n_{\text{ph}}}$$

The mean energy after the scattering is thus obtained to be

$$\begin{aligned} \langle \epsilon_1 \rangle &= P \sqrt{\frac{dN_{\text{scatt}}}{dt}} \\ &= \gamma^2 \left(1 + \frac{1}{3} \beta^2 \right) \epsilon \end{aligned}$$

- The net power that is lost by the electron and then converted into radiation is

$$P_{\text{IC}} \equiv \frac{dE_1}{dt} - \frac{dE^{\text{initial}}}{dt} = c\sigma_T U_{\text{ph}} \left[\gamma^2 \left(1 + \frac{1}{3}\beta^2 \right) - 1 \right]$$

$$\therefore P_{\text{IC}} = \frac{4}{3}c\sigma_T\gamma^2\beta^2U_{\text{ph}}$$

$$\leftarrow \qquad \gamma^2 - 1 = \gamma^2\beta^2$$

- Net energy gain (transfer) by nonrelativistic thermal electrons with temperature T :

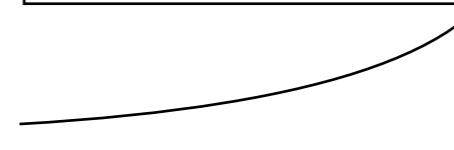
The fractional photon energy gain for a single electron velocity is

$$\frac{\Delta\epsilon}{\epsilon} = \frac{\langle h\nu_1 \rangle - h\nu}{h\nu} = \frac{4}{3}\gamma^2\beta^2$$

The total energy gain can be obtained by integrating over the Maxwell velocity distribution and assuming that $\gamma = 1$.

$$\left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle \simeq \frac{4}{3} \langle \beta^2 \rangle = \frac{4kT}{m_e c^2}$$

$$\langle \beta^2 \rangle = \langle v^2/c^2 \rangle = 3kT/m_e c^2$$



Correction to the Thomson-limit:

We note that the above result was obtained, assuming that the photon energy is not lost in the electron's rest frame. Now, we will make a correction to the energy gain by considering the photon energy loss due to the electron recoil. In the electron's rest frame, the photon energy will be lost:

$$\epsilon'_1 \approx \epsilon' \left[1 - \frac{\epsilon'}{m_e c^2} (1 - \cos \Theta) \right] \rightarrow \left\langle \frac{\Delta \epsilon'}{\epsilon'} \right\rangle \equiv \left\langle \frac{\epsilon'_1 - \epsilon'}{\epsilon'} \right\rangle = -\frac{\epsilon'}{m_e c^2} \quad : \text{after making an angle average}$$

Hence, **net energy transfer** (including the correction term) is

$$\boxed{\left\langle \frac{\Delta \epsilon}{\epsilon} \right\rangle = \frac{4kT - \epsilon}{m_e c^2}}$$

Note that if the electrons have high enough temperature relative to incident photons, the photons gain energy. This is the inverse Compton scattering.

If $\epsilon > 4kT$, on the other hand, energy is transferred from photons to electrons.

[Repeated Scattering: The Compton y Parameter]

- We restrict our considerations to situations in which the Thomson limit applies: $\gamma\epsilon \ll mc^2$
- **Compton y parameter**, to determine whether a photon will significantly change its energy in traversing the medium:

$$y \equiv \left(\begin{array}{l} \text{average fractional} \\ \text{energy change per} \\ \text{scattering} \end{array} \right) \times \left(\begin{array}{l} \text{mean number of} \\ \text{scatterings} \end{array} \right)$$

When $y \gtrsim 1$, the total photon energy and spectrum will be significantly altered; whereas for $y \ll 1$, the total energy is not much changed.

- **Average fractional energy gain per scattering** (for a thermal distribution of electrons)
 - (a) Consider first the nonrelativistic limit.

$$\left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle = \frac{4kT}{m_e c^2}$$

(b) In the ultrarelativistic limit ($\gamma \gg 1, \beta \approx 1$), ignoring the energy transfer in the electron rest frame,

$$\frac{P_{\text{compt}}}{|dE_1^{\text{loss}}/dt|} = \frac{4/3\sigma_T c \gamma^2 \beta^2 U_{\text{ph}}}{\sigma_T c U_{\text{ph}}} = \frac{4}{3} \gamma^2 \beta^2 \rightarrow (\Delta\epsilon)_R \approx \frac{4}{3} \gamma^2 \epsilon$$

For a thermal distribution of ultrarelativistic electrons,

$$\langle \gamma^2 \rangle = \frac{\langle \epsilon^2 \rangle}{(m_e c^2)^2} = 12 \left(\frac{kT}{m_e c^2} \right)^2 \rightarrow (\Delta\epsilon)_R \approx 16\epsilon \left(\frac{kT}{m_e c^2} \right)^2$$

Note that the relativistic Maxwell distribution (Maxwell-Jüttner distribution) is given by

$$f(\gamma) = \frac{\gamma^2 \beta}{\theta K_2(1/\theta)} e^{-\gamma/\theta}$$

where $\beta = \sqrt{1 - 1/\gamma^2}$, $\theta = \frac{kT}{mc^2}$, and

K_2 is the modified Bessel function.

- **Mean number of scatterings**,

Recall that, for a pure scattering medium,

$$\left(\begin{array}{c} \text{mean number of} \\ \text{scatterings} \end{array} \right) \approx \text{Max}(\tau_{\text{es}}, \tau_{\text{es}}^2)$$

where $\tau_{\text{es}} \approx \rho \kappa_{\text{es}} R$

$$\rho = m_p n_H (n_H = n_e)$$

$$\kappa_{\text{es}} = \frac{\sigma_T}{m_p} = 0.40 \text{ cm}^2 \text{ g}^{-1} \text{ for ionized hydrogen}$$

R = size of the finite medium

$$y_{\text{NR}} = \frac{4kT}{m_e c^2} \text{Max}(\tau_{\text{es}}, \tau_{\text{es}}^2)$$

$$y_R = 16\epsilon \left(\frac{kT}{m_e c^2} \right)^2 \text{Max}(\tau_{\text{es}}, \tau_{\text{es}}^2)$$

[Repeated Scattering: Spectra and Comptonization]

- We have already shown that a power-law spectrum may be a natural consequence of a power-law distribution of electrons.
- **We will show that a power-law photon distribution can also be produced from repeated scattering off a nonpower-law electron distribution.**

Let A = the mean amplification of photon energy per scattering

$$A \equiv \frac{\epsilon_1}{\epsilon} \sim \frac{4}{3} \langle \gamma^2 \rangle = 16 \left(\frac{kT}{m_e c^2} \right)^2 \quad \text{for thermal electron distribution}$$

mean photon energy = ϵ_i

(number) intensity = $I(\epsilon_i)$ at ϵ_i

After k scattering, the photon energy will be $\epsilon_k \sim \epsilon_i A^k$.

For an optically thin scattering medium ($\tau_{\text{es}} < 1$), the probability of a photon undergoing k scattering before escaping the medium is $p_k(\tau_{\text{es}}) \sim \tau_{\text{es}}^k$.

The emergent intensity at energy ϵ_k is given by

$$I(\epsilon_k) \sim I(\epsilon_i) \tau_{\text{es}}^k \sim I(\epsilon_i) \tau_{\text{es}}^{\ln(\epsilon_k/\epsilon_i)/\ln A} = I(\epsilon_i) \left(\frac{\epsilon_k}{\epsilon_i} \right)^{\ln \tau_{\text{es}} / \ln A}$$

$$\begin{aligned} \tau^{\ln(\epsilon_k/\epsilon_i)/\ln A} &= x \\ [\ln(\epsilon_k/\epsilon_i) / \ln A] \ln \tau &= \ln x \\ \ln(\epsilon_k/\epsilon_i)^{\ln \tau / \ln A} &= \ln x \\ (\epsilon_k/\epsilon_i)^{\ln \tau / \ln A} &= x \end{aligned}$$

$\therefore I(\epsilon_k) \sim I(\epsilon_i) \left(\frac{\epsilon_k}{\epsilon_i} \right)^{-\alpha}$ where $\alpha \equiv -\frac{\ln \tau_{\text{es}}}{\ln A}$

→ power-law shape

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- Total Compton power in the output spectrum is given by

$$P \propto \int I(\epsilon_k) d\epsilon_k = I(\epsilon_i) \epsilon_i \left[\int x^{-\alpha} dx \right]$$

The factor in square brackets is approximately **the factor by which the initial power $I(\epsilon_i) \epsilon_i$ is amplified in energy.**

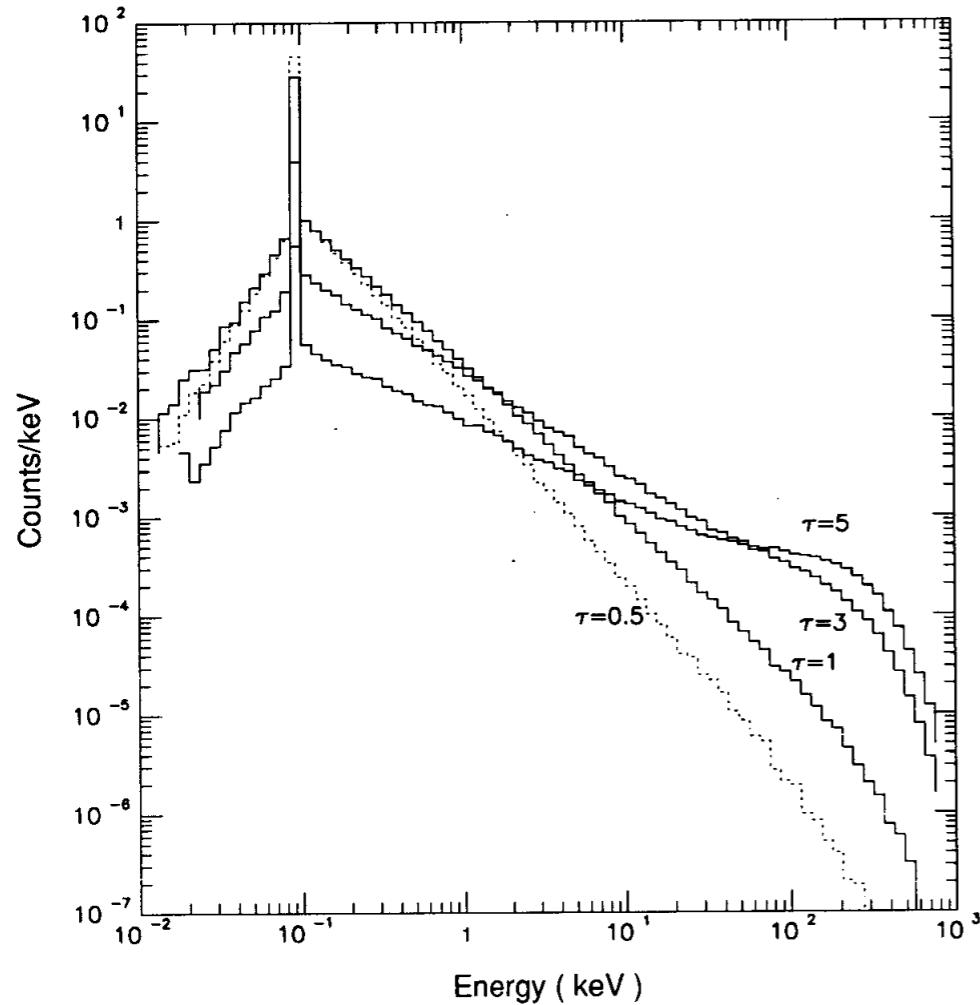
Clearly, this amplification will be important if $\alpha \ll 1$. Therefore, **energy amplification of a soft photon input spectrum is important when**

$$\begin{aligned} \alpha &= \frac{-\ln \tau_{\text{es}}}{\ln A} \lesssim 1 \quad \rightarrow \quad \ln (\tau_{\text{es}} A) \gtrsim 0 \\ &\rightarrow \quad y = A \tau_{\text{es}} \sim 16 \left(\frac{kT}{m_e c^2} \right)^2 \tau_{\text{es}} \gtrsim 1 \end{aligned}$$

