Astrophysics

Lecture 08
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선광일 (Kwang-Il Seon)
UST / KASI

- Thus we obtain the two results:

$$\boxed{\begin{aligned} \frac{dP_e}{d\Omega} &= \gamma^2 (1+\beta\mu')^3 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4 (1-\beta\mu)^3} \frac{dP'}{d\Omega'} \\ \frac{dP_r}{d\Omega} &= \gamma^4 (1+\beta\mu')^4 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4 (1-\beta\mu)^4} \frac{dP'}{d\Omega'} \end{aligned}}$$

  P_r is the power actually measured by an observer. It has the expected symmetry property of yielding the inverse transformation by interchanging primed and unprimed variables, along with a change of sign of β.

  P_e is used in the discussion of emission coefficient.

  In practice, **the distinction between emitted and received power is often not important, since they are equal in an average sense for stationary distributions of particles.**

- Beaming effect:

  If the radiation is isotropic in the particle's frame, then the angular distribution in the observer's frame will be highly peaked in the forward direction for highly relativistic velocities.

  The factor $\gamma^{-4} (1-\beta\mu)^{-4}$ is sharply peaked near $\theta \approx 0$ with an angular scale of order $1/\gamma$.

$$\gamma^{-4} (1-\beta\mu)^{-4} \approx \gamma^{-4} \left[1 - \left(1 - \frac{1}{2\gamma^2}\right)\left(1 - \frac{\theta^2}{2}\right)\right]^{-4} = \gamma^{-4} \left(\frac{1}{2\gamma^2} + \frac{\theta^2}{2}\right)^{-4} = \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^4$$

- Angular distribution of the dipole emission from a slowly moving particle in the instantaneous rest frame is

$$\frac{dP'}{d\Omega'} = \frac{q^2 a'^2}{4\pi c^3} \sin^2 \Theta'$$

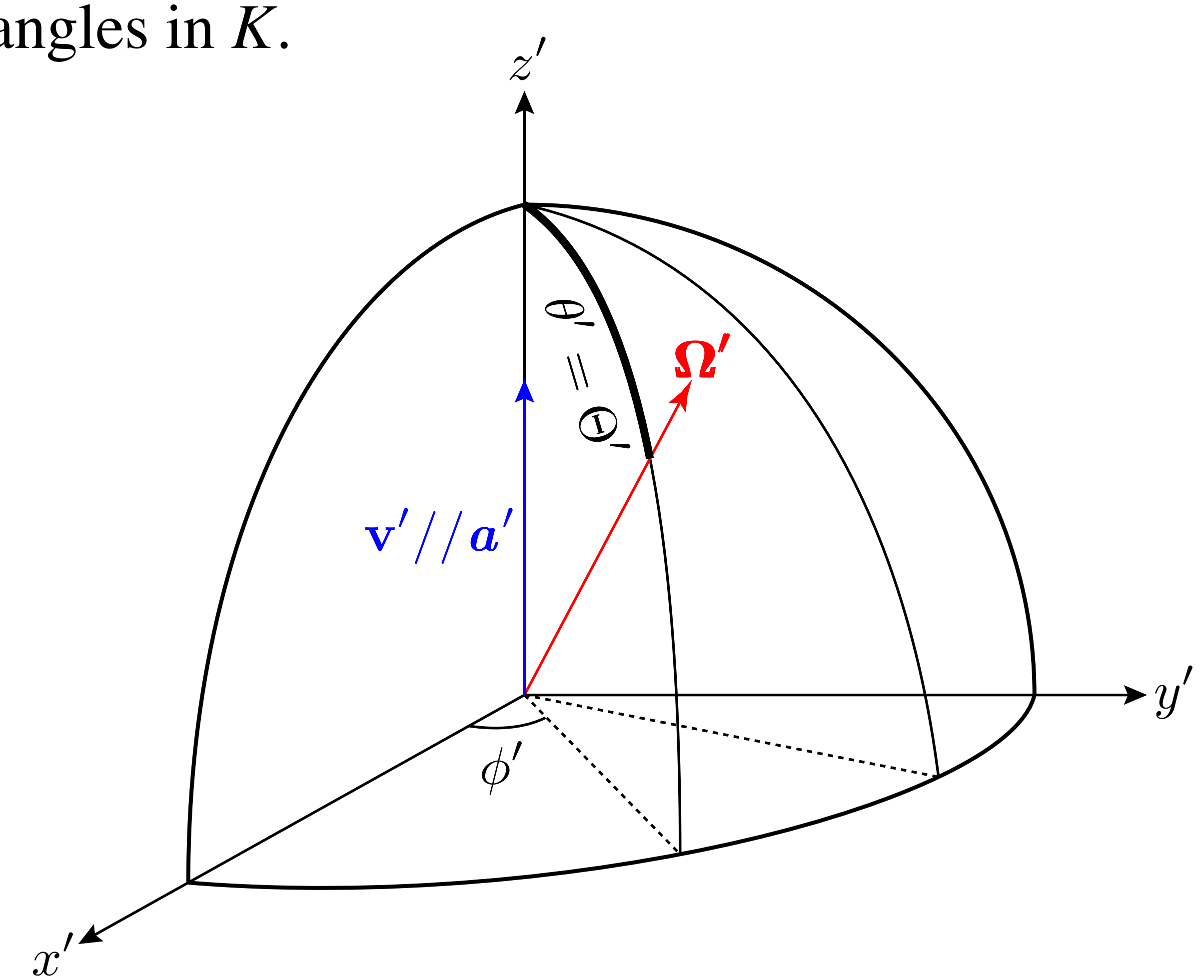
Θ' = the angle between the acceleration and the direction of emission.

Using $a'_{\parallel} = \gamma^3 a_{\parallel},\ a'_{\perp} = \gamma^2 a_{\perp}$ and $\frac{dP_r}{d\Omega} = \gamma^{-4}(1-\beta\mu)^{-4}\frac{dP'}{d\Omega'}$, we obtain

$$\boxed{\frac{dP_r}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{\left(\gamma^2 a_{\parallel}^2 + a_{\perp}^2\right)}{(1-\beta\mu)^4}\sin^2\Theta'}$$

To use this formula, we must relate Θ' to the angles in K.

(1) Acceleration parallel to velocity: $\Theta' = \theta',\ a_{\perp} = 0$



$$\sin^2\Theta' = 1-\mu'^2 = 1-\left(\frac{\mu-\beta}{1-\beta\mu}\right)^2 = \frac{1-\mu^2}{\gamma^2(1-\beta\mu)^2}$$

$$\longrightarrow \quad \frac{dP_{r\parallel}}{d\Omega} = \frac{q^2 a_{\parallel}^2}{4\pi c^3}\frac{1-\mu^2}{(1-\beta\mu)^6}$$

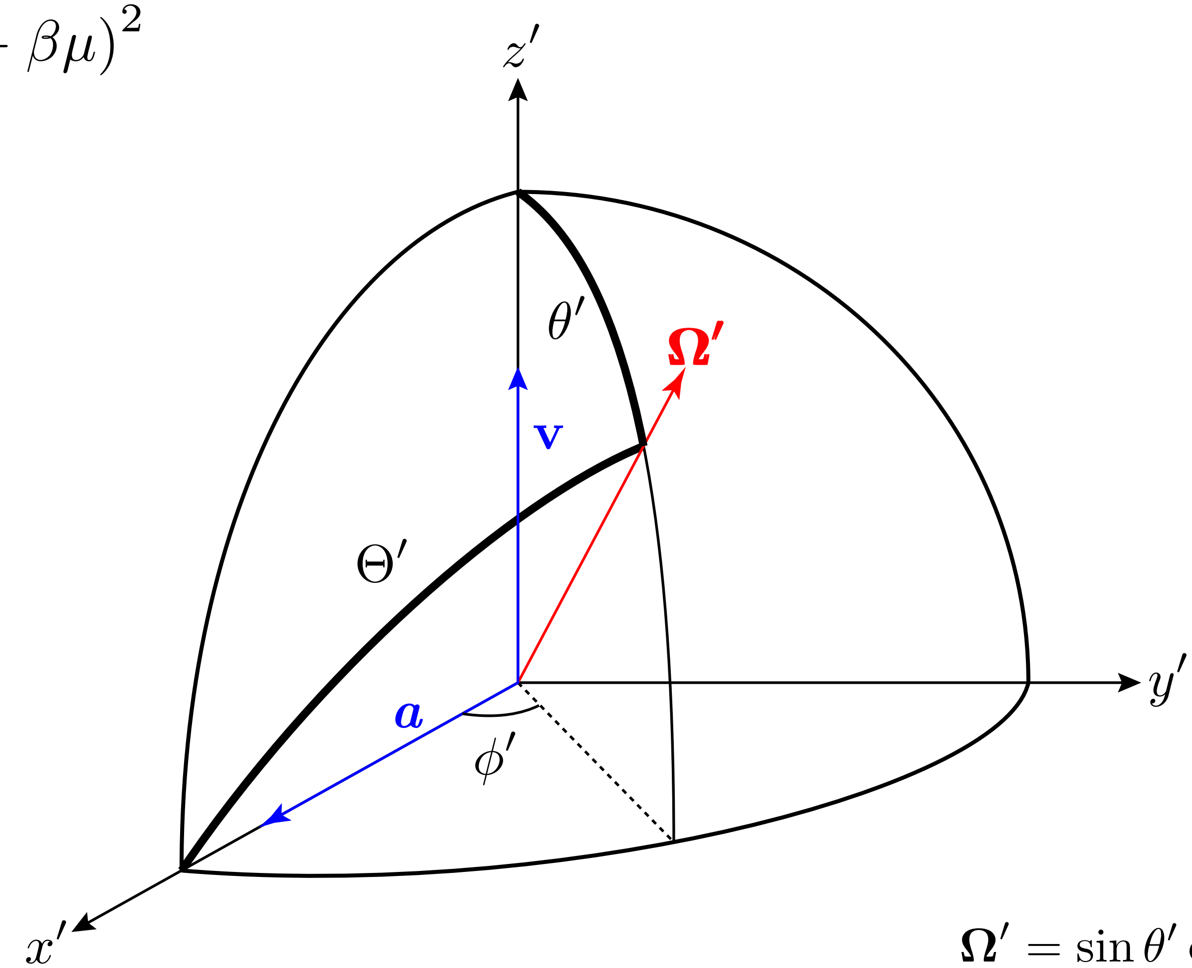
(2) Acceleration perpendicular to velocity: $\cos\Theta' = \sin\theta' \cos\phi'$ (when a is in x-direction in the figure)

$$\begin{aligned}\sin^2\Theta' &= 1-\cos^2\Theta' \\ &= 1-\sin^2\theta'\cos^2\phi' \\ &= 1-\frac{\sin^2\theta}{\gamma^2(1-\beta\cos\theta)^2}\cos^2\phi \\ &= 1-\frac{\left(1-\mu^2\right)\cos^2\phi}{\gamma^2\left(1-\beta\mu\right)^2}\end{aligned}$$

$$\longrightarrow \quad \frac{dP_{r\perp}}{d\Omega} = \frac{q^2 a_\perp^2}{4\pi c^3}\frac{1}{\left(1-\beta\mu\right)^4}\left[1-\frac{\left(1-\mu^2\right)\cos^2\phi}{\gamma^2\left(1-\beta\mu\right)^2}\right]$$

aberration of light

$$\sin\theta' = \frac{\sin\theta}{\gamma(1-\beta\cos\theta)}$$
$$\phi' = \phi$$

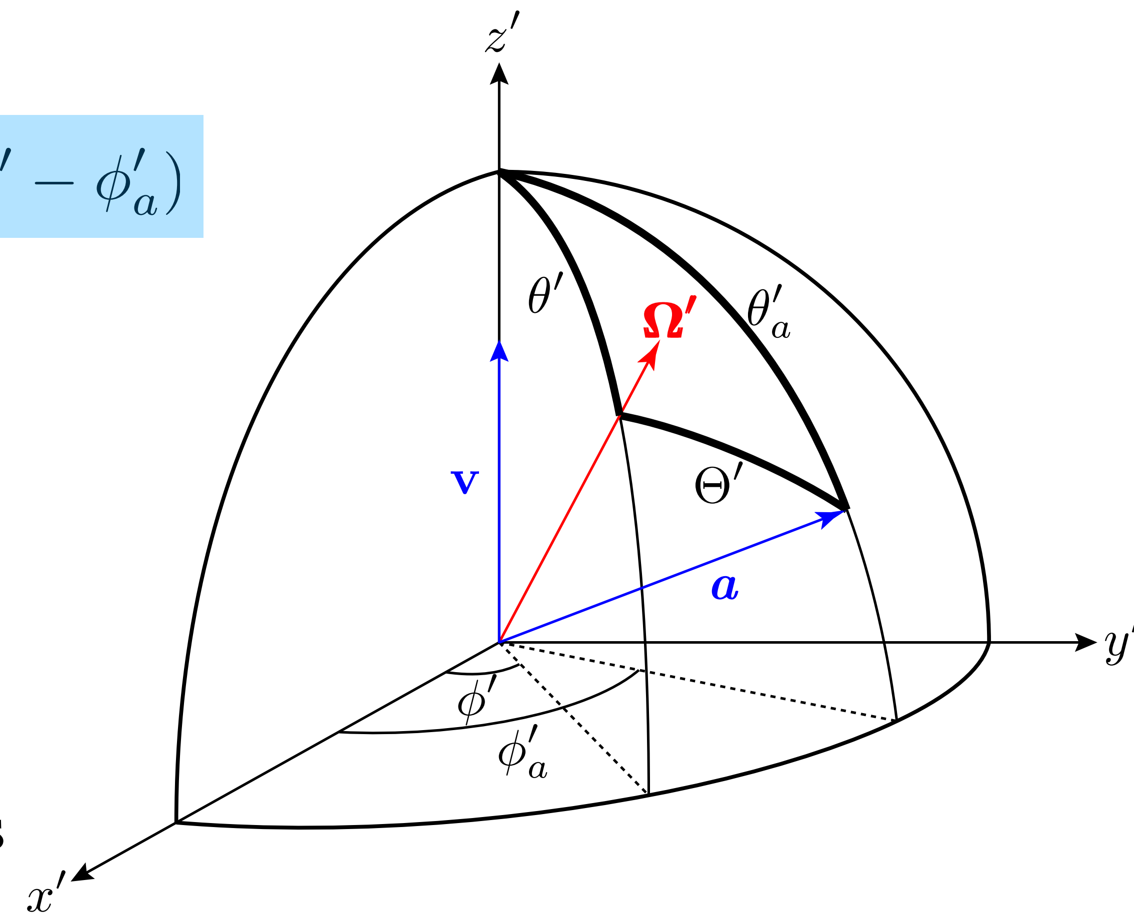


$$\mathbf{\Omega}' = \sin\theta'\cos\phi'\hat{\mathbf{x}} + \sin\theta'\sin\phi'\hat{\mathbf{y}} + \cos\theta'\hat{\mathbf{z}}$$
$$\rightarrow \quad \cos\Theta' = \mathbf{\Omega}'\cdot\hat{\mathbf{x}} = \sin\theta'\cos\phi'$$

