

Interstellar Medium (ISM)

Week 4

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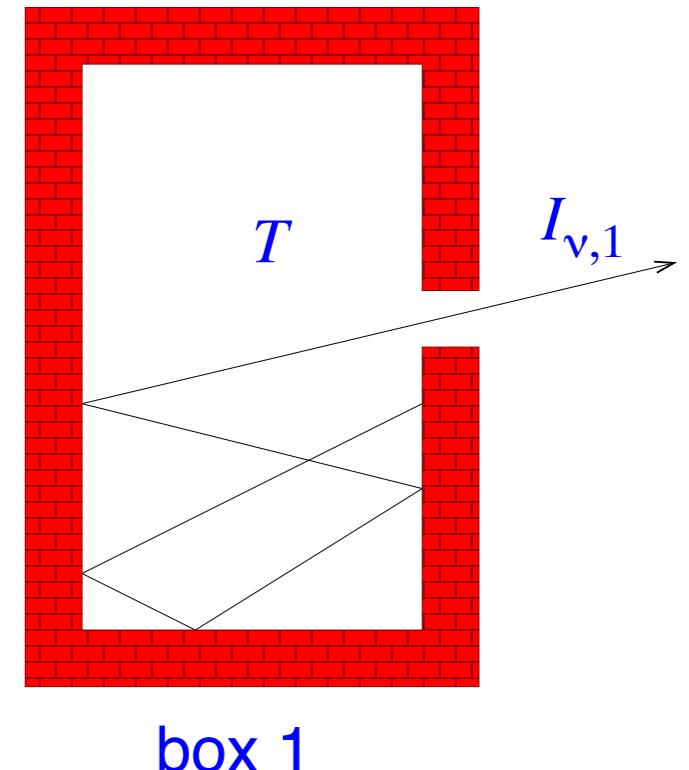
Thermal equilibrium

- In general, equilibrium means a state of balance.
 - ***Thermal equilibrium refers to steady states of temperature, which defines the average energy of material or photons.***
 - In a state of (complete) ***thermodynamic equilibrium (TE)***, no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system. ***In TE, matter and radiation are in equilibrium at the same temperature T.***
 - If the material is (locally) in thermodynamic equilibrium at a well-defined temperature T , ***it is said to be in local thermodynamic equilibrium (LTE) even if it is not in equilibrium with the radiation field.***

Blackbody

- Imagine a container bounded by opaque walls with a very small hole.

- ***Photons will be scattered and absorbed many times, (and eventually trapped and completely absorbed in the box).*** Under such conditions, the particles and photons continually share their kinetic energies. In perfect thermal equilibrium, ***the average particle kinetic energy will be equal to the average photon energy, and a unique temperature T can be defined.***

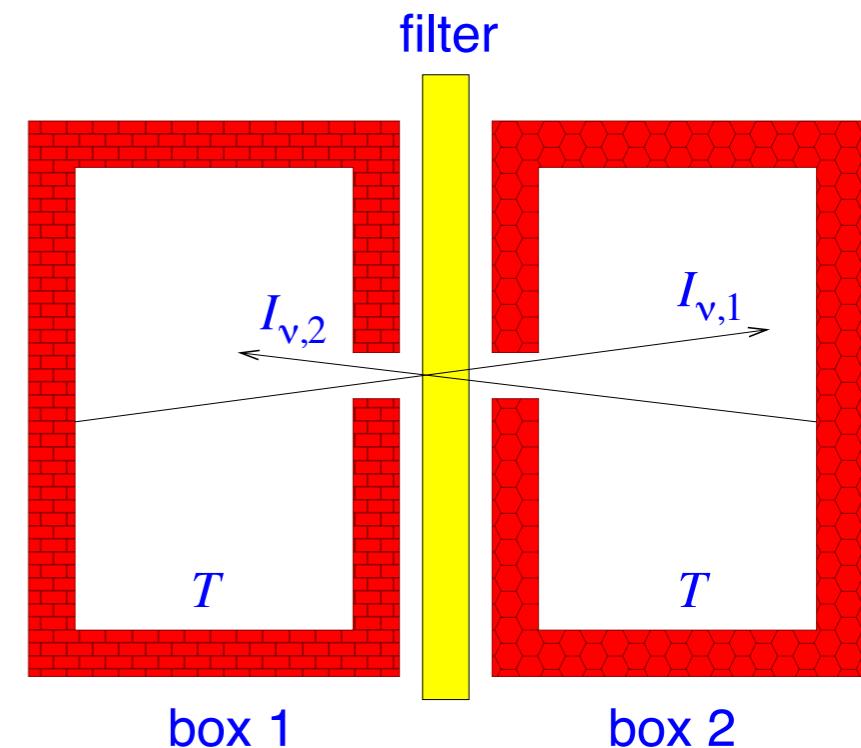


- A **blackbody** is an idealized physical body that absorbs all incident radiation regardless of frequency or angle of incidence (i.e., perfect absorber). The above cavity can be regarded to be a blackbody.
- Radiation from a blackbody in thermal equilibrium is called the **blackbody radiation**.

Blackbody radiation if the universal function.

- Now, consider another cavity (box 2), also at the same temperature, but made of different material or shape and connect two cavities with a filter transparent only in the narrow frequency range ν and $\nu + d\nu$.

- In equilibrium at T , radiation should transfer no net energy from one cavity to the other. Otherwise, one cavity will cool down and the other heats up; this violates the second law of thermodynamics.
- Therefore, the intensity or spectrum that passes through the holes should be a universal function of T and should be isotropic.
- The intensity and spectrum of the radiation emerging from the hole should be independent of the wall material (e.g., wood, copper, or concrete, etc) and any absorbing material that may be inside the cavity.



- The universal function is called the Planck function $B_\nu(T)$.
- This is the **blackbody radiation**.

Kirchhoff's Law in TE

- In (full) thermodynamic equilibrium at temperature T , by definition, we know that

$$\frac{dI_\nu}{ds} = 0 \quad \text{and} \quad I_\nu = B_\nu(T)$$

- We also note that

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu$$

- Then, we can obtain ***the Kirchhoff's law for a system in TE:***

$$\frac{j_\nu(T)}{\kappa_\nu(T)} = B_\nu(T)$$

- This is remarkable because it connects the properties $j_\nu(T)$ and $\kappa_\nu(T)$ of any kind of matter to the single universal spectrum $B_\nu(T)$.

Kirchhoff's Law in LTE

- Recall that Kirchhoff's law was derived for a system in thermodynamic equilibrium.
- ***Kirchhoff's law applies not only in TE but also in LTE:***
 - Recall that $B_\nu(T)$ is independent of the properties of the radiating /absorbing material.
 - In contrast, both $j_\nu(T)$ and $\kappa_\nu(T)$ depend only on the materials in the cavity and on the temperature of that material; they do not depend on the ambient radiation field or its spectrum.
 - Therefore, the Kirchhoff's law should be true even for the case of LTE.
 - ***In LTE, the ratio of the emission coefficient to the absorption coefficient is a function of temperature and frequency only.***
 - This generalized version of Kirchhoff's law is an exceptionally valuable tool for calculating the emission coefficient from the absorption coefficient or vice versa.

Implications of Kirchhoff's Law

- A good absorber is a good emitter, and a poor absorber is a poor emitter. (In other words, a good reflector must be a poor absorber, and thus a poor emitter.)

$$j_\nu = \kappa_\nu B_\nu(T) \rightarrow j_\nu \text{ increases as } \kappa_\nu \text{ increases}$$

- It is not possible to thermally radiate more energy than a blackbody, at equilibrium.

$$j_\nu < B_\nu(T) \text{ because } \kappa_\nu < 1$$

- The radiative transfer equation in LTE can be rewritten:

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T)$$

- ***Blackbody radiation vs. Thermal radiation***

- ***Blackbody radiation*** means $I_\nu = B_\nu(T)$. An object for which the intensity is the Planck function is emitting blackbody radiation.
- ***Thermal radiation is defined to be radiation emitted by “matter” in LTE***. Thermal radiation means $S_\nu = B_\nu(T)$. An object for which the source function is the Planck function is emitting thermal radiation.
- ***Thermal radiation becomes blackbody radiation only for optically thick media.***

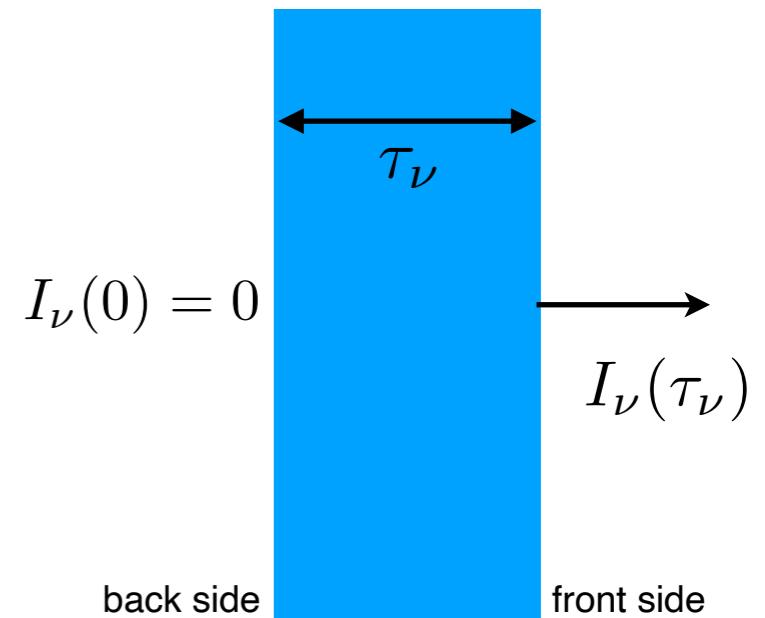
Blackbody Radiation vs. Thermal Radiation

- To see the difference between thermal and blackbody radiation,
 - Consider a slab of material with optical depth τ_ν that is producing thermal radiation.
 - If no light is falling on the back side of the slab, the intensity that is measured on the front side of the slab is

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu}) \\ I_\nu(0) = 0 \quad \longrightarrow \quad S_\nu = B_\nu &= B_\nu (1 - e^{-\tau_\nu}) \end{aligned}$$

- If the slab is optical thick at frequency ν ($\tau_\nu \gg 1$), then

$$I_\nu = B_\nu \quad \text{as } \tau_\nu \rightarrow \infty$$



- If the slab is optically thin ($\tau_\nu \ll 1$), then

$$I_\nu \approx \tau_\nu B_\nu \ll B_\nu \quad \text{as } \tau_\nu \ll 1$$

This indicates that the radiation, although it is thermal, will not be blackbody radiation.

Thermal radiation becomes blackbody radiation only for optical thick media.

The Planck Spectrum (Quantum Mechanics)

- There is no perfect blackbody.
 - However, the cosmic microwave background comes quite close; stars can sometimes be usefully approximated as blackbodies.
 - By the end of the 19th century, the blackbody spectrum was fairly well known empirically, from laboratory studies. In 1900, Max Planck, using his idea of quantized energies, derived the blackbody spectrum.
- How to calculate the blackbody spectrum?
 - Intensity spectrum is related to the energy density:

$$J_\nu = \frac{c}{4\pi} u_\nu$$

- Energy density =
Number density of photon states x Average energy of each state
- Number density of photon states =
number of states per solid angle per volume per frequency

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- (1) Number density of photon state:
 - Consider a photon propagating in direction \mathbf{n} inside a box with dimensions L_x, L_y, L_z in x, y, z directions.

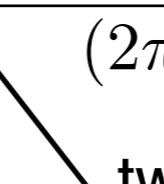
- wave vector: $\mathbf{k} = \frac{2\pi}{\lambda} \mathbf{n} = \frac{2\pi\nu}{c} \mathbf{n}$

- If each dimension of the box is much longer than a wavelength, the photon can be represented by standing wave in the box.
- Number of nodes in each direction: $n_x = L_x/\lambda = k_x L_x / 2\pi$
- Number of node changes in a wave number interval (if $n_i \gg 1$):

$$\Delta n_x = \frac{L_x \Delta k_x}{2\pi}$$

- Number of states in 3D wave vector element $\Delta k_x \Delta k_y \Delta k_z = d^3 k$:

$$\Delta N = \Delta n_x \Delta n_y \Delta n_z = 2 \frac{L_x L_y L_z d^3 k}{(2\pi)^3} = 2 \frac{V d^3 k}{(2\pi)^3}$$


 two independent polarizations

-
- the density of states (**number of states per solid angle per volume per frequency**):

$$d^3k = k^2 dk d\Omega = \frac{(2\pi)^3 \nu^2 d\nu d\Omega}{c^3}$$

$$\rightarrow \rho_s = \frac{dN}{V d\nu d\Omega} = \frac{2\nu^2}{c^3}$$

- (2) Average energy of each state:

- Each state may contain n photons of energy $h\nu$.
- The energy of the state is $E_n = nh\nu$.
- The probability of a state of energy E_n is proportional to $e^{-\beta E_n}$, where $\beta = (k_B T)^{-1}$ and k_B is the Boltzmann's constant. (from statistical mechanics)
- Therefore, the average energy is:

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = -\frac{\partial}{\partial \beta} \ln \left(\sum_{n=0}^{\infty} e^{-\beta E_n} \right)$$

$$\sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} (e^{-\beta h\nu})^n = (1 - e^{-\beta h\nu})^{-1}$$

$$\langle E \rangle = \frac{h\nu e^{-\beta h\nu}}{1 - e^{-\beta h\nu}} = \frac{h\nu}{\exp(h\nu/k_B T) - 1}$$

- Energy density, integrated over all solid angle:

$$u_\nu = 4\pi\rho_s \langle E \rangle = \frac{8\pi h\nu^3/c^3}{\exp(h\nu/k_B T) - 1}$$

- Planck Law:

$$J_\nu = \frac{c}{4\pi} u_\nu \rightarrow B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/k_B T) - 1} \text{ or } B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda k_B T) - 1}$$

$B_\nu d\nu = B_\lambda d\lambda$

See “Fundamentals of Statistical and Thermal Physics” (Frederick Reif) or “Astrophysical Concepts” (Harwit) for more details.

