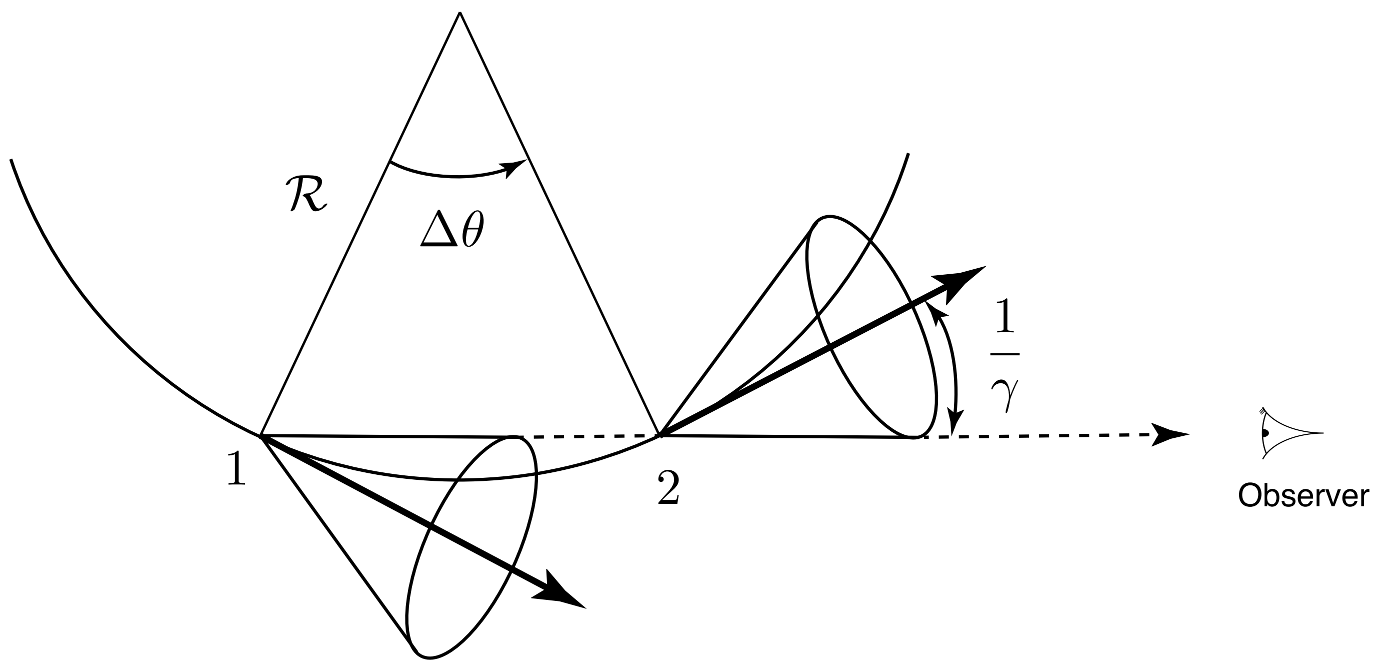
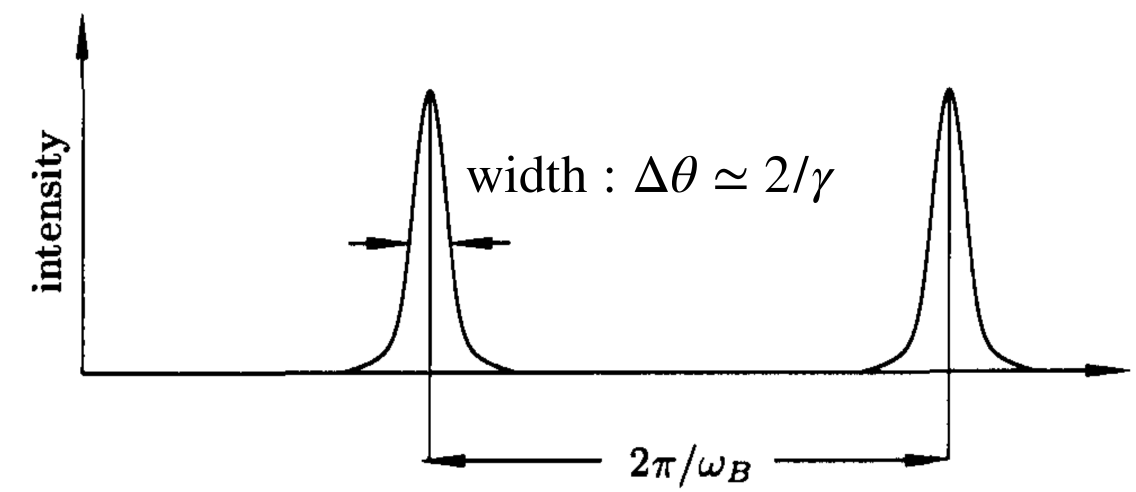
KAIST Astrophysics (PH481) - Part 1

Week 2b
Sep. 11 (Wed), 2019

Kwang-il Seon (선광일)
Korea Astronomy & Space Science Institute (KASI)
University of Science and Technology (UST)

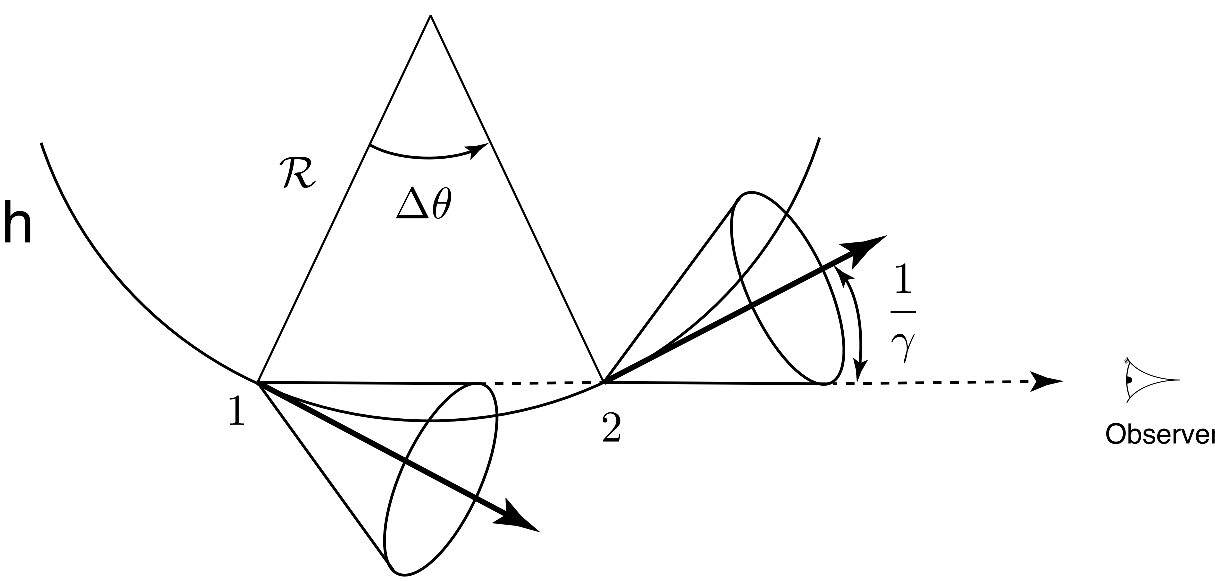
Spectrum of Synchrotron Radiation: A Qualitative Discussion





- Because of beaming effects the emitted radiation fields appear to be concentrated in a narrow set of directions about the particle's velocity.
  - The observer will see a pulses of radiation confined to a time interval much smaller than the gyration period.
  - The cone of emission has an angular width $\sim 1/\gamma$. Therefore, the observer will see emission over the angular range of $\Delta\theta \simeq 2/\gamma$.
  - The radiation appears beamed toward the direction of the observer in ***a series of pulses spaced in time (period)*** $2\pi/\omega_{\rm B}$ ***apart***, but with ***each pulse lasting only*** $\Delta\theta \simeq 2/\gamma$.
  - The spectrum will thus be spread over a much broader frequency range than on of order $\omega_{\rm B}$.

- To Fourier analyze the pulse shape, we need to **calculate the interval of the arrival times of the pulse corresponding to $\Delta\theta \sim 2/\gamma$.**
  - Let's consider an instantaneous rest frame of the electron.



$\Delta s = \mathcal{R}\Delta\theta$ = the pathlength from point 1 to 2
$\mathcal{R}$ = the radius of curvature of the path
$\Delta t = \Delta s/v$ = time interval from point 1 to 2
$|\Delta \mathbf{v}| = v\Delta\theta$ = velocity change

  - From the equation of motion, we find the curvature radius:

$$\gamma m_e \frac{\Delta \mathbf{v}}{\Delta t} = \frac{e}{c}\mathbf{v}\times\mathbf{B}$$

$$\gamma m_e \frac{v\Delta\theta}{\Delta s/v} = \frac{e}{c} vB\sin\alpha \rightarrow \boxed{\mathcal{R} = \frac{\Delta s}{\Delta\theta} = \frac{v}{\omega_{\rm B}\sin\alpha}}$$

Note that the radius of curvature is different from the gyroradius, which is the projected radius of the curvature radius.

  - Therefore the path length is given by

$$\boxed{\Delta s = \mathcal{R}(2/\gamma) = \frac{2v}{\gamma\omega_{\rm B}\sin\alpha} = \frac{2v}{\omega_{\rm L}\sin\alpha}}$$

- Time interval that the particle passes from point 1 to 2:

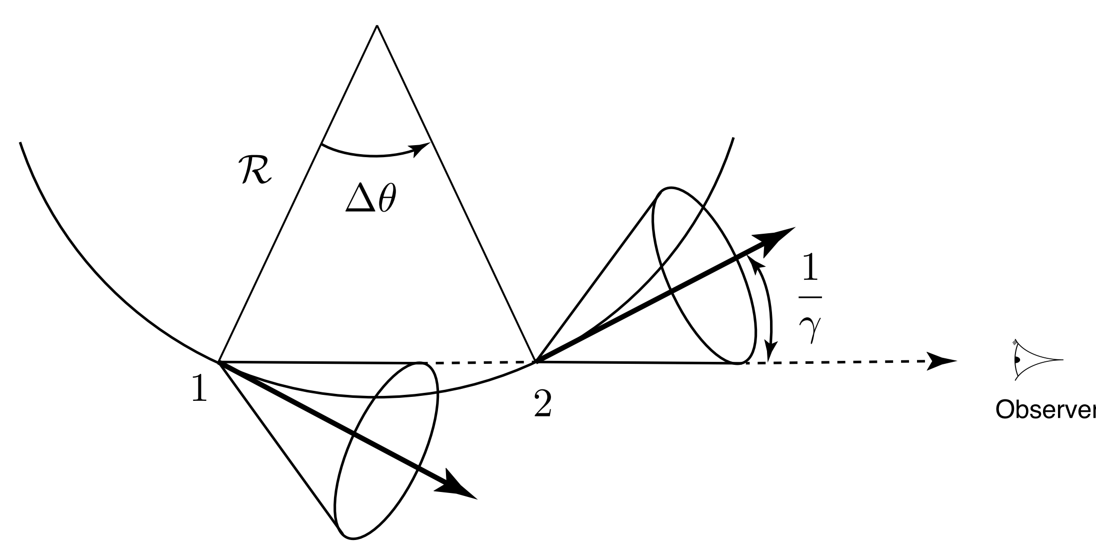
$$\Delta t = t_2 - t_1 = \frac{\Delta s}{v} \simeq \frac{2}{\omega_{\mathrm{L}} \sin\alpha}$$

- However, this is the time interval for the particle to travel from point 1 to 2. We need to calculate the interval of the arrival times of the pulse measured in the observer frame.
- Note that point 2 is closer than point 1 by $\Delta s/c$. Therefore, the difference of the arrival times of the pulse is

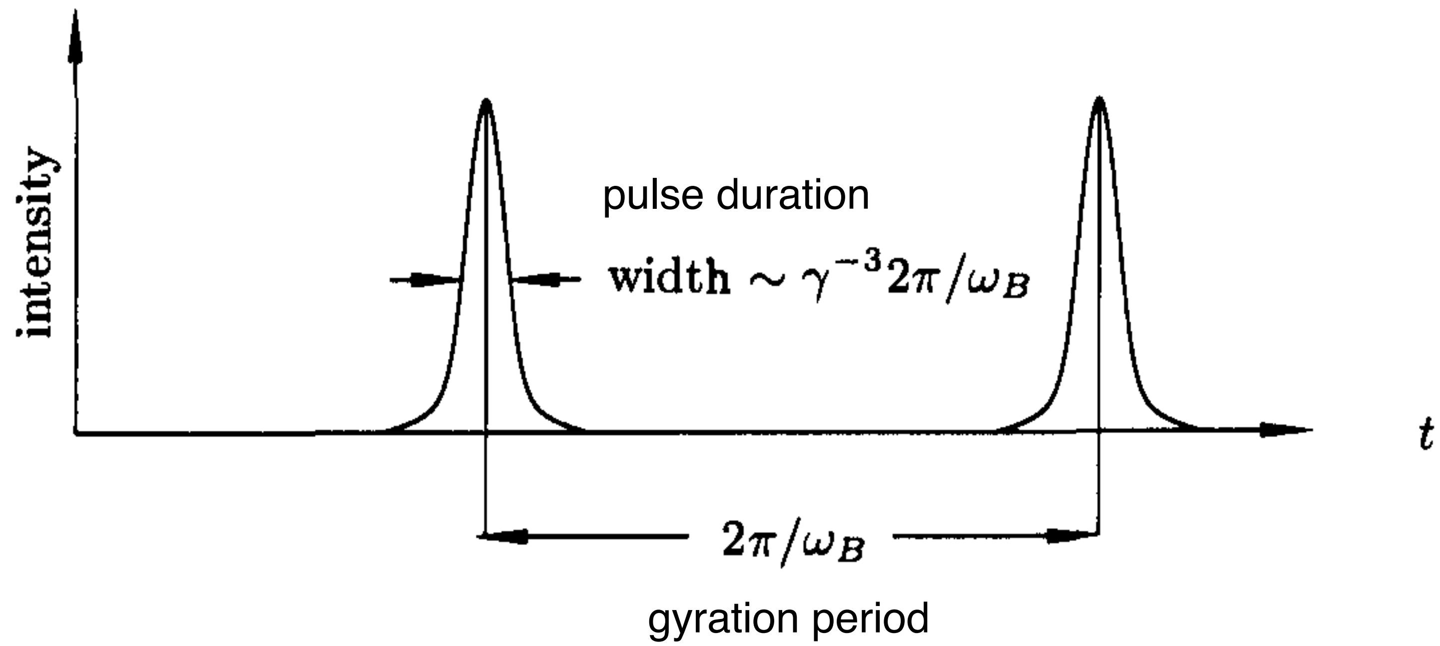
$$\Delta t^A = t_2^A - t_1^A = \Delta t - \frac{\Delta s}{c} = \Delta t \left(1 - \frac{v}{c}\right) \approx \frac{1}{\gamma^2 \omega_{\mathrm{L}} \sin\alpha} \qquad \leftarrow \quad 1 - \frac{v}{c} \approx \frac{1}{2\gamma^2}$$

