

Astrophysics

Lecture 10

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Synchrotron Emission from Crab Nebula



Crab Nebula ($d \sim 2$ kpc) caused by SN explosion in 1054 A.D. - composite image. Chandra X-ray [blue], HST optical [red and yellow], Spitzer infrared [purple]. X-ray image is smaller than others as extremely energetic electrons emitting X-rays radiate away their energy more quickly than lower-energy electrons emitting in optical and infrared [Credits: NASA]

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- Emission as synchrotron radiation of relativistic electrons, characteristic frequency:

$$\nu_c \simeq \frac{1}{2\pi} \gamma^2 \frac{eB}{m_e c} \simeq 280 \left(\frac{B}{10^{-4} \text{ G}} \right) \gamma^2 \text{ Hz}$$

for average $B \sim 10^{-4}$ G in Crab Nebula

- **Optical Emission:**

Optical emission (HST) at $\nu \sim 5 \times 10^{14}$ Hz requires electrons with $\gamma \sim 10^6$.

Cooling time scale $t_{\text{cool}} \sim 2500 (10^6/\gamma)$ yr $\gtrsim t_{\text{age}}$ age of Nebula.

- **X-ray Emission:**

Chandra (ACIS, 0.2-10 keV) X-ray emission $\nu \sim 10^{17}$ Hz requires $\gamma \sim 10^7$, electrons cool quicker by a factor ~ 10 .

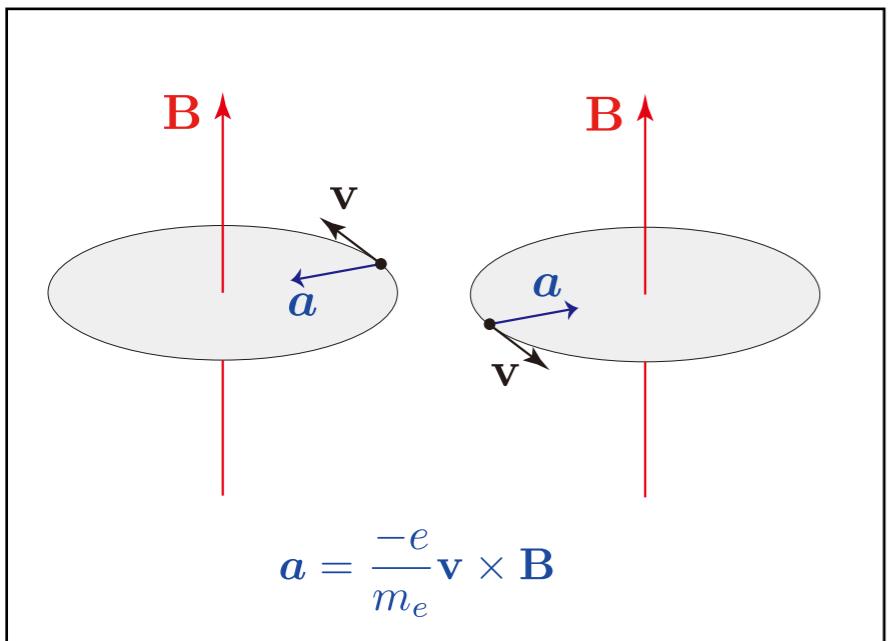
X-ray emission spatially less extended.

$t_{\text{cool}} < t_{\text{age}} \sim 950$ yr of Nebula, need continuous supply of fresh electrons.

- **Radio Emission:**

Crab Nebula also bright in radio (NRAO, $\nu \sim 5 \times 10^9$ Hz), less energetic electrons needed, $\gamma \sim 5 \times 10^3$, size constrained by the age of Nebula

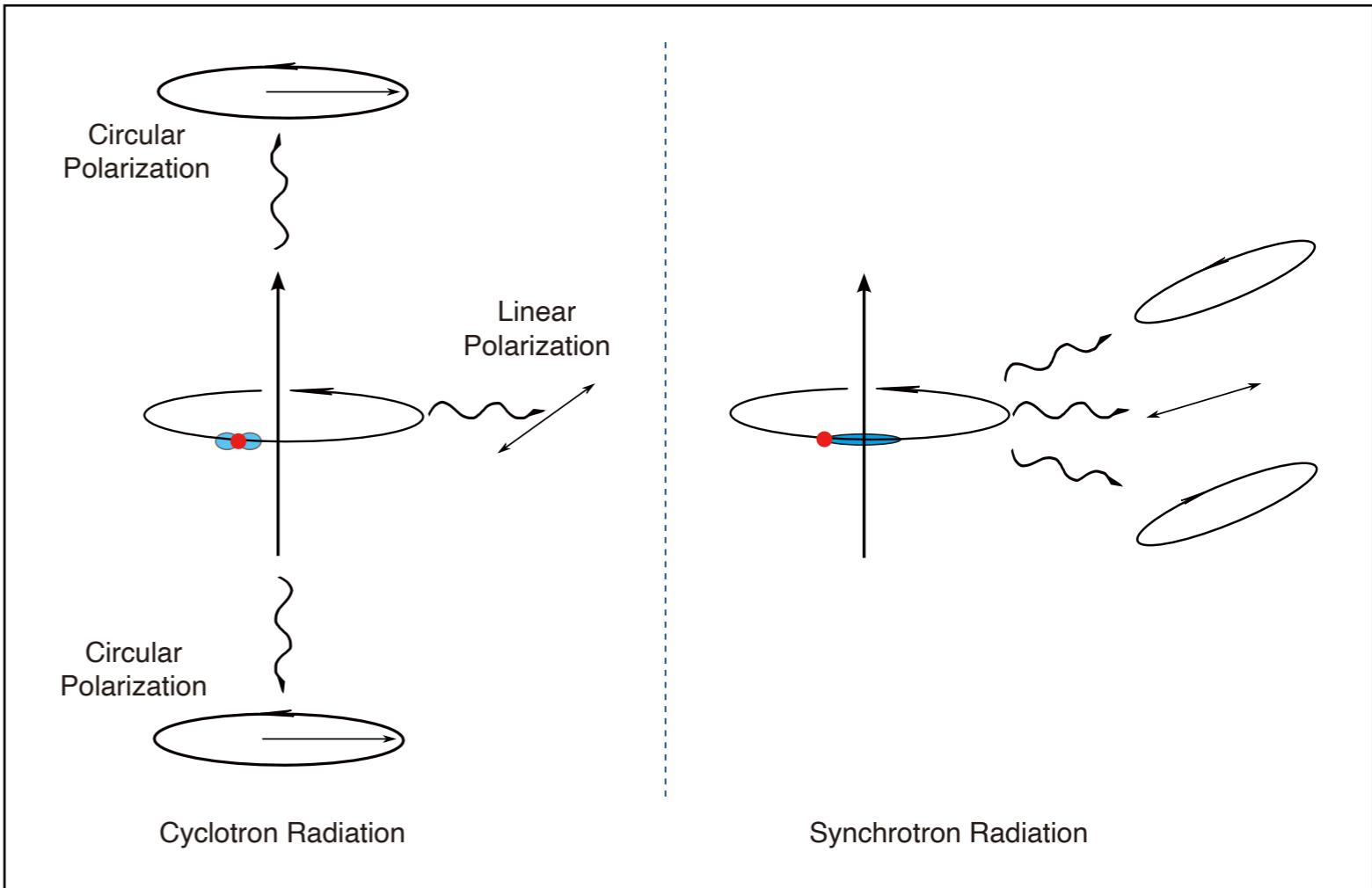
Polarization of Synchrotron Radiation



Electrons rotate counterclockwise when viewed from the positive tip of the **B** axis.

