Astrophysics [Part I]

Lecture 1
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Overview

Radiative processes

- link astrophysical systems with astronomical observables
- cover many areas of physics and astrophysics (electrodynamics, quantum mechanics, statistical mechanics, relativity...)

Textbooks

- Radiative Processes in Astrophysics (George Rybicki & Alan Lightman)
- 천체물리학: 복사와 기체역학 (구본철, 김웅태)
- The Physics of Interstellar and Intergalactic Medium (Bruce T. Draine)
- The Physics of Astrophysics, Volume 1 Radiation (Frank H. Shu)

Chapter 1.
Fundamentals of Radiative Transfer

Electromagnetic Radiation

Particle/wave duality

classically: electromagnetic waves

- speed of light: $c = 3 \times 10^{10} \mathrm{cm\ s}^{-1}$
- wavelength and frequency: $\lambda = c/\nu$

quantum mechanically: photons

- quanta: massless, spin-1 particles (boson)
- Plank: $E = h\nu = hc/\lambda \; (h = 6.625 \times 10^{-27} \; \mathrm{ergs})$
- Einstein: $E^2 = (m_\gamma c^2)^2 + (pc)^2$

$$p = E/c$$

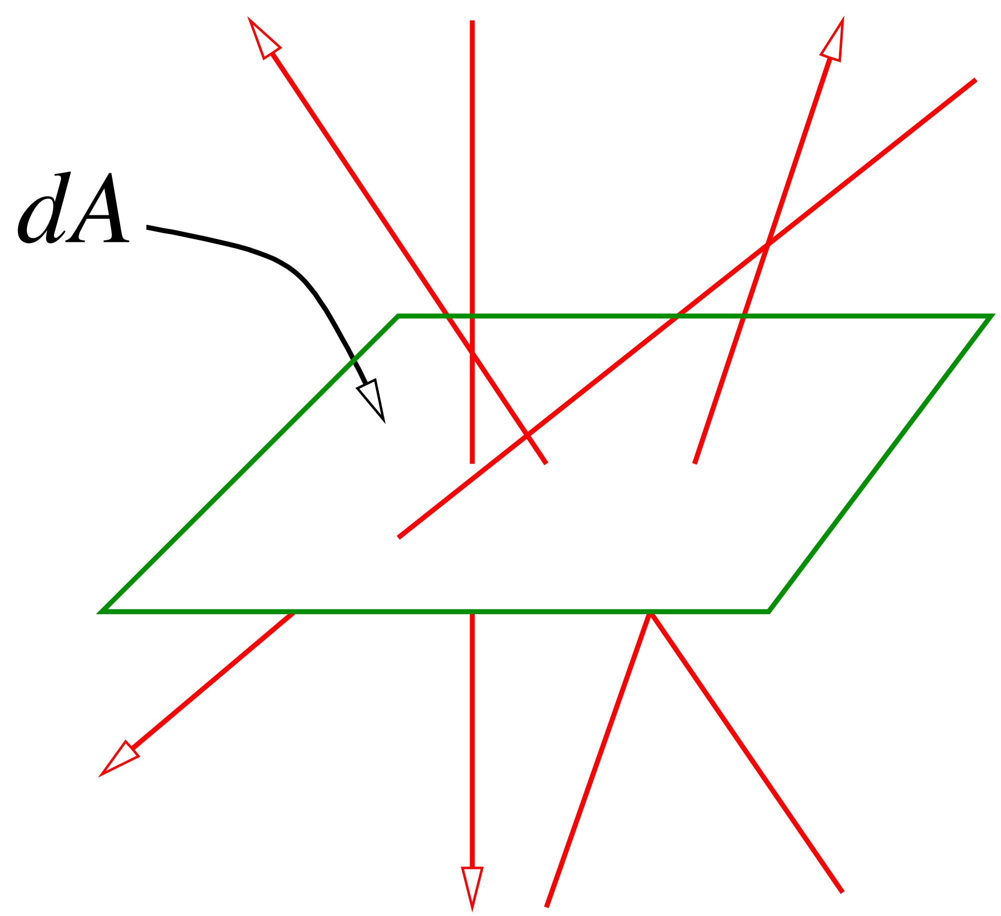
Energy flux (or energy flux density)

Definition

- Energy flux, F, is defined as the energy dE passing through an element of area dA in time interval dt

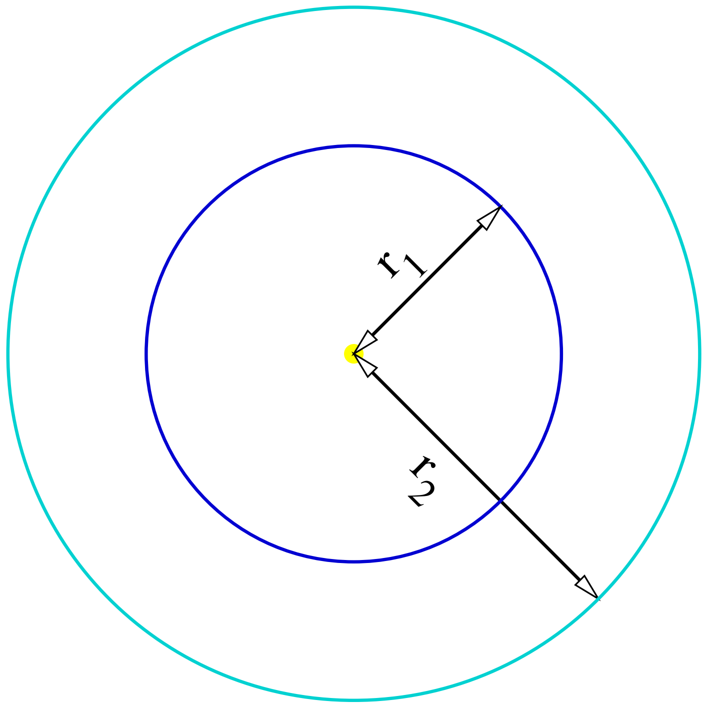
$$dE = F dA dt$$

- F depends on the orientation of elements of dA and the frequency (or wavelength).
- Unit: $\mathrm{erg\ cm^{-2}\ s^{-1}}$



Inverse Square Law

Flux from an isotropic radiation source, i.e., a source emitting equal amounts of energy in all directions.



- Because of energy conservation, flux through two shells around the source must be the same.

$$4\pi r_1^2 F(r_1) = 4\pi r_2^2 F(r_2)$$

- Therefore, we obtain the inverse square law.

$$F = \frac{\text{const.}}{r^2}$$

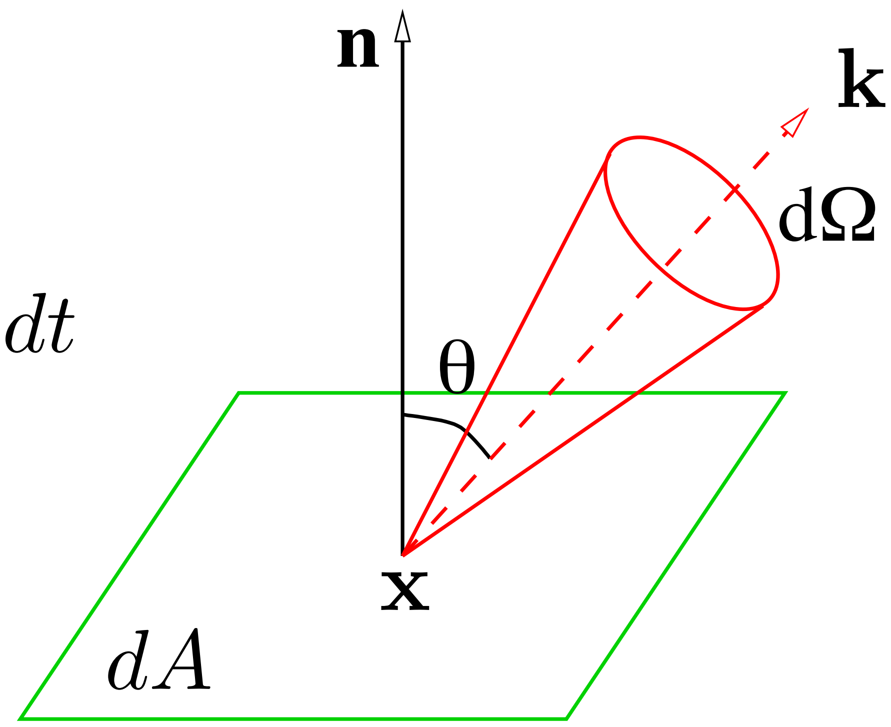
(Specific) Intensity or (Surface) Brightness

Flux = a measure of the energy carried by all rays passing through a given area

Intensity = the energy carried along by individual rays.

- Let dE_ν be the amount of radiant energy which crosses in time dt the area dA with unit normal $\mathbf{n}$ in a direction within solid angle $d\Omega$ centered about $\mathbf{k}$ with photon frequency ν between $\nu + d\nu$.
- The monochromatic specific intensity I_ν is then defined by the equation.

$$\begin{aligned} dE_\nu &= I_\nu(\mathbf{k}, \mathbf{x}, t) dA_\mathbf{k} d\Omega d\nu dt \\ &= I_\nu(\mathbf{k}, \mathbf{x}, t) \cos\theta dA d\Omega d\nu dt \end{aligned}$$



- area normal to $\mathbf{k}$: $dA_\mathbf{k} = dA\cos\theta$
- Unit: $\mathrm{erg\ s^{-1} cm^{-2} sr^{-1} Hz^{-1}}$

Note

Moments of intensity

- intensity : scalar (amplitude of the differential flux)
- differential flux : vector
- momentum flux (radiation pressure) : tensor

Intensity can be defined as per wavelength interval.

$$\begin{gathered} I_\nu |d\nu| = I_\lambda |d\lambda| \\ \nu I_\nu = \lambda I_\lambda \end{gathered} \quad \leftarrow \left| \frac{d\nu}{d\lambda} \right| = \frac{c}{\lambda^2} = \frac{\nu}{\lambda}$$

Integrated intensity is defined as the intensity over all frequencies.

$$I = \int_0^\infty I_\nu d\nu = \int_0^\infty I_\lambda d\lambda$$

Constancy of Specific Intensity in Free Space

Consider a bundle of rays and any two points along the rays. Construct areas dA_1 and dA_2 normal to the rays at these points.

Consider the energy carried by the rays passing through both areas. Because energy is conserved,

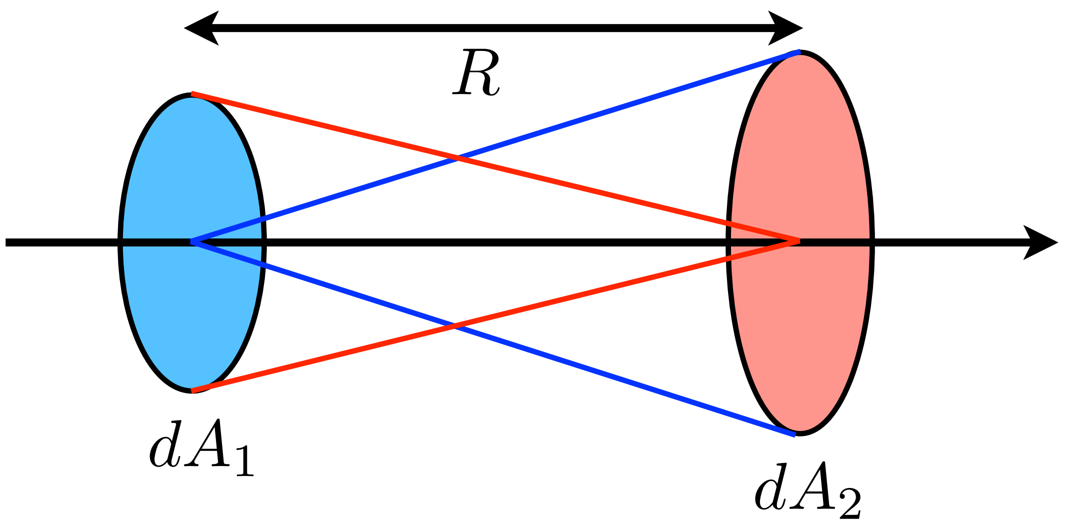
$$dE_1 = I_{\nu_1} dA_1 dt d\Omega_1 d\nu_1 = dE_2 = I_{\nu_2} dA_2 dt d\Omega_2 d\nu_2$$

Here, $d\Omega_1$ is the solid angle subtended by dA_2 at dA_1 and so forth.