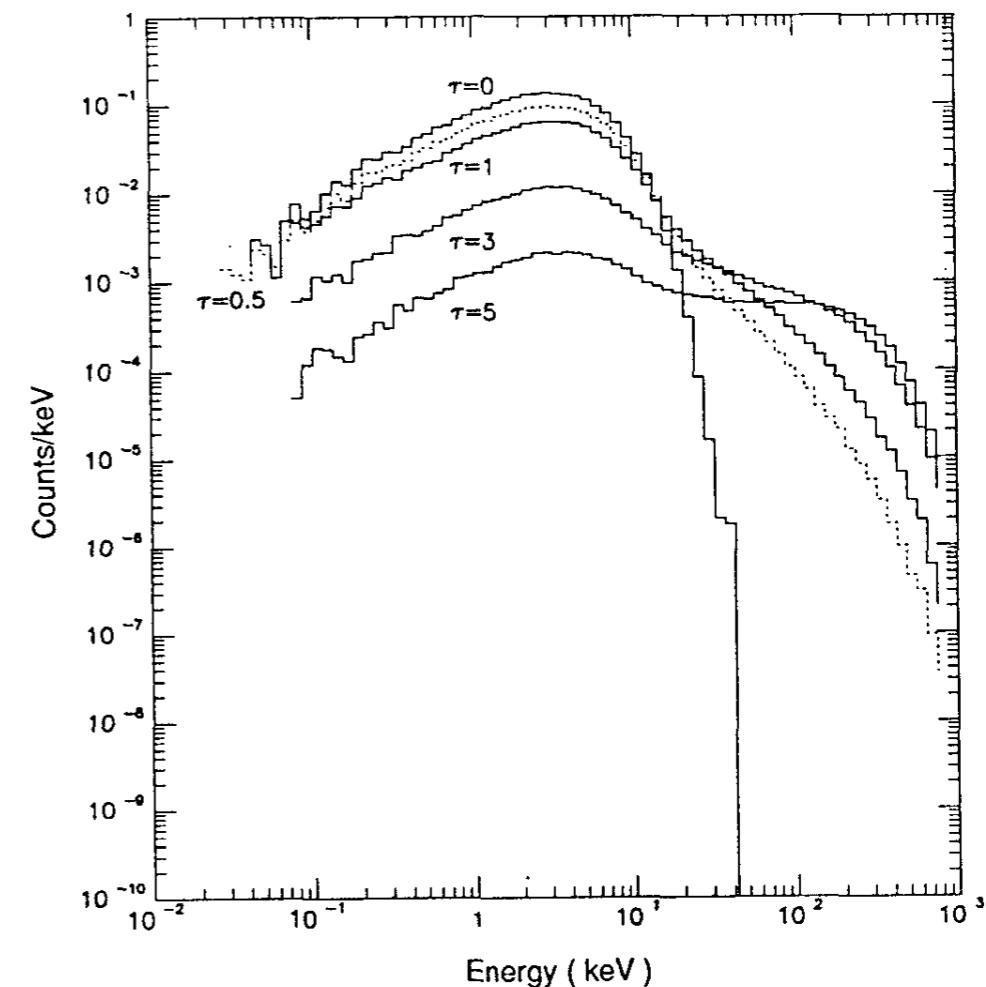
- **Comptonization**

The change in the spectrum of light due to scattering from electrons is referred to as Comptonization.

- Examples of Comptonized Spectra



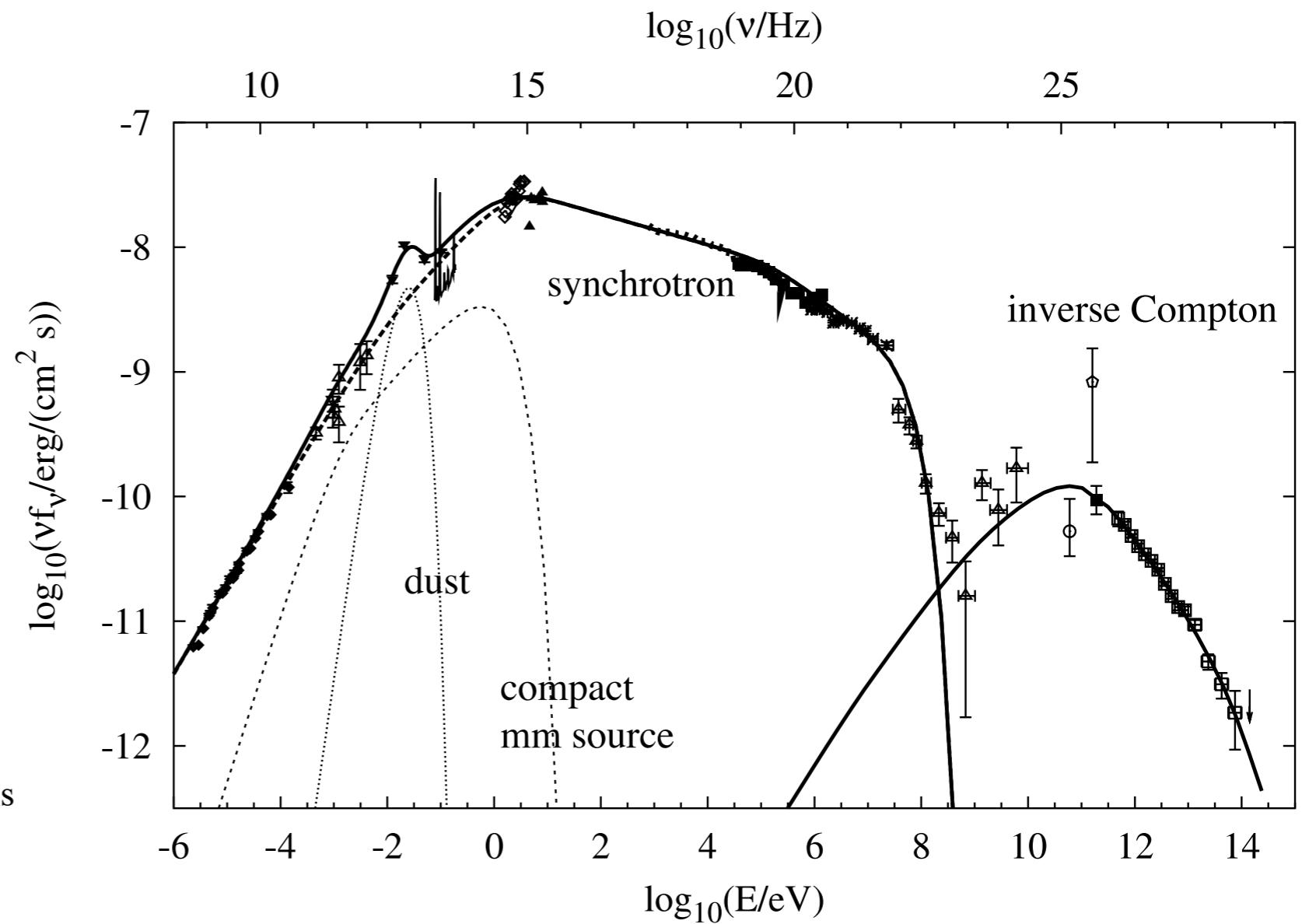
Comptonized spectra of monochromatic photons with $E_0 = 0.1$ keV emerging from a spherical shell plasma. The electron temperature is taken as 100 keV. The individual spectrum is normalized by the input photons (Seon et al. 1994, JKAS).



Comptonized spectra of a blackbody of $kT_b = 2$ keV emerging from a spherical shell plasma. The electron temperature is taken as 100 keV (Seon et al. 1994, JKAS)

