# Astrophysics [Part I]

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# Compton Scattering

# [Scattering from Electrons at Rest]

- **Thomson Scattering**

$$\epsilon = \epsilon_1$$
$$\frac{d\sigma_T(\Omega)}{d\Omega} = \frac{1}{2} r_0^2 \left(1 + \cos^2\theta\right)$$
$$\sigma_T = \frac{8\pi}{3} r_0^2$$

$$\epsilon = \text{energy of the incident photon}$$
$$\epsilon_1 = \text{energy of the scattered photon}$$
$$r_0 = \frac{e^2}{m_e c^2}$$

When $\epsilon = \epsilon_1$, the scattering is called **coherent or elastic**.

- **Compton scattering**:

However, a photon carries momentum $h\nu/c$ and energy $h\nu$. Quantum effects appear in two ways.

(1) The scattering will no longer be elastic $(\epsilon \neq \epsilon_1)$ because of the recoil of the electron.

(2) The cross sections are altered by the quantum effects.

- Conservation of momentum and energy (for the case in which the electron is initially at rest)

Let the initial and final four-momenta of the photon: $\vec{P}_{\gamma i} = (\epsilon/c)(1,\ \mathbf{n}_i),\ \vec{P}_{\gamma f} = (\epsilon_1/c)(1,\ \mathbf{n}_f)$

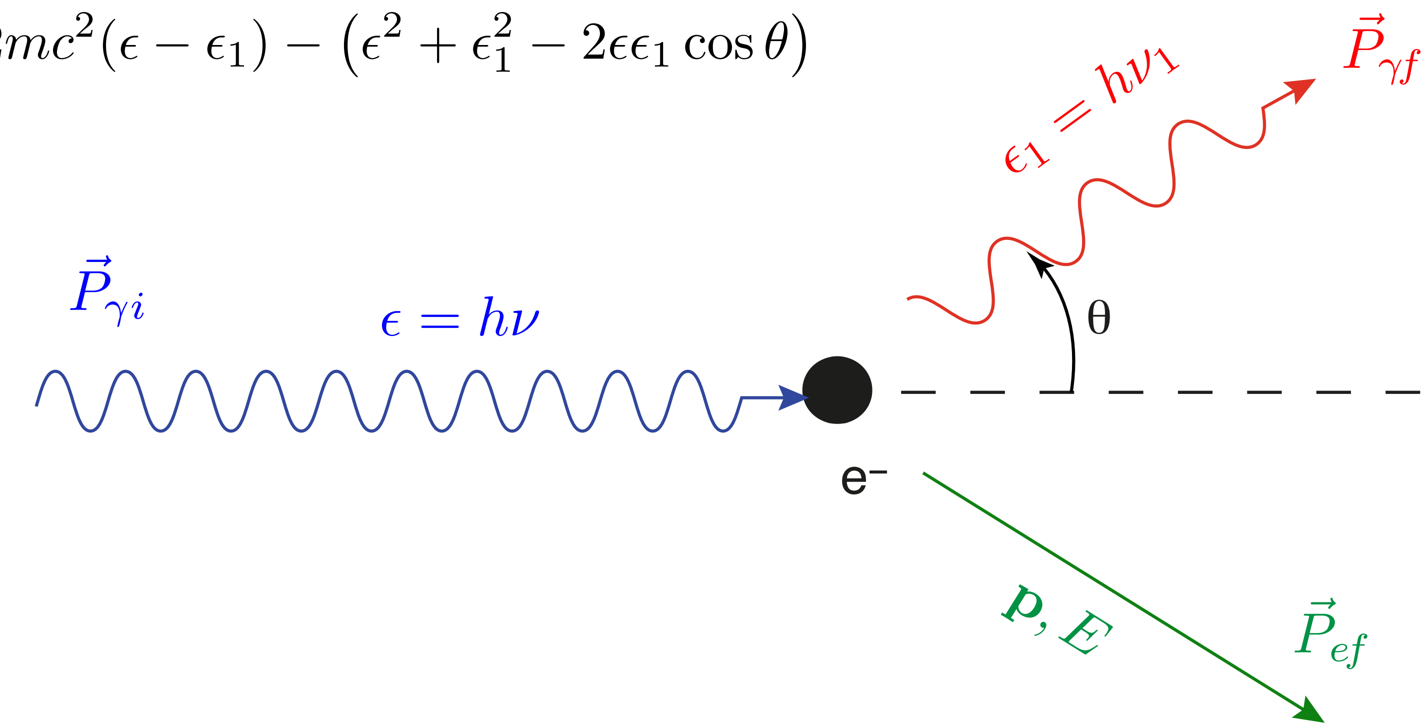
The initial and final momenta of the electron are: $\vec{P}_{ei} = (mc,\ \mathbf{0}),\ \vec{P}_{ef} = (E/c,\ \mathbf{p})$

Then, the conservation of momentum and energy is expressed by

$$\vec{P}_{ei} + \vec{P}_{\gamma i} = \vec{P}_{ef} + \vec{P}_{\gamma f}$$

- Rearranging terms and squaring gives $\left|\vec{P}_{ef}\right|^2 = \left|\vec{P}_{ei} + \vec{P}_{\gamma i} - \vec{P}_{\gamma f}\right|^2$

$$\left|\vec{P}_{ef}\right|^2 c^2 = \left|\vec{P}_{ei} + \vec{P}_{\gamma i} - \vec{P}_{\gamma f}\right|^2 c^2$$
$$E^2 - |\mathbf{p}|^2 c^2 = \left(mc^2 + \epsilon - \epsilon_1\right)^2 - |\epsilon \mathbf{n}_i - \epsilon_1 \mathbf{n}_f|^2$$
$$(mc^2)^2 = (mc^2)^2 + \epsilon^2 + \epsilon_1^2 - 2\epsilon\epsilon_1 + 2mc^2(\epsilon - \epsilon_1) - \left(\epsilon^2 + \epsilon_1^2 - 2\epsilon\epsilon_1 \cos\theta\right)$$
$$0 = mc^2\epsilon - \epsilon_1\left(\epsilon + mc^2 - \epsilon\cos\theta\right)$$