(3) In general

$$\cos\Theta' = \mu'\mu'_a + \left(1-\mu'^2\right)^{1/2}\left(1-\mu_a'^2\right)^{1/2}\cos\left(\phi'-\phi'_a\right)$$

$$\mathbf{r}_1 = (\sin\theta_1\cos\phi_1,\ \ \sin\theta_1\sin\phi_1,\ \ \cos\theta_1)$$
$$\mathbf{r}_2 = (\sin\theta_2\cos\phi_2,\ \ \sin\theta_2\sin\phi_2,\ \ \cos\theta_2)$$
$$\begin{aligned}\cos\Theta &= \mathbf{r}_1\cdot\mathbf{r}_2 \\ &= \mu_1\mu_2 + \sqrt{1-\mu_1^2}\sqrt{1-\mu_2^2}\,(\cos\phi_1\cos\phi_2+\sin\phi_1\sin\phi_2) \\ &= \mu_1\mu_2 + \sqrt{1-\mu_1^2}\sqrt{1-\mu_2^2}\cos(\phi_1-\phi_2)\end{aligned}$$

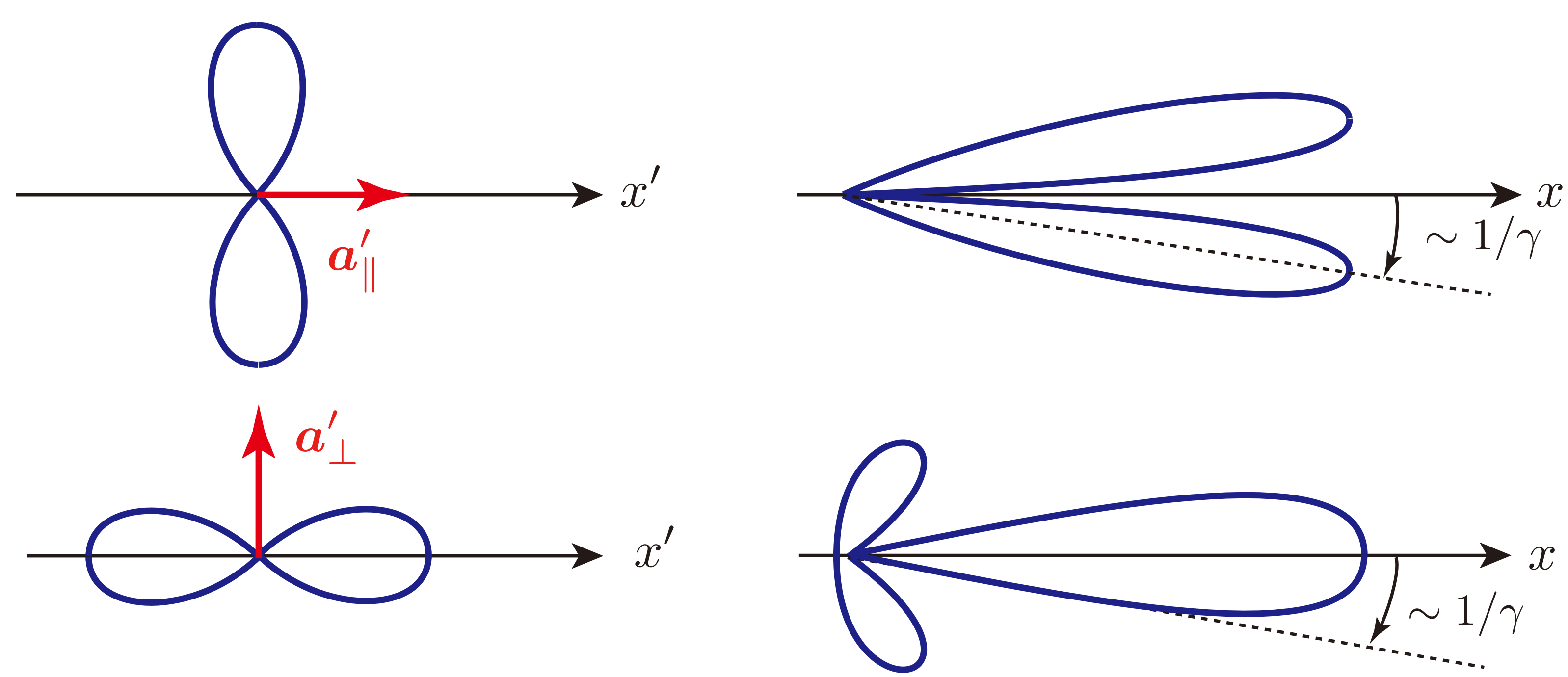


- In the extreme relativistic limit, the radiation becomes strongly peaked in the forward direction.

particle's rest frame: **observer's frame:**

parallel acceleration:

perpendicular acceleration:



[Invariant Phase Volumes and Specific Intensity]

- **Phase volume**

Consider a group of particles that occupy a slight spread in position and in momentum at a particular time. In a rest frame comoving with the particles, they occupy a spatial volume element $d^3\mathbf{x}' = dx'dy'dz'$ and a momentum volume element $d^3\mathbf{p}' = dP'_x dP'_y dP'_z$.

$$\begin{aligned} d^3\mathbf{x}' &= dx'dy'dz' \\ d^3\mathbf{p}' &= dp'_x dp'_y dp'_z \end{aligned} \longrightarrow$$

phase volume in the comoving frame:

$$d\mathcal{V}' \equiv d^3\mathbf{x}' d^3\mathbf{p}' = dx'dy'dz'dp'_x dp'_y dp'_z$$

In the observer's frame, $dx = \frac{1}{\gamma} dx'$ (length contraction), $dy = dy'$, $dz = dz'$

$$dp_x = \gamma \left(dp'_x + \beta dP'_0 \right), \quad dp_y = dp'_y, \quad dp_z = dp'_z$$

We note that $dP'_0 = 0 + \mathcal{O}(dp'^2_x)$ because the velocities are near zero in the comoving frame and the energy is quadratic in velocity. Therefore, we have $dp_x = \gamma dp'_x$. By combining all these, it is found that **the phase-space volume is invariant.**

$$\boxed{d\mathcal{V}' \equiv d^3\mathbf{x}' d^3\mathbf{p}' = d^3\mathbf{x} d^3\mathbf{p} \equiv d\mathcal{V}}$$

: Lorentz invariant

This contains no reference to particle mass, and therefore it has applicability to photons. **The phase space density is an invariant**, since the number of particles within the phase volume element is a countable quantity and itself invariant.

$$\boxed{f \equiv \frac{dN}{d\mathcal{V}} = \frac{dN}{d^3\mathbf{x} d^3\mathbf{p}}}$$

: Lorentz invariant

- **Specific Intensity and Source Function**

From the definition of the energy density per unit solid angle per frequency range.

$$u_\nu(\Omega)d\Omega d\nu d^3\mathbf{x} = h\nu\frac{dN}{d\mathcal{V}}d^3\mathbf{p}d^3\mathbf{x} = h\nu\frac{dN}{d\mathcal{V}}p^2dpd\Omega d^3\mathbf{x} \longrightarrow u_\nu(\Omega) = h\nu\frac{dN}{d\mathcal{V}}p^2\frac{dp}{d\nu} = h\nu\left(\frac{h\nu}{c}\right)^2\left(\frac{h}{c}\right)\frac{dN}{d\mathcal{V}} \quad \leftarrow \quad p = \frac{h\nu}{c}$$

Since $u_\nu(\Omega) = I_\nu/c$, we find that $\boxed{\frac{I_\nu}{\nu^3} = \text{ Lorentz invariant}}$

Because the source function occurs in the transfer equation as the difference $I_\nu - S_\nu$, the source function must have the same transformation properties as the intensity.

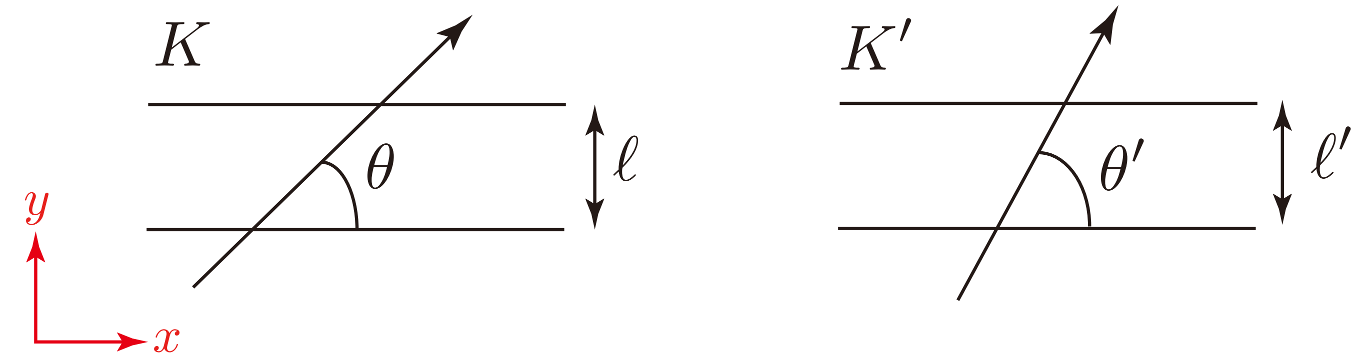
- **Optical Depth, Absorption Coefficient and Emission Coefficient**

The optical depth must be an invariant, since $\exp(-\tau)$ gives the fraction of photons passing through the material, and this involves simple counting.

$$\boxed{\tau = \text{ Lorentz invariant}}$$

- **Absorption Coefficient and Emission Coefficient**

Consider the optical depth in two frames:



Then, the optical depth is

$$\tau_\nu = \frac{\ell\alpha_\nu}{\sin\theta} = \frac{\ell}{\nu\sin\theta}\nu\alpha_\nu = \text{ Lorentz invariant}$$

Note that $\nu\sin\theta$ is proportional to the y component of the photon four-momentum $\vec{k} = (\omega/c,\ \mathbf{k})$.

Both k_y and ℓ are the same in both frames, being perpendicular to the motion. Therefore, we have

$$\nu\sin\theta \propto k_y, \quad k_y = k_y', \quad \ell' = \ell \qquad \boxed{\nu\alpha_\nu = \text{ Lorentz invariant}}$$

Finally, we obtain the transformation of the emission coefficient from the definition of the source function:

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu} \qquad \boxed{\frac{j_\nu}{\nu^2} = \text{ Lorentz invariant}}$$

- **Number density**

$$n = \frac{dN}{dV} = \frac{dN}{d^3\mathbf{x}} = \frac{dN'}{\gamma^{-1} d^3\mathbf{x}'} = \gamma n'$$

$$\boxed{n = \gamma n'}$$

Bremsstrahlung

[Bremsstrahlung]

- ***Bremsstrahlung*** **(= "breaking radiation" or free-free emission)**

  Coulomb collisions between electrons and ions in a hot ionized gas (plasma) give rise to photons because electrons are accelerated by the Coulomb forces and thereby emit radiation, thus losing energy. The German word bremsstrahlung means "braking radiation."

  These are free-free transitions because they are transitions from one free (unbound) state of the atom to another such state.