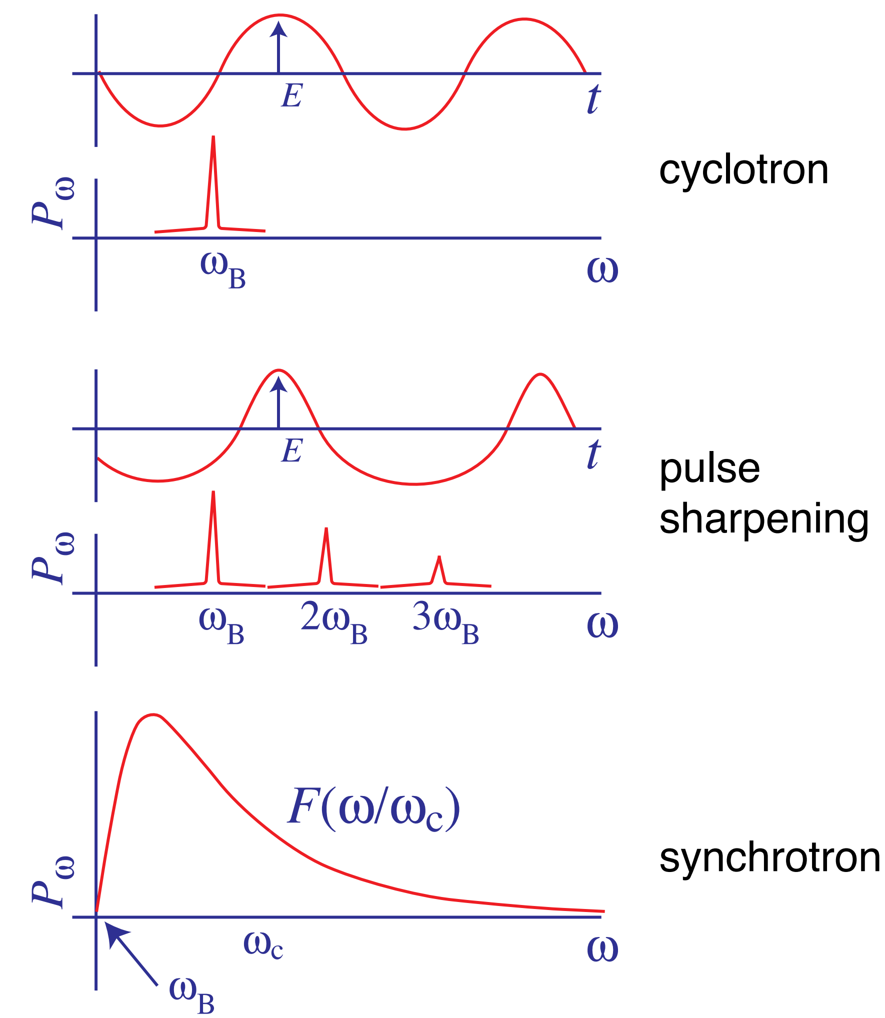
$$\boxed{\Delta t^A = t_2^A - t_1^A \approx \frac{1}{\gamma^2 \omega_{\mathrm{L}} \sin\alpha} = \frac{1}{\gamma^3 \omega_{\mathrm{B}} \sin\alpha}}$$

The width of the observed pulses is smaller than the gyration by a factor γ^3.



- Temporal pattern of received pulses:





- We define a critical frequency:

$$\omega_c \equiv \frac{3}{2}\gamma^2 \omega_{\rm L} \sin\alpha = \frac{3}{2}\gamma^3 \omega_{\rm B} \sin\alpha$$

The factor 3/2 is from the accurate calculation. (See Pacholczyk 1970, Radio Astrophysics. Nonthermal Processes in Galactic and Extragalactic Sources)

- From the properties of Fourier transformation, we expect that ***the spectrum will be fairly broad, within the frequency range of*** $\omega_{\rm B} \lesssim \omega \lesssim \omega_c$.

Spectrum of Synchrotron Radiation: Real Spectrum

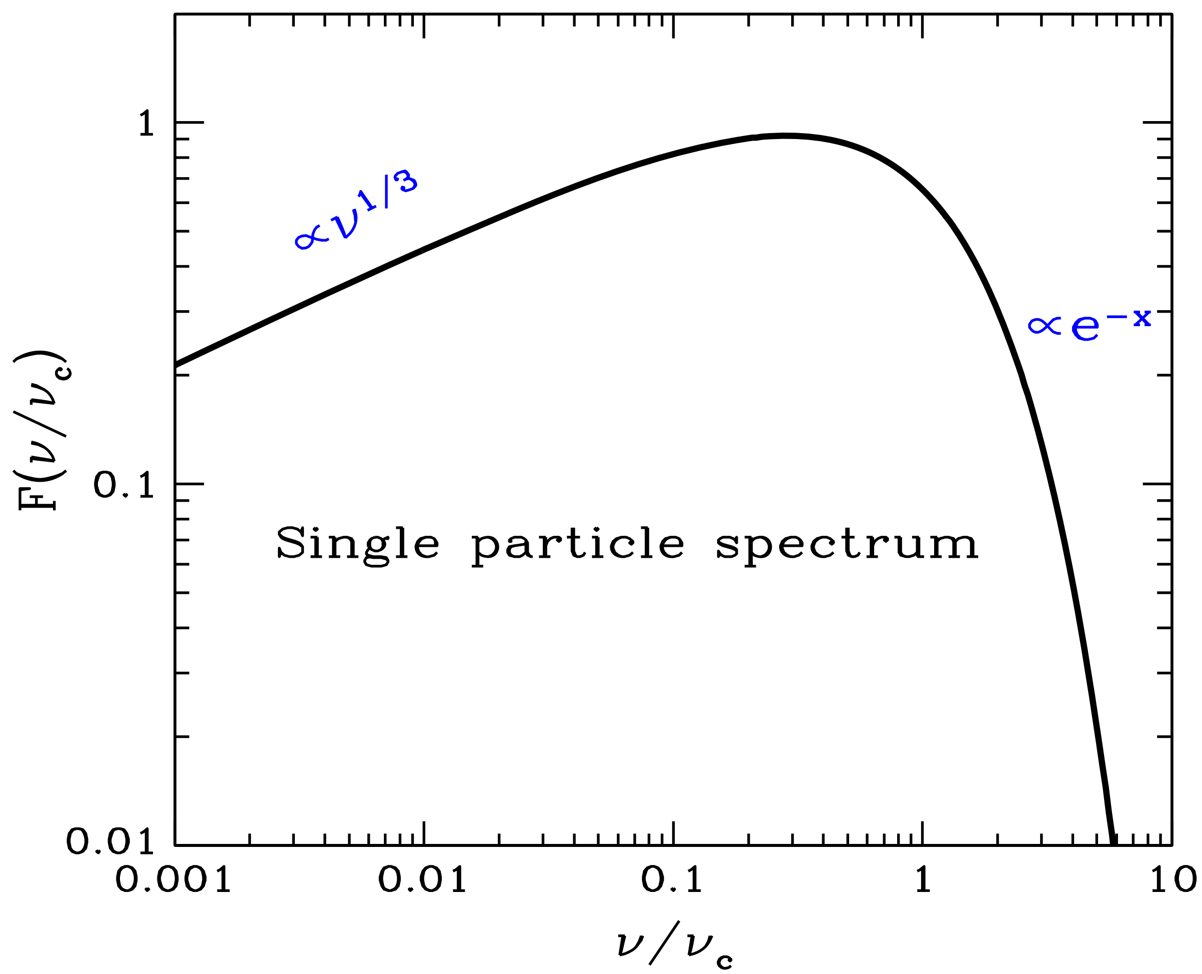
- After a lengthy calculation, the power per unit frequency emitted by an electron of given Lorentz factor γ and pitch angle α is found to be:

$$P_s(\nu,\gamma,\alpha) = \frac{\sqrt{3}e^3 B\sin\alpha}{m_e c^2} F(\nu/\nu_c)$$

$$F(\nu/\nu_c) \equiv \frac{\nu}{\nu_c}\int_{\nu/\nu_c}^{\infty} K_{5/3}(y)dy$$

Here, $K_{5/3}(y)$ is the modified Bessel function of order 5/3.

- Therefore, the emitted spectrum is a function of ν/ν_c.



Spectral Index for Power-law Electron Distribution

- Often the number density of particles with energies between E and $E + dE$ can be approximately expressed in the form:

$$N(E)dE = CE^{-p}dE \quad (E_1 \le E \le E_2) \quad \text{or} \quad N(\gamma)d\gamma = C\gamma^{-p}d\gamma \quad (\gamma_1 \le \gamma \le \gamma_2)$$

- The total power radiated per unit volume per unit frequency by such a distribution is given by

$$\begin{aligned} P_{\text{tot}}(\omega) &= \int_{\gamma_1}^{\gamma_2} N(\gamma)P(\omega)d\gamma \\ &\propto \int_{\gamma_1}^{\gamma_2} \gamma^{-p} B F\left(\frac{\omega}{\omega_c}\right) d\gamma \\ &\propto B\left(\frac{\omega}{B}\right)^{-(p-1)/2} \int_{x_1}^{x^2} x^{-(p-3)/2} F(x)dx \end{aligned}$$

$$\text{Recall } \omega_c \equiv \frac{3}{2}\gamma^2\omega_{\text{L}} \sin\alpha \propto \gamma^2 B$$

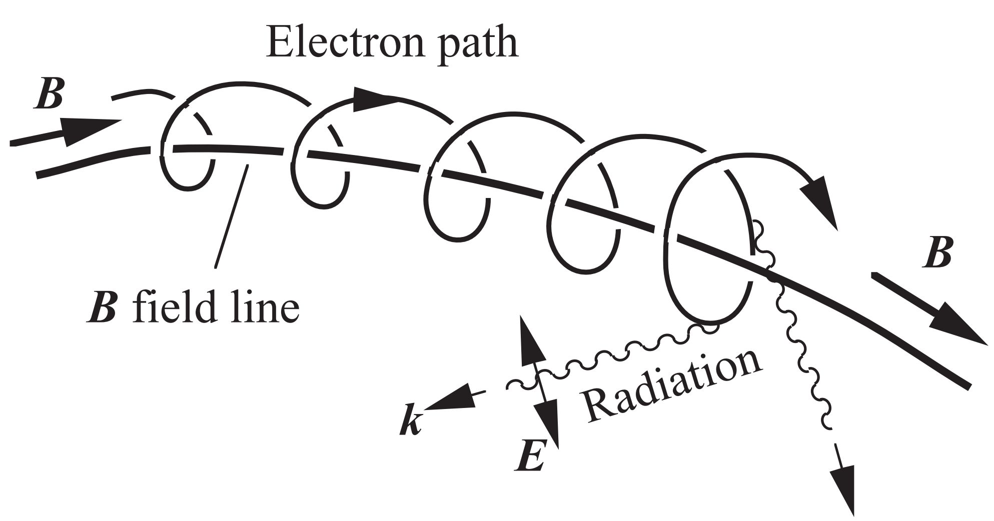
$$\leftarrow \text{ set } x = \frac{\omega}{\omega_c} \propto \gamma^{-2}\left(\frac{\omega}{B}\right)$$

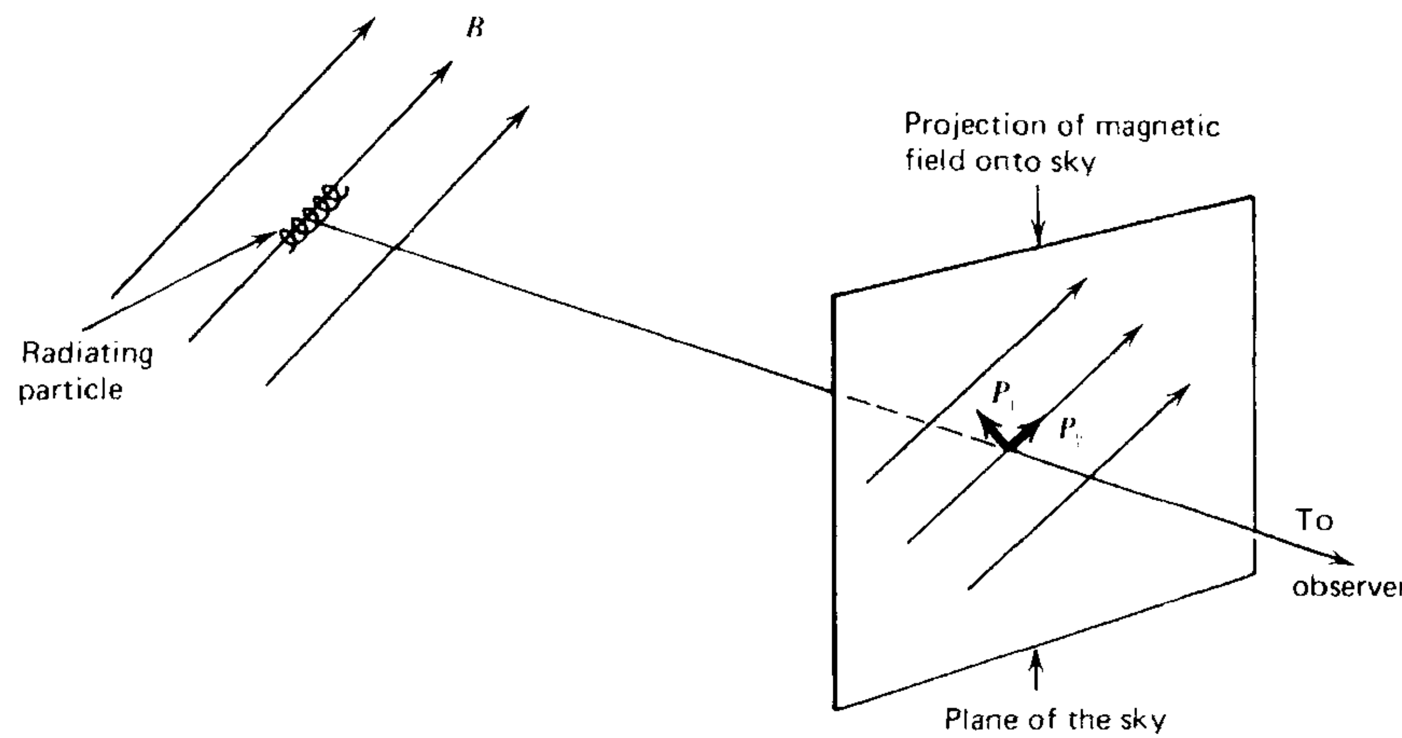
- Then, the spectrum is also a power law and the power-law spectral index *s* is related to the particle distribution index *p* by