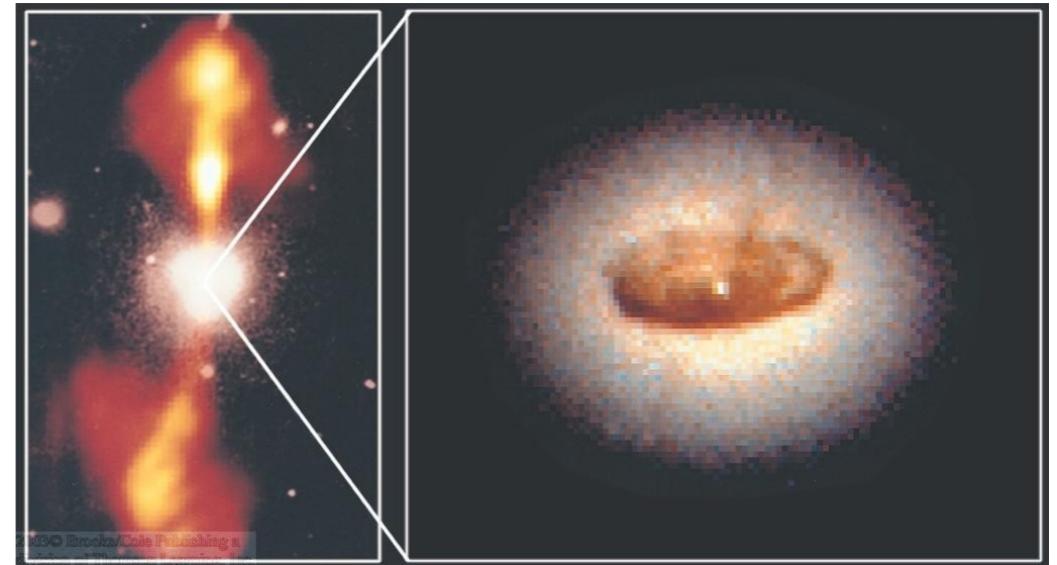
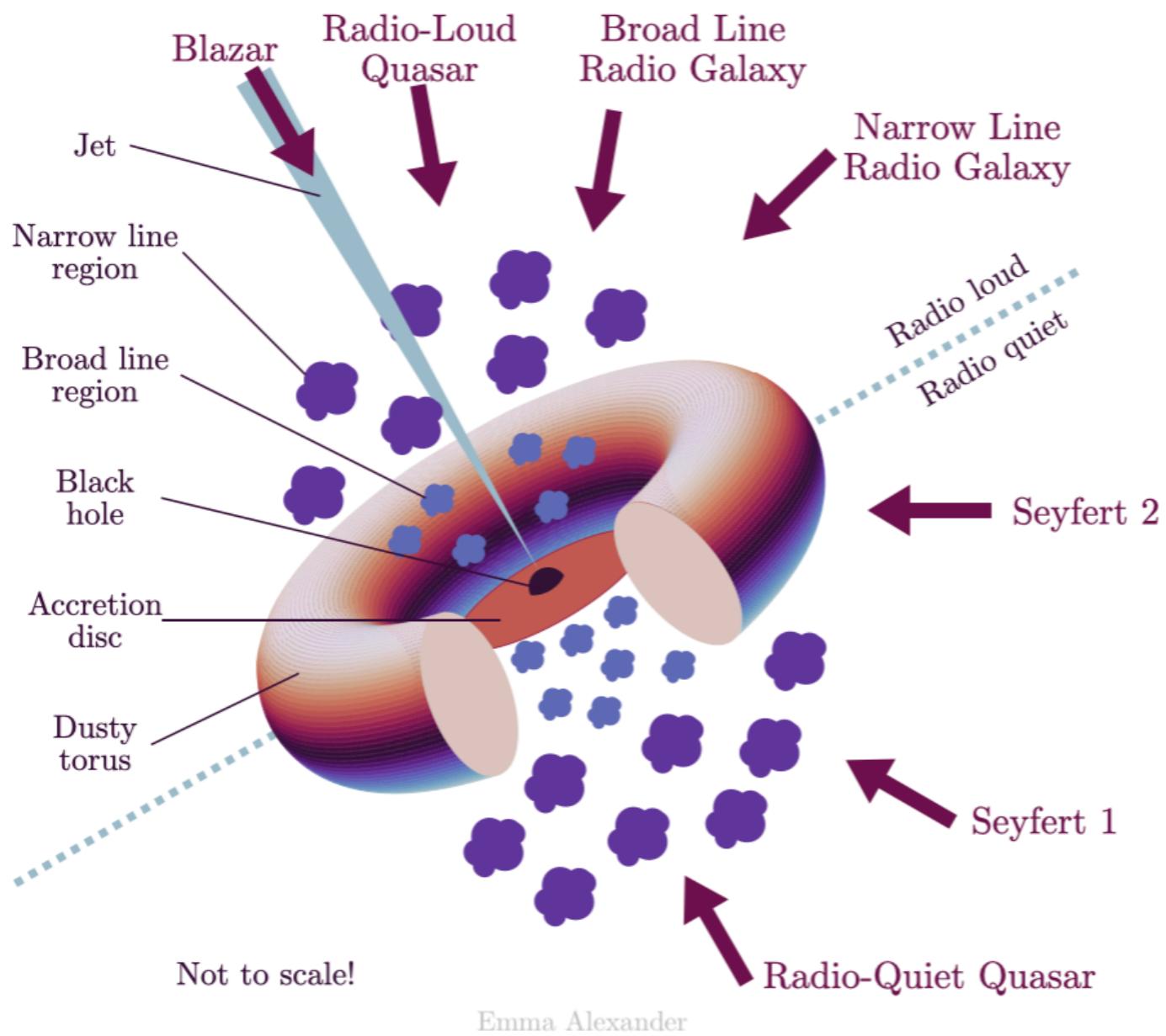
[Synchrotron self-Compton (SSC) emission]

- The modification of the photon spectrum by Compton scattering is called **Comptonization**.
- Relativistic electrons in the presence of a magnetic field will surely emit synchrotron radiation at some level. The photons will undergo inverse Compton scattering by the very same electrons that emitted them in the first place. Such scattering must take place before the synchrotron photon leaves the source region. This is the **synchrotron self-Compton (SSC) process**.
- **Crab nebula**



Active Galactic Nuclei

- A Unified Model for AGN



HST image of the Dust Torus in NGC 4261

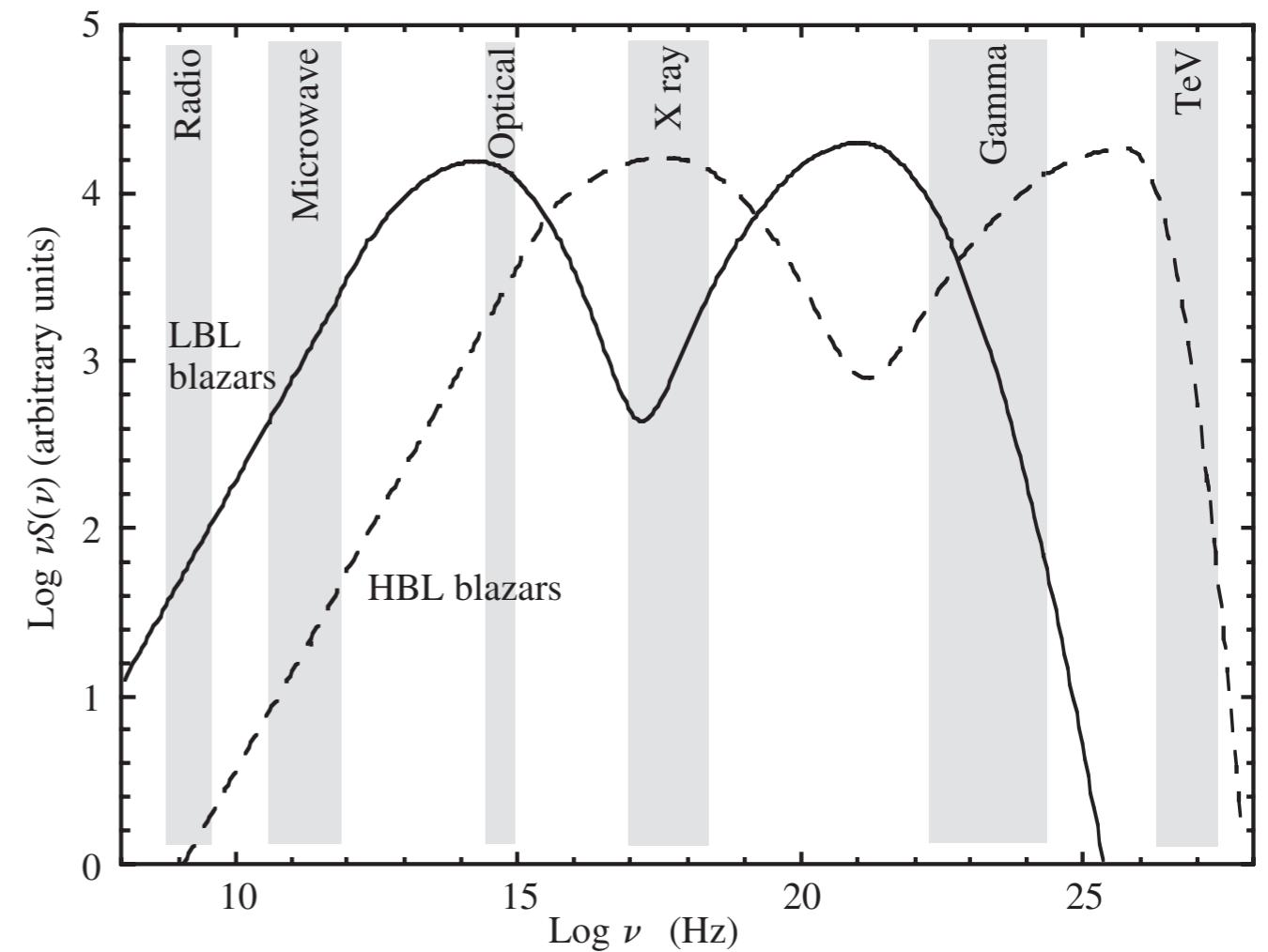
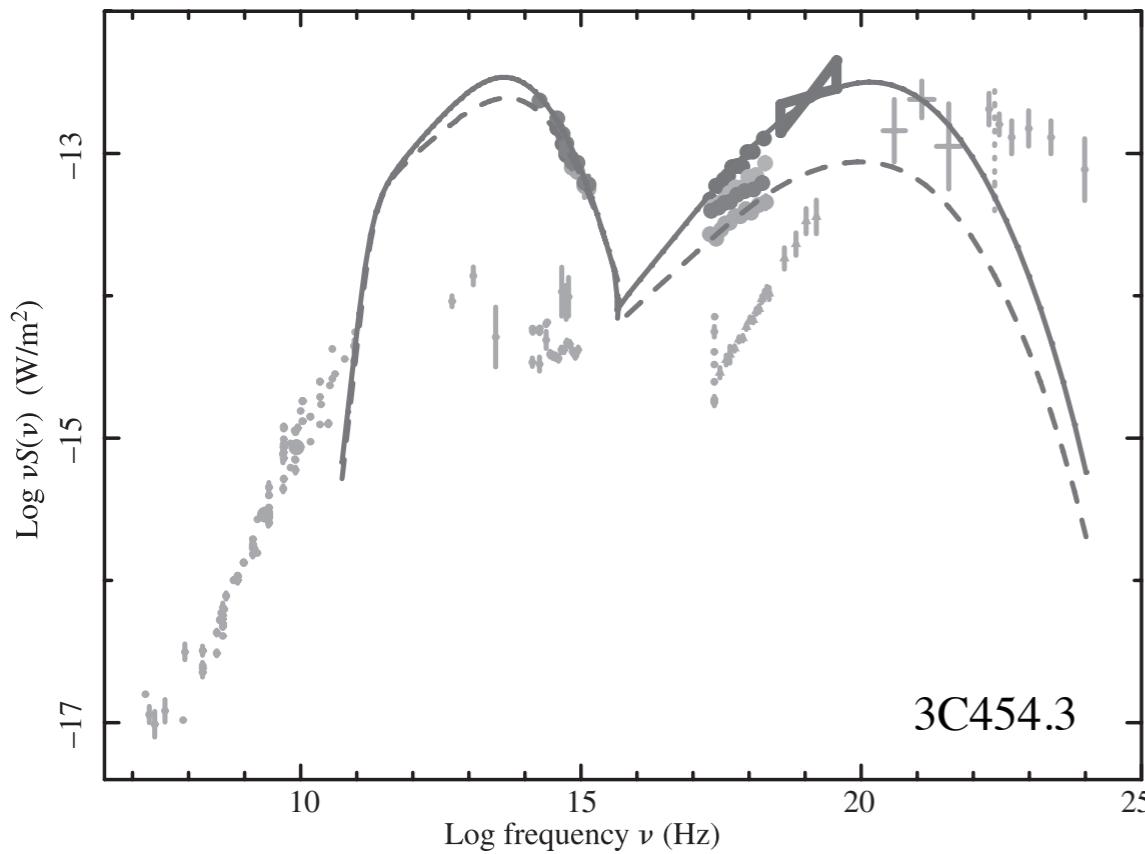
Unified model of AGN adapted from Urry & Padovani (1995).

