Astrophysics

Lecture 11

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[Inverse Compton Power for Single Scattering]

- **Assumptions:**

  (1) Isotropic distributions of photons and electrons.

  (2) The change in energy of the photon in the rest frame is negligible.

  (Thomson scattering is applicable in the electron's rest frame). $\epsilon_1' \approx \epsilon'$ or $h\nu_1' = h\nu'$

- **Total Inverse Compton Power**:

  We first note that the power is invariant because the energy and time transform in the same way:

$$\frac{dE_1}{dt} = \frac{dE_1'}{dt'} \quad \rightarrow \quad P = P'$$

Therefore, the **power contained in the scattered radiation** in the laboratory frame can be expressed in terms of the quantities in the electron's rest frame.

$$P' = c\sigma_{\mathrm{T}} \int \epsilon_1' f_{\mathrm{ph}}'(\mathbf{p}')d^3\mathbf{p}'$$

Here, $f_{\mathrm{ph}}'(\mathbf{p}')$ is the number density of photons with momentum between $\mathbf{p}'$ and $\mathbf{p} + d\mathbf{p}'$ and $cf_{\mathrm{ph}}'(\mathbf{p}')d^3\mathbf{p}'$ is the incident photon flux in the electrons' rest frame.

$$f_{\mathrm{ph}} = \frac{dN_{\mathrm{ph}}}{d^3\mathbf{x}d^3\mathbf{p}}$$

The number density is invariant because the number of particles and the phase space volume element are Lorentz invariants.

$$f_{\mathrm{ph}}'(\mathbf{p}') = f_{\mathrm{ph}}(\mathbf{p})$$

Recall that **the momentum volume element transforms in the same way as the energy under Lorentz transforms**. Therefore,

$$d^3\mathbf{p}' = \gamma\left(1 - \beta\cos\theta\right) d^3\mathbf{p}$$

The scattered photon energy in the electron rest frame can be expressed in terms of the incident photon energy in the lab frame:

$$\begin{aligned}\epsilon_1' &= \epsilon' \quad \text{(Thomson scattering)} \\ &= \gamma\left(1 - \beta\cos\theta\right)\epsilon \quad \text{(Lorentz transform)}\end{aligned}$$

Then the power contained in the scattered radiation is

$$\begin{aligned}P = P' &= c\sigma_{\mathrm{T}}\gamma^2 \int \epsilon\left(1 - \beta\cos\theta\right)^2 f_{\mathrm{ph}}(\mathbf{p}) d^3\mathbf{p} \\ &= c\sigma_{\mathrm{T}}\gamma^2 \int\int \epsilon\left(1 - \beta\cos\theta\right)^2 f_{\mathrm{ph}}(p) p^2 dp d\Omega \\ &= c\sigma_{\mathrm{T}}\gamma^2 \left[\int \epsilon f_{\mathrm{ph}}(p) p^2 dp\,(4\pi)\right]\left[\int \left(1 - \beta\cos\theta\right)^2 \frac{d\Omega}{4\pi}\right]\end{aligned}$$

For an isotropic distribution of photons, integrating over θ, we have the last term:

$$\left\langle\left(1 - \beta\cos\theta\right)^2\right\rangle = 1 + \frac{1}{3}\beta^2 \qquad \leftarrow \quad \langle\cos\theta\rangle = 0, \quad \left\langle\cos^2\theta\right\rangle = 1/3$$

Finally, we obtain the power contained in the scattered radiation given by

$$\boxed{P = c\sigma_{\mathrm{T}}\gamma^2\left(1+\frac{1}{3}\beta^2\right)U_{\mathrm{ph}}}$$

where

$$\boxed{U_{\mathrm{ph}} = \int \epsilon f_{\mathrm{ph}}(p)p^2 dp\,(4\pi) = \int \epsilon f_{\mathrm{ph}}(\mathbf{p})\,d^3\mathbf{p}}$$

is the initial photon energy density.

- **Mean photon energy after the inverse Compton scattering**:

Now, we consider the ***monochromatic*** radiation field with energy $\epsilon = h\nu =$ constant. Then, the initial photon energy density is

$$U_{\mathrm{ph}} = \epsilon\int f_{\mathrm{ph}}(\mathbf{p})d^3\mathbf{p} = \epsilon n_{\mathrm{ph}} \quad \longleftarrow \quad n_{\mathrm{ph}} \equiv \int f_{\mathrm{ph}} d^3\mathbf{p} = \int \frac{dN_{\mathrm{ph}}}{d^3\mathbf{x}d^3\mathbf{p}}d^3\mathbf{p}$$

The scattering power can be expressed by the multiplication of the number scattering events per unit time and the mean photon energy after scattering.

$$P = \langle\epsilon_1\rangle \times \frac{dN_{\mathrm{scatt}}}{dt}$$

The **scattering rate** in the lab frame is related to that in the electron's rest frame.

$$\frac{dN_{\rm scatt}}{dt} = \frac{1}{\gamma}\frac{dN'_{\rm scatt}}{dt'}$$ because the number of event is invariant: $dN_{\rm scatt} = dN'_{\rm scatt}$

The scattering rate in the electron's rest frame is calculated, as done for the scattering power:

$$\begin{aligned}\frac{dN'_{\rm scatt}}{dt'} &= c\sigma_{\rm T}\int f'_{\rm ph}(\mathbf{p}')d^3\mathbf{p}' = c\sigma_{\rm T}\gamma\int (1-\beta\cancel{\cos\theta})\, f_{\rm ph}(\mathbf{p})d^3\mathbf{p} \\ &= c\sigma_{\rm T}\gamma n_{\rm ph}\end{aligned}$$

Then, we obtain the scattering rate:

$$\boxed{\frac{dN_{\rm scatt}}{dt} = c\sigma_{\rm T} n_{\rm ph}}$$

The mean energy after the scattering is thus obtained to be

$$\boxed{\begin{aligned}\langle\epsilon_1\rangle &= P \bigg/ \frac{dN_{\rm scatt}}{dt} \\ &= \gamma^2\left(1+\frac{1}{3}\beta^2\right)\epsilon\end{aligned}}$$

- **The net power that is lost by the electron and then converted into radiation** is

$$P_{\rm IC} \equiv \frac{dE_1}{dt} - \frac{dE^{\rm initial}}{dt} = c\sigma_T U_{\rm ph}\left[\gamma^2\left(1+\frac{1}{3}\beta^2\right)-1\right]$$

$$\boxed{\therefore \quad P_{\rm IC} = \frac{4}{3}c\sigma_T\gamma^2\beta^2 U_{\rm ph}} \quad \longleftarrow \quad \gamma^2 - 1 = \gamma^2\beta^2$$

- **Net energy gain (transfer) by nonrelativistic thermal electrons with temperature *T***:

The fractional photon energy gain for a single electron velocity is

$$\frac{\Delta\epsilon}{\epsilon} = \frac{\langle h\nu_1\rangle - h\nu}{h\nu} = \frac{4}{3}\gamma^2\beta^2$$

The **total energy gain** can be obtained by integrating over the Maxwell velocity distribution and assuming that $\gamma = 1$.

$$\boxed{\left\langle\frac{\Delta\epsilon}{\epsilon}\right\rangle \simeq \frac{4}{3}\left\langle\beta^2\right\rangle = \frac{4kT}{m_e c^2}}$$

$$\left\langle\beta^2\right\rangle = \left\langle v^2/c^2\right\rangle = 3kT/mc^2$$

Correction to the Thomson-limit:

We note that the above result was obtained, assuming that the photon energy is not lost in the electron's rest frame. Now, we will make a correction to the energy gain by considering the photon energy loss due to the electron recoil. In the electron's rest frame, the photon energy will be lost:

$$\epsilon_1' \approx \epsilon' \left[1 - \frac{\epsilon'}{mc^2} (1 - \cos\Theta) \right] \quad \rightarrow \quad \left\langle \frac{\Delta\epsilon'}{\epsilon'} \right\rangle \equiv \left\langle \frac{\epsilon_1' - \epsilon'}{\epsilon'} \right\rangle = -\frac{\epsilon'}{mc^2} \quad \text{: after making an angle average}$$

Hence, **net energy transfer** (including the correction term) is

$$\boxed{\left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle = \frac{4kT - \epsilon}{m_e c^2}}$$

Note that if the electrons have high enough temperature relative to incident photons, the photons gain energy. This is the inverse Compton scattering.

If $\epsilon > 4kT$, on the other hand, energy is transferred from photons to electrons.

[Repeated Scattering: The Compton y Parameter]

- We restrict our considerations to situations in which the Thomson limit applies: $\gamma\epsilon \ll mc^2$
- **Compton *y* parameter**, to determine whether a photon will significantly change its energy in traversing the medium:

$$y \equiv \left(\begin{array}{l} \text{average fractional} \\ \text{energy change per} \\ \text{scattering} \end{array} \right) \times \left(\begin{array}{l} \text{mean number of} \\ \text{scatterings} \end{array} \right)$$

When $y \gtrsim 1$, the total photon energy and spectrum will be significantly altered; whereas for $y \ll 1$, the total energy is not much changed.

- **Average fractional energy gain per scattering** (for a thermal distribution of electrons)

(a) Consider first the nonrelativistic limit.

$$\left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle = \frac{4kT}{m_e c^2}$$

(b) In the ultrarelativistic limit $(\gamma \gg 1,\ \beta \approx 1)$, ignoring the energy transfer in the electron rest frame,

$$\frac{P_{\rm compt}}{\left|dE_1^{\rm loss}/dt\right|} = \frac{4/3\sigma_T c\gamma^2\beta^2 U_{\rm ph}}{\sigma_T c U_{\rm ph}} = \frac{4}{3}\gamma^2\beta^2 \quad \rightarrow \quad (\Delta\epsilon)_{\rm R} \approx \frac{4}{3}\gamma^2\epsilon$$

For a thermal distribution of ultrarelativistic electrons,

$$\left\langle\gamma^2\right\rangle = \frac{\left\langle\epsilon^2\right\rangle}{(mc^2)^2} = 12\left(\frac{kT}{mc^2}\right)^2 \longrightarrow \boxed{(\Delta\epsilon)_{\rm R} \approx 16\epsilon\left(\frac{kT}{mc^2}\right)^2}$$

- **Mean number of scatterings**,

  Recall that, for a pure scattering medium,

$$\left(\begin{array}{l}\text{mean number of}\\ \text{scatterings}\end{array}\right) \approx {\rm Max}(\tau_{\rm es},\ \tau_{\rm es}^2)$$

where
$$\tau_{\rm es} \approx \rho\kappa_{\rm es}R$$
$$\rho = m_p n_H (n_H = n_e)$$
$$\kappa_{\rm es} = \frac{\sigma_T}{m_p} = 0.40\ {\rm cm}^2\ {\rm g}^{-1}\ \text{for ionized hydrogen}$$
$$R = \ \text{size of the finite medium}$$

- **Compton y parameter**:

$$\boxed{y_{\rm NR} = \frac{4kT}{mc^2}{\rm Max}(\tau_{\rm es},\ \tau_{\rm es}^2)} \qquad \boxed{y_{\rm R} = 16\epsilon\left(\frac{kT}{mc^2}\right)^2 {\rm Max}(\tau_{\rm es},\ \tau_{\rm es}^2)}$$

[Repeated Scattering: Spectra and Power]

- We have already shown that a power-law spectrum may be a natural consequence of a power-law distribution of electrons.
- **We will show that a power-law photon distribution can also be produced from repeated scattering off a nonpower-law electron distribution.**

Let A = the mean amplification of photon energy per scattering

$$A \equiv \frac{\epsilon_1}{\epsilon} \sim \frac{4}{3}\langle\gamma^2\rangle$$

$$= 16\left(\frac{kT}{mc^2}\right)^2 \quad \text{for thermal electron distribution}$$

mean photon energy = ϵ_i

(number) intensity = $I(\epsilon_i)$ at ϵ_i

After k scattering, the photon energy will be $\epsilon_k \sim \epsilon_i A^k$.