$$\text{gyrofrequency : } \omega_B = \frac{eB}{\gamma m_e c}$$

$$\text{gyroradius : } r_B = \frac{v_\perp}{\omega_B} \quad (v_\perp = v \sin \alpha)$$



[left] **Non-relativistic cyclotron motion.** When viewed in orbital plane, radiation is 100% linearly polarized with electric vector oscillating perpendicular to magnetic field \mathbf{B} . Viewed from along \mathbf{B} , emission is 100% circularly polarized. **Note that the electric vector depends only on the observer's direction and is independent of the pitch angle.**

[right] For **relativistic motion**, radiation is beamed into direction of motion. The emission for a single electron is effectively confined to within a small angle $1/\gamma$ of \mathbf{v} (The electric field depends both on the observer's direction and pitch angle). The fourth Stokes parameter is an odd function of the angle between \mathbf{n} and \mathbf{v} . The number of electrons passing with an pitch angle α is the same as that with $-\alpha$. These two components of circular polarization effectively cancel almost, whereas linear polarization largely survives.

Compton Scattering

Thomson & Compton Scattering

- The simplest interaction between photons and free electrons is scattering.
 - Thomson scattering:** When the energy of the incoming photons (as seen in the coming frame of the electron) is small with respect to the electron rest mass-energy, the process is called Thomson scattering.

$$\epsilon = \epsilon_1$$

$$\frac{d\sigma_T(\Omega)}{d\Omega} = \frac{1}{2} r_0^2 (1 + \cos^2 \theta)$$

$$\sigma_T = \frac{8\pi}{3} r_0^2$$

ϵ = energy of the incident photon

ϵ_1 = energy of the scattered photon

$$r_0 = \frac{e^2}{m_e c^2}$$

Thomson scattering condition in the rest frame:

$$\epsilon' \ll m_e c^2 = 0.5 \text{MeV}$$

- When $\epsilon = \epsilon_1$, the scattering is called **coherent or elastic**.
- Compton scattering:** As the energy of the incoming photons is comparable or greater than the electron rest mass-energy, it is called Compton scattering and a quantum treatment is necessary (Klein-Nishina regime).

[Compton Scattering: Scattering from Electrons at Rest]

- **Compton scattering:**

However, a photon carries momentum $h\nu/c$ and energy $h\nu$.

Quantum effects appear in two ways.

- (1) The scattering will no longer be elastic ($\epsilon_1 \neq \epsilon_2$) because of the recoil of the electron.
- (2) The cross sections are altered by the quantum effects.

- Conservation of momentum and energy (for the case in which **the electron is initially at rest**)

Let the initial and final four-momenta of the photon:

$$\vec{P}_{\gamma i} = (\epsilon/c)(1, \mathbf{n}_i), \quad \vec{P}_{\gamma f} = (\epsilon_1/c)(1, \mathbf{n}_f)$$

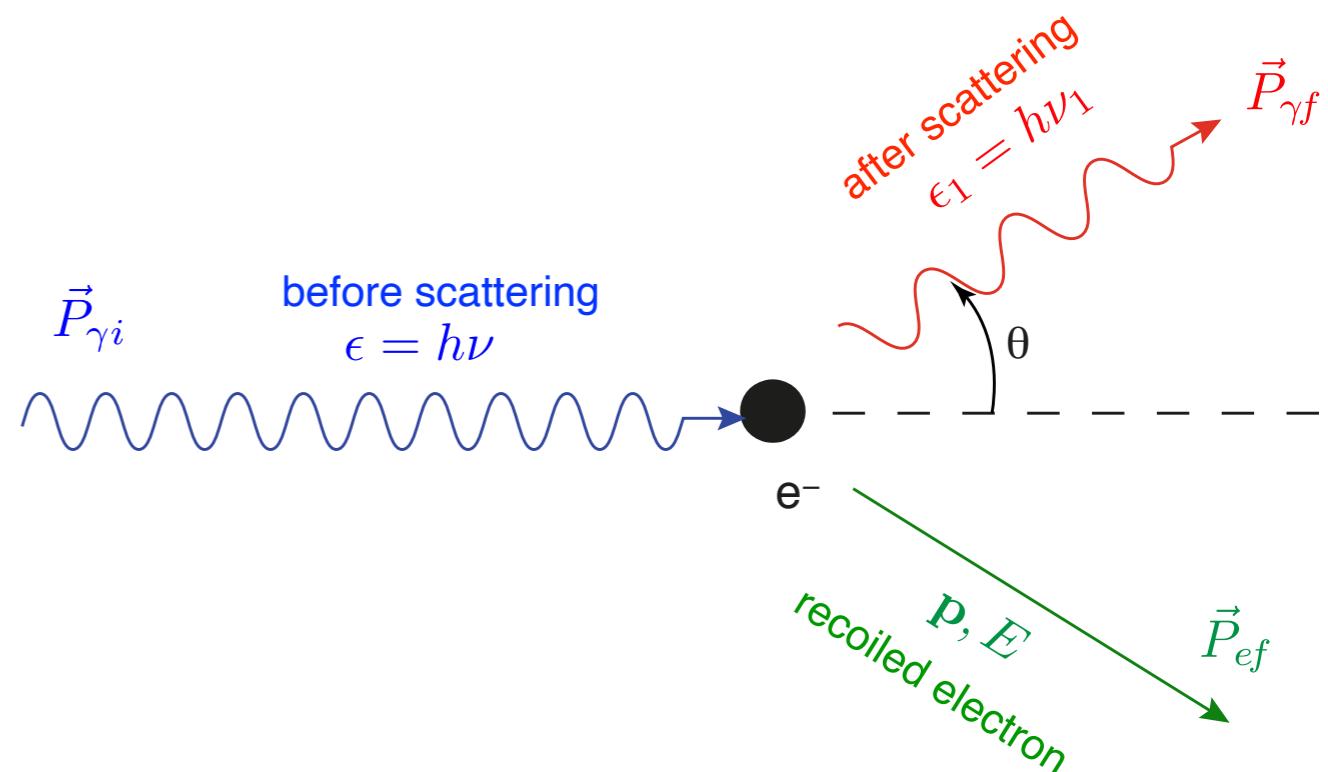
The initial and final momenta of the electron are:

$$\vec{P}_{ei} = (mc, \mathbf{0}), \quad \vec{P}_{ef} = (E/c, \mathbf{p})$$

Then, the conservation of momentum and energy is expressed by

$$\vec{P}_{ei} + \vec{P}_{\gamma i} = \vec{P}_{ef} + \vec{P}_{\gamma f}$$

Here, e and γ denote the electron and photon, respectively. i and f represent the initial and final states.