$$\epsilon_1 = \frac{\epsilon}{1 + \frac{\epsilon}{mc^2}(1 - \cos\theta)}$$

In terms of wavelength, $\lambda_1 - \lambda = \frac{h}{mc}(1 - \cos\theta)$

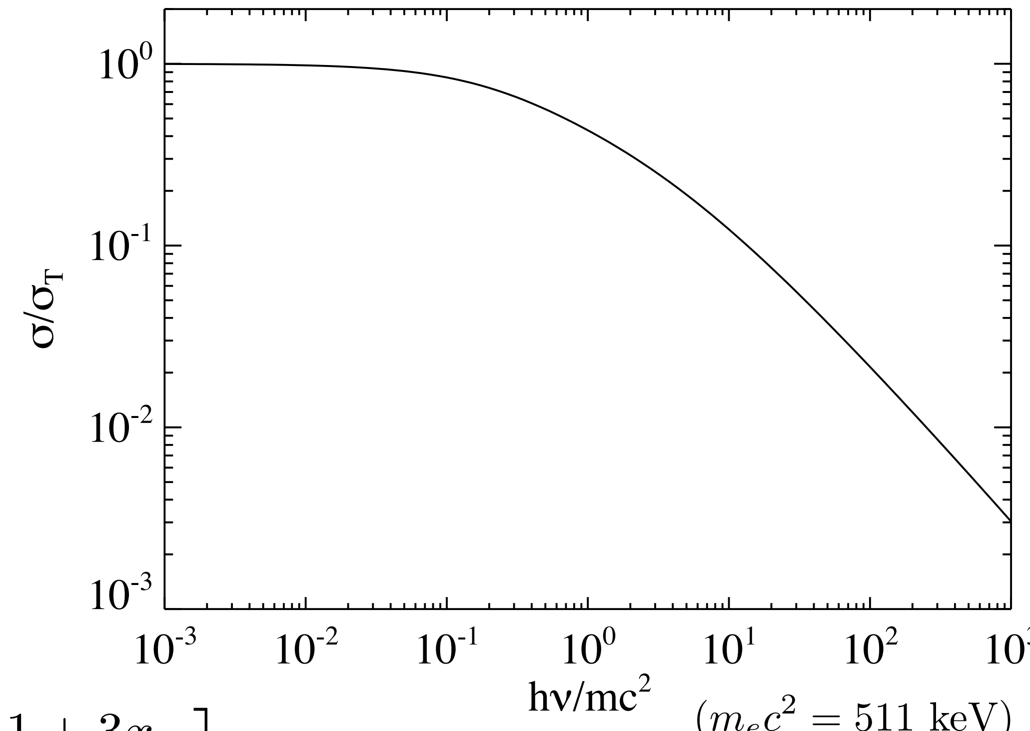
**Compton wavelength**: $\lambda_c \equiv \frac{h}{mc} = 0.02426\text{Å}$ for electrons

There is a wavelength change of the order of $\lambda_c$ upon scattering.

For long wavelengths $\lambda \gg \lambda_c$ (i.e., $h\nu \ll mc^2$), the scattering is closely elastic.

- **Klein-Nishina formula** (the differential cross section for unpolarized radiation, QED)

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{16\pi}\frac{\epsilon_1^2}{\epsilon^2}\left(\frac{\epsilon}{\epsilon_1} + \frac{\epsilon}{\epsilon_1} - \sin^2\theta\right)$$




$(m_e c^2 = 511 \text{ keV})$

- Total cross section:

$$\sigma = 2\pi\int_{-1}^{1}\frac{d\sigma}{d\Omega}d\cos\theta$$
$$= \frac{3\sigma_T}{4}\left[\frac{1+x}{x^3}\left\{\frac{2x(1+x)}{1+2x} - \ln(1+2x)\right\} + \frac{\ln(1+2x)}{2x} - \frac{1+3x}{(1+2x)^2}\right]$$

where $x \equiv \dfrac{h\nu}{mc^2}$

Compton scattering becomes less efficient at high energies.

- Approximations:

nonrelativisitic regime: $\sigma \approx \sigma_T\left(1 - 2x + \dfrac{26x^2}{5} + \cdots\right), \quad x \ll 1$

extreme relativisitic regime: $\sigma \approx \dfrac{3}{8}\sigma_T\dfrac{1}{x}\left(\ln 2x + \dfrac{1}{2}\right), \quad x \gg 1$

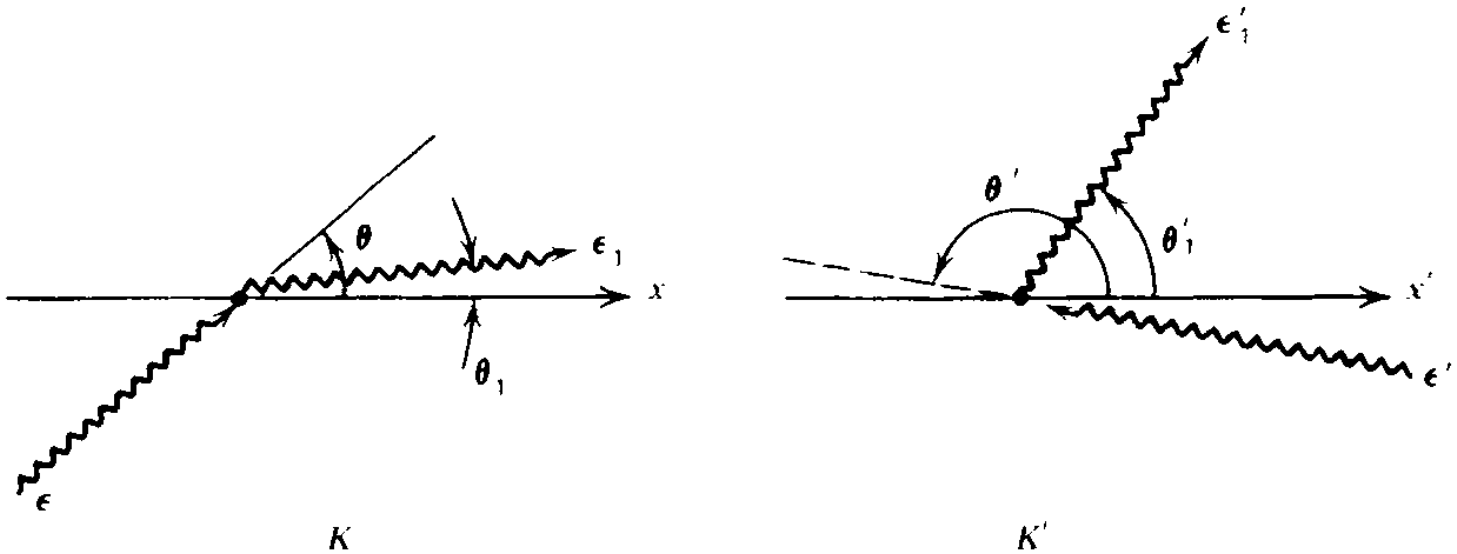
# [Scattering from Electrons in Motion]

- **Inverse Compton Scattering**: Whenever the moving electron has sufficient kinetic energy compared to the photon, net energy may be transferred from the electron to the photon.