For an optically thin scattering medium ($\tau_{\rm es} < 1$), the probability of a photon undergoing k scattering before escaping the medium is $p_k(\tau_{\rm es}) \sim \tau_{\rm es}^k$.

The emergent intensity at energy ϵ_k is given by

$$I(\epsilon_k) \sim I(\epsilon_i)\tau_{\rm es}^k \sim I(\epsilon_i)\tau_{\rm es}^{\ln(\epsilon_k/\epsilon_i)/\ln A} = I(\epsilon_i)\left(\frac{\epsilon_k}{\epsilon_i}\right)^{\ln\tau_{\rm es}/\ln A}$$

$$\tau^{\ln(\epsilon_k/\epsilon_i)/\ln A} = x$$
$$[\ln(\epsilon_k/\epsilon_i)/\ln A]\ln\tau = \ln x$$
$$\ln(\epsilon_k/\epsilon_i)^{\ln\tau/\ln A} = \ln x$$
$$(\epsilon_k/\epsilon_i)^{\ln\tau/\ln A} = x$$

$$\therefore\ I(\epsilon_k) \sim I(\epsilon_i)\left(\frac{\epsilon_k}{\epsilon_i}\right)^{-\alpha} \quad \text{where} \quad \alpha \equiv \frac{-\ln\tau_{\rm es}}{\ln A}$$

$\longrightarrow$ power-law shape

- Total Compton power in the output spectrum is given by

$$P \propto \int I(\epsilon_k) d\epsilon_k = I(\epsilon_i)\epsilon_i \left[\int x^{-\alpha} dx\right]$$

The factor in square brackets is approximately **the factor by which the initial power** $I(\epsilon_i)\epsilon_i$ **is amplified** in energy.

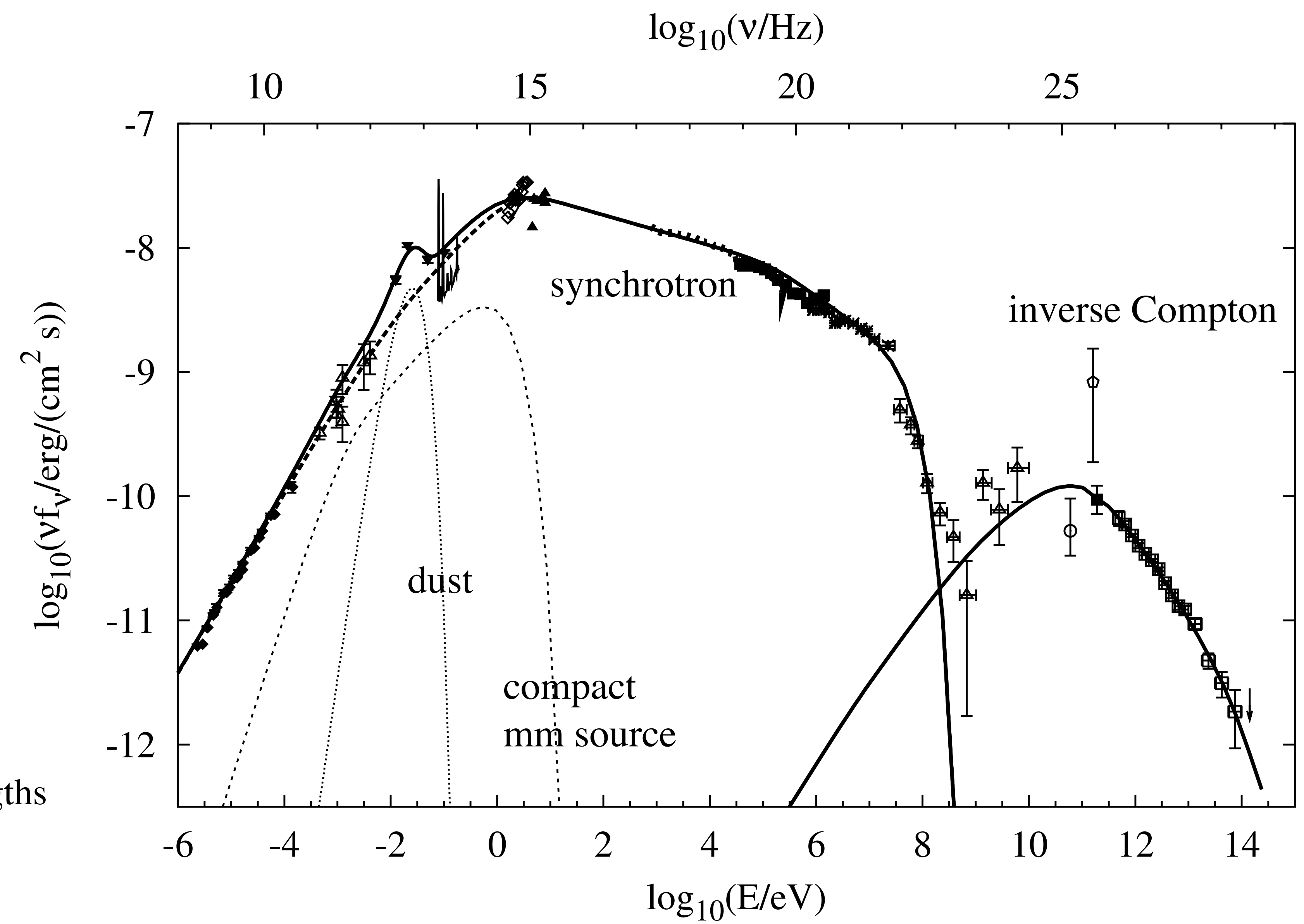
Clearly, this amplification will be important if $\alpha \ll 1$. Therefore, **energy amplification of a soft photon input spectrum is important when**

$$\alpha = \frac{-\ln \tau_{\rm es}}{\ln A} \lesssim 1 \quad \rightarrow \quad \ln\left(\tau_{\rm es} A\right) \gtrsim 0$$

$$\rightarrow \quad y = A\tau_{\rm es} \sim 16 \left(\frac{kT}{mc^2}\right)^2 \tau_{\rm es} \gtrsim 1$$

[Synchrotron self-Compton (SSC) emission]

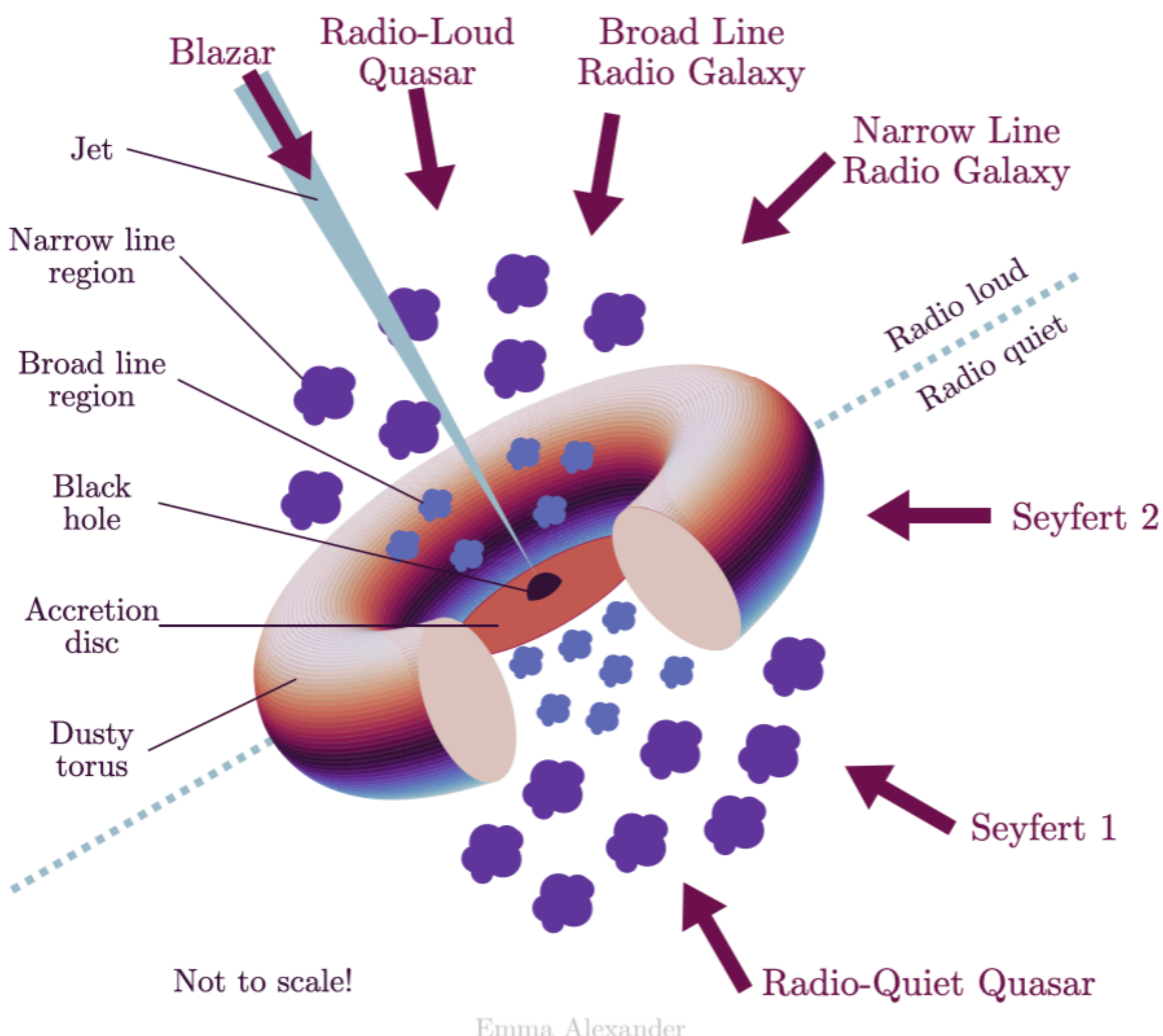
- The modification of the photon spectrum by Compton scattering is called **Comptonization**.
- Relativistic electrons in the presence of a magnetic field will surely emit synchrotron radiation at some level. The photons will undergo inverse Compton scattering by the very same electrons that emitted them in the first place. Such scattering must take place before the synchrotron photon leaves the source region. This is the **synchrotron self-Compton (SSC) process**.
- **Crab nebula**



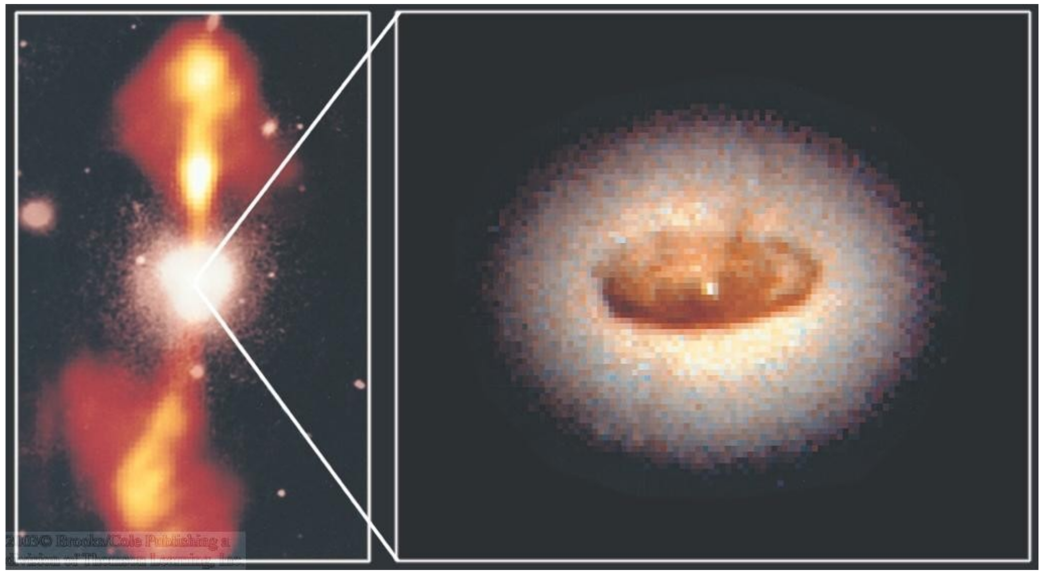
Dots: modified blackbody with $T = 46$ K.
Thin dashed line: emission at mm wavelengths
Thick dashed line: synchrotron emission

Active Galactic Nuclei

- A Unified Model for AGN



Unified model of AGN adapted from Urry & Padovani (1995).