$$d\Omega_1 = dA_2/R^2$$

$$d\Omega_2 = dA_1/R^2 \rightarrow I_{\nu_1} = I_{\nu_2}$$

$$d\nu_1 = d\nu_2$$



radiative transfer equation in free space:

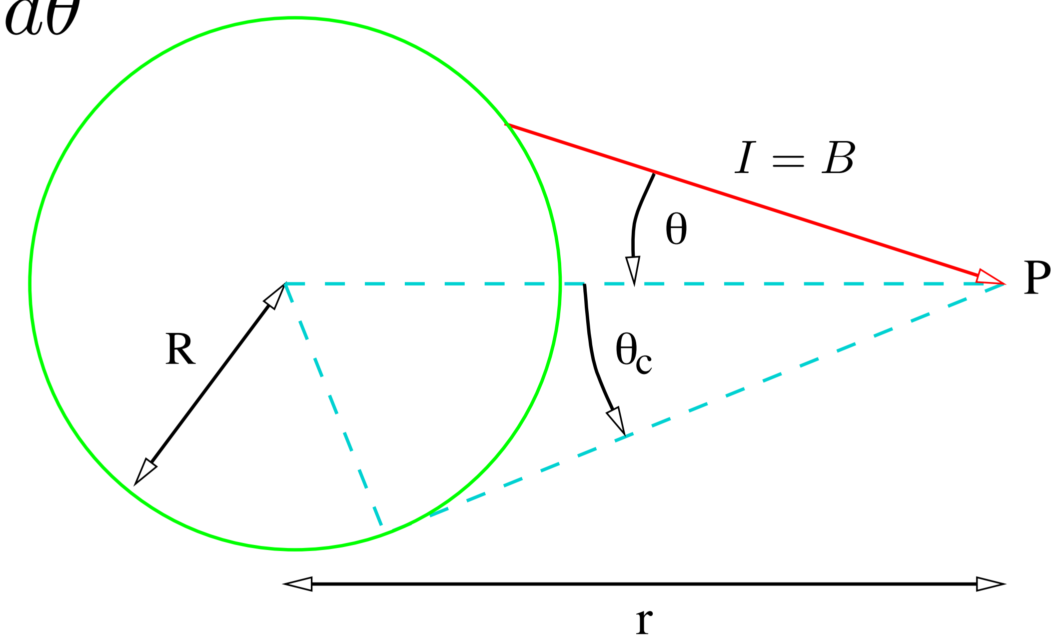
$$\frac{dI_\nu}{ds} = 0$$

Inverse Square Law for a Uniformly Bright Sphere

Let's calculate the flux at P from a sphere of uniform brightness B

$$F = \int I \cos\theta d\Omega = B \int_0^{\pi} d\phi \int_0^{\theta_c} \cos\theta \sin\theta d\theta$$

$$= \pi B(1 - \cos^2\theta_c) = \pi B \sin^2\theta_c$$



$$\sin\theta_c = \frac{R}{r} \rightarrow F = \pi B \left(\frac{R}{r}\right)^2$$

Therefore, there is no conflict between the constancy of intensity and the inverse square law.

Note

- The flux at a surface of uniform brightness B is $F = \pi B$.
- For stellar atmosphere, the astrophysical flux is defined by F/π.

(Specific) Energy Density

Consider a bundle of rays passing through a volume element dV in a direction Ω.

Then, the energy density per unit solid angle is defined by

$$dE = u_\nu(\Omega) dV d\Omega d\nu$$

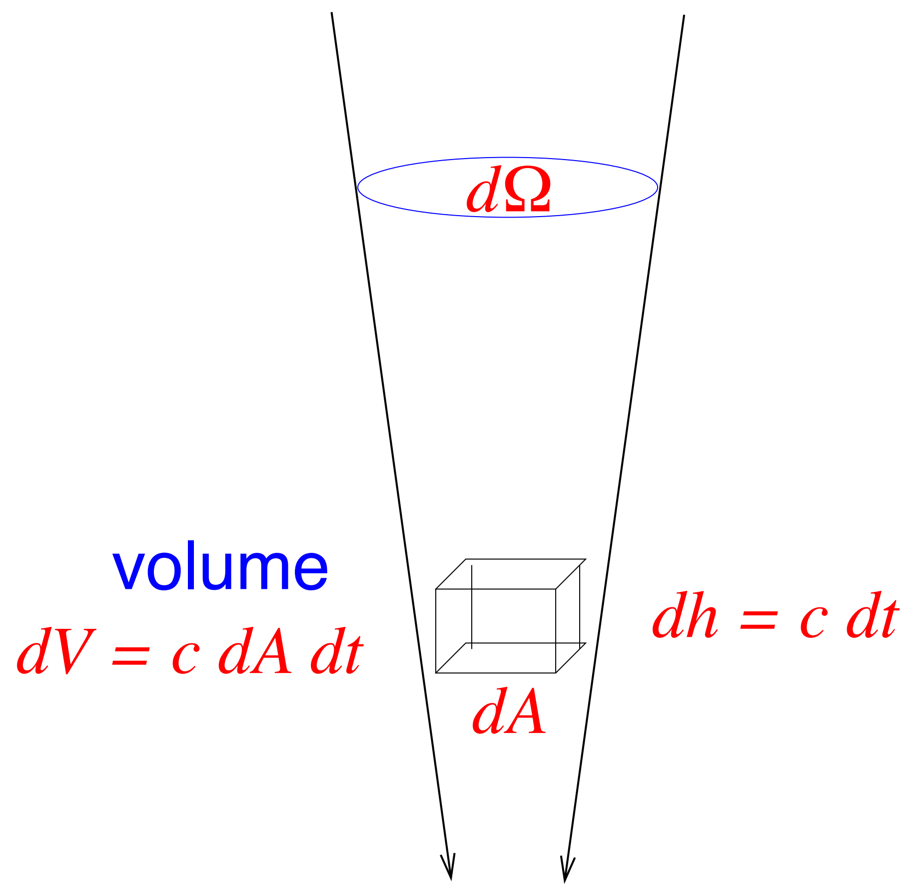
Since radiation travels at velocity c, $dV = dA(cdt)$

the definition of the intensity

$$dE = I_\nu dA dt d\Omega d\nu$$

Therefore,

$$u_\nu(\Omega) = I_\nu(\Omega)/c$$



Energy Density and Mean Intensity

Integrating over all solid angle, we obtain

$$u_\nu = \int u_\nu(\Omega) d\Omega = \frac{1}{c} \int I_\nu d\Omega$$

Mean intensity is defined by

$$J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$$

Then, the energy density is

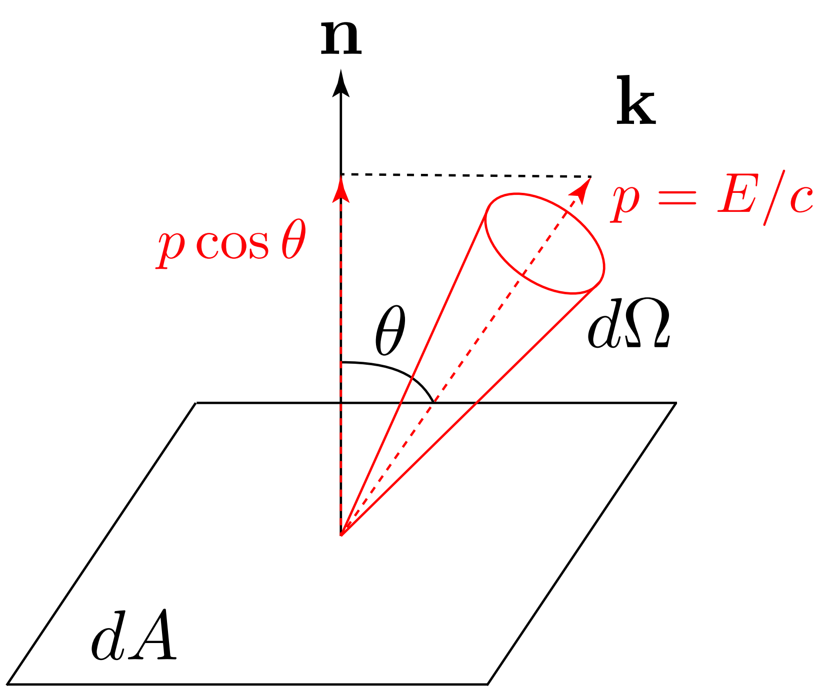
$$u_\nu = \frac{4\pi}{c} J_\nu$$

Total energy density is obtained by integrating over all frequencies.

$$u = \int u_\nu d\nu = \frac{4\pi}{c} \int J_\nu d\nu$$

Momentum Flux: Radiation Pressure (due to absorption)

- momentum of a photon: $p = E/c$
- force: $F = \dfrac{\Delta p}{\Delta t}$
- radiation pressure = force per unit area
- radiation pressure due to energy flux propagating within solid angle $d\Omega$ and with frequency between $(\nu, \nu + d\nu)$:



$$p_\nu d\Omega d\nu = \frac{\Delta F_\nu}{\Delta A} = \frac{1}{\Delta A}\frac{\Delta E_\nu/c}{\Delta t}\cos\theta$$
$$= \frac{1}{c} I_\nu \cos^2\theta d\Omega d\nu$$

Integrating over solid angle,

$$P\nu = \frac{1}{c}\int I_\nu \cos^2\theta d\Omega$$

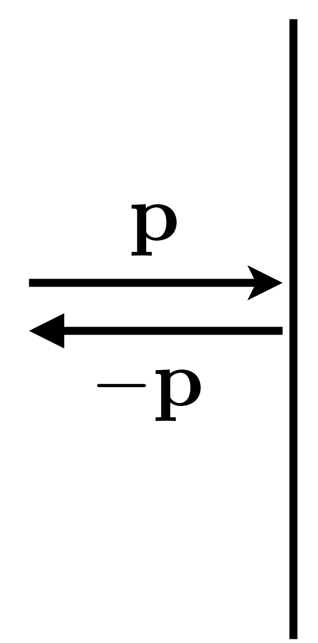
The first cosine factor is due to the area normal to $\mathbf{k}$ and the second one is due to the projection of the differential flux vector to the normal vector $\mathbf{n}$.

Radiation Pressure (due to reflection)

Consider a reflecting enclosure containing an isotropic radiation field.

Each photon transfers twice its normal component of momentum on reflection. Thus, we have

$$
\begin{aligned}
p_\nu &= \frac{2}{c}\int I_\nu \cos^2\theta d\Omega \\
&= \frac{2}{c}J_\nu \int \cos^2\theta d\Omega \\
&= \frac{4\pi}{c}J_\nu \int_0^1 \mu^2 d\mu \\
&= \frac{1}{3}u_\nu
\end{aligned}
$$



$$\Delta p = 2p$$

The angular integration yields $p = \frac{1}{3}u$

Radiative Transfer Equation

As a ray passes through matter, energy may be added, subtracted, or scattered from it by emission, absorption, or scattering.

The intensity will not in general remain constant.

We need to derive the radiative transfer equation.

Emission coefficient and Emissivity

- (monochromatic) spontaneous emission coefficient j_ν = the energy emitted per unit time per unit solid angle and per unit volume

$$dE = j_\nu dV d\Omega dt d\nu \quad (j_\nu : \mathrm{erg\ cm^{-3}\ s^{-1}\ sr^{-1}\ Hz^{-1}})$$

- (angle integrated) emissivity ϵ_ν = the energy emitted spontaneously per unit frequency per unit time per unit mass. For isotropic emission,

$$dE = \epsilon_\nu \rho dV dt d\nu \frac{d\Omega}{4\pi} \quad (\epsilon_\nu : \mathrm{erg\ g^{-1}\ s^{-1}\ Hz^{-1}})$$

- Then, we obtain $\quad j_\nu = \dfrac{\epsilon_\nu \rho}{4\pi} \quad \text{or} \quad \displaystyle\int j_\nu d\Omega = \epsilon_\nu \rho$

- In going a distance ds, a beam of cross section dA travels through a volume $dV = dAds$. Thus the intensity added to the beam is by spontaneous emission is:

$$dI_\nu = j_\nu ds$$

Absorption Coefficient

Consider the medium with particle number density $n\ (\mathrm{cm}^{-3})$, each having effective absorbing area (cross section) $\sigma_\nu\ (\mathrm{cm}^2)$

- number of absorbers = $ndAds$
- total absorbing area = $n\sigma_\nu dAds = dA_{\mathrm{abs}}$

energy taken out of beam

$$-dI_\nu dAd\Omega dtd\nu = I_\nu(n\sigma_\nu dAds)d\Omega dtd\nu$$
$$\to dI_\nu = -n\sigma_\nu I_\nu ds = -\alpha_\nu I_\nu ds$$

Absorption coefficient $\alpha_\nu\ (\mathrm{cm}^{-1})$ is defined by

$$\begin{aligned}\alpha_\nu &= n\sigma_\nu \\ &= \rho\kappa_\nu\end{aligned}$$

where $\rho\ (\mathrm{g\ cm}^{-3})$ is the mass density and $\kappa_\nu\ (\mathrm{cm}^2\ \mathrm{g}^{-1})$ is called the mass absorption coefficient or the opacity coefficient.