From Doppler shift formulas

- from lab to electron rest frame

$$\epsilon' = \epsilon\gamma(1 - \beta\cos\theta)$$




lab frame

electron rest frame

- scattering

$$\epsilon_1' = \frac{\epsilon'}{1 + \frac{\epsilon'}{mc^2}(1 - \cos\Theta')}$$
$$\approx \epsilon'\left[1 - \frac{\epsilon'}{mc^2}(1 - \cos\Theta')\right] \quad (\text{if } \epsilon' \ll mc^2)$$

$\mathbf{n}'$ = direction vector of incident photon in the electron rest frame

$\mathbf{n}_1'$ = direction vector of scattered photon in the electron rest frame

$$\mathbf{n}' = (\sin\theta'\cos\phi',\ \ \sin\theta'\sin\phi',\ \ \cos\theta')$$
$$\mathbf{n}_1' = (\sin\theta_1'\cos\phi_1',\ \ \sin\theta_1'\sin\phi_1',\ \ \cos\theta_1')$$
$$\cos\Theta' \equiv \mathbf{n}'\cdot\mathbf{n}_1'$$

$$\cos\Theta' = \cos\theta_1'\cos\theta' + \sin\theta'\sin\theta_1'\cos(\phi' - \phi_1') \longleftarrow$$

($\Theta'$ = scattering angle in the electron rest frame)

- from electron to lab frame

$$\epsilon_1 = \epsilon_1'\gamma(1 + \beta\cos\theta)$$

# [Scattering from Electrons in Motion]

In the case of relativistic electrons, $\gamma^2 - 1 \gg h\nu/mc^2$, the energies of the photon before scattering, in the electron rest frame, and after the scattering in the lab frame are in the approximate ratios

$$\epsilon \ : \ \epsilon' \ : \ \epsilon_1 \ \approx \ 1 \ : \ \gamma \ : \ \gamma^2$$

providing that the condition for Thomson scattering in the rest frame is met $(\epsilon' \approx \epsilon\gamma \ll mc^2)$.

Therefore, **the inverse Compton scattering converts a low-energy photon to a high-energy photon by a factor of order $\gamma^2$.**

# [Inverse Compton Power for Single Scattering]

- Assumptions:

(1) isotropic distributions of photons and electrons.

(2) The change in energy of the photon in the rest frame is negligible

(Thomson scattering is applicable in the electron's rest frame). $\epsilon_1' \approx \epsilon'$

- **Total power scattered in the electron's rest frame**:

$$\frac{dE_1'}{dt'} = c\sigma_T \int \epsilon_1' n_\epsilon' d\epsilon'$$

where $n_\epsilon' d\epsilon'$ is the number density of incident photons.

- Recall: $\dfrac{dE_1}{dt} = \dfrac{dE_1'}{dt'}$ since energy and time transforms in the same way.

$d^3\mathbf{p} = \gamma d^3\mathbf{p}'$ transforms in the same way as energy.

$n_p \equiv \dfrac{dN}{d\mathcal{V}} \left( = \dfrac{d^6 N}{d^3\mathbf{x} d^3\mathbf{p}} \right)$ is a Lorentz invariant.

$n_p d^3\mathbf{p} = n_\epsilon d\epsilon$ The number densities of incident photons, represented in terms of momentum and energy, transforms in the same way as energy.

$$\boxed{\therefore \quad \frac{n_\epsilon d\epsilon}{\epsilon} = \frac{n_\epsilon' d\epsilon'}{\epsilon'}}$$

- Thus we have the results

$\epsilon_1' \approx \epsilon'$ **<u>Thomson scattering assumption in the rest frame</u>**

$$\frac{dE_1}{dt} = \frac{dE_1'}{dt'} = c\sigma_T \int \epsilon_1' n_\epsilon' d\epsilon' = c\sigma_T \int \epsilon'^2 \frac{n_\epsilon' d\epsilon'}{\epsilon'} = c\sigma_T \int \epsilon'^2 \frac{n_\epsilon d\epsilon}{\epsilon} \quad \leftarrow \quad \epsilon' = \epsilon\gamma(1-\beta\cos\theta)$$
$$= c\sigma_T \gamma^2 \int (1-\beta\cos\theta)^2 \, \epsilon n_\epsilon d\epsilon$$

For an isotropic distribution of photons,

$$\left\langle (1-\beta\cos\theta)^2 \right\rangle = 1 + \frac{1}{3}\beta^2 \qquad \leftarrow \quad \langle\cos\theta\rangle = 0, \quad \left\langle \cos^2\theta \right\rangle = 1/3$$

Therefore, we obtain the **total power scattered in the lab frame**:

$$\frac{dE_1}{dt} = c\sigma_T\gamma^2 \left(1 + \frac{1}{3}\beta^2\right) U_{\rm ph}$$

where $U_{\rm ph} \equiv \int \epsilon n_\epsilon d\epsilon$ is the initial photon energy density.

**The rate of decrease of the total initial photon energy** is

$$\frac{dE_1^{\rm loss}}{dt} = -c\sigma_T \int \epsilon n_\epsilon d\epsilon = -c\sigma_T U_{\rm ph}$$

- Thus, **the net power lost by the electron, and converted into increased radiation**, is

$$P_{\rm compt} \equiv \frac{dE_1}{dt} - \left|\frac{dE_1^{\rm loss}}{dt}\right| = c\sigma_T U_{\rm ph}\left[\gamma^2\left(1+\frac{1}{3}\beta^2\right)-1\right]$$

$$\boxed{\therefore \quad P_{\rm compt} = \frac{4}{3}c\sigma_T\gamma^2\beta^2 U_{\rm ph}} \longleftarrow \qquad \gamma^2 - 1 = \gamma^2\beta^2$$

- When the energy transfer in the electron rest frame is not neglected, the power is given by

$$P_{\rm compt} = \frac{4}{3}c\sigma_T\gamma^2\beta^2 U_{\rm ph}\left[1-\frac{63}{10}\frac{\gamma\left\langle\epsilon^2\right\rangle}{mc^2\left\langle\epsilon\right\rangle}\right] \qquad \text{(cf. Blumenthal \& Gould, 1970)}$$

- Note that the above equation allows energy to be either given or taken from the photons.

- Recall that the formula for the synchrotron power emitted by each electron is

$$P_{\mathrm{synch}} = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_{\mathrm{B}}$$

Therefore, $\boxed{\dfrac{P_{\mathrm{synch}}}{P_{\mathrm{compt}}} = \dfrac{U_{\mathrm{B}}}{U_{\mathrm{ph}}}}$

The radiation losses due to synchrotron emission and to inverse Compton effect are in the same ratio as the magnetic field energy density and photon energy density.

- Let $N(\gamma)\mathrm{d}\gamma$ be the number of electrons per unit volume. Then, the total Compton power per unit volume is

$$P_{\mathrm{tot}} = \int P_{\mathrm{compt}} N(\gamma) d\gamma$$

(1) Power-law distribution of relativistic electrons ($\beta \sim 1$)

$$N(\gamma) = \begin{cases} C\gamma^{-p}, & \gamma_{\min} \le \gamma \le \gamma_{\max} \\ 0, & \text{otherwise} \end{cases} \longrightarrow \quad P_{\mathrm{tot}} = \frac{4}{3}\sigma_T c U_{\mathrm{ph}} C(3-p)^{-1}\left(\gamma_{\min}^{3-p} - \gamma_{\max}^{3-p}\right)$$

(2) Thermal distribution of nonrelativistic electrons ($\gamma \sim 1$)

$$\langle \beta^2 \rangle = \langle v^2/c^2 \rangle = 3kT/mc^2 \quad \longrightarrow \quad P_{\mathrm{tot}} = \left(\frac{4kT}{mc^2}\right)\sigma_T c n_e U_{\mathrm{ph}}$$

$\gamma \approx 1$

$\longrightarrow$ fractional photon energy gain

# [Inverse Compton Spectra for Single Scattering]

- Approach: (1) Determine the spectrum for the scattering of photons of a single energy off electrons of a single energy, and then (2) Average over the actual distribution of photons and electrons.