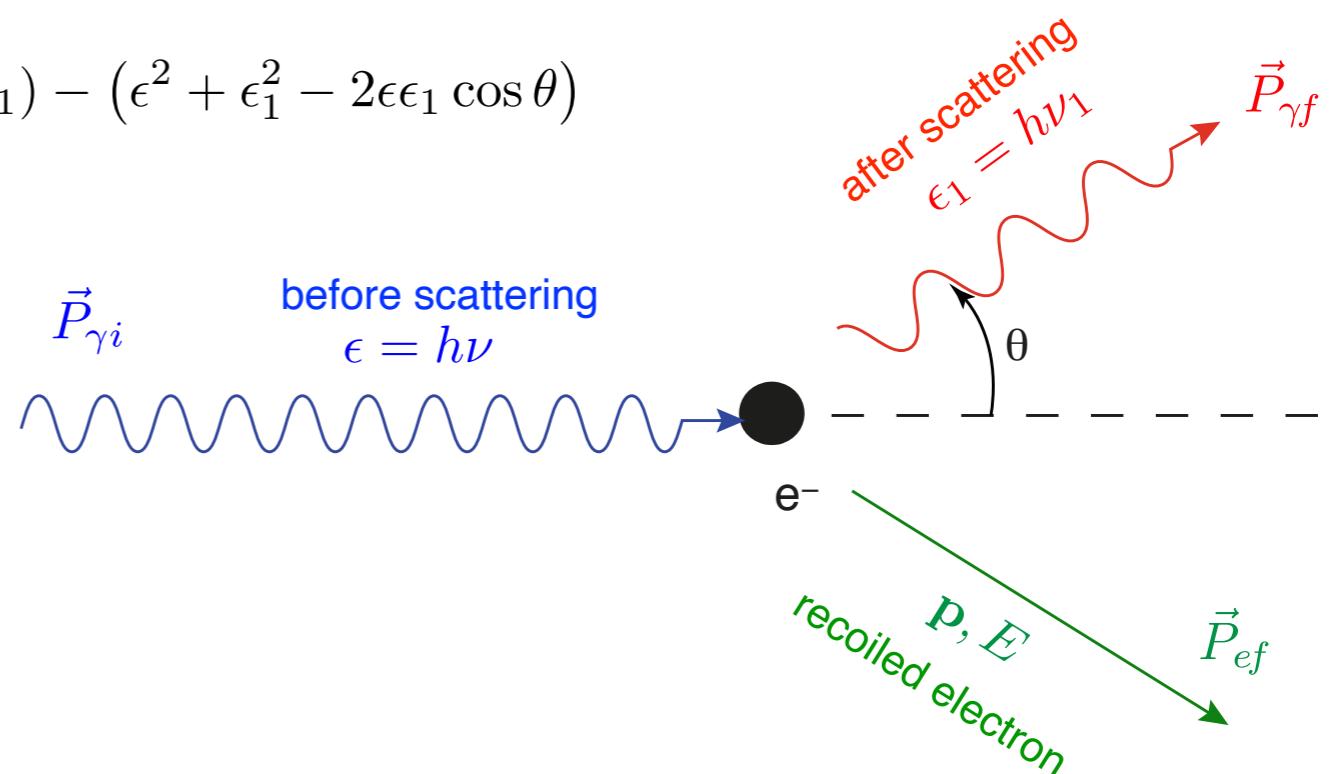
- Rearranging terms and squaring gives $\left| \vec{P}_{ef} \right|^2 = \left| \vec{P}_{ei} + \vec{P}_{\gamma i} - \vec{P}_{\gamma f} \right|^2$

$$\left| \vec{P}_{ef} \right|^2 c^2 = \left| \vec{P}_{ei} + \vec{P}_{\gamma i} - \vec{P}_{\gamma f} \right|^2 c^2$$

$$E^2 - |\mathbf{p}|^2 c^2 = (mc^2 + \epsilon - \epsilon_1)^2 - |\epsilon \mathbf{n}_i - \epsilon_1 \mathbf{n}_f|^2$$

$$(mc^2)^2 = (mc^2)^2 + \epsilon^2 + \epsilon_1^2 - 2\epsilon\epsilon_1 + 2mc^2(\epsilon - \epsilon_1) - (\epsilon^2 + \epsilon_1^2 - 2\epsilon\epsilon_1 \cos\theta)$$

$$0 = mc^2\epsilon - \epsilon_1 (\epsilon + mc^2 - \epsilon \cos\theta)$$



$$\epsilon_1 = \frac{\epsilon}{1 + \frac{\epsilon}{mc^2} (1 - \cos\theta)}$$

In terms of wavelength, $\lambda_1 - \lambda = \frac{h}{mc} (1 - \cos\theta)$

Compton wavelength: $\lambda_c \equiv \frac{h}{mc} = 0.02426 \text{ \AA}$ for electrons

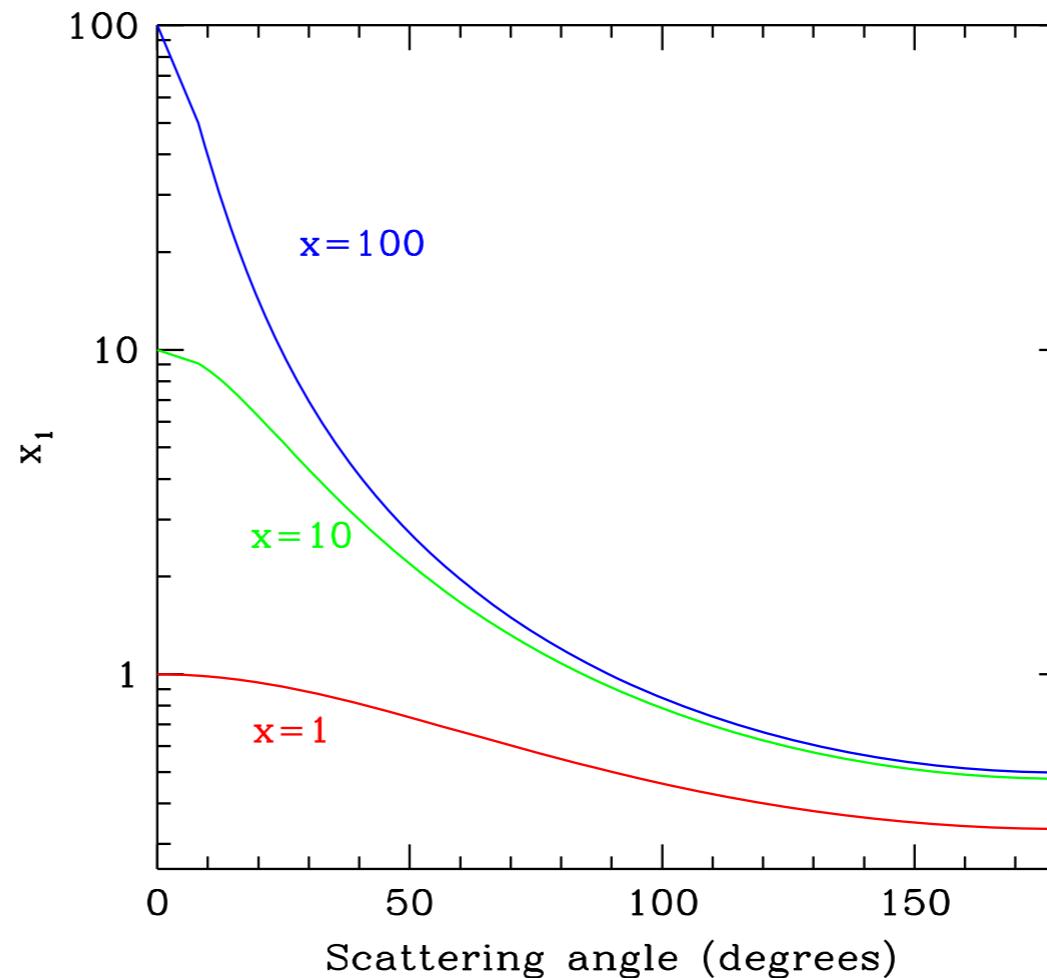
There is a wavelength change of the order of λ_c upon scattering.