HST image of the Dust Torus in NGC 4261

Active Galaxies = galaxies with extremely violent energy release in their nuclei (pl. of nucleus). Active Galactic Nucleus means the compact region of an Active Galaxy.

BLR, Broad Line Region : produces very broad lines. (velocity of 1,000-10,000 km/s)

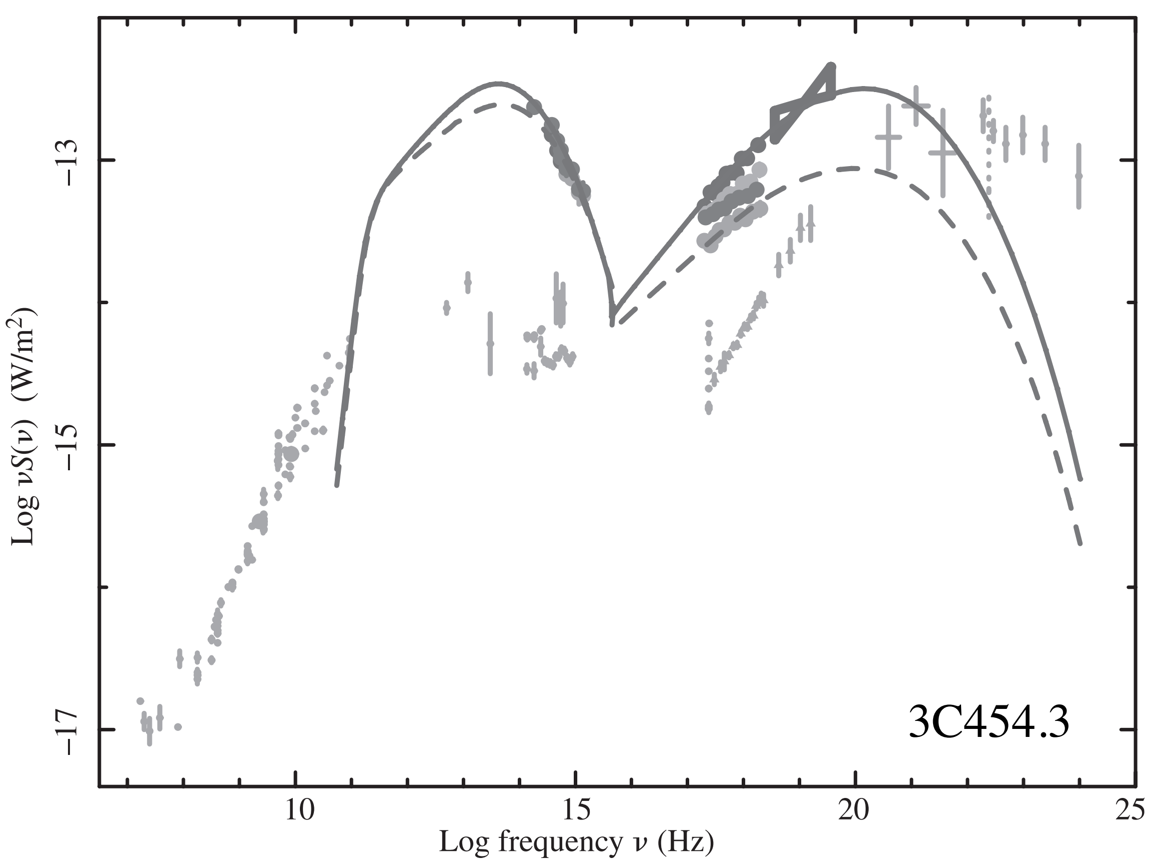
NLR, Narrow Line Region : produces relatively narrow lines (velocity of 100 km/s).

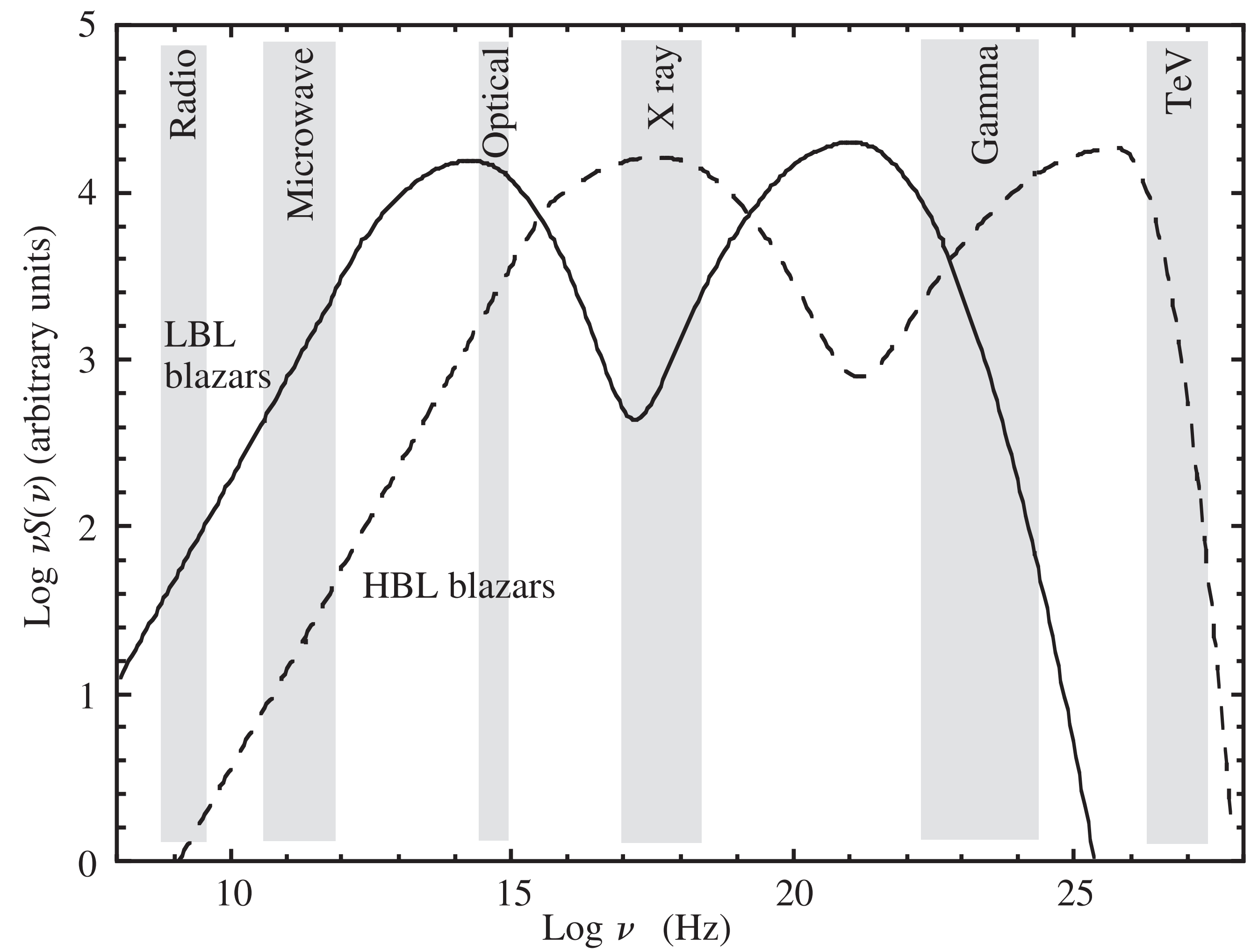
Seyfert 1 shows strong broad emission lines while Seyfert 2 does not.

- **Blazars**: If the observer view is more or less normal to the accretion disk, the action close to the core becomes visible. The observer considered to lie within the jet beam. Such objects are known as blazars or as BL Lacertae objects.
- Blazars have SEDs that are typically two peaked. The peak at lower frequency is attributed to synchrotron radiation and the one at higher frequency to IC scattering.

  The lower-energy case (LBL blazar) extends from the radio to the gamma-ray bands but is quiet in the TeV band. The higher-energy case (HBL blazar) reaches TeV energies but is quiet in the radio range.

LBL: Low-frequency peaked BL Lacs
HBL: High-frequency peaked BL Lacs





[Sunyaev-Zeldovich effect]

This part taken from
[Bradt, Astrophysical Processes]

- The **Sunyaev-Zeldovich effect** is Comptonization of the Cosmic Microwave Background (CMB), the distortion of the blackbody spectrum (T = 2.73 K) of the CMB owing to the Inverse Compton (IC) scattering of the CMB photons by the energetic electrons in the galaxy clusters.

  Thermal SZ effects, where the CMB photons interact with thermal electrons that have high energies due to their "high" temperature.

  Kinematic SZ effects (Ostriker-Vishniac effect), a second-order effect where the CMB photons interact with electrons that have high energies due to their bulk motion (peculiar motion). The motions of galaxies and clusters of galaxies relative to the Hubble flow are called peculiar velocities. The plasma electrons in the cluster also have this velocity. The energies of the CMB photons that scattered by the electrons reflect this motion.

  Determinations of the peculiar velocities of clusters enable astronomers to map out the growth of large-scale structure in the universe. This topic is fundamental importance, and the kinetic SZ effect is a promising method for approaching it.

- Thermal SZ effect

- The net effect of the IC scattering on the photon spectrum is obtained by multiplying the photon number spectrum by the kernel $K(\nu/\nu_0)$ and integrating over the spectrum.

$$N_{\text{scatt}}(\nu) = \int_0^\infty N(\nu_0) K(\nu/\nu_0) d\nu_0$$

The net effect is that the BB spectrum is shifted to the right and distorted (Figure (c)).

Observations of the CMB are most easily carried out in the low-frequency Rayleigh-Jeans region of the spectrum $\left(k\nu \ll kT_{\text{CMB}}\right)$.