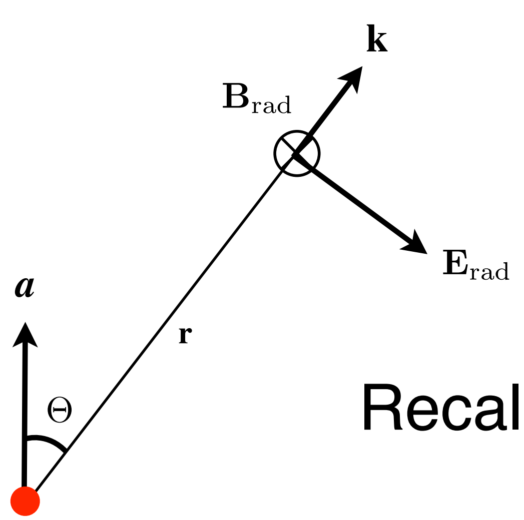
$$P_{\text{tot}}(\nu) \propto B^{s+1}\nu^{-s} \quad \text{where} \quad s = \frac{p-1}{2}$$

Polarization of Synchrotron Radiation

- In general, the radiation from a single charge will be elliptically polarized.
  - The electric field is ***in the same plane as the acceleration vector, which is perpendicular to the magnetic field.***
  - For any reasonable distribution of particles that varies smoothly with pitch angle, the elliptical component will cancel out as emission cones will contribute equally from both sides of the line of sight.
  - Thus, on average, **the radiation will be partially linearly polarized perpendicular to the magnetic field.**



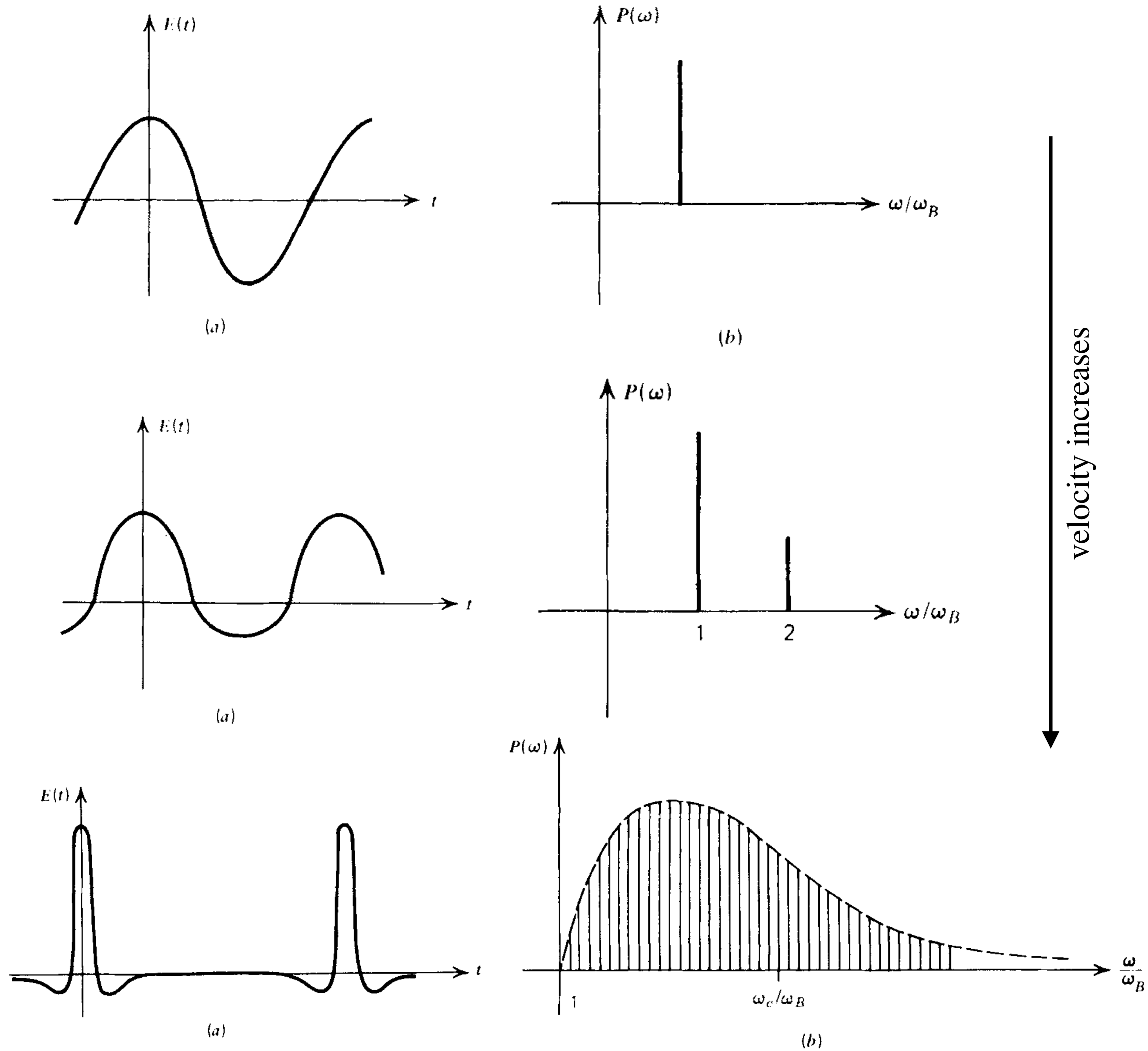




Recall that $\mathbf{E}_{\rm rad} \propto \mathbf{k} \times (\mathbf{k} \times \boldsymbol{a})$.

Transition from Cyclotron to Synchrotron Emission

- For low energies, the electric field components vary sinusoidally with the same frequency as the gyration in the magnetic field. The spectrum consists of a single line.
- When v/c increases, higher harmonics of the fundamental frequency begin to contribute.
- For very relativistic velocities, the originally sinusoidal form of *E(t)* has now become a series of sharp pulses, which is repeated at time intervals $2\pi/\omega_B$. The spectrum now involves a great number of harmonics, the envelope of which approaches the form of the function *F(x)*.



Sychrotron Self-Absorption

- All emission processes have their absorption counterpart, and the synchrotron emission is no exception. Since it is not thermal emission, we need to use Einstein coefficients to derive α_ν.
  - We can derive that the dependencies of the absorption coefficient (see Textbook of Rybicki & Lightman):

$$\boxed{\alpha_\nu \propto B^{(p+2)/2} \nu^{-(p+4)/2}}$$

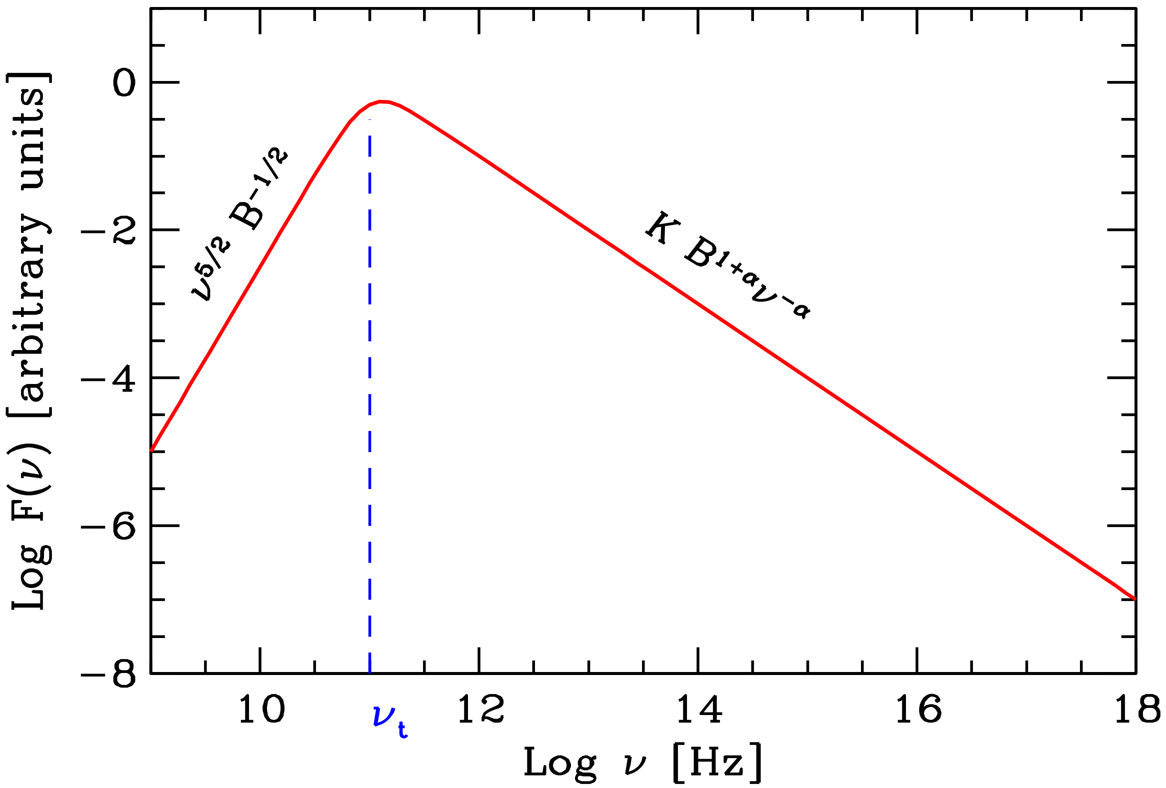
  - This indicates that **the synchrotron emission is optically thick at low frequencies and optically thin at high frequencies.**
  - The source function is $$S_\nu = \frac{j_\nu}{\alpha_\nu} \propto B^{-1/2} \nu^{5/2}$$
  - At low frequencies (optically thick emission)

$$I_\nu = S_\nu \propto B^{-1/2} \nu^{5/2}$$

  - At high frequencies (optically thin emission)

$$I_\nu = \int j_\nu ds \propto B^{(p+1)/2} \nu^{-(p-1)/2}$$

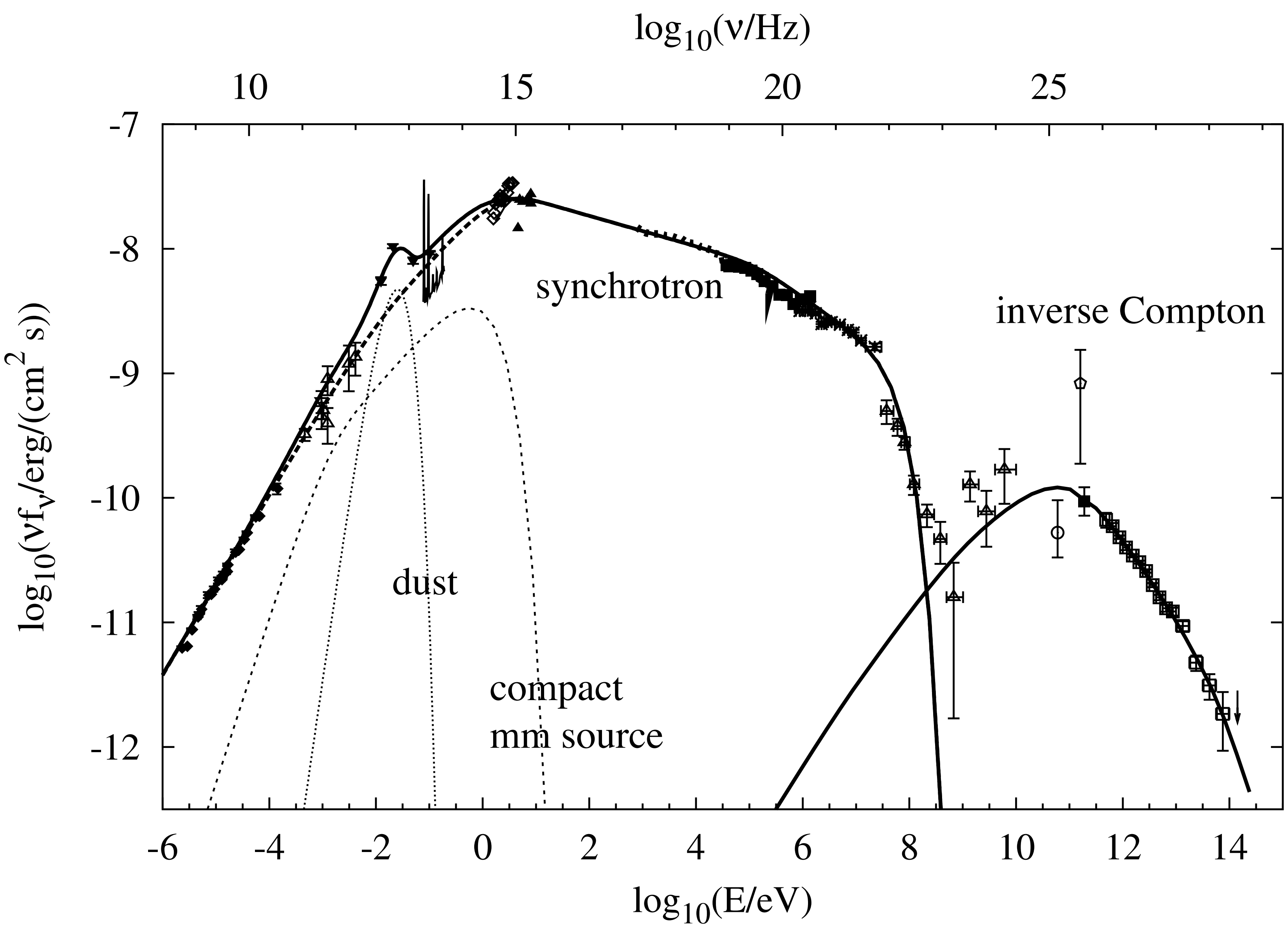
  - Observations of the self-absorption part could determine B.
  - Observations of the thin part can then determine the proportional constant K and the electron slope p.