For long wavelengths $\lambda \gg \lambda_c$ (i.e., $h\nu \ll mc^2$), the scattering is closely elastic.

$$\frac{\epsilon_1}{m_e c^2} = \frac{\epsilon/m_e c^2}{1 + (\epsilon/m_e c^2)(1 - \cos \theta)}$$

$$x = \frac{\epsilon}{m_e c^2}$$

$$x_1 = \frac{\epsilon_1}{m_e c^2}$$



Scattered photons energies as a function of the scattering angle, for different incoming photon energies.

Note that, for $x \gg 1$ and for large scattering angle, the scattered photon energies becomes $x_1 \sim 1/2$, independent of the initial photon energy x .

- **Klein-Nishina formula** (the differential cross section for unpolarized radiation, QED)

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{16\pi} \frac{\epsilon_1^2}{\epsilon^2} \left(\frac{\epsilon}{\epsilon_1} + \frac{\epsilon}{\epsilon_1} - \sin^2 \theta \right)$$

Total cross section:

$$\begin{aligned}\sigma &= 2\pi \int_{-1}^1 \frac{d\sigma}{d\Omega} d\cos\theta \\ &= \frac{3\sigma_T}{4} \left[\frac{1+x}{x^3} \left\{ \frac{2x(1+x)}{1+2x} - \ln(1+2x) \right\} + \frac{\ln(1+2x)}{2x} - \frac{1+3x}{(1+2x)^2} \right]\end{aligned}$$

$$\text{where } x \equiv \frac{h\nu}{mc^2}$$

Compton scattering becomes less efficient at high energies.

$$(m_e c^2 = 511 \text{ keV})$$

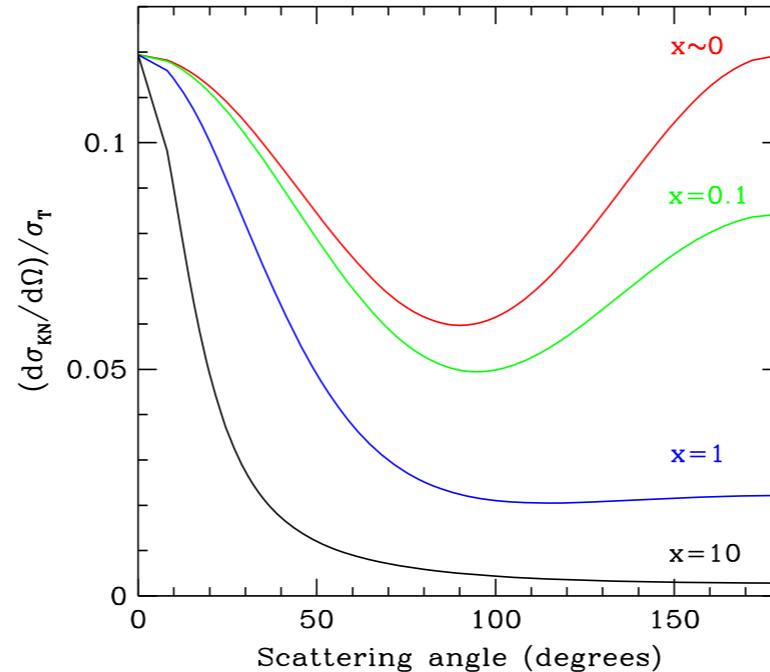
Approximations:

- nonrelativistic regime:

$$\sigma \approx \sigma_T \left(1 - 2x + \frac{26x^2}{5} + \dots \right), \quad x \ll 1$$

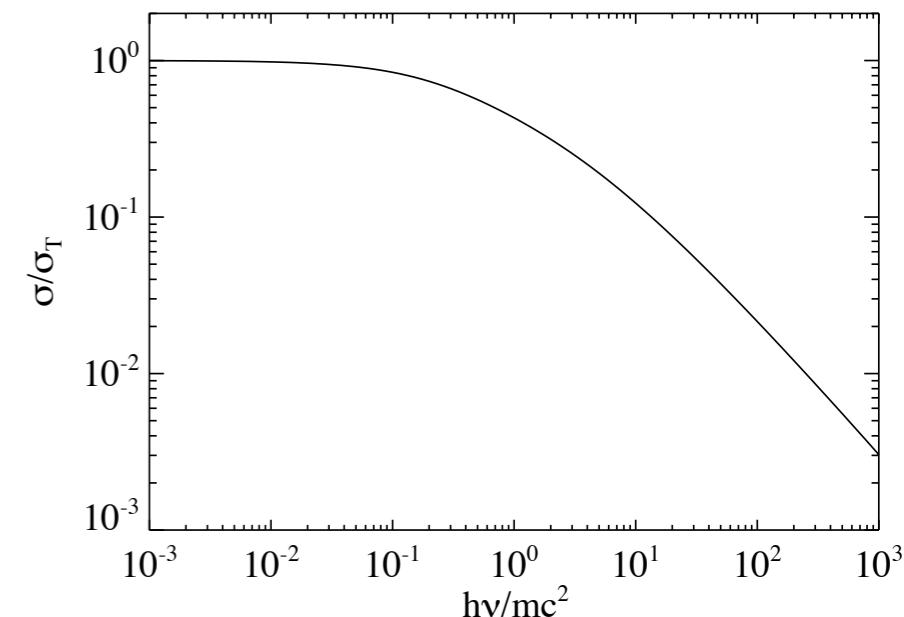
- extreme relativistic regime:

$$\sigma \approx \frac{3}{8} \sigma_T \frac{1}{x} \left(\ln 2x + \frac{1}{2} \right), \quad x \gg 1$$



Note that the scattering becomes preferentially forward as the energy of the photon increases

$$x = \frac{h\nu}{m_e c^2}$$



[Inverse Compton Scattering: Scattering from Electrons in Motion]

- **Inverse Compton Scattering:** Whenever the moving electron has sufficient kinetic energy compared to the photon, net energy may be transferred from the electron to the photon.
- What is the energy of photon after the inverse Compton scattering?
 (1) In the frame K' comoving with electron, the incoming photon energy is