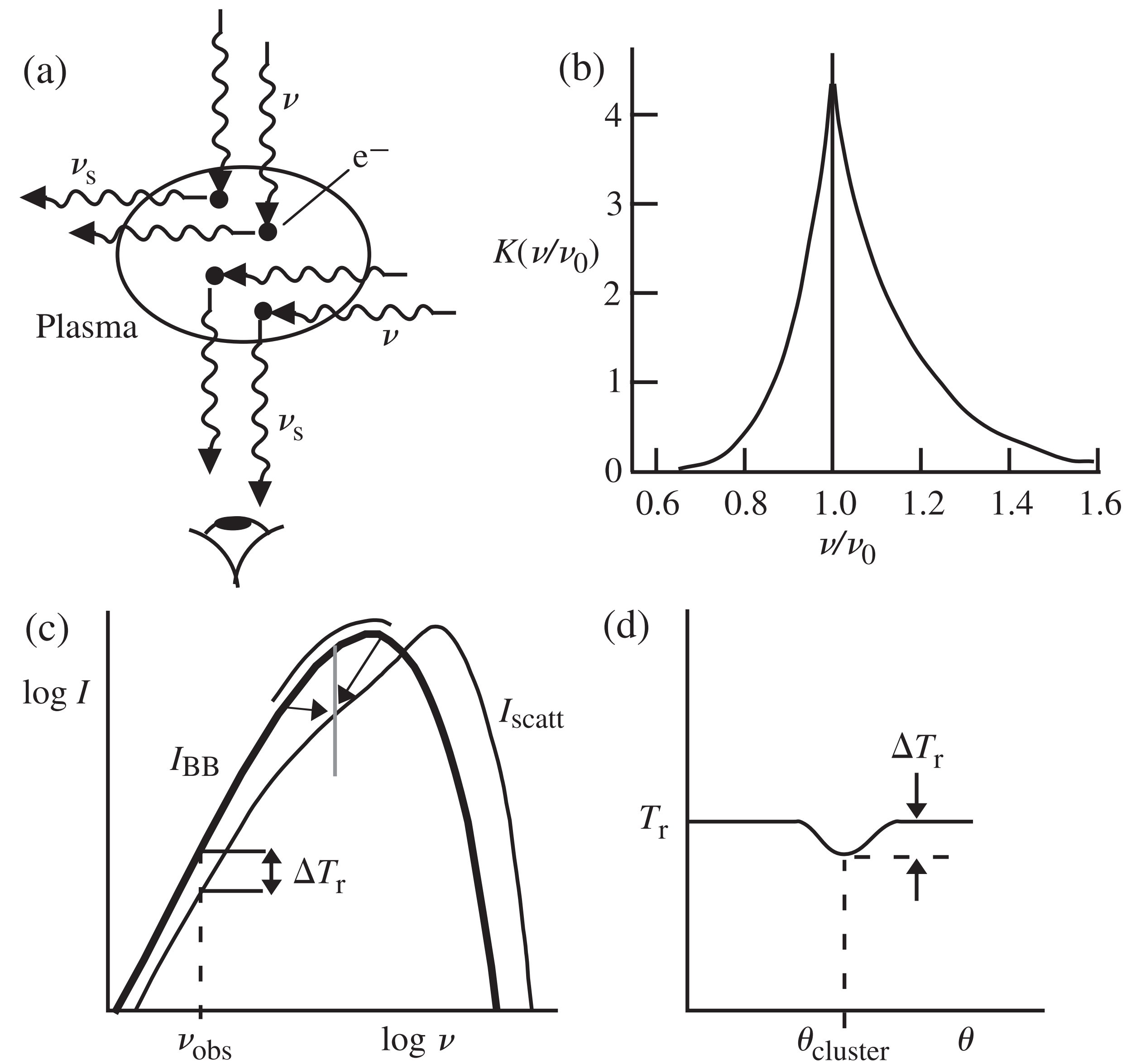
Measurement of the CMB temperature as a function of position on the sky would thus exhibit **antenna temperature dips in the directions of clusters** that contain hot plasmas (Figures (c) and (d)).

Note that the scattered spectrum is not a BB spectrum. **The effective temperature increases.** But, the total number of photons detected in a given time over the entire spectrum remains constant.

In Rayleigh-Jeans regime, the intensity decreases at a fixed frequency.

The result of such scatterings for an initial blackbody photon spectrum is shown in the following figure for the value:

$$\frac{kT_e}{mc^2}\tau = 0.5$$



- **Change of the BB temperature**

In the Rayleigh-Jeans region, $$I(\nu) = \frac{2\nu^2}{c^2} k_{\rm B} T_{\rm CMB}$$

If the spectrum is shifted parallel to itself on a log-log plot, the fractional frequency change of a scattered photon is constant.

$$\varepsilon = \frac{\Delta\nu}{\nu} = \frac{\nu' - \nu}{\nu} = \text{constant} \quad \text{or} \quad \nu' = \nu(1+\varepsilon) \quad \longrightarrow \quad d\nu' = d\nu(1+\varepsilon)$$

Total photon number is conserved: $$N'(\nu')d\nu' = N(\nu)d\nu \;\rightarrow\; \frac{I'(\nu')}{h\nu'}d\nu' = \frac{I(\nu)}{h\nu}d\nu$$

Compton-scattered = initial

$$\therefore \quad I'(\nu') = I(\nu) = I\left(\frac{\nu'}{1+\varepsilon}\right)$$

$$\rightarrow \quad I'(\nu) = I\left(\frac{\nu}{1+\varepsilon}\right) = \frac{2\nu^2}{c^2(1+\varepsilon)^2} k_{\rm B} T_{\rm CMB}$$

$$\frac{\Delta I}{I} = \frac{I'(\nu) - I(\nu)}{I(\nu)} = \frac{1}{(1+\varepsilon)^2} - 1 \approx -2\varepsilon = -2\frac{\Delta\nu}{\nu}$$

We compare the spectrum at one frequency ν in ***the cluster direction*** with that at the same frequency ***in an off-cluster direction.***

$I'(\nu)$ in the cluster direction (Compton-scattered)

$I(\nu)$ in an off-cluster direction (Initial)

$$\frac{\Delta T_{\rm CMB}}{T_{\rm CMB}} = \frac{\Delta I}{I} \approx -2\epsilon = -2\frac{\Delta\nu}{\nu}$$

The properly calculated result is $$\varepsilon = \frac{\Delta\nu}{\nu} = \frac{k_{\rm B} T_{\rm CMB}}{m_e c^2}\tau \,. \quad \longrightarrow \quad \frac{\Delta T_{\rm CMB}}{T_{\rm CMB}} \approx -2\frac{k_{\rm B} T_{\rm CMB}}{m_e c^2}\tau$$

(see homework)

$R_c \sim 10^{24}$ cm (~ 320 pc)

A typical cluster have an average electron density of $\sim 2.5 \times 10^{-3}$ cm^{-3}, a core radius of $R_c \sim 10^{24}$ cm (~ 320 pc), and an electron temperature of $k_{\mathrm{B}}T \approx 5\,\mathrm{eV}$.

A typical optical depth is thus

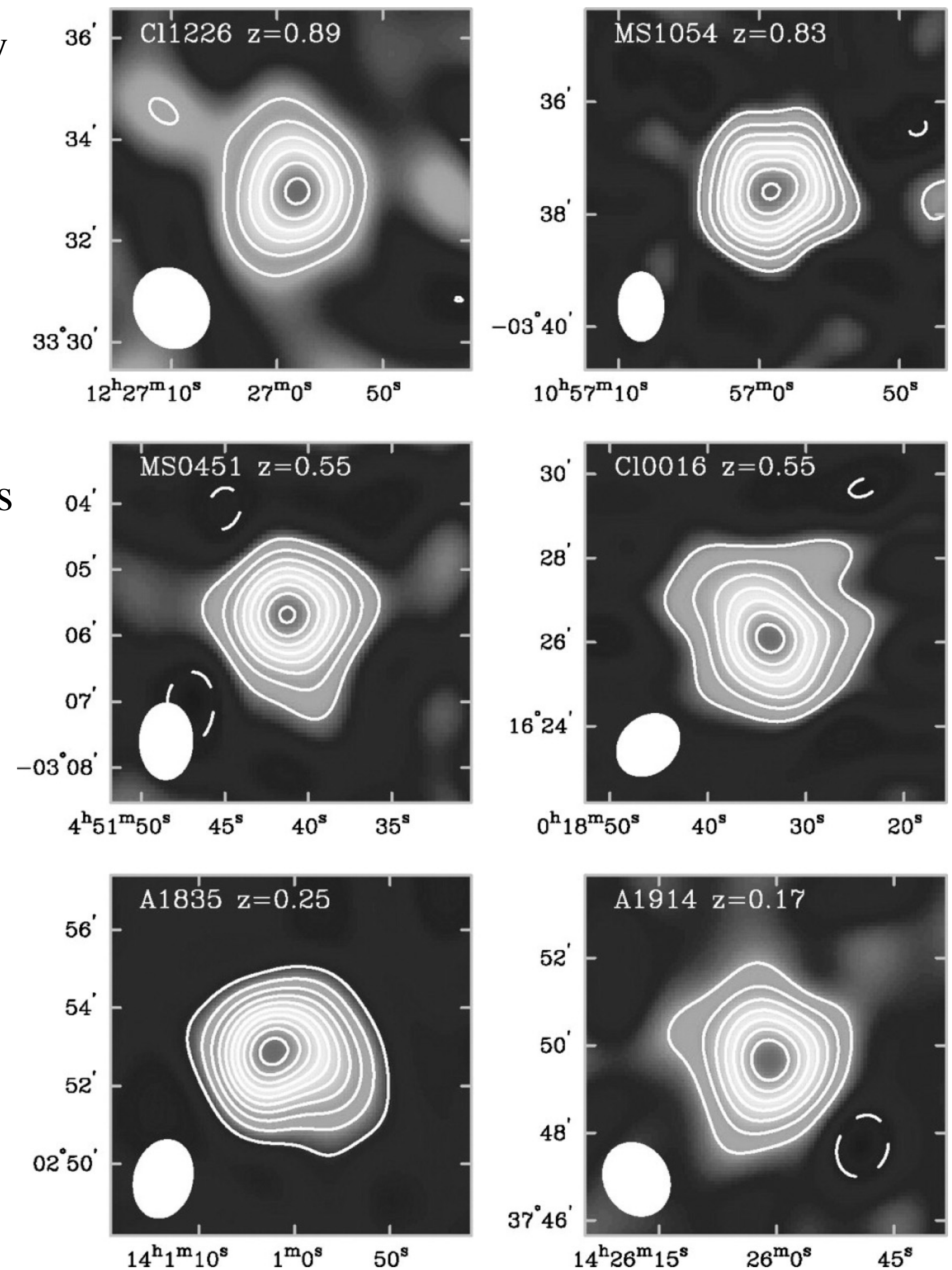
$$\tau \approx 3\sigma_{\mathrm{T}} n_e R_c \approx 0.005$$

Then, the expected antenna temperature change is

$$\frac{\Delta T_{\mathrm{CMB}}}{T_{\mathrm{CMB}}} \approx -1 \times 10^{-4}$$

$$\boxed{\Delta T_{\mathrm{CMB}} \approx -0.3 \text{ mK for } T_{\mathrm{CMB}} = 2.7 \text{ K}}$$

This effect has been measured in dozens of clusters.