- **Bremsstrahlung due to the collision of identical particles (electron-electron, proton-proton) is zero in the dipole approximation (in absence of external forces)**, because the dipole moment is simply proportional to the center of mass (a constant of motion).
  - Because the accelerations of the two particles are equal in magnitude but opposite in direction. Their radiated electric fields are equal in magnitude but opposite in sign, so that the net radiated electric field approaches zero at distances much larger than the collision impact parameter.

$$\sum e_i \mathbf{r}_i = e \sum \mathbf{r}_i \propto m \sum \mathbf{r}_i = \sum m_i \mathbf{r}_i = 0 \qquad \text{center of mass}$$

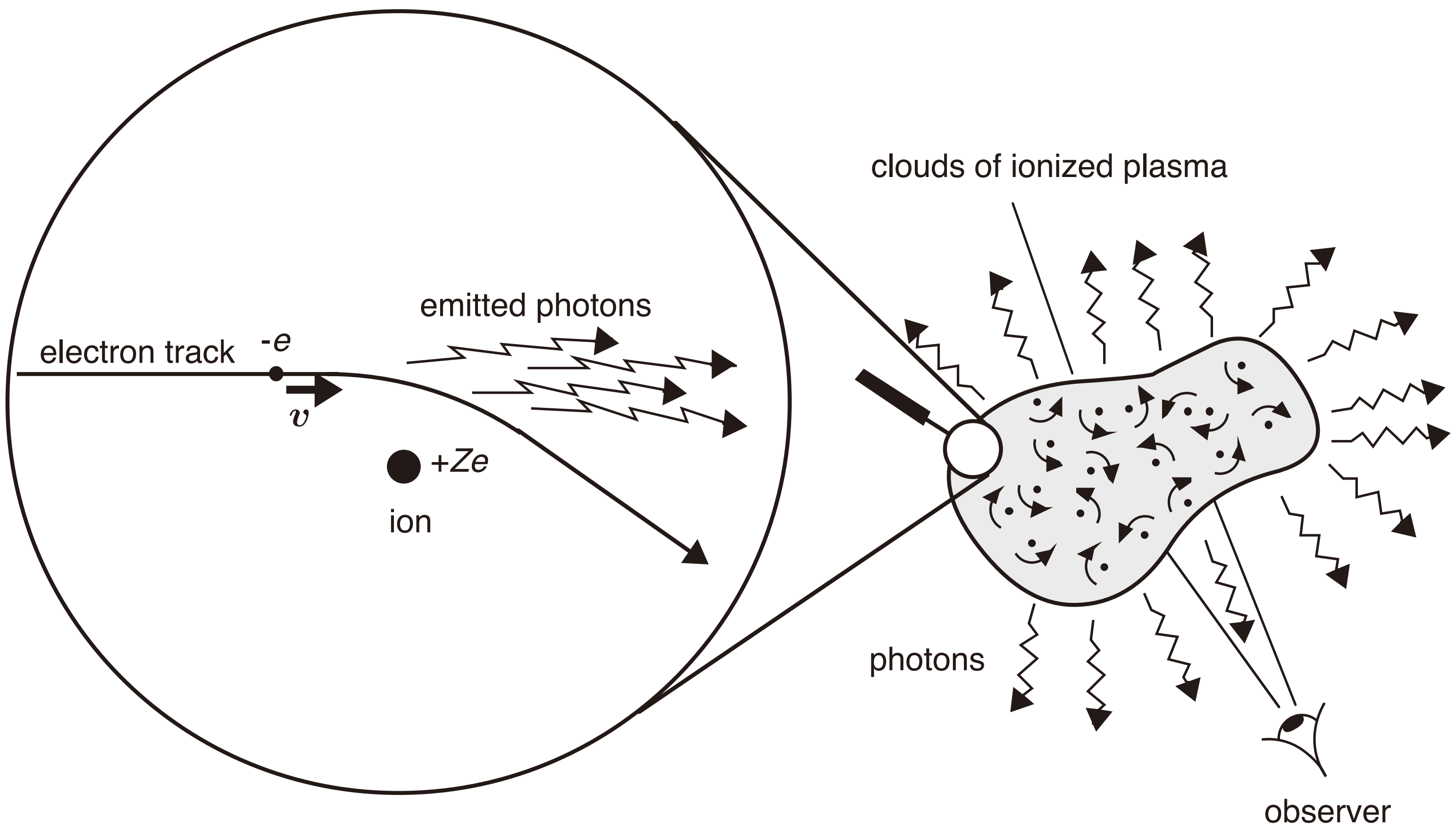
- The bottom line is that **only the electron-ion collisions are important, and only the electrons radiate significantly.**

- A full understanding of this process requires a quantum treatment. However, a classical treatment is justified in some regimes, and the formulas so obtained have the correct functional dependence for most of the physical parameters.

- ***Thermal Bremsstrahlung*** is the radiation produced by thermal electrons distributed according to the Maxwell velocity distribution. Thus, *it is thermal emission*, because it is produced by a source whose emitting particles are in local thermodynamic equilibrium (LTE).

- **Electron-ion collisions**

Consider the collision of a single electron with a single ion. The velocities of the ions are on average much less than those of the electrons because both components are in thermal equilibrium. Furthermore, in a given collision, the acceleration of the ion is much less than that of the electron owing to momentum conservation or Newton's third law. Thus, **we can consider the electron to be traveling through a region of static electric fields and neglect radiation from accelerating ions.**



Cloud of plasma (ionized gas) giving rise to photons owing to the collisions of the electrons and ions. The electrons are accelerated and thus emit radiation in the form of photons.

[Bradt, Astrophysics Processes]

- **Small angle scattering + perturbation**

  Consider bremsstrahlung radiated from a plasma of temperature T and densities $n_e\,(\mathrm{cm}^{-3})$ electrons with charge -e and $n_i\,(\mathrm{cm}^{-3})$ ions with charge Ze.

  We calculate an important ratio:

$$
\begin{aligned}
\frac{\text{Coulomb potential energy}}{\text{thermal kinetic energy}} \approx \frac{Ze^2/\langle r\rangle}{kT} &\approx \frac{Ze^2\, n_e^{1/3}}{kT} \\
&= 1.670\times 10^{-7}\ Z\ \left(\frac{n_e}{1\ \mathrm{cm}^{-3}}\right)^{1/3} \frac{10^4\ K}{T} \\
&\ll 1
\end{aligned}
$$

  for typical $n_e < 1\,\mathrm{cm}^{-3}$ and $T \sim 10^4 - 10^8\,\mathrm{K}$.

  Therefore, **Coulomb interaction is only a perturbation on the thermal motions of the electrons.**

- Although the classical trajectory of a free electron encountering an ion has an exact representation in terms of hyperbola, the expressions that emerge from such a treatment become quite complex.
- The electron moves rapidly enough so that ***the deviation of its path from a straight line is negligible.***

- Assumptions

The plasma is assumed to be completely ionized and to consist of elections of charge $-e$ and ions of charge $+Ze$, where Z is the atomic number.

In electron-ion bremsstrahlung, we treat the electron as moving in a fixed Coulomb field of the ion, since ***the relative accelerations are inversely proportional to the masses.*** Therefore, ***radiation from ions can be ignored***.

$$|F_i| = |F_e| \quad \rightarrow \quad \boxed{\frac{a_i}{a_e} = \frac{m_e}{m_i} \sim (1800)^{-1} < 10^{-3}}$$

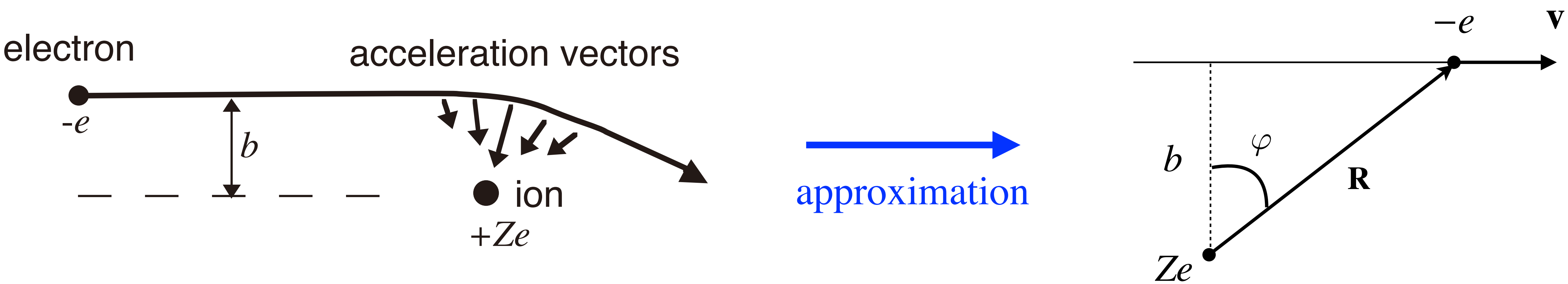
The electrons emit photons with energies $h\nu$ substantially less than their own kinetic energies.

The energy loss of an electron, integrated over an entire collision, is only a small part of the electron kinetic energy.

[Emission from single particle collisions]

- **Small-angle scattering** approximation:

  The electron moves rapidly enough so that the deviation of its path from a straight line is negligible.



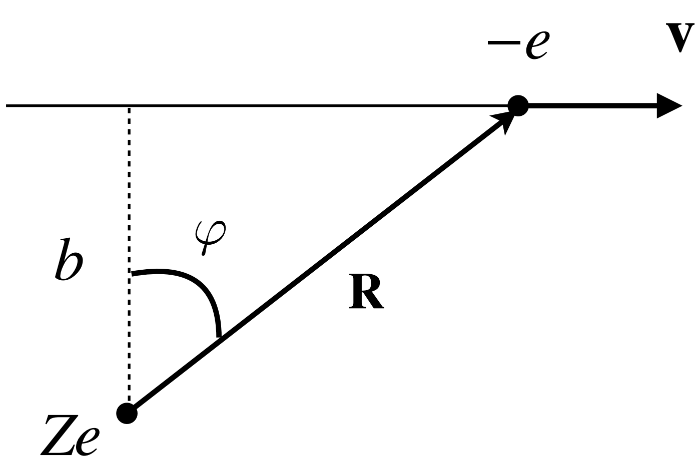
The impact parameter b of a given collision is defined to be the projected distance of the closest approach of the electron to the ion for a given encounter.

If the electron loses only a small fraction of its (large) kinetic energy, the trajectory will be deflected only slightly from straight line, and the closest approach will be approximately equal to the impact parameter b.

In a plasma of randomly moving particles, the impact parameter will differ from collision to collision. For our purposes, the ion may be considered to be infinitesimally small.

- **Small-angle scattering approximation:**

The interaction of a free electron with a free ion is described by the following equation:



dipole moment: $\mathbf{d} = -e\mathbf{R}$

$$\ddot{\mathbf{d}} = -e\dot{\mathbf{v}}$$

$$m_e \left|\dot{\mathbf{v}}\right| = \frac{Ze^2}{b^2 + v^2 t^2} \quad \text{and} \quad \dot{\mathbf{v}} \perp \mathbf{v}, \quad \frac{v}{c} \ll 1$$

Take the Fourier transform of the second derivative of the dipole moment.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \ddot{\mathbf{d}} e^{i\omega t} dt = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\mathbf{v}} e^{i\omega t} dt \longrightarrow$$

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} \ddot{\mathbf{d}} e^{i\omega t} dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d^2}{dt^2}\left[\int_{-\infty}^{\infty} \bar{\mathbf{d}}(\omega') e^{-i\omega' t} d\omega'\right] e^{i\omega t} dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(-\omega'^2\right)\left[\int_{-\infty}^{\infty} \bar{\mathbf{d}}(\omega') e^{-i\omega' t} d\omega'\right] e^{i\omega t} dt \\
&= -\int_{-\infty}^{\infty} \omega'^2 \bar{\mathbf{d}}(\omega') \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt\right] d\omega' \\
&= -\int_{-\infty}^{\infty} \omega'^2 \bar{\mathbf{d}}(\omega') \delta\left(\omega' - \omega\right) d\omega' \\
&= -\omega^2 \bar{\mathbf{d}}(\omega)
\end{aligned}$$

$$-\omega^2 \bar{\mathbf{d}}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\mathbf{v}} e^{i\omega t} dt$$

$$\boxed{\bar{\mathbf{d}}(\omega) = \frac{e}{2\pi\omega^2}\int_{-\infty}^{\infty} \dot{\mathbf{v}} e^{i\omega t} dt}$$

(a-1) a rough estimation

Collision time: We assume that the interaction between the electron and the ion happens only when the electron passes close to the ion. The characteristic time τ is

$$\tau = \frac{b}{v}$$

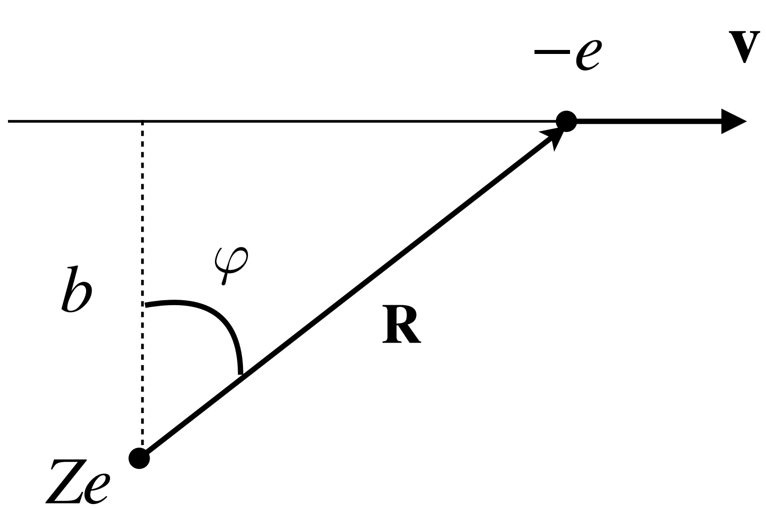
Then, the radiation consists of a single pulse at an angular frequency of $\omega \approx 1/\tau = v/b$. The characteristic frequency of the radiation due to a single collision is thus

$$\nu = \frac{\omega}{2\pi} \approx \frac{v}{2\pi b}$$

The smaller the impact parameter, the higher the emitted frequency.

For $\omega\tau \gg 1$, the exponential in the integral oscillates rapidly, and the integral is small.

For $\omega\tau \ll 1$, the exponential is essentially unity, so we may write



$$\bar{\mathbf{d}}(\omega) = \frac{e}{2\pi\omega^2}\int_{-\infty}^{\infty} \dot{\mathbf{v}} e^{i\omega t} dt$$

$$\bar{\mathbf{d}}(\omega) \approx \begin{cases} \dfrac{e}{2\pi\omega^2}\Delta\mathbf{v} & \text{if} \quad \omega\tau \ll 1 \\ 0 & \text{if} \quad \omega\tau \gg 1 \end{cases}$$

Here, $\Delta\mathbf{v}$ is the change of velocity during the collision.