- Assumptions:

  (1) Both the photons and electrons have isotropic distributions; the scattered photons are then also isotropically distributed.

  (2) Thomson scattering in the rest frame: $\gamma\epsilon_0 \ll mc^2, \quad \epsilon_0' \approx \epsilon_1'$

  (3) Isotropic scattering in the rest frame: $\dfrac{d\sigma'}{d\Omega'} = \dfrac{1}{4\pi}\sigma_T$

  Even with these assumptions, we obtain the correct qualitative behavior of the results.

- We will use an intensity and emission coefficient based on photon number rather than energy.

  $I\, dAdtd\Omega d\epsilon$

  = number of photons crossing are $dA$ in time $dt$ within solid angle $d\Omega$ and energy range $d\epsilon$

- Isotropic and monoenergetic photon field:

  in the observer frame, $I(\epsilon) = F_0\delta(\epsilon - \epsilon_0)$

  in the electron rest frame, $I'(\epsilon', \mu') = F_0\left(\dfrac{\epsilon'}{\epsilon}\right)^2\delta(\epsilon - \epsilon_0) \longleftarrow \quad \dfrac{I}{\nu^2} = \text{Lorentz invariant}$

From the Doppler formula $\epsilon = \epsilon'\gamma(1+\beta\mu')$, the incident intensity is

$$I'(\epsilon',\mu') = \left(\frac{\epsilon'}{\epsilon}\right)\delta(\epsilon-\epsilon_0) = \left(\frac{\epsilon'}{\epsilon_0}\right)^2 F_0\delta\left(\gamma\epsilon'(1+\beta\mu')-\epsilon_0\right)$$
$$= \left(\frac{\epsilon'}{\epsilon_0}\right)^2 \frac{F_0}{\gamma\beta\epsilon'}\delta\left(\mu' - \frac{\epsilon_0-\gamma\epsilon'}{\gamma\beta\epsilon'}\right)$$

**Emission coefficient in the rest frame**:

(under the elastic scattering assumption, $\epsilon_1' = \epsilon'$)

Let $N$ = density of an electron beam, i.e., $N = \int n_p d^3\mathbf{p} = \int \frac{dN}{d\mathcal{V}} d^3\mathbf{p}$.

The emission coefficient is

$$j'(\epsilon_1') = N'\sigma_T \frac{1}{4\pi} 2\pi \int_{-1}^{+1} I'(\epsilon_1',\mu')d\mu'$$
$$= \begin{cases} \dfrac{N'\sigma_T\epsilon_1' F_0}{2\epsilon_0^2\gamma\beta} & \text{if } \left|\dfrac{\epsilon_0-\gamma\epsilon_1'}{\gamma\beta\epsilon_1'}\right| \le 1 \quad \left(\text{or equivalently } \dfrac{\epsilon_0}{\gamma(1+\beta)} \le \epsilon_1' \le \dfrac{\epsilon_0}{\gamma(1-\beta)}\right) \\ 0, & \text{otherwise} \end{cases}$$

## **Emission coefficient in the observer's frame**:

Recall that $\dfrac{j}{\epsilon} = \text{Lorentz invariant} \quad \rightarrow \quad j'(\epsilon') = \dfrac{\epsilon'}{\epsilon} j(\epsilon)$

$$\epsilon_1' = \epsilon_1 \gamma (1 - \beta\mu_1)$$

$$N d^3\mathbf{x} = N' d^3\mathbf{x}', \quad d^3\mathbf{x} = \frac{d^3\mathbf{x}'}{\gamma}, \quad N = \gamma N'$$

$$\begin{aligned} j(\epsilon_1, \mu_1) &= \frac{\epsilon_1}{\epsilon_1'} j'(\epsilon_1') \\ &= \begin{cases} \dfrac{N' \sigma_T \epsilon_1 F_0}{2\epsilon_0^2 \gamma\beta}, & \text{if } \dfrac{\epsilon_0}{\gamma(1+\beta)} \leq \epsilon_1' \leq \dfrac{\epsilon_0}{\gamma(1-\beta)} \\ \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$$\therefore \quad j(\epsilon_1, \mu_1) = \begin{cases} \dfrac{N \sigma_T \epsilon_1 F_0}{2\epsilon_0^2 \gamma^2 \beta}, & \text{if } \dfrac{\epsilon_0}{\gamma^2(1+\beta)(1-\beta\mu_1)} \leq \epsilon_1 \leq \dfrac{\epsilon_0}{\gamma^2(1-\beta)(1-\beta\mu_1)} \\ \\ 0, & \text{otherwise} \end{cases}$$

- The above results hold for a beam of electrons.

  For an isotropic distribution of electrons, the emission coefficient is obtained by averaging over the angle between the electron and scattered photon:

$$j(\epsilon_1) = \frac{1}{2}\int_{-1}^{+1} j(\epsilon_1, \mu_1) d\mu_1$$

  The integrand is nonzero only for a certain interval of $\mu_1$ :