Spectrum of blackbody radiation

- The frequency dependence of blackbody radiation is given by the ***Planck function***:

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/k_B T) - 1} \quad \text{or} \quad B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda k_B T) - 1}$$

$h = 6.63 \times 10^{-27}$ erg s (Planck's constant)

$k_B = 1.38 \times 10^{-16}$ erg K⁻¹ (Boltzmann's constant)

- Energy density:***

$$u_\nu(T) = \frac{4\pi}{c} B_\nu(T) = \frac{8\pi h\nu^3/c^3}{\exp(h\nu/k_B T) - 1}$$

Note that the textbook Ryden's "Interstellar and Intergalactic Medium" use the symbol $\varepsilon_\nu(T)$ to denote the energy density.

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- Photon occupation number:
 - The photon occupation number is dimensionless, and is simply **the average number of photons per mode per polarization.**

$$n_\gamma(\nu) = \frac{1}{4\pi\rho_s} \frac{u_\nu}{h\nu} = \frac{\langle E \rangle}{h\nu}$$

$$\rho_s = \frac{2\nu^2}{c^3}$$

$$n_\gamma(\nu) = \frac{c^2}{2h\nu^3} I_\nu$$

- If the radiation field is a blackbody, the photon occupation number is given by

$$n_\gamma(\nu; T) = \frac{1}{\exp(h\nu/k_B T) - 1}$$

Bose-Einstein statistics

Stefan-Boltzmann Law

- Emergent flux is proportional to T^4 .

$$F = \pi \int B_\nu(T) d\nu = \pi B(T)$$

←

$$B(T) = \int B_\nu(T) d\nu = \frac{ac}{4\pi} T^4 = \frac{\sigma}{\pi} T^4$$

$$F = \sigma T^4$$

Stephan – Boltzmann constant : $\sigma = \frac{2\pi^5 k_B^4}{15c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^2 \text{ s}^{-1} \text{ K}^{-4} \text{ sr}^{-1}$

- Total energy density (*another form of the Stefan-Boltzmann law*)

$$u = \frac{4\pi}{c} \int B_\nu(T) d\nu = \frac{4\pi}{c} B(T)$$

$$u(T) = \left(\frac{T}{3400 \text{ K}} \right)^4 \text{ erg cm}^{-3}$$

$$u = aT^4$$

radiation constant : $a \equiv \frac{4\sigma}{c} = 7.57 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$

Rayleigh-Jeans Law & Wien Law

Rayleigh-Jeans Law (low-energy limit)

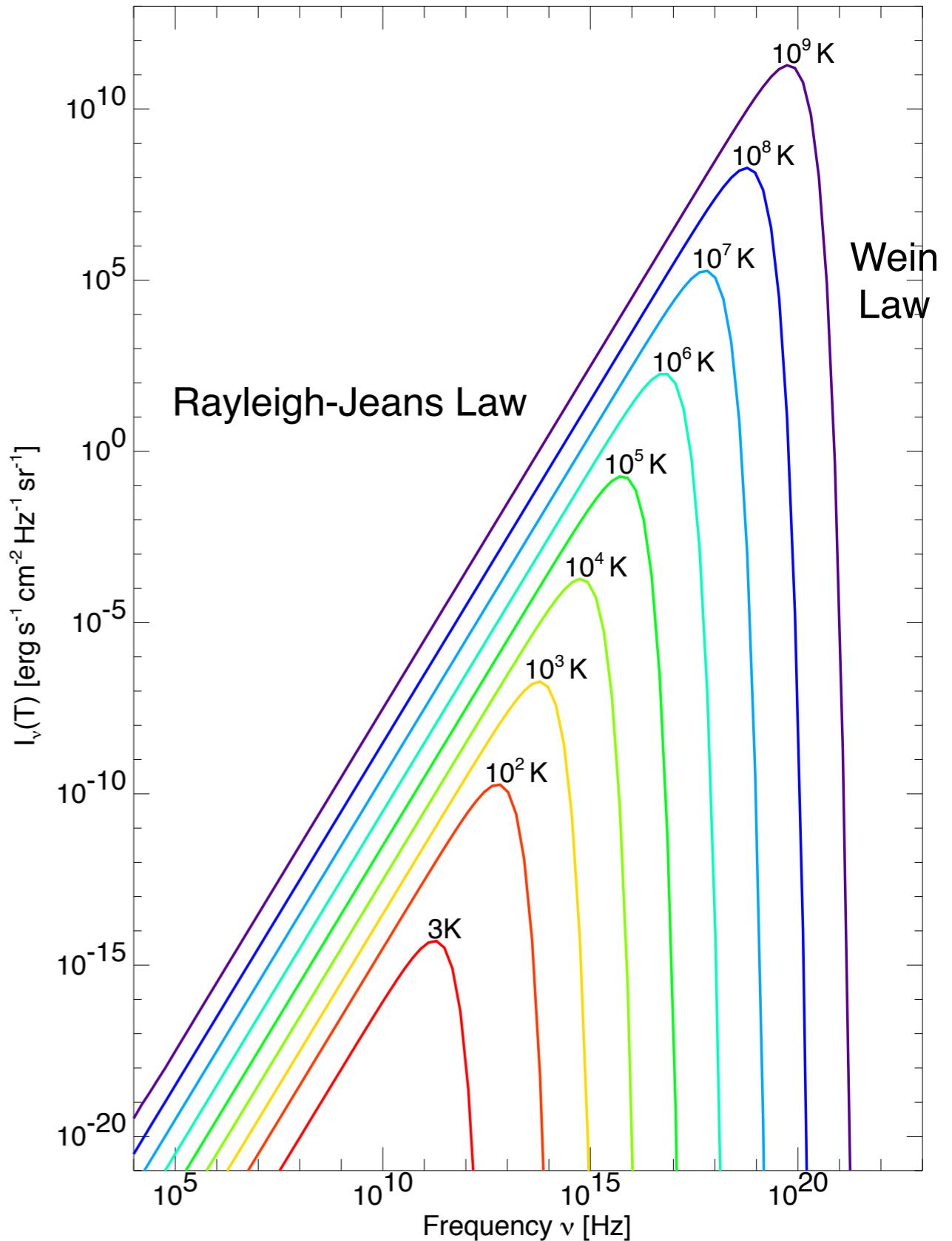
$$h\nu \ll k_B T \quad (\nu \ll 2 \times 10^{10} (T/1\text{ K}) \text{ Hz})$$

$$I_\nu^{RJ}(T) = \frac{2\nu^2}{c^2} k_B T$$

Wien Law (high-energy limit)

$$h\nu \gg k_B T$$

$$I_\nu^W(T) = \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{k_B T}\right)$$



Characteristic Temperatures

- **Brightness Temperature:**

- The brightness temperature is defined to be the temperature such that a blackbody at that temperature would have specific intensity:

$$I_\nu = B_\nu(T_b) \rightarrow T_b(\nu) = \frac{h\nu/k_B}{\ln [1 + 2h\nu^3/(c^2 I_\nu)]}$$

- **Antenna Temperature:**

- Radio astronomers are used to working at very low frequencies. They define the antenna temperature as being the brightness temperature in the Rayleigh-Jeans tail.

$$I_\nu = \frac{2\nu^2}{c^2} k_B T_b \rightarrow T_A \equiv \frac{c^2}{2k_B \nu^2} I_\nu$$

- Radiative transfer equation in the RJ limit:

- ▶ If the matter has its energy levels populated according to an excitation temperature $T_{\text{exc}} \gg h\nu/k_B$, then the source function is given by $S_\nu(T_{\text{exc}}) = (2\nu^2/c^2) k_B T_{\text{exc}}$ from the generalized Kirchhoff's law.

$$\frac{dT_A}{d\tau_\nu} = -T_A + T_{\text{exc}} \quad \text{if } h\nu \ll k_B T_{\text{exc}}$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

- ▶ Then, RT equation becomes

$$T_A = T_A(0)e^{-\tau_\nu} + T_{\text{exc}}(1 - e^{-\tau_\nu}) \quad \text{if } T_{\text{exc}} \text{ is constant.}$$

- **Color Temperature:**
 - By fitting the spectrum to a blackbody curve without regarding to vertical scale (absolute intensity scale), a color temperature T_c is obtained.
 - The color temperature correctly gives the temperature of a blackbody source of unknown absolute scale.
- **Effective Temperature:**
 - The effective temperature of a source is obtained by equating the actual flux F to the flux of a blackbody at temperature T_{eff} .

$$F = \int \int I_\nu \cos \theta d\nu d\Omega = \sigma T_{\text{eff}}^4$$

Stefan-Boltzmann law

- **Excitation Temperature:**
 - The excitation temperature of level u relative to level ℓ is defined by
- Boltzmann distribution

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{E_{u\ell}}{k_B T_{\text{exc}}}\right) \rightarrow T_{\text{exc}} \equiv \frac{E_{u\ell}/k_B}{\ln\left(\frac{n_\ell/g_\ell}{n_u/g_u}\right)} \quad (E_{u\ell} \equiv E_u - E_\ell)$$
- Radio astronomers studying the 21 cm line sometimes use the term “**spin temperature**” T_{spin} for excitation temperature.

Neutral Medium

- Intrinsic Line Profile
- Excitation & De-excitation
- Collisional Excitation & De-excitation
- Optical/UV Absorption Line
- Curve of Growth

Line Profile: Classical model

- **Lorentz Oscillator Model** to describe the interaction between atoms and electric fields
 - The electron (with a small mass) is bound to the nucleus of the atom (with a much larger mass) by a force that behaves according to Hooke's Law (a spring-like force).
 - An applied electric field would then interact with the charge of the electron, causing “stretching” or “compression” of the spring.
 - **The electron’s equation of motion:**

$$m\ddot{\mathbf{x}} = -k\mathbf{x} + \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{rad}}$$

$k = m\omega_0^2$, where k = spring constant

ω_0 = natural (fundamental or resonant) frequency

\mathbf{F}_{ext} = external force, driving force, or external electric field

\mathbf{F}_{rad} = radiation reaction force (radiation damping)
the damping of a charge’s motion which arises because of
the emission of radiation

[1] Spontaneous Emission : Damping, Free Oscillator

- **Undriven Harmonically Bound Particles** (free oscillator)
 - Since an oscillating electron represents a continuously accelerating charge, the electron will radiate energy.
 - The energy radiated away must come from the particle's own energy (energy conservation). In other words, **there must be a force acting on a particle by virtue of the radiation it produces. This is called the *radiation reaction force*.**
 - Let's derive the formula for the radiation reaction force from the fact that the energy radiated must be compensated for by the work done against the radiation reaction force.
 - On one hand, the radiative loss rate of energy, averaged over one cycle of the oscillating dipole, can be represented by the radiative reaction force:

$$\frac{dW}{dt} = \langle \mathbf{F}_{\text{rad}} \cdot \dot{\mathbf{x}} \rangle$$

- On the other hand, from the Larmor's formula for a dipole, the radiative loss will be:

$$\frac{dW}{dt} = -\frac{2e^2 \langle |\ddot{\mathbf{x}}|^2 \rangle}{3c^3}$$

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} e^2 |\bar{x}(\omega)|^2$$

angular frequency : $\omega = 2\pi\nu$

[1] Spontaneous Emission : Abraham-Lorentz formula

$$\therefore \langle \mathbf{F}_{\text{rad}} \cdot \dot{\mathbf{x}} \rangle = -\frac{2e^2 \langle |\ddot{\mathbf{x}}|^2 \rangle}{3c^3}$$

Here, $\langle |\ddot{\mathbf{x}}|^2 \rangle \equiv \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}} dt$ where τ is the oscillation period.

$$= \frac{1}{\tau} \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \Big|_{-\tau/2}^{\tau/2} - \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt$$

We assume that the initial and final states are the same: $\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}(-\tau/2) = \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}(\tau/2)$

Then,

$$\langle |\ddot{\mathbf{x}}|^2 \rangle = -\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt = -\langle \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \rangle \rightarrow \langle \mathbf{F}_{\text{rad}} \cdot \dot{\mathbf{x}} \rangle = \frac{2e^2 \langle \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \rangle}{3c^3}$$

Therefore, we can obtain

$$\mathbf{F}_{\text{rad}} = \frac{2e^2 \ddot{\mathbf{x}}}{3c^3} : \text{Abraham-Lorentz formula}$$

- **Abraham-Lorentz formula:**

$$\mathbf{F}_{\text{rad}} = \frac{2e^2 \ddot{\mathbf{x}}}{3c^3}$$

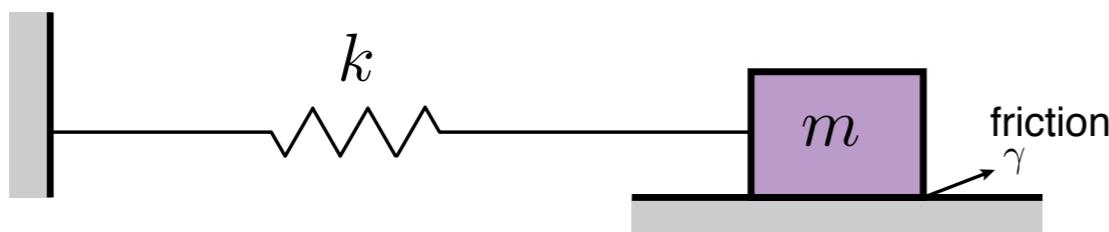
- This formula depends on the derivative of acceleration. This increases the degree of the equation of motion of a particle and can lead to some nonphysical behavior if not used properly and consistently.
- For a simple harmonic oscillator with a frequency ω_0 , we can avoid the difficulty by using

$$\ddot{\mathbf{x}} = -\omega_0^2 \dot{\mathbf{x}} \quad \leftarrow \quad \begin{aligned} x(t) &= x_0 \cos(\omega_0 t) & \ddot{x}(t) &= -\omega_0^2 x(t) \\ \dot{x}(t) &= -\omega_0 x_0 \sin(\omega_0 t) & \ddot{x}(t) &= -\omega_0^2 \dot{x}(t) \end{aligned}$$

- ***This is a good assumption as long as the energy is to be radiated on a time scale that is long compared to the period of oscillation ($\gamma \ll \omega_0$)***. In this regime, ***radiation reaction may be considered as a perturbation on the particle's motion***.