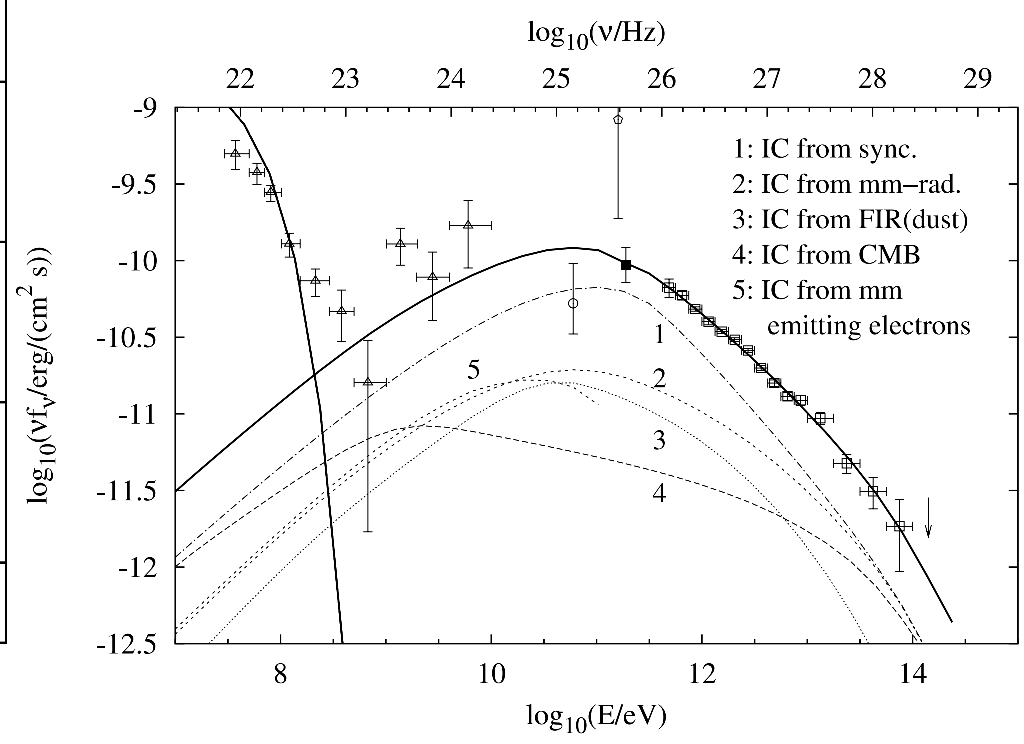


Astrophysical Example

- Crab nebula

Dots: modified blackbody with $T = 46$ K.
Thin dashed line: emission at mm wavelengths
Thick dashed line: synchrotron emission





Aharonian et al. (2004)

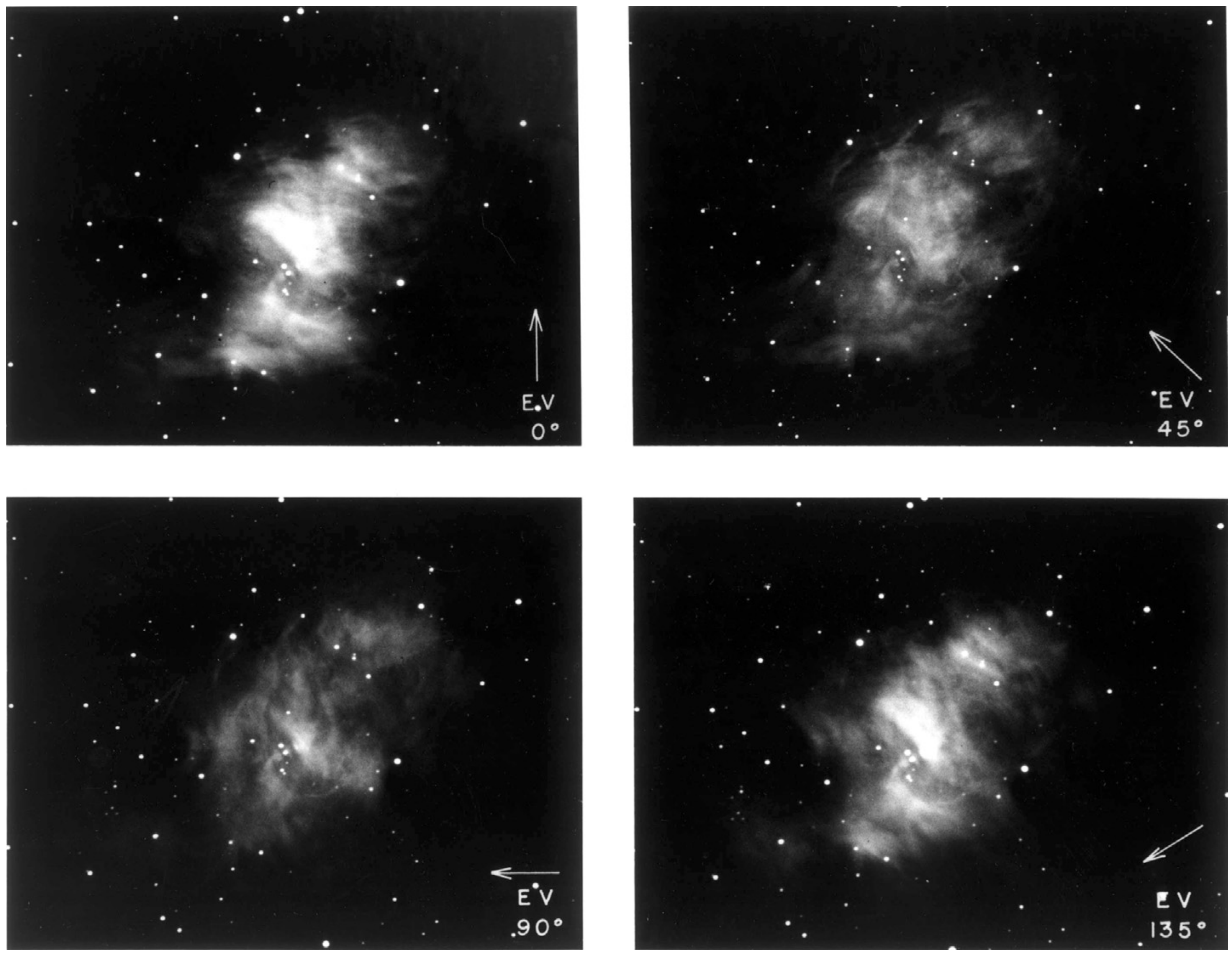


Fig. 8.3: Photographs of the Crab nebula in polarized light with the polarizer at different orientations. The arrows show the directions or planes of the transmitted transverse electric vector. Note the changing brightness pattern from photo to photo. The nebula has angular size $4' \times 6'$ and is $\sim$6 000 LY distant from the solar system. North is up and east to the left. The pulsar is the southwest (lower right) partner of the doublet at the center of the nebula. [Palomar Observatory/CalTech]

Compton Scattering

Thomson & Compton Scattering

- The simplest interaction between photons and free electrons is scattering.
  - **Thomson scattering:** When the energy of the incoming photons (as seen in the coming frame of the electron) is small with respect to the electron rest mass-energy, the process is called Thomson scattering.

$$\epsilon = \epsilon_1$$

$$\frac{d\sigma_T(\Omega)}{d\Omega} = \frac{1}{2} r_0^2 \left(1 + \cos^2\theta\right)$$

$$\sigma_T = \frac{8\pi}{3} r_0^2$$

$\epsilon =$ energy of the incident photon

$\epsilon_1 =$ energy of the scattered photon

$$r_0 = \frac{e^2}{m_e c^2}$$

Thomson scattering condition in the rest frame:

$\epsilon' \ll m_e c^2 = 0.5\mathrm{MeV}$

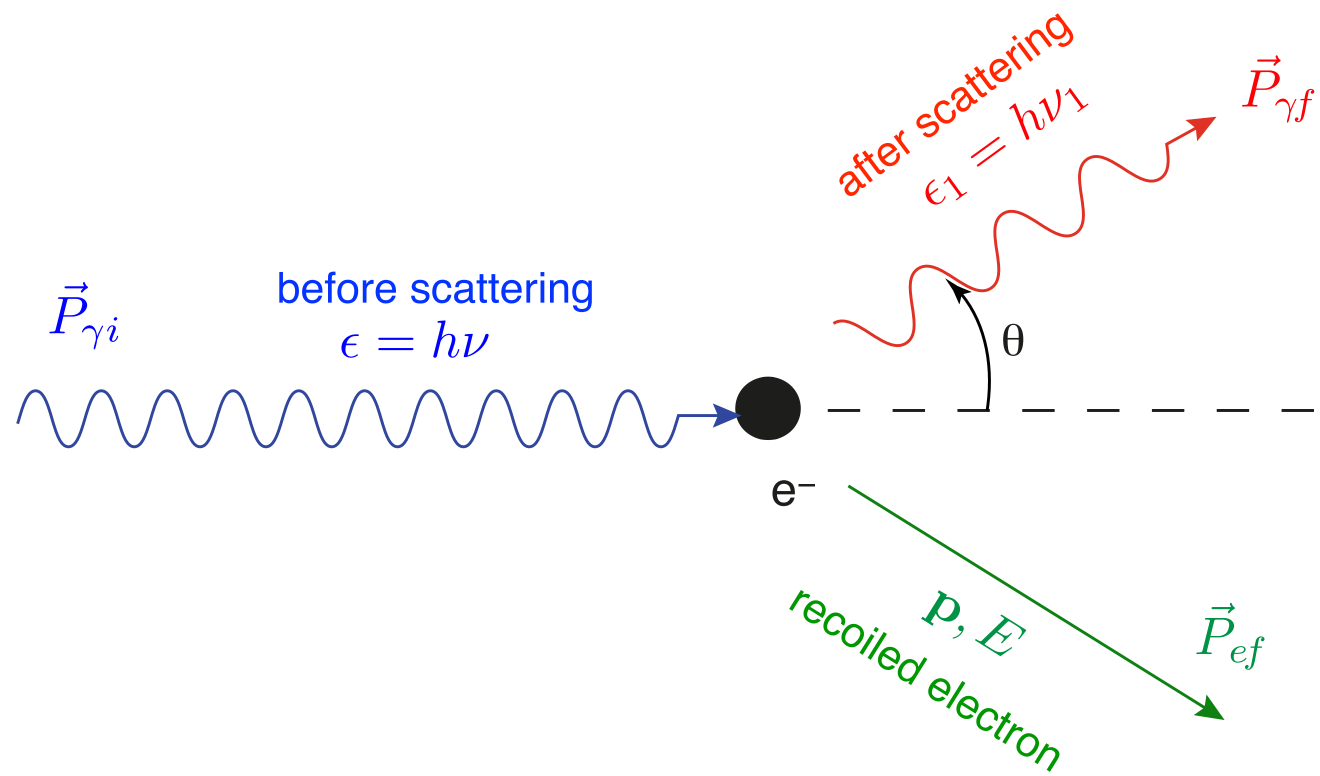
  - When $\epsilon = \epsilon_1$, the scattering is called **coherent or elastic**.

  - **Compton scattering:** As the energy of the incoming photons is comparable or greater than the electron rest mass-energy, it is called Compton scattering and a quantum treatment is necessary (Klein-Nishina regime).

Scattered Frequency

- Compton scattering (scattering of a photon by a non-moving electron)
  - The scattering will no longer be elastic ($\epsilon \neq \epsilon_1$) because of the recoil of the electron.
  - The cross sections are altered by the quantum effects.

From the energy and momentum conservation, we can derive the energy (or wavelength) of the scattered photon:



Energy (frequency):

$$\epsilon_1 = \frac{\epsilon}{1 + \frac{\epsilon}{mc^2}(1 - \cos\theta)}$$

Wavelength:

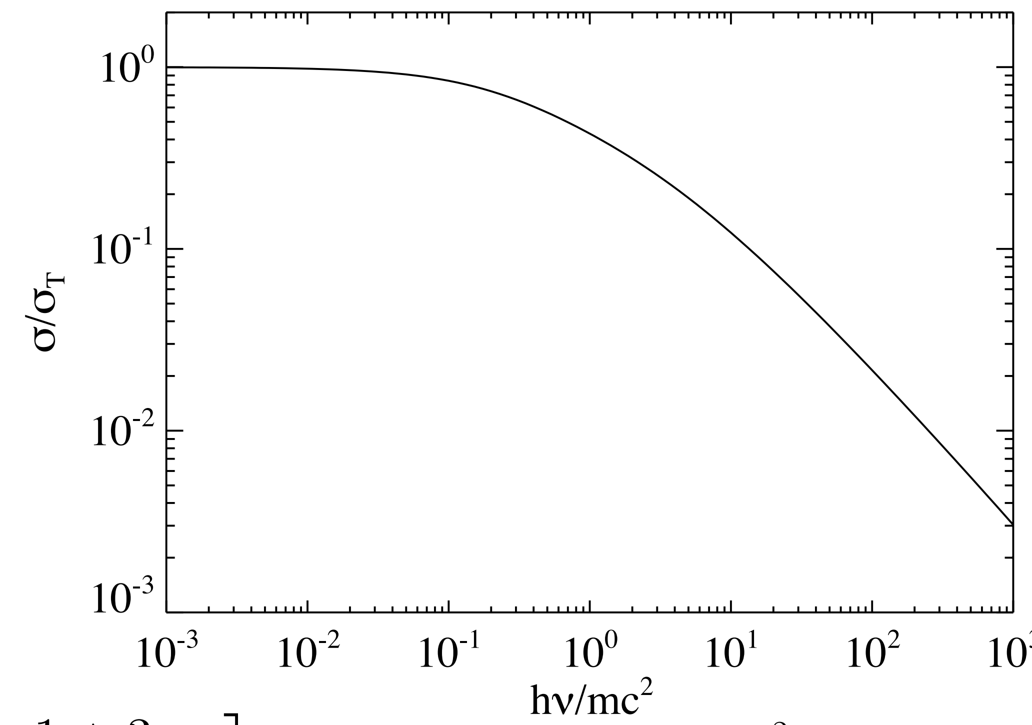
$$\lambda_1 - \lambda = \frac{h}{mc}(1 - \cos\theta)$$

Compton wavelength: $\lambda_c \equiv \dfrac{h}{mc} = 0.02426\text{Å}$ for electron.

There is a wavelength change of the order of λ_c upon scattering.
For long wavelengths $\lambda \gg \lambda_c$ (ie, $h\nu \ll mc^2$), the scattering is closely elastic.

Cross-section

- **Klein-Nishina formula** (the differential cross section for unpolarized radiation, QED)



$(m_e c^2 = 511 \text{ keV})$

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{16\pi}\frac{\epsilon_1^2}{\epsilon^2}\left(\frac{\epsilon}{\epsilon_1} + \frac{\epsilon}{\epsilon_1} - \sin^2\theta\right)$$

- Total cross section:

$$\sigma = 2\pi\int_{-1}^{1}\frac{d\sigma}{d\Omega}d\cos\theta$$

$$= \frac{3\sigma_T}{4}\left[\frac{1+x}{x^3}\left\{\frac{2x(1+x)}{1+2x} - \ln(1+2x)\right\} + \frac{\ln(1+2x)}{2x} - \frac{1+3x}{(1+2x)^2}\right]$$

where $\quad x \equiv \dfrac{h\nu}{mc^2}$

Compton scattering becomes less efficient at high energies.