(a-2)

- **Spectrum of the emitted radiation by a single electron**:

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\left|\bar{d}(\omega)\right|^2 = \begin{cases} \dfrac{2e^2}{3\pi c^3}|\Delta \mathbf{v}|^2 & \text{if} \quad \omega\tau \ll 1 \\ 0 & \text{if} \quad \omega\tau \gg 1 \end{cases}$$

Let us now estimate $\Delta \mathbf{v}$. Since the path is almost linear, the change in velocity is predominantly normal to the path.

$$F_\perp = F\cos\varphi$$
$$\leftarrow \quad F = Ze^2/R^2, \quad \cos\varphi = b/R, \quad R = (b^2 + v^2t^2)^{1/2}$$
$$\therefore \quad F_\perp = Ze^2 b/(b^2 + v^2t^2)^{1/2}$$

$$\Delta v \approx \Delta v_\perp = \frac{1}{m_e}\int F_\perp dt$$
$$= \frac{Ze^2}{m_e}\int_{-\infty}^{\infty}\frac{b}{(b^2+v^2t^2)^{3/2}}dt = \frac{2Ze^2}{m_e bv}$$

$$\int_{-\infty}^{\infty}\frac{dx}{(b^2+x^2)^{3/2}} = \left.\frac{x}{b^2(b^2+x^2)^{1/2}}\right|_{-\infty}^{\infty} = \frac{2}{b^2}$$

Thus for small angle scatterings, **the emission from a single collision** is

$$\boxed{\frac{dW(b)}{d\omega} = \begin{cases} \dfrac{8Z^2e^6}{3\pi c^3 m_e^2 v^2 b^2} & \text{if} \quad b \ll v/\omega \\ 0 & \text{if} \quad b \gg v/\omega \end{cases}}$$

low frequency

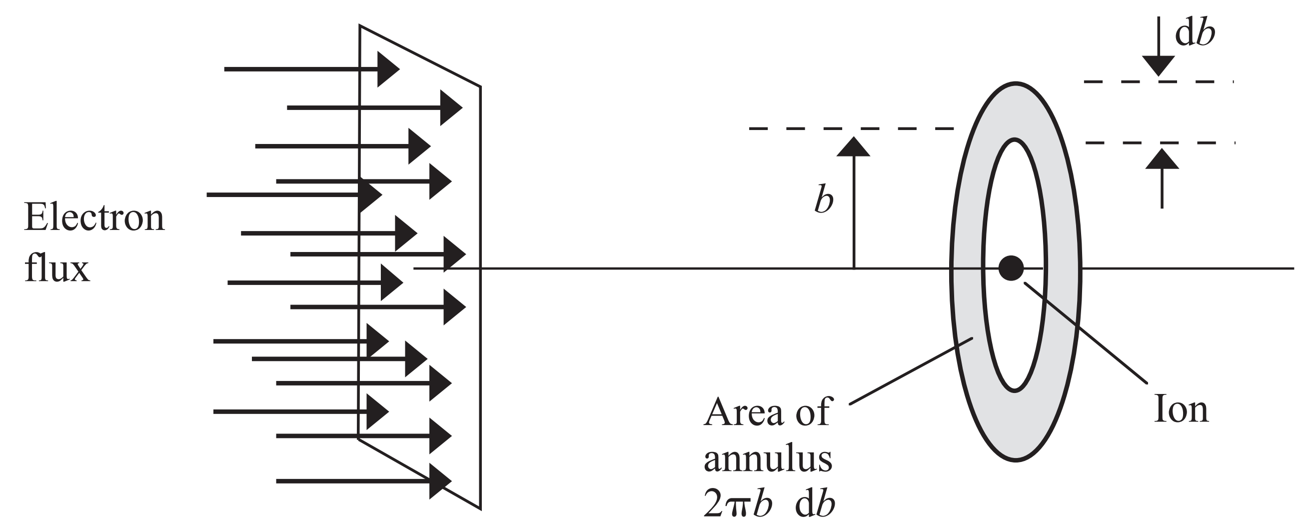
high frequency

(a-3)

- **Total spectrum for a medium** with ion density n_i, electron density n_e and for a fixed electron speed.

  - flux of electrons (per unit area per unit time) incident on one ion = $n_e v$
  - element of area = $2\pi b db$
  - a good approximation is obtained in low-frequency regimes $b \ll v/\omega$:

$$\frac{dW}{d\omega dV dt} = n_e n_i v \int_{b_{\min}}^{\infty} \frac{dW(b)}{d\omega} 2\pi b db$$
$$= \frac{16e^6}{3c^3 m_e^2 v} n_e n_i Z^2 \int_{b_{\min}}^{b_{\max}} \frac{db}{b}$$
$$= \frac{16e^6}{3c^3 m_e^2 v} n_e n_i Z^2 \ln\left(\frac{b_{\max}}{b_{\min}}\right)$$



[Bradt (2008) Astrophysics Processes]

- **Upper limit:** The upper limit is uncertain, but it is of order of $b_{\max} \sim v/\omega$. Since $b_{\max}$ occurs inside the logarithm, its precise value is not very important.

  The integral is negligible for $b \gg b_{\max} \sim v/\omega$.

(b-1) a better calculation

$$\bar{\mathbf{d}}(\omega) = \frac{e}{2\pi\omega^2}\int_{-\infty}^{\infty} \dot{\mathbf{v}} e^{i\omega t} dt$$

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\left|\bar{d}(\omega)\right|^2$$

$$m_e\left|\dot{\mathbf{v}}\right| = \frac{Ze^2}{b^2+v^2t^2}$$

$$\begin{aligned}
\frac{dW}{d\omega} &= \frac{8\pi\omega^4}{3c^3}\left|\bar{d}(\omega)\right|^2 = \frac{8\pi\omega^4}{3c^3}\frac{e^2}{4\pi^2\omega^4}\left|\int_{-\infty}^{\infty}\dot{\mathbf{v}}e^{i\omega t}dt\right|^2 \\
&= \frac{2e^2}{3c^3\pi}\frac{Z^2e^4}{m_e^2}\left|\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{b^2+v^2t^2}dt\right|^2 \\
&= \frac{2Z^2e^6}{3m_e^2c^3\pi}\left|\int_0^{\infty}\frac{e^{i\omega t}+e^{-i\omega t}}{b^2+v^2t^2}dt\right|^2 \\
&= \frac{2Z^2e^6}{3m_e^2c^3\pi}4\left|\int_0^{\infty}\frac{\cos\omega t}{b^2+v^2t^2}dt\right|^2 \\
&= \frac{2Z^2e^6}{3m_e^2c^3\pi}4\left|\frac{\pi}{2bv}e^{-\omega b/v}\right|^2 \\
&= \frac{2Z^2e^6}{3m_e^2c^3\pi}\left(\frac{\pi}{bv}\right)^2 e^{-2\omega b/v}
\end{aligned}$$

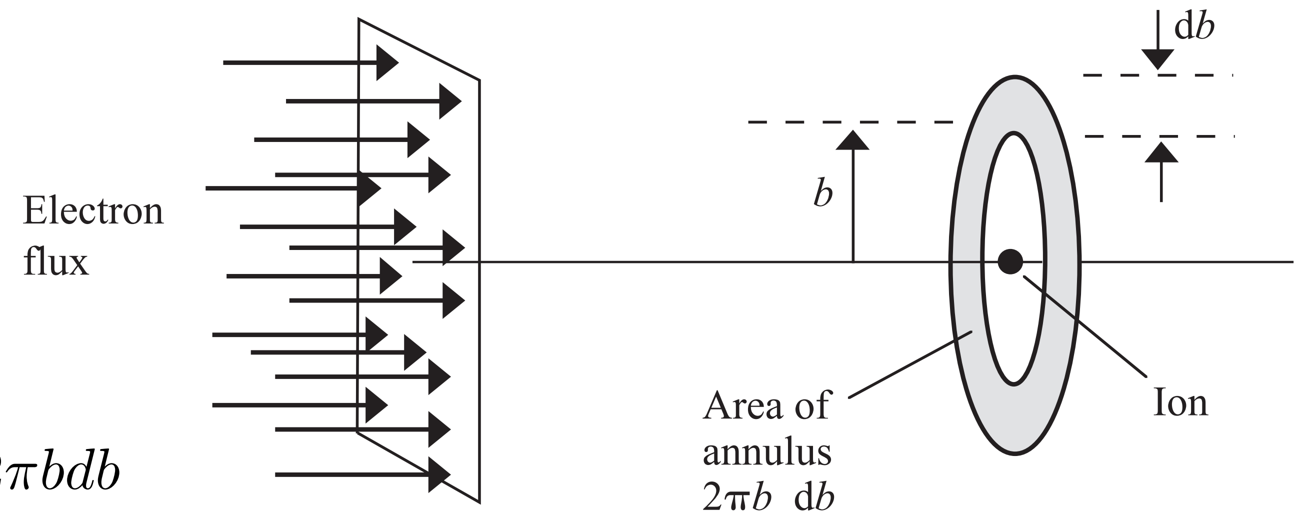
$$\boxed{\frac{dW(b)}{d\omega} = \frac{2Z^2e^6}{3m_e^2c^3\pi}\left(\frac{\pi}{bv}\right)^2 e^{-2\omega b/v}}$$

$$\boxed{\frac{dW(b)}{d\nu} = \frac{4Z^2e^6}{3m_e^2c^3}\left(\frac{\pi}{bv}\right)^2 e^{-4\pi\nu b/v}}$$

These equations represent the ***single particle spectrum*** resulting from a single collision.

(b-2)

- **Total spectrum for a medium** with ion density n_i, electron density n_e and for a fixed electron speed.
  - flux of electrons (per unit area per unit time) incident on one ion = $n_e v$
  - element of area = $2\pi b db$
  - a good approximation is obtained in low-frequency regimes $b \ll v/\omega$:



[Bradt (2008) Astrophysics Processes]

$$
\begin{aligned}
\frac{dW(v,\nu)}{d\nu dV dt} &= n_e n_i v \int_{b_{\min}}^{\infty} \frac{dW(b)}{d\nu} 2\pi b db \\
&= n_e n_i v \int_{b_{\min}}^{\infty} \frac{4Z^2e^6}{3m_e^2c^3} \left(\frac{\pi}{bv}\right)^2 e^{-4\pi\nu b/v} 2\pi b db \\
&= n_e n_i \frac{8\pi^3 Z^2 e^6}{3m_e^2c^3 v} \int_{b_{\min}}^{\infty} \frac{e^{-4\pi\nu b/v}}{b} db \\
&= n_e n_i \frac{8\pi^3 Z^2 e^6}{3m_e^2c^3 v} E_1(\xi_{\min})
\end{aligned}
$$

Here, the exponential integral is defined as

$$E_1(\xi_{\min}) = \int_{\xi_{\min}}^{\infty} \frac{e^{-\xi}}{\xi} d\xi \quad \text{and} \quad \xi_{\min} = 4\pi\nu b_{\min}/v$$

E_1 behaves like a negative exponential for large values and like a logarithm for small values. E_1 can be bracketed by as follows:

$$\frac{1}{2}e^{-x}\ln\left(1+\frac{2}{x}\right) < E_1(x) < e^{-x}\ln\left(1+\frac{1}{x}\right)$$

- **Lower limits for b_{min}**

  What should we choose for b_{min} or ξ_{min}?

  (1) small-angle approximation: To protect our perturbation procedure from large-angle deflections, we can choose by

  $$\Delta v/v \sim (Ze^2/b)/(m_e v^2/2) < 1 \quad \rightarrow \quad \boxed{b_{\text{min}} > b_{\text{min}}^{(1)} \equiv Ze^2/m_e v^2}$$

  (2) from the uncertainty principle:

  $$\Delta x \Delta p \geq \hbar \quad \rightarrow \boxed{b_{\text{min}} > b_{\text{min}}^{(2)} \equiv \hbar/m_e v \;\; \text{(de Broglie wavelength)}}$$

  When $\quad b_{\text{min}}^{(1)} \gg b_{\text{min}}^{(2)}$ or $\dfrac{1}{2}m_e v^2 \ll Z^2\text{Ry}$ $\quad \left(\text{Ry} \equiv \dfrac{m_e c^4}{2\hbar^2} = 13.6 \text{ eV} = \text{Rydberg energy for H atom}\right)$

  a classical description of the scattering process is valid. Then, $\quad b_{\text{min}} = b_{\text{min}}^{(1)}$

  When $\quad b_{\text{min}}^{(1)} \ll b_{\text{min}}^{(2)}$ or $\dfrac{1}{2}m_e v^2 \gg Z^2\text{Ry}$

  the uncertainty principle plays an important role. Then, $\quad b_{\text{min}} = b_{\text{min}}^{(2)}$

  In summary,

  $$\boxed{\therefore \;\; b_{\text{min}} = \max\left(b_{\text{min}}^{(1)}, b_{\text{min}}^{(2)}\right)}$$

- For any regime the exact results are conveniently stated in terms of correction factor or **Gaunt factor**. Precise expression of the Gaunt factor comes from QED (Quantum Electrodynamics) computation.

- For any regime the exact results are conveniently stated in terms of correction factor or **Gaunt factor**. Precise expression of the Gaunt factor comes from QED (Quantum Electrodynamics) computation.

$$\boxed{4\pi j_\omega(v,\omega) = \frac{dW}{d\omega dV dt} = \frac{16\pi e^6}{3\sqrt{3}c^3 m_e^2 v} n_e n_i Z^2 g_{\rm ff}(v,\omega)}$$

$$g_{\rm ff}(v,\omega) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right)$$

Typically $g_{\rm ff} \approx 1$ to a few.

Tables and plots are available by Bressaard and van de Hulst (1962) and Karzas and Latter (1961).

[Thermal Bremsstrahlung Emission]

- We now average the above single-speed expression over a thermal distribution of electron speeds.

$$f(\mathbf{v})d^3\mathbf{v} = \left(\frac{m_e}{2\pi kT}\right)^{3/2} e^{-m_e v^2/2kT} d^3\mathbf{v} = \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} e^{-m_e v^2/2kT} v^2 dv$$

At frequency ν, the incident velocity must be at least such that $\frac{1}{2}m_e v^2 \geq h\nu$, because otherwise a photon of energy $h\nu$ could not be created.