- **Blazars:** If the observer view is more or less normal to the accretion disk, the action close to the core becomes visible. The observer considered to lie within the jet beam. Such objects are known as blazars or as BL Lacertae objects.
- Blazars have SEDs that are typically two peaked. The peak at lower frequency is attributed to synchrotron radiation and the one at higher frequency to IC scattering.

The lower-energy case (LBL blazar) extends from the radio to the gamma-ray bands but is quiet in the TeV band. The higher-energy case (HBL blazar) reaches TeV energies but is quiet in the radio range.

LBL: Low-frequency peaked BL Lacs

HBL: High-frequency peaked BL Lacs



[Sunyaev-Zeldovich effect]

This part taken from
[Bradt, Astrophysical Processes]

- The **Sunyaev-Zeldovich effect** is Comptonization of the Cosmic Microwave Background (CMB), the distortion of the blackbody spectrum ($T = 2.73$ K) of the CMB owing to the Inverse Compton (IC) scattering of the CMB photons by the energetic electrons in the galaxy clusters.

Thermal SZ effects, where the CMB photons interact with thermal electrons that have high energies due to their “high” temperature.

Kinematic SZ effects (Ostriker-Vishniac effect), a second-order effect where the CMB photons interact with electrons that have high energies due to their bulk motion (peculiar motion). The motions of galaxies and clusters of galaxies relative to the Hubble flow are called peculiar velocities. The plasma electrons in the cluster also have this velocity. The energies of the CMB photons that scattered by the electrons reflect this motion.

Determinations of the peculiar velocities of clusters enable astronomers to map out the growth of large-scale structure in the universe. This topic is fundamental importance, and the kinetic SZ effect is a promising method for approaching it.

- Thermal SZ effect

- The net effect of the IC scattering on the photon spectrum is obtained by multiplying the photon number spectrum by the kernel $K(\nu/\nu_0)$ and integrating over the spectrum.

$$N_{\text{scatt}}(\nu) = \int_0^\infty N(\nu_0) K(\nu/\nu_0) d\nu_0$$

The net effect is that the BB spectrum is shifted to the right and distorted (Figure (c)).

Observations of the CMB are most easily carried out in the low-frequency Rayleigh-Jeans region of the spectrum ($h\nu \ll kT_{\text{CMB}}$).

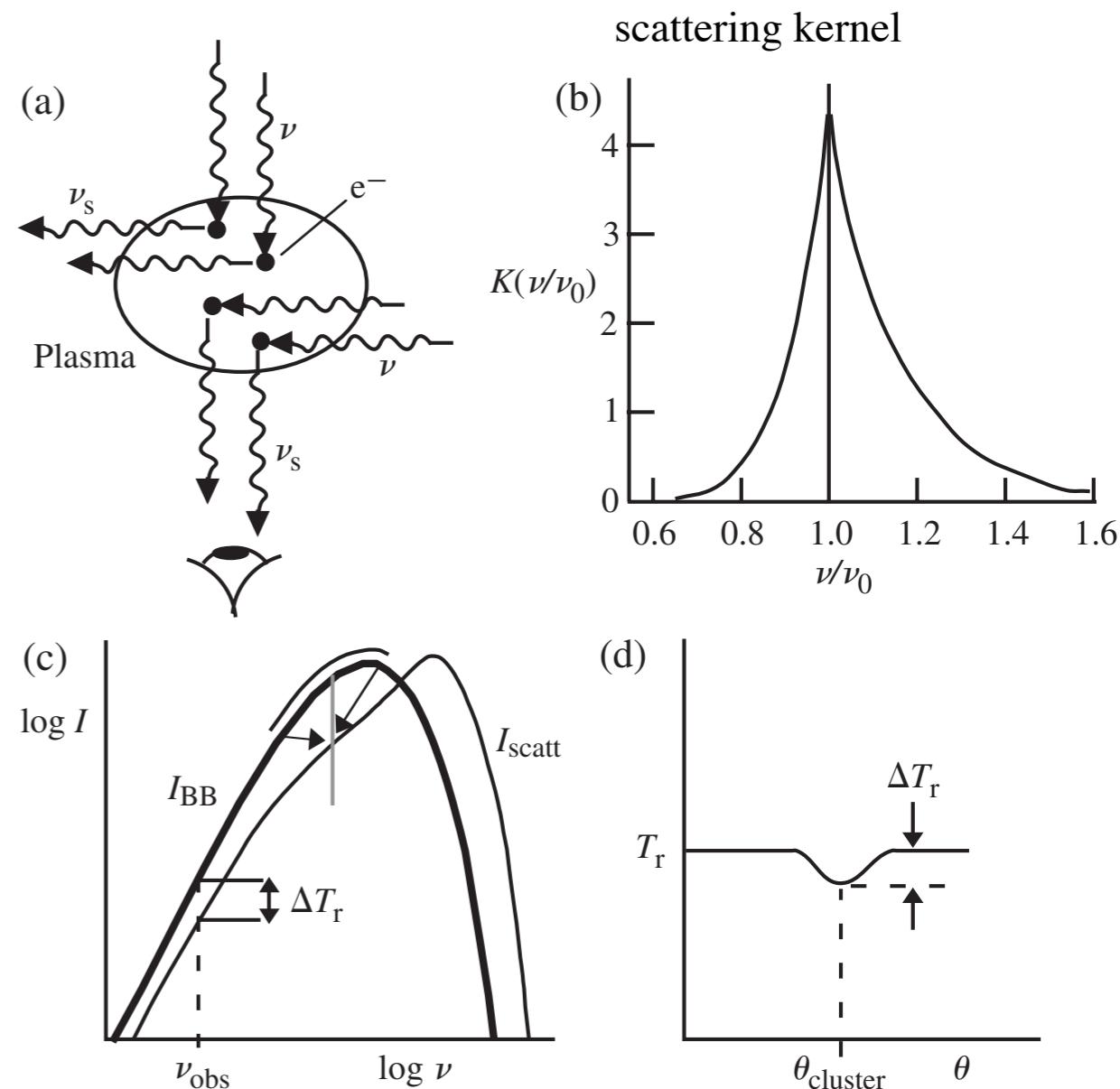
Measurement of the CMB temperature as a function of position on the sky would thus exhibit **antenna temperature dips in the directions of clusters** that contain hot plasmas (Figures (c) and (d)).

Note that the scattered spectrum is not a BB spectrum. **The effective temperature increases.** But, the total number of photons detected in a given time over the entire spectrum remains constant.

In Rayleigh-Jeans regime, the intensity decreases at a fixed frequency.

The result of such scatterings for an initial blackbody photon spectrum is shown in the following figure for the value:

$$\frac{kT_e}{m_e c^2} \tau = 0.5$$



- **Change of the BB temperature**

In the Rayleigh-Jeans region,

$$I(\nu) = \frac{2\nu^2}{c^2} k_B T_{\text{CMB}}$$

If the spectrum is shifted parallel to itself on a log-log plot, the fractional frequency change of a scattered photon is constant.

$$\varepsilon = \frac{\Delta\nu}{\nu} = \frac{\nu' - \nu}{\nu} = \text{constant} \quad \text{or} \quad \nu' = \nu(1 + \varepsilon) \quad \longrightarrow \quad d\nu' = d\nu(1 + \varepsilon) \quad \Rightarrow \quad \frac{d\nu'}{\nu'} = \frac{d\nu}{\nu}$$

Total photon number is conserved: $N'(\nu')d\nu' = N(\nu)d\nu \rightarrow \frac{I'(\nu')}{h\nu'}d\nu' = \frac{I(\nu)}{h\nu}d\nu$

$$\therefore I'(\nu') = I(\nu) = I\left(\frac{\nu'}{1 + \varepsilon}\right)$$

Compton-scattered = initial

$$\rightarrow I'(\nu) = I\left(\frac{\nu}{1 + \varepsilon}\right) = \frac{2\nu^2}{c^2(1 + \varepsilon)^2} k_B T_{\text{CMB}}$$

$$\frac{\Delta I}{I} = \frac{I'(\nu) - I(\nu)}{I(\nu)} = \frac{1}{(1 + \varepsilon)^2} - 1 \approx -2\varepsilon = -2\frac{\Delta\nu}{\nu}$$

We compare the spectrum at one frequency ν in ***the cluster direction*** with that at the same frequency ***in an off-cluster direction***.

$I'(\nu)$ in the cluster direction (Compton-scattered)
 $I(\nu)$ in an off-cluster direction (Initial)

$$\frac{\Delta T_{\text{CMB}}}{T_{\text{CMB}}} = \frac{\Delta I}{I} \approx -2\varepsilon = -2\frac{\Delta\nu}{\nu}$$

compare with $y = \frac{4k_B T_e}{m_e c^2} \tau$ in slide 8

The properly calculated result is
 (see homework)

$$\varepsilon = \frac{\Delta\nu}{\nu} = \frac{k_B T_e}{m_e c^2} \tau . \quad \longrightarrow$$

$$\frac{\Delta T_{\text{CMB}}}{T_{\text{CMB}}} \approx -2 \frac{k_B T_e}{m_e c^2} \tau$$

A typical cluster have an average electron density of $\sim 2.5 \times 10^{-3} \text{ cm}^{-3}$, a core radius of $R_c \sim 10^{24} \text{ cm}$ ($\sim 320 \text{ kpc}$), and an electron temperature of $k_B T_e \approx 5 \text{ keV}$.