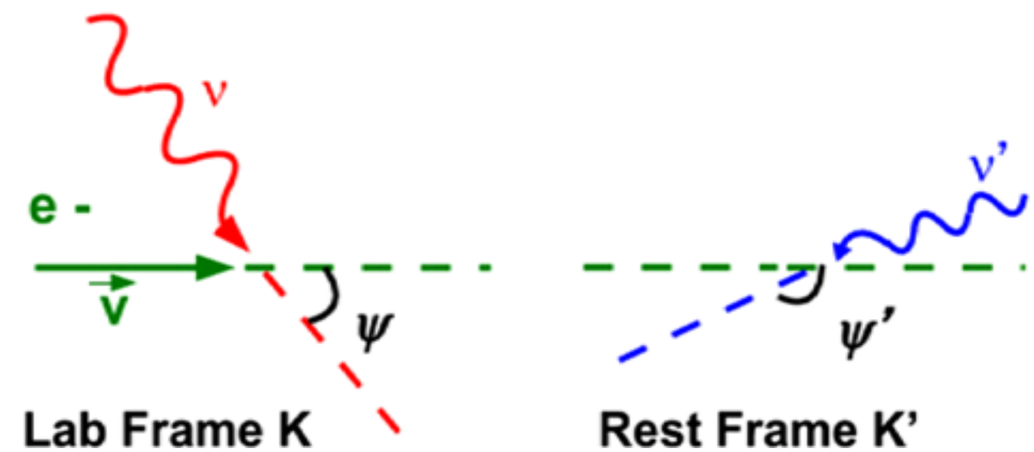
- Approximations:

non-relativisitic regime: $\quad \sigma \approx \sigma_T\left(1 - 2x + \dfrac{26x^2}{5} + \cdots\right), \quad x \ll 1$

extreme relativisitic regime: $\quad \sigma \approx \dfrac{3}{8}\sigma_T\dfrac{1}{x}\left(\ln 2x + \dfrac{1}{2}\right), \quad x \gg 1$

Inverse Compton Scattering: Scattering from Electrons in Motion

- **Inverse Compton Scattering**: Whenever the moving electron has sufficient kinetic energy compared to the photon, net energy may be transferred from the electron to the photon.
  - What is the energy of photon after the inverse Compton scattering?



(1) In the frame K' coming withe electron, the incoming photon energy is

$$\boxed{\epsilon' = \epsilon\gamma(1 - \beta\cos\psi)}$$

Here, ψ is the angle between the electron velocity and the photon direction in the lab frame.

(2) In the electron rest frame, we assume the Thomson regime so that no change in the photon energy.

$$\boxed{\epsilon'_1 = \epsilon'}$$

Here, $'$ (prime) denotes a quantity measured in the electron rest frame. the subscript 1 (one) is for a quantity after scattering.

(3) Going back to the electron rest frame, the energy of the scattered photon is

$$\begin{aligned} \epsilon_1 &= \epsilon'_1\gamma(1 + \beta\cos\psi'_1) \\ &\simeq \epsilon\gamma^2(1 + \beta\cos\psi'_1)(1 - \beta\cos\psi) \end{aligned}$$

Here, ψ'_1 is the scattered angle of the photon in the electron rest frame.

- In summary,

$$\epsilon' \approx \epsilon\gamma$$
$$\epsilon'_1 \approx \epsilon'$$
$$\epsilon_1 \approx \epsilon'_1\gamma \approx \epsilon'\gamma \approx \epsilon\gamma^2$$

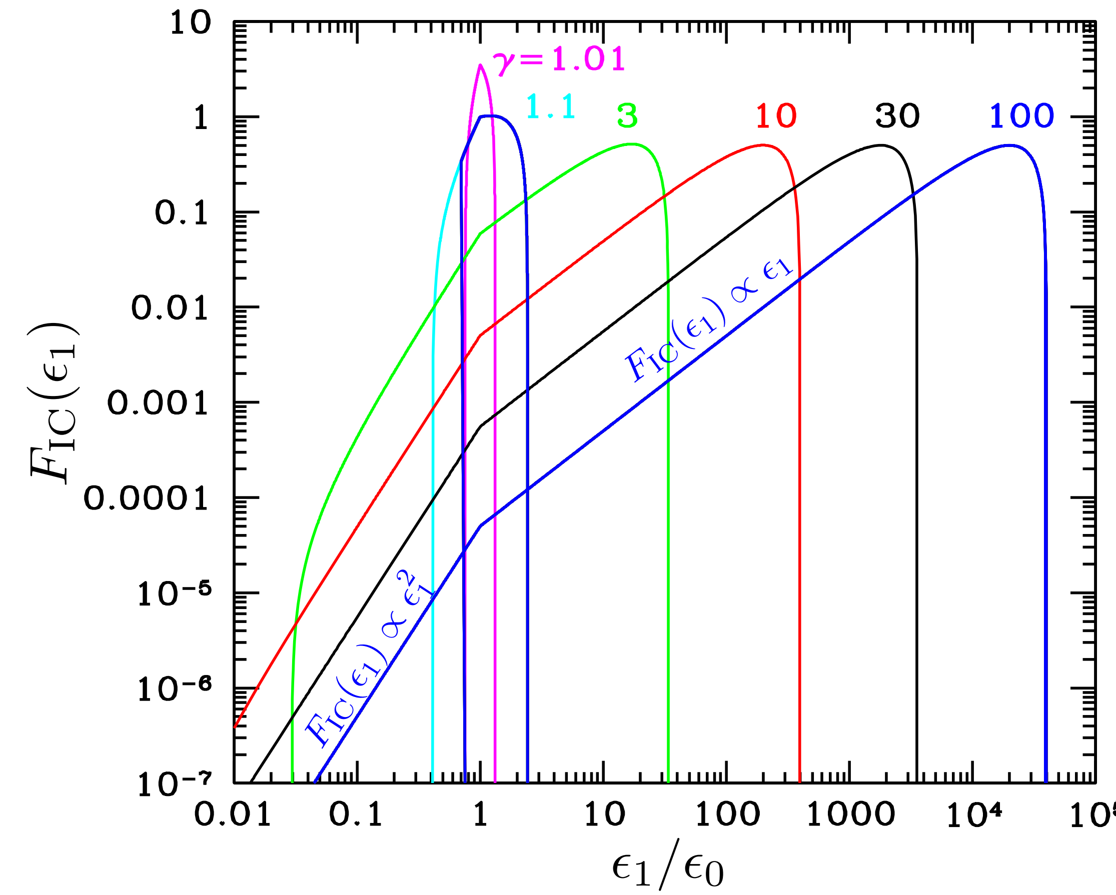
- The energies of the photon (1) before scattering in the lab frame, (2) in the electron rest frame, and (3) after the scattering in the lab frame are in the approximate ratios of

$$\epsilon \,:\, \epsilon' \,:\, \epsilon_1 \,\approx\, 1 \,:\, \gamma \,:\, \gamma^2$$

The inverse Compton scattering converts a low-energy photon to a high-energy photon by a factor of order γ^2.

- For an isotropic radiation, the average energy of scattered photon is found to be

$$\langle \epsilon_1 \rangle = \frac{4}{3}\gamma^2\epsilon$$ after a single scattering



Spectrum emitted by the Inverse Compton process by electrons of different γ (as labeled) scattering an isotropic monochromatic radiation field of dimensionless energy ϵ_0

(taken from Ghisellini's book)

Inverse Compton Spectra for relativistic electrons

- For a photon beam with a monochromatic energy ϵ_0 (initial energy), the spectrum of photons after a single scattering by energetic electrons is a function of ϵ/ϵ_c, i.e., $F(\epsilon/\epsilon_c)$.
  - Here, the characteristic frequency is $\epsilon_c = \gamma^2 \epsilon_0$.

- **<u>Power-law distribution of relativistic electrons</u>:**
  - If electrons follow a power-law velocity distribution $N(\gamma) = C\gamma^{-p}$, then the photon spectrum can be obtained by integrating $F(\epsilon/\epsilon_c)$ over the velocity distribution:

$$P(\epsilon) = \int F(\epsilon/\epsilon_c) N(\gamma) d\gamma$$

  - Setting $x \equiv \frac{\epsilon}{\epsilon_c} = \frac{\epsilon}{\gamma^2 \epsilon_0}$, we obtain $P(\epsilon) \propto \left(\frac{\epsilon}{\epsilon_0}\right)^{-(p-1)/2} \int F(x) x^{-(p+3)/2} dx.$

Here, we use $\gamma = \left(\frac{\epsilon}{\epsilon_0}\right)^{1/2} x^{-1/2}$ and $|d\gamma| = \frac{1}{2}\left(\frac{\epsilon}{\epsilon_0}\right)^{1/2} x^{-3/2} |dx|$

  - Finally, we obtain a power-law spectrum, as for Syncrotron radiation:

$$\boxed{P(\epsilon) \propto \epsilon^{-s}, \quad \text{where } s = (p-1)/2}$$

Comptonization: Repeated Scattering

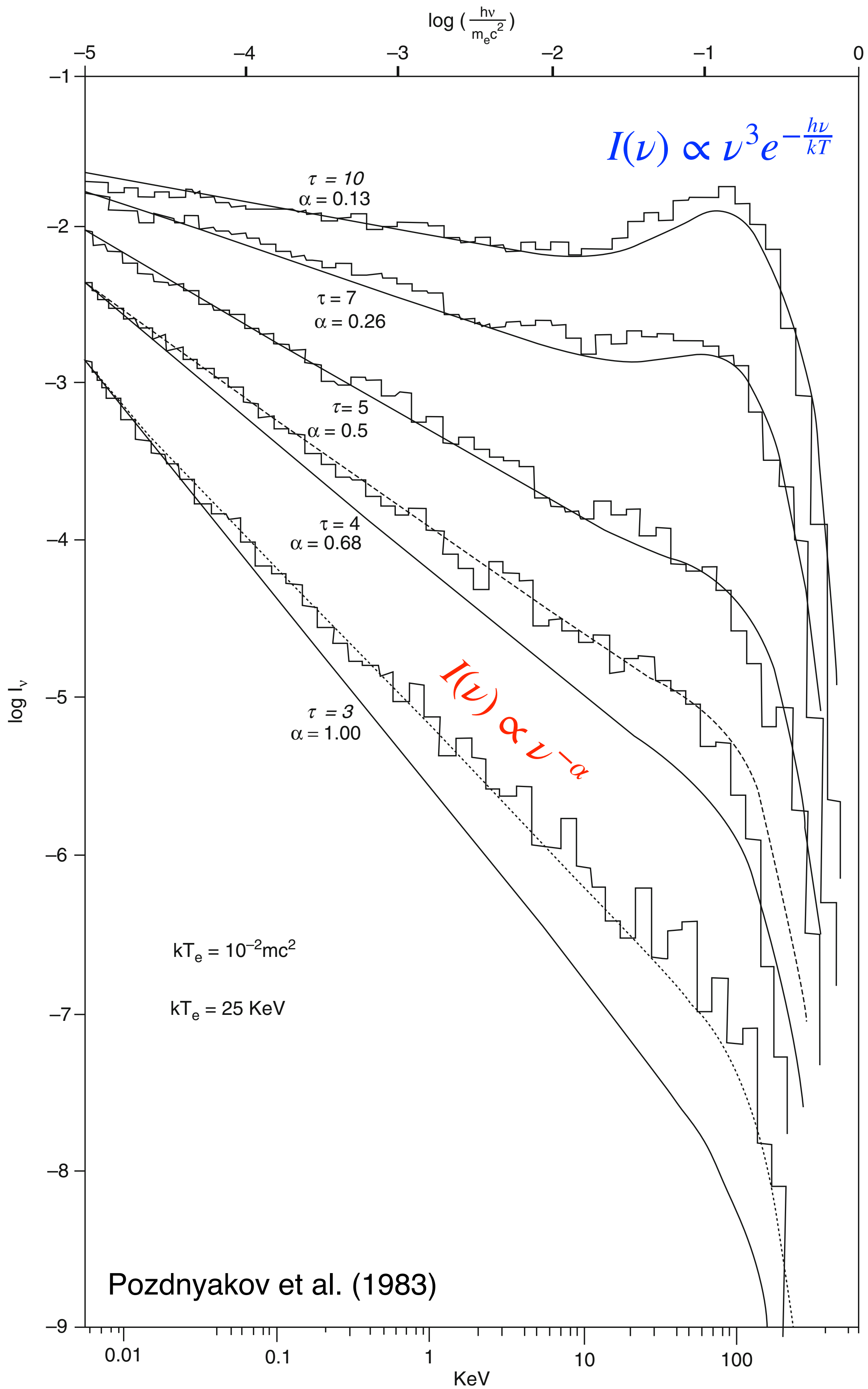
- A power-law spectrum is a natural consequence of a power-law distribution of electrons.
- **We will show that a power-law photon distribution can also be produced from repeated scattering off a nonpower-law electron distribution.**
  - Let A = the mean amplification of photon energy per scattering, i.e.,

$$A \equiv \frac{\epsilon}{\epsilon_0} \sim \frac{4}{3}\gamma^2$$

  - After k scattering, the photon energy will be $\epsilon_k \approx \epsilon_0 A^k$.
  - For an optically thin scattering medium ($\tau_{\rm es} < 1$), the probability of a photon undergoing *k* scattering before escaping the medium is $P_k(\tau_{\rm es}) \sim \tau_{\rm es}^k$.
  - The emergent intensity at energy ϵ_k is given by

$$I(\epsilon_k) \sim I(\epsilon_0) P_k(\tau_{\rm es}) \sim I(\epsilon_0) \tau_{\rm es}^{\ln(\epsilon_k/\epsilon_0)/\ln A} = I(\epsilon_0) \left(\frac{\epsilon_k}{\epsilon_0}\right)^{\ln \tau_{\rm es}/\ln A}$$

$$I(\epsilon_k) \sim I(\epsilon_0) \left(\frac{\epsilon_k}{\epsilon_0}\right)^{-\alpha}, \quad \text{where} \quad \alpha \equiv \frac{-\ln \tau_{\rm es}}{\ln A}$$ $\longrightarrow$ power-law shape



The modification of the photon spectrum by Compton scattering is called **Comptonization**.

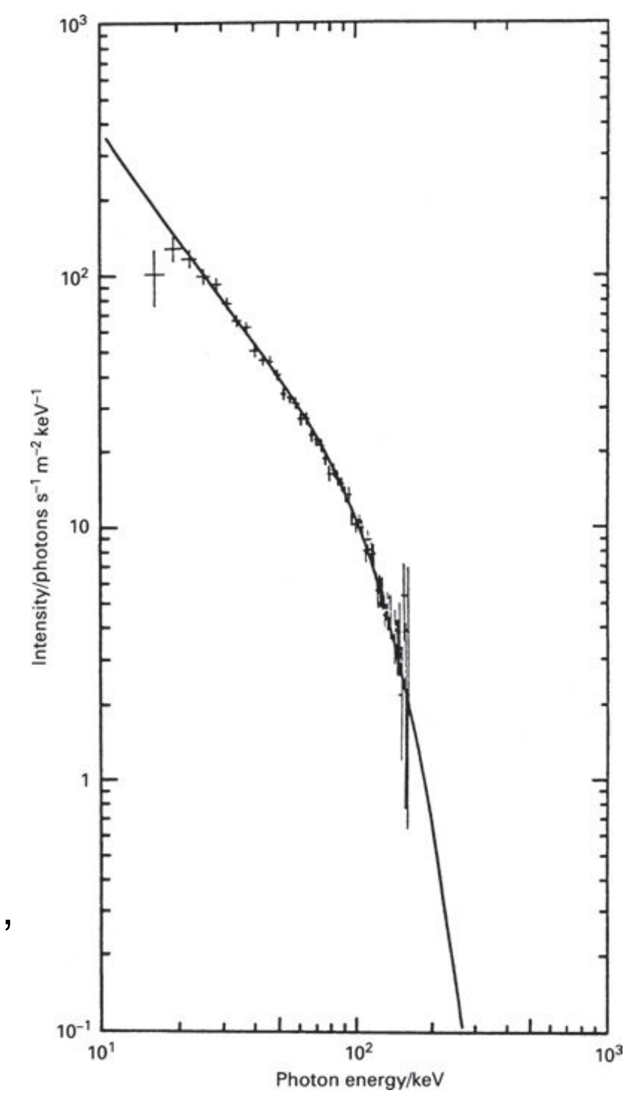
The Comptonization of low-frequency photons in a spherical plasma with $kT_e = 25$ keV.