Interferometric images at 30 GHz of six clusters of galaxies. The solid white contours indicate negative decrements to the CMB.
(Carlstrom et al. 2002, ARAA, 40, 643)

- Hubble Constant

- **The SZ effect can be used to estimate the Hubble constant.** A value of the Hubble constant can be obtained for a given galaxy (cluster) only if one has independent measures of a recession speed v and a distance d of a galaxy.

$$H_0 = \frac{v}{d}$$

First, the recession speed is readily obtained from the spectral redshift.

Second, how to mesure the distance:

X-ray observations:

$$I(\nu, T_e) = C\frac{g(\nu, T_e)}{T_e^{1/2}}\exp(-h\nu/kT_e)n_e^2(2R)$$

Here, R is the cluster radius.

S-Z CMB decrement:

$$\frac{\Delta T_{\rm CMB}}{T_{\rm CMB}} = -2\frac{kT_e}{m_e c^2}\tau = -2\frac{kT_e}{m_e c^2}\left(\sigma_{\rm T} n_e 2R\right)$$

The radio and X-ray measurements yield absolute values of **(1) the electron density** n_e **and (2) cluster radius** R without a priori knowledge of the cluster distance.

Imaging of the cluster in the radio or X-ray band yields the angular size of the cluster θ. Then the distance d to the cluster is obtained by

$$d = \frac{R}{\theta}$$

The SZ effect (at radio frequencies) in conjunction with X-ray measurements can give distances to clusters of galaxies. This can be used to derive the Hubble constant.

[Kompaneets Equation]

- The Kompaneets equation describes the time evolution of the distribution of photon occupancies in the case where photons and electrons are interacting through Compton scattering.
- Boltzmann transport equation

$$\frac{\partial n(\omega)}{\partial t} = c \int d^3p \int d\Omega \frac{d\sigma}{d\Omega} \left[f_e(\mathbf{p})' n(\omega')(1+n(\omega)) - f_e(\mathbf{p}) n(\omega)(1+n(\omega'))\right]$$

In $1 + n(\omega)$, the "1" for spontaneous Compton scattering, and the $n(\omega)$ for stimulated Compton scattering.

The Boltzmann equation may be expanded to second order in the small energy transfer, yielding an approximation called the *Fokker-Plank equation*. For photons scattering off a nonrelativistic, thermal distribution of electrons, the Fokker-Plank equation was first derived by A. S. Kompaneets (1957) and is known as the Kompaneets equation.

$$\frac{\partial n(\omega)}{\partial t} = \left(\frac{k_{\rm B}T}{mc^2}\right) \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} + n + n^2\right)\right] \quad \text{where} \quad x \equiv \frac{\hbar\omega}{k_{\rm B}T}, \quad \text{and} \quad t_c \equiv (n_e \sigma_T c)\, t$$

- For the complete derivation, see the books "X-ray spectroscopy in Astrophysics (eds. van Paradijs)", pages 213-218, and the book "High Energy Astrophysics (Katz)", pages 103-110.
- Monte Carlo Simulation of Compton scattering: see "Pozdnyakov, Sobol, and Suyaev (1983, Soviet Scientific Reviews, vol. 2, 189-331)" (1983ASPRv...2..189P)

8. Plasma Effects

- So far we have assumed our propagation medium to be a vacuum.
- **Dispersion**: The electrons and magnetic fields of the ionized interstellar medium (ISM) alter the character of radio waves propagating through it. The effect of the electron is to slow the propagation speed of the low-frequency components of a signal relative to that of the higher frequencies. This phenomenon is known as dispersion.
- Roughly speaking a **plasma** is a globally neutral partially or completely ionized gas.

[Dispersion in Cold, Isotropic Plasma]

- Maxwell's equations for a vacuum can be used for a plasma if the charge and current densities ρ and $\mathbf{j}$ due to the plasma are explicitly included.
- **<u>Plasma Dispersion Relation</u>**

If we assume a space and time variation of all quantities of the form $\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]$, Maxwell's equations become

Gauss's law	$\nabla \cdot \mathbf{E} = 4\pi\rho$
Faraday's law	$\nabla \cdot \mathbf{B} = 0$
Ampere's law	$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$
Gauss's law for magnetic field	$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$

$\longrightarrow$

$$\boxed{\begin{aligned} i\mathbf{k}\cdot\mathbf{E} &= 4\pi\rho \\ i\mathbf{k}\cdot\mathbf{B} &= 0 \\ i\mathbf{k}\times\mathbf{E} &= i\frac{\omega}{c}\mathbf{B} \\ i\mathbf{k}\times\mathbf{B} &= \frac{4\pi}{c}\mathbf{j} - i\frac{\omega}{c}\mathbf{E} \end{aligned}}$$

Let us assume that the our plasma consists of electrons with density n. The ions are very much less mobile than electrons and thus are neglected. We also assume that there is no external magnetic field; thus the plasma is isotropic. Then, the equation of motion of electrons when there is no external magnetic field is

$$m\dot{\mathbf{v}} = -e\left(\mathbf{E} + \cancel{\frac{\mathbf{v}}{c}\times\mathbf{B}}\right) = -e\mathbf{E} \quad\longrightarrow\quad \begin{aligned} -im\omega\mathbf{v} &= -e\mathbf{E} \\ \mathbf{v} &= \frac{e\mathbf{E}}{i\omega m} \end{aligned}$$

<u>Current density:</u> $\mathbf{j} = -ne\mathbf{v} = \sigma\mathbf{E}$ where the $\boxed{\textbf{conductivity is } \sigma \equiv \frac{ine^2}{\omega m}}$

The charge conservation equation (continuity equation) gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \qquad -i\omega\rho + i\mathbf{k}\cdot\mathbf{j} = 0 \;\rightarrow\; \rho = \frac{1}{\omega}\mathbf{k}\cdot\mathbf{j} = \frac{\sigma}{\omega}\mathbf{k}\cdot\mathbf{E}$$

Using the expressions for **j** and ρ, Maxwell's equations become

$$\begin{aligned} \mathbf{j} &= \sigma\mathbf{E} \\ \rho &= \frac{\sigma}{\omega}\mathbf{k}\cdot\mathbf{E} \end{aligned} \longrightarrow \begin{aligned} i\mathbf{k}\cdot\mathbf{E} &= 4\pi\rho \\ i\mathbf{k}\cdot\mathbf{B} &= 0 \\ i\mathbf{k}\times\mathbf{E} &= i\frac{\omega}{c}\mathbf{B} \\ i\mathbf{k}\times\mathbf{B} &= \frac{4\pi}{c}\mathbf{j} - i\frac{\omega}{c}\mathbf{E} \end{aligned} \longrightarrow \begin{aligned} i\mathbf{k}\cdot\epsilon\mathbf{E} &= 0 \\ i\mathbf{k}\cdot\mathbf{B} &= 0 \\ i\mathbf{k}\times\mathbf{E} &= i\frac{\omega}{c}\mathbf{B} \\ i\mathbf{k}\times\mathbf{B} &= -i\frac{\omega}{c}\epsilon\mathbf{E} \end{aligned}$$

These equations are now "source-free" and can be solved in the same way as before. We find again that $\mathbf{k},\ \mathbf{E},\ \mathbf{B}$ form a mutually orthogonal right-hand vector triad.

The **dispersion relation** connecting k and ω is given by

$$\begin{aligned} i\mathbf{k}\times\mathbf{B} &= -i\frac{\omega}{c}\epsilon\mathbf{E} \\ i\mathbf{k}\times\left(\frac{c}{\omega}\mathbf{k}\times\mathbf{E}\right) &= -i\frac{\omega}{c}\epsilon\mathbf{E} \\ \mathbf{k}\,(\cancel{\mathbf{k}\cdot\mathbf{E}}) - \mathbf{E}\,(\mathbf{k}\cdot\mathbf{k}) &= -\frac{\omega^2}{c^2}\epsilon\mathbf{E} \end{aligned} \longrightarrow \boxed{\begin{aligned} c^2k^2 &= \epsilon\omega^2 \\ &= \omega^2 - \omega_p^2 \\ \omega^2 &= \omega_p^2 + c^2k^2 \end{aligned}}$$

where the **dielectric constant**:

$$\begin{aligned} \epsilon \equiv 1 - \frac{4\pi\sigma}{i\omega} &= 1 - \frac{4\pi n e^2}{m\omega^2} \\ &= 1 - \left(\frac{\omega_p}{\omega}\right)^2 \end{aligned}$$

plasma frequency:

$$\begin{aligned} \omega_p &\equiv \sqrt{\frac{4\pi n e^2}{m}} \\ &= 5.63\times 10^4\left(n/\mathrm{cm}^{-3}\right)^{1/2}\ \mathrm{s}^{-1} \end{aligned}$$

- We now see immediately that for $\omega < \omega_p$, the wavenumber is imaginary ($k^2 < 0$).

$$k = \frac{i}{c}\sqrt{\omega_p^2 - \omega^2}$$

In this case, the wave amplitude decreases as $e^{-|k|r}$ on a scale of the order of $2\pi/|k| \approx 2\pi c/\omega_p$.

Thus the plasma frequency ω_p defines **a plasma cutoff frequency below which there is no electromagnetic propagation**.

For instance, Earth ionosphere prevents extraterrestrial radiation at frequencies less than ~ 1 MHz from being observed at the earth's surface ($n \sim 10^4$ cm^{-3}). The existence of the plasma cutoff yields an important method of probing the ionosphere.

Method of probing the ionosphere:

Let a pulse of radiation in a narrow range about ω be directed straight upward from the earth's surface.

When there is a layer at which the density n is large enough to make $\omega < \omega_p$, the pulse will be totally reflected from the layer.

The time delay of the pulse provides information on the height of the layer.

By making such measurements at many different frequencies, the electron density can be determined as a function of height.

- **Group and Phase Velocity**

When $\omega > \omega_p$, waves do propagate without damping with phase velocity.

Phase velocity

$$v_{\rm ph} \equiv \frac{\omega}{k} = \frac{\omega}{\sqrt{\omega^2 - \omega_p^2/c}} = \frac{c}{n_r}$$

where ***the index of refraction*** is defined to be $n_r \equiv \sqrt{\epsilon} = \sqrt{1 - \frac{\omega_p^2}{\omega}}$

The phase velocity always exceeds the speed of light: $v_{\rm ph} > c$

Group velocity

$$\omega^2 = \omega_p^2 + c^2k^2 \quad \longrightarrow \quad v_{\rm g} \equiv \frac{\partial\omega}{\partial k} = \frac{c^2k}{\omega} = c\sqrt{1 - \frac{\omega_p^2}{\omega}} \quad \to \quad v_{\rm g} < c$$

The group velocity is always less than c. The wave energy travels at the group velocity, as does any modulation of the wave (information coding).

- **Pulsar**

An important application of the formula for group velocity is to pulsars. Each individual pulse from the pulsar has a spectrum covering a wide band of frequency. Thus, the pulse will be dispersed by its interaction with the interstellar plasma, since each small range of frequencies travels at a slightly different group velocity and will reach earth at a slightly different time.

Let d = pulsar distance to the pulsar

The time required for a pulse to reach earth at frequency ω is $t_p = \int_0^d \frac{ds}{dv_g}$.