The integration path would be 2-3 times the core radius. A typical optical depth is thus

$$\tau \approx 3\sigma_T n_e R_c \approx 0.005$$

Then, the expected antenna temperature change is

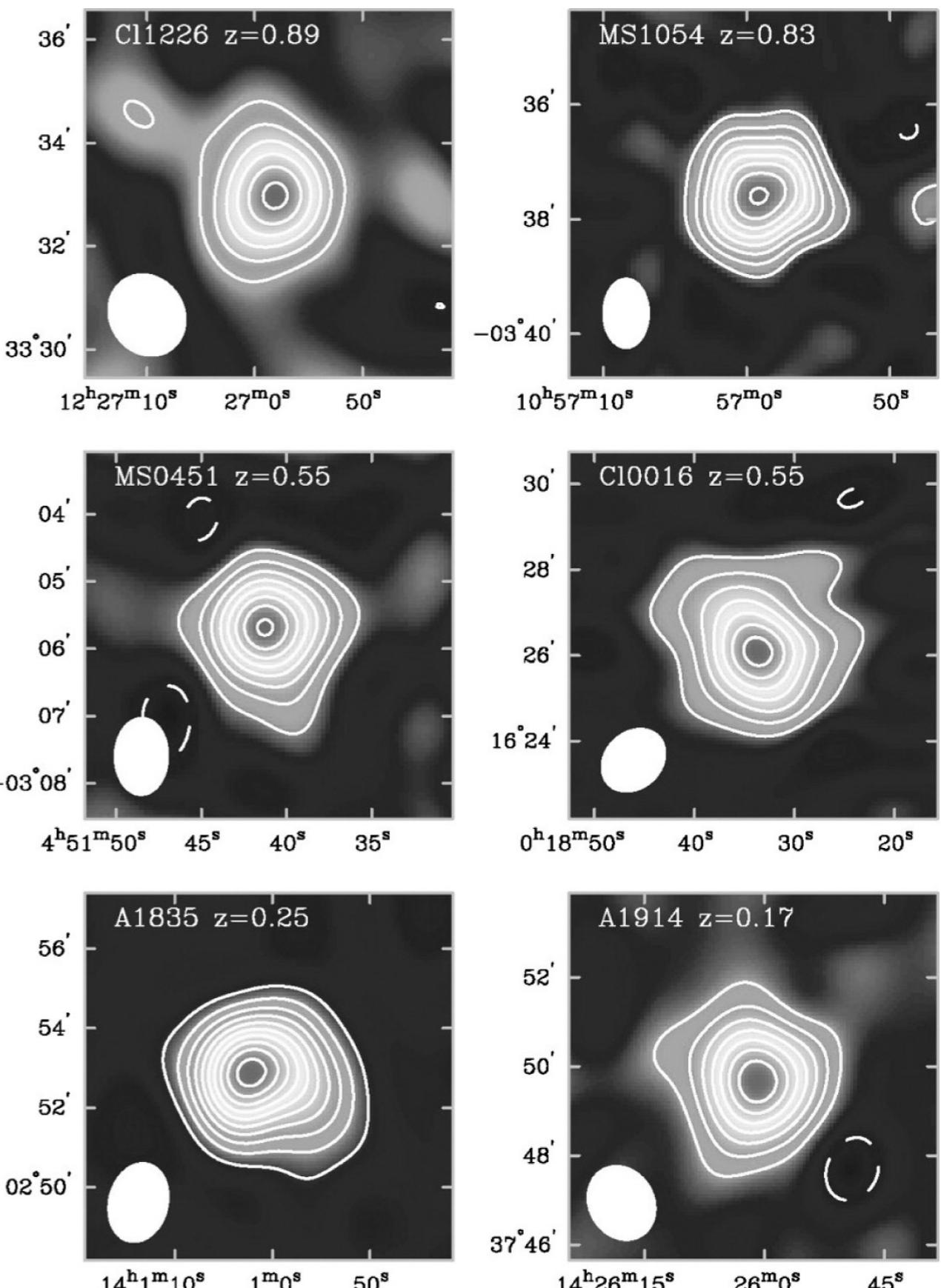
$$\frac{\Delta T_{\text{CMB}}}{T_{\text{CMB}}} \approx -1 \times 10^{-4}$$

$$\Delta T_{\text{CMB}} \approx -0.3 \text{ mK for } T_{\text{CMB}} = 2.7 \text{ K}$$

This effect has been measured in dozens of clusters.

Interferometric images at 30 GHz of six clusters of galaxies. The solid white contours indicate negative decrements to the CMB.

(Carlstrom et al. 2002, ARA&A, 40, 643)



- Hubble Constant

- **The SZ effect can be used to estimate the Hubble constant.** A value of the Hubble constant can be obtained for a given galaxy (cluster) only if one has independent measures of a recession speed v and a distance d of a galaxy.

$$H_0 = \frac{v}{d}$$

First, the recession speed is readily obtained from the spectral redshift.

Second, how to measure the distance:

X-ray observations:

$$I(\nu, T_e) = C \frac{g(\nu, T_e)}{T_e^{1/2}} \exp(-h\nu/kT_e) n_e^2(2R) \quad \text{Here, } R \text{ is the cluster radius.}$$

S-Z CMB decrement:

$$\frac{\Delta T_{\text{CMB}}}{T_{\text{CMB}}} = -2 \frac{kT_e}{m_e c^2} \tau = -2 \frac{kT_e}{m_e c^2} (\sigma_T n_e 2R)$$

The radio and X-ray measurements yield absolute values of **(1) the electron density n_e and (2) cluster radius R** without a priori knowledge of the cluster distance.

Imaging of the cluster in the radio or X-ray band yields the angular size of the cluster θ . Then the distance d to the cluster is obtained by

$$d = \frac{R}{\theta}$$

The SZ effect (at radio frequencies) in conjunction with X-ray measurements can give distances to clusters of galaxies. This can be used to derive the Hubble constant.

[Kompaneets Equation]

- The Kompaneets equation describes the time evolution of the distribution of photon occupancies in the case where photons and electrons are interacting through Compton scattering.
- Boltzmann transport equation

$$\frac{\partial n(\omega)}{\partial t} = c \int d^3 p \int d\Omega \frac{d\sigma}{d\Omega} [f_e(\mathbf{p}') n(\omega') (1 + n(\omega)) - f_e(\mathbf{p}) n(\omega) (1 + n(\omega'))]$$

In $1 + n(\omega)$, the “1” for spontaneous Compton scattering, and the $n(\omega)$ for stimulated Compton scattering.

The Boltzmann equation may be expanded to second order in the small energy transfer, yielding an approximation called the ***Fokker-Plank equation***. For photons scattering off a nonrelativistic, thermal distribution of electrons, the Fokker-Plank equation was first derived by A. S. Kompaneets (1957) and is known as the Kompaneets equation.

$$\frac{\partial n(\omega)}{\partial t} = \left(\frac{k_B T}{mc^2} \right) \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} + n + n^2 \right) \right] \quad \text{where } x \equiv \frac{\hbar\omega}{k_B T}, \text{ and } t_c \equiv (n_e \sigma_T c) t$$

- For the complete derivation, see the books “X-ray spectroscopy in Astrophysics (eds. van Paradijs)”, pages 213-218, and the book “High Energy Astrophysics (Katz)”, pages 103-110.
- Monte Carlo Simulation of Compton scattering: see “[Pozdnyakov, Sobol, and Suyaev \(1983, Soviet Scientific Reviews, vol. 2, 189-331\)](#)” and “[Seon et al. \(1994, JKAS, 27, 45\)](#).”

8. Plasma Effects

Introduction

- So far we have assumed our propagation medium to be a vacuum.
- **Dispersion:** The electrons and magnetic fields of the ionized interstellar medium (ISM) alter the character of radio waves propagating through it. The effect of the electron is to slow the propagation speed of the low-frequency components of a signal relative to that of the higher frequencies. **This phenomenon, the variation of transmission speed with frequency through a plasma, is known as dispersion.**
 - A pulse of radiation emitted by a radio pulsar will contain a wide range of radio frequencies (e.g., 100-400 MHz). During the pulse's passage through the ISM, the higher frequencies propagate faster than the lower ones. A receiver on earth tuned to 100 MHz will detect the pulse substantially delayed relative to the arrival time at 400 MHz.
 - Radio pulsars emit sequential pulses of radiation, and so the dispersion is repeated again and again. One could not observe dispersion with a steady source; all frequencies would arrive at all times.
- Roughly speaking a **plasma** is a globally neutral partially or completely ionized gas.

[Dispersion in Cold, Isotropic Plasma]

- Maxwell's equations for a vacuum can be used for a plasma if the charge and current densities ρ and \mathbf{j} due to the plasma are explicitly included.
- **Plasma Dispersion Relation**

If we assume a space and time variation of all quantities of the form $\exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$, Maxwell's equations become

Gauss's law Gauss's law for magnetic field Ampere's law Faraday's law	$\nabla \cdot \mathbf{E} = 4\pi\rho$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$ $\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$		$i\mathbf{k} \cdot \mathbf{E} = 4\pi\rho$ $i\mathbf{k} \cdot \mathbf{B} = 0$ $i\mathbf{k} \times \mathbf{E} = i\frac{\omega}{c} \mathbf{B}$ $i\mathbf{k} \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} - i\frac{\omega}{c} \mathbf{E}$
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Let us assume that our plasma consists of electrons with density n . The ions are very much less mobile than electrons and thus are neglected. We also assume that there is no external magnetic field; thus the plasma is isotropic. Then, the equation of motion of electrons when there is no external magnetic field is

$$m_e \dot{\mathbf{v}} = -e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) = -e\mathbf{E} \quad \longrightarrow \quad -im_e\omega\mathbf{v} = -e\mathbf{E}$$

$$\mathbf{v} = \frac{e\mathbf{E}}{i\omega m_e}$$

Current density: $\mathbf{j} = -nev = \sigma\mathbf{E}$ where the **conductivity** is $\sigma \equiv \frac{ine^2}{\omega m_e}$.

The charge conservation equation (continuity equation) gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

$$-i\omega\rho + ik \cdot j = 0 \rightarrow \rho = \frac{1}{\omega} \mathbf{k} \cdot \mathbf{j} = \frac{\sigma}{\omega} \mathbf{k} \cdot \mathbf{E}$$

Using the expressions for \mathbf{j} and ρ , Maxwell's equations become

$$\begin{aligned} \mathbf{j} &= \sigma \mathbf{E} \\ \rho &= \frac{\sigma}{\omega} \mathbf{k} \cdot \mathbf{E} \end{aligned}$$

$$\begin{aligned} ik \cdot \mathbf{E} &= 4\pi\rho \\ ik \cdot \mathbf{B} &= 0 \\ ik \times \mathbf{E} &= i\frac{\omega}{c} \mathbf{B} \\ ik \times \mathbf{B} &= \frac{4\pi}{c} \mathbf{j} - i\frac{\omega}{c} \mathbf{E} \end{aligned}$$



$$\begin{aligned} ik \cdot \epsilon \mathbf{E} &= 0 \\ ik \cdot \mathbf{B} &= 0 \\ ik \times \mathbf{E} &= i\frac{\omega}{c} \mathbf{B} \\ ik \times \mathbf{B} &= -i\frac{\omega}{c} \epsilon \mathbf{E} \end{aligned}$$

These equations are now “source-free” and can be solved in the same way as before. We find again that \mathbf{k} , \mathbf{E} , \mathbf{B} form a mutually orthogonal right-hand vector triad.