We then rewrite the radiation reaction force as

$$\mathbf{F}_{\text{rad}} = -\frac{2e^2 \omega_0^2}{3c^3} \dot{\mathbf{x}} = -m\gamma \dot{\mathbf{x}}, \quad \gamma \equiv \frac{2e^2 \omega_0^2}{3mc^3} : \text{ damping constant}$$



$$m\ddot{\mathbf{x}} + k\mathbf{x} + m\gamma \dot{\mathbf{x}} = 0$$

This is the equation for a string-mass system subject to friction damping.

- Therefore, the equation of motion of the electron in a Lorentz atom is

$$\ddot{\mathbf{x}} + \gamma \dot{\mathbf{x}} + \omega_0^2 \mathbf{x} = 0$$

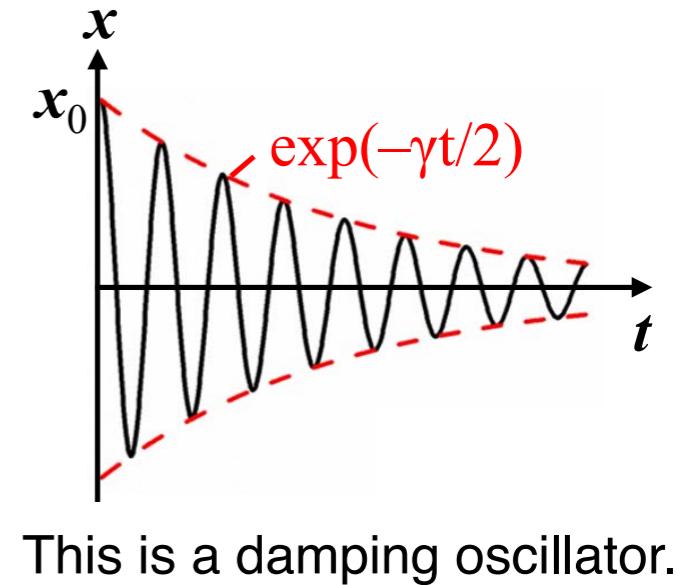
- This equation may be solved by assuming that $x(t) \propto e^{\alpha t}$.

$$\begin{aligned}\alpha^2 + \gamma\alpha + \omega_0^2 &= 0 \rightarrow \alpha = -(\gamma/2) \pm \sqrt{(\gamma/2)^2 - \omega_0^2} \\ &= -\gamma/2 \pm i\omega_0 + \mathcal{O}(\gamma^2/\omega_0^2)\end{aligned}$$

Here, we assumed $\gamma \ll \omega_0$.

- Assuming initial conditions: $x(0) = x_0$, $\dot{x}(0) = 0$ at $t = 0$
- we have

$$x(t) = \frac{1}{2}x_0 \left[e^{-(\gamma/2 - i\omega_0)t} + e^{-(\gamma/2 + i\omega_0)t} \right] = x_0 e^{-\gamma t/2} \cos \omega_0 t \quad \longrightarrow$$



- Power spectrum:

$$\bar{x}(\omega) = \frac{1}{2\pi} \int_0^\infty x(t) e^{i\omega t} dt = \frac{x_0}{4\pi} \left[\frac{1}{\gamma/2 - i(\omega + \omega_0)} + \frac{1}{\gamma/2 - i(\omega - \omega_0)} \right]$$

- This becomes large in the vicinity of $\omega = \omega_0$ and $\omega = -\omega_0$.
- We are ultimately interested only in positive frequencies, and only in regions in which the values become large. Therefore, we obtain

$$\bar{x}(\omega) \approx \frac{x_0}{4\pi} \frac{1}{\gamma/2 - i(\omega - \omega_0)}, \quad |\bar{x}(\omega)|^2 = \left(\frac{x_0}{4\pi} \right)^2 \frac{1}{(\omega - \omega_0)^2 + (\gamma/2)^2}$$

- Spontaneous Emission: Line profile

- Recall the Larmor's formula:

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} e^2 |\bar{x}(\omega)|^2$$

- Energy radiated per unit frequency:

$$\begin{aligned}\frac{dW}{d\omega} &= \frac{8\pi\omega^4}{3c^3} \frac{e^2 x_0^2}{(4\pi)^2} \frac{1}{(\omega - \omega_0)^2 + (\gamma/2)^2} = \frac{1}{2} m \left(\frac{\omega^4}{\omega_0^2} \right) x_0^2 \frac{\gamma/2\pi}{(\omega - \omega_0)^2 + (\gamma/2)^2} \\ &\approx \frac{1}{2} m \omega_0^2 x_0^2 \frac{\gamma/2\pi}{(\omega - \omega_0)^2 + (\gamma/2)^2}\end{aligned}$$

- For a harmonic oscillator, note that the equation of motion is $\mathbf{F} = -k\mathbf{x} = -m\omega_0^2\mathbf{x}$, spring constant is $k = m\omega_0^2$, and the potential energy (energy stored in spring) is $(1/2)kx_0^2$.

- From

$$\int_{-\infty}^{\infty} \frac{\gamma/2\pi}{(\omega - \omega_0)^2 + (\gamma/2)^2} d\omega = \frac{1}{\pi} \tan^{-1} \{ 2(\omega - \omega_0)/\gamma \} \Big|_{-\infty}^{\infty} = 1$$

- Note that the total emitted energy is equal to the initial potential energy of the oscillator:

$$W = \int_0^{\infty} \frac{dW}{d\omega} d\omega = \frac{1}{2} k x_0^2$$

- Profile of the emitted spectrum:

$$\phi(\omega) = \frac{\gamma/2\pi}{(\omega - \omega_0)^2 + (\gamma/2)^2}$$

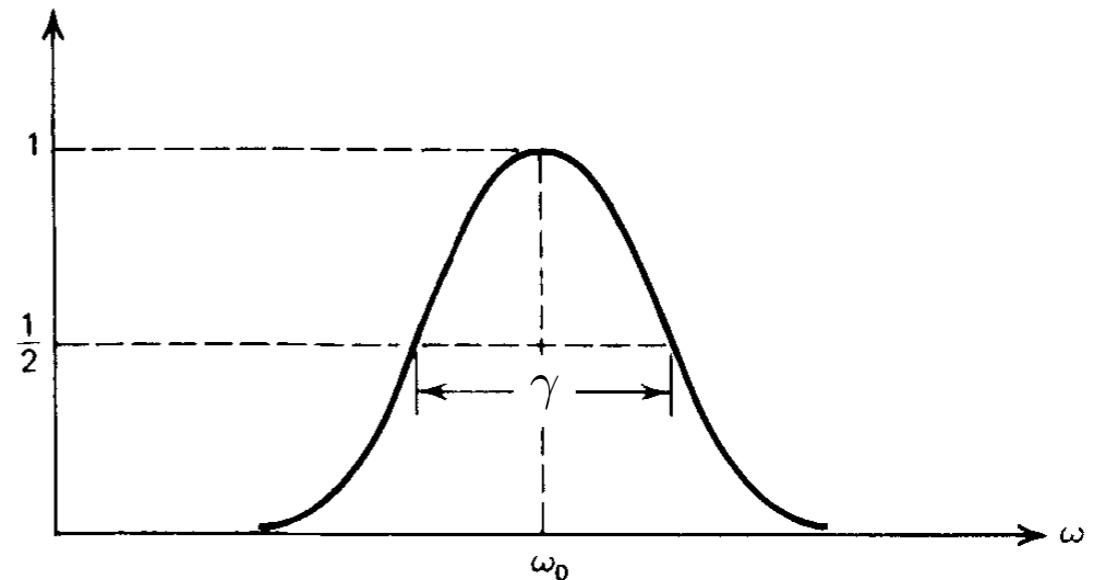
This is the Lorentz (natural) line profile.

- Damping constant is the full width at half maximum (FWHM).

$$\phi(\omega) = \frac{\gamma/2\pi}{(\omega - \omega_0)^2 + (\gamma/2)^2}$$

$$\phi(\nu) = \frac{\gamma/4\pi^2}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2}$$

Note $\phi(\omega)d\omega = \phi(\nu)d\nu$



- The line width $\Delta\omega = \gamma$ is a universal constant when expressed in terms of wavelength:

$$\lambda = \frac{2\pi c}{\omega}$$

$$\Delta\lambda = 2\pi c \frac{\Delta\omega}{\omega^2} = 2\pi c \frac{2}{3} \frac{r_e}{c} \quad \leftarrow \quad \left(\Delta\omega = \gamma = \frac{2}{3} r_e \frac{\omega_0^2}{c} \right)$$

$$= \frac{4}{3} \pi r_e$$

$$= 1.2 \times 10^{-4} \text{ Å}$$

However, in Quantum Mechanics, the line width is not a universal constant.

[2] Absorption/Scattering : Driven Oscillator

- **Driven Harmonically Bound Particles** (forced oscillators)

- Electron's equation of motion (electric charge = $-e$): $\mathbf{F}_{\text{ext}} = -e\mathbf{E}_0 e^{i\omega t}$

$$\ddot{\mathbf{x}} + \gamma \dot{\mathbf{x}} + \omega_0^2 \mathbf{x} = -\frac{e\mathbf{E}_0}{m} e^{i\omega t}$$



- A particular solution for this inhomogeneous differential equation:

$$\mathbf{x} = \mathbf{x}_0 e^{i\omega t} \equiv |\mathbf{x}_0| e^{i(\omega t + \delta)} \rightarrow (-\omega^2 + i\omega\gamma + \omega_0^2) \mathbf{x}_0 e^{i\omega t} = -\frac{e\mathbf{E}_0}{m} e^{i\omega t}$$

$$\mathbf{x}_0 = \frac{(e/m)\mathbf{E}_0}{(\omega^2 - \omega_0^2) - i\omega\gamma}$$

$$\mathbf{x}_0 = |\mathbf{x}_0| e^{i\delta} \propto (\omega^2 - \omega_0^2) + i\omega\gamma \rightarrow \delta = \tan^{-1} \left(\frac{\omega\gamma}{\omega^2 - \omega_0^2} \right)$$

The response is slightly out of phase with respect to the imposed field.