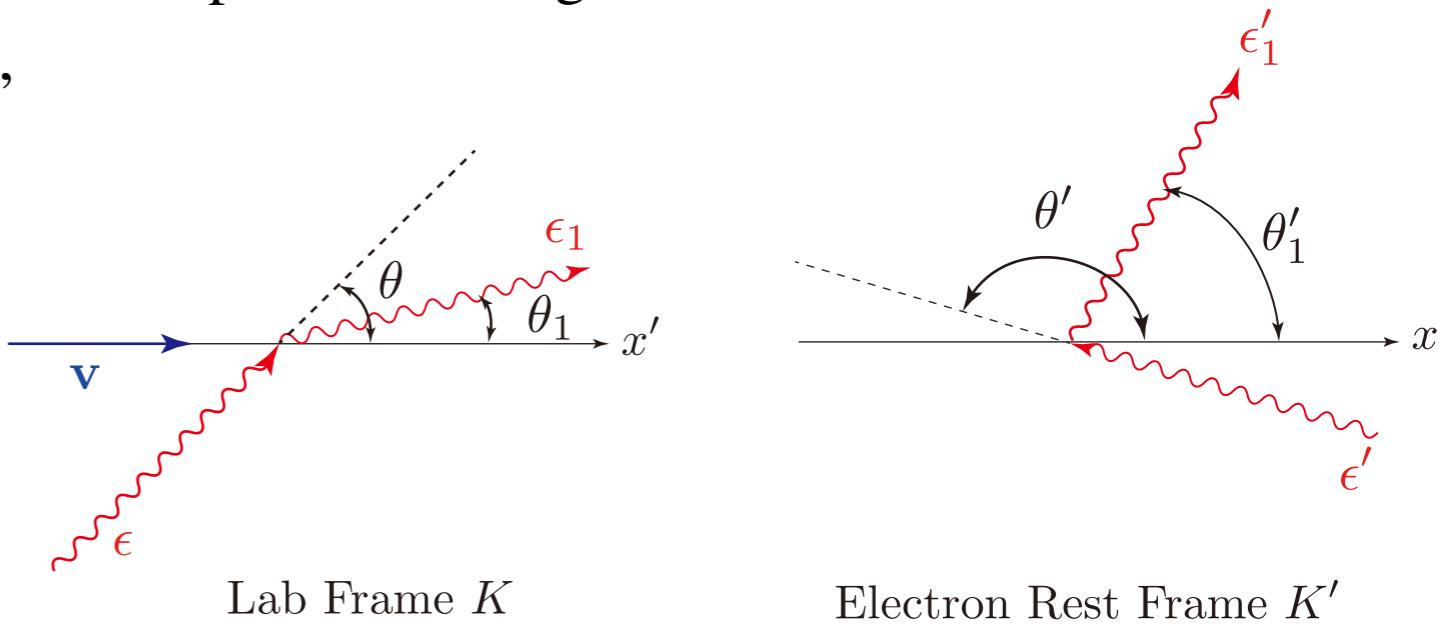
$$\epsilon' = \epsilon\gamma(1 - \beta \cos \theta)$$

Here, θ is the angle between the electron velocity and the photon direction in the lab frame.

- (2) In the electron rest frame, we assume the Thomson regime so that no change in the photon energy.

$$\begin{aligned} \epsilon'_1 &= \frac{\epsilon'}{1 + \frac{\epsilon'}{mc^2}(1 - \cos \Theta')} \\ &\approx \epsilon' \left[1 - \frac{\epsilon'}{mc^2}(1 - \cos \Theta') \right] \quad (\text{if } \epsilon' \ll mc^2) \\ &\approx \epsilon' \quad (\text{Thomson scattering condition}) \end{aligned}$$

Thomson scattering condition in the rest frame:
 $\epsilon' \ll m_e c^2 = 0.5 \text{ MeV}$



Lab Frame K

Electron Rest Frame K'

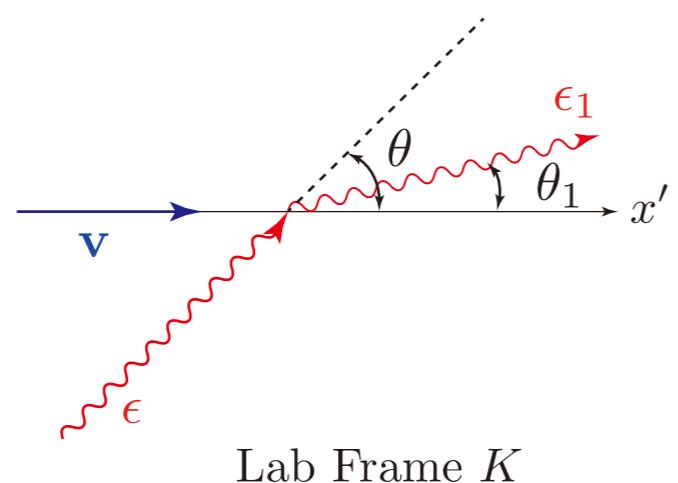
\mathbf{n}' = direction vector of incident photon in the electron rest frame
 \mathbf{n}'_1 = direction vector of scattered photon in the electron rest frame
 $\mathbf{n}' = (\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta')$
 $\mathbf{n}'_1 = (\sin \theta'_1 \cos \phi'_1, \sin \theta'_1 \sin \phi'_1, \cos \theta'_1)$
 $\cos \Theta' \equiv \mathbf{n}' \cdot \mathbf{n}'_1$
 $= \cos \theta'_1 \cos \theta' + \sin \theta' \sin \theta'_1 \cos(\phi' - \phi'_1)$
 $(\Theta' = \text{scattering angle in the electron rest frame})$

(3) Going back to the lab frame, the energy of the scattered photon is

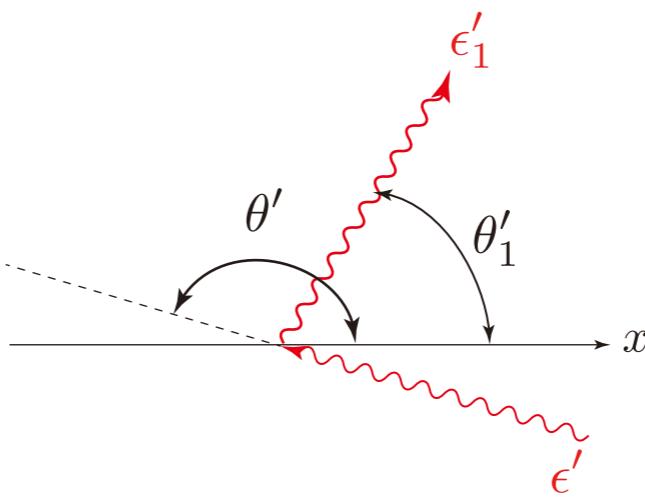
$$\begin{aligned}\epsilon_1 &= \epsilon'_1 \gamma (1 + \beta \cos \theta'_1) \\ &\approx \epsilon' \gamma (1 + \beta \cos \theta'_1) \\ &= \epsilon \gamma^2 (1 + \beta \cos \theta'_1) (1 - \beta \cos \theta)\end{aligned}$$

$$\begin{aligned}&\leftarrow \epsilon'_1 \approx \epsilon' \text{ (Thomson scattering)} \\ &\leftarrow \epsilon' = \epsilon \gamma (1 - \beta \cos \theta)\end{aligned}$$

Here, θ'_1 is the scattered angle of the photon in the electron rest frame.