The plasma frequency in ISM is usually quite low (~ 10^3 Hz), so we can assume that $\omega \gg \omega_p$. Then, the travel time is given by

$$\frac{1}{v_g} = \frac{1}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} \approx \frac{1}{c}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right) \longrightarrow \quad t_p \approx \frac{d}{c} + \frac{2\pi e^2}{c m_e \omega^2}\int_0^d n ds$$

= transit time for a vacuum + plasma correction

- **Dispersion measure**:

What is usually measured is **the rate of change of arrival time with respect to frequency**.

$$\boxed{\frac{dt_p}{d\omega} = -\frac{4\pi e^2}{c m_e \omega^3}\mathcal{DM} \quad \text{where} \quad \mathcal{DM} \equiv \int_0^d n ds}$$ is the dispersion measure of the ray.

The arrival interval of pulsar signal between two frequencies ν_1 and ν_2:

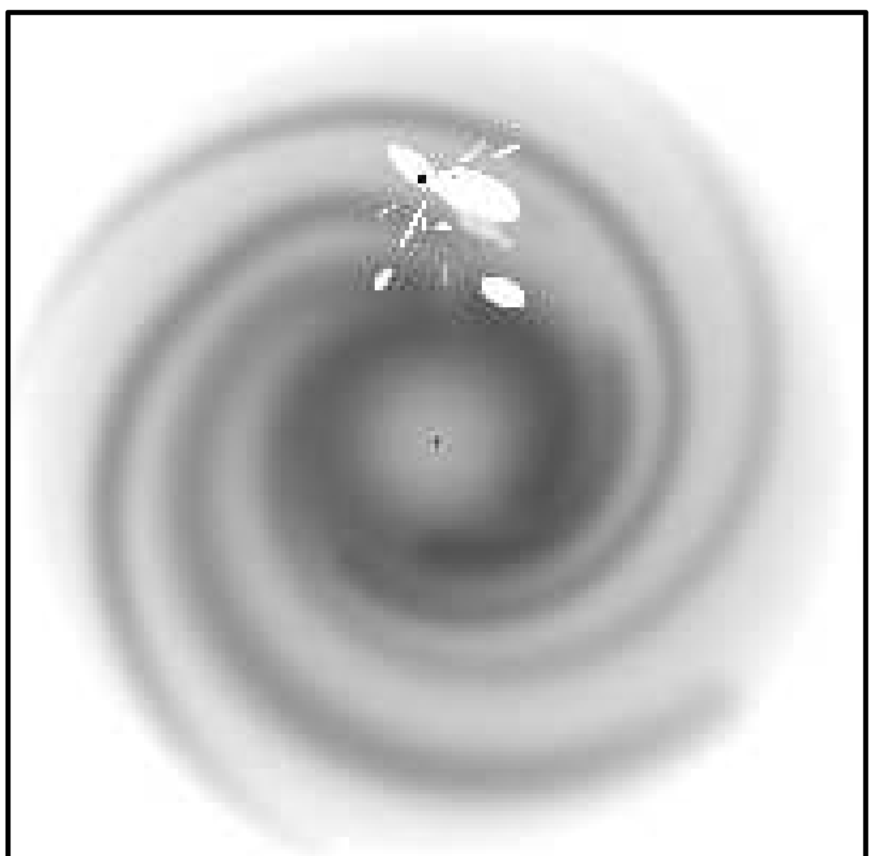
$$\boxed{\Delta t_p = \frac{e^2}{2\pi c m_e}\left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2}\right)\mathcal{DM} = 4.15 \times 10^{15}\left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2}\right)\frac{\mathcal{DM}}{\text{cm}^{-3}\ \text{pc}}\ \text{sec}}$$

If one has idea of pulsar distance, one can use pulsar data to map free electron density.

Taylor & Cordes (1993, ApJ, 411, 674)

Cordes & Lazio (2003, arXiv:astro-ph/0207156)

Schnitzeler (2012, MNRAS, 427, 664)



Cordes & Lazio (2003)

FIG. 2.— Electron density corresponding to the best fit model plotted as a grayscale with logarithmic levels on a 30×30 kpc x-y plane at z=0 and centered on the Galactic center. The most prominent large-scale features are the spiral arms, a thick, tenuous disk, a molecular ring component. A Galactic center component appears as a small dot. The small-scale, lighter features represent the local ISM and underdense regions required for by some lines of sight with independent distance measurements. The small dark region embedded in one of the underdense, ellipsoidal regions is the Gum Nebula and Vela supernova remnant.

[Propagation along a Magnetic Field; Faraday Rotation]

- Now we extend the discussion of plasma propagation effects by considering the effect of an external, fixed magnetic field $\mathbf{B}_0$.

  The properties of the waves will then depend on the direction of propagation relative to the magnetic field direction. For this reason the plasma is called ***anisotropic***.

  Because of the magnetic field, a new frequency enters the problem, namely, the cyclotron frequency.

$$\text{cyclotron frequency} \qquad \omega_{\mathrm{B}} = \frac{eB_0}{m_e c} = 1.67 \times 10^7 \ (B_0/\mathrm{G}) \ \mathrm{s}^{-1}$$

$$\hbar\omega_{\mathrm{B}} = 1.16 \times 10^{-8} \ (B_0/\mathrm{G}) \ \mathrm{eV}$$

  If the fixed magnetic field $\mathbf{B}_0$ is much stronger than the field strengths of the propagating wave, then the equation of motion of an electron is approximately

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - \frac{e}{c}\mathbf{v} \times \mathbf{B}_0$$

  For simplicity, assume that the wave propagates along the fixed field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, and assume that the wave is circularly polarized and sinusoidal.

$$\mathbf{E}_{\pm}(t) = Ee^{-i\omega t}\left(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}\right)$$

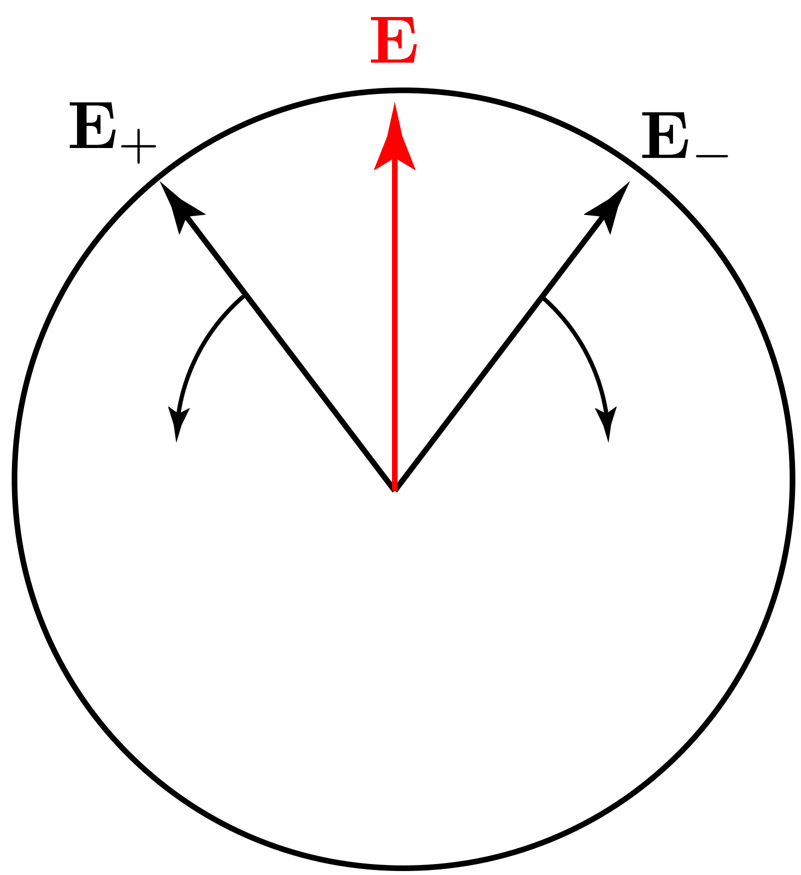
  Here, $\pm$ denotes right and left circular polarization.

  Note that Rybiki & Lightman define oppositely the RCP and LCP.

Bases vectors for circularly rotating electric field

The bases vectors $\mathbf{E}_{\pm}$ rotates counterclockwise (+) or clockwise (-) when viewed from the tip of the wave vector.

$$
\begin{aligned}
Re\left(\mathbf{E}_{\pm}\right) &= Re\left(\left(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}\right) E e^{-i\omega t}\right) \\
&= Re\left(\left(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}\right) E \left(\cos\omega t - i\sin\omega t\right)\right) \\
&= (\cos\omega t)\hat{\mathbf{x}} \pm (\sin\omega t)\hat{\mathbf{y}} \\
&= \cos(\pm\omega t)\hat{\mathbf{x}} + \sin(\pm\omega t)\hat{\mathbf{y}}
\end{aligned}
$$



Now, we solve the equation of motion for an electron and obtaine the dielectric constant.

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - \frac{e}{c}\mathbf{v}\times\mathbf{B}_0 \qquad \frac{d\mathbf{v}_\parallel}{dt} = 0 \qquad \mathbf{v}_\parallel = \text{constant}$$

Set $\mathbf{v}_\perp = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$

$$(-i\omega)e^{-i\omega t}(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}) = -\frac{e}{m}Ee^{-i\omega t}(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) - \frac{e}{mc}e^{-i\omega t}(v_yB_0\hat{\mathbf{x}} - v_xB_0\hat{\mathbf{y}})$$

$$\longrightarrow \quad v_x = -\frac{ie}{m\omega}E - \frac{ieB_0}{mc\omega}v_y, \quad v_y = \pm\frac{e}{m\omega}E + \frac{ieB_0}{mc\omega}v_x$$

$$(1)\ v_x = -\frac{ie}{m\omega}E - \frac{ieB_0}{mc\omega}\left(\pm\frac{e}{m\omega}E + \frac{ieB_0}{mc\omega}v_x\right)$$

$$\rightarrow \left(1 - \frac{e^2B_0^2}{m^2c^2\omega^2}\right)v_x = -\frac{ie}{m\omega}E\left(1 \pm \frac{eB_0}{mc\omega}\right)$$

$$\rightarrow \left(\omega^2 - \omega_{\rm B}^2\right)v_x = -\frac{ie}{m}E\left(\omega \pm \omega_{\rm B}\right)$$

$$\rightarrow v_x = -\frac{ie}{m}E\frac{1}{\omega \mp \omega_{\rm B}}$$

$$(2)\ v_y = \pm\frac{e}{m\omega}E + \frac{ieB_0}{mc\omega}\left(-\frac{ie}{m\omega}E - \frac{ieB_0}{mc\omega}v_y\right)$$

$$\rightarrow \left(1 - \frac{e^2B_0^2}{m^2c^2\omega^2}\right)v_y = \pm\frac{e}{m\omega}E\left(1 \pm \frac{eB_0}{mc\omega}\right)$$

$$\rightarrow \left(\omega^2 - \omega_{\rm B}^2\right)v_y = \pm\frac{e}{m}E\left(\omega \pm \omega_{\rm B}\right)$$

$$\rightarrow v_y = \pm\frac{e}{m}E\frac{1}{\omega \mp \omega_{\rm B}}$$

$$\therefore\ \mathbf{v} = \frac{-ie}{m\left(\omega \mp \omega_{\rm B}\right)}\mathbf{E}(t)$$

Then, the current density and conductivity are given by

$$\boxed{\begin{aligned}&\mathbf{j} \equiv -ne\mathbf{v} = \sigma\mathbf{E}\\ &\text{where } \sigma_{\rm R,L} \equiv \frac{ine^2}{m_e\left(\omega \mp \omega_{\rm B}\right)}\end{aligned}}$$

Dielectric constant

$$\epsilon \equiv 1 - \frac{4\pi\sigma}{i\omega} = 1 - \frac{4\pi n e^2}{m_e \omega \left(\omega \mp \omega_{\mathrm{B}}\right)}$$

$$\epsilon_{\mathrm{R,L}} = 1 - \frac{\omega_p^2}{\omega \left(\omega \mp \omega_{\mathrm{B}}\right)}$$

Dispersion relation
$$\omega^2 = k^2 c^2 + \frac{\omega_p^2 \omega}{\omega \mp \omega_{\mathrm{B}}}$$

Phase velocity
$$v_p \equiv \frac{\omega}{k} = \frac{c}{n_r}$$

$$n_r = \sqrt{1 - \frac{\omega_p^2}{\omega \left(\omega \mp \omega_{\mathrm{B}}\right)}}$$

Right (+) and left (-) circularly polarized waves travel with different speeds.