The Radiative Transfer Equation

Without scattering term,

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

Including scattering term, we obtain an integrodifferential equation.

$$\boldsymbol{\Omega} \cdot \nabla I_\nu = -\alpha_\nu^{\mathrm{ext}} I_\nu + j_\nu + \alpha_\nu^{\mathrm{sca}} \int \phi_\nu(\boldsymbol{\Omega}, \boldsymbol{\Omega}') I_\nu(\boldsymbol{\Omega}') d\Omega'$$

- scattering coefficient $\alpha_\nu^{\mathrm{sca}}$ (cm^{-1})
- scattering phase function $\int \phi_\nu(\boldsymbol{\Omega}, \boldsymbol{\Omega}') d\Omega = 1$
- for isotropic scattering $\phi_\nu(\boldsymbol{\Omega}, \boldsymbol{\Omega}') = \dfrac{1}{4\pi}$

Stimulated emission:

- We consider "absorption" to include both "true absorption" and stimulated emission, because both are proportional to the intensity of the incoming beam (unlike spontaneous emission).

Emission Only & Absorption Only

For pure emission, $\alpha_\nu = 0$

$$\frac{dI_\nu}{ds} = j_\nu \quad \rightarrow \quad I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')ds'$$

- The brightness increase is equal to the emission coefficient integrated along the line of sight.

For pure absorption, $j_\nu = 0$

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \quad \rightarrow \quad I_\nu(s) = I_\nu(s_0)\exp\left[-\int_{s_0}^{s} \alpha_\nu(s')ds'\right]$$

- The brightness decreases along the ray by the exponential of the absorption coefficient integrated along the line of sight.

Optical Depth & Source Function

Optical depth:

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')ds' = \int_{s_0}^{s} n(s')\sigma_\nu ds' = \int_{s_0}^{s} \rho(s')\kappa_\nu ds'$$

- Then, for pure absorption, $I_\nu(s) = I_\nu(s_0)e^{-\tau_\nu}$
- A medium is said to be optically thick if $\tau_\nu > 1$
- A medium is said to be optically thin if $\tau_\nu < 1$

Source function: $S_\nu = \dfrac{j_\nu}{\alpha_\nu}$

- The radiative transfer equation can now be written

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

Mean Free Path

- From the exponential absorption law, the probability of a photon absorbed between optical depths τ_ν and $\tau_\nu + d\tau_\nu$:

$$|dI_\nu| = \left|\frac{dI_\nu}{d\tau_\nu}\right| d\tau_\nu \;\; \& \;\; |dI_\nu| \propto P(\tau_\nu) d\tau_\nu \quad \rightarrow \quad P(\tau_\nu) = e^{-\tau_\nu}$$

- The mean optical depth traveled is thus equal to unity:

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu P(\tau_\nu) d\tau_\nu = \int_0^\infty \tau_n u e^{-\tau_\nu} d\tau_\nu = 1$$

- The mean free path is defined as the average distance a photon can travel through an absorbing material without being absorbed. In a homogeneous medium, the mean free path is determined by

$$\langle \tau_\nu \rangle = \alpha_\nu \ell_\nu = 1 \quad \rightarrow \quad \ell_\nu = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}$$

- A local mean path at a point in an inhomogeneous material can be also defined.

Formal Solution of the RT equation

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

$$\frac{d}{d\tau_\nu}\left(e^{\tau_\nu} I_\nu\right) = e^{\tau}_{\nu} S_\nu$$

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu$$

The solution is easily interpreted as the sum of two terms:

- the initial intensity diminished by absorption
- the integrated source diminished by absorption.

Relaxation

For a constant source function, the solution becomes

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right) \end{aligned}$$

"Relaxation"

- $I_\nu > S_\nu \quad \rightarrow \quad \dfrac{dI_\nu}{d\tau_\nu} < 0, \quad \text{then } I_\nu \text{ tends to decrease along the ray}$

  $I_\nu < S_\nu \quad \rightarrow \quad \dfrac{dI_\nu}{d\tau_\nu} > 0, \quad \text{then } I_\nu \text{ tends to increase along the ray}$
- The source function is the quantity that the specific intensity tries to approach, and does approach if given sufficient optical depth.

$$\text{As} \quad \tau_\nu \rightarrow \infty, \quad I_\nu \rightarrow S_\nu$$

Radiation Force

- radiation flux vector in direction $\mathbf{n}$:

$$\mathbf{F}_\nu = \int I_\nu \mathbf{n} d\Omega$$

- the vector momentum per unit area per unit time per unit path length absorbed by the medium is

$$\mathsf{F} = \frac{1}{c} \int \alpha_\nu \mathbf{F}_\nu d\nu \quad \leftarrow \quad n\sigma_\nu dA ds \frac{\mathbf{F}_\nu}{c}$$

- This is the force per unit volume imparted onto the medium by the radiation field. The force per unit mass of material is given by

$$\mathsf{f} = \frac{\mathsf{F}}{\rho} = \frac{1}{c} \int \kappa_\nu \mathbf{F}_\nu d\nu$$

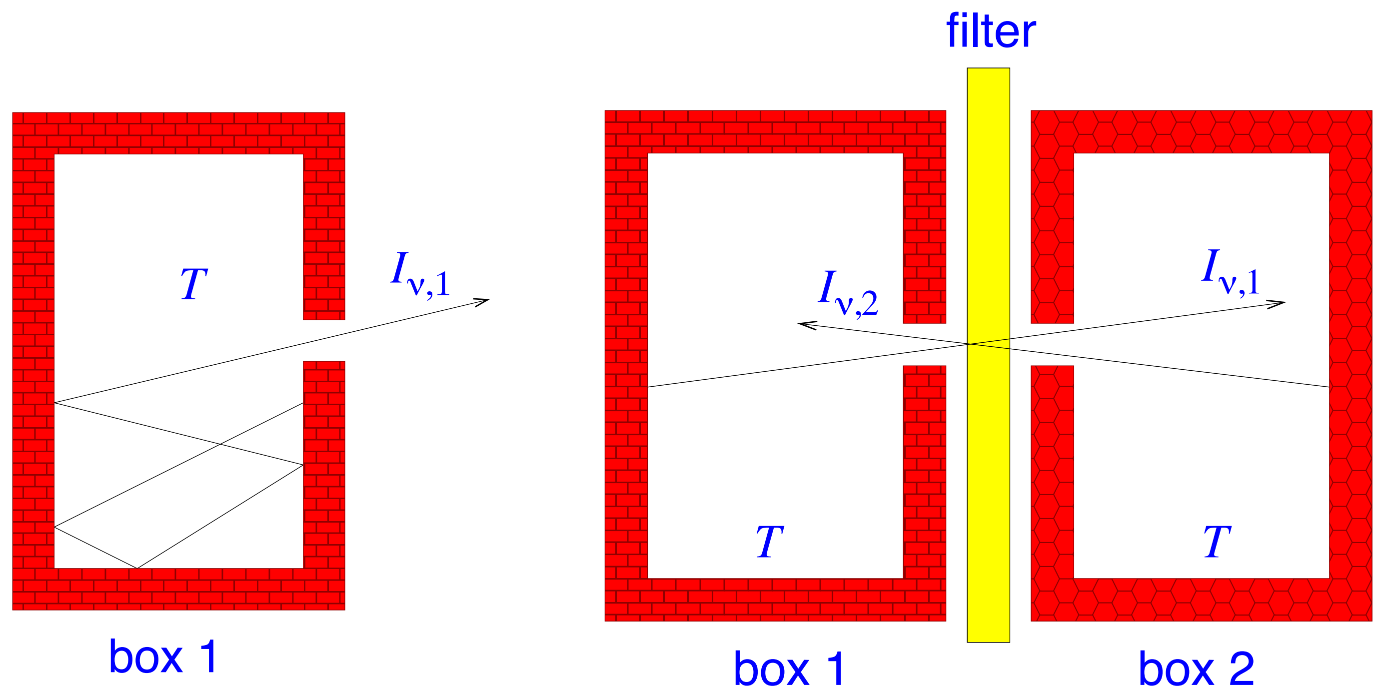
- Home work: derive the Eddington luminosity (problem 1.4)

Thermal equilibrium

- Equilibrium means a state of balance.
- Thermal equilibrium refers to steady states of temperature, which may be spatial or temporal.
- In a state of (complete) thermodynamic equilibrium, no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system.
- When the material is in thermodynamic equilibrium, and ***only the radiation field is allowed to depart from its TE***, we refer to the state of the system as being in local thermodynamic equilibrium (LTE).
- A blackbody is an idealized physical body that absorbs all incident radiation regardless of frequency or angle of incidence (i.e., perfect absorber).
- A blackbody in thermal equilibrium emits the blackbody radiation, which is itself in thermal equilibrium.
- Thermal radiation is radiation emitted by “matter” in thermal equilibrium.

Universal function

- In equilibrium, radiation field in box doesn't change.



- Now, consider another enclosure (box 2), also at the same temperature, but made of different material or shape. If their escaping intensities are different, energy will flow spontaneously between the two boxes. This violets the second law of thermodynamics. Therefore, the escaping intensity should be a universal function of T and should be isotropic. The universal function is called the Planck function $B_\nu(T)$.

Kirchhoff Law (천문학백과사전)

키르히호프 법칙(Kirchhoff Law) [편집 | 원본 편집]

키르히호프가 1859년에 열복사에 대한 법칙을 발표하였다. 키르히호프의 법칙은 고체상태 물체의 방출율과 흡수율의 비는 물체의 종류 또는 특성에는 관계가 없고 온도에만 의존하는 흑체복사로 주어진다는 것이다.

주변의 빛과 열적 평행상태에 있는 어떤 물체를 고려한다면, 그 물체에 흡수된 빛의 스펙트럼은 물체에서 방출되는 스펙트럼과 같아야 한다. 그렇지 않다면 열적 평행상태에 도달할 수 없다. 주파수 v 에서 빛의 스펙트럼을 $S_\nu(T)$ 라고 표현하고 물체의 흡수율은 κ_ν, 그리고 방출율은 ϵ_ν 이라고 나타낸다면, $\epsilon_\nu = \kappa_\nu S_\nu(T)$ 또는 $\epsilon_\nu/\kappa_\nu = S_\nu(T)$ 을 만족해야 한다. 이때 흡수율 κ_ν 과 방출율 ϵ_ν 은 물체의 특성에만 관계된다. 반면에 $S_\nu(T)$ 은 복사광의 특성에만 관계되기 때문에 $S_\nu(T)$ 은 물체와는 무관해야 한다. 즉, 방출율과 흡수율의 비는 물질의 특성과 무관하게 일정해야 한다. 이 식은 흡수를 잘 하는 물체는 방출도 잘한다는 것을 의미한다.