Lab Frame K

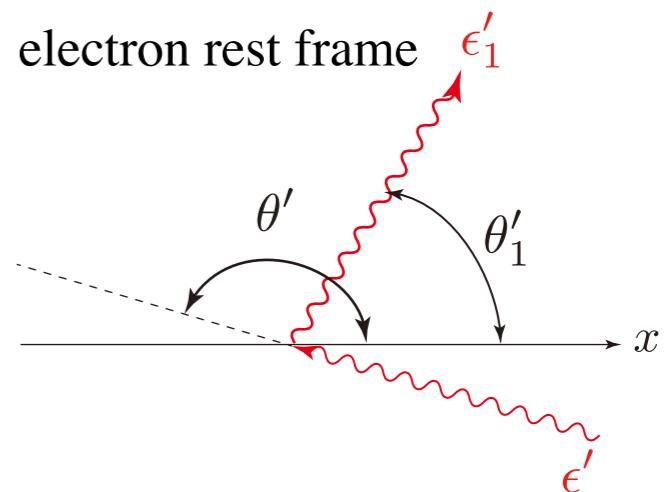


Electron Rest Frame K'

$$\epsilon_1 = \epsilon \gamma^2 (1 + \beta \cos \theta'_1) (1 - \beta \cos \theta)$$

$$= \epsilon \frac{1 - \beta \cos \theta}{1 - \beta \cos \theta_1}$$

$$\leftarrow \cos \theta'_1 = \frac{\cos \theta_1 - \beta}{1 - \beta \cos \theta_1} \text{ (aberration)}$$



- Let's assume isotropic distribution of photons.

In the electron rest frame, most photons will be incident toward the electron, i.e., $\pi - \theta' \lesssim 1/\gamma$.

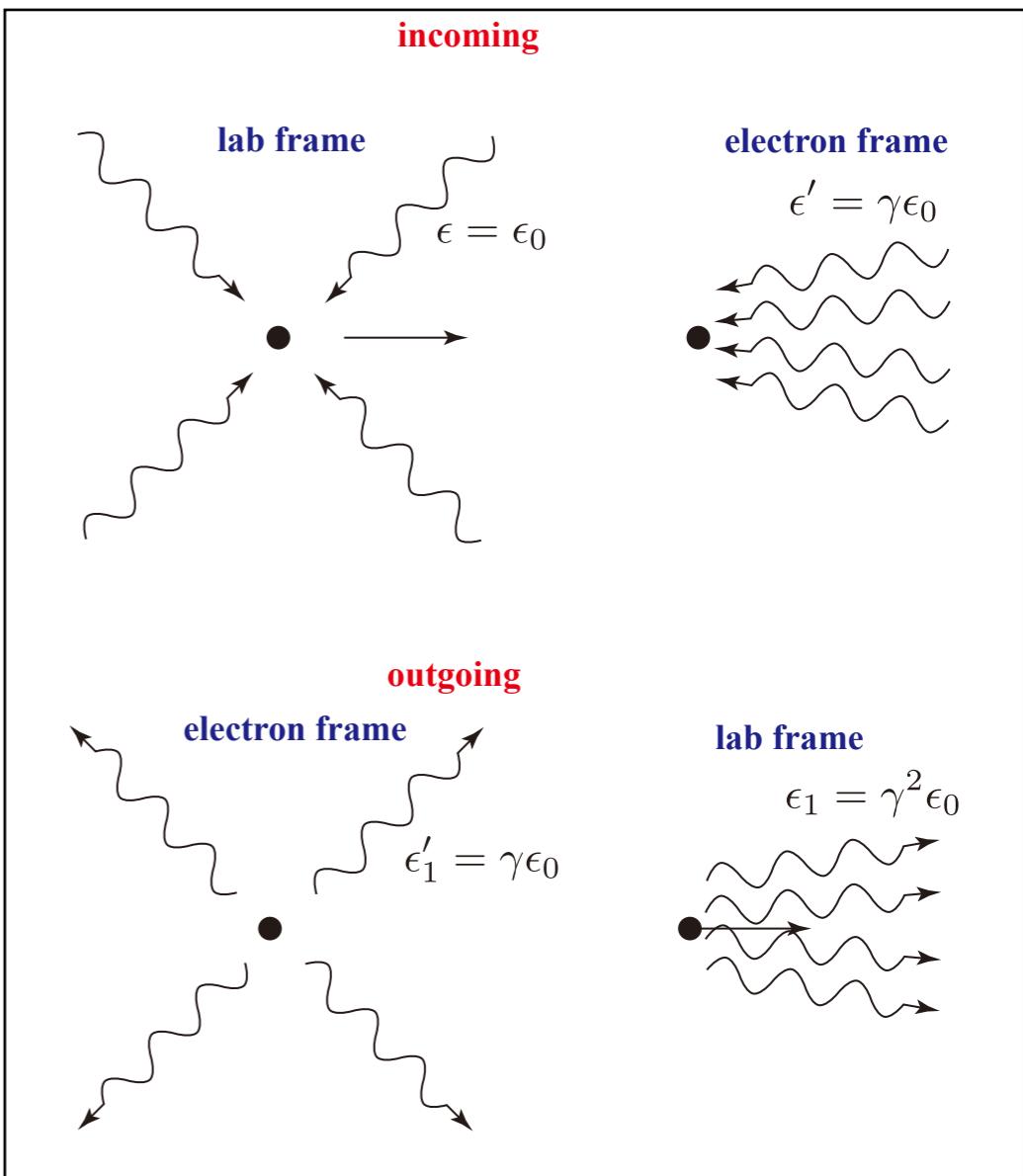
Thomson scattering is symmetric with respect to forward and backward scattering. Therefore, in the lab frame, the scattered photon will be mostly concentrated within a narrow angle: $\theta_1 \lesssim 1/\gamma$. Assuming that $\cos \theta_1 \approx \beta$, we obtain

$$\epsilon_1 \approx \epsilon \frac{1 - \beta \cos \theta}{1 - \beta^2} = (1 - \beta \cos \theta) \gamma^2 \epsilon \rightarrow \boxed{\langle \epsilon_1 \rangle \approx \gamma^2 \epsilon}$$

(because $\langle \cos \theta \rangle = 0$)

See the following slides for a precise formula.

$$\boxed{\langle \epsilon_1 \rangle = \gamma^2 \left(1 + \frac{\beta^2}{3}\right) \epsilon}$$



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- The energies of the photon before scattering, in the electron rest frame, and after the scattering in the lab frame are in the approximate ratios

The inverse Compton scattering converts a low-energy photon to a high-energy photon by a factor of order γ^2

$$\epsilon : \epsilon' : \epsilon_1 \approx 1 : \gamma : \gamma^2$$

[Inverse Compton Power for Single Scattering]

- Assumptions:
 - (1) Isotropic distributions of photons and electrons.
 - (2) The change in energy of the photon in the rest frame is negligible.
(Thomson scattering is applicable in the electron's rest frame). $\epsilon'_1 \approx \epsilon'$

- **Total power scattered in the electron's rest frame:**

$$\frac{dE'_1}{dt'} = c\sigma_T \int \epsilon'_1 n'_\epsilon d\epsilon' \quad \text{where } n'_\epsilon d\epsilon' \text{ is the number density of incident photons.}$$

- Recall: $\frac{dE_1}{dt} = \frac{dE'_1}{dt'}$ since energy and time transforms in the same way.

$d^3\mathbf{p} = \gamma d^3\mathbf{p}'$ transforms in the same way as energy.