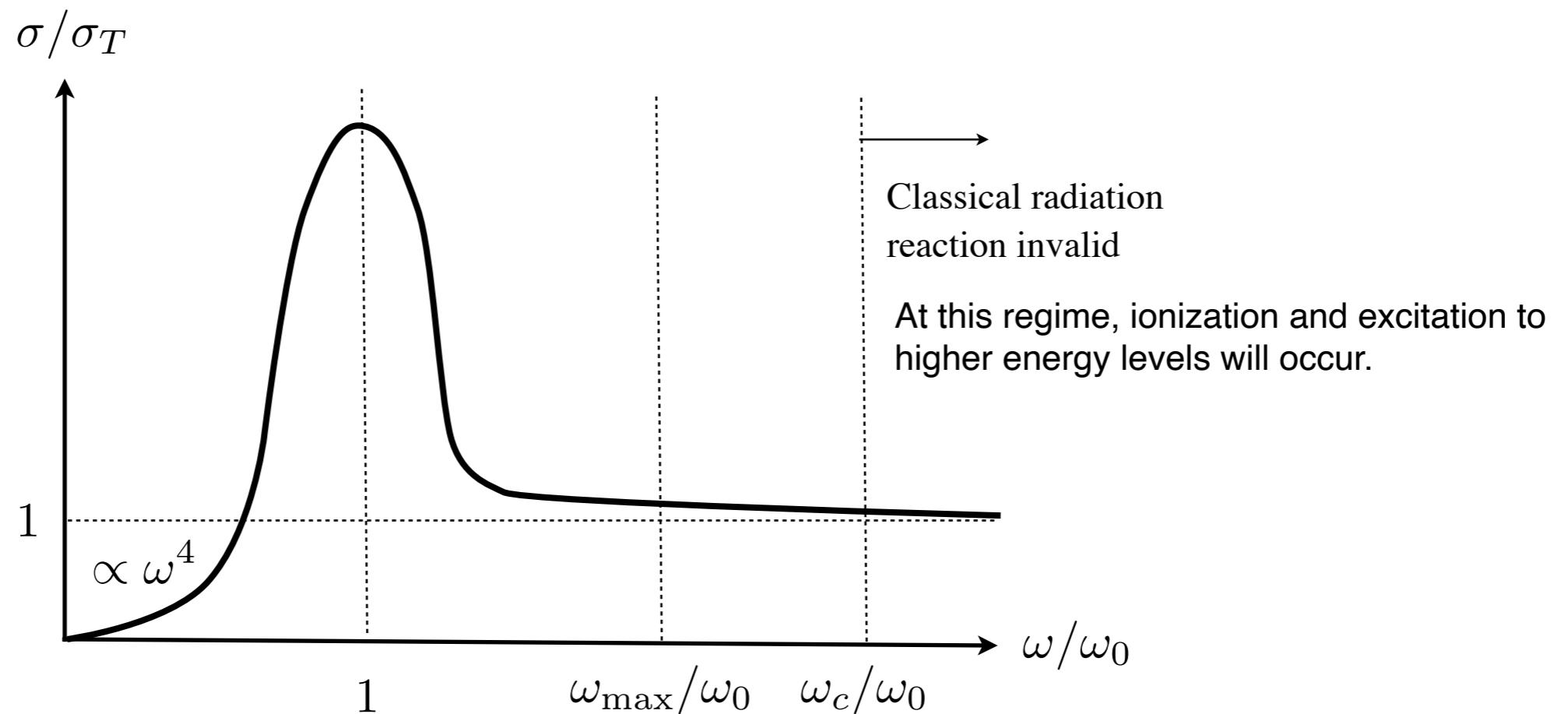
- Time-averaged total power radiated is given by

$$\begin{aligned} P &= \left\langle \frac{dW}{dt} \right\rangle = \frac{2e^2 \langle |\ddot{\mathbf{x}}|^2 \rangle}{3c^3} = \frac{e^2 \omega^4 |\mathbf{x}_0|^2}{3c^3} \\ &= \frac{e^4 E_0^2}{3m^2 c^3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2} \end{aligned}$$

- Scattering cross section:

$$\sigma_{\text{sca}} \equiv \frac{\langle P \rangle}{\langle S \rangle}, \quad \langle S \rangle = \frac{c}{8\pi} E_0^2 \quad \longrightarrow \quad \sigma_{\text{sca}}(\omega) = \frac{8\pi e^4}{3m^2 c^4} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}$$

$$= \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}$$



- Limiting Cases of Interest

(a) $\omega \gg \omega_0$ (Thomson scattering by free electron)

$$\sigma_{\text{sca}} = \sigma_T = \frac{8\pi}{3} r_e^2$$

- ▶ At high incident energies, the binding becomes negligible. Therefore, this corresponds to the case of a free electron.

(b) $\omega \ll \omega_0$ (Rayleigh scattering by bound electron)

$$\sigma_{\text{sca}} = \sigma_T \left(\frac{\omega}{\omega_0} \right)^4 = \sigma_T \left(\frac{\lambda_0}{\lambda} \right)^4$$

- ▶ Rayleigh scattering refers to the ***scattering of light by particles smaller than the wavelength of the light.***
- ▶ The strong wavelength dependence of the scattering means that shorter (blue) wavelengths are scattered more strongly than longer wavelengths.
- ▶ (blue color of the sky) The dependence results in the indirect blue light coming from all regions of the sky.
- ▶ (red color of the sun at sunset) Conversely, glancing toward the Sun, the colors that were not scattered away - the longer wavelengths such as red and yellow light - are directly visible, giving the Sun itself a slightly yellowish color.
- ▶ However, viewed from space, the sky is black and the Sun is white.

- Absorption/Scattering : Line Profile

(c) $\omega \approx \omega_0$ (resonance scattering of line radiation)

$$\begin{aligned}\sigma_{\text{sca}}(\omega) &\approx \sigma_T \frac{\omega_0^4}{(\omega - \omega_0)^2(2\omega_0)^2 + (\omega_0\gamma)^2} \\ &= \sigma_T \frac{\omega_0^2/4}{(\omega - \omega_0)^2 + (\gamma/2)^2} \\ \sigma_T \frac{\omega_0^2}{4} &= \frac{8\pi}{3} \left(\frac{e^2}{mc^2} \right)^2 \times \frac{1}{4} \times \left(\gamma \frac{3}{2} \frac{mc^3}{e^2 \omega_0^2} \right) = 2\pi^2 \frac{e^2}{mc} (\gamma/2\pi)\end{aligned}$$

Note that $\nu = 2\pi\omega$ and $\sigma_\nu = \sigma_\omega/2\pi$.

$$\begin{aligned}\sigma_\omega &= \frac{2\pi^2 e^2}{mc} \frac{\gamma/2\pi}{(\omega - \omega_0)^2 + (\gamma/2\pi)^2} \\ \sigma_\nu &= \frac{\pi e^2}{mc} \frac{\gamma/4\pi^2}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2}\end{aligned}$$

- ▶ In the neighborhood of the resonance, ***the shape of the absorption/scattering cross-section is the same as the (spontaneous) emission line profile from the free oscillator. We already obtained the same conclusion, in the previous lecture.***
- ▶ Total scattering cross section is
- **Resonance line**
 - A spectral line caused by an electron jumping ***between the ground state and the first energy level in an atom or ion.*** It is the longest wavelength line produced by a jump to or from the ground state.
 - Because the majority of electrons are in the ground state in many astrophysical environments, and because the energy required to reach the first level is the least needed for any transition, resonance lines are the strongest lines in the spectrum for any given atom or ion.

- ***In the quantum theory of spectral lines,***

we obtain similar formulas, which are conveniently stated in terms of the classical results as

$$\sigma_\nu = f_{nn'} \frac{\pi e^2}{mc} \frac{\gamma/4\pi^2}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2}$$

$$\int_0^\infty \sigma_\nu d\nu = f_{nn'} \frac{\pi e^2}{mc}$$

where $f_{nn'}$ is called the **oscillator strength** or **f-value** for the transition between states n and n' .

Selected Resonance Lines^a with $\lambda < 3000 \text{ \AA}$

	Configurations	ℓ	u	$E_\ell/hc(\text{ cm}^{-1})$	$\lambda_{\text{vac}}(\text{ \AA})$	$f_{\ell u}$
C IV	$1s^2 2s - 1s^2 2p$	$^2S_{1/2}$	$^2P_{1/2}^o$	0	1550.772	0.0962
		$^2S_{1/2}$	$^2P_{3/2}^o$	0	1548.202	0.190
N V	$1s^2 2s - 1s^2 2p$	$^2S_{1/2}$	$^2P_{1/2}^o$	0	1242.804	0.0780
		$^2S_{1/2}$	$^2P_{3/2}^o$	0	1242.821	0.156
O VI	$1s^2 2s - 1s^2 2p$	$^2S_{1/2}$	$^2P_{1/2}^o$	0	1037.613	0.066
		$^2S_{1/2}$	$^2P_{3/2}^o$	0	1037.921	0.133
C III	$2s^2 - 2s 2p$	1S_0	$^1P_1^o$	0	977.02	0.7586
C II	$2s^2 2p - 2s 2p^2$	$^2P_{1/2}^o$	$^2D_{3/2}^o$	0	1334.532	0.127
		$^2P_{3/2}^o$	$^2D_{5/2}^o$	63.42	1335.708	0.114
N III	$2s^2 2p - 2s 2p^2$	$^2P_{1/2}^o$	$^2D_{3/2}^o$	0	989.790	0.123
		$^2P_{3/2}^o$	$^2D_{5/2}^o$	174.4	991.577	0.110
CI	$2s^2 2p^2 - 2s^2 2p 3s$	3P_0	$^3P_1^o$	0	1656.928	0.140
		3P_1	$^3P_2^o$	16.40	1656.267	0.0588
		3P_2	$^3P_2^o$	43.40	1657.008	0.104
N II	$2s^2 2p^2 - 2s 2p^3$	3P_0	$^3D_1^o$	0	1083.990	0.115
		3P_1	$^3D_2^o$	48.7	1084.580	0.0861
		3P_2	$^3D_3^o$	130.8	1085.701	0.0957
NI	$2s^2 2p^3 - 2s^2 2p^2 3s$	$^4S_{3/2}^o$	$^4P_{5/2}$	0	1199.550	0.130
		$^4S_{3/2}^o$	$^4P_{3/2}$	0	1200.223	0.0862
OI	$2s^2 2p^4 - 2s^2 2p^3 3s$	3P_2	$^3S_1^o$	0	1302.168	0.0520
		3P_1	$^3S_1^o$	158.265	1304.858	0.0518
		3P_0	$^3S_1^o$	226.977	1306.029	0.0519
Mg II	$2p^6 3s - 2p^6 3p$	$^2S_{1/2}$	$^2P_{1/2}^o$	0	2803.531	0.303
		$^2S_{1/2}$	$^2P_{3/2}^o$	0	2796.352	0.608
Al III	$2p^6 3s - 2p^6 3p$	$^2S_{1/2}$	$^2P_{1/2}^o$	0	1862.790	0.277
		$^2S_{1/2}$	$^2P_{3/2}^o$	0	1854.716	0.557

Table 9.4 in [Draine]

See also Table 9.3

[3] Line Broadening Mechanisms

- **Atomic levels are not infinitely sharp**, nor are the lines connecting them.
 - (1) Doppler (Thermal) Broadening
 - (2) Natural Broadening
 - (3) Collisional Broadening
 - (4) Thermal Doppler + Natural Broadening
- **[1] Doppler (Thermal) Broadening**
 - The simplest mechanism for line broadening in the Doppler effect. An atom is in thermal motion, so that the frequency of emission or absorption in its own frame corresponds to a different frequency for an observer.
 - Each atom has its own Doppler shift, so that the net effect is to spread the line out, but not to change its total strength.
 - The change in frequency associated with an atom with velocity component v_z along the line of sight (say, z axis) is, to lowest order in v_z/c , given by

$$\nu - \nu_0 = \nu_0 \frac{v_z}{c}$$

Recall Doppler shift: $\left[\frac{\nu}{\nu_0} = \frac{1}{\gamma(1 - \beta \cos \theta)} \rightarrow \nu \approx \nu_0 (1 + \beta \cos \theta) \rightarrow \nu - \nu_0 = \frac{\nu_0 v_z}{c} \right]$

- Here, ν_0 is the rest-frame frequency.

-
- We need to consider the velocity distribution of atoms. The number of atoms having velocities in the range $(v_z, v_z + dv_z)$ is proportional to

$$f(v_z)dv_z = \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv_z^2}{2kT}\right) dv_z = \frac{1}{\sqrt{2\pi}v_{\text{rms}}} \exp\left(-\frac{v_z^2}{2v_{\text{rms}}^2}\right) dv_z$$

- From the Doppler shift formula, we have

$$v_z = \frac{c(\nu - \nu_0)}{\nu_0} \rightarrow dv_z = \frac{cd\nu}{\nu_0}$$

- Therefore, the strength of the emission is proportional to

$$\exp\left(-\frac{mv_z^2}{2kT}\right) dv_z = \frac{c}{\nu_0} \exp\left[-\frac{mc^2(\nu - \nu_0)^2}{2\nu_0^2 kT}\right] d\nu$$

- Then, the normalized profile function is

$$\left(v_{\text{rms}} = \sqrt{\frac{kT}{m}}\right)$$

$$\phi(\nu) = \frac{1}{\Delta\nu_D \sqrt{\pi}} e^{-(\nu - \nu_0)^2 / (\Delta\nu_D)^2} \quad \text{where } \Delta\nu_D = \nu_0 \frac{v_{\text{th}}}{c} \text{ is the Doppler width.}$$

$$= \frac{\nu_0}{c} \sqrt{\frac{2kT}{m}} = \nu_0 \frac{\sqrt{2}v_{\text{rms}}}{c}$$

-
- Numerical value of the velocity broadening is

$$v_{\text{th}} = \left(\frac{2k_{\text{B}}T}{m} \right)^{1/2} = 1.3 \text{ km s}^{-1} \left(\frac{T}{100 \text{ K}} \right)^{1/2} \left(\frac{m}{m_{\text{H}}} \right)^{-1/2}$$

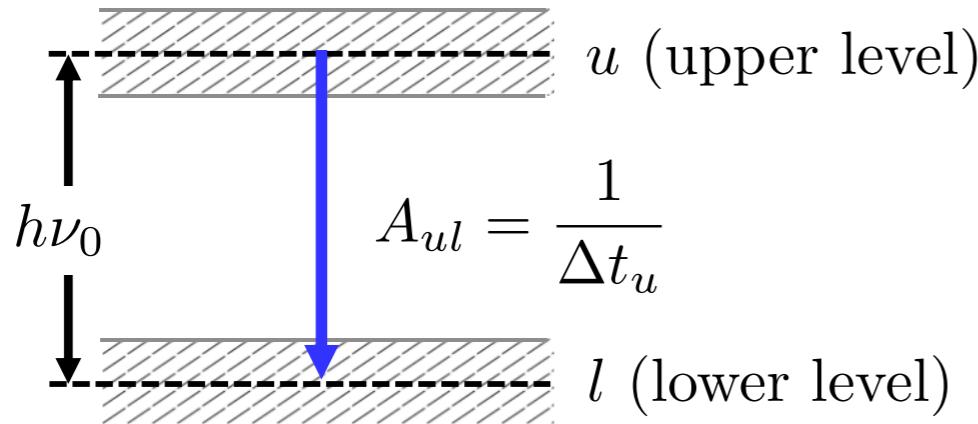
- In addition to thermal motions, there can be turbulent velocities associated with macroscopic velocity fields. The turbulent motions are accounted for by an effective Doppler width.

$$\Delta\nu_{\text{D}} = \nu_0 \frac{b}{c}$$
$$b \equiv (v_{\text{th}}^2 + v_{\text{turb}}^2)^{1/2}$$

where v_{turb} is $\sqrt{2}$ times a root-mean-square measure of the turbulent velocities. This assumes that the turbulent velocities also have a Gaussian distribution.