- The solid curves are analytic solutions of the so-called "Kompaneets equation," which is an approximate equation to describe the Comptonization.
- The histograms are the results of Monte-Carlo simulations.

The spectrum is in general power-law. As optical depth increases, the Wien peak develops at energy $h\nu \simeq kT_e$:

Recall that

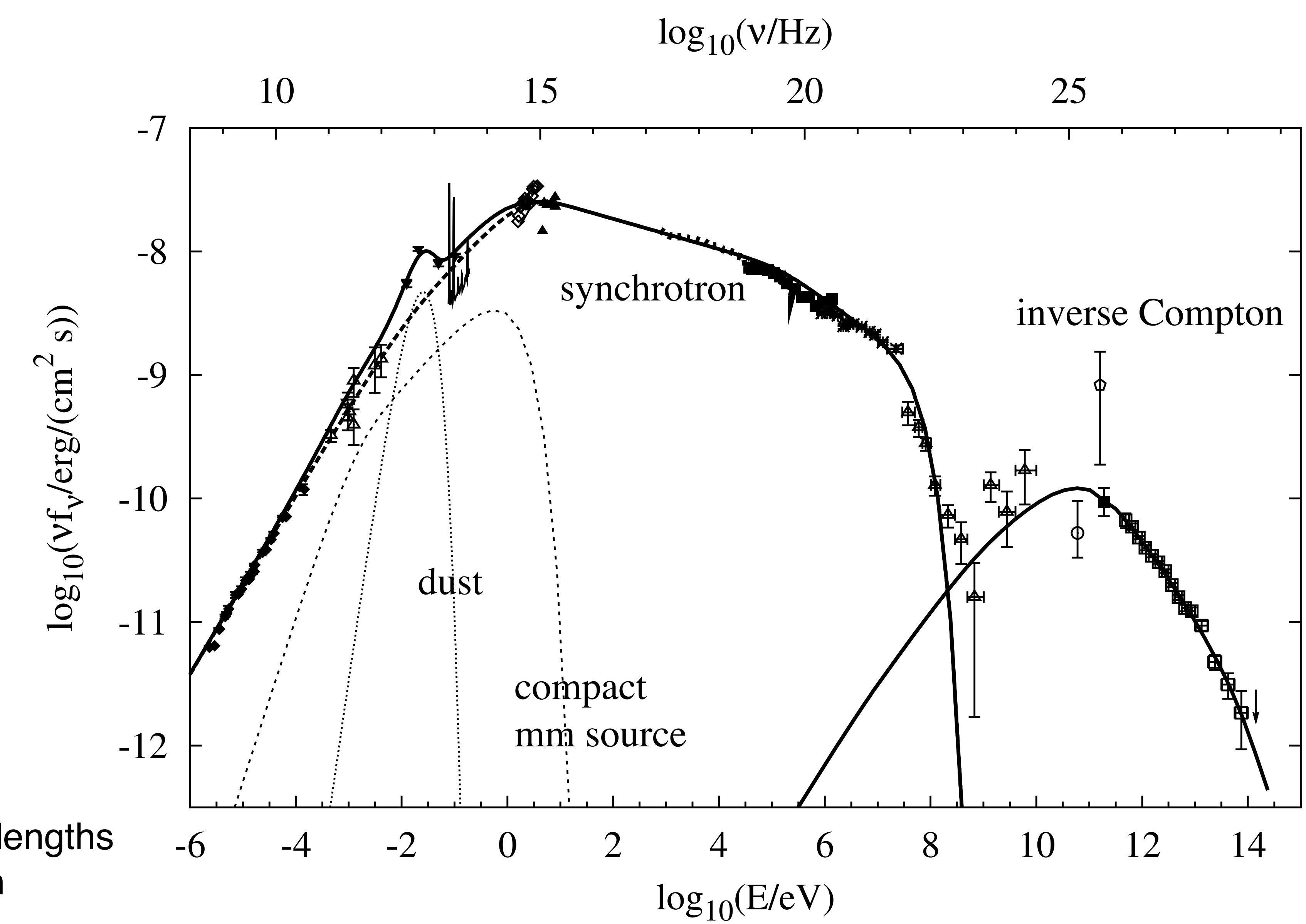
$$\text{as } \tau \gg 1,\ I(\nu) \to B_\nu(T)$$
$$\propto \nu^3 e^{-\frac{h\nu}{kT}} \text{ for } h\nu \gg kT$$



The X-ray spectrum of Cyg X-1, a stellar-mass blackhole candidate.

Synchrotron self-Compton (SSC) emission

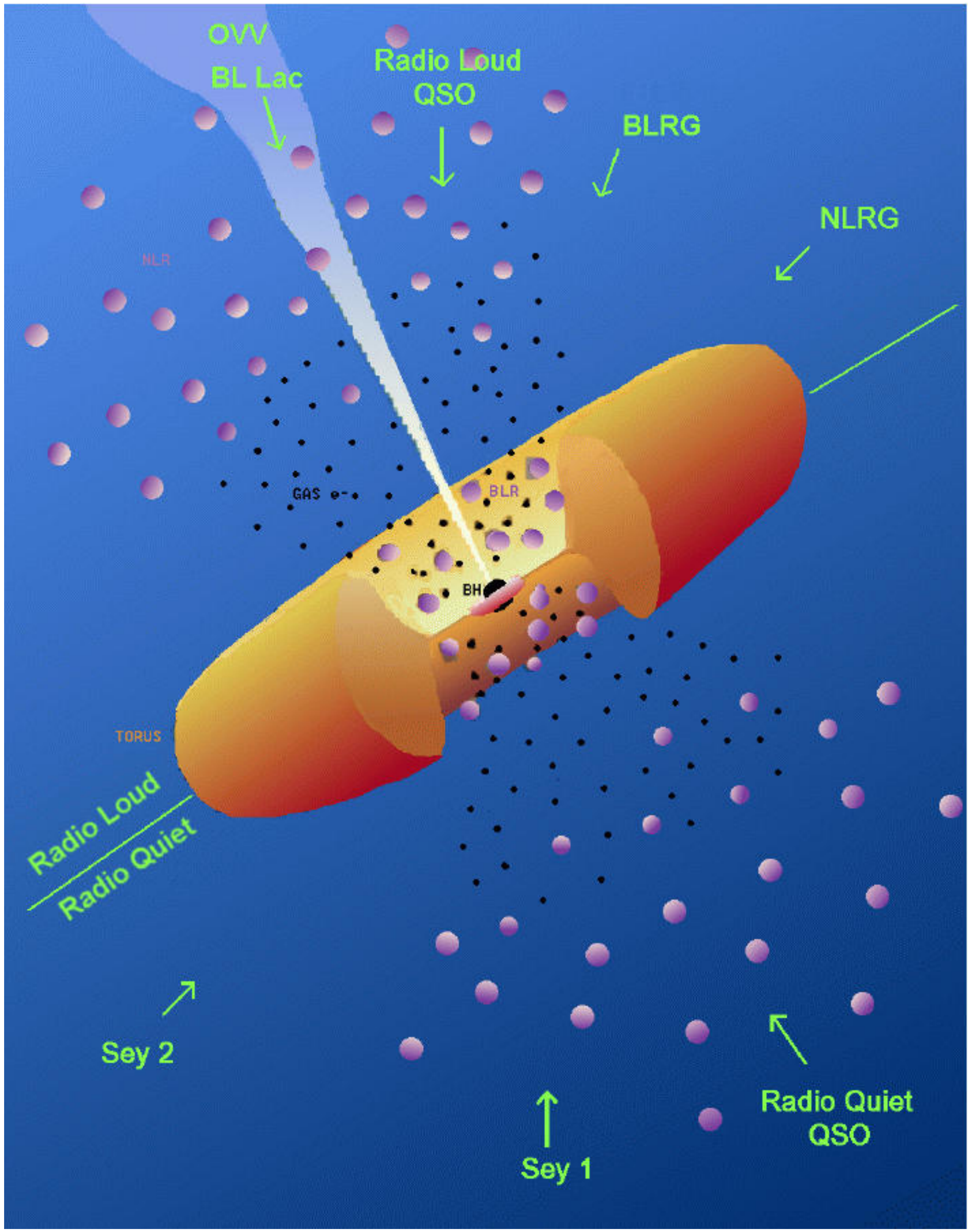
- Relativistic electrons in the presence of a magnetic field will surely emit synchrotron radiation at some level. The photons will undergo inverse Compton scattering by the very same electrons that emitted them in the first place. Such scattering must take place before the synchrotron photon leaves the source region. This is the **synchrotron self-Compton (SSC) process**.
- **Crab nebula**

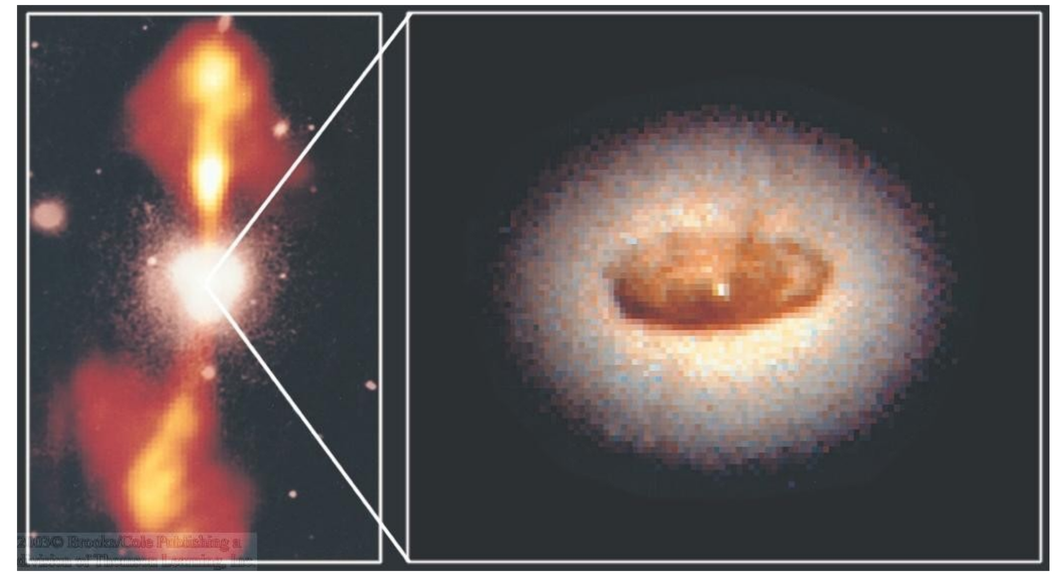


Dots: modified blackbody with $T = 46$ K.
Thin dashed line: emission at mm wavelengths
Thick dashed line: synchrotron emission

Active Galactic Nuclei

- A Unified Model for AGN





HST image of the Dust Torus in NGC 4261

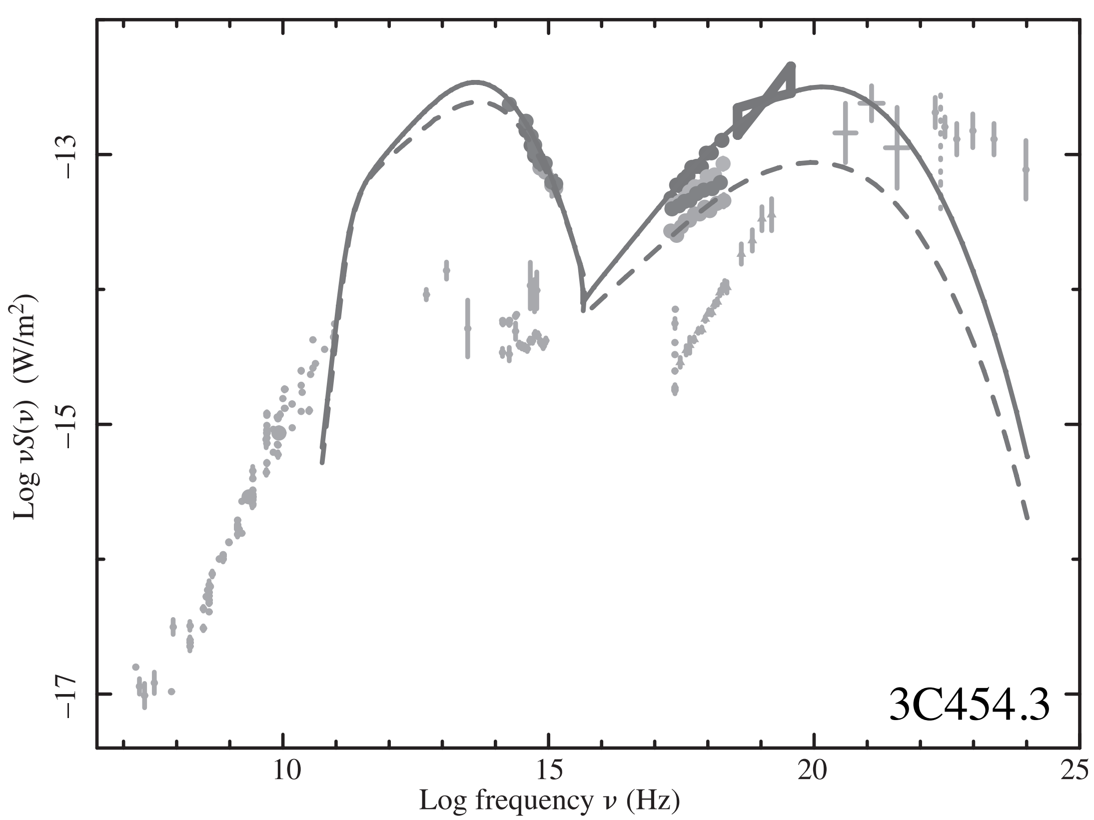
Active Galaxies = galaxies with extremely violent energy release in their nuclei (pl. of nucleus). Active Galactic Nucleus means the compact region of an Active Galaxy.