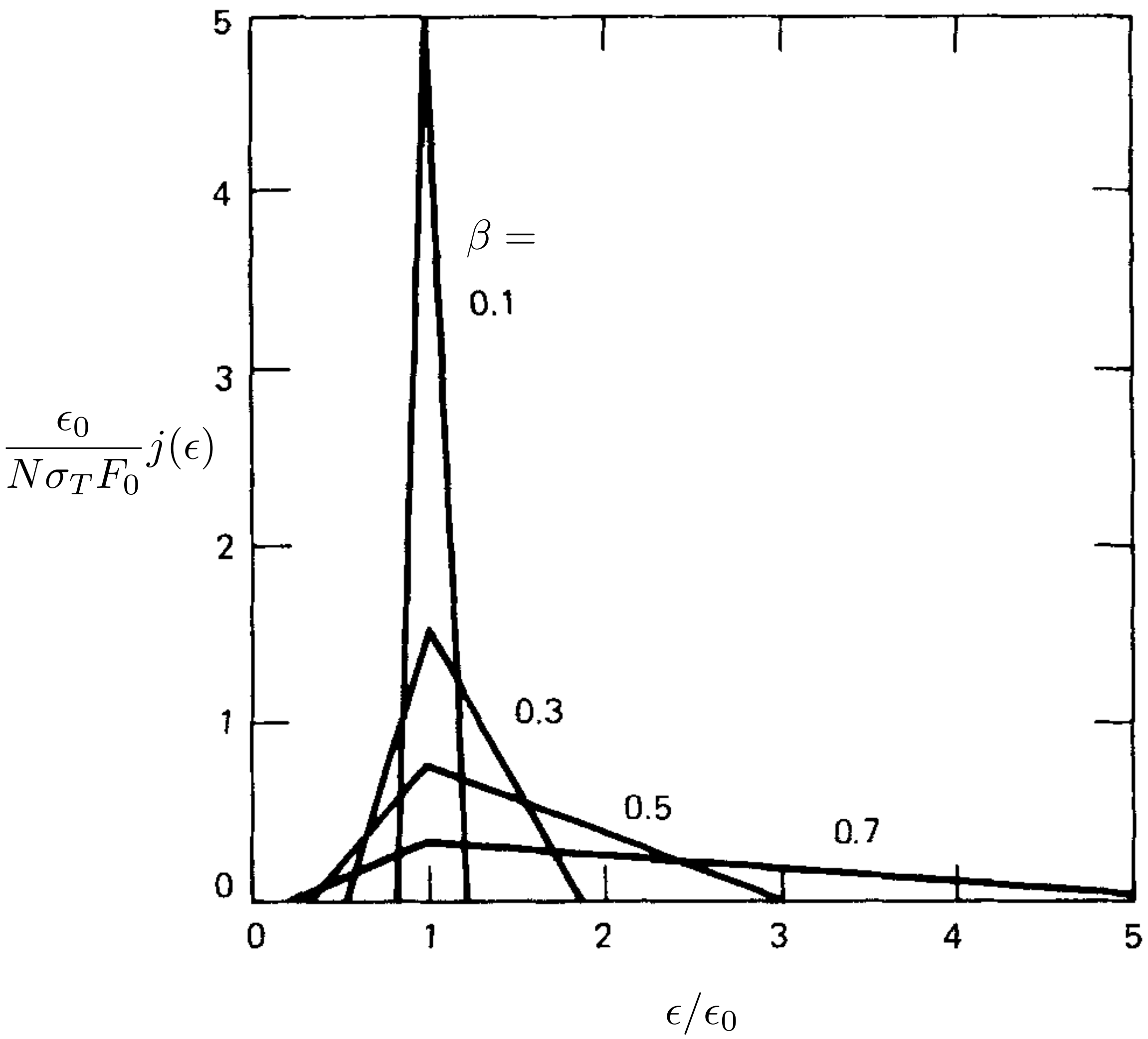
$$\frac{\epsilon_0}{\gamma^2(1+\beta)(1-\beta\mu_1)} \leq \epsilon_1 \leq \frac{\epsilon_0}{\gamma^2(1-\beta)(1-\beta\mu_1)} \quad \rightarrow \quad \frac{1}{\beta}\left[1 - \frac{\epsilon_0}{\epsilon_1}(1+\beta)\right] \leq \mu_1 \leq \frac{1}{\beta}\left[1 - \frac{\epsilon_0}{\epsilon_1}(1-\beta)\right]$$

  Since $-1 \leq \mu_1 \leq 1$, the nonzero interval becomes:

$$\begin{aligned} -1 \leq \mu_1 \leq \frac{1}{\beta}\left[1 - \frac{\epsilon_0}{\epsilon_1}(1-\beta)\right], &\quad \text{for} \quad \frac{1-\beta}{1+\beta} \leq \frac{\epsilon_1}{\epsilon_0} \leq 1 \\ \frac{1}{\beta}\left[1 - \frac{\epsilon_0}{\epsilon_1}(1+\beta)\right] \leq \mu_1 \leq 1, &\quad \text{for} \quad 1 \leq \frac{\epsilon_1}{\epsilon_0} \leq \frac{1+\beta}{1-\beta} \end{aligned}$$

The result is:

$$j(\epsilon_1) = \frac{N\sigma_T F_0}{4\epsilon_0\gamma^2\beta^2} \begin{cases} (1+\beta)\dfrac{\epsilon_1}{\epsilon_0} - (1-\beta), & \dfrac{1-\beta}{1+\beta} \leq \dfrac{\epsilon_1}{\epsilon_0} \leq 1 \\ (1+\beta) - \dfrac{\epsilon_1}{\epsilon_0}(1-\beta), & 1 \leq \dfrac{\epsilon_1}{\epsilon_0} \leq \dfrac{1+\beta}{1-\beta} \\ 0, & \text{otherwise} \end{cases}$$




Note that

(1) For small $\beta$ the curves are symmetrical about the initial photon energy.

(2) As $\beta$ increases, the portion of the curve for $\epsilon \gg \epsilon_0$ becomes more and more dominant.

Note

$$\int_0^\infty j(\epsilon_1)d\epsilon_1 = N\sigma_T F_0$$

: the rate of photon scattering per unit volume, per unit solid angle (the conservation of number of photons upon scattering)

$$\int_0^\infty j(\epsilon_1)(\epsilon_1 - \epsilon_0)d\epsilon_1 = N\sigma_T \frac{4}{3}\gamma^2\beta^2\epsilon_0 F_0$$

: the average increase in photon energy for single scattering

Recall that the scattered power due to a single electron was obtained to be:

$$P_{\rm compt} = \frac{4}{3}c\sigma_T\gamma^2\beta^2 U_{\rm ph}$$

Setting $\int_0^\infty j(\epsilon_1)(\epsilon_1 - \epsilon_0)d\epsilon_1 = NP_{\rm compt}$

We obtain $U_{\rm ph} = \frac{1}{c}\epsilon_0 F_0$

For extreme relativistic case $(\beta \approx 1,\ \gamma \gg 1)$, the right portion of the spectrum dominant.

$$\begin{aligned} j(\epsilon_1) &= \frac{N\sigma_T F_0}{4\epsilon_0\gamma^2\beta^2}\left[(1+\beta) - \frac{\epsilon_1}{\epsilon_0}(1-\beta)\right], \qquad 1 \le \frac{\epsilon_1}{\epsilon_0} \le \frac{1+\beta}{1-\beta} \\ &= \frac{N\sigma_T F_0}{4\epsilon_0\gamma^2}\frac{1+\beta}{\beta^2}\left(1 - \frac{\epsilon_1}{\epsilon_0}\frac{1-\beta}{1+\beta}\right) \\ &\approx \frac{N\sigma_T F_0}{4\epsilon_0\gamma^2}2\left(1 - \frac{\epsilon_1}{\epsilon_0}\frac{1}{4\gamma^2}\right) \end{aligned}$$