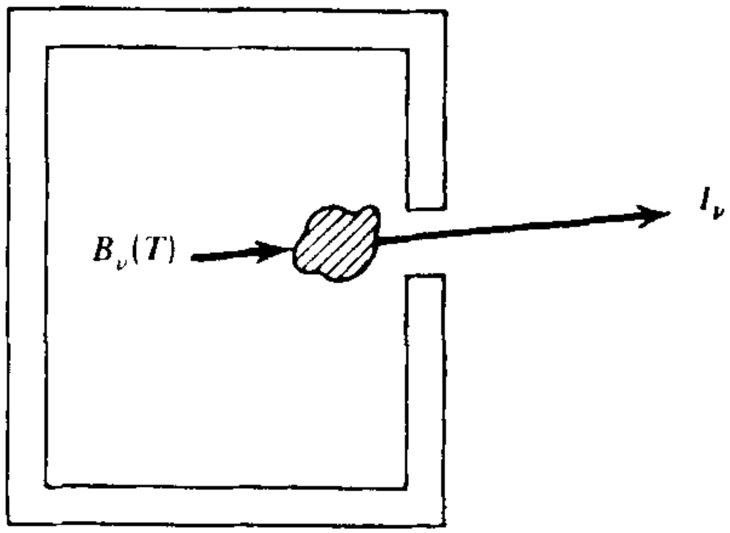
모든 주파수의 빛을 완벽하게 흡수하는 이상적인 흑체는 존재하지 않는다. 하지만, 모든 입사광을 완벽하게 흡수하는 가상의 흑체를 생각한다면 $\kappa_\nu = 1$ 이 될 것이므로 이 물체에서 방출되는 스펙트럼은 $\epsilon_\nu = S_\nu(T)$ 가 되어야 한다. 따라서, $S_\nu(T)$는 흑체에서 방출되는 스펙트럼인 $B_\nu(T)$ 이 되어야 한다. 앞에서 언급했듯이 방출율과 흡수율의 비 ϵ_ν/κ_ν 는 물체의 특성과 무관해야 하므로 물체가 흑체이든 아니든 상관없이 방출율과 흡수율의 비 $S_\nu = \epsilon_\nu/\kappa_\nu$ 는 항상 흑체복사 스펙트럼이 되어야 한다. 즉, 물체의 특성과 관계없이 아래의 방정식을 만족해야 한다.

$$\epsilon_\nu/\kappa_\nu = B_\nu(T).$$

이 방정식이 키르히호프의 법칙이다. 천문학에서 키르히호프의 법칙은 별 빛이 성간먼지에 흡수된 후 재 방출되는 빛을 계산하기 위해 사용된다.

Kirchhoff's Law

- Consider an element of some thermally emitting material at temperature T.
- Put this into a blackbody enclosure at the same temperature.
- Let the source function of the material be S_ν .



$$\text{if } S_\nu > B_\nu \quad \rightarrow \quad I_\nu > B_\nu$$
$$\text{if } S_\nu < B_\nu \quad \rightarrow \quad I_\nu < B_\nu$$
$$\therefore \; S_\nu = B_\nu$$

- But, the presence of the material cannot alter the radiation, since the new configuration is also a blackbody enclosure at T.
- Kirchhoff's Law: in LTE, the ratio of the emission coefficient to the absorption coefficient is a function of temperature and frequency only.

$$j_\nu = \alpha_\nu B_\nu(T) \;\rightarrow\; \text{Kirchhoff's Law}$$

$$\text{Note}: j_\nu = B_\nu(T) \;\; \text{if} \;\; \alpha_\nu = 1 \text{ (perfect absorber, i.e., blackbody)}$$

Implications of Kirchhoff's Law

- A good absorber is a good emitter, and a poor absorber is a poor emitter. A good reflector must be a poor absorber, and thus a poor emitter.
- It is not possible to thermally radiate more energy than a blackbody, at equilibrium.

$$j_\nu \leq B_\nu(T)$$

- The radiative transfer equation in LTE:

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T)$$

- Note:

blackbody radiation means $I_\nu = B_\nu(T)$

thermal radiation means $S_\nu = B_\nu(T)$

Thermal radiation becomes blackbody radiation only for optically thick media.

Thermodynamics

- First Law of Thermodynamics: heat is energy in transit.

$$dQ = dU + pdV$$

where Q is heat and U is total (internal) energy.

- Second Law of Thermodynamics: heat is entropy.

$$dS = \frac{dQ}{T}$$

where S is entropy.

See "Fundamentals of Statistical and Thermal Physics" (Federick Reif)

Thermodynamics of Blackbody Radiation

$$
\begin{aligned}
dS = \frac{dU}{T} + p\frac{dV}{T} &= \frac{1}{T}d(uV) + \frac{1}{3}\frac{u}{T}dV \\
&= \frac{V}{T}\frac{du}{dT}dT + \frac{u}{T}dV + \frac{1}{3}\frac{u}{T}dV \\
&= \frac{V}{T}\frac{du}{dT}dT + \frac{4}{3}\frac{u}{T}dV
\end{aligned}
$$

$$\therefore \frac{\partial S}{\partial T} = \frac{V}{T}\frac{du}{dT} \quad \text{and} \quad \frac{\partial S}{\partial V} = \frac{4u}{3T}$$

$$\frac{\partial^2 S}{\partial T \partial V} \quad \rightarrow \quad \frac{1}{T}\frac{du}{dT} = -\frac{4u}{3T^2} + \frac{4}{3T}\frac{du}{dT}$$

$$\boxed{\therefore \frac{du}{dT} = \frac{4u}{T} \quad \text{or} \quad \frac{du}{u} = 4\frac{dT}{T}}$$

- Stefan-Boltzmann law:

$$\mathrm{Stenfan-Boltzmann\ law:}\quad u(T) = aT^4 \;\leftarrow\; \log u = 4\log T + \log a$$

$$u(T) = \left(\frac{T}{3400\ K}\right)^4 \ \mathrm{erg\ cm^{-3}}$$

- total energy density:

$$u = \frac{4\pi}{c}\int B_\nu(T)d\nu = \frac{4\pi}{c}B(T)$$

- the integrated Planck function

$$B(T) = \int B_\nu(T)d\nu = \frac{ac}{4\pi}T^4 = \frac{\sigma}{\pi}T^4 \qquad \left(a = \frac{4\sigma}{c}\right)$$

- emergent flux (another form of the Stefan-Boltzmann law)

$$F = \int F_\nu d\nu = \pi\int B_\nu d\nu = \pi B(T)$$
$$= \sigma T^4$$

Stefan-Boltzmann constant

Entropy of Blackbody Radiation

- Entropy:

$$dS = \frac{V}{T} 4aT^3 dT + \frac{4a}{3} T^3 dV \quad \rightarrow \quad S = \frac{4}{3} a T^3 V$$

- Entropy density:

$$s = S/V = \frac{4}{3} T^3$$

- The law of adiabatic expansion for blackbody radiation:

$$T_{\rm ad} \propto V^{-1/3}$$

$$p_{\rm ad} \propto T_{\rm ad}^4 \propto V^{-4/3}$$

Thus, we have the adiabatic index for blackbody radiation:

$$\gamma = \frac{4}{3} \quad \leftarrow \quad pV^{\gamma} = \text{constant}$$

$$\gamma = \frac{5}{3} \quad \text{for a monatomic gas}$$
$$= \frac{7}{5} \quad \text{for a diatomic gas}$$

The Planck Spectrum (Quantum Mechanics)

How to calculate the Blackbody spectrum?

- Intensity spectrum is related to the energy density:

$$J_\nu = \frac{c}{4\pi} u_\nu$$

- Energy density = Number density of photon states

x Average energy of each state

- Density of photon states = **number of states per solid angle per volume per frequency**

The Planck Spectrum (Quantum Mechanics)

(1) Number density of photon state:

- Consider a photon propagating in direction $\mathbf{n}$ inside a box with dimensions L_x, L_y, L_z in x, y, z directions.
- wave vector: $\mathbf{k} = \frac{2\pi}{\lambda}\mathbf{n} = \frac{2\pi\nu}{c}\mathbf{n}$
- If each dimension of the box is much longer than a wavelength, the photon can be represented by standing wave in the box.
- number of nodes in each direction: $n_x = k_x L_x / 2\pi$
- number of node changes in a wave number interval (if $n_i \gg 1$):

$$\Delta n_x = \frac{L_x \Delta k_x}{2\pi}$$

- number of states in 3D wave vector element $\Delta k_x \Delta k_y \Delta k_z = d^3k$:

$$\Delta N = \Delta n_x \Delta n_y \Delta n_z = 2\frac{L_x L_y L_z d^3k}{(2\pi)^3} = 2\frac{V d^3k}{(2\pi)^3}$$

two independent polarizations

- the density of states (**number of states per solid angle per volume per frequency**):

$$d^3k = k^2 dk d\Omega = \frac{(2\pi)^3 \nu^2 d\nu d\Omega}{c^3}$$

$$\rightarrow \quad \rho_s = \frac{dN}{V d\nu d\Omega} = \frac{2\nu^2}{c^3}$$

(2) Average energy of each state:

- Each state may contain n photons of energy $h\nu$. The energy of the state is $E_n = nh\nu$.
- The probability of a state of energy E_n is proportional to $e^{-\beta E_n}$, where $\beta = (k_{\rm B}T)^{-1}$ and $k_{\rm B}$ is the Boltzmann's constant. (from statistical mechanics)
- Therefore, the average energy is:

$$\langle E \rangle = \frac{\Sigma_{n=0}^{\infty} E_n e^{-\beta E_n}}{\Sigma_{n=0}^{\infty} e^{-\beta E_n}} = -\frac{\partial}{\partial \beta} \ln \left(\Sigma_{n=0}^{\infty} e^{-\beta E_n} \right)$$

$$\Sigma_{n=0}^{\infty} e^{-\beta E_n} = \Sigma_{n=0}^{\infty} \left(e^{-\beta h\nu}\right)^n = \left(1 - e^{-\beta h\nu}\right)^{-1}$$

$$\langle E \rangle = \frac{h\nu e^{-\beta h\nu}}{1 - e^{-\beta h\nu}} = \frac{h\nu}{\exp(h\nu/k_{\mathrm{B}}T) - 1}$$

Average number of photons (occupation number):

$$n_\nu = \langle E \rangle / h\nu = \frac{1}{\exp(h\nu/k_{\mathrm{B}}T) - 1} \rightarrow \mathrm{Bose-Einstein\ statistics}$$

Energy density:

$$u_\nu(\Omega) = \rho_s \langle E \rangle = \frac{2h\nu^3/c^3}{\exp(h\nu/k_{\mathrm{B}}T) - 1}$$

Planck Law:

$$B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/k_{\mathrm{B}}T) - 1} \text{ or } B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda k_{\mathrm{B}}T) - 1}$$

Stefan-Boltzmann constant & Riemann zeta function

$$\text{Bose integral}: \quad I_n = \int_0^\infty dx \frac{x^n}{e^x - 1} = \int_0^\infty dx x^n \sum_{i=0}^\infty e^{-(i+1)x}$$

$$= \sum_{i=0}^\infty \frac{1}{(i+1)^{n+1}} \int_0^\infty dy y^n e^{-y} \quad (y \equiv (i+1)x)$$

$$= \zeta(n+1)\Gamma(n+1)$$

$$\int_0^\infty B_\nu(T) d\nu = (2h/c^2)(k_\mathrm{B} T/h)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

$$= \frac{2k_\mathrm{B}^4 T^4}{c^2 h^3} \zeta(4)\Gamma(4) = \frac{2k_\mathrm{B}^4 T^4}{c^2 h^3} \frac{\pi^4}{90} 6$$

$$= \frac{2\pi^4 k_\mathrm{B}^4}{15 c^2 h^3} T^4$$

$$\therefore \sigma = \frac{2\pi^5 k_\mathrm{B}^4}{15 c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{ deg}^{-4} \text{ s}^{-1}$$

$$a = \frac{8\pi^5 k_\mathrm{B}^4}{15 c^3 h^3} = 7.56 \times 10^{-15} \text{ erg cm}^{-3} \text{ deg}^{-4}$$

Rayleigh-Jeans Law & Wien Law

Rayleigh-Jeans Law (low-energy limit)

$$h\nu \ll k_{\rm B}T \ (\nu \ll 2 \times 10^{10}{\rm Hz}(T/1{\rm K})) \quad \rightarrow \quad I_\nu^{RJ}(T) = \frac{2\nu^2}{c^2} k_{\rm B}T$$