$n_p \equiv \frac{dN}{d\mathcal{V}} \left(= \frac{d^6 N}{d^3 \mathbf{x} d^3 \mathbf{p}} \right)$ is a Lorentz invariant. (density in the phase space)

$n_p d^3\mathbf{p} = n_\epsilon d\epsilon$ The number densities of incident photons, represented in terms of
 $n_\epsilon d\epsilon = \gamma n'_\epsilon d\epsilon'$ momentum and energy, transforms in the same way as energy.

$$\therefore \frac{n_\epsilon d\epsilon}{\epsilon} = \frac{n'_\epsilon d\epsilon'}{\epsilon'}$$

- Thus we have the results

$$\frac{dE_1}{dt} = \frac{dE'_1}{dt'} = c\sigma_T \int \epsilon'_1 n'_\epsilon d\epsilon' = c\sigma_T \int \epsilon'^2 \frac{n'_\epsilon d\epsilon'}{\epsilon'} = c\sigma_T \int \epsilon'^2 \frac{n_\epsilon d\epsilon}{\epsilon}$$

$$= c\sigma_T \gamma^2 \int (1 - \beta \cos \theta)^2 \epsilon n_\epsilon d\epsilon \quad \leftarrow \quad \epsilon' = \epsilon \gamma (1 - \beta \cos \theta)$$

$\epsilon'_1 \approx \epsilon' \quad \text{Thomson scattering assumption in the rest frame}$

For an isotropic distribution of photons, integrating over θ , we have

$$\langle (1 - \beta \cos \theta)^2 \rangle = 1 + \frac{1}{3}\beta^2 \quad \leftarrow \quad \langle \cos \theta \rangle = 0, \quad \langle \cos^2 \theta \rangle = 1/3$$

Therefore, we obtain the **total power scattered in the lab frame**:

$$\frac{dE_1}{dt} = c\sigma_T \gamma^2 \left(1 + \frac{1}{3}\beta^2\right) U_{\text{ph}}$$

where $U_{\text{ph}} \equiv \int \epsilon n_\epsilon d\epsilon$ is the initial photon energy density.

Note that the **rate of decrease of the total initial photon energy** is

$$\frac{dE_1^{\text{loss}}}{dt} = -c\sigma_T \int \epsilon n_\epsilon d\epsilon = -c\sigma_T U_{\text{ph}}$$

The incident power and scattered power in the lab frame can be represented in term of the rate of scattering (per unit time) and the initial and final photon energies.

$$\left| \frac{dE_1^{\text{loss}}}{dt} \right| = \epsilon \frac{dN_{\text{scatt}}}{dt}$$

$$\frac{dE_1}{dt} = \langle \epsilon_1 \rangle \frac{dN_{\text{scatt}}}{dt}$$

Then, the mean photon energy after scattering will be

$$\langle \epsilon_1 \rangle = \gamma^2 \left(1 + \frac{1}{3}\beta^2\right) \epsilon$$

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- Thus, **the net power lost by the electron, and converted into increased radiation, is**

$$P_{\text{compt}} \equiv \frac{dE_1}{dt} - \left| \frac{dE_1^{\text{loss}}}{dt} \right| = c\sigma_T U_{\text{ph}} \left[\gamma^2 \left(1 + \frac{1}{3}\beta^2 \right) - 1 \right]$$

$$\therefore P_{\text{compt}} = \frac{4}{3}c\sigma_T\gamma^2\beta^2U_{\text{ph}}$$



$$\gamma^2 - 1 = \gamma^2\beta^2$$

- **When the energy transfer in the electron rest frame is not neglected,** the power is given by

$$P_{\text{compt}} = \frac{4}{3}c\sigma_T\gamma^2\beta^2U_{\text{ph}} \left[1 - \frac{63}{10} \frac{\gamma \langle \epsilon^2 \rangle}{mc^2 \langle \epsilon \rangle} \right] \quad (\text{cf. Blumenthal \& Gould, 1970})$$

Note that the above equation allows energy to be either given or taken from the photons.

- Recall that the formula for the synchrotron power emitted by each electron is

$$P_{\text{synch}} = \frac{4}{3} \sigma_T c \gamma^2 \beta^2 U_B$$

Therefore,

$$\frac{P_{\text{synch}}}{P_{\text{compt}}} = \frac{U_B}{U_{\text{ph}}}$$

The radiation losses due to synchrotron emission and to inverse Compton effect are in the same ratio as the magnetic field energy density and photon energy density.

- Let $N(\gamma)d\gamma$ be the number of electrons per unit volume. Then, the total Compton power per unit volume is

$$P_{\text{tot}} = \int P_{\text{compt}} N(\gamma) d\gamma$$

- (1) Power-law distribution of relativistic electrons ($\beta \sim 1$)

$$N(\gamma) = \begin{cases} C\gamma^{-p}, & \gamma_{\min} \leq \gamma \leq \gamma_{\max} \\ 0, & \text{otherwise} \end{cases} \longrightarrow P_{\text{tot}} = \frac{4}{3} \sigma_T c U_{\text{ph}} C (3-p)^{-1} (\gamma_{\min}^{3-p} - \gamma_{\max}^{3-p})$$

- (2) Thermal distribution of nonrelativistic electrons ($\gamma \sim 1$) of number density n_e .

$$\langle \beta^2 \rangle = \langle v^2/c^2 \rangle = 3kT/mc^2 \quad \longrightarrow \quad P_{\text{tot}} = \left(\frac{4kT}{mc^2} \right) \sigma_T c n_e U_{\text{ph}}$$

$\gamma \approx 1$

Note that $\frac{4kT}{mc^2}$ is the fractional photon energy gain because $\frac{dE^{\text{loss}}}{dt} = -c\sigma_T U_{\text{ph}}$.

[Spectrum of single-scattered photons by power-law electrons]

- We will first show that a power-law photon distribution is produced from a power-law electron distribution.

The photon energy increases after a single scattering by a factor proportional to γ^2 .

$$\langle \epsilon_1 \rangle = \gamma^2 \left(1 + \frac{1}{3} \beta^2 \right) \epsilon = \frac{4}{3} \gamma^2 \epsilon \quad \text{as } \beta \rightarrow 1$$

Therefore, the (energy) spectrum scattered power per unit volume per unit energy due to electrons with a power-law distribution is

$$\begin{aligned}
 \frac{dE}{dV dt d\epsilon_1} &= \int N(\gamma) \overset{\text{energy}}{\epsilon_1} \delta(\epsilon_1 - (4/3)\epsilon\gamma^2) d\gamma \\
 &\propto \int_{\gamma_1}^{\gamma_2} \epsilon_1 \gamma^{-p} \delta(\epsilon_1 - x) \frac{d\gamma}{dx} dx \\
 &\propto \int_{\gamma_1}^{\gamma_2} \epsilon_1 \gamma^{-p} \delta(\epsilon_1 - x) \frac{\gamma}{2x} dx \\
 &\propto \int_{\gamma_1}^{\gamma_2} \epsilon_1 x^{-(p-1)/2-1} \delta(\epsilon_1 - x) dx \\
 &\propto \epsilon_1^{-(p-1)/2}
 \end{aligned}$$

$$\frac{dE}{dV dt d\epsilon_1} \propto \epsilon_1^{-(p-1)/2}$$

The resulting spectrum has the same power-law slope as that of the synchrotron.

[Repeated Scattering: The Compton y Parameter]

- We restrict our considerations to situations in which the Thomson limit applies: $\gamma\epsilon \ll mc^2$
- **Compton y parameter**, to determine whether a photon will significantly change its energy in traversing the medium:

$$y \equiv \left(\begin{array}{l} \text{average fractional} \\ \text{energy change per} \\ \text{scattering} \end{array} \right) \times \left(\begin{array}{l} \text{mean number of} \\ \text{scatterings} \end{array} \right)$$

When $y \gtrsim 1$, the total photon energy and spectrum will be significantly altered; whereas for $y \ll 1$, the total energy is not much changed.