BLR, Broad Line Region : produces very broad lines. (velocity of 1,000-10,000 km/s)

NLR, Narrow Line Region : produces relatively narrow lines (velocity of 100 km/s).

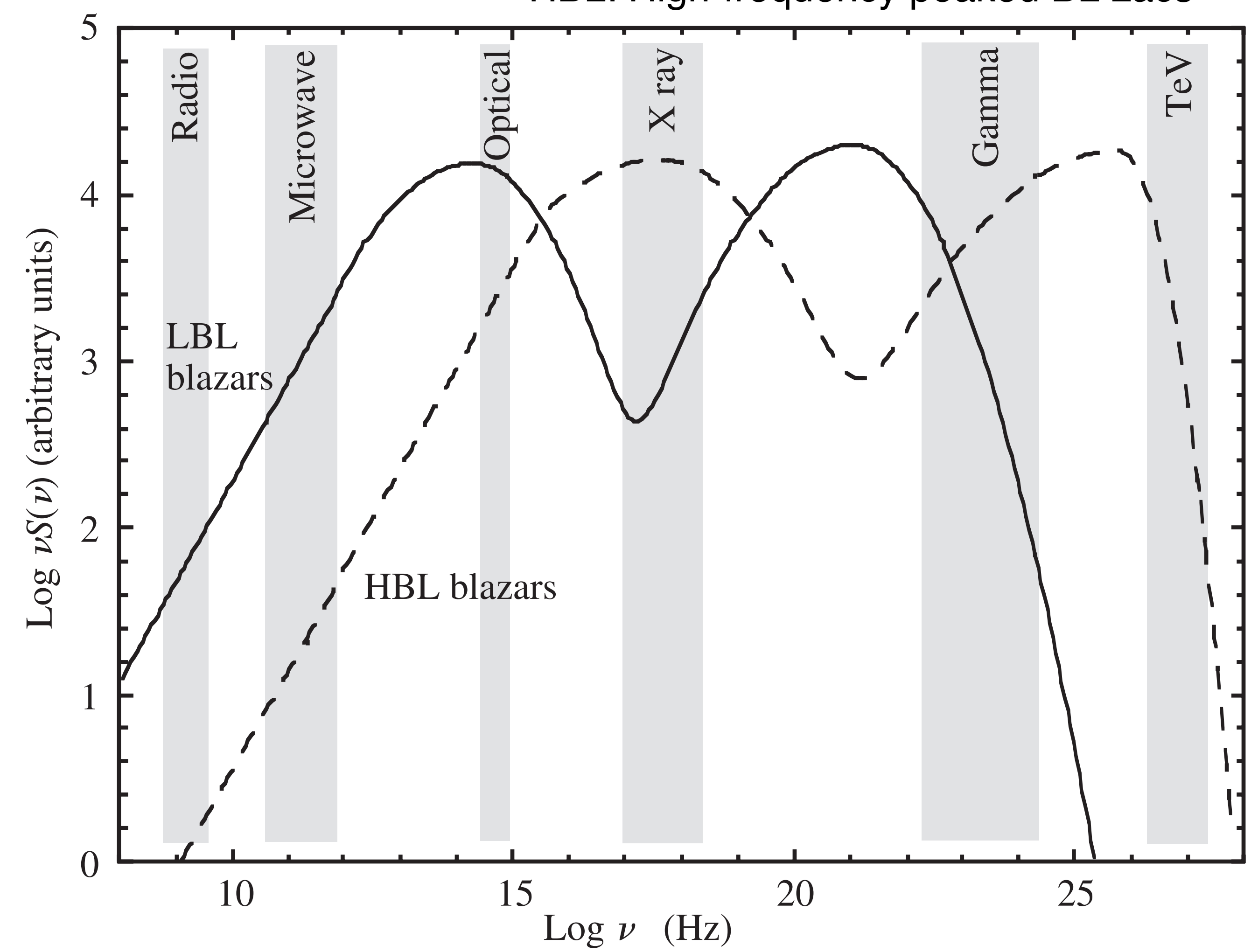
Prof. Sang-Sung Lee will explain more about AGN.

- **Blazars**: If the observer view is more or less normal to the accretion disk, the action close to the core becomes visible. The observer considered to lie within the jet beam. Such objects are known as blazars or as BL Lacertae objects.
- Blazars have SEDs that are typically two-peaked. The peak at lower frequency is attributed to synchrotron radiation and the one at higher frequency to IC scattering.
  - The lower-energy case (LBL blazar) extends from the radio to the gamma-ray bands but is quiet in the TeV band. The higher-energy case (HBL blazar) reaches TeV energies but is quiet in the radio range.



One popular hypothesis is that LBLs are viewed at smaller angles than HBLs, so that the difference is purely an orientation effect. However, someone believes that there should an intrinsic difference between them.

LBL: Low-frequency peaked BL Lacs
HBL: High-frequency peaked BL Lacs



Sunyaev-Zeldovich effect

- The **Sunyaev-Zeldovich effect** is the distortion of the blackbody spectrum (T = 2.73 K) of the CMB owing to the IC scattering of the CMB photons by the energetic electrons in the galaxy clusters.

  - Thermal SZ effects, where the CMB photons interact with electrons that have high energies due to their temperature.

  - Kinematic SZ effects (Ostriker-Vishniac effect), a second-order effect where the CMB photons interact with electrons that have high energies due to their bulk motion (peculiar motion). The motions of galaxies and clusters of galaxies relative to the Hubble flow are called peculiar velocities. The plasma electrons in the cluster also have this velocity. The energies of the CMB photons that scattered by the electrons reflect this motion.

  - Determinations of the peculiar velocities of clusters enable astronomers to map out the growth of large-scale structure in the universe. This topic is fundamental importance, and the kinetic SZ effect is a promising method for approaching it.

- Thermal SZ effect

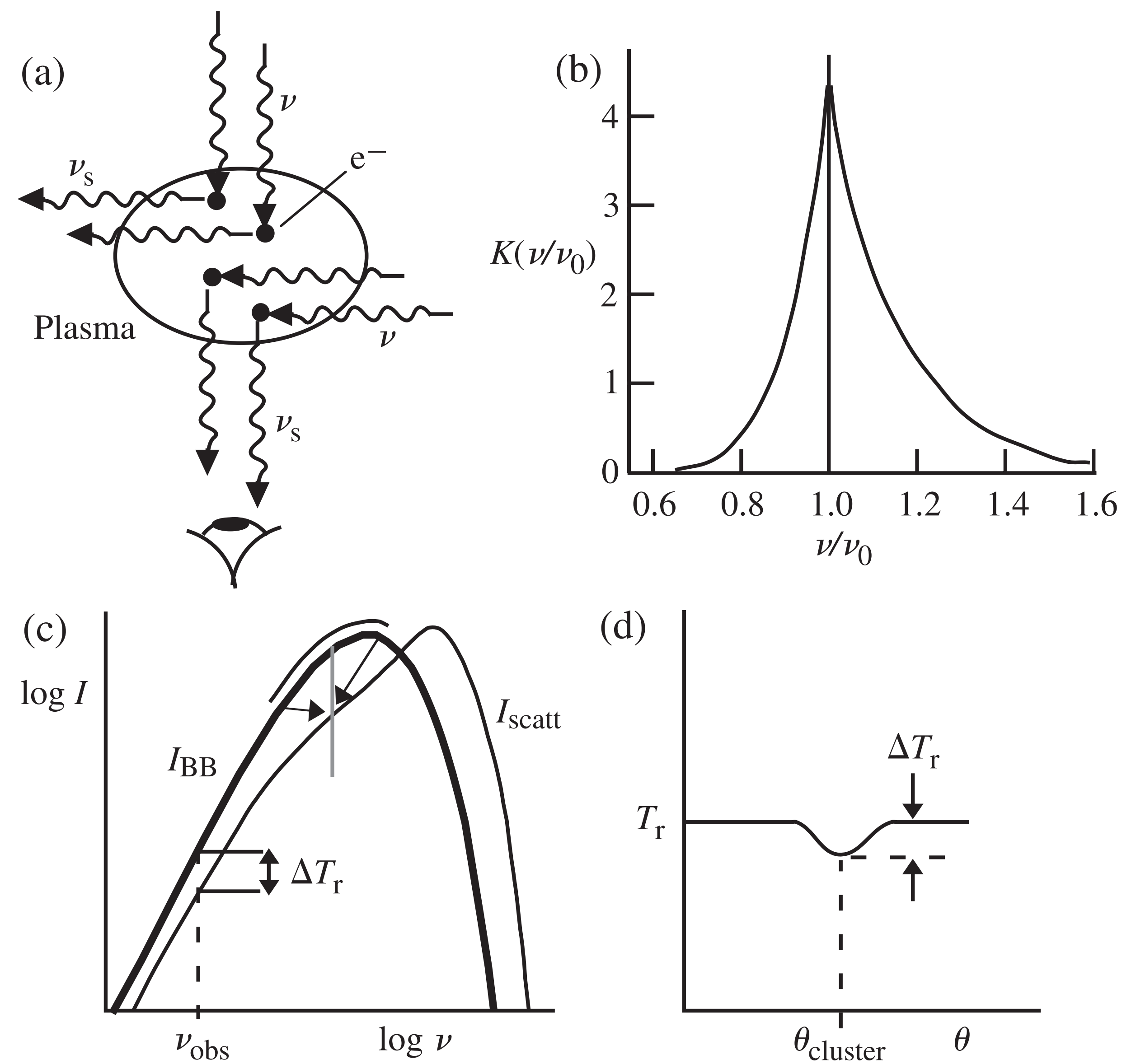
- The net effect of the IC scattering on the photon spectrum is obtained by multiplying the photon number spectrum by the kernel $K(\nu/\nu_0)$ and integrating over the spectrum.

$$N_{\mathrm{scatt}}(\nu) = \int_0^{\infty} N(\nu_0) K(\nu/\nu_0) d\nu_0$$

- The net effect is that the BB spectrum is shifted to the right and distorted.
- Observations of the CMB are most easily carried out in the low-frequency Rayleigh-Jeans region of the spectrum $(h\nu \ll kT_{\mathrm{CMB}})$.
- Measurement of the CMB temperature as a function of position on the sky would thus exhibit antenna temperature dips in the directions of clusters that contain hot plasmas.
- Note that the scattered spectrum is not a BB spectrum. The effect temperature increases. But, the total number of photons detected in a given time over the entire spectrum remains constant.

The result of such scatterings for an initial blackbody photon spectrum is shown in the following figure for the value:

$$\frac{kT_e}{mc^2}\tau = 0.5$$



- **Change of the BB temperature**
  - In the Rayleigh-Jeans region, $$I(\nu) = \frac{2\nu^2}{c^2} k_{\rm B} T_{\rm CMB}$$
  - If the spectrum is shifted parallel to itself on a log-log plot, the fractional frequency change of a scattered photon is constant.

$$\varepsilon = \frac{\Delta\nu}{\nu} = \frac{\nu' - \nu}{\nu} = \text{constant} \quad \text{or} \quad \nu' = \nu(1+\varepsilon) \longrightarrow d\nu' = d\nu(1+\varepsilon)$$

  - Total photon number is conserved: $$N'(\nu')d\nu' = N(\nu)d\nu \quad \rightarrow \quad \frac{I'(\nu')}{h\nu'} d\nu' = \frac{I(\nu)}{h\nu} d\nu$$

$$\therefore \quad I'(\nu') = I(\nu) \qquad I(\nu) = I\left(\frac{\nu'}{1+\varepsilon}\right) = \frac{2\nu'^2}{c^2(1+\varepsilon)^2} k_{\rm B} T_{\rm CMB}$$

$$\frac{\Delta I}{I} = \frac{I' - I}{I} = \frac{1}{(1+\varepsilon)^2} - 1 \approx -2\varepsilon = -2\frac{\Delta\nu}{\nu}$$

$$\frac{\Delta T_{\rm CMB}}{T_{\rm CMB}} = \frac{\Delta I}{I} \approx -2\epsilon = -2\frac{\Delta\nu}{\nu}$$

  - The properly calculated result is $\varepsilon = \frac{\Delta\nu}{\nu} = \frac{k_{\rm B} T_{\rm CMB}}{mc^2}\tau$. $\longrightarrow$ $$\boxed{\frac{\Delta T_{\rm CMB}}{T_{\rm CMB}} \approx -2\frac{k_{\rm B} T_{\rm CMB}}{mc^2}\tau}$$

A typical cluster have
an average electron density of $\sim 2.5 \times 10^{-3}\ \mathrm{cm}^{-3}$,
a core radius of $R_c \sim 10^{24}\ \mathrm{cm}\ (\sim 320\ \mathrm{pc})$,
and an electron temperature of $k_{\mathrm{B}} T_e \approx 5\ \mathrm{keV}$.