• [2] Natural Broadening

- The intrinsic line width of a line is due to ***the Heisenberg uncertainty principle***. If an energy level u has a lifetime Δt , then uncertainty (spread) in energy ΔE must be $\Delta E \sim \hbar/\Delta t$ ($\hbar = h/2\pi$), and the resulting spread in the frequency of emitted photons is $\Delta\nu = \Delta E/h$.



A_{ul} = decay rate
= decay probability per unit time, Einstein A coefficient.

ΔE_u = uncertainty in energy of u

Δt_u = the uncertainty in time of occupation of u

$\Delta\nu_u$ = uncertainty in frequency

$$= \Delta E_u/h = 1/(2\pi\Delta t_u) = A_{ul}/(2\pi) \longrightarrow$$

$$\phi_\nu = \frac{\gamma/4\pi^2}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2}$$

In terms of the line width $\Delta\nu_u$, the line profile can be rewritten as

$$\phi_\nu = \frac{1}{2\pi} \frac{\Delta\nu_u/2}{(\nu - \nu_0)^2 + (\Delta\nu_u/2)^2}$$

FWHM of the Lorentz function:
 $\Delta\nu_u = \gamma/2\pi$

Therefore, γ is equivalent to the Einstein A-coefficient., i.e., $\gamma = A_{ul}$.

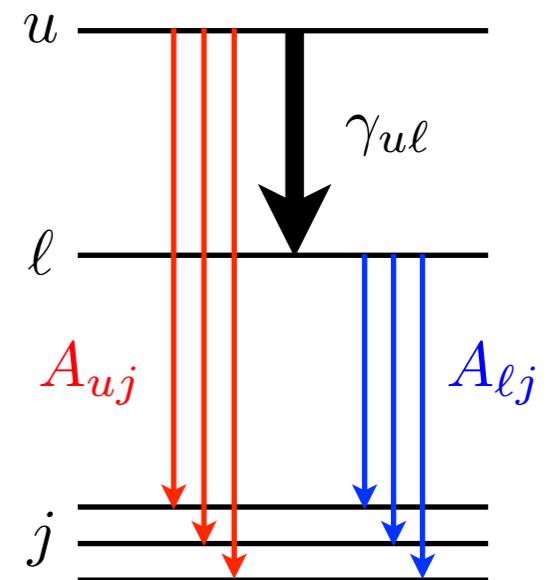
- The intrinsic line width is $\gamma = A_{ul}$.
 - This means forbidden lines are intrinsically narrower than permitted lines.
 - For instance, the permitted Ly α line has $A_{ul}/\nu_{ul} \sim 3 \times 10^{-7}$, while the forbidden [O III] 5007Å has a tiny width $A_{ul}/\nu_{ul} \sim 3 \times 10^{-17}$.
 - The intrinsic line width of [O III] 5007Å is equivalent to the Doppler broadening of

$$\Delta v \sim 3 \times 10^{-17} c \sim 10 \text{ nm s}^{-1} \sim 30 \text{ cm yr}^{-1}$$

- For a multiple-level absorber, the upper and lower can both be broadened by transitions to other levels.

$$\gamma_{ul} = \sum_{E_j < E_u} A_{uj} + \sum_{E_j < E_\ell} A_{\ell j}$$

- For Ly α ($n = 1-2$), $\gamma_{ul} = A_{21} = 6.3 \times 10^8 \text{ s}^{-1}$
 $\Delta\nu/\nu \sim 4 \times 10^{-8}$
- For H α ($n = 2-3$), $\gamma_{ul} = A_{32} + A_{31} + A_{21} = 8.9 \times 10^8 \text{ s}^{-1}$
 $\Delta\nu/\nu \sim 3 \times 10^{-7}$

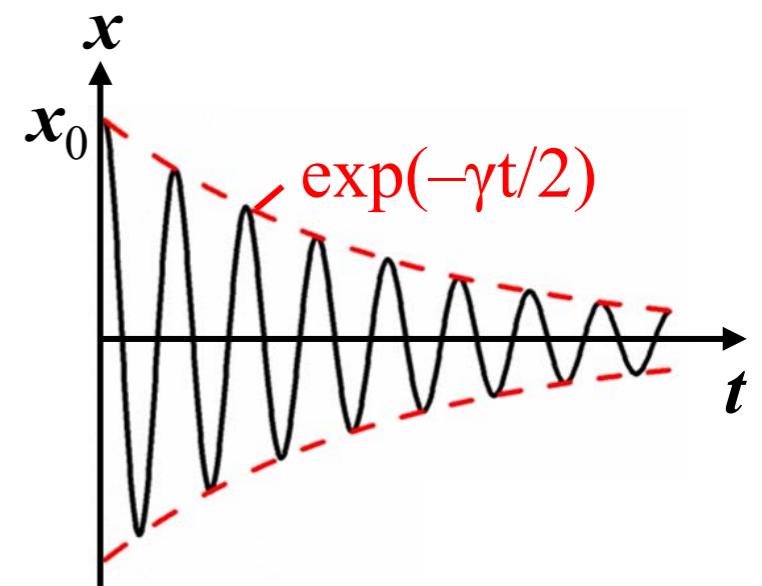


Suppose that the electric field is of the form $e^{-\gamma t/2}$ and then the energy decays proportional to $e^{-\gamma t}$.

We then have an emitted spectrum determined by the decaying sinusoid type of electric field.

Its Fourier transform (spectral profile) is a Lorentz (or natural, or Cauchy) profile:

$$\phi_\nu = \frac{\gamma/4\pi^2}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2}$$



Semiclassical (Weissokpf-Woolley) Picture of Quantum Levels

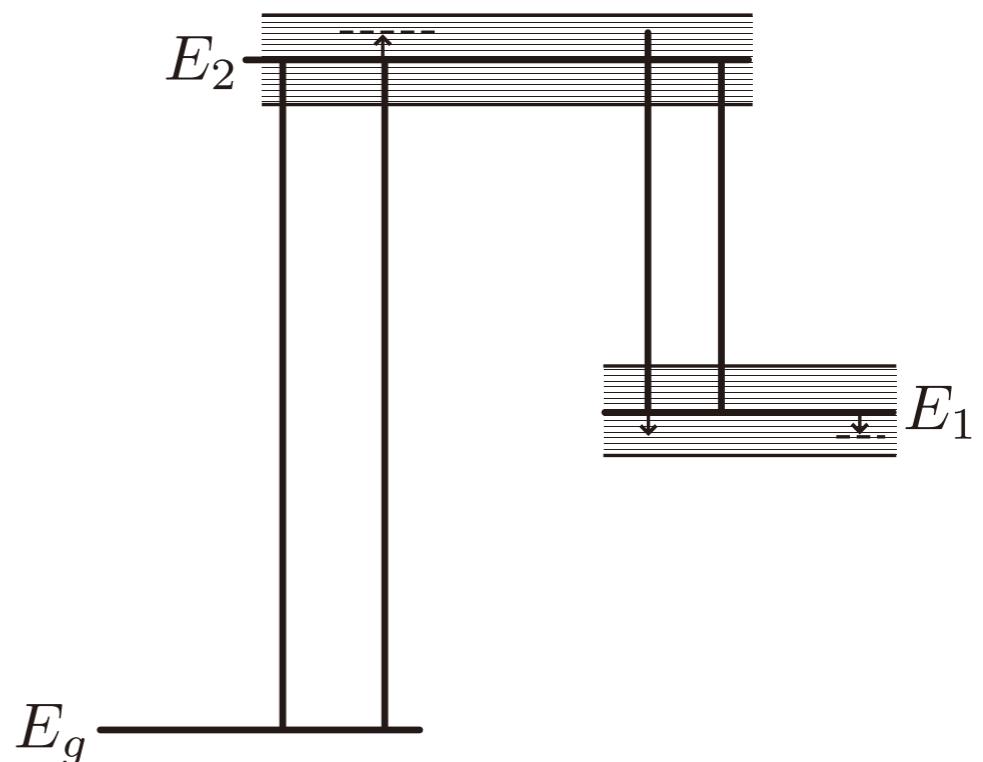
- In the semiclassical picture, each level is viewed as a continuous distribution of sublevels with energies close to the energy of the level (E_n).

The distribution of sublevels are explained by the Heisenberg Uncertainty Principle. The level has a lifetime $\Delta t = 1/A$ (A = Einstein A coefficient) and a spread in energy about $\Delta E \approx \hbar/\Delta t = \hbar A$.

$$\Delta E \Delta t \approx \hbar$$

The ground level has no spread in energy

because $\Delta t = \infty$.



The atom is in a definite sublevel of some level.

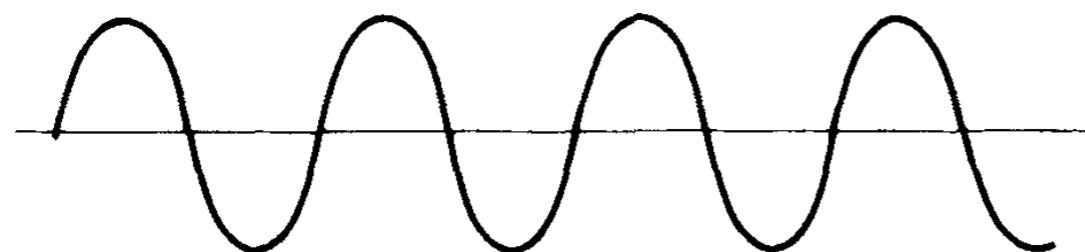
A transition in a spectral line is considered to be an instantaneous transition between a definite sublevel of an initial level to a definite sublevel of a final level.

The energy spread of a sublevel is described by a Lorentzian profile.

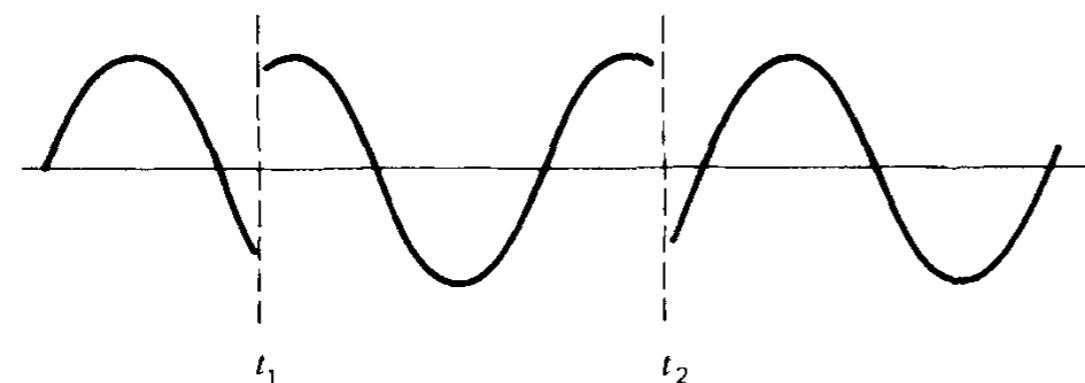
• [3] Collisional Broadening (or Pressure Broadening)

- The Lorentz profile applies even to certain types of collisional broadening mechanisms.
- If the atom suffers collisions with other particles while it is emitting, the phase of the emitted radiation can be altered suddenly. If the phase changes completely randomly at the collision times, then information about the emitting frequencies is lost.
- If the collisions occur with frequency ν_{col} , that is, each atom experiences ν_{col} collisions per unit time on the average, then the profile is

$$\phi_\nu = \frac{\Gamma/4\pi^2}{(\nu - \nu_0)^2 + (\Gamma/4\pi)^2} \quad \text{where} \quad \Gamma = \gamma + 2\nu_{\text{col}}$$



purely sinusoidal



random phase interruptions
by atomic collisions

• [4] Voigt profile : Thermal + Natural broadening

- Atoms shows both a Lorentz profile plus the Doppler effect. In this case, we can write the profile as an average of the Lorentz profile over the various velocity states of the atom:

Maxwell distribution: $f_{v_z} = \frac{1}{\pi^{1/2} (2kT/m)^{1/2}} \exp(-mv_z^2/2kT) \longrightarrow f_y = \frac{1}{\pi^{1/2}} \exp(-y^2)$

$$v_{\text{th}} \equiv \sqrt{\frac{2kT}{m}}, \quad y \equiv \frac{v_z}{v_{\text{th}}}$$

To interact with an atom with velocity v_z , the photon central frequency should be $\nu_0 + \nu_0(v_z/c)$. Then, the Lorentz profile at the frequency $\nu' = \nu - [\nu_0 + \nu_0(v_z/c)] = (\nu - \nu_0) - \nu_0(v_{\text{th}}/c)y$ is supposed to be multiplied with the Maxwell distribution.