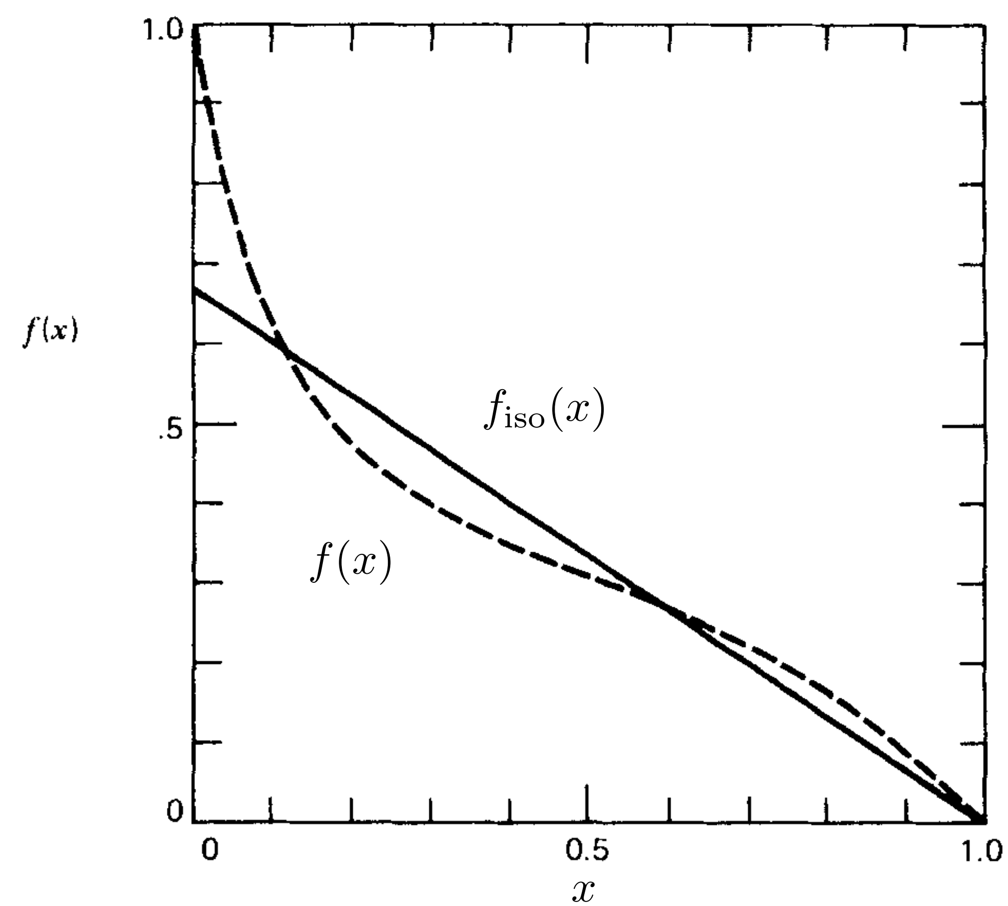
$$j(\epsilon_1) \approx \frac{3N\sigma_T F_0}{4\epsilon_0\gamma^2} f_{\rm iso}(x) \quad \text{where} \quad x \equiv \frac{\epsilon_1}{4\gamma^2\epsilon_0}$$

$$f_{\rm iso} \equiv \begin{cases} \frac{2}{3}(1-x), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Note: $1 \le \frac{\epsilon_1}{\epsilon_0} \le \frac{1+\beta}{1-\beta} \quad \rightarrow \quad 1 \le \frac{\epsilon_1}{\epsilon_0} \lesssim 4\gamma^4 \quad \rightarrow \quad 0 \lesssim x \lesssim 1$

When the exact angular dependence
in the differential cross section is included,

$$f(x) = 2x\ln x + x + 1 - 2x^2, \quad (0 < x < 1)$$



- **<u>Power law distribution of relativistic electrons</u>:**

$N(\gamma) = C\gamma^{-p}$ : electron distribution

$n_\epsilon = \dfrac{4\pi I(\epsilon)}{c}$ : photon number density for the isotropic intensity

Total scattered power per volume per energy is

$$
\begin{aligned}
\frac{dE}{dV\,dtd\epsilon_1} &= 4\pi\epsilon_1 j(\epsilon_1) \\
&= \frac{3c\sigma_T}{4}\int d\epsilon\left(\frac{\epsilon_1}{\epsilon}\right) n_\epsilon \int_{\gamma_1}^{\gamma_2} d\gamma\left(C\gamma^{-p-2}\right) f(x) \\
&= 3\sigma_T c C 2^{p-2}\epsilon_1^{-(p-1)/2}\int d\epsilon\epsilon^{(p-1)/2} n_\epsilon \int_{x_1}^{x_2} dx x^{(p-1)/2} f(x)
\end{aligned}
$$

Suppose that $\gamma_2 \gg \gamma_1$ and that $n_\epsilon$ peaks at some value $\bar{\epsilon}$.

Then $x_1 \equiv \epsilon_1/(4\gamma_1^2\epsilon) \ \rightarrow\ 0$ and the second integral is independent of $\epsilon_1$.

$x_2 \equiv \epsilon_2/(4\gamma_2^2\epsilon) \ \rightarrow\ \infty$

The spectral index is to be identical to the case of synchrotron emission.

$$
\boxed{\frac{dE}{dV\,dtd\epsilon_1} \propto \epsilon_1^{-(p-1)/2}}
$$

# [Repeated Scattering: The Compton y Parameter]

- We restrict our considerations to situations in which the Thomson limit applies: $\gamma\epsilon \ll mc^2$
- **<u>Compton y parameter</u>**, to determine whether a photon will significantly change its energy in traversing the medium:

$$y \equiv \left( \begin{array}{l} \text{average fractional} \\ \text{energy change per} \\ \text{scattering} \end{array} \right) \times \left( \begin{array}{l} \text{mean number of} \\ \text{scatterings} \end{array} \right)$$

When $y \gtrsim 1$, the total photon energy and spectrum will be significantly altered; whereas for $y \ll 1$, the total energy is not much changed.

- **<u>Average fractional energy change per scattering</u>** (for a thermal distribution of electrons)

<u>Consider first the nonrelativistic limit</u>.

$$\epsilon_1' \approx \epsilon' \left[ 1 - \frac{\epsilon'}{mc^2} (1 - \cos\Theta) \right] \quad \rightarrow \quad \left\langle \frac{\Delta\epsilon'}{\epsilon'} \right\rangle \equiv \left\langle \frac{\epsilon_1' - \epsilon'}{\epsilon'} \right\rangle = -\frac{\epsilon'}{mc^2} \quad \text{: angle average}$$

In the lab frame to lowest order, this must be of the form

$$\left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle = -\frac{\epsilon}{mc^2} + \alpha \frac{kT}{mc^2}$$

To calculate $\alpha$, **image that the photons and electrons are in complete equilibrium but interact only through scattering**.