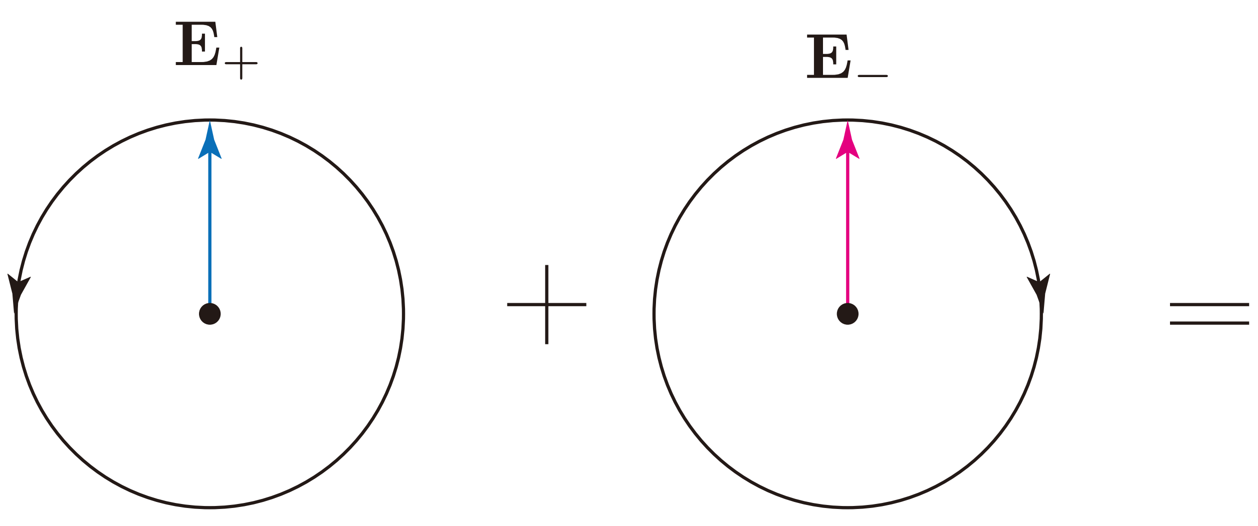
Speed difference sense is $v_{\mathrm{R}} > v_{\mathrm{L}}$.

These waves travel with different velocities. Therefore, a plane polarized wave, which is a linear superposition of a right-hand and a left-hand polarized wave, will not keep a constant plane of polarization, but this plane will rotate as it propagates. This effect is called Faraday rotation.

- **<u>Faraday Rotation</u>**

If the incident radiation is ***circularly polarized*** (either R or L), then the radiation will encounter different dispersion than unmagnetized case. But, the radiation will still remain circularly polarized.

If the incident radiation is ***linearly polarized***, i.e., a linear superposition of a right-hand and a left-hand polarized wave, then ***<u>the line of polarization will rotate as it propagates</u>***. This effect is called Faraday rotation.



$$
\begin{aligned}
\mathbf{E}(t) &= Ee^{-i\omega t}\hat{\mathbf{x}} \\
&= \frac{1}{2}\left[(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + (\hat{\mathbf{x}} - i\hat{\mathbf{y}})\right] Ee^{-i\omega t} \\
&= \frac{1}{2}\left(\boldsymbol{\varepsilon}_+ + \boldsymbol{\varepsilon}_-\right) Ee^{-i\omega t}
\end{aligned}
$$

Decomposition of linear polarization into components of right and left circular polarization

- The phase angle ϕ after traveling a distance d is, in general, if the wave number is not constant along the path, is given by

$$\phi_{\mathrm{R,L}} = \int_0^d k_{\mathrm{R,L}} ds$$

Assume that $\omega \gg \omega_p$ and $\omega \gg \omega_{\mathrm{B}}$

$$k_{\mathrm{R,L}} = \frac{\omega}{c}\sqrt{\epsilon_{\mathrm{R,L}}} = \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{\omega^2\left(1 \mp \omega_{\mathrm{B}}/\omega\right)}\right]^{1/2} \approx \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{2\omega^2}\left(1 \pm \frac{\omega_{\mathrm{B}}}{\omega}\right)\right]$$
$$= \frac{\omega}{c} - \frac{\omega_p^2}{2c\omega} \mp \frac{\omega_p^2\omega_{\mathrm{B}}}{2c\omega^2} = k_0 \mp \Delta k \quad \left(\Delta k \equiv k_{\mathrm{L}} - k_{\mathrm{R}} = \frac{\omega_p^2\omega_{\mathrm{B}}}{c\omega_2}\right)$$

Consider an electromagnetic wave that starts off linearly polarized in the x-direction at the source.

$$\mathbf{E}(t) = Ee^{-i\omega t}\hat{\mathbf{x}} = \frac{1}{2}\left[(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + (\hat{\mathbf{x}} - i\hat{\mathbf{y}})\right]Ee^{-i\omega t}$$

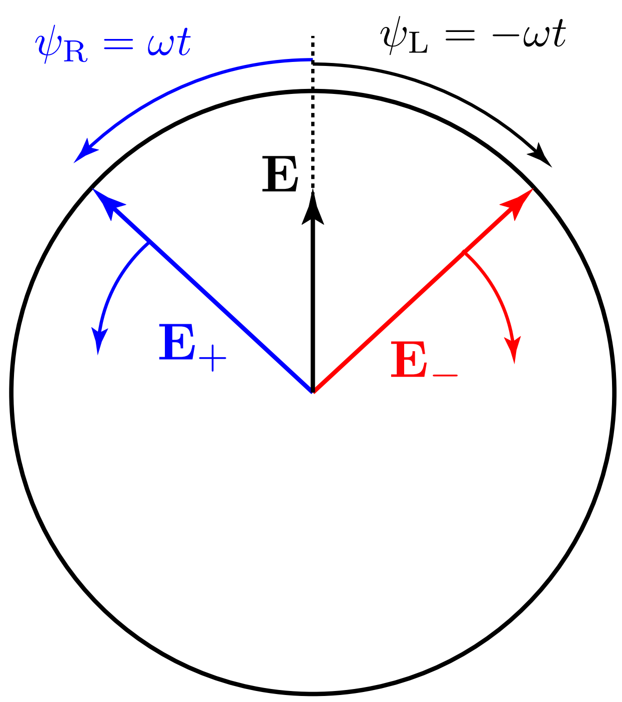
Let’s define

$$\int_0^d k_{\mathrm{R,L}} ds = \int_0^d k_0 ds \mp \int_0^d \Delta k ds \equiv \phi \mp \varphi$$

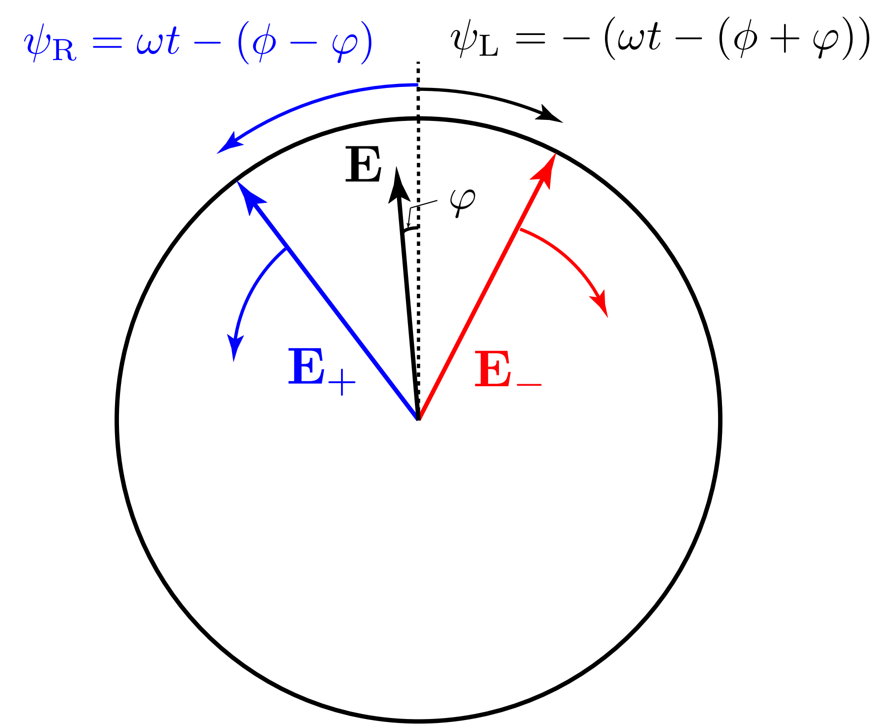
Then, after propagating a distance d through a magnetized plasma toward the observer, the electric field will be

$$\mathbf{E}(t) = \frac{1}{2}\left[(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\,e^{i(\phi-\varphi)} + (\hat{\mathbf{x}} - i\hat{\mathbf{y}})\,e^{i(\phi+\varphi)}\right]Ee^{-i\omega t}$$
$$= (\hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\sin\varphi)\,Ee^{i(\phi-\omega t)}$$

- The resulting electric field is also linearly polarized, but the polarization angle is rotated counterclockwise by angle φ (when viewed at a fixed position).



Before passing through the medium



After passing through the medium

- Radiation that starts linearly polarized in a certain direction is rotated by the *Faraday effect* through an angle φ after propagating a distance *d* through a magnetized plasma.

$$\boxed{\varphi = \frac{1}{2}\int_0^d \Delta k ds = \frac{1}{2}\int_0^d \frac{\omega_p^2 \omega_{\rm B}}{c\omega^2} ds = \frac{2\pi e^3}{m^2 c^2 \omega^2}\int_0^d n B_{\parallel} ds}$$

- We cannot, of course, generally measure the absolute rotation angle, since we do not know the intrinsic polarization direction of the radiation when it started from the source.

However, since φ varies with frequency (as ω^{-2}), we can determine the value of integral $\int nB_{\|}ds$ by making measurements at several frequencies. This can give information about the interstellar magnetic field.