This cutoff in the lower limit of the integration over electron velocities is called a ***photon discreteness effect***.

$$\frac{dW}{dV\,dt\,d\omega} = \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} \int_{v_{\min}}^{\infty} \frac{dW(v,\omega)}{dV\,dt\,d\omega} v^2 e^{-m_e v^2/2kT} dv \qquad \left(\text{where} \quad v_{\min} \equiv \sqrt{\frac{2h\nu}{m_e}}\right)$$

$$= \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} \frac{16\pi e^6}{3^{3/2} c^3 m_e^2} n_i n_e Z^2 \int_{v_{\min}}^{\infty} \frac{g_{\text{ff}}(v,\omega)}{v} v^2 e^{-m_e v^2/2kT} dv$$

$$= \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} \frac{16\pi e^6}{3^{3/2} c^3 m_e^2} n_i n_e Z^2 \int_{v_{\min}}^{\infty} g_{\text{ff}}(v,\omega) e^{-m_e v^2/2kT} d(v^2/2)$$

The exponential factor can be written as

$$\exp\left(-\frac{m_e v^2}{2kT}\right) = \exp\left(-\frac{m_e v_{\min}^2}{2kT}\right) \exp\left(-\frac{m_e(v^2 - v_{\min}^2)}{2kT}\right) = \exp\left(-\frac{h\nu}{kT}\right) \exp\left(-\frac{m_e(v^2 - v_{\min}^2)}{2kT}\right)$$

$$\boxed{\frac{dW}{dV\,dt\,d\omega} = \left(\frac{m_e}{kT}\right)^{3/2}\left(\frac{2}{\pi}\right)^{1/2}\frac{16\pi e^6}{3^{3/2}c^3m_e^2}n_in_eZ^2e^{-h\nu/kT}\left(\frac{m_e}{kT}\right)^{-1}\int_0^\infty g_{\rm ff}(v,\omega)e^{-u}du}$$

$$\left(\text{where}\quad u \equiv \frac{m_e(v^2-v_{\rm min}^2)}{2kT}\right)$$

Or, using the more precise formula,

$$\left(\text{where}\quad v_{\rm min} \equiv \sqrt{\frac{2h\nu}{m_e}}\right)$$

$$\begin{aligned}
\frac{dW}{dV\,dt\,d\nu} &= \left(\frac{m_e}{kT}\right)^{3/2}\left(\frac{2}{\pi}\right)^{1/2}\int_{v_{\rm min}}^\infty \frac{dW(v,\nu)}{dV\,dt\,d\nu}v^2e^{-m_ev^2/2kT}dv \\
&= n_en_i\left(\frac{m_e}{kT}\right)^{3/2}\left(\frac{2}{\pi}\right)^{1/2}\frac{8\pi^3Z^2e^6}{3m_e^2c^3}\int_{v_{\rm min}}^\infty E_1(\xi_{\rm min})ve^{-m_ev^2/2kT}dv \\
&= n_en_i\left(\frac{m_e}{kT}\right)^{3/2}\left(\frac{2}{\pi}\right)^{1/2}\frac{8\pi^3Z^2e^6}{3m_e^2c^3}\int_{v_{\rm min}}^\infty E_1(\xi_{\rm min})e^{-m_ev^2/2kT}d(v^2/2) \\
&= n_en_i\left(\frac{m_e}{kT}\right)^{3/2}\left(\frac{2}{\pi}\right)^{1/2}\frac{8\pi^3Z^2e^6}{3m_e^2c^3}\left(\frac{kT}{m_e}\right)\int_{x_{\rm min}}^\infty E_1(\xi_{\rm min})e^{-x}dx \\
&= n_en_i\left(\frac{2m_e}{\pi kT}\right)^{1/2}\frac{8\pi^3Z^2e^6}{3m_e^2c^3}\int_{x_{\rm min}}^\infty E_1(\xi_{\rm min})e^{-x}dx
\end{aligned}$$

$$\boxed{\frac{dW}{dV\,dt\,d\nu} = n_en_i\left(\frac{2m_e}{\pi kT}\right)^{1/2}\frac{8\pi^3Z^2e^6}{3m_e^2c^3}\int_{x_{\rm min}}^\infty E_1(\xi_{\rm min})e^{-x}dx \qquad \begin{aligned}\text{where}\quad x_{\rm min} &= m_ev_{\rm min}^2/2kT = h\nu/kT \\ \xi_{\rm min} &= 4\pi\nu b_{\rm min}/v = 4\pi\nu b_{\rm min}\sqrt{\frac{m_e}{2kT}}x^{-1/2}\end{aligned}}$$

In terms of $\nu = \omega/2\pi$, the volume emissivity ($\varepsilon_\nu^{\rm ff} = 4\pi j_\nu^{\rm ff}$) is

$$\varepsilon_\nu^{\rm ff} \equiv 2\pi \frac{dW}{dV\,dt\,d\omega} = 2\pi \left(\frac{m_e}{kT}\right)^{1/2} \left(\frac{2}{\pi}\right)^{1/2} \frac{16\pi e^6}{3^{3/2} c^3 m_e^2} n_i n_e Z^2 e^{-h\nu/kT} \int_0^\infty g_{\rm ff}(v,\omega) e^{-u} du$$

$$= \left(\frac{2}{kT}\right)^{1/2} \frac{32\pi^{3/2} e^6}{3^{3/2} c^3 m_e^{3/2}} n_i n_e Z^2 e^{-h\nu/kT} \int_0^\infty g_{\rm ff}(v,\omega) e^{-u} du$$

$$= \frac{2^5 \pi e^6}{3 m_e c^3} \left(\frac{2\pi}{3 k m_e}\right)^{1/2} n_i n_e Z^2 T^{-1/2} e^{-h\nu/kT} \overline{g_{\rm ff}}$$

$$\boxed{\varepsilon_\nu^{\rm ff} = 6.8 \times 10^{-38} n_i n_e Z^2 T^{-1/2} e^{-h\nu/kT} \overline{g_{\rm ff}}}$$

$$\overline{g_{\rm ff}} \equiv \int_0^\infty g_{\rm ff}(v,\omega) e^{-u} du$$

$$g_{\rm ff}(v,\omega) = (\sqrt{3}/\pi) \ln\left(b_{\rm max}/b_{\rm min}\right)$$

where $\overline{g_{\rm ff}}$ is the velocity-averaged free-free Gaunt factor.

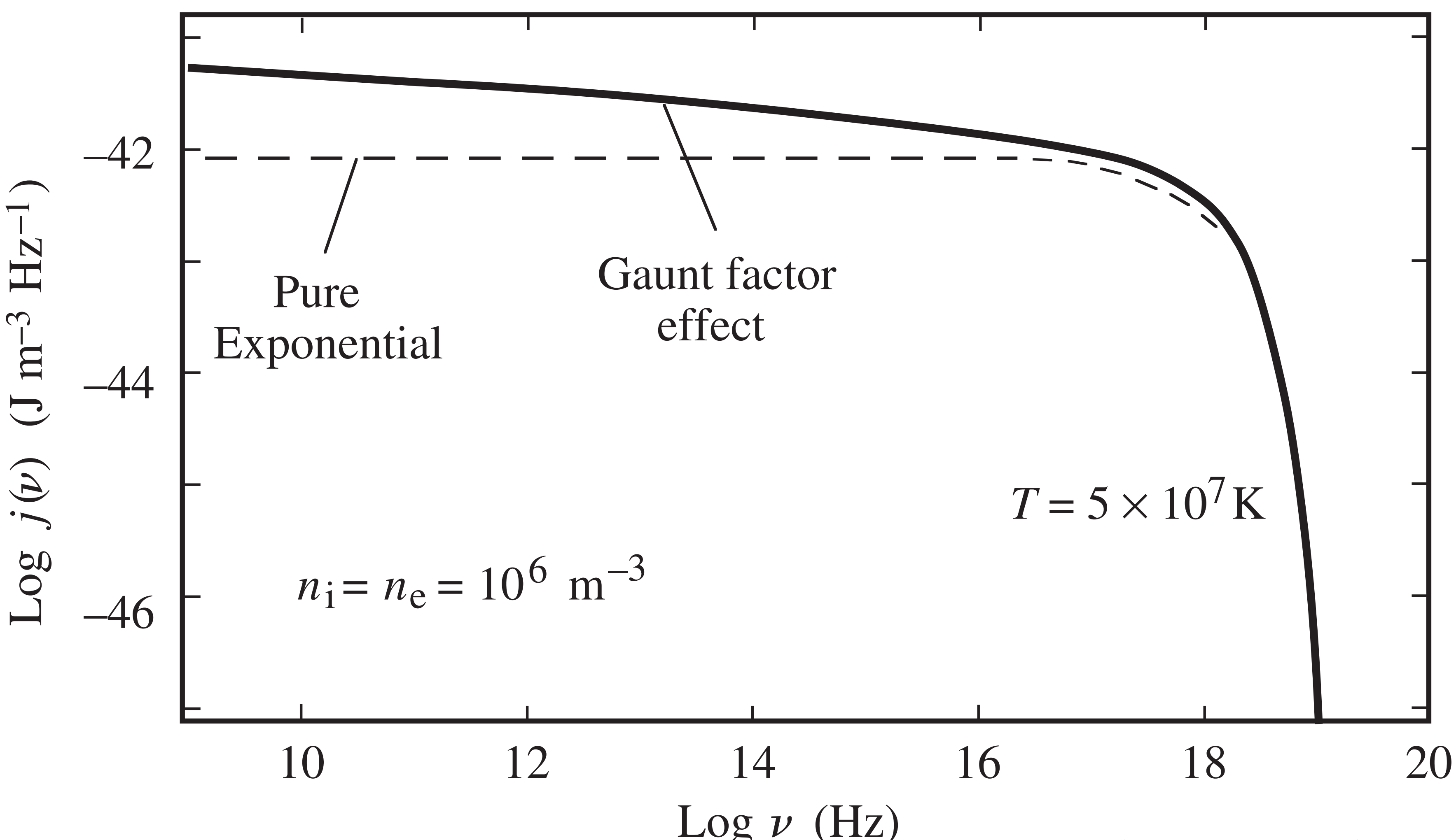
Summing over all ion species gives the emissivity:

$$\boxed{\varepsilon_\nu^{\rm ff} = 6.8 \times 10^{-38} \sum_i n_i n_e Z^2 T^{-1/2} e^{-h\nu/kT} \overline{g_{\rm ff}} \quad ({\rm erg\ s^{-1}\ cm^{-3}\ Hz^{-1}})}$$

Note that main frequency dependence is $\varepsilon_\nu^{\rm ff} \propto \exp\left(-h\nu/kT\right)$, which shows a "flat spectrum" with a cut off at $\nu \sim kT/h$. The cut-off of the spectrum can be used to determine the temperature of hot plasma.

For a hydrogen plasma ($Z = 1$) with $T > 3 \times 10^5$ K at low frequencies ($h\nu \ll kT$) Gaunt factor is given by

$$\overline{g_{\rm ff}} = \frac{\sqrt{3}}{\pi} \ln\left(\frac{2.25kT}{h\nu}\right)$$



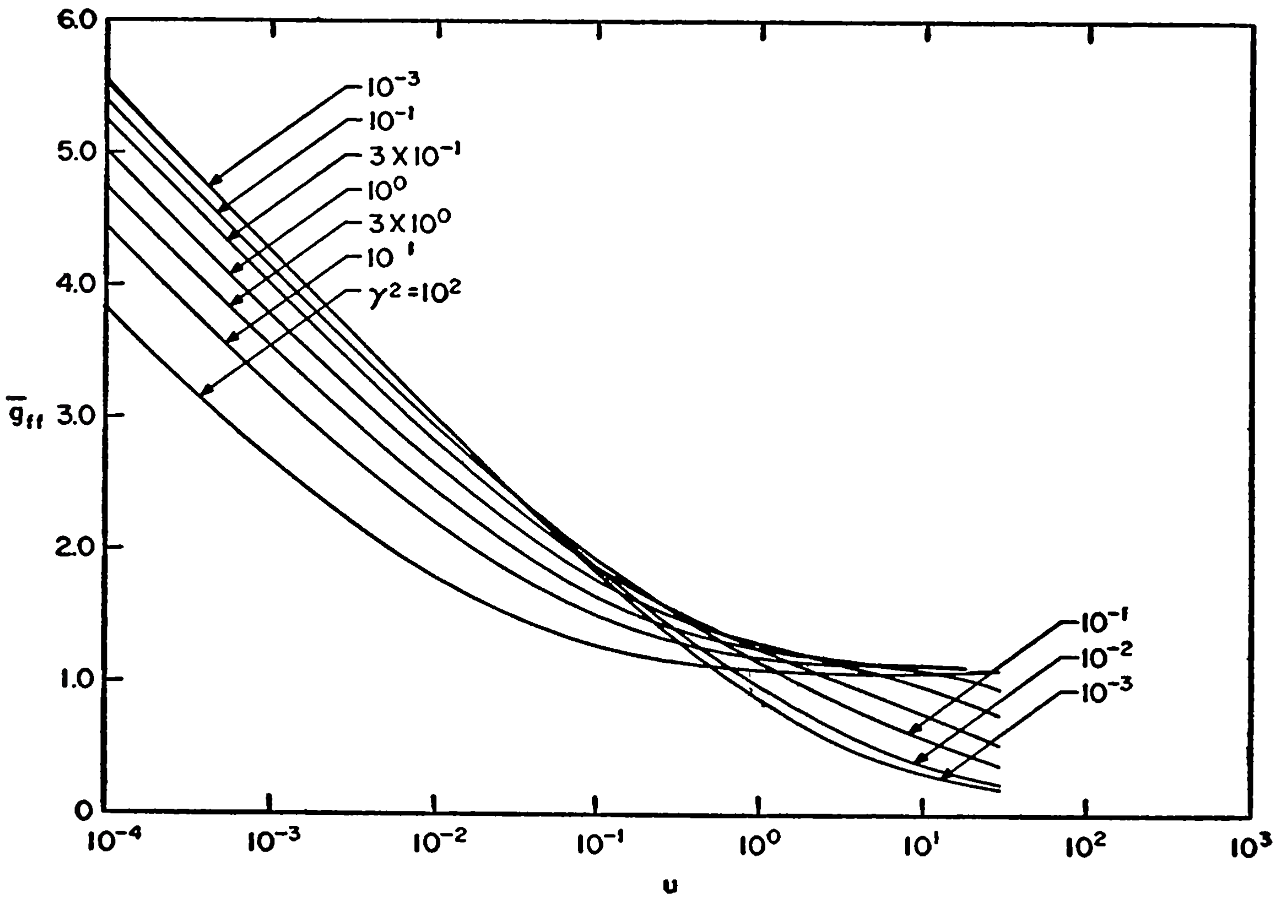
[Bradt, Astrophysics Processes]