Assume that the photon density is sufficiently small that stimulated processes can be neglected. Then, we obtain the Wien's law for the photon distribution:

$$n_\epsilon = K\epsilon^2 \exp\left(-\frac{\epsilon}{kT}\right)$$

We have the averages

$$\langle \epsilon \rangle \equiv \int \epsilon n_\epsilon d\epsilon \Big/ \int n_\epsilon d\epsilon = 3kT$$

$$\langle \epsilon^2 \rangle \equiv \int \epsilon^2 n_\epsilon d\epsilon \Big/ \int n_\epsilon d\epsilon = 12(kT)^2$$

For this case, no net energy can be transferred from photons to electrons, so

$$\Delta\epsilon = 0 = -\frac{\langle \epsilon^2 \rangle}{mc^2} + \alpha\frac{kT}{mc^2}\langle \epsilon \rangle = \frac{3kT}{mc^2}(\alpha - 4)kT \quad \rightarrow \quad \alpha = 4$$

Thus for nonrelativisitic electrons in thermal equilibrium, the energy transfer per scattering is

$$\boxed{(\Delta\epsilon)_{\mathrm{NR}} = \frac{\epsilon}{mc^2}(4kT - \epsilon)}$$

Note that if the electrons have high enough temperature relative to incident photons, the photons gain energy. This is the inverse Compton scattering.

If $\epsilon > 4kT$, on the other hand, energy is transferred from photons to electrons.

- <u>In the ultrarelativistic limit $(\gamma \gg 1,\ \beta \approx 1)$</u>, ignoring the energy transfer in the electron rest frame,

$$\frac{P_{\rm compt}}{\left|dE_1^{\rm loss}/dt\right|} = \frac{4/3\sigma_T c\gamma^2\beta^2 U_{\rm ph}}{\sigma_T c U_{\rm ph}} = \frac{4}{3}\gamma^2\beta^2 \quad \rightarrow \quad (\Delta\epsilon)_{\rm R} \approx \frac{4}{3}\gamma^2\epsilon$$

For a thermal distribution of ultrarelativistic electrons,

$$\langle\gamma^2\rangle = \frac{\langle\epsilon^2\rangle}{(mc^2)^2} = 12\left(\frac{kT}{mc^2}\right)^2 \longrightarrow \boxed{(\Delta\epsilon)_{\rm R} \approx 16\epsilon\left(\frac{kT}{mc^2}\right)^2}$$

- **<u>Mean number of scatterings</u>**,

Recall that, for a pure scattering medium,

$$\left(\begin{array}{l}\text{mean number of}\\ \text{scatterings}\end{array}\right) \approx {\rm Max}(\tau_{\rm es},\ \tau_{\rm es}^2)$$

where
$$\tau_{\rm es} \sim \rho\kappa_{\rm es}R$$
$$\kappa_{\rm es} = \frac{\sigma_T}{m_p} = 0.40\ {\rm cm^2\ g^{-1}}\ \text{for ionized hydrogen}$$
$$R = \text{ size of the finite medium}$$

- **<u>Compton *y* parameter</u>**:

$$\boxed{y_{\rm NR} = \frac{4kT}{mc^2}{\rm Max}(\tau_{\rm es},\ \tau_{\rm es}^2) \qquad\qquad y_{\rm R} = 16\epsilon\left(\frac{kT}{mc^2}\right)^2{\rm Max}(\tau_{\rm es},\ \tau_{\rm es}^2)}$$

# [Repeated Scattering: Spectra and Power]

- A power-law spectrum may be a natural consequence of a power-law distribution of electrons.
- **We will show that a power-law photon distribution can also be produced from repeated scattering off a nonpower-law electron distribution.**

Let A = the mean amplification of photon energy per scattering

$$A \equiv \frac{\epsilon_1}{\epsilon} \sim \frac{4}{3}\langle\gamma^2\rangle$$

$$= 16\left(\frac{kT}{mc^2}\right)^2 \qquad \text{for thermal electron distribution}$$

mean photon energy = $\epsilon_i$

intensity = $I(\epsilon_i)$ at $\epsilon_i$

After $k$ scattering, the photon energy will be $\epsilon_k \sim \epsilon_i A^k$.

For an optically thin scattering medium ($\tau_{\rm es} < 1$), the probability of a photon undergoing $k$ scattering before escaping the medium is $p_k(\tau_{\rm es}) \sim \tau_{\rm es}^k$ .

<u>The emergent intensity at energy $\epsilon_k$ is given by</u>

$$I(\epsilon_k) \sim I(\epsilon_i)\tau_{\rm es}^k \sim I(\epsilon_i)\tau_{\rm es}^{\ln(\epsilon_k/\epsilon_i)/\ln A} = I(\epsilon_i)\left(\frac{\epsilon_k}{\epsilon_i}\right)^{\ln\tau_{\rm es}/\ln A}$$

$$\therefore\ I(\epsilon_k) \sim I(\epsilon_i)\left(\frac{\epsilon_k}{\epsilon_i}\right)^{-\alpha} \quad \text{where} \quad \alpha \equiv \frac{-\ln\tau_{\rm es}}{\ln A} \longrightarrow \text{power-law shape}$$

- Total Compton power in the output spectrum is given by

$$P \propto \int I(\epsilon_k) d\epsilon_k = I(\epsilon_i)\epsilon_i \left[ \int x^{-\alpha} dx \right]$$

The factor in square brackets is approximately **the factor by which the initial power** $I(\epsilon_i)\epsilon_i$ **is amplified** in energy.