Rotation measure is defined by

$$\boxed{\varphi = \frac{2\pi e^3}{m^2c^2\omega^2}\mathcal{RM} = \frac{e^3\lambda^2}{2\pi m^2c^4}\mathcal{RM}, \ \text{where } \mathcal{RM} \equiv \int_0^d nB_{\|}ds}$$

However, ***the field changes direction often along the line of sight and this method gives only a lower limit to actual field magnitudes.***

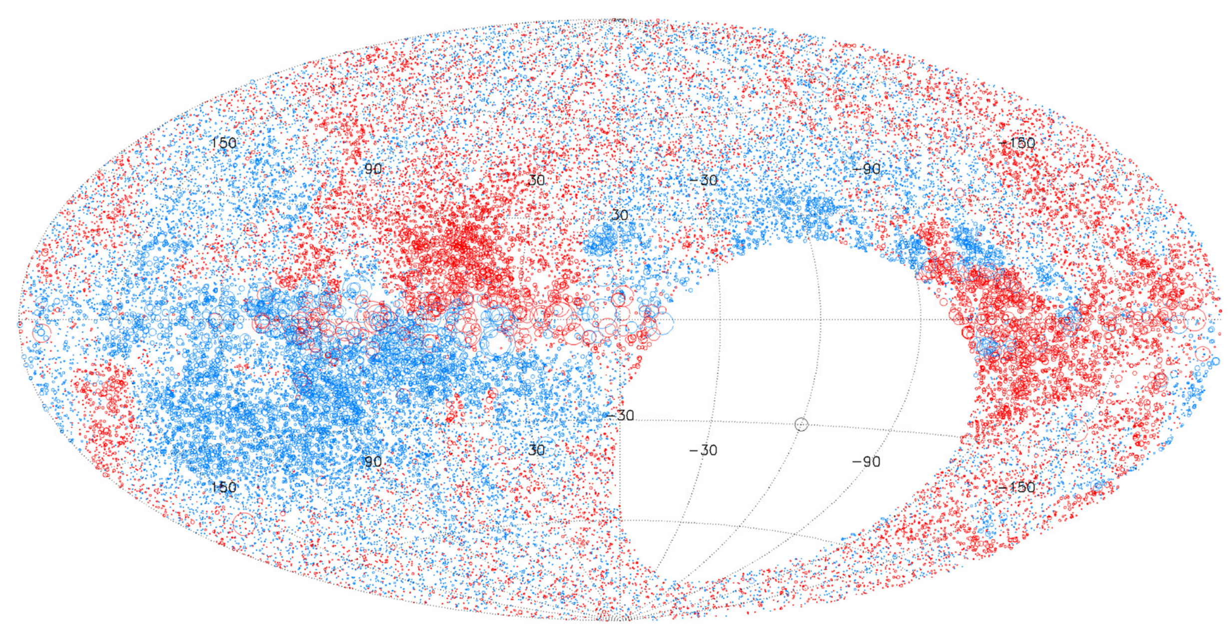
For measurements toward sources (pulsars) where the dispersion measure (DM) is also known, we can derive an estimate of the mean field strength along the line of sight.

$$\langle B_{\|}\rangle = \frac{\mathcal{RM}}{\mathcal{DM}}$$

Radio astronomers have concluded that

$$\langle n_e\rangle \approx 0.03 \text{ cm}^{-3}$$

$$\langle B_{\|}\rangle \approx 3\mu\text{G}$$



(Taylor, Stil, & Sunstrum 2009, ApJ, 702, 1230)
Red circles are positive RM and blue circles are negative.
The size of the circle scales linearly with magnitude of RM.

[Plasma Effects in High-Energy Emission Processes]

- Maxwell equations in dielectric medium:

$$
\begin{aligned}
\nabla \cdot (\epsilon \mathbf{E}) &= 4\pi\rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial(\epsilon \mathbf{E})}{\partial t}
\end{aligned}
\qquad
\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} \\
\mathbf{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}
\end{aligned}
$$

These equations formally result from Maxwell's equation in vacuum by the substitutions.

$$
\begin{aligned}
\mathbf{E} &\rightarrow \sqrt{\epsilon}\mathbf{E} \\
c &\rightarrow c/\sqrt{\epsilon} \\
\mathbf{B} &\rightarrow \mathbf{B} \\
e &\rightarrow e/\sqrt{\epsilon} \\
\phi &\rightarrow \sqrt{\epsilon}\phi \\
\mathbf{A} &\rightarrow \mathbf{A}
\end{aligned}
$$

These equations may be solved in the same manner as before for the retarded and Lienard-Wiechert potentials.

- Cherenkov Radiation

- Radiation from relativistic charges moving in a plasma with $n_r \equiv \sqrt{\epsilon} > 1$.

  In this case, the velocity of the charges can exceed the phase velocity:

$$v_p = \frac{c}{n_r} < v < c \quad \rightarrow \quad \beta n_r > 1$$

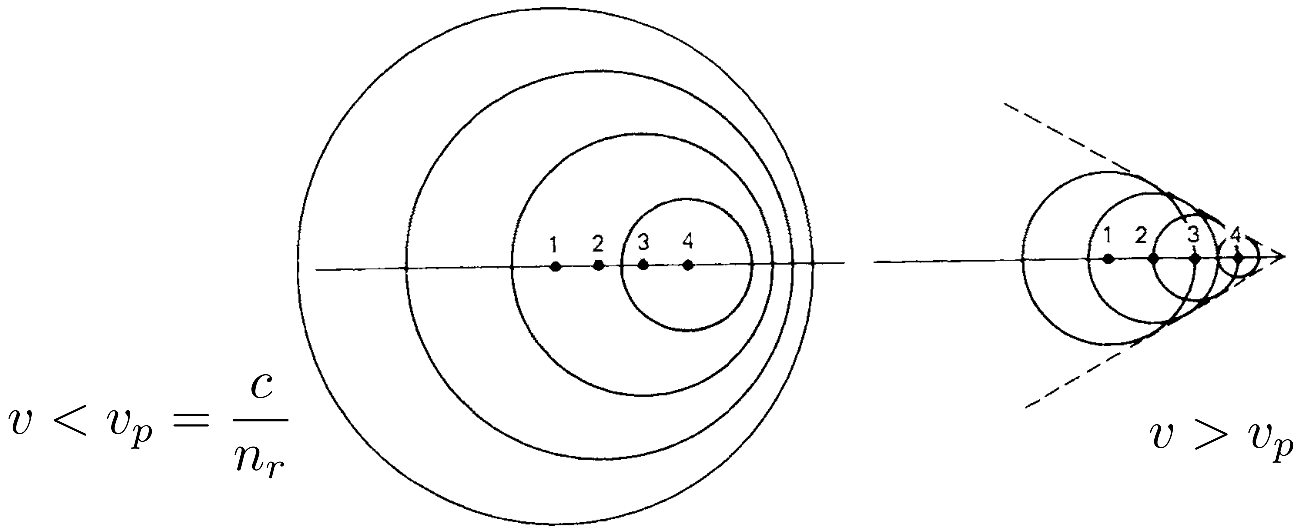
The beaming term of the Lienard-Wiechert potentials can vanish for an angle θ such that $\cos\theta = (n_r\beta)^{-1}$.

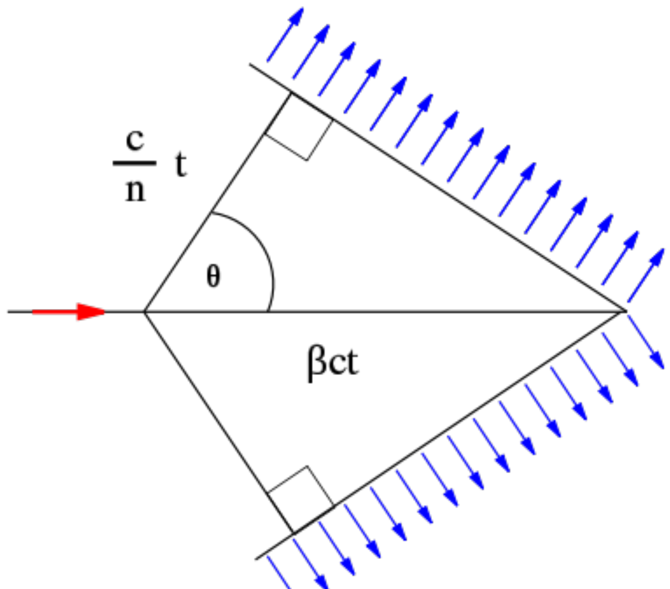
$$\kappa = 1 - (v/c)\cos\theta \quad \rightarrow \quad \kappa = 1 - \beta n_r \cos\theta$$

The potentials become infinite at certain places. In consequence, the uniformly moving particle can now radiate.

Cherenkov cone: Outside the cone, points feel no potentials yet. Inside the cone, each point is intersected by two spheres. The resulting radiation is called Cherenkov radiation.

A common analogy is the sonic boom of a supersonic aircraft or bullet.





- Razin-Tsytovich Effect

- When $n_r < 1$, Cherenkov radiation cannot occur.

The critical angle defining the beaming effect in a vacuum was shown to be $\theta_b \sim 1/\gamma = \sqrt{1-\beta^2}$. But in a plasma we have instead

$$\theta_b \sim \sqrt{1 - n_r^2 \beta^2}$$

If $n_r \ll 1$, and $\beta \sim 1$,

$$\theta_b \sim \sqrt{1 - n_r^2} = \sqrt{1 - \left(1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_{\rm B})}\right)} \approx \frac{\omega_p}{\omega}$$

<u>If $\omega < \gamma\omega_p$, $\theta_b > 1/\gamma$ and the beaming effect is suppressed.</u>

<u>Below the frequency $\gamma\omega_p$, the synchrotron spectrum will be cut off because of the suppression of beaming. This is called the Razin-Tsytovich effect.</u>

As frequencies increase, θ_b decreases until it becomes of order of the vacuum value $1/\gamma$, and therefore the vacuum results apply.

Therefore, the plasma medium effect is unimportant when $\omega \gg \gamma\omega_p$.

Homework (due: 11/23)

- The shift $\Delta I = I' - I$ of any photon spectrum at a given frequency due to single scattering from a Maxwell distribution of electrons is given by the formula (Sunyaev & Zeldovich, Comments on Astrophysics and Space Physics, 4, 173)

$$\Delta I_\nu = I'(\nu) - I(\nu) = y\nu \frac{d}{d\nu}\left[\nu^4 \frac{d}{d\nu}\left(\nu^{-3} I(\nu)\right)\right] \quad \text{where} \quad \begin{aligned} y &\equiv \frac{kT_e}{m_e c^2}\tau \\ \tau &= \int n_e \sigma_{\mathrm{T}} d\ell \\ \sigma_{\mathrm{T}} &= 6.65 \times 10^{-25}\,\mathrm{cm}^2 \end{aligned}$$

This is valid for $y \ll 1$ and $kT_e \gg h\nu$. Here, T_e is the electron temperature.

(a) Calculate the shift ΔI_ν for the blackbody spectrum with a radiation temperature T_r.

Hint: $$I(\nu) = B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT_r} - 1}$$

Define $x = h\nu/kT_r$, represent the equation in terms of x, take derivatives, and then give your results in terms of x, y, and $I(\nu)$.

The result should be $$\Delta I_\nu = yI(\nu)\frac{xe^x}{e^x - 1}\left(x\frac{e^x + 1}{e^x - 1} - 4\right)$$

(b) Find the limiting expressions for the Rayleigh-Jeans regime ($x \ll 1$) and Wien regime ($x \gg 1$). Show that the results are given by

$$\begin{aligned} \Delta I_\nu &= -2yI(\nu) & \text{for } x \ll 1 \\ &= y\,(x-4)\,xI(\nu) & \text{for } x \gg 1 \end{aligned}$$

Compare the result for $x \ll 1$ with that given in this lecture note.

(c) Sketch or plot a log-log plot of $I(\nu)$ and $I'(\nu)$, for instance, for $y = 0.15$ and $T_r = 2.7$ K.