- Gaunt Factor

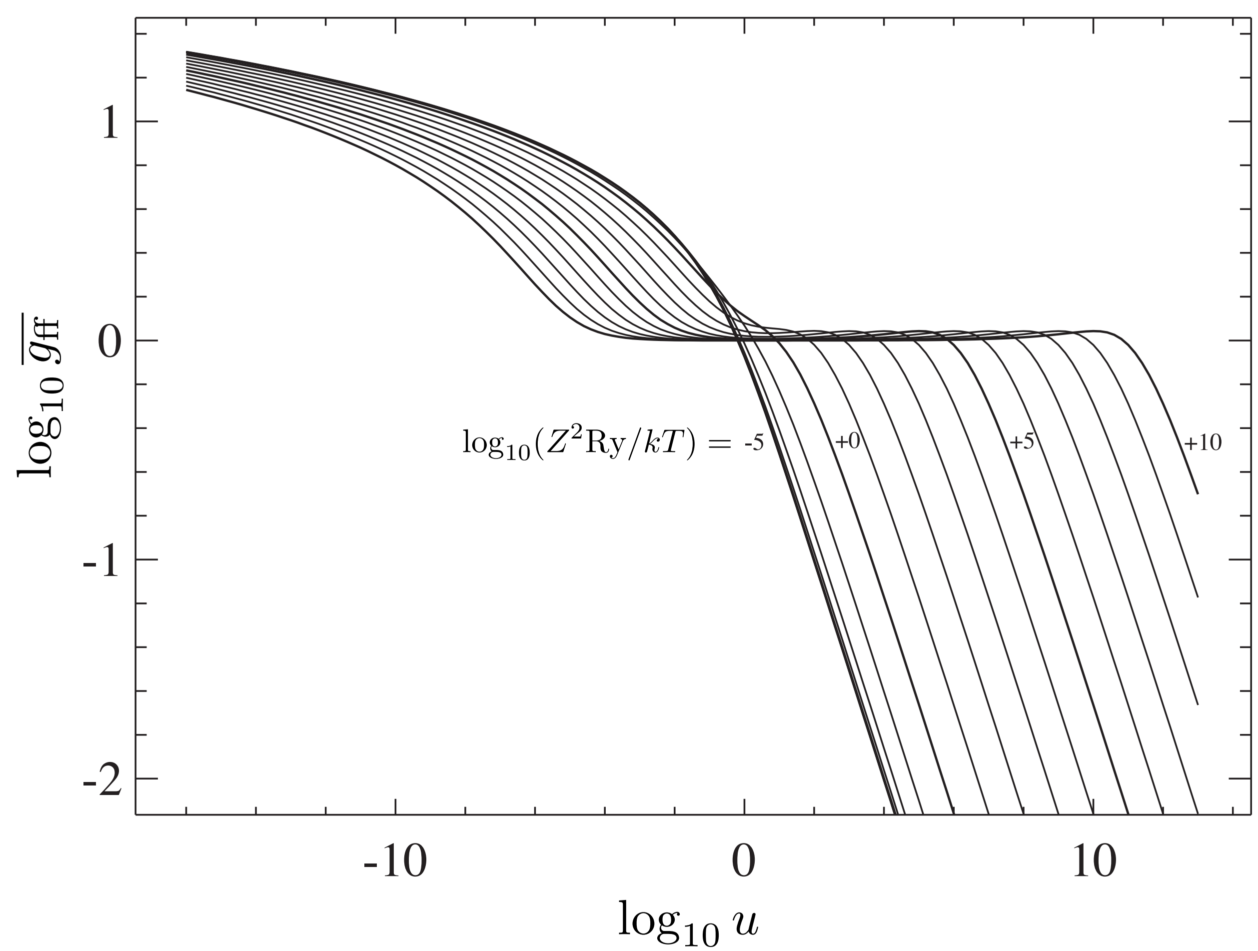
- Note that the values of Gaunt factor for $u = h\nu/kT \gg 1$ are not important, since the spectrum cuts off for these values.

$$\overline{g_{\rm ff}} \sim \begin{cases} 1 & \text{for } u \sim 1 \\ 1-5 & \text{for } 10^{-4} < u < 1 \end{cases}$$

Karzas & Latter (1961, ApJS, 6, 167)



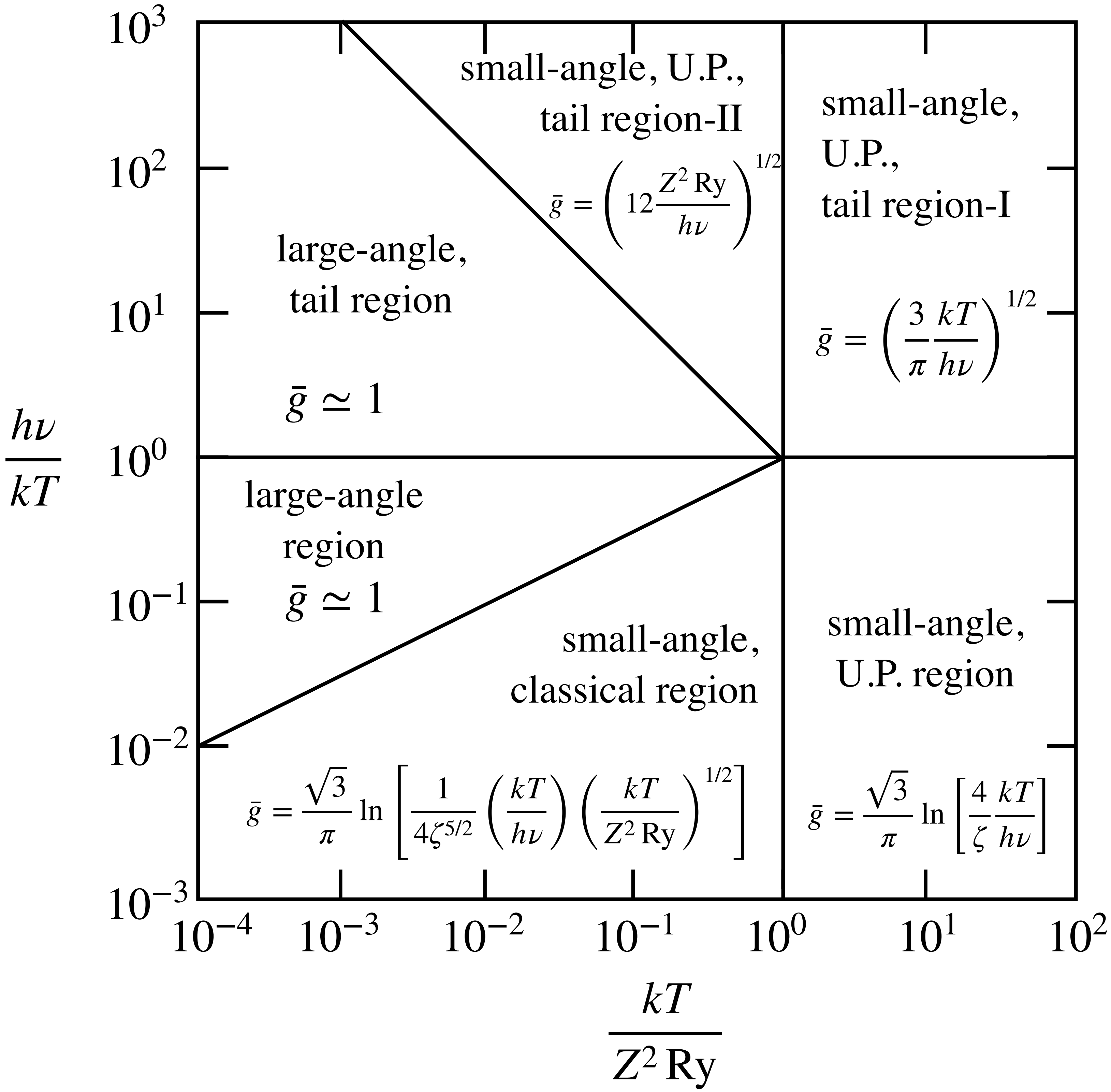
van Hoof et al. (2014, MNRAS, 444, 420)



$$u = h\nu/kT = 4.8 \times 10^{11}\nu/T$$

$$\gamma^2 = Z^2\mathrm{Ry}/kT = 1.58 \times 10^5 Z^2/T$$

- Novikov & Thorne (1973, in Black Holes, Les Houches)



U.P. = Uncertainty principle

- To obtain the formulas for **non-thermal bremsstrahlung**, one needs to know the actual distributions of velocities, and the formula for emission from a single-speed electron must be averaged over that distribution. One also must have the appropriate Gaunt factors.
- Integrated Bremsstrahlung emission per unit volume:

$$\begin{aligned}
\varepsilon^{\rm ff} \equiv \int \varepsilon^{\rm ff}(\nu)d\nu &= \frac{2^5\pi e^6}{3m_ec^3}\left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-1/2}n_in_eZ^2\int e^{-h\nu/kT}\overline{g_{\rm ff}}d\nu \\
&= \frac{2^5\pi e^6}{3m_ec^3}\left(\frac{2\pi}{3km_e}\right)^{1/2}\left(\frac{kT^{1/2}}{h}\right)n_in_eZ^2\int_0^\infty e^{-u}\overline{g_{\rm ff}}du \\
&= \left(\frac{2\pi kT}{3m_e}\right)^{1/2}\frac{2^5\pi e^6}{3hm_ec^3}n_in_eZ^2\overline{g_{\rm B}}
\end{aligned}$$

$$\boxed{\varepsilon^{\rm ff}\left(\equiv\frac{dW}{dtdV}\right) = 1.42\times 10^{-27}n_in_eZ^2T^{1/2}\overline{g_{\rm B}}\ \text{erg cm}^{-3}\ \text{s}^{-1} \longrightarrow \varepsilon_{\rm ff}\propto T^{1/2}}$$

where frequency average of the velocity averaged Gaunt factor:

$$\begin{aligned}\overline{g_{\rm B}} &= \int_0^\infty e^{-u}\overline{g_{\rm ff}}du \quad (u = h\nu/kT) \\ &= 1.3\pm 0.2\end{aligned}$$

$$\overline{g_{\rm B}} \approx 1 + \frac{0.44}{1+0.058\left[\ln(T/10^{5.4}Z^2K)\right]^2}$$

$$\text{for } 10^{4.2}\ \text{K} \le T/Z^2 \le 10^{8.2}\ \text{K},\ \text{Draine (2011)}$$

[Thermal Bremsstrahlung (free-free) Absorption]

- ***Free-free absorption is an inverse process of the free-free emission***, which we get by "running the film backward."
- Absorption of radiation by free electrons moving in the field of ions:

For thermal system, Kirchoff's law says:

$$\frac{1}{4\pi}\frac{dW}{dV\,dt\,d\nu} = j_\nu^{\rm ff} = \alpha_\nu^{\rm ff} B_\nu(T) \qquad B_\nu(T) = \left(2h\nu^3/c^2\right)\left[\exp(h\nu/kT) - 1\right]^{-1}$$

We have then

$$\boxed{\begin{aligned}\alpha_\nu^{\rm ff} &= \frac{4e^6}{3m_e hc}\left(\frac{2\pi}{3km_e}\right)^{1/2} n_i n_e Z^2 T^{-1/2}\nu^{-3}\left(1 - e^{-h\nu/kT}\right)\overline{g_{\rm ff}} \\ &= 3.7\times10^8\; n_i n_e Z^2 T^{-1/2}\nu^{-3}\left(1 - e^{-h\nu/kT}\right)\overline{g_{\rm ff}}\quad({\rm cm}^{-1})\end{aligned}}$$

For $h\nu \gg kT$, $\alpha_\nu^{\rm ff} = 3.7\times10^8 n_i n_e Z^2 T^{-1/2}\nu^{-3}\overline{g_{\rm ff}}\quad({\rm cm}^{-1})$ $\longrightarrow$ $\tau_\nu \propto \alpha_\nu^{\rm ff} \propto \nu^{-3}$ for $h\nu \gg kT$

For $h\nu \ll kT$, $\alpha_\nu^{\rm ff} = \dfrac{4e^6}{3m_e kc}\left(\dfrac{2\pi}{3km_e}\right)^{1/2} n_i n_e Z^2 T^{-3/2}\nu^{-2}\overline{g_{\rm ff}}$ $\longrightarrow$ $\tau_\nu \propto \alpha_\nu^{\rm ff} \propto \nu^{-2}$ for $h\nu \ll kT$

$$= 0.018 n_i n_e Z^2 T^{-3/2}\nu^{-2}\overline{g_{\rm ff}}$$

$1 - e^{-h\nu/kT} \approx 1 - (1 - h\nu/kT) = h\nu/kT$

Bremsstrahlung self-absorption: **The medium becomes always optically thick at sufficiently small frequency.** Therefore, the free-free emission is absorbed inside plasma at small frequencies.