The **dispersion relation** connecting k and ω is given by

$$\begin{aligned} ik \times \mathbf{B} &= -i\frac{\omega}{c} \epsilon \mathbf{E} \\ ik \times \left(\frac{c}{\omega} \mathbf{k} \times \mathbf{E}\right) &= -i\frac{\omega}{c} \epsilon \mathbf{E} \\ k(k \cdot \mathbf{E}) - \mathbf{E}(k \cdot k) &= -\frac{\omega^2}{c^2} \epsilon \mathbf{E} \end{aligned}$$



$$\begin{aligned} c^2 k^2 &= \epsilon \omega^2 \\ &= \omega^2 - \omega_p^2 \\ \omega^2 &= \omega_p^2 + c^2 k^2 \end{aligned}$$

where the **dielectric constant**:

$$\begin{aligned} \epsilon &\equiv 1 - \frac{4\pi\sigma}{i\omega} = 1 - \frac{4\pi n e^2}{m_e \omega^2} \\ &= 1 - \left(\frac{\omega_p}{\omega}\right)^2 \end{aligned}$$

plasma frequency:

$$\begin{aligned} \omega_p &\equiv \sqrt{\frac{4\pi n e^2}{m_e}} \\ &= 5.63 \times 10^4 (n/\text{cm}^{-3})^{1/2} \text{ s}^{-1} \end{aligned}$$

-
- We now see immediately that for $\omega < \omega_p$, the wavenumber is imaginary ($k^2 < 0$).

$$k = \frac{i}{c} \sqrt{\omega_p^2 - \omega^2} \quad e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \propto e^{-|k|r}$$

In this case, the wave amplitude decreases as $e^{-|k|r}$ on a scale of the order of $1/|k| \approx c/\omega_p$.

Thus the plasma frequency ω_p defines **a plasma cutoff frequency below which there is no electromagnetic propagation.**

For instance, Earth ionosphere prevents extraterrestrial radiation at frequencies less than ~ 1 MHz from being observed at the earth's surface ($n \sim 10^4 \text{ cm}^{-3}$). The existence of the plasma cutoff yields an important method of probing the ionosphere.

Radio astronomy from the earth's surface can not be carried out at frequencies $\nu \lesssim 9$ MHz. Such radio waves from outer space are reflected back to space by the ionosphere.

Method of probing the ionosphere:

Let a pulse of radiation in a narrow range about ω be directed straight upward from the earth's surface.

When there is a layer at which the density n is large enough to make $\omega < \omega_p$, the pulse will be totally reflected from the layer.

The time delay of the pulse provides information on the height of the layer.

By making such measurements at many different frequencies, the electron density can be determined as a function of height.

- **Radio Broadcast**

AM radio ($\sim 500\text{-}1700\text{ kHz}$) can propagate long distances via such reflections from the bottom of the ionosphere. This is particularly effective at night when the relatively low “D” layer of the ionosphere becomes neutral (and disappears); then reflection can take place at higher altitudes that have large values of n_e .

In contrast, FM radio at $\sim 100\text{ MHz}$ travels right through the ionosphere into outer space; it sounds better, but one has to be close enough to the station that the signal can be received more or less directly without ionosphere reflections.

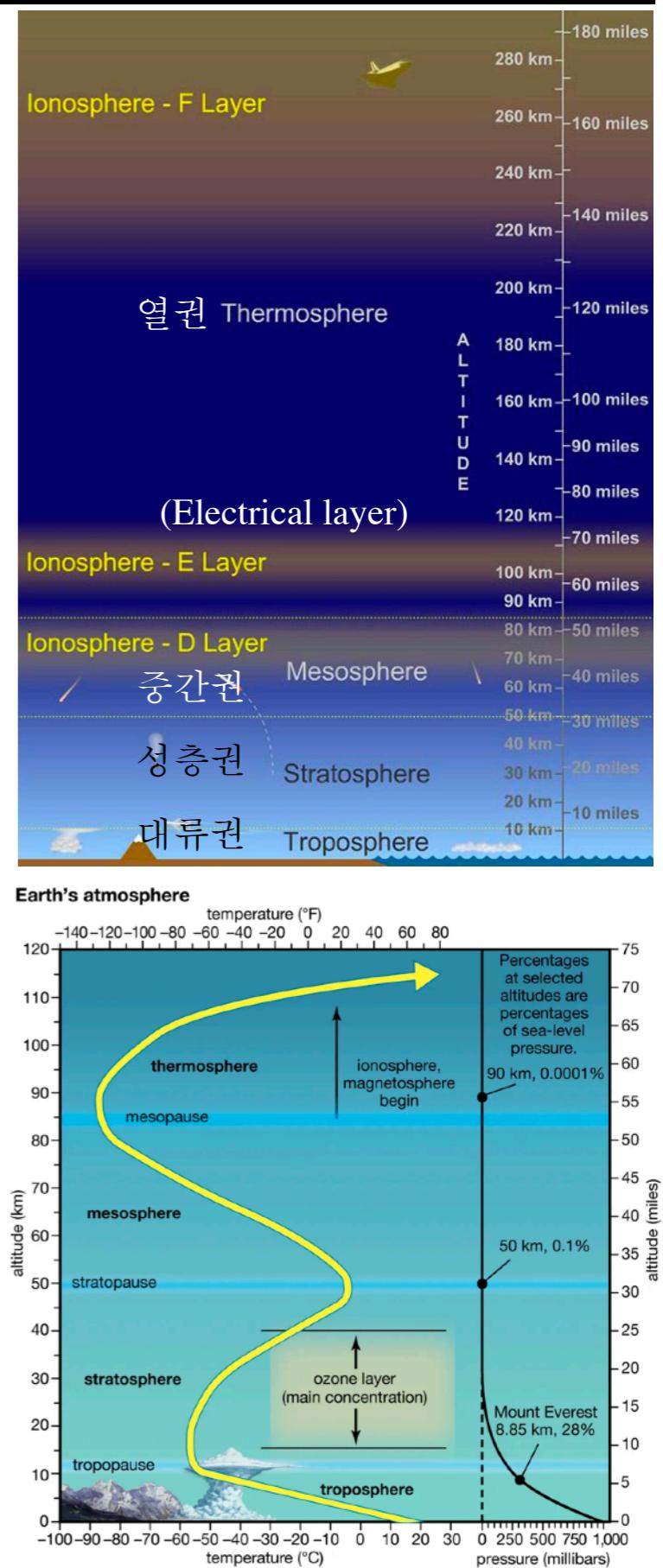
The ionosphere is between 80 and $\sim 600\text{ km}$ where EUV and X-ray solar radiation ionize the atoms and molecules thus creating a layer of electrons.

- **Interstellar cutoff**

In interstellar space, the number densities of ions are typically much less ($n_e \sim 1\text{ cm}^{-3}$); thus

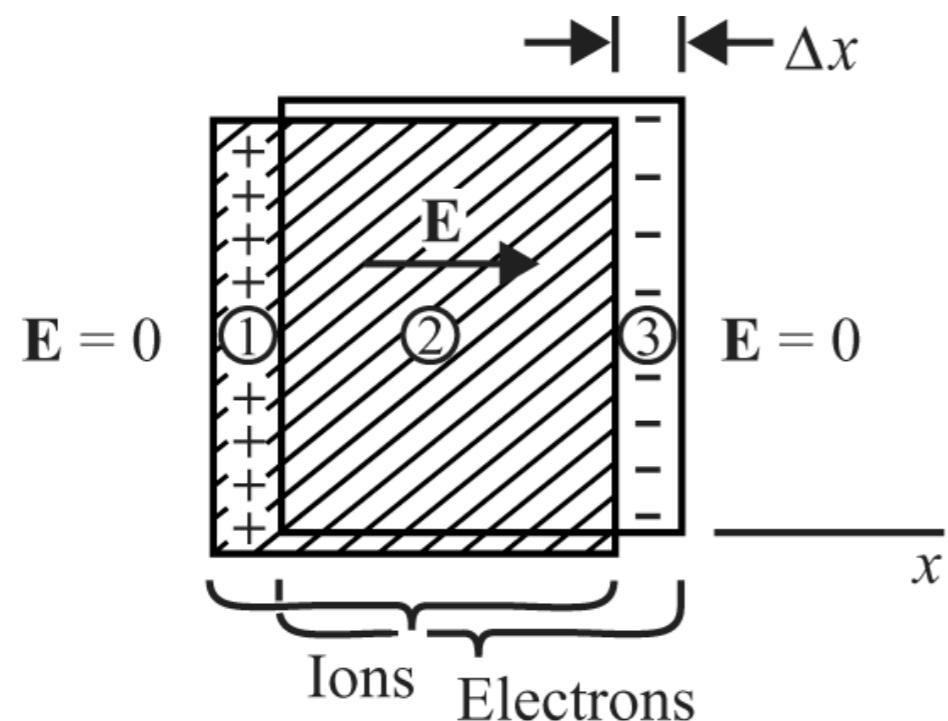
$$\nu_p \approx 300\text{--}1000\text{ Hz}$$

This is well below the frequencies that penetrate the ionosphere ($\nu_p \gtrsim 9\text{ MHz}$). Consequently, **radio waves that can penetrate the atmosphere can easily travel through much of interstellar space. This makes galactic and extragalactic radio astronomy possible.**



[Plasma Frequency]

- If the electrons in a uniform, homogeneous plasma are displaced from their equilibrium position, an electric field arises because of charge separation. This electric field produces a restoring force on the displaced electrons. Since the magnitude of the charge imbalance is directly proportional to the displacement, the restoring force is given by Hooke's law, $F = -k\Delta x$ and the system behaves as a harmonic oscillator. The resulting oscillations are called **electron plasma oscillations**. The oscillation frequency is called the **electron plasma frequency**.



$$\nabla \cdot \mathbf{E} = 4\pi\rho \rightarrow E = 4\pi n e \Delta x$$

n = electron density

$$m_e \frac{d^2 \Delta x}{dt^2} = -eE = -4\pi n e^2 \Delta x$$

$$\frac{d^2 \Delta x}{dt^2} + \frac{4\pi n e^2}{m_e} \Delta x = 0$$

$$\frac{d^2 \Delta x}{dt^2} + \omega_p^2 \Delta x = 0$$

where $\omega_p \equiv \frac{4\pi n e^2}{m_e}$

[Phase Velocity and Group Velocity]