- **Average fractional energy change per scattering** (for a thermal distribution of electrons)
 - (a) Consider first the nonrelativistic limit.

$$\epsilon'_1 \approx \epsilon' \left[1 - \frac{\epsilon'}{mc^2} (1 - \cos \Theta) \right] \rightarrow \left\langle \frac{\Delta\epsilon'}{\epsilon'} \right\rangle \equiv \left\langle \frac{\epsilon'_1 - \epsilon'}{\epsilon'} \right\rangle = -\frac{\epsilon'}{mc^2} : \text{angle average}$$

In the lab frame to lowest order, this must be of the form

$$\left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle = -\frac{\epsilon}{mc^2} + \alpha \frac{kT}{mc^2}$$

To calculate α , **imagine that the photons and electrons are in complete equilibrium but interact only through scattering.**

Assume that the photon density is sufficiently small that stimulated processes can be neglected. Then, we obtain the Wien's law for the photon distribution:

$$n_\epsilon = K \epsilon^2 \exp\left(-\frac{\epsilon}{kT}\right)$$

We have the averages

$$\langle \epsilon \rangle \equiv \int \epsilon n_\epsilon d\epsilon / \int n_\epsilon d\epsilon = 3kT$$

$$\langle \epsilon^2 \rangle \equiv \int \epsilon^2 n_\epsilon d\epsilon / \int n_\epsilon d\epsilon = 12(kT)^2$$

For this case, no net energy can be transferred from photons to electrons, so

$$\Delta\epsilon = 0 = -\frac{\langle \epsilon^2 \rangle}{mc^2} + \alpha \frac{kT}{mc^2} \langle \epsilon \rangle = \frac{3kT}{mc^2}(\alpha - 4)kT \rightarrow \alpha = 4$$

Thus for nonrelativistic electrons in thermal equilibrium, the energy transfer per scattering is

$$(\Delta\epsilon)_{\text{NR}} = \frac{\epsilon}{mc^2}(4kT - \epsilon)$$

Note that if the electrons have high enough temperature relative to incident photons, the photons gain energy. This is the inverse Compton scattering.

If $\epsilon > 4kT$, on the other hand, energy is transferred from photons to electrons.

(b) In the ultrarelativistic limit ($\gamma \gg 1, \beta \approx 1$), ignoring the energy transfer in the electron rest frame,

$$\frac{P_{\text{compt}}}{|dE_1^{\text{loss}}/dt|} = \frac{4/3\sigma_T c \gamma^2 \beta^2 U_{\text{ph}}}{\sigma_T c U_{\text{ph}}} = \frac{4}{3} \gamma^2 \beta^2 \rightarrow (\Delta\epsilon)_R \approx \frac{4}{3} \gamma^2 \epsilon$$

For a thermal distribution of ultrarelativistic electrons,

$$\langle \gamma^2 \rangle = \frac{\langle \epsilon^2 \rangle}{(mc^2)^2} = 12 \left(\frac{kT}{mc^2} \right)^2 \longrightarrow (\Delta\epsilon)_R \approx 16\epsilon \left(\frac{kT}{mc^2} \right)^2$$

- **Mean number of scatterings**,

Recall that, for a pure scattering medium,

$$\left(\begin{array}{c} \text{mean number of} \\ \text{scatterings} \end{array} \right) \approx \text{Max}(\tau_{\text{es}}, \tau_{\text{es}}^2)$$

where $\tau_{\text{es}} \sim \rho \kappa_{\text{es}} R$

$$\kappa_{\text{es}} = \frac{\sigma_T}{m_p} = 0.40 \text{ cm}^2 \text{ g}^{-1} \text{ for ionized hydrogen}$$

R = size of the finite medium

- **Compton y parameter**:

$$y_{\text{NR}} = \frac{4kT}{mc^2} \text{Max}(\tau_{\text{es}}, \tau_{\text{es}}^2)$$

$$y_R = 16\epsilon \left(\frac{kT}{mc^2} \right)^2 \text{Max}(\tau_{\text{es}}, \tau_{\text{es}}^2)$$

[Repeated Scattering: Spectra and Power]

- We have already shown that a power-law spectrum may be a natural consequence of a power-law distribution of electrons.
- **We will show that a power-law photon distribution can also be produced from repeated scattering off a nonpower-law electron distribution.**

Let A = the mean amplification of photon energy per scattering

$$\begin{aligned} A &\equiv \frac{\epsilon_1}{\epsilon} \sim \frac{4}{3} \langle \gamma^2 \rangle \\ &= 16 \left(\frac{kT}{mc^2} \right)^2 \quad \text{for thermal electron distribution} \end{aligned}$$

mean photon energy = ϵ_i

(number) intensity = $I(\epsilon_i)$ at ϵ_i

After k scattering, the photon energy will be $\epsilon_k \sim \epsilon_i A^k$.

For an optically thin scattering medium ($\tau_{\text{es}} < 1$), the probability of a photon undergoing k scattering before escaping the medium is $p_k(\tau_{\text{es}}) \sim \tau_{\text{es}}^k$.

The emergent intensity at energy ϵ_k is given by

$$I(\epsilon_k) \sim I(\epsilon_i) \tau_{\text{es}}^k \sim I(\epsilon_i) \tau_{\text{es}}^{\ln(\epsilon_k/\epsilon_i)/\ln A} = I(\epsilon_i) \left(\frac{\epsilon_k}{\epsilon_i} \right)^{\ln \tau_{\text{es}} / \ln A}$$

$$\begin{aligned} \tau^{\ln(\epsilon_k/\epsilon_i)/\ln A} &= x \\ [\ln(\epsilon_k/\epsilon_i) / \ln A] \ln \tau &= \ln x \\ \ln(\epsilon_k/\epsilon_i)^{\ln \tau / \ln A} &= \ln x \\ (\epsilon_k/\epsilon_i)^{\ln \tau / \ln A} &= x \end{aligned}$$

$\therefore I(\epsilon_k) \sim I(\epsilon_i) \left(\frac{\epsilon_k}{\epsilon_i} \right)^{-\alpha}$ where $\alpha \equiv -\frac{\ln \tau_{\text{es}}}{\ln A}$ \longrightarrow power-law shape

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- Total Compton power in the output spectrum is given by

$$P \propto \int I(\epsilon_k) d\epsilon_k = I(\epsilon_i) \epsilon_i \left[\int x^{-\alpha} dx \right]$$

The factor in square brackets is approximately **the factor by which the initial power $I(\epsilon_i)\epsilon_i$ is amplified** in energy.

Clearly, this amplification will be important if $\alpha \ll 1$. Therefore, **energy amplification of a soft photon input spectrum is important when**

$$\alpha = \frac{-\ln \tau_{\text{es}}}{\ln A} \lesssim 1 \rightarrow \ln (\tau_{\text{es}} A) \gtrsim 0$$

$$\rightarrow y = A\tau_{\text{es}} \sim 16 \left(\frac{kT}{mc^2} \right)^2 \tau_{\text{es}} \gtrsim 1$$