[Overall Spectral Shape]

- An approximate formula for the free-free Gaunt factor is given by Draine (2011).

$$\overline{g_{\rm ff}} \approx 6.155(Z\nu_9)^{-0.118}T_4^{0.177} \quad (0.14 < Z\nu_9/T_4^{3/2} < 250) \quad \text{where } \nu_9 = \nu/10^9 \text{ Hz}, \quad T_4 = T/10^4 \text{ K}$$

- Emission and absorption coefficients:

$$\boxed{\begin{aligned} j_\nu &= \frac{1}{4\pi}\varepsilon_\nu \approx 3.35\times10^{-40} n_i n_e Z^{1.882} T_4^{-0.323} \nu_9^{-0.118} \text{ erg cm}^{-3}\text{ s}^{-1}\text{ Hz}^{-1}\text{ sr}^{-1} \\ \alpha_\nu &= \frac{j_\nu}{B_\nu} \approx 3.37\times10^{-7} n_i n_e Z^{1.882} T_4^{-1.323} \nu_9^{-2.118} \text{ pc}^{-1} \end{aligned}}$$

- Optical depth:

$$\tau_\nu = \int \alpha_\nu ds \approx 3.37\times10^{-7} Z^{1.882} T_4^{-1.323} \nu_9^{-2.118} \left(\frac{n_i}{n_p}\right)\left[\frac{\text{EM}}{\text{cm}^{-6}\text{pc}}\right] \quad \text{where } \boxed{\overset{\text{emission measure}}{\text{EM} \equiv \int n_e n_p ds}}$$

- SED (Spectral Energy Density) from a uniform sphere

$$\text{for } \tau_\nu \gg 1,\ h\nu \ll kT \longrightarrow I_\nu = S_\nu = B_\nu \qquad F_\nu = \pi B_\nu \left(\frac{R}{d}\right)^2 \propto \nu^2 \quad \text{(Rayleigh-Jeans Law)}$$

$$\text{for } \tau_\nu \ll 1,\ h\nu \gg kT \longrightarrow I_\nu = \int j_\nu ds \qquad F_\nu = 4\pi j_\nu \left(\frac{4\pi R^3}{3}\right)\frac{1}{4\pi d^2} \propto \nu^{-0.1}$$

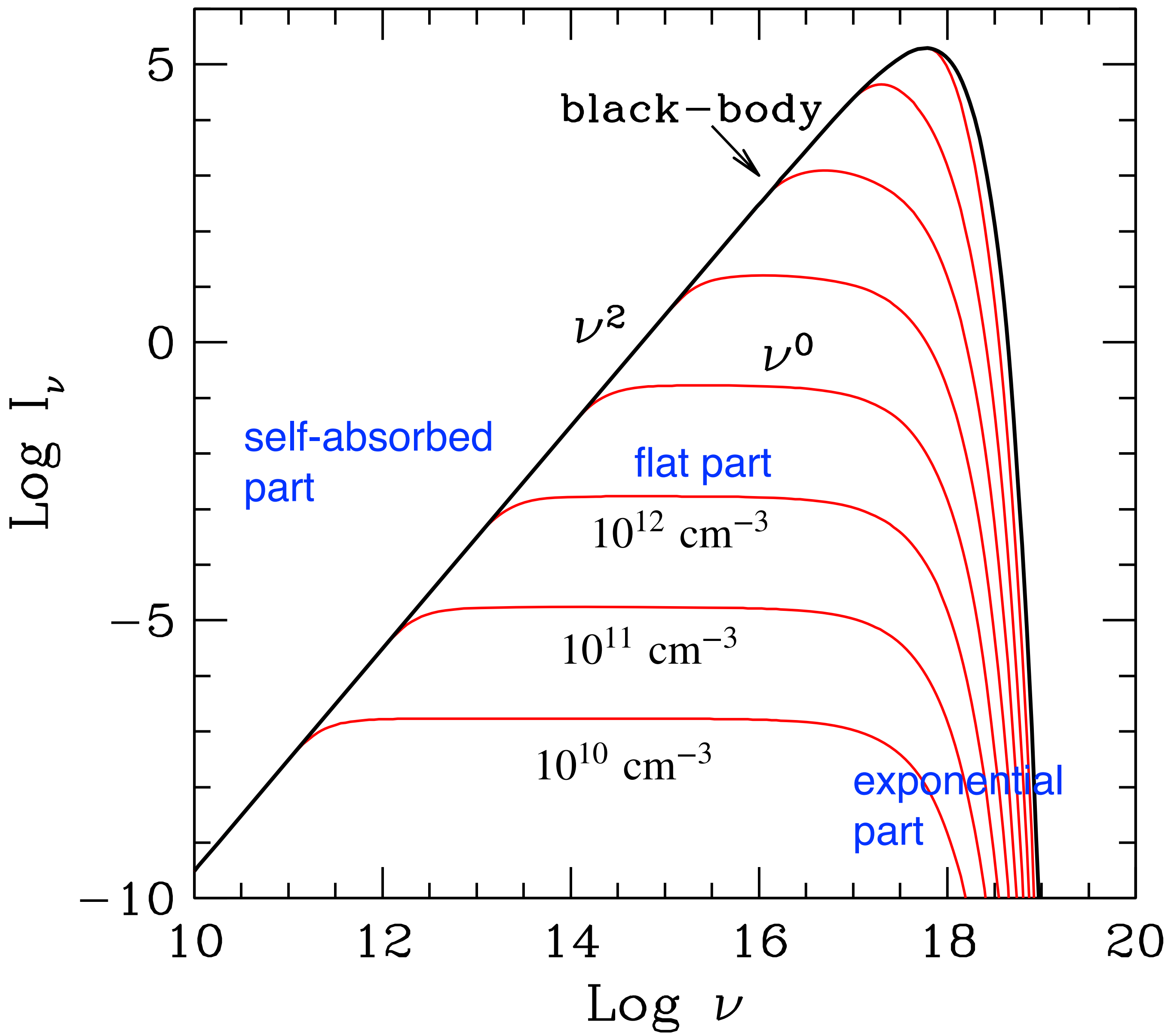
- Spectral shape
  - At low frequencies (optically thick emission),

$$I_\nu = S_\nu = B_\nu \propto \nu^2$$

  - At high frequencies (optically thin emission),

$$I_\nu = \int j_\nu ds \propto e^{-h\nu/kT}$$
$$= \text{constant if } h\nu \ll kT$$

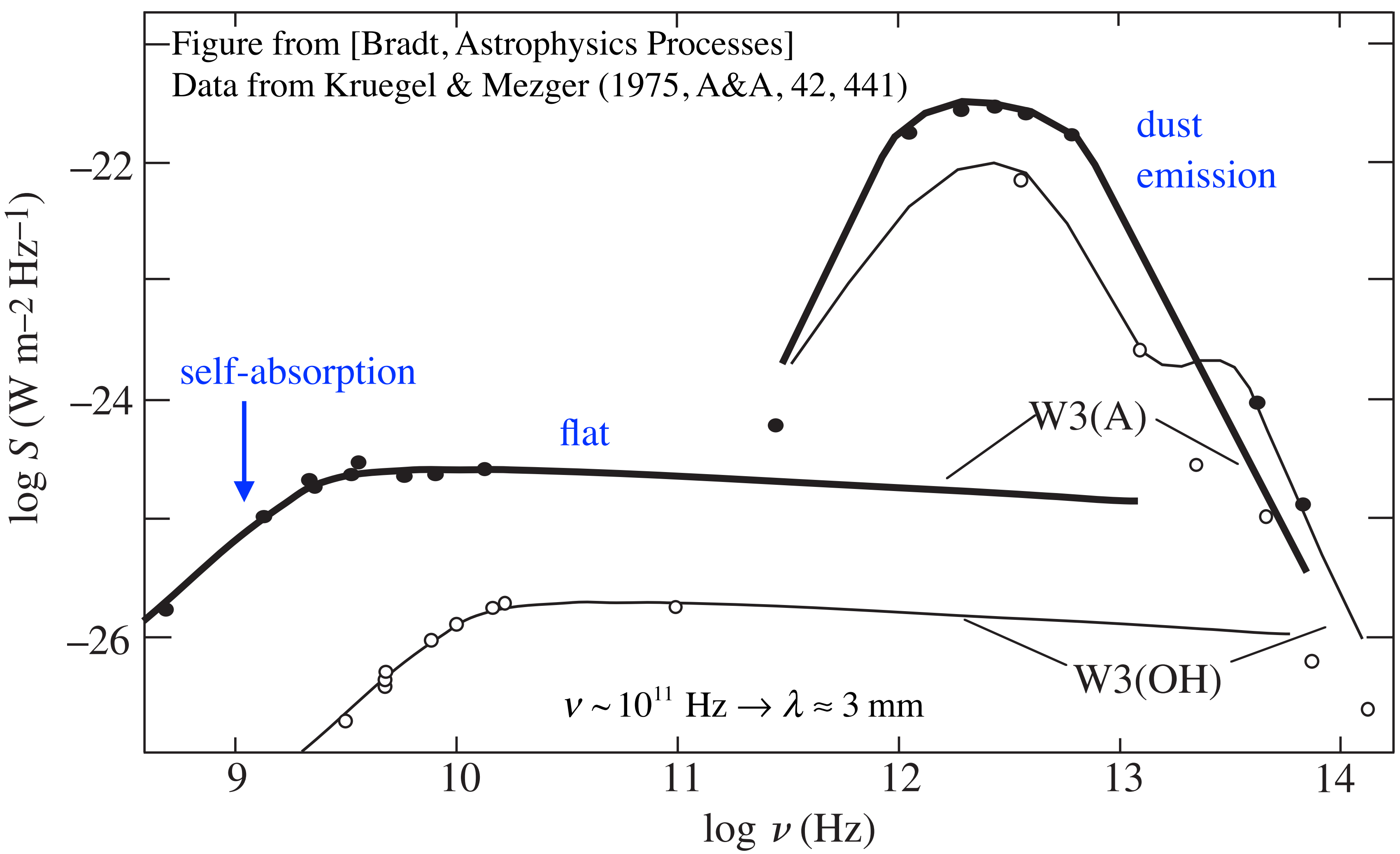
- This spectrum shows the bremsstrahlung intensity from a source of radius $R = 10^{15}$ cm and temperature $T = 10^7$ K.
  - The Gaunt factor is set to unity for simplicity.
  - The density $n_e = n_p$ varies from 10^{10} cm^{-3} to 10^{18} cm^{-3} increasing by a factor 10 for each curve.
  - As the density increases, the optical depth also increases and the spectrum approaches the blackbody one.



Astronomical Examples - H II regions

- The radio spectra of H II regions clearly show the flat spectrum of an optically thin thermal source. The bright stars in the H II regions emit copiously in the UV and thus ionize the hydrogen gas.
- Continuum spectra of two H II regions, W3(A) and W3(OH):

  Note a flat thermal bremsstrahlung (radio), a low-frequency cutoff (radio, self absorption), and a large peak at high frequency (infrared, $10^{12} - 10^{13}$ Hz, 300-30 μm) due to heated, but still "cold" dust grains in the nebula.



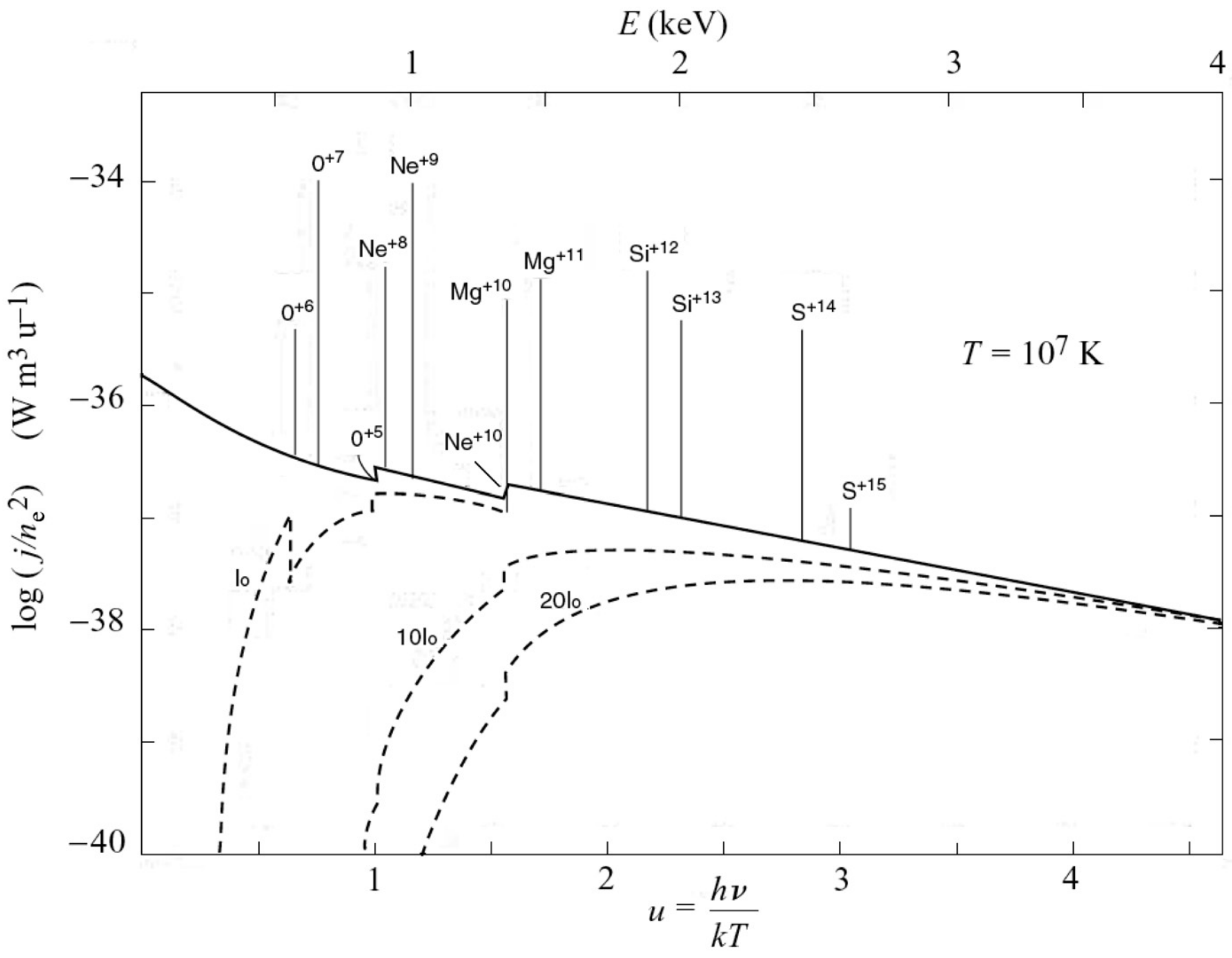
The term H II is pronounced "H two" by astronomers. "H" indicates hydrogen, and "II" is the Roman numeral for 2.