Change of variables for the Lorentz function: $\phi_y^L = \phi_\nu^L \left| \frac{d\nu}{dy} \right| = \phi_\nu^L \times \left(\nu_0 \frac{v_{\text{th}}}{c} \right)$

Let $\Delta\nu_D \equiv \nu_0 \frac{v_{\text{th}}}{c}$, $u \equiv \frac{\nu - \nu_0}{\Delta\nu_D} = \frac{\nu - \nu_0}{\nu_0} \frac{c}{v_{\text{th}}}$, $a = \frac{\Gamma/4\pi}{\Delta\nu_D}$

$$\begin{aligned} \phi(\nu) &= \int_{-\infty}^{\infty} \phi_y^L f_y dy \\ &= \int_{-\infty}^{\infty} \left(\nu_0 \frac{v_{\text{th}}}{c} \right) \frac{\Gamma/4\pi^2}{[(\nu - \nu_0) - \nu_0(v_{\text{th}}/c)y]^2 + (\Gamma/4\pi)^2} \left(\frac{1}{\pi^{1/2}} \right) \exp(-y^2) dy \\ &= \frac{a}{\pi^{3/2} \Delta\nu_D} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(u - y)^2 + a^2} dy \end{aligned}$$

- The profile can be written using the Voigt function.

$$\phi(\nu) = \frac{1}{\Delta\nu_D \sqrt{\pi}} H(u, a)$$

Here, a is a ratio of the intrinsic broadening to the thermal broadening.

u is a measure of how far you are from the line center, in units of thermal broadening parameter.

In terms of Doppler velocity, u can be expressed as

$$u = \frac{\nu - \nu_0}{\Delta\nu_D} = \frac{\nu - \nu_0}{\nu_0} \frac{c}{v_{\text{th}}}$$

In the velocity term,

$$u = \frac{v}{v_{\text{th}}}, \text{ where } v = \frac{\nu - \nu_0}{\nu_0} c$$

Voigt-Hjerting function:

$$H(u, a) \equiv \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(u - y)^2 + a^2}$$

$$a \equiv \frac{\Gamma}{4\pi\Delta\nu_D}$$

$$u \equiv \frac{\nu - \nu_0}{\Delta\nu_D}$$

$$\Delta\nu_D = \frac{\nu_0}{c} \sqrt{\frac{2kT}{m}}$$

Including the turbulent motion

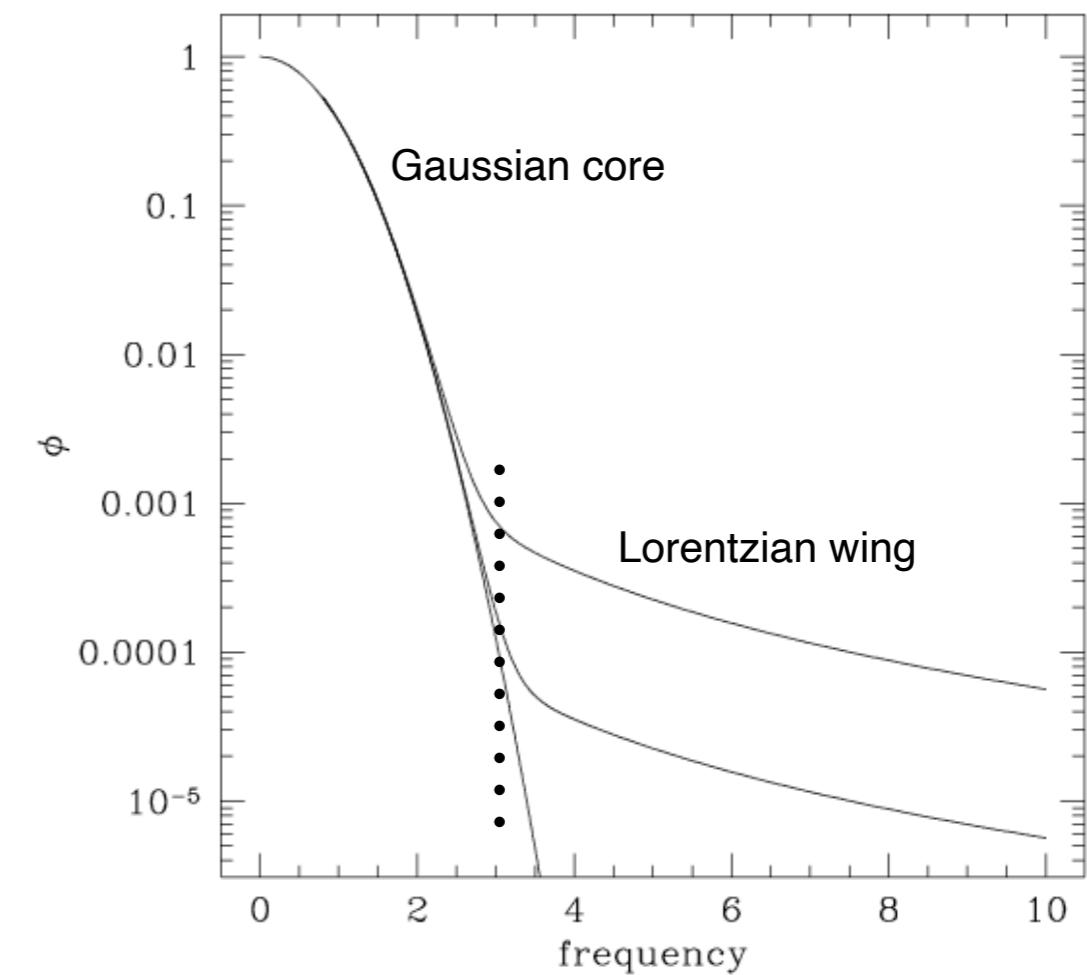
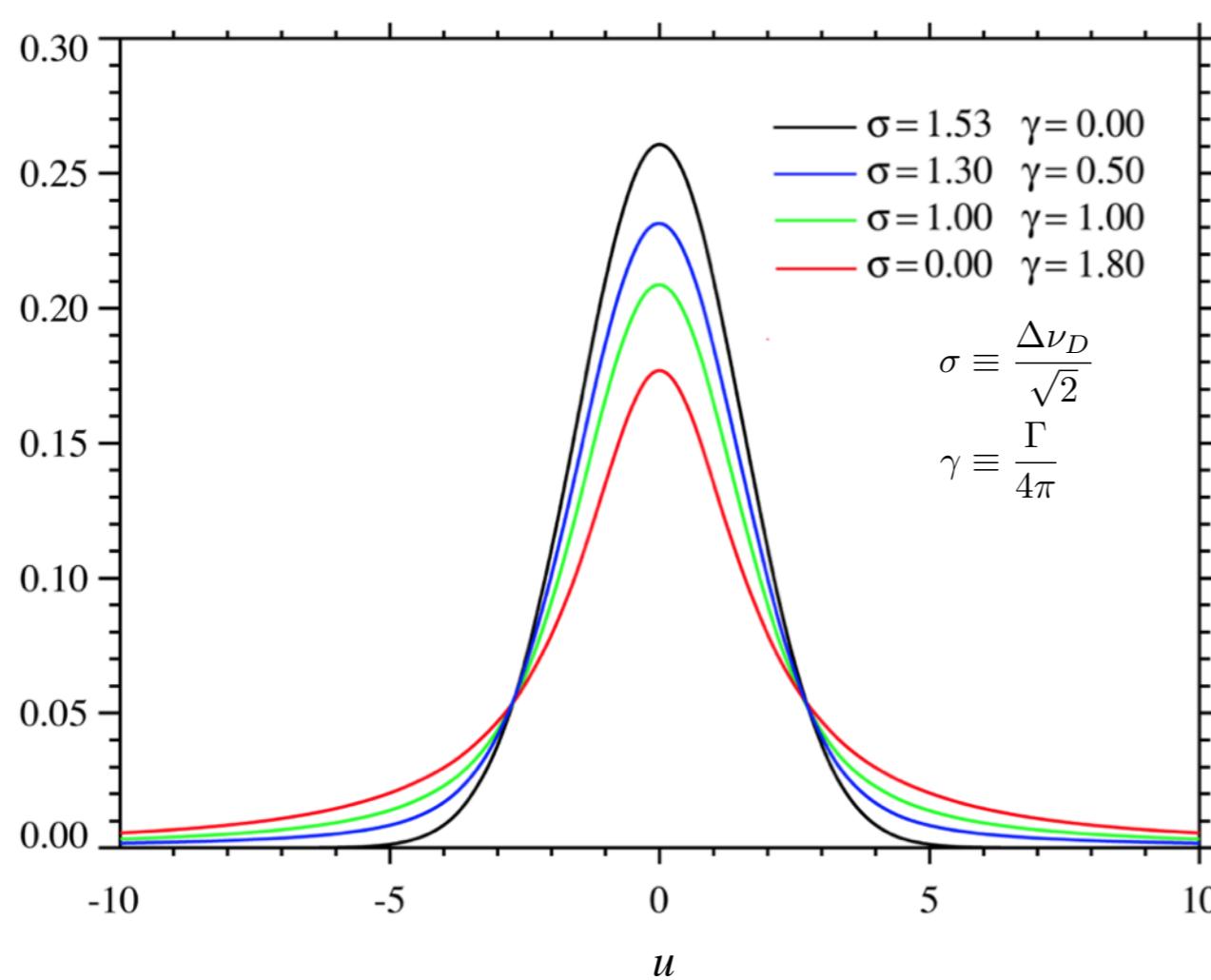
$$\Delta\nu_D = \nu_0 \frac{v_{\text{th}}}{c} \rightarrow \Delta\nu_D = \nu_0 \frac{b}{c}$$

$$\text{where } b = \sqrt{v_{\text{th}}^2 + v_{\text{turb}}^2}, \quad v_{\text{th}} = \sqrt{\frac{2kT}{m}}$$

Properties of Voigt Function

- For small a , the “core” of the line is dominated by the Gaussian (Doppler) profile, whereas the “wings” are dominated by the Lorentz profile.

$$H(a, u) \equiv \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(u - y)^2 + a^2}$$



- In most case, $a \ll 1$. For Ly α at $T = 100$ K, $a \sim 0.05$.

- Line center:

$$H(0, a) = \exp(a^2) \operatorname{Erfc}(a) \approx 1 - \frac{2}{\sqrt{\pi}}a + a^2 - \mathcal{O}(a^3)$$

- Taylor series Expansion of the Voigt function :

$$H(u, a) \equiv \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(u - y)^2 + a^2}$$

- Near the line center ($u \rightarrow 0$), the contribution to the integral is dominated by $y = u$. Therefore,

$$H(u, a) \simeq \frac{a}{\pi} e^{-u^2} \int_{-\infty}^{\infty} \frac{dy}{y^2 + a^2} = e^{-u^2}$$

which is known as the Doppler core.

- In the line wings away from the core ($u \gg 1$), the integral is dominated by $y \sim 0$ because of the rapidly decreasing function in the numerator.

$$H(u, a) \simeq \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{u^2} = \frac{a}{\pi} \frac{\sqrt{\pi}}{u^2} = \frac{a}{\sqrt{\pi} u^2}$$

- In summary, we obtain the Voigt function in a Taylor series expansion around $a = 0$.

$$H(u, a) \approx H(u, 0) + a \left. \frac{dH}{da} \right|_{a=0} \approx e^{-u^2} + a \frac{1}{\sqrt{\pi} u^2}$$

- The first term represents the Gaussian core, provided by the thermal broadening, and the second term represents the Lorentizan damping wing.
- Transition from Doppler core to damping wing can be found by solving:

$$e^{u^2} = \frac{\sqrt{\pi}}{a} u^2 \quad \rightarrow \quad u^2 = \ln \left(\frac{\sqrt{\pi}}{a} \right) + \ln u^2 \quad \text{for hydrogen}$$

$$b = 13 \text{ km s}^{-1} (T/10^4 \text{ K})^{1/2}$$

- The solution for this transcendental equation for Ly α is

$$u^2 \approx 10.31 + \ln \left[\left(\frac{6.265 \times 10^8 \text{ s}^{-1}}{\gamma_{ul}} \right) \left(\frac{1215.67 \text{ \AA}}{\lambda_{ul}} \right) \left(\frac{b}{10 \text{ km s}^{-1}} \right) \right]$$

provided that the quantity in square brackets is not very large or very small. The damping wing for $|u| \gtrsim 3.2$ or velocity shifts $|v| \gtrsim 3.2 (b/10 \text{ km s}^{-1})$.