A typical optical depth is thus

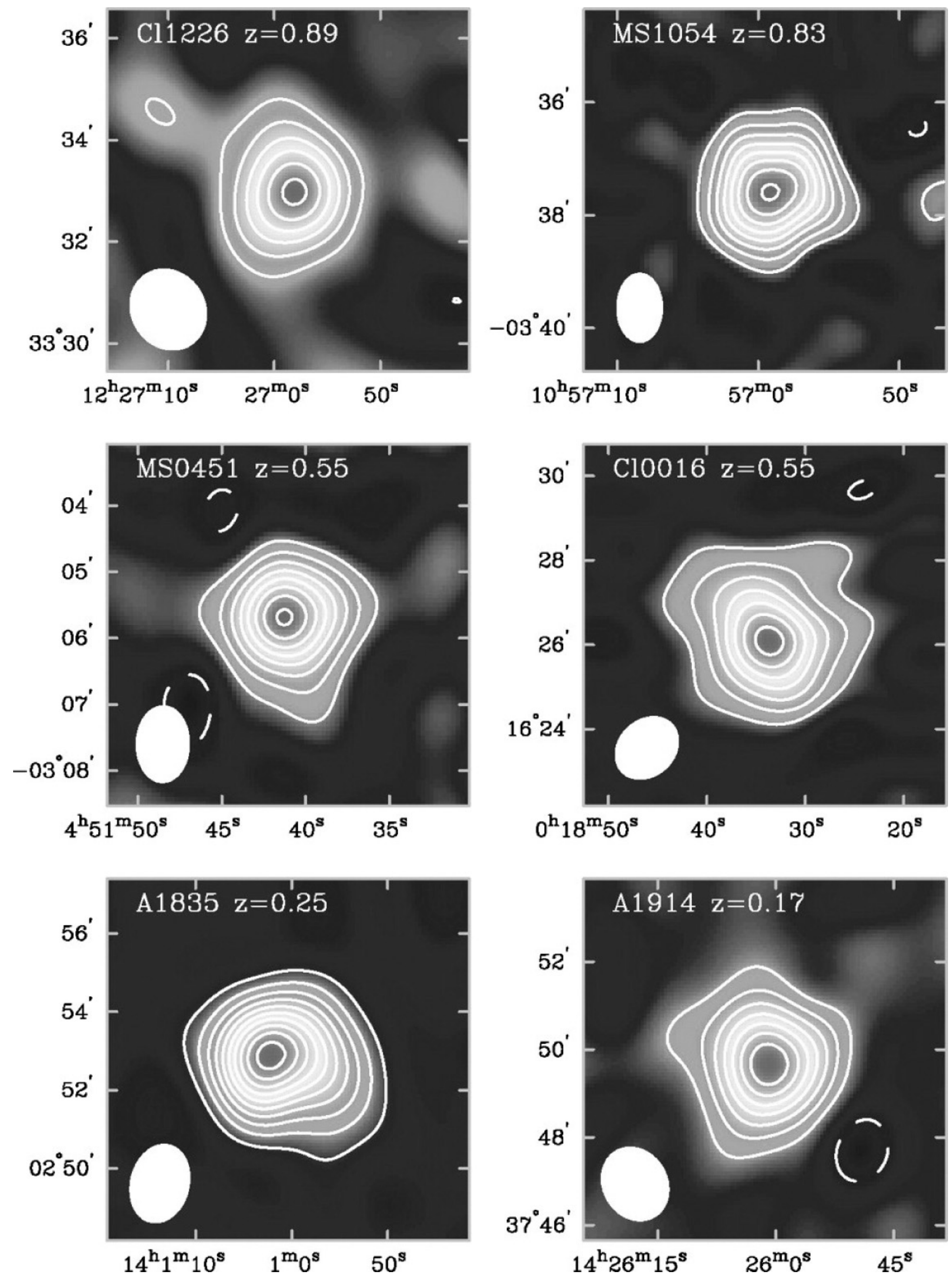
$$\tau \approx 3 \sigma_T n_e R_e \approx 0.005$$

The expected antenna temperature change is

$$\frac{\Delta T_{\mathrm{CMB}}}{T_{\mathrm{CMB}}} \approx -1 \times 10^{-4}$$

$$\Delta T_{\mathrm{CMB}} \approx -0.3\ \mathrm{mK\ for}\ T_{\mathrm{CMB}} = 2.7\ \mathrm{K}$$

This effect has been measured in dozens of clusters.



Interferometric images at 30 GHz of six clusters of galaxies. The solid white contours indicate negative decrements to the CMB. (Carlstrom et al. 2002, ARAA, 40, 643)

- Hubble Constant

- A value of the Hubble constant is obtained for a given galaxy only if one has independent measures of a recession speed v and a distance d of a galaxy.

$$H_0 = \frac{v}{d}$$

- Recession speed is readily obtained from the spectral redshift
- Distance:
  - X-ray observations: $$I(\nu, T_e) = C\frac{g(\nu, T_e)}{T_e^{1/2}} \exp(-h\nu/kT_e) n_e^2 (2R)$$
  - S-Z CMB decrement: $$\frac{\Delta T_{\rm CMB}}{T_{\rm CMB}} = -2\frac{kT_e}{mc^2}\tau = -2\frac{kT_e}{mc^2}\left(\sigma_T n_e 2R\right)$$
  - The radio and X-ray measurements yield absolute values of the electron density n_e and cluster radius R without a priori knowledge of the cluster distance.
  - Imaging of the cluster in the radio or X-ray band yields the angular size of the cluster θ. Then the distance *d* to the cluster is obtained by

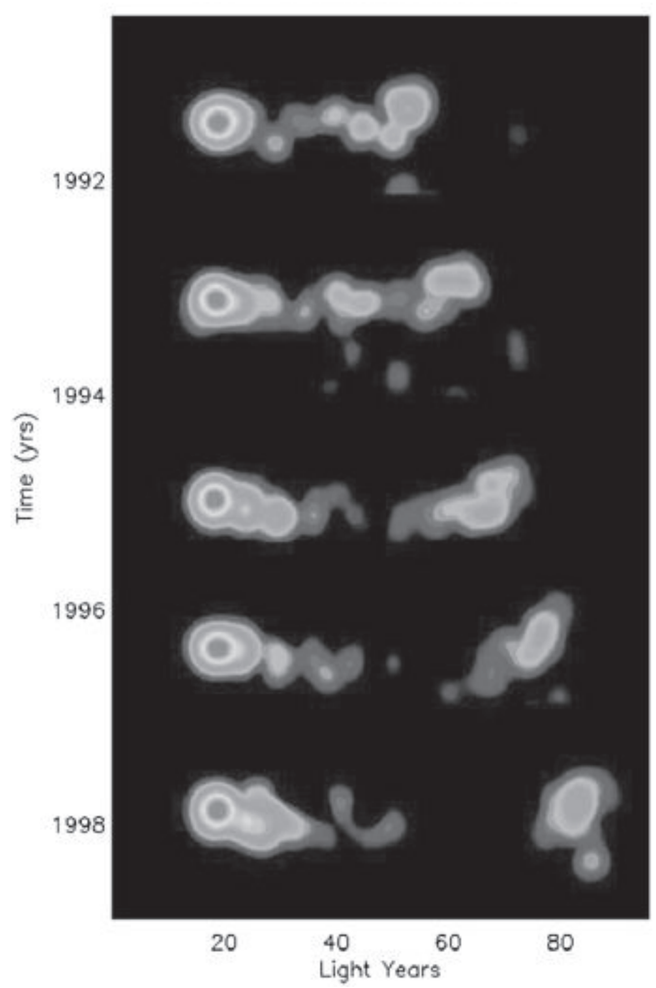
$$d = \frac{R}{\theta}$$

- The SZ effect (at radio frequencies) in conjunction with X-ray measurements can give distances to clusters of galaxies. This can be used to derive the Hubble constant.

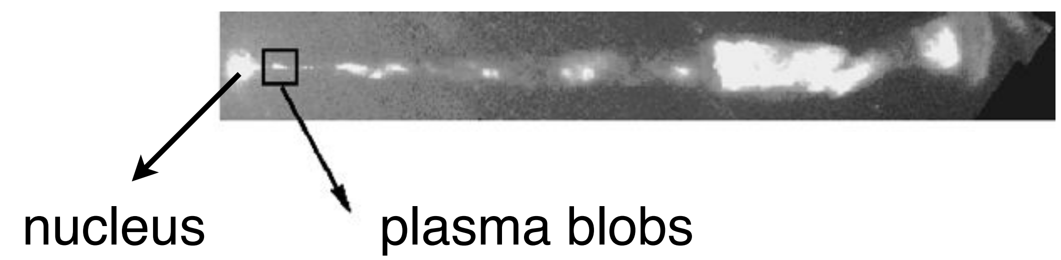
Homework: Superluminal motion of relativistic jets

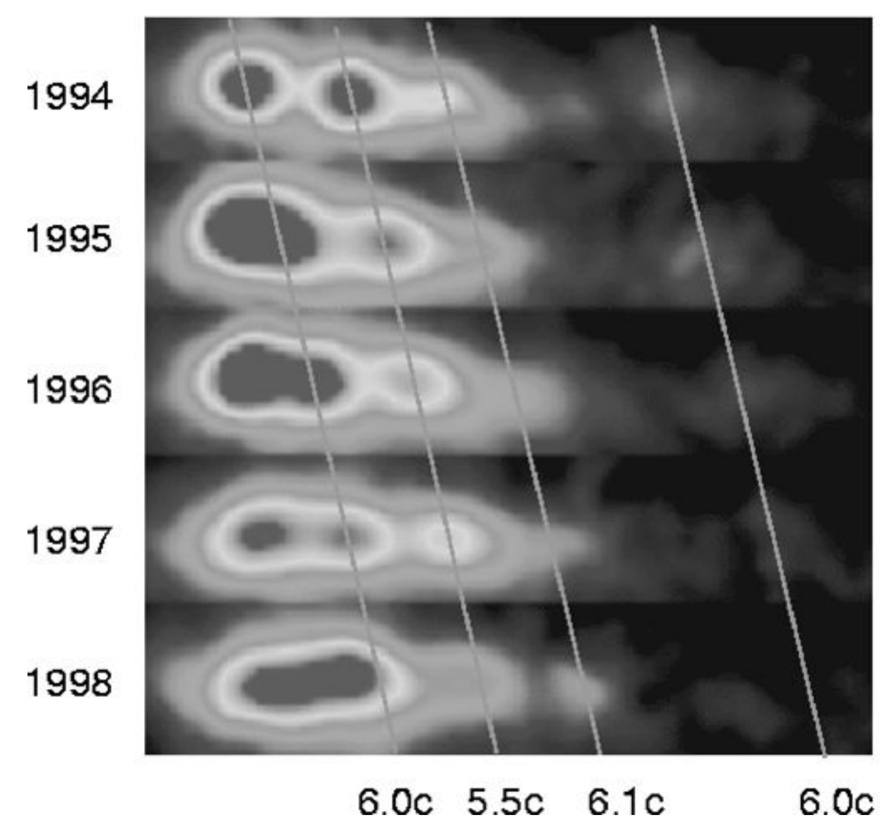
Deadline: Sep. 23 (Mon)

- Background
  - In astronomy, superluminal motion is the apparently faster-than-light motion seen in some radio galaxies, quasars, and recently also in some galactic sources called micro quasars. All of these source are thought to contain a black hole, responsible for the ejection of mass at high velocities.
  - In 1966, Martin Rees pointed out that “an object moving relativistically in suitable directions may appear to a distant observer to have a transverse velocity much greater than the speed of light.

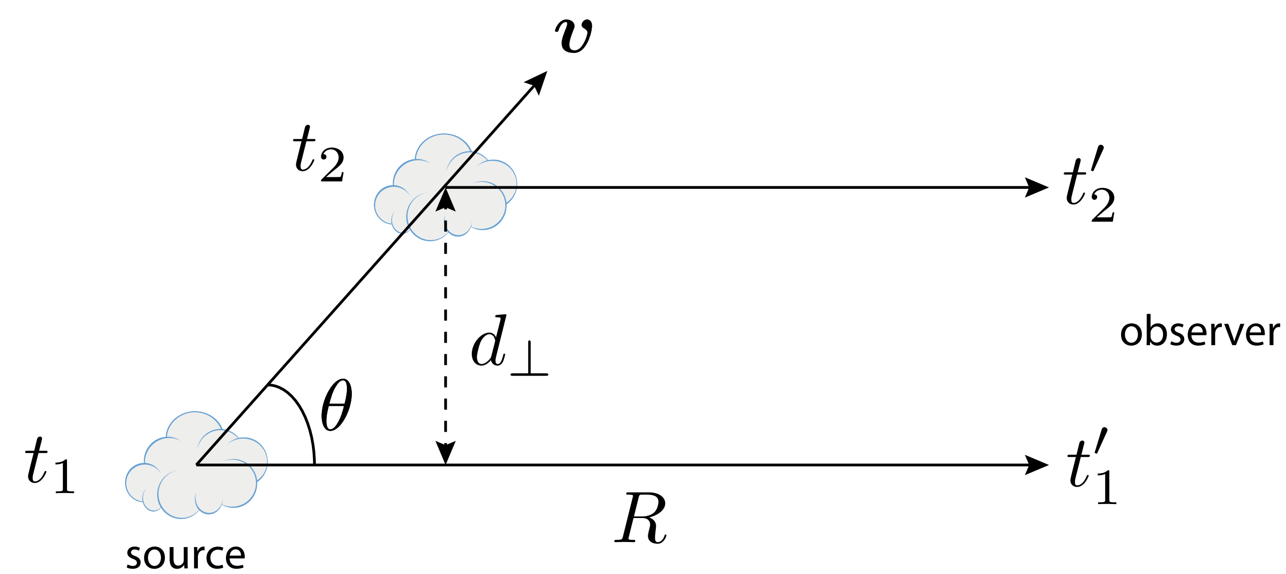


The superluminal expansion of 3C 279 is shown over a six year period.





The inner regions of the jet of M87 as observed by the Hubble Space Telescope through the period 1992-1998 (Biretta et al. 1999). The optical knots are observed to move out from the nucleus at speeds up to 6c.



- Let's imagine a plasma cloud is ejected from a source, moving with a velocity v along a direction that makes an angle θ $(0 \leq \theta \leq \pi/2)$ with line line of sight.

    - Suppose that the cloud starts at an instant t_1; the photons emitted at this time reach Earth at $t_1' = t_1 + R/c$, where R is the distance to the source from Earth.
    - At t_2, the cloud is at a distance $v(t_2 - t_2)$ from the source. Its distance from the source projected on the sky is $d_\perp = v(t_2 - t_1)\sin\theta$, whereas its distance from Earth is $R - v(t_2 - t_1)\cos\theta$.
    - Therefore, the photons that left at time t_2 reach Earth at the time $t_2' = t_2 + \left[R - v(t_2 - t_1)\cos\theta\right]/c$.
    - Thus, to an observer on Earth, the cloud appears to have moved the distance $d_\perp$ in the time interval $t_2' - t_1'$.

- [Questions]

  (1) Show that the "apparent" velocity $v_{\rm app} \equiv d_\perp/(t_2' - t_1')$ of the cloud on the sky is

$$\mathrm{Eq.(1):} \qquad \beta_{\rm app} = \frac{\beta\sin\theta}{1 - \beta\cos\theta}, \quad \text{where } \beta_{\rm app} = v_{\rm app}/c, \text{ and } \beta = v/c$$

(2) For a given velocity β, show that the condition for the apparent velocity to exceed the speed of light ($\beta_{\rm app} > 1$) is

$$\mathrm{Eq.(2)}: \quad \sin\theta + \cos\theta \geq \frac{1}{\beta}$$

(3) Plot or sketch the above condition and demonstrate that Eq.(2) indicates that if the inclination θ has favorable values, we can have superluminal motion ($v_{\rm app} \gg c$).

(4) Take a square on both sides of Eq. (2) and show that the condition for the superluminal motion is

$$\mathrm{Eq.(3)}: \quad \sin 2\theta \geq \frac{1}{\beta^2} - 1$$

(5) Note that $\sin 2\theta$ is symmetric about $\theta = \pi/4$. Setting $\theta = \pi/4 + x$, show that Eq.(3) is equivalent to the following condition.

$$\mathrm{Eq.(4)}: \quad \left|\theta - \frac{\pi}{4}\right| \leq \frac{1}{2}\cos^{-1}\left(\frac{1}{\beta^2} - 1\right)$$

(6) From Eq.(3) or (4), show that there is a limit on β below which the source will never appear superluminal. The limit is $\beta_{\rm min} = 1/\sqrt{2}$. This result indicates that the cloud should move relativistically at least at a velocity of ~ 71% of the speed of light to show the superluminal motion.

(7) Differentiate the right hand side of Eq.(1) with respect to θ and calculate the angle $\theta_{\rm max}$ at which the apparent velocity is maximal. What is the maximal, apparent velocity $\beta_{\rm app}^{\rm max}$?