- Originally derived by assuming the classical equipartition energy

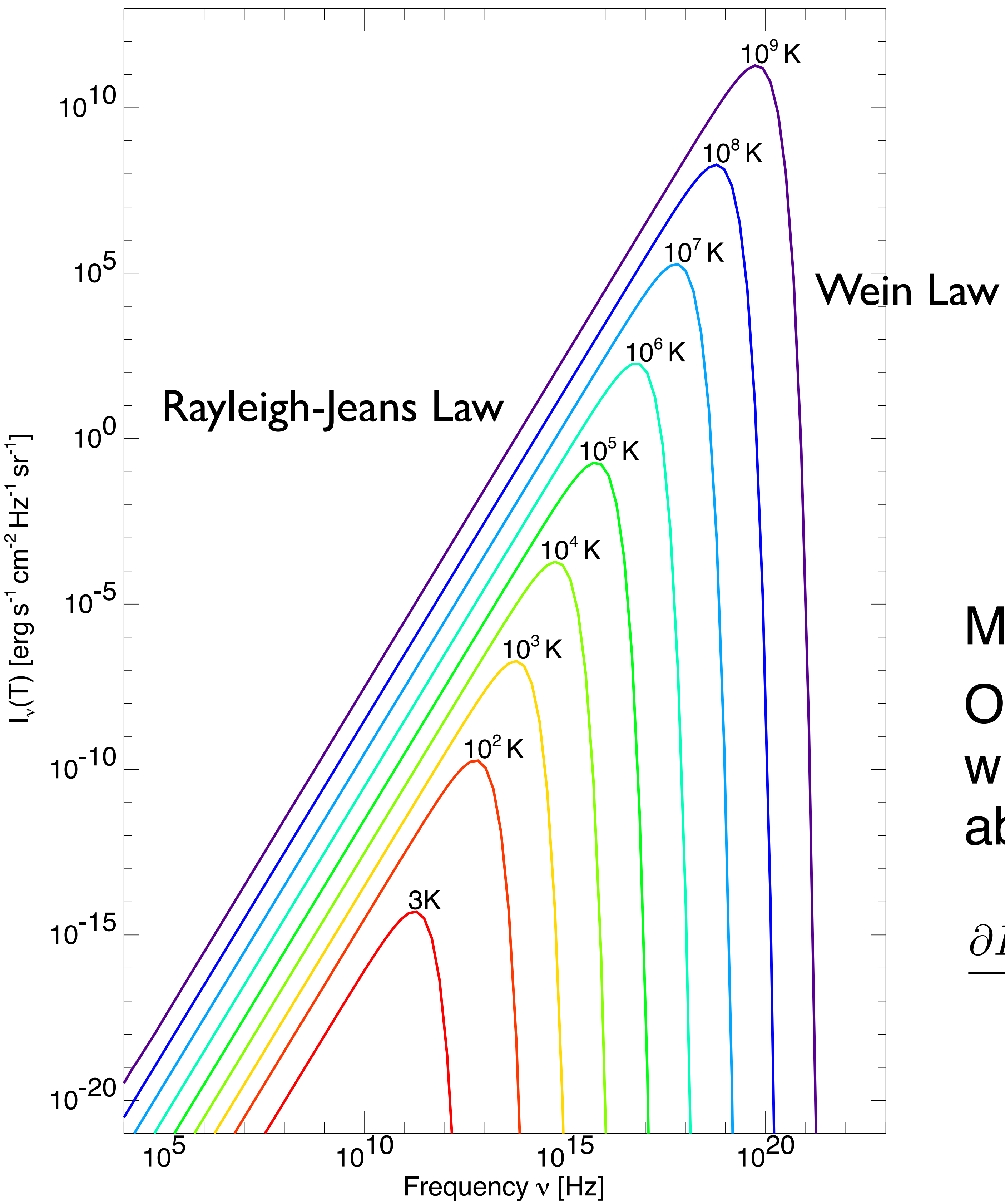
$$\langle E \rangle = 2 \times (1/2) k_{\rm B}T$$

- ultraviolet catastrophe: if the equation is applied to all frequencies, the total amount of energy would diverge. $\int \nu^2 d\nu \rightarrow \infty$

Wien Law (high-energy limit)

$$h\nu \gg k_{\rm B}T \quad \rightarrow \quad I_\nu^{W}(T) = \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{k_{\rm B}T}\right)$$

Monotonicity with Temperature



Monotonicity:

Of two blackbody curves, the one with higher temperature lies entirely above the other.

$$\frac{\partial B_\nu(T)}{\partial T} = \frac{2h^2\nu^4}{c^2 k_\mathrm{B} T^2} \frac{\exp(h\nu/k_\mathrm{T})}{[\exp(h\nu/k_\mathrm{B}T) - 1]^2} > 0$$

Wien Displacement Law

Frequency at which the peak occurs:

$$\frac{\partial B_\nu}{\partial \nu}|_{\nu=\nu_{\rm max}} = 0 \quad \rightarrow \quad x = 3(1-e^{-x}), \text{ where } x = h\nu_{\rm max}/k_{\rm B}T$$

$$h\nu_{\rm max} = 2.82 k_{\rm B}T \quad \text{or} \quad \frac{\nu_{\rm max}}{T} = 5.88 \times 10^{10} \text{ Hz deg}^{-1}$$

Wavelength at which the peak occurs:

$$\frac{\partial B_\lambda}{\partial \lambda}|_{\lambda=\lambda_{\rm max}} = 0 \quad \rightarrow \quad y = 5(1-e^{-y}), \text{ where } y = hc/(\lambda_{\rm max}k_{\rm B}T)$$

$$y = 4.97 \quad \text{and} \quad \lambda_{\rm max}T = 0.290 \text{ cm deg}$$

Be aware $\nu_{\rm max} \neq c/\lambda_{\rm max}$

Characteristic Temperatures

Brightness Temperature:

$$I_\nu = B_\nu(T_b)$$

- The definition is used especially in radio astronomy, where the RJ law is usually applicable. In the RJ limit,

$$T_b = \frac{c^2}{2\nu^2 k_{\mathrm{B}}} I_\nu$$

- radiative transfer equation in the RJ limit:

$$\frac{dT_b}{d\tau_\nu} = -T_b + T \quad (T = \text{the temperature of the material})$$

$$T_b = T_b(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu}) \quad \text{if } T \text{ is constant.}$$

- In the Wien region, the concept is not so useful.

Color Temperature:

- By fitting the spectrum to a blackbody curve without regarding to vertical scale (absolute intensity scale), a color temperature T_c is obtained.
- The color temperature correctly gives the temperature of a blackbody source of unknown absolute scale.

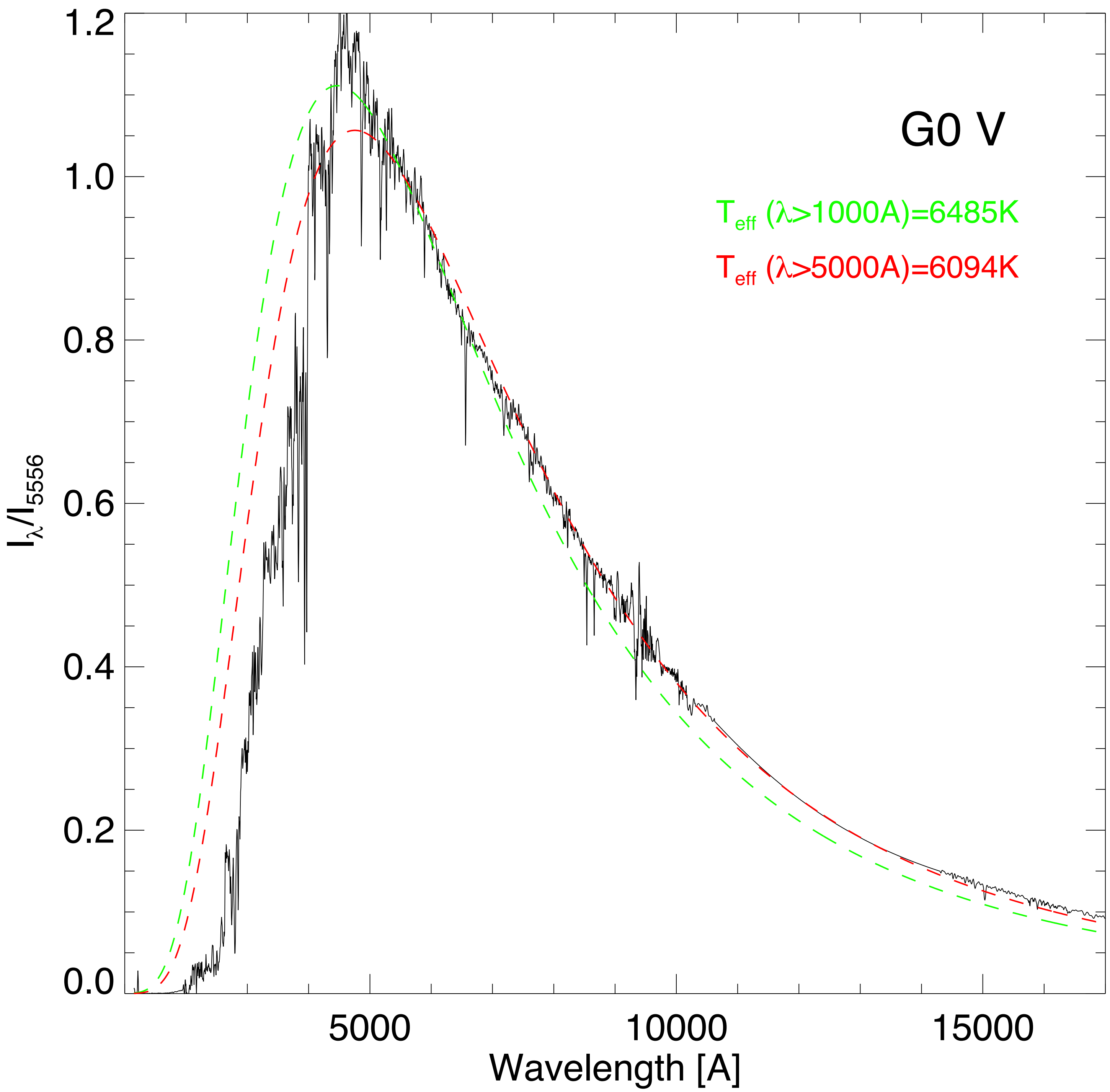
Effective Temperature:

- The effective temperature of a source is obtained by equating the actual flux F to the flux of a blackbody at temperature $T_{\rm eff}$.

$$F = \int \cos\theta I_\nu d\nu d\Omega = \sigma T_{\rm eff}^4$$

(Excitation Temperature)

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} \exp\left(-\frac{(E_u - E_l)}{k_{\rm B} T_{\rm ex}}\right)$$

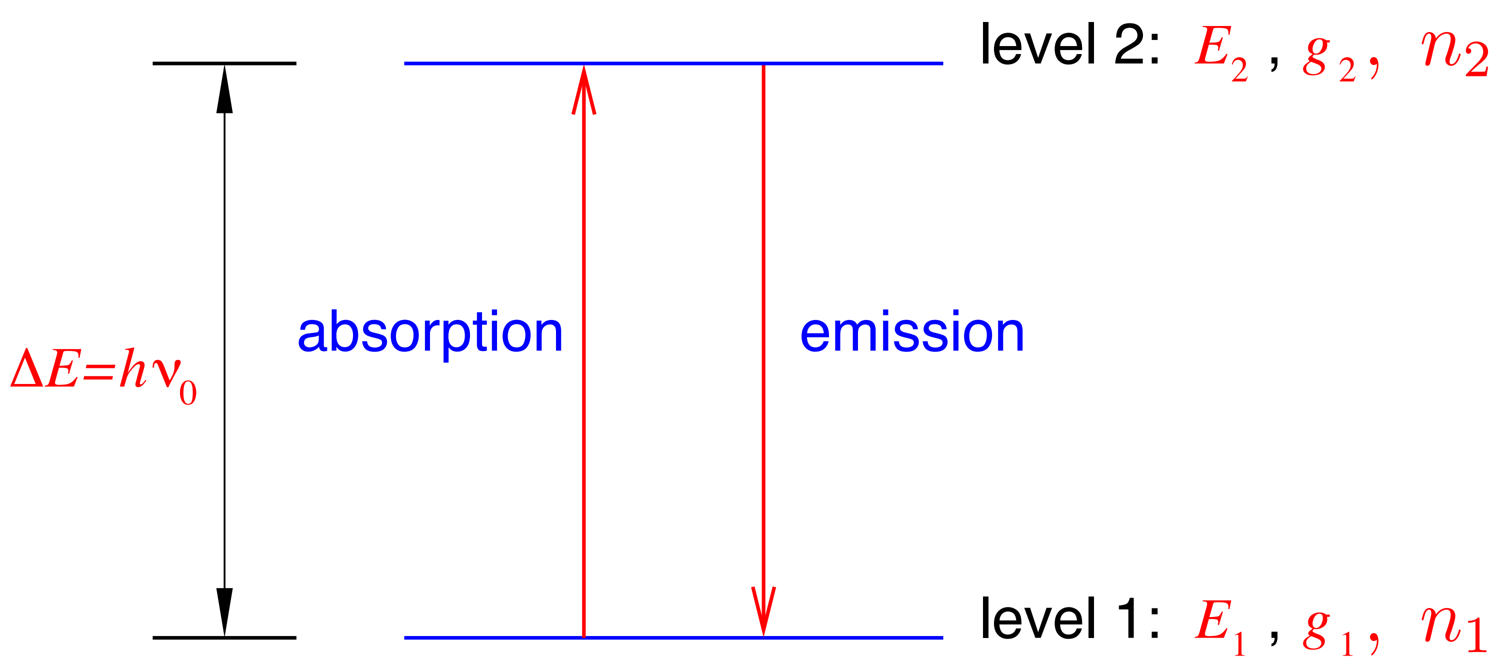


G0V spectrum (Pickles 1998, PASP, 110, 863)

(Note that the solar spectral type is G2V.)