Atomic Processes

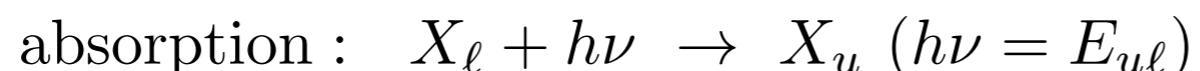
- **Excitation and de-excitation (Transition)**
 - ▶ Radiative excitation (photoexcitation; photoabsorption)
 - ▶ Radiative de-excitation (spontaneous emission and stimulated emission)
 - ▶ Collisional excitation
 - ▶ Collisional de-excitation
- **Emission Line**
 - ▶ Collisionally-excited emission lines
 - ▶ Recombination lines (recombination following photoionization or collisional ionization)
- **Ionization**
 - ▶ Photoionization and Auger-ionization
 - ▶ Collisional Ionization (Direct ionization and Excitation-autoionization)
- **Recombination**
 - ▶ Radiative recombination \Leftrightarrow Photoionization
 - ▶ Dielectronic Recombination (not dielectric!)
 - ▶ Three-body recombination \Leftrightarrow Direct collisional ionization
- **Charge exchange**

Radiative Excitation and De-excitation (Absorption and Emission)

- Three Radiative Transitions and Einstein Coefficients

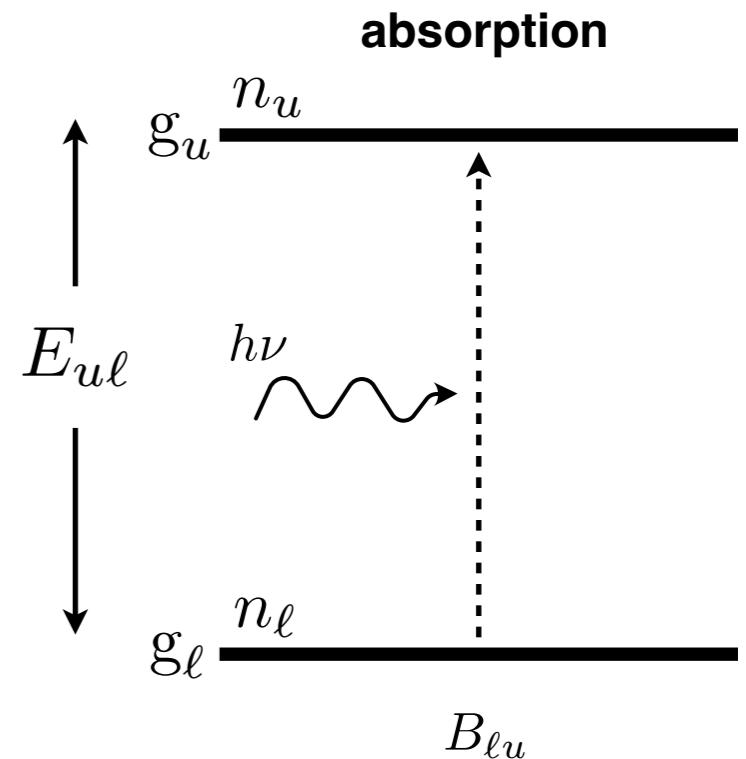
- Absorption:**

- If an absorber (atom, ion, molecule, or dust grain) X is in a lower level ℓ and there is radiation present with photons having an energy equal to $E_{u\ell}$. The absorber can absorb a photon and undergo an upward transition.



- The rate per volume at which the absorbers absorb photons will be proportional to both the energy density u_ν of photons of the appropriate energy and the number density n_ℓ of absorbers in the lower level ℓ .

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = - \left(\frac{dn_\ell}{dt} \right)_{\ell \rightarrow u} = n_\ell B_{\ell u} u_\nu$$



- The proportionality constant $B_{\ell u}$ is the **Einstein B coefficient** for the upward transition $\ell \rightarrow u$.

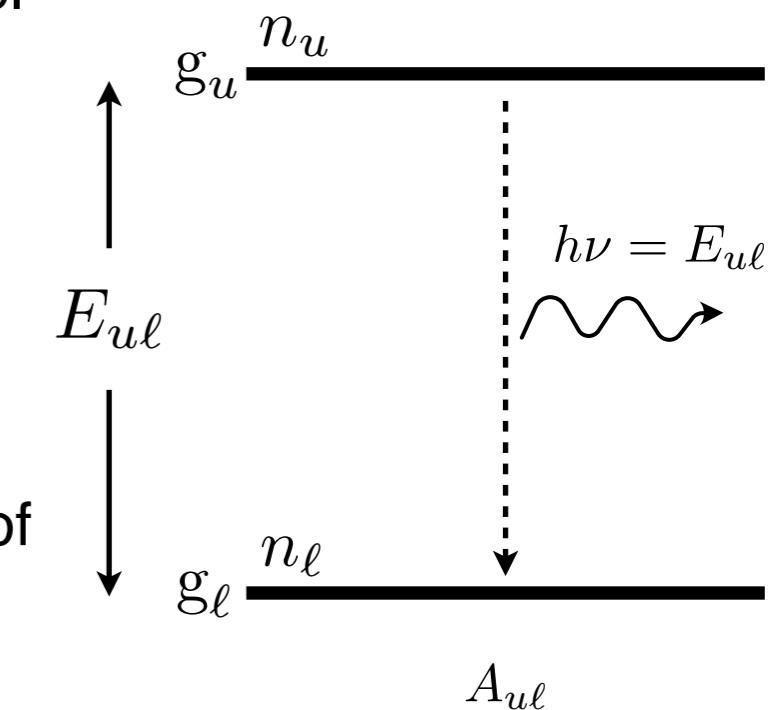
- **Emission:**

- ▶ An absorber X in an excited level u can decay to a lower level ℓ with emission of a photon. There are two ways this can happen:

spontaneous emission : $X_u \rightarrow X_\ell + h\nu$ ($h\nu = E_{ul}$)

stimulated emission : $X_u + h\nu \rightarrow X_\ell + 2h\nu$ ($h\nu = E_{ul}$)

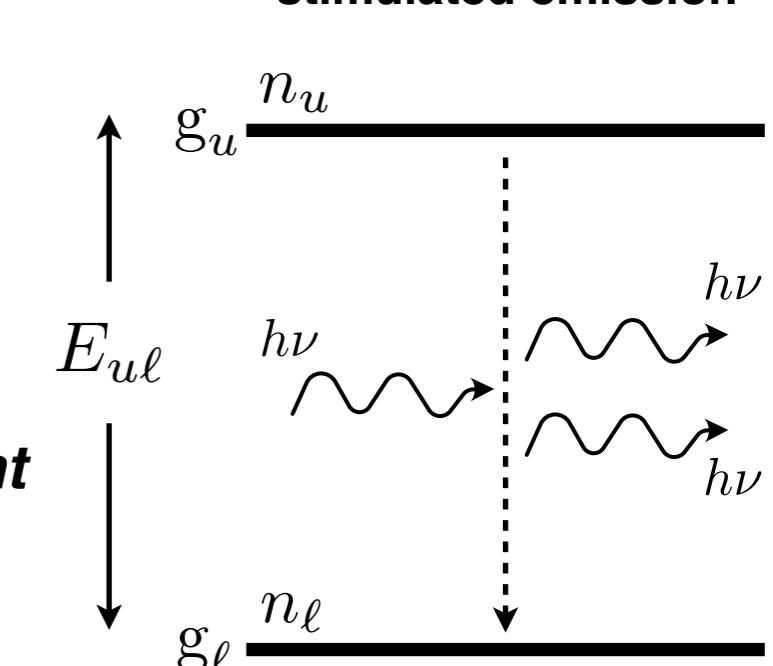
spontaneous emission



- ▶ **Spontaneous emission** is a random process, independent of the presence of a radiation field.
- ▶ **Stimulated emission** occurs if photons of the identical frequency, polarization, and direction of propagation are already present, and the rate of stimulated emission is proportional to the energy density u_ν of these photons.

$$\left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell} = - \left(\frac{dn_u}{dt} \right)_{u \rightarrow \ell} = n_u (A_{ul} + B_{ul} u_\nu)$$

stimulated emission



- ▶ The probability per unit time A_{ul} is the **Einstein A coefficient** for spontaneous transition. The coefficient B_{ul} is the **Einstein B coefficient** for the downward transition $u \rightarrow \ell$.

B_{ul}

Relations between the Einstein coefficients

- The three Einstein coefficients are not mutually independent.
- ***In thermal equilibrium***, the radiation field becomes the “blackbody” radiation field and the two levels must be populated according to the Boltzmann distribution.

$$(u_\nu)_{\text{TE}} = \frac{4\pi}{c} B_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}$$

$$\left(\frac{n_u}{n_\ell} \right)_{\text{TE}} = \frac{g_u}{g_\ell} e^{-E_{u\ell}/k_B T} \quad \text{Here, } E_{u\ell} = h\nu.$$

- The net rate of change of level u should be equal to zero, in TE.

$$\begin{aligned} \frac{dn_u}{dt} &= \left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} + \left(\frac{dn_u}{dt} \right)_{u \rightarrow \ell} \\ &= n_\ell B_{\ell u} u_\nu - n_u (A_{u\ell} + B_{u\ell} u_\nu) \\ &= 0 \end{aligned}$$

$$n_\ell B_{\ell u} u_\nu - n_u (A_{u\ell} + B_{u\ell} u_\nu) = 0$$

$$(n_\ell B_{\ell u} - n_u B_{u\ell}) u_\nu = n_u A_{u\ell}$$

$$\begin{aligned} u_\nu &= \frac{n_u A_{u\ell}}{n_\ell B_{\ell u} - n_u B_{u\ell}} \\ &= \frac{(n_u A_{u\ell}) / (n_\ell B_{\ell u})}{1 - (n_u B_{u\ell}) / (n_\ell B_{\ell u})} \\ &= \frac{(g_u/g_\ell) e^{-h\nu/kT} (A_{u\ell}/B_{\ell u})}{1 - (g_u/g_\ell) e^{-h\nu/kT} (B_{u\ell}/B_{\ell u})} \quad \leftarrow \quad \frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} e^{-h\nu_{u\ell}/kT_{\text{exc}}} \\ &= \frac{(g_u/g_\ell) (A_{u\ell}/B_{\ell u})}{e^{h\nu/kT} - (g_u/g_\ell) (B_{u\ell}/B_{\ell u})} \end{aligned}$$

Comparing the above eq. with Planck function,

$$u_\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

we can immediately recognize that the following relations should be satisfied.

$$(g_u/g_\ell) (A_{u\ell}/B_{\ell u}) = \frac{8\pi h\nu^3}{c^3}$$

$$(g_u/g_\ell) (B_{u\ell}/B_{\ell u}) = 1$$

Only one coefficient is independent.

[Note] If there is no stimulated emission ($B_{u\ell} = 0$), the only way to make the left eq. consistent with the Planck function is to assume $h\nu/kT \gg 1$ (Wien's regime). Therefore, the stimulated emission is negligible in the Wien's regime. In other words, the stimulated emission term is required in the Rayleigh-Jean regime.

In summary, we obtained the following relations between the Einstein coefficients.

$$A_{u\ell} = \frac{8\pi h\nu^3}{c^3} B_{u\ell}$$

$$B_{\ell u} = \frac{g_u}{g_\ell} B_{u\ell}$$

$$B_{u\ell} = \frac{c^3}{8\pi h\nu^3} A_{u\ell}$$

$$B_{\ell u} = \frac{g_u}{g_\ell} \frac{c^3}{8\pi h\nu^3} A_{u\ell}$$

- We note the Einstein coefficients are intrinsic properties of the absorbing material, irrelevant to the assumption of TE. Hence, **the relations between the Einstein coefficients should hold in any condition.**
- Using the relation, we can rewrite the downward and upward transition rates:

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = n_\ell \frac{g_u}{g_\ell} \frac{c^3}{8\pi h\nu^3} A_{u\ell} u_\nu \quad \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell} = n_u A_{u\ell} \left(1 + \frac{c^3}{8\pi h\nu^3} u_\nu \right)$$

- It is helpful to use a dimensionless quantity, the photon occupation number:

$$n_\gamma \equiv \frac{c^2}{2h\nu^3} I_\nu \quad \xrightarrow{\text{averaging over directions}} \quad \langle n_\gamma \rangle = \frac{c^2}{2h\nu^3} J_\nu = \frac{c^3}{8\pi h\nu^3} u_\nu$$

- Then, the above transition rates are simplified:

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = n_\ell \frac{g_u}{g_\ell} A_{u\ell} \langle n_\gamma \rangle \quad \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell} = n_u A_{u\ell} (1 + \langle n_\gamma \rangle)$$

- The photon occupation number determines the relative importance of stimulated and spontaneous emission: stimulated emission is important only when $\langle n_\gamma \rangle \gg 1$.