Astronomers use "I" for neutral atoms, "II" for singly-ionized, "III" for doubly-ionized, etc.

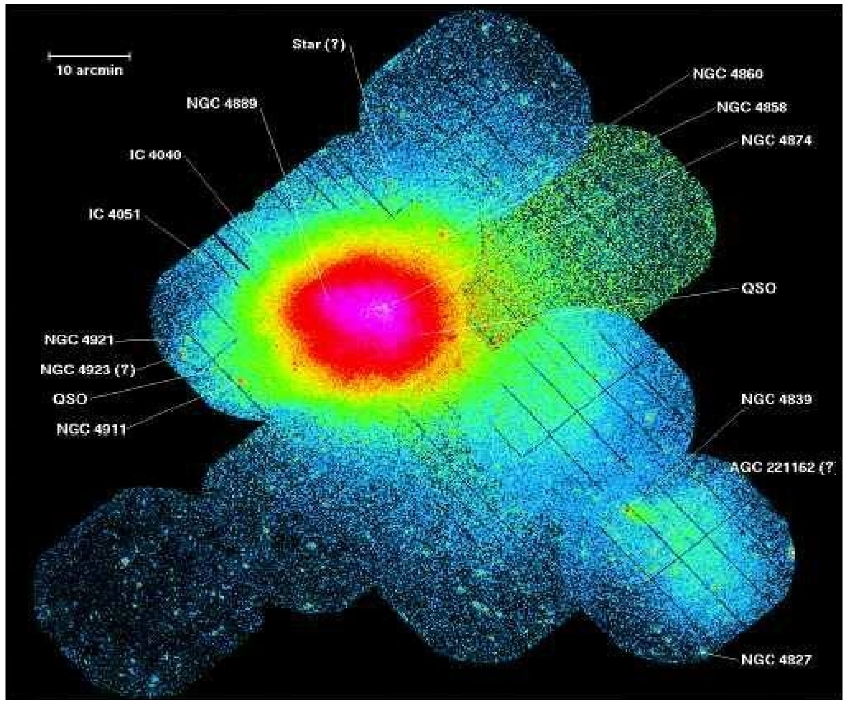
An H II region in the Large Magellanic Cloud (observed with MUSE, VLT)

Astronomical Examples - X-ray emission

- Theoretical spectrum for a plasma of temperature 10^7 K that takes into account quantum effects.
  - Comparison with real spectra from clusters of galaxies allows one to deduce the actual amounts of different elements and ionized species in the plasma as well as its temperature.
  - It is only in the present millennium that X-ray spectra taken from satellites (e.g., Chandra and the XMM Newton satellite) have had sufficient resolution to distinguish these narrow lines The dashed lines show the effect of X-ray absorption by interstellar gas [Bradt, Astrophysics Processes].



Coma cluster (z = 0.0232), size ~ 1 Mpc



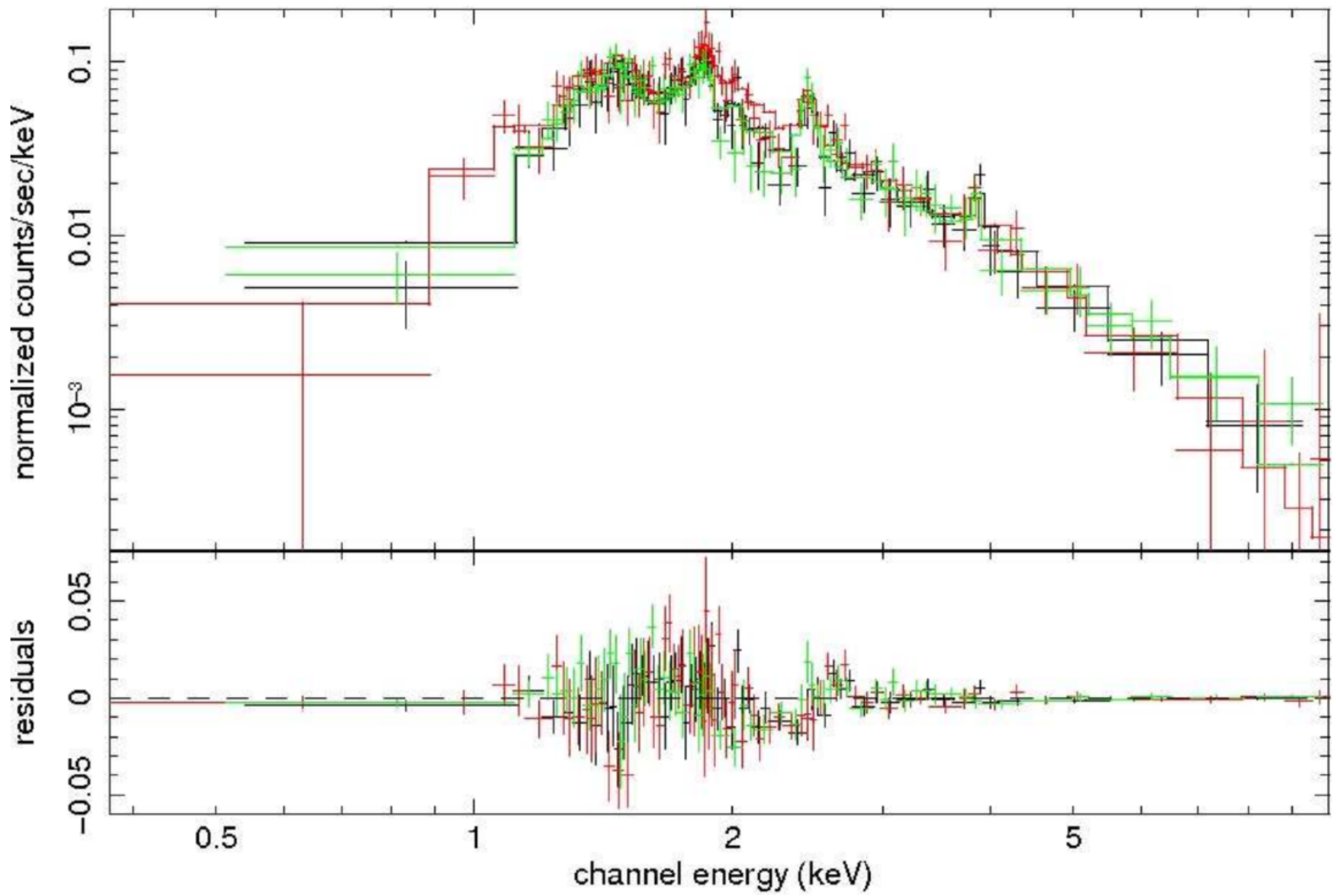
Astronomical Examples - Supernova Remnants

- SNR G346.6-0.2

  X-ray spectra of the SNR from three of the four telescopes on-board Suzaku (represented by green, red and black).

  The underlying continuum is thermal bremsstrahlung, while the spectral features are due to elements such as Mg, S, Si, Ca and Fe.

  The roll over in the spectrum at low and high energies is due to a fall in the detector response, which is forward-modeled together with the spectrum.



Sezer et al. (2011, MNRAS, 415, 301)

Cooling function

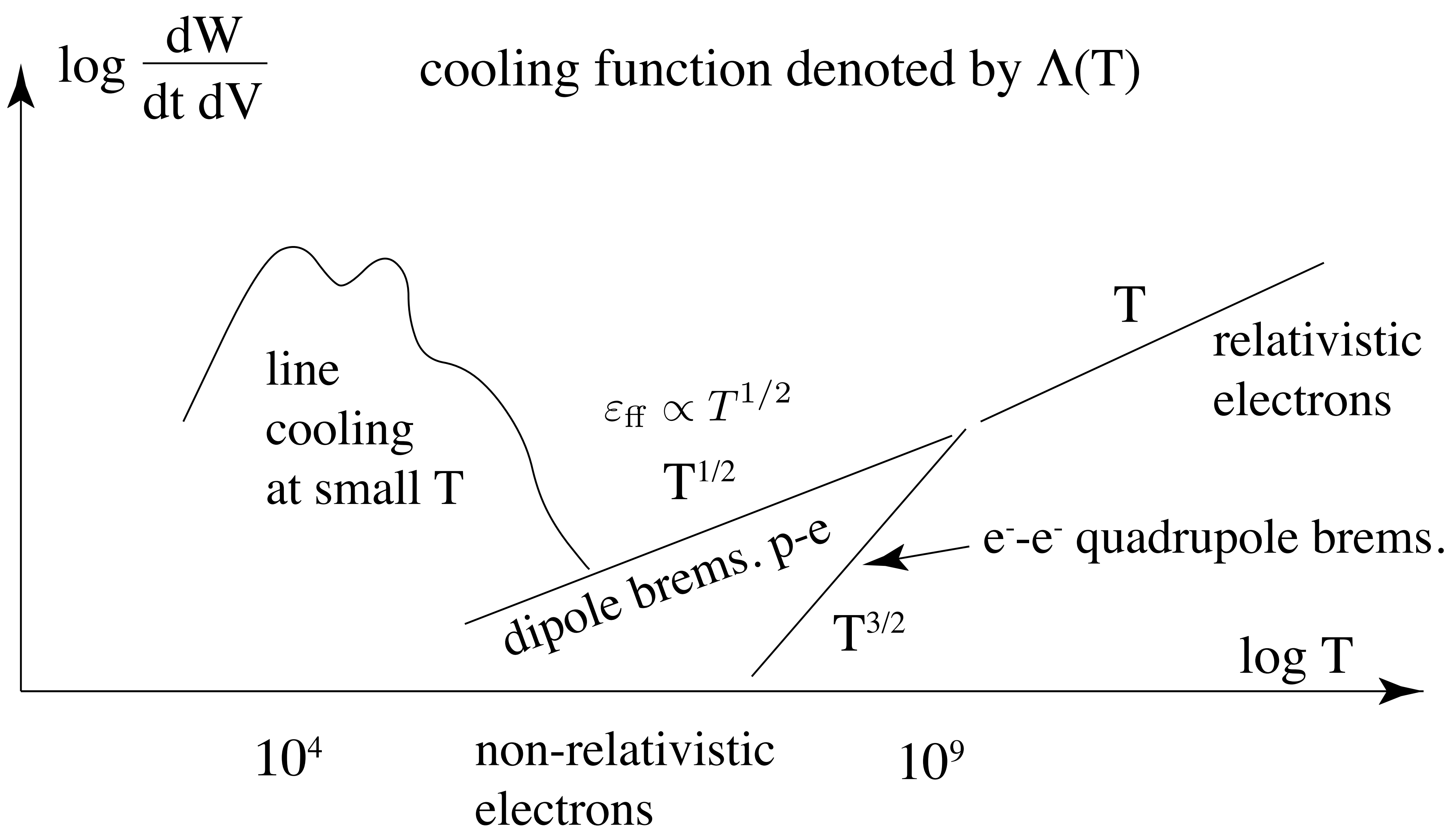


Figure from the Lecture Note of J. Poutanen

[Relativistic Bremsstrahlung]*

- Normally, the ions move rather slowly in comparison to the electrons.

  However, in a frame of reference in which electron is initially at rest, the ion appears to move rapidly toward the electron. **The electrostatic field of the ion appears to the electron to be a pulse of electromagnetic radiation. This radiation then Compton (or Thompson) scatters off the electron to produce emitted radiation.** Transforming back to the rest frame of the ion (or lab frame) we obtain the bremsstrahlung emission of the electron. **Relativistic bremsstrahlung can be regarded as the Compton scattering of the virtual quanta of the ion's electrostatic field as seen in the electron's frame.**

- In the (primed) electron rest frame, the spectrum of the pulse of the virtual quanta:

$$\frac{dW'}{dA'd\omega'} = \frac{q^2}{\pi^2 b'^2 v^2}\left(\frac{b'\omega'}{\gamma v}\right)^2 K_1^2\left(\frac{b'\omega'}{\gamma v}\right) = \frac{(Ze)^2}{\pi^2 b'^2 v^2}\left(\frac{b'\omega'}{\gamma v}\right)^2 K_1^2\left(\frac{b'\omega'}{\gamma c}\right) \quad \leftarrow \ v \approx c$$

(in the ultrarelativistic limit)

  In the low-frequency limit, the scattered radiation is

$$\frac{dW}{d\omega} = \sigma_T \frac{dW'}{dA'd\omega'} \qquad \left(\sigma_T = \frac{2\pi}{3}\frac{e^4}{m_e^2 c^4}\right)$$

  Transverse lengths are unchanged, $b = b'$, and $\omega = \gamma\omega'(1+\beta\cos\theta')$. The scattering is forward-backward symmetric, we therefore have the averaged relation $\omega = \gamma\omega'$.

- For a plasma with a single-speeds

$$\begin{aligned}\frac{dW}{dV\,dt\,d\omega} &= n_e n_i v \int_{b_{\min}}^{\infty} \frac{dW(b)}{d\omega} 2\pi b\,db \\ &= \frac{16Z^2e^6}{3c^3m_e^2} n_e n_i \int_{b_{\min}}^{b_{\max}} \left(\frac{b\omega}{\gamma^2 c}\right) K_1\left(\frac{b\omega}{\gamma^2 c}\right) db \\ &= \frac{16Z^2e^6}{3c^3m_e^2} n_e n_i \ln\left(\frac{0.68\gamma^2 c}{\omega b_{\min}}\right)\end{aligned}$$

- For a Maxwell distribution of electrons, a useful approximate expression for the frequency integrated power is given by Novikov & Thorne (1973).

$$\varepsilon_\nu^{\mathrm{ff}} = 1.4 \times 10^{-27} n_i n_e Z^2 T^{1/2} \overline{g_{\mathrm{B}}} \left(1 + 4.4 \times 10^{-10} T\right) \quad (\mathrm{erg\ s^{-1}\ cm^{-3}})$$

See also Itoh et al. (2000, ApJS, 128, 125), Zekovic (2013, arXiv:1310.5639v1)

- At higher frequencies Klein-Nishina corrections must be used.