The Einstein Coefficients

- Consider a system with two discrete energy levels (E_1, E_2) and degeneracies (g_1, g_2) . Let (n_1, n_2) be the number densities of atoms in levels (1, 2).

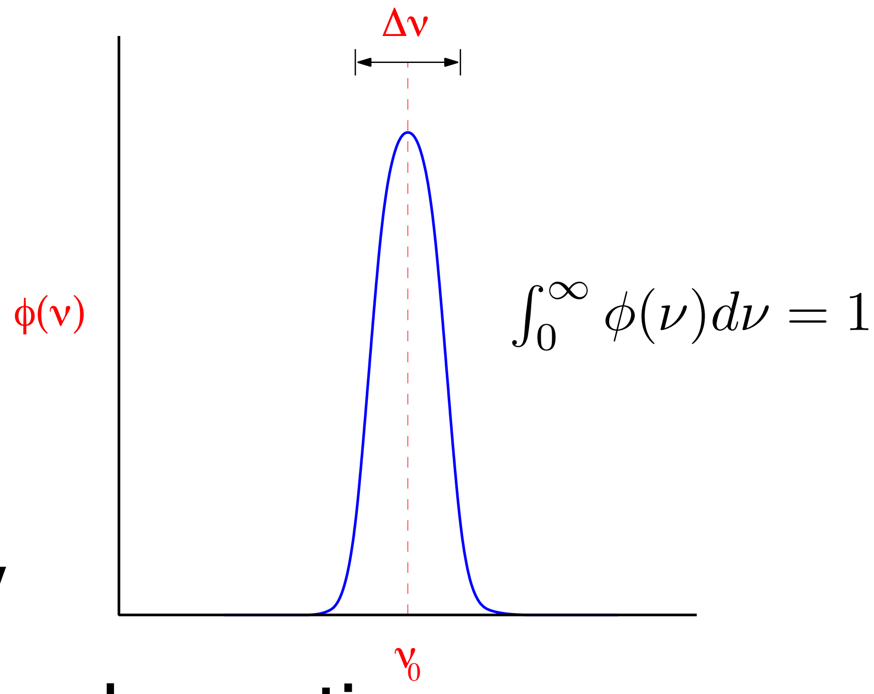


Spontaneous Emission from level 2 to level 1:

The Einstein A-coefficient A_{21} is the transition probability per unit time for spontaneous emission.

Absorption from level 1 to level 2 occurs in the presence of photons of energy $h\nu_0$.

- The absorption probability per unit time is proportional to the density of photons (or to the mean intensity) at frequency ν_0 .
- In general, the energy difference between the two levels have finite width which can be described by a line profile function $\phi(\nu)$.



- The Einstein B-coefficient B_{12} is defined by

$B_{12}\bar{J} =$ transition probability per unit time for absorption

where $\bar{J} = \int_0^\infty J_\nu \phi(\nu) d\nu.$

Stimulated emission from level 2 to level 1:

- Another Einstein B-coefficient is defined by

  $B_{21}\bar{J} =$ transition probability per unit time for stimulated emission.

- Einstein found that to derive Planck's law another process was required that was proportional to radiation field and caused emission of a photon.
- The stimulated emission is precisely coherent (same direction and frequency, etc) with the photon that induced the emission.

Note:

- Be aware that the energy density is often used instead of intensity to define the Einstein B-coefficients.

Relations between Einstein Coefficients

- In TE, total absorption rate = total emission rate:

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

$$\rightarrow \bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}$$

- Populations of the atomic states follow the Boltzmann distribution.

$$\frac{n_1}{n_2} = \frac{g_1 \exp(-E_1/k_\mathrm{B}T)}{g_2 \exp(-E_2/k_\mathrm{B}T)} = \frac{g_1}{g_2} \exp(h\nu_0/k_\mathrm{B}T).$$

- Therefore,

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_0/k_\mathrm{B}T) - 1}$$

- In TE, $J_\nu = B_\nu$ for all temperatures. We must have the following Einstein relations:

$$g_1 B_{12} = g_2 B_{21}$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

Einstein relations:

$$\boxed{\begin{aligned} g_1 B_{12} &= g_2 B_{21} \\ A_{21} &= \frac{2h\nu^3}{c^2} B_{21} \end{aligned}}$$

- If we can determine any one of the coefficients, these relations allow us to determine the other two.
- These connect atomic properties (A_{21}, B_{21}, B_{12}) and have no reference to the temperature. Thus, the relations must hold whether or not the atoms are in TE.
  - ✦ If the relations were only for TE, the relations would contain the dependence on T.
- Without stimulated emission, Einstein could not get Planck's law, but only Wien's law.
  - ✦ When $h\nu \gg k_\mathrm{B}T$ (Wien's limit), level 2 is very sparsely populated relative to level 1. Then, stimulated emission is unimportant compared to absorption.

Radiative Transfer Equation in terms of Einstein Coefficients

Emission coefficient:

- assumption: the line profile function of the emitted radiation is the same profile as for the absorption $\phi(\nu)$.
- energy emitted in volume dV, solid angle $d\Omega$, frequency range $d\nu$, and time dt:

$$j_\nu dV d\Omega d\nu dt = (h\nu/4\pi) n_2 A_{21} dV d\Omega \phi(\nu) d\nu dt$$

  Here, note that each atom emits an energy $h\nu$ distributed over solid angle 4π.

- Then, the emission coefficient is given by

$$\boxed{j_\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu)}$$

Absorption coefficient:

- energy absorbed out of a beam in frequency range $d\nu$, solid angle $d\Omega$, time dt, and volume dV

$$\alpha_\nu I_\nu dV\,dt d\Omega d\nu = (h\nu/4\pi) n_1 B_{12} I_\nu dV\,dt d\Omega \phi(\nu) d\nu$$

- Then, the absorption coefficient (uncorrected for stimulated emission) is given by

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \phi(\nu)$$

- What about the stimulated emission? It is proportional to the intensity, in close analogy to the absorption process. Thus, the stimulated emission can be treated as negative absorption. The absorption coefficient, corrected for stimulate emission, is

$$\boxed{\alpha_\nu = \frac{h\nu}{4\pi} \phi(\nu) \left(n_1 B_{12} - n_2 B_{21} \right)}$$

Source function:

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$$

- Using the Einstein relations, the absorption coefficient and source function can be written

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \left(1 - \frac{g_1 n_2}{g_2 n_1}\right)$$

$$\boxed{S_\nu = \frac{2h\nu^3}{c^2} \left(\frac{g_2 n_1}{g_1 n_2} - 1\right)^{-1} \quad \rightarrow \; \text{generalized Kirchhoff's law}}$$

Thermal Emission (LTE)

- If the matter is in TE with itself (but not necessarily with the radiation), we have the Boltzmann distribution. The matter is said to be in LTE.

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left(h\nu / k_\mathrm{B} T\right)$$

- In LTE, we obtain the absorption coefficient and the Kirchhoff's law:

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \left[1 - \exp\left(-\frac{h\nu}{k_\mathrm{B} T}\right)\right] \phi(\nu)$$

$$\boxed{S_\nu = B_\nu(T) \ \rightarrow \ \text{Kirchhoff's law in LTE}}$$

- The Kirchhoff's law holds even in LTE condition.

Normal & Inverted Populations

Normal populations:

- In LTE, $$\frac{n_2 g_1}{n_1 g_2} = \exp\left(-\frac{h\nu}{k_\mathrm{B}T}\right) < 1 \;\rightarrow\; \frac{n_1}{g_1} > \frac{n_2}{g_2}$$
- The normal populations is usually satisfied even when the material is out of thermal equilibrium. This is called "normal populations."

Inverted populations: $$\frac{n_1}{g_1} < \frac{n_2}{g_2}$$

- In this case, the absorption coefficient is negative and the intensity increases along a ray.
- Such a system is said to be a **maser** (**m**icrowave **a**mplification by **s**timulated **e**mission of **r**adiation; also **laser** for **l**ight...).
- The amplification can be very large. A negative optical depth of -100 leads to an amplification by a factor of $e^{100} = 10^{43}$.

Scattering Effects: Pure Scattering

- Assumptions

  isotropic scattering: scattered equally into equal solid angles

  coherent scattering (elastic or monochromatic scattering): the total amount of radiation scattered per unit frequency is equal to the total amount absorbed in the same frequency range.

  Thompson scattering (scattering from non-relativistic electrons) is nearly coherent.

- scattering coefficient

  In the textbook,
  the scattering coefficient is denoted by σ_ν .

$$j_\nu = \alpha_\nu^{\rm sca} \int \Phi_\nu(\mathbf{\Omega}, \mathbf{\Omega}') I_\nu(\mathbf{\Omega}') d\Omega'$$

$$= \alpha_\nu^{\rm sca} \frac{1}{4\pi} \int I_\nu d\Omega = \alpha_\nu^{\rm sca} J_\nu$$

- source function

$$S_\nu = J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$$

- radiative transfer equation

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\rm sca}(I_\nu - J_\nu)$$

This is an integro-differential equation, and cannot be solved by the formal solution.

$\rightarrow$ Rosseland approximation, Eddington approximation, or random walks

Random Walks (in infinite medium)

- Random walks: let's consider a single photon rather than a beam of photons (i.e., ray).
- In an infinite, homogeneous medium, net displacement of the photon after *N* free paths is zero, because the average displacement, being a vector, must be zero.

$$\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \cdots + \mathbf{r}_N \quad \rightarrow \quad \langle \mathbf{R} \rangle = 0$$

- root mean square net displacement:

$$\begin{aligned} \ell_*^2 \equiv \left\langle \mathbf{R}^2 \right\rangle &= \left\langle \mathbf{r}_1^2 \right\rangle + \left\langle \mathbf{r}_2^2 \right\rangle + \left\langle \mathbf{r}_3^2 \right\rangle + \cdots + \left\langle \mathbf{r}_N^2 \right\rangle \\ &\quad + 2 \left\langle \mathbf{r}_1 \cdot \mathbf{r}_2 \right\rangle + 2 \left\langle \mathbf{r}_1 \cdot \mathbf{r}_3 \right\rangle + \cdots \\ &= N\ell^2 \end{aligned}$$

$\leftarrow$ Note $\left\langle \mathbf{r}_i^2 \right\rangle \approx \ell^2, \ \ \langle \mathbf{r}_i \cdot \mathbf{r}_j \rangle = 0 \ (i \neq j)$

$$\therefore \ell_* = \sqrt{N}\ell$$

The cross terms involve averaging the cosine of the angle between the directions before and after scattering, and this vanishes for isotropic scattering and for any scattering with front-back symmetry (Thompson or Rayleigh scattering)

Random Walks (in finite medium)

- <u>In a finite medium</u>, a photon generated somewhere within the medium will scatter until it escapes completely.
- For regions of large optical depth, the mean number of scatterings to escape is roughly determined by $\ell_* \approx L$ (the typical size of the medium).

$$\ell_* = \sqrt{N}\ell \ \approx \ L \rightarrow N \approx L^2/\ell^2 = L^2 \left(n\sigma_\nu^{\rm sca}\right)^2$$

$$\therefore \ N \approx \tau^2 \ (\tau \gg 1)$$

- For regions of small optical depth, the probability of scatterings within τ is $1 - e^{-\tau} \approx \tau$.

$$\therefore \ N \approx \tau \qquad (\tau \ll 1)$$

- For any optical thickness, the mean number of scatterings is

$$N \approx \tau^2 + \tau \quad \text{or} \quad N \approx \max(\tau, \ \tau^2)$$

- However, for Lyα scattering, $N \approx \tau$ as $\tau \gg 1$.

Combined Scattering and Absorption

- The RT equation to the case of combined absorption and scattering, in which scattering is coherent isotropic.