Absorption and Emission Coefficients in terms of Einstein coefficients

- The Einstein coefficients are useful means of analyzing absorption and emission processes. However, we often find it even more useful to use cross section because **the cross section has a natural geometric meaning.**
- (pure) Absorption cross section:**

- The number density of photons per unit frequency interval is $u_\nu/h\nu$. Let $\sigma_{\ell u}(\nu)$ be the cross section for absorption of photons for the transition $\ell \rightarrow u$. Then, the absorption rate is

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} = n_\ell \int d\nu \sigma_{\ell u}(\nu) c \frac{u_\nu}{h\nu} \approx n_\ell u_\nu \frac{c}{h\nu_{ul}} \int d\nu \sigma_{\ell u}(\nu)$$

- Here, we assumed that $u_\nu/h\nu$ do not vary appreciably over the profile of the cross section. Therefore, we derive a simple relation between the absorption cross section and the Einstein B coefficient:

$$\int d\nu \sigma_{\ell u}(\nu) = \frac{h\nu_{ul}}{c} B_{\ell u} = \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{ul}^2} A_{ul}$$

- If the cross section has a normalized profile of ϕ_ν , we can write the absorption cross section as follows:

$$\sigma_{\ell u}(\nu) = \frac{h\nu_{ul}}{c} B_{\ell u} \phi_\nu = \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{ul}^2} A_{ul} \phi_\nu \quad \text{with} \quad \int \phi_\nu d\nu = 1$$

- **(effective) Absorption Coefficient**

- We note that the stimulated emission is proportional to the energy density of ambient radiation field. In the radiative transfer equation, it is convenient to include the stimulated emission term in the absorption coefficient as a negative absorption.

$$\left(\frac{dn_u}{dt} \right)_{\ell \rightarrow u} - \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell}^{\text{stimulated}} = n_\ell B_{\ell u} u_\nu - n_u B_{u \ell} u_\nu$$

$$= n_\ell B_{\ell u} u_\nu - n_u \left(\frac{g_\ell}{g_u} B_{\ell u} \right) u_\nu$$

- Therefore, we may define the cross section for stimulated emission and the net (effective) absorption coefficient as follows:

$$\sigma_{u\ell} = \frac{g_\ell}{g_u} \sigma_{\ell u}$$

$$\kappa_\nu = n_\ell \sigma_{\ell u} - n_u \sigma_{u\ell}$$

$$= n_\ell \sigma_{\ell u} \left(1 - \frac{n_u/n_\ell}{g_u/g_\ell} \right)$$

pure absorption coefficient

- Using the definition of the excitation temperature, we can rewrite them:

$$\kappa_\nu = n_\ell \sigma_{\ell u} \left[1 - \exp \left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}} \right) \right] \quad \text{or} \quad \sigma_\nu^{\text{eff}} = \sigma_{\ell u} \left[1 - \exp \left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}} \right) \right]$$

- **Emission coefficient (Emissivity)**

- The emissivity is defined as the power radiated per unit frequency per unit solid angle per unit volume.
- The line emissivity can be expressed in terms of the spontaneous downward transition rate:

$$4\pi \int d\nu j_\nu = h\nu_{u\ell} \left(\frac{dn_\ell}{dt} \right)_{u \rightarrow \ell}^{\text{spontaneous}}$$

- Comparing with the definition of the Einstein A coefficient, we obtain:
- $$\int d\nu j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell}$$
- If the emission line has a normalized profile of ϕ_ν , we can write the emissivity as follows:

$$j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell} \phi_\nu \quad \text{with} \quad \int d\nu \phi_\nu = 1$$

-
- The correction factor for the stimulated emission in absorption coefficient:

- For Ly α line,

$$h\nu_{u\ell} = 10.2 \text{ eV} \rightarrow 1 - \exp\left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}}\right) = 1 - \exp\left(-\frac{1.1837 \times 10^5 \text{ K}}{T_{\text{exc}}}\right)$$

$$\simeq 1 \quad \text{for } T_{\text{exc}} \approx T_{\text{gas}} < 1 \times 10^5 \text{ K}$$

- ▶ The stimulated emission is negligible.

- For 21 cm line,

$$h\nu_{u\ell} = 6 \mu\text{eV} \rightarrow 1 - \exp\left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}}\right) = 1 - \exp\left(-\frac{0.068 \text{ K}}{T_{\text{exc}}}\right)$$

$$\simeq \frac{0.068 \text{ K}}{T_{\text{exc}}} \ll 1 \quad \text{for } T_{\text{exc}} \approx T_{\text{gas}} \sim 100 \text{ K}$$

- ▶ The correction for stimulated emission is very important. **We, therefore, need to take into account the stimulated emission in dealing with the 21 cm line.**

- Two limiting cases:

- At radio and sub-mm frequencies, the upper levels are often appreciably populated, and it is important to include both spontaneous and stimulated emission.
- When we consider propagation of optical, UV, or X-ray radiation in cold ISM, the upper levels of atoms and ions usually have negligible populations, and stimulated emission can be neglected.

- **Source Function:**

$$\begin{aligned}
 S_\nu &= \frac{j_\nu}{\kappa_\nu} \\
 &= \frac{n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell} \phi_\nu^{\text{emiss}}}{n_\ell \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell} \phi_\nu^{\text{abs}} [1 - \exp(-h\nu_{u\ell}/k_B T_{\text{exc}})]} && \leftarrow \frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp(-h\nu_{u\ell}/k_B T_{\text{exc}}) \\
 &= \frac{2h\nu_{u\ell}^3}{c^2} \frac{1}{\exp(h\nu_{u\ell}/k_B T_{\text{exc}}) - 1} && \leftarrow \phi_\nu^{\text{emiss}} = \phi_\nu^{\text{abs}}
 \end{aligned}$$

- This is called the **generalized Kirchhoff's law**.
- **The intrinsic profiles for absorption and emission are the same.**
 - ▶ The source function should approach the Planck function in LTE ($T_{\text{exc}} = T_{\text{kinetic}}$). For this to be true, the intrinsic profile of emission line should be the same as that of absorption line.
 - ▶ We will show, later, that the intrinsic emission and absorption profiles are, indeed, the same, using a semi-classical model for an atom.

Oscillator Strength

- In the previous slides, we characterized the absorption cross section by the Einstein A coefficient. Equivalently, we can express the cross section in terms of the oscillator strength for the absorption transition $\ell \rightarrow u$, defined by the relation:

$$\int \sigma_{\ell u}(\nu) d\nu = \frac{\pi e^2}{m_e c} f_{\ell u} \quad \rightarrow \quad \sigma_{\ell u}(\nu) = \frac{\pi e^2}{m_e c} f_{\ell u} \phi_\nu$$

- Here, the factor $\frac{\pi e^2}{m_e c}$ is the cross-section for a classical oscillator model.
- The oscillator strength is the factor which corrects the classical result. The quantum mechanical process can be interpreted as being due to a (fractional) number f of equivalent classical electron oscillators of the same frequency.
- The Einstein A coefficient is related to the absorption oscillator strength of the upward transition by

$$A_{u\ell} = \frac{8\pi^2 e^2 \nu_{u\ell}^2}{m_e c^3} \frac{g_\ell}{g_u} f_{\ell u} = \left(\frac{0.8167 \text{ cm}}{\lambda_{u\ell}} \right)^2 \frac{g_\ell}{g_u} f_{\ell u} [\text{s}^{-1}]$$

- For 21.1 cm line, $g_u = 3$, $g_\ell = 1$ ($g_F = 2F + 1$) $f_{\ell u} = 5.75 \times 10^{-12}$
 $A_{u\ell} = 2.88 \times 10^{-15} [\text{s}^{-1}] = (11 \text{ Myr})^{-1}$
 - For Ly α (1215.67 Å) line, $g_u = 3$, $g_\ell = 1$ ($g_L = 2L + 1$) $f_{\ell u} = 0.4164$ for $1^2S_{1/2} \rightarrow 2^2P$
 $A_{u\ell} = 6.265 \times 10^8 [\text{s}^{-1}]$
- $f_{\ell u} = 0.27760$ for ${}^2S_{1/2} \rightarrow {}^2P_{3/2}$
 $= 0.13881$ for ${}^2S_{1/2} \rightarrow {}^2P_{1/2}$

Maser Lines

- Population inversion
 - Under some conditions, a process may act to “pump” an excited state u by either collisional or radiative excitation of a higher level u' that then decays to populate level u . If this pumping process is rapid enough (relative to the processes that depopulate u), it may be possible for the relative level populations between u and ℓ to satisfy the inequality (also to have a negative excitation temperature).

$$n_u > \frac{g_u}{g_\ell} n_\ell \quad \rightarrow \quad T_{\text{exc}, u\ell} < 0.$$

- When this population inversion occurs, stimulated emission is stronger than pure absorption, and ***the radiation is amplified as it propagates***. Then, the effective absorption coefficient, optical depth, and attenuation factor are

$$\kappa_\nu = \sigma_{\ell u} \left(1 - \frac{n_u/g_u}{n_\ell/g_\ell} \right) < 0, \quad \tau_\nu = \int \kappa_\nu ds < 0, \quad e^{-\tau_\nu} > 1$$

- Maser
 - Such population inversion have been observed for microwave transitions of H I, OH, H₂O, and SiO, and hence we speak of ***maser (microwave amplification by stimulated emission of radiation)*** emission.

-
- Observational properties
 - If $|k_B T_{\text{exc}, u\ell}| \gg h\nu$, the RT equation becomes
$$\begin{aligned} T_A &= T_A(0)e^{-\tau_\nu} + T_{\text{exc}}(1 - e^{-\tau_\nu}) \\ &= (T_A(0) + |T_{\text{exc}}|)e^{|\tau_\nu|} - |T_{\text{exc}}| \end{aligned}$$
 - The factor $e^{|\tau_\nu|}$ is in some cases very large - some OH and H₂O masers have been observed to have $T_A > 10^{11}$ K.
 - We note that
 - ▶ $e^{|\tau_\nu|}$ is more strongly peaked on the sky than $|\tau_\nu|$ - the angular size of the maser is less than the actual transverse dimension of the maser region.
 - ▶ $e^{|\tau_\nu|}$ is more strongly in ν than $|\tau_\nu|$ - the maser line is narrower than the actual velocity distribution of the maser species.
 - Some maser can be very bright, allowing the use of interferometry, as well as observations of sources at large distances.
 - ▶ This has enabled measurements of proper motion of maser spots in star-forming regions of the MW, as well as in material orbiting a supermassive black hole in the spiral galaxy NGC 4258 (Hernstein et al. 1999).

Homework (due date: 04/17)

from Draine's problems

- [Q5] 1 A local HI cloud is interposed between us and the cosmic microwave background with temperature $T_{\text{CMB}} = 2.7255 \text{ K}$. Suppose that the HI in the cloud has a spin temperature $T_{\text{spin}} = 50 \text{ K}$, and that the optical depth at line-center (of the 21 cm line) is $\tau = 0.1$. The cloud is extended. We observe the cloud with a radio telescope.
- What will be the (absolute) brightness temperature T_B at line-center of the 21 cm line? Express your answer in deg K. You may assume that $h\nu \ll kT_B$.
 - What will be the (absolute) intensity at line-center of the 21 cm line? Express your answer in Jy sr^{-1} .
- 2 Consider a photon of frequency $h\nu$ entering a slab of material containing two-level atoms with excitation temperature T_{ul} . At the frequency of the photon, let the optical depth of the slab be τ .
- Let P_{abs} be the probability that the original photon will undergo absorption before exiting from the slab. Give an expression for P_{abs} in terms of τ and $h\nu/kT_{ul}$. Hint: here, τ is the optical depth for pure absorption.
 - Consider a photon that crossed the slab without being absorbed. Let $P_{\text{stim.em.}}$ be the probability that the incident photon will stimulate emission of one or more photons. Give an expression for $P_{\text{stim.em.}}$ in terms of τ and $h\nu/kT_{ul}$.
- 3 Suppose that we have a molecule with three energy levels – denoted 0, 1, 2 – ordered according to increasing energy, $E_0 < E_1 < E_2$. Let g_0, g_1, g_2 be the degeneracies of the levels. Suppose that there is radiation present with $h\nu = E_2 - E_0$, due to an external source plus emission in the $2 \rightarrow 0$ transition.
- Let ζ_{02} be the absorption probability per unit time for a molecule in level 0, with a transition to level 2. Let A_{20}, A_{21} , and A_{10} be the Einstein A coefficients for decays $2 \rightarrow 0$, $2 \rightarrow 1$, and $1 \rightarrow 0$ by spontaneous emission of a photon. Ignore collisional processes.
- Hint: this mean no external radiation field for $2 \rightarrow 1$ and $1 \rightarrow 0$
- Ignoring possible absorption of photons in the $2 \rightarrow 1$ and $1 \rightarrow 0$ transitions, obtain an expression for the ratio n_1/n_0 , where n_i is the number density of molecules in level i .
 - How large must ζ_{02} be for this molecule to act as a maser in the $1 \rightarrow 0$ transition?
 - Is it possible for this system to have maser emission in the $2 \rightarrow 1$ transition? If so, what conditions must be satisfied?
- Hint: find a relation between A_{21} and A_{10} to make the solution self-consistent.

[Q6] Voigt profile

- We want to derive an approximate formula for the transition point from the Gaussian core to the Lorentz wing, which is defined by

$$u^2 = \ln(\sqrt{\pi}/a) + \ln u^2 \quad \text{or} \quad x = \ln(\sqrt{\pi}/a) + \ln x, \text{ where } x \equiv u^2$$

The above equation can be expressed in the form:

$$x = g(x) \text{ where } g(x) = \ln x + \ln(\sqrt{\pi}/a)$$

This equation can be solved using “**Fixed Point Iteration Method.**” Starting from any initial point x_0 , the following recursive process gives an approximate solution of the equation.

$$x_{n+1} = g(x_n)$$

- (1) Find a numerical solution x_* for Ly α line with $b = 1.0 \text{ km s}^{-1}$, which is appropriate for Ly α in the CNM with $T \sim 100 \text{ K}$.
- (2) Let’s denote the width parameter as a_* for $b = 1.0 \text{ km s}^{-1}$. This means that

$$x_* = \ln x_* + \ln(\sqrt{\pi}/a_*)$$

Now, for any parameter a which is different from a_* , you may express the constant term in $g(x)$ as follows:

$$\ln(\sqrt{\pi}/a) = \ln(a_*/a) + \ln(\sqrt{\pi}/a_*)$$

To find the solution for $a \neq a_*$ (but, $a \approx a_*$), choose an initial guess to be $x_0 = x_*$. Show that the solution for any a can be expressed as (after only a single iteration):

$$x_1 = x_* + \ln(a_*/a)$$

Insert numerical values into the above equation and compare it with Eq. (2.39) in Ryden's book (our textbook).

- (3) Repeat the above procedure for $b = 10 \text{ km s}^{-1}$, and show that the results are exactly the same as our solution in this lecture note and Eq. (6.42) in Draine's book.

[Note]

- An approximate solution for the damping optical depth, defined by $x = C \ln(x/\ln 2)$, can be obtained using “Fixed Point Iteration Method.”