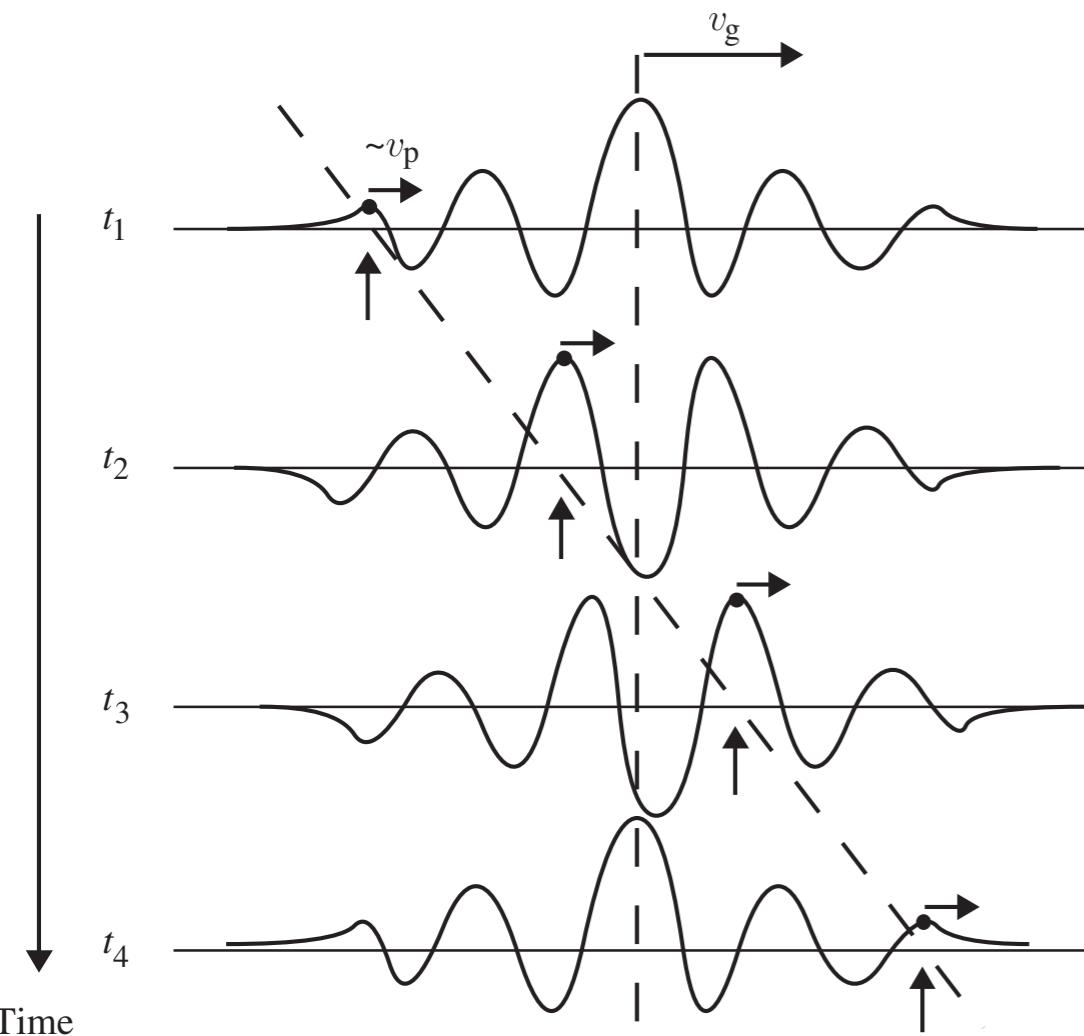
- Dispersion relation, phase velocity, and group velocity:

dispersion relation = a function which gives ω as a function of k .

phase velocity = the rate at which the phase of the wave propagates in space.

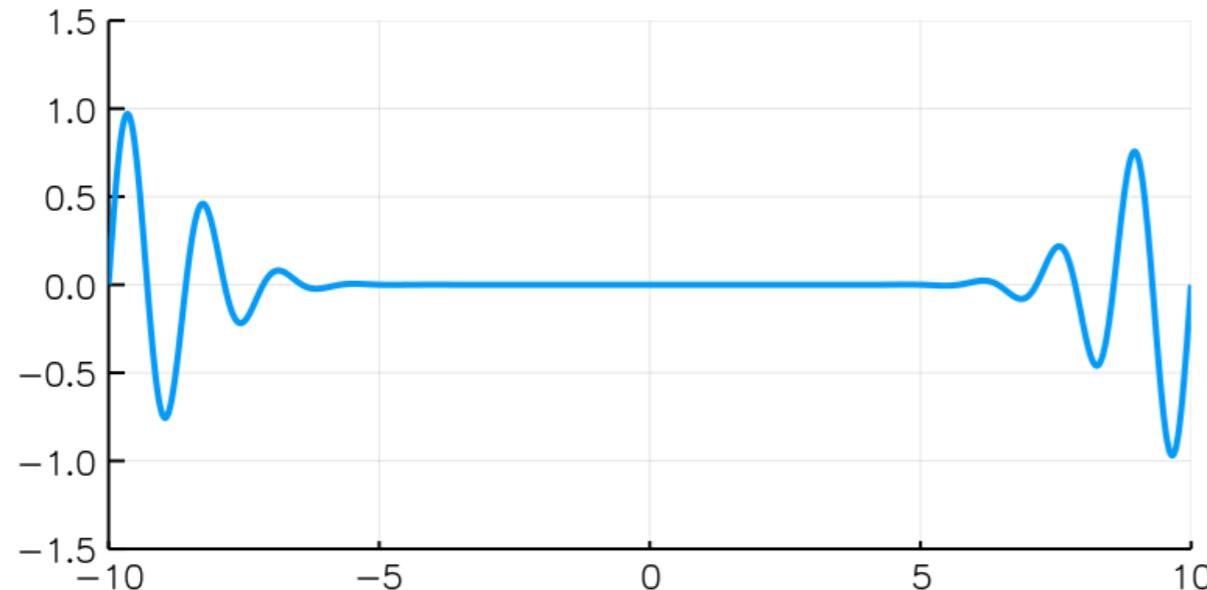
group velocity = the velocity with which the overall shape of the waves' amplitudes (modulation or envelope of the wave) propagates through space.

$$\text{In vacuum, } \omega = ck, \quad v_{\text{ph}} \equiv \frac{\omega}{k} = c, \quad v_g \equiv \frac{\partial \omega}{\partial k} = c$$



A wavelet or wave packet shown at four different times. The packet as a whole moves to the right at group velocity v_g . The interference between the component waves that make up the packet, with their different phase velocities, causes a single crest (arrows and dots) to gradually overtake the centroid of the packet. Relative to the medium, the overtaking crest moves at approximately the median phase velocity.

Figure 11.3. [Bradt] Astrophysics Processes



https://en.wikipedia.org/wiki/Group_velocity

- Assume that a wave packet E is almost monochromatic, so that its Fourier component is nonzero only in the vicinity of a central wavenumber k_0 . Then, linearization gives:

$$\begin{aligned}\omega(k) &\approx \omega_0 + (k - k_0) \left. \frac{\partial \omega}{\partial k} \right|_{k=k_0} \\ &= \omega_0 + (k - k_0)v_g\end{aligned}$$



$$\begin{aligned}E(x, t) &= \int dk \int d\omega \bar{E}(k, \omega) e^{i(kx - \omega t)} \\ &\approx e^{it(v_g k_0 - \omega_0)} \int dk \bar{E}(k, \omega_0) e^{ik(x - v_g t)}\end{aligned}$$

where $v_g \equiv \left. \frac{\partial \omega}{\partial k} \right|_{k=k_0}$

$$|E(x, t)| = |E(x - v_g t, 0)|$$

- Therefore, the envelope of the wavepacket travels at velocity $v_g = (\partial \omega / \partial k)_{k=k_0}$.

- **Group and Phase Velocity in terms of the index of refraction**

Phase velocity

$$v_{\text{ph}} \equiv \frac{\omega}{k} = \frac{\omega}{\sqrt{\omega^2 - \omega_p^2}/c} = \frac{c}{n_r}$$

where **the index of refraction** is defined to be

$$n_r \equiv \sqrt{\epsilon} = \sqrt{1 - \omega_p^2/\omega^2}$$

When $\omega > \omega_p$, waves do propagate without damping with phase velocity and the phase velocity always exceeds the speed of light ($v_{\text{ph}} > c$).

Group velocity

Group velocity is the speed with which a pulse of radiation containing a band of frequencies will propagate through space.

$$\omega^2 = \omega_p^2 + c^2 k^2$$

$$\longrightarrow v_g \equiv \frac{\partial \omega}{\partial k} = \frac{c^2 k}{\omega} = c \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \rightarrow v_g < c$$

$$\boxed{v_{\text{ph}} = \frac{c}{n_r}}$$

$$\boxed{v_g = c n_r}$$

The group velocity is always less than c . The wave energy travels at the group velocity, as does any modulation of the wave (information coding). **The information content moves at the group velocity.**

The lower frequencies travel more slowly than higher frequencies. For $\omega < \omega_p$, the index of refraction becomes imaginary, leading to attenuation. If the frequency ω is lowered toward ω_p and increasing, the group velocity decreases toward zero. If the frequency ω is well above ω_p and increasing, the group velocity approaches c . As the density is lowered ($n \rightarrow 0$; become more closely a vacuum), ω_p decreases and the group velocity more closely approaches the vacuum velocity c .

[Dispersion Measure]

- **Pulsar**

An important application of the formula for group velocity is to pulsars. Each individual pulse from the pulsar has a spectrum covering a wide band of frequency. Thus, the pulse will be dispersed by its interaction with the interstellar plasma, since each small range of frequencies travels at a slightly different group velocity and will reach earth at a slightly different time.

Let d = pulsar distance to the pulsar

The time required for a pulse to reach earth at frequency ω is $t_p = \int_0^d \frac{ds}{v_g}$.

The plasma frequency in ISM is usually quite low ($\sim 10^3$ Hz), so we can assume that $\omega \gg \omega_p$. Then, the travel time is given by

$$\frac{1}{v_g} = \frac{1}{c} \left(1 - \frac{\omega_p^2}{\omega^2} \right)^{-1/2} \approx \frac{1}{c} \left(1 + \frac{1}{2} \frac{\omega_p^2}{\omega^2} \right) \rightarrow t_p \approx \frac{d}{c} + \frac{2\pi e^2}{cm_e \omega^2} \int_0^d n ds$$

= transit time for a vacuum + plasma correction

- **Dispersion measure:**

What is usually measured is **the rate of change of arrival time with respect to frequency**.

$$\frac{dt_p}{d\omega} = -\frac{4\pi e^2}{cm_e \omega^3} \mathcal{DM} \quad \text{where } \mathcal{DM} \equiv \int_0^d n ds$$

is the dispersion measure of the ray.
This is the electron column density.

The arrival interval of pulsar signal between two frequencies ν_1 and ν_2 :

$$\Delta t_p = \frac{e^2}{2\pi cm_e} \left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2} \right) \mathcal{DM} = 4.15 \times 10^{15} \left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2} \right) \frac{\mathcal{DM}}{\text{cm}^{-3} \text{ pc}} \text{ sec}$$

- **Crab nebula**

Observations of the pulses from the Crab pulsar show that the delay between frequencies 100 MHz and 400 MHz is 23.5 s, which yields a value for the column density expressed as

$$\int_0^D n_e ds = 1.86 \times 10^{20} \text{ cm}^{-2}. \text{ electron column density to Crab pulsar}$$

Note that astronomers use the mixed units pc cm^{-3} to denote the dispersion measure (DM), but cm^{-2} for the column density.

Because the Crab pulsar is about 2 kpc ($= 6.2 \times 10^{21} \text{ cm}$), the average electron density is

$$\langle n_e \rangle_{\text{abs}} = \frac{1}{D} \int_0^D n_e ds = 0.03 \text{ electrons cm}^{-3}$$

This is relatively low density in the context of the ISM. It lies between the densities of the Warm Ionized Medium (WIM) and Hot Ionized Medium (HIM).