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\text{abs}}(I_\nu - B_\nu) - \alpha_\nu^{\text{sca}}(I_\nu - J_\nu)$$
$$= -(\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}})(I_\nu - S_\nu) = -\alpha_\nu^{\text{ext}}(I_\nu - S_\nu)$$
$$\text{where} \quad S_\nu \equiv \frac{\alpha_\nu^{\text{abs}} B_\nu + \alpha_\nu^{\text{sca}} J_\nu}{\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}}} \quad \text{and} \quad \alpha_\nu^{\text{ext}} \equiv \alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}}$$

- Source function is an weighted average of the two source functions.
- extinction coefficient: $\alpha_\nu^{\text{ext}} \equiv \alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}}$
- optical depth: $d\tau_\nu \equiv \alpha_\nu^{\text{ext}} ds$
- If a matter element is deep inside a medium (i.e., in TE),

$$J_\nu = B_\nu \;\rightarrow\; S_\nu = B_\nu$$

- If the element is isolate in free space, $J_\nu = 0 \;\rightarrow\; S_\nu = \alpha_\nu^{\text{abs}} B_\nu / \alpha_\nu^{\text{ext}}$

- generalized mean free path:

$$\ell_\nu = \left(\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}\right)^{-1}$$

- probability of a (random walk) step ending in absorption:

$$\epsilon_\nu = \alpha_\nu^{\rm abs} / \alpha_\nu^{\rm ext}$$

- probability for scattering (known as the single-scattering albedo)

$$a_\nu = 1 - \epsilon_\nu = \alpha_\nu^{\rm sca} / \alpha_\nu^{\rm ext}$$

- source function (for coherent isotopic scattering case) :

$$\begin{aligned} S_\nu &= \epsilon_\nu B_\nu + (1 - \epsilon_\nu) J_\nu \\ &= (1 - a_\nu) B_\nu + a_\nu J_\nu \end{aligned}$$

Random Walks with Scattering and Absorption

- In an infinite medium, every photon is eventually absorbed.
- Since a random walk can be terminated with probability $\epsilon \; (= \alpha^{\rm abs}/\alpha^{\rm ext})$ at the end of each free path, the mean number of free paths before absorption is given by

mean number of free paths x probability of termination = 1

$$N\epsilon = 1 \;\rightarrow\; N = 1/\epsilon$$

- diffusion length (thermalization length, effective mean path, or effective free path): a measure of the net displacement between the points of creation and destruction of a typical photon.

$$\begin{aligned}
\ell_* &\approx \sqrt{N}\ell = \ell/\sqrt{\epsilon} \\
&\approx \left(\alpha_\nu^{\rm ext}\right)^{-1} \sqrt{\alpha_\nu^{\rm ext}/\alpha_\nu^{\rm abs}} \\
&\approx \left(\alpha_\nu^{\rm abs}\alpha_\nu^{\rm ext}\right)^{-1/2}
\end{aligned}$$

- In a finite medium:
- The behavior depends on whether its size *L* is larger or smaller than the effective free path ℓ_* .
- effective optical thickness: $\tau_* = L/\ell_* \approx \sqrt{\tau_{\rm abs}\left(\tau_{\rm abs} + \tau_{\rm sca}\right)} = \sqrt{\tau_{\rm abs}\tau_{\rm ext}}$

$$\text{where } \tau_{\rm abs} \equiv \alpha_\nu^{\rm abs} L,\ \ \tau_{\rm sca} \equiv \alpha_\nu^{\rm sca} L,\ \ \tau_{\rm ext} \equiv \alpha_\nu^{\rm ext}$$

- If effectively thin or translucent $\left(\tau_* \ll 1,\ L \ll \ell_*\right)$, most photons will escape the medium before being destroyed.

  luminosity of thermal source with volume V is

$$L_\nu = 4\pi j_\nu V = 4\pi\alpha_\nu^{\rm abs} B_\nu V \quad (\tau_* \ll 1)$$

- If effectively thick, we expect $I_\nu \to B_\nu,\ S_\nu \to B_\nu$, and only the photons emitted within an effective path length of the boundary will have a reasonable chance of escaping before being absorbed.

$$L_\nu = \pi\alpha_\nu^{\rm abs} B_\nu A \ell_* = \pi\sqrt{\epsilon_\nu} B_\nu A \qquad (F = \pi B \text{ at surface of the source})$$

Approximate Solutions

How to solve the radiative transfer equation:

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\rm ext}(I_\nu - S_\nu)$$

$$S_\nu = (1-\epsilon_\nu)J_\nu + \epsilon_\nu B_\nu \text{ and } \epsilon = \alpha_\nu^{\rm abs}/\alpha_\nu^{\rm ext}$$

We will learn two approximations to solve the equation.

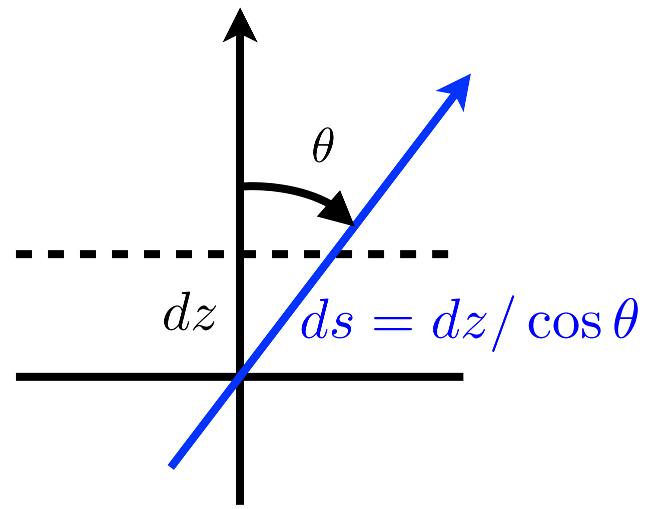
- Rosseland approximation
- Eddington approximation

Radiative Diffusion: (1) Rosseland Approximation

- Imagine a plane-parallel medium (in which ρ, T depend only on depth *z*).

$$ds = \frac{dz}{\cos\theta} = \frac{dz}{\mu} \;\rightarrow\; \mu\frac{\partial I_\nu(z,\mu)}{\partial z} = -\alpha_\nu^{\rm ext}(I_\nu - S_\nu)$$

$$I_\nu(z,\mu) = S_\nu - \frac{\mu}{\alpha_\nu^{\rm ext}}\frac{\partial I_\nu}{\partial z}$$



- "zeroth" approximation: when the point in question is deep in the material, <u>all quantities changes slowly on the scale of a mean free path</u> $(l_* = 1/\alpha_\nu^{\rm ext})$ and the derivative term above is very small.

$$I_\nu^{(0)}(z,\mu) \approx S_\nu^{(0)}(T)$$

This is independent of the angle. $\therefore\; J_\nu^{(0)} = S_\nu^{(0)}$ and $I_\nu^{(0)} = S_\nu^{(0)} = B_\nu$

- "first" approximation:

$$I_\nu^{(1)}(z,\mu) \approx S_\nu^{(0)} - \frac{\mu}{\alpha_\nu^{\rm ext}}\frac{\partial I_\nu^{(0)}}{\partial z} = B_\nu(T) - \frac{\mu}{\alpha_\nu^{\rm ext}}\frac{\partial B_\nu(T)}{\partial z} \rightarrow \text{linear in } \mu$$

- net specific flux along z : the angle-independent part of the intensity does not contribute to the flux.

$$F_\nu(z) = \int I_\nu^{(1)}(z,\mu)\cos\theta d\Omega = 2\pi \int_{-1}^{+1} I_\nu^{(1)}(z,\mu)\mu d\mu$$

$$= -\frac{2\pi}{\alpha_\nu^{\rm ext}}\frac{\partial B_\nu}{\partial z}\int_{-1}^{+1}\mu^2 d\mu$$

$$= -\frac{4\pi}{3\alpha_\nu^{\rm ext}}\frac{\partial B_\nu(T)}{\partial T}\frac{\partial T}{\partial z}$$

- total integrated flux: $$F(z) = \int_0^\infty F_\nu(z)d\nu = -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty \frac{1}{\alpha_\nu^{\rm ext}}\frac{\partial B_\nu}{\partial T}d\nu$$

let's define the Rosseland mean absorption coefficient

$$\frac{1}{\alpha_R} \equiv \frac{\int_0^\infty \frac{1}{\alpha_\nu^{\rm ext}}\frac{\partial B_\nu}{\partial T}d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu}$$

use

$$\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \frac{\partial}{\partial T} \int_0^\infty B_\nu d\nu = \frac{\partial \left(\sigma T^4/\pi\right)}{\partial T} = \frac{4\sigma T^3}{\pi}$$

Then, we obtain the Rosseland approximation to radiative flux

$$F(z) = -\frac{16\sigma T^3}{3\alpha_R} \frac{\partial T}{\partial z} \to -\chi \nabla T$$

which is also called the equation of radiative diffusion.

- The flux equation can be interpreted as a heat conduction with an "effective heat conductivity," $\chi = 16\sigma T^3/3\alpha_R$.
- At which frequencies the Rosseland mean becomes important?

The mean involves a weighted average of $1/\alpha_\nu^{\mathrm{ext}}$ so that frequencies at which the extinction coefficient is small (transparent) tend to dominate.

The weighting function $\partial B_\nu/\partial T$ has a shape similar to that of the Planck function, but it peaks at $h\nu_{\mathrm{max}} = 3.8 k_{\mathrm{B}} T$, instead of $h\nu_{\mathrm{max}} = 2.8 k_{\mathrm{B}} T$.

Radiative Diffusion: (2) Eddition Approximation

- In Eddington approximation, the intensities are assumed to approach isotropy, and not necessarily their thermal values.

  In the Rosseland approximation, the intensities approach the Planck function at large effective depths.

- Near isotropy can be introduced by assuming that the intensity is linear in μ. (frequency is suppressed for convenience)

$$I(\tau, \mu) = a(\tau) + b(\tau)\mu$$

- Let us take the first three moments.

mean intensity: $$J \equiv \frac{1}{2}\int_{-1}^{+1} I d\mu = a$$

flux: $$H \equiv \frac{1}{2}\int_{-1}^{+1} \mu I d\mu = \frac{b}{3}$$

radiation pressure: $$K \equiv \frac{1}{2}\int_{-1}^{+1} \mu^2 I d\mu = \frac{a}{3}$$

$$\rightarrow \boxed{K = \frac{1}{3}J}$$

Eddington approximation

Compare with the following equations for the isotropic radiation.

$$p = \frac{1}{3}u \quad \left(p \equiv \frac{1}{c}\int I \cos^2\theta d\Omega,\ u(\mathbf{\Omega}) = \frac{1}{c}I\right)$$

optical depth and the transfer equation: $d\tau(z) \equiv -\alpha^{\rm ext} dz, \quad \mu \dfrac{\partial I}{\partial \tau} = I - S$

Note: source function is independent to μ (because $S = (1-\epsilon)J + \epsilon B$).

Integrate the above equation and obtain the following equations.

$$\frac{1}{2}\int_{-1}^{+1} d\mu \left(\mu \frac{\partial I}{\partial \tau} = I - S\right) \rightarrow \frac{\partial H}{\partial \tau} = J - S$$

$$\frac{1}{2}\int_{-1}^{+1} d\mu\mu \left(\mu \frac{\partial I}{\partial \tau} = I - S\right) \rightarrow \frac{\partial K}{\partial \tau} = H \rightarrow \frac{1}{3}\frac{\partial J}{\partial \tau} = H$$

The two equations can be combined to yield:

$$\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = J - S \quad \rightarrow \boxed{\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = \epsilon(J - B)}$$

Let us define a new optical depth $\tau_* \equiv \sqrt{3\epsilon}\tau = \sqrt{3\tau_{\rm abs}(\tau_{\rm abs} + \tau_{\rm sca})}$

The radiative equation is then $\boxed{\dfrac{\partial^2 J}{\partial \tau_*^2} = J - B}$

This equation is sometimes called the radiative diffusion equation. Given the temperature structure of the medium, $B(\tau)$, the equation can be solved for J.

To solve the second order differential equation, we need two boundary conditions. The boundary conditions can be provided in several ways. One way to do is to use two-stream approximation, in which the entire radiation field is represented by radiation at just two angles, i.e., $\mu = \pm\mu_0$:

$$I(\tau,\mu) = I^+(\tau)\delta(\mu-\mu_0) + I^-(\tau)\delta(\mu+\mu_0)$$

The two terms denote the outward and inward intensities. Then, the three moments are

$$J = \frac{1}{2}\left(I^+ + I^-\right)$$

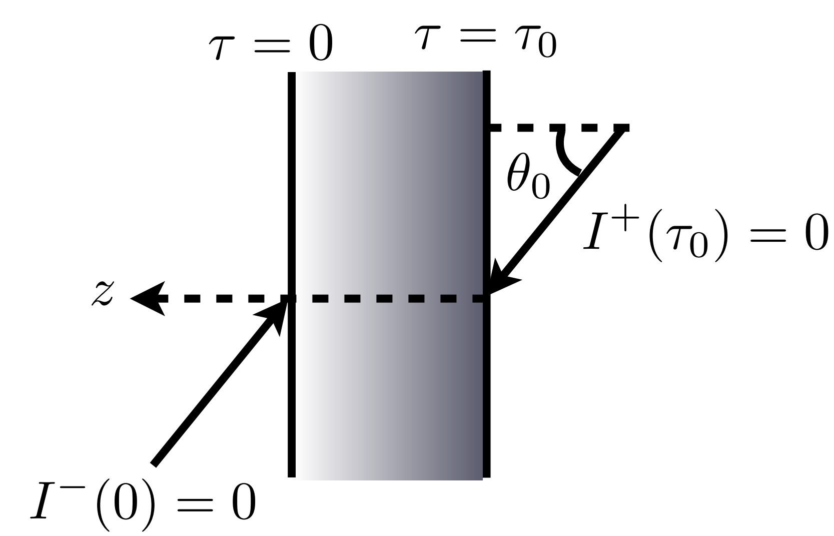
$$H = \frac{1}{2}\mu_0\left(I^+ - I^-\right) \quad \rightarrow \text{ we obtain } \mu_0 = \frac{1}{\sqrt{3}} \text{ in order to satisty } K = \frac{1}{3}J$$

$$K = \frac{1}{2}\mu_0^2\left(I^+ + I^-\right) \qquad (\theta_0 = \cos^{-1}\mu_0 = 54.74^\circ)$$

Using $H = \dfrac{1}{3}\dfrac{\partial J}{\partial \tau}$, we obtain $\boxed{I^+ = J + \dfrac{1}{3}\dfrac{\partial J}{\partial \tau}, \quad I^- = J - \dfrac{1}{3}\dfrac{\partial J}{\partial \tau}}$.

Suppose the medium extends from $\tau = 0$ to $\tau = \tau_0$ and there is no incident radiation. Then, we obtain two boundary conditions:

$$I^+(\tau_0) = 0 \text{ and } I^-(0) = 0 \quad \rightarrow \quad \boxed{\begin{aligned} \frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau} &= J \quad \text{at} \quad \tau = 0 \\ \frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau} &= -J \quad \text{at} \quad \tau = \tau_0 \end{aligned}}$$



Iteration Method

- Recall
$$\frac{dI(s)}{ds} = -\alpha^{\rm ext} I(s) + \alpha^{\rm sca} \int \Phi(\Omega,\Omega') I(s,\Omega') d\Omega' + j(s)$$
or
$$\frac{dI(\tau)}{d\tau} = -I(\tau) + a \int \Phi(\Omega,\Omega') I(\tau,\Omega') d\Omega' + \mathsf{S}(\tau) \qquad \left(d\tau \equiv \alpha^{\rm ext} ds,\ \mathsf{S}(\tau) \equiv \frac{j(\tau)}{\alpha^{\rm ext}} \right)$$

- Let I_0 be the intensity of photons that come directly from the source, I_1 the intensity of photons that have been scattered once by dust, and I_n the intensity after *n* scatterings. Then,
$$I(s) = \sum_{n=0}^{\infty} I_n(s)$$

- The intensities I_n satisfy the equations.
$$\frac{dI_0(\tau)}{d\tau} = -I_0(\tau) + \mathsf{S}(\tau)$$
$$\frac{dI_n(\tau)}{d\tau} = -I_n(\tau) + a \int \Phi(\Omega,\Omega') I_{n-1}(\tau,\Omega') d\Omega'$$
$$\equiv -I_n(\tau) + \mathsf{S}_{n-1}(\tau) \qquad \left(\mathsf{S}_{n-1}(\tau) \equiv a \int \Phi(\Omega,\Omega') I_{n-1}(\tau,\Omega') d\Omega' \right)$$

- Then, the formal solutions are:
$$\rightarrow \quad \begin{aligned} I_0(\tau) &= e^{-\tau} I_0(0) + \int_0^{\tau} e^{-(\tau-\tau')} \mathsf{S}(\tau') d\tau' \\ I_n(\tau) &= e^{-\tau} I_n(0) + \int_0^{\tau} e^{-(\tau-\tau')} \mathsf{S}_{n-1}(\tau') d\tau' \end{aligned}$$

Approximation: (1) application to the edge-on galaxies

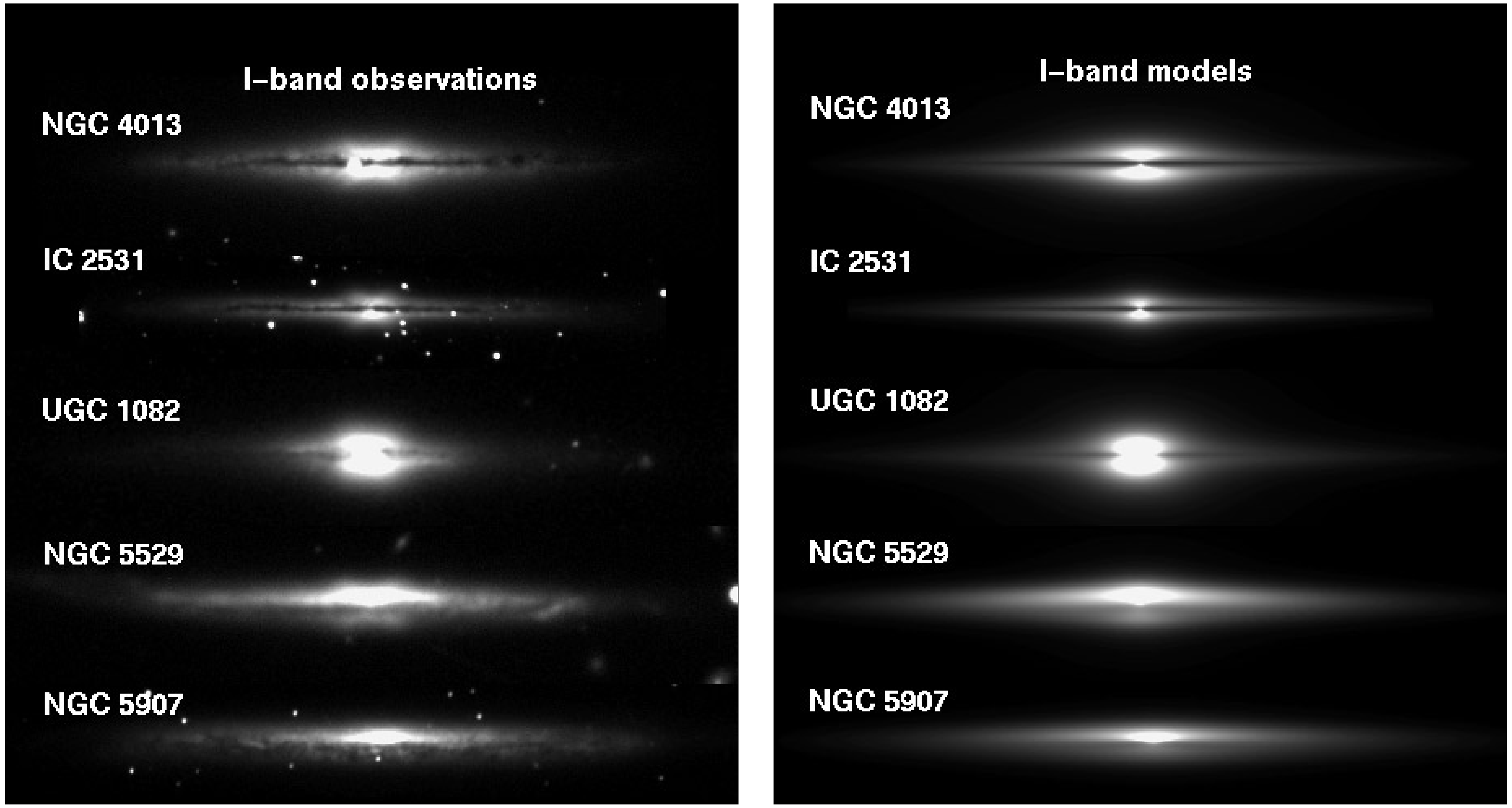
The solution can be further simplified by assuming that

$$\frac{I_n}{I_{n-1}} \approx \frac{I_1}{I_0} \quad (n \geq 2)$$

Then, the infinite series becomes

$$I_n \approx I_0 \sum_{n=0}^{\infty} \left(\frac{I_1}{I_0}\right)^n = \frac{I_0}{1 - I_1/I_0}$$

Kylafis & Bahcall (1987) and Xilouris et al. (1997, 1998, 1999) applied this approximation to model the dust radiative transfer process in the edge-on galaxies.



Xilouris et al. (1999)

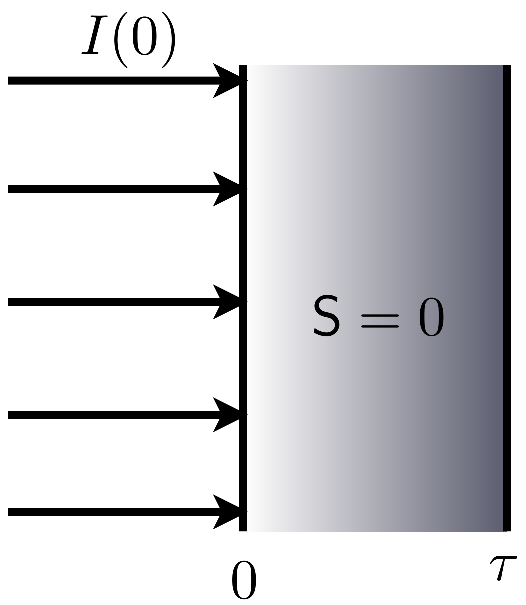
Approximation: (2) solution for the forward scattering

- Assume the very strong forward-scattering

$$\Phi(\Omega, \Omega') = \delta(\Omega' - \Omega)$$
$$\rightarrow \quad \mathsf{S}_{n-1}(\tau) = a I_{n-1}(\tau)$$

- The iterative solutions are:

$$I_0(\tau) = e^{-\tau} I_0(0)$$
$$\rightarrow \quad \mathsf{S}_0(\tau) = a I_0(\tau) = a e^{-\tau} I_0(0)$$
$$I_1(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_0(\tau') d\tau' = (a\tau) e^{-\tau} I_0(0)$$
$$\rightarrow \quad \mathsf{S}_1(\tau) = a I_1(\tau) = (a^2 \tau) e^{-\tau} I_0(0)$$
$$I_2(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_1(\tau') d\tau' = \frac{(a\tau)^2}{2} I_0(0)$$
$$\rightarrow \quad \mathsf{S}_2(\tau) = a I_2(\tau) = (a^3 \tau^2) e^{-\tau} I_0(0)$$
$$I_3(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_2(\tau') d\tau' = \frac{(a\tau)^3}{3 \times 2} I_0(0)$$
$$\vdots$$
$$\rightarrow \quad \mathsf{S}_{n-1}(\tau) = a I_{n-1}(\tau) = (a^n \tau^{n-1}) e^{-\tau} I_0(0)$$
$$I_n(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_{n-1}(\tau') d\tau' = \frac{(a\tau)^n}{n!} I_0(0)$$



The final solutions are:

$$I^{\rm direc}(\tau) = e^{-\tau} I(0)$$
$$I^{\rm scatt}(\tau) = \sum_{n=1}^{\infty} I_n(\tau) = \sum_{n=1}^{\infty} \frac{(a\tau)^n}{n!} e^{-\tau} I(0)$$
$$= (e^{a\tau} - 1) e^{-\tau} I(0)$$
$$\approx a\tau e^{-\tau} I(0) \quad \text{if} \quad a\tau \ll 1$$
$$\approx a\tau I(0) \quad \text{if} \quad \tau \ll 1$$

$$I^{\rm tot}(\tau) = I^{\rm direc}(\tau) + I^{\rm scatt}(\tau)$$
$$= e^{-(1-a)\tau} I(0)$$
$$= e^{-\tau_{\rm abs}} I(0)$$