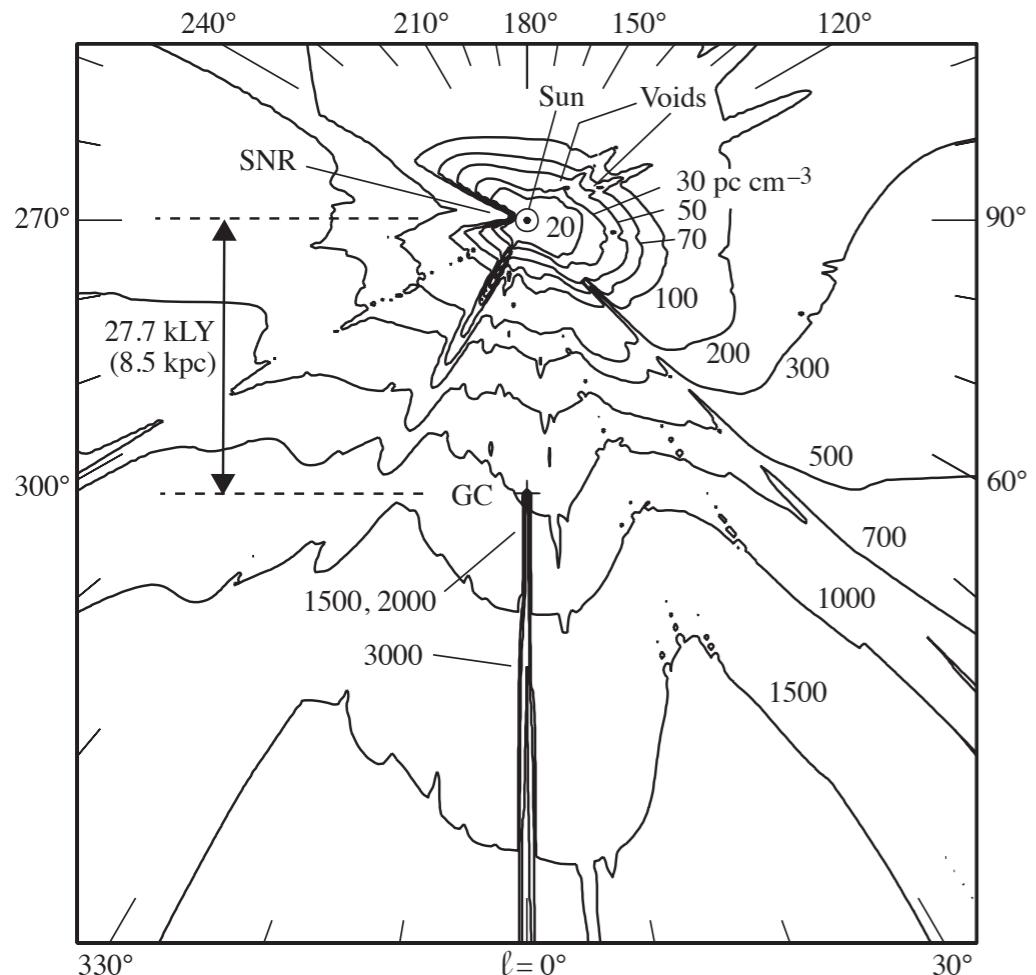
The 23-s time lag from 400 MHz to 100 MHz is very long compared with the 33-ms interval between pulses from the Crab pulsar. At earth, each pulse is “heard” as a frequency that continuously decreases.

- **Galactic model of electron density**

The number of known radio pulsars is now over 2000, and the DM has been measured for almost all of them. In some cases, the distance to the pulsar is known from independent evidence such as parallax measurements, H I absorption, or because the distance is associated with an object of known distance such as a supernova remnant. The DM in these cases provides an average value of the electron density along the path.

$$\mathcal{DM} = \int n_e ds \quad \mathcal{EM} = \int n_e^2 ds$$

The DM and other results (such as H α) have led to a model of electron density n_e in the Galaxy. The best current model for the electron density n_e for the Galactic plane is known as NE2001 because it is constructed from data available through the year 2001.



Contours of DM in units of pc cm $^{-3}$ in the Galactic plane viewed from above the plane.

The sun and Galactic center (GC, cross) are marked; their separation is 8.5 pc.

Strong local enhancements of n_e , such as the GC at $\ell = 0$ or the Gum and Vela nebulae (SNR) at $\ell \approx 260^\circ$, are reflected in the contours at distances beyond the enhancements. Similarly, voids such as those $\ell \approx 120^\circ$ produce “fingers” in the contours at greater distance

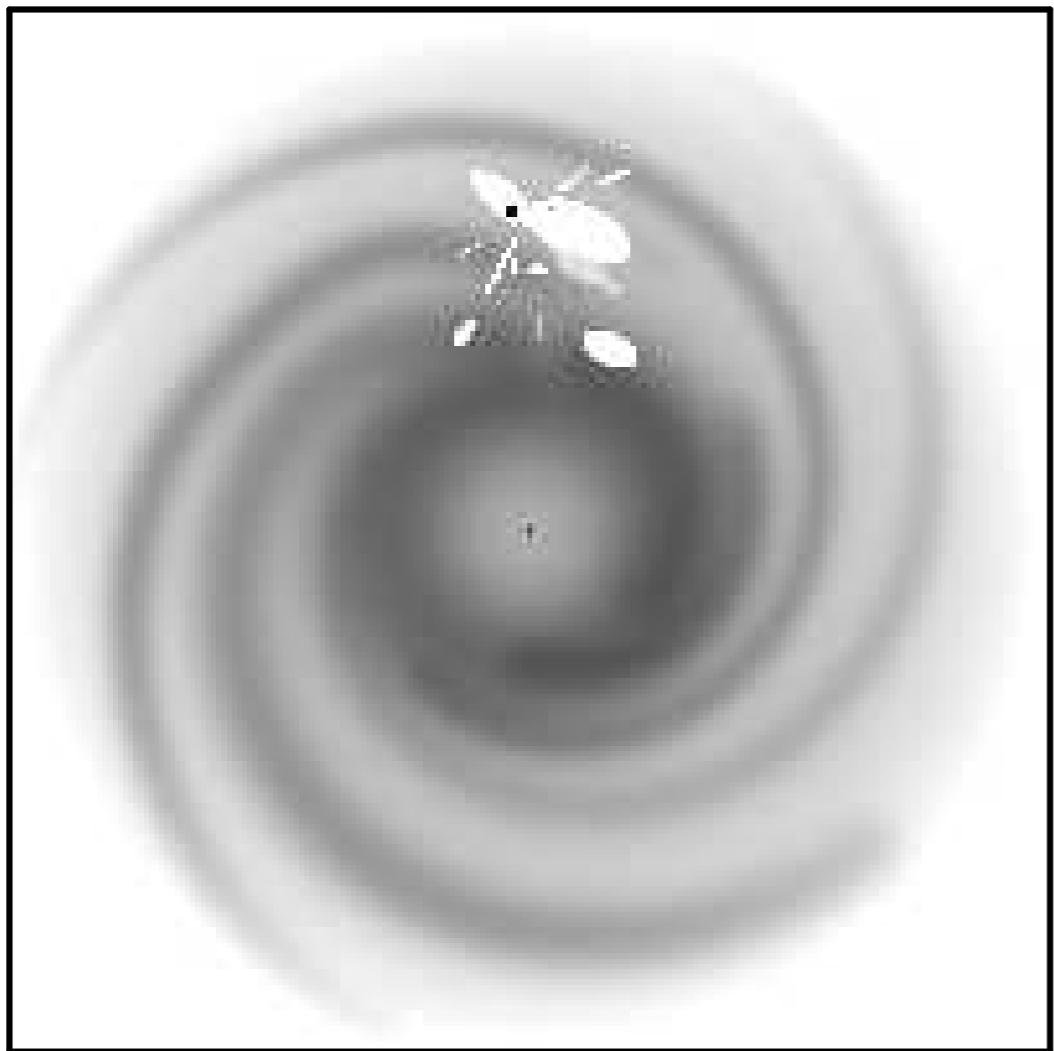
Cordes & Lazio 2003, arXiv:0207156

NE2001 - <http://hosting.astro.cornell.edu/~cordes/NE2001/>
pyne2001 - <https://pypi.org/project/pyne2001/>

Taylor & Cordes (1993, ApJ, 411, 674)

Cordes & Lazio (2003, arXiv:astro-ph/0207156)

Schnitzeler (2012, MNRAS, 427, 664)



Cordes & Lazio (2003)

FIG. 2.— Electron density corresponding to the best fit model plotted as a grayscale with logarithmic levels on a 30×30 kpc x-y plane at $z=0$ and centered on the Galactic center. The most prominent large-scale features are the spiral arms, a thick, tenuous disk, a molecular ring component. A Galactic center component appears as a small dot. The small-scale, lighter features represent the local ISM and underdense regions required for some lines of sight with independent distance measurements. The small dark region embedded in one of the underdense, ellipsoidal regions is the Gum Nebula and Vela supernova remnant.

Homework (due date: 11/27)

[Q11] The shift $\Delta I = I' - I$ of any photon spectrum at a given frequency due to single scattering from a Maxwell distribution of electrons is given by the formula ([Sunyaev & Zeldovich, 1972, Comments on Astrophysics and Space Physics, 4, 173](#))

$$\Delta I_\nu = I'(\nu) - I(\nu) = y\nu \frac{d}{d\nu} \left[\nu^4 \frac{d}{d\nu} (\nu^{-3} I(\nu)) \right]$$

$$y \equiv \frac{kT_e}{m_e c^2} \tau$$

where $\tau = \int n_e \sigma_T d\ell$

$$\sigma_T = 6.65 \times 10^{-25} \text{ cm}^2$$

This is valid for $y \ll 1$ and $kT_e \gg h\nu$. Here, T_e is the electron temperature.

(a) Calculate the shift ΔI_ν for the blackbody spectrum with a radiation temperature T_r .

Hint: $I(\nu) = B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT_r} - 1}$

Define $x = h\nu/kT_r$, represent the equation in terms of x , take derivatives, and then give your results in terms of x , y , and $I(\nu)$.

The result should be $\Delta I_\nu = yI(\nu) \frac{xe^x}{e^x - 1} \left(x \frac{e^x + 1}{e^x - 1} - 4 \right)$

(b) Find the limiting expressions for the Rayleigh-Jeans regime ($x \ll 1$) and Wien regime ($x \gg 1$). Show that the results are given by

$$\begin{aligned}\Delta I_\nu &= -2yI(\nu) && \text{for } x \ll 1 \\ &= y(x-4)xI(\nu) && \text{for } x \gg 1\end{aligned}$$

Compare the result for $x \ll 1$ with that given in this lecture note.

(c) Sketch or plot a log-log plot of $I(\nu)$ and $I'(\nu) = I(\nu) + \Delta I_\nu$, for instance, for $y = 0.15$ and $T_r = 2.7$ K.

Note that $I'(\nu)$ is not a derivative of $I(\nu)$.