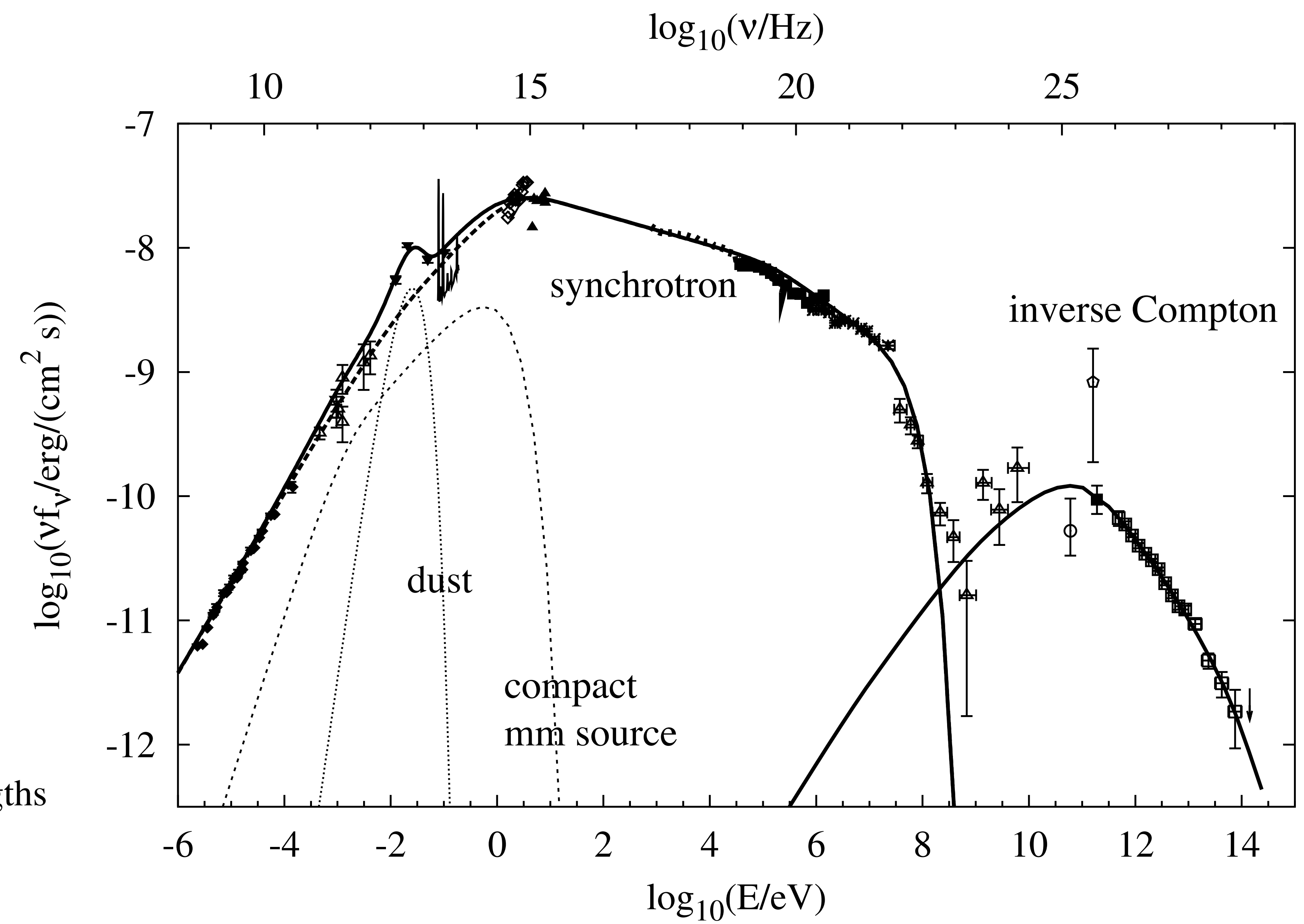
Clearly, this amplification will be important if $\alpha \ll 1$. Therefore, **energy amplification of a soft photon input spectrum is important when**

$$\alpha = \frac{-\ln \tau_{\rm es}}{\ln A} \lesssim 1 \quad \rightarrow \quad \ln(\tau_{\rm es} A) \gtrsim 0$$

$$\rightarrow \quad y = A\tau_{\rm es} \sim 16 \left( \frac{kT}{mc^2} \right)^2 \tau_{\rm es} \gtrsim 1$$

# [Synchrotron self-Compton (SSC) emission]

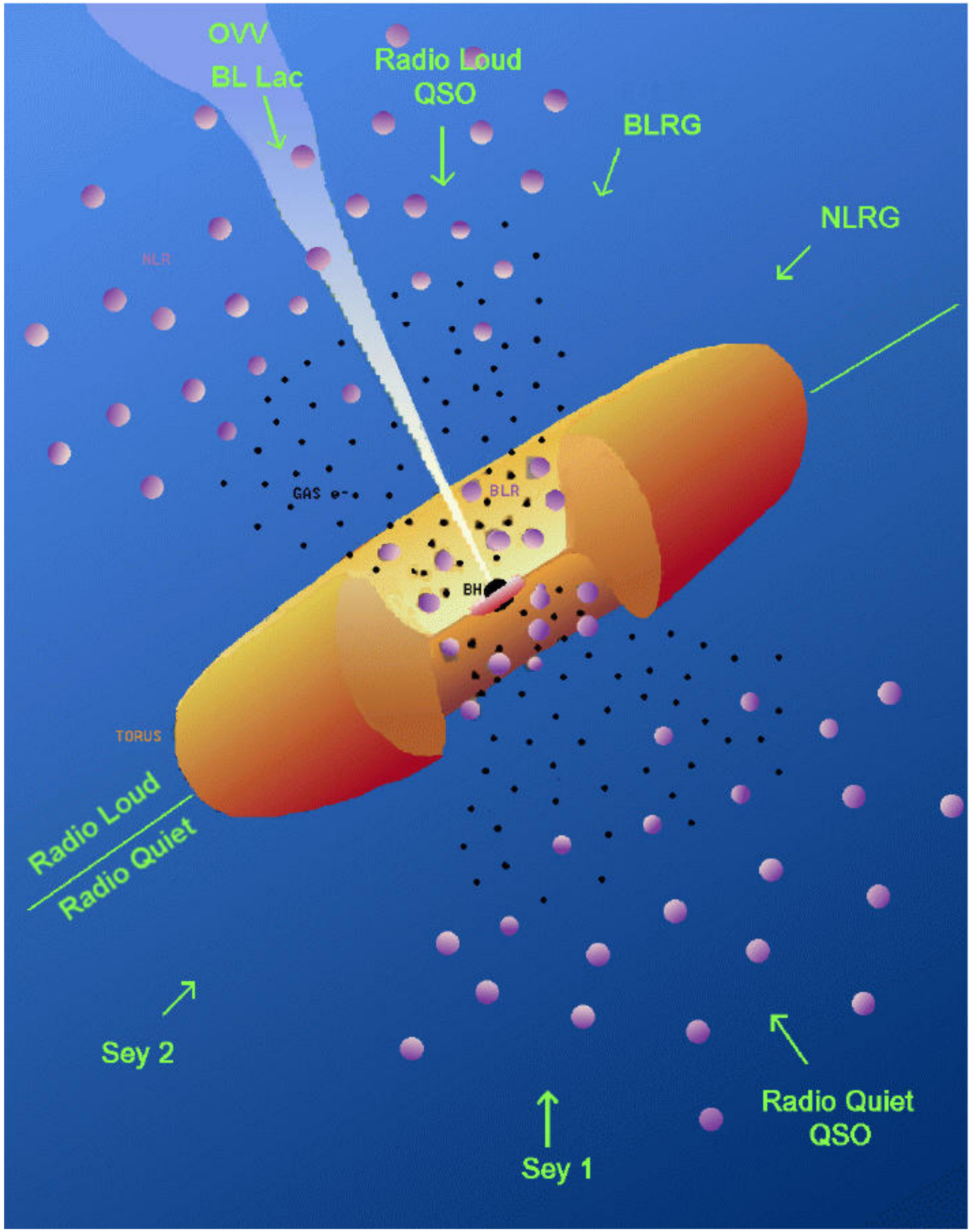
- The modification of the photon spectrum by Compton scattering is called **Comptonization**.
- Relativistic electrons in the presence of a magnetic field will surely emit synchrotron radiation at some level. The photons will undergo inverse Compton scattering by the very same electrons that emitted them in the first place. Such scattering must take place before the synchrotron photon leaves the source region. This is the **synchrotron self-Compton (SSC) process**.
- **Crab nebula**




Dots: modified blackbody with $T = 46$ K.
Thin dashed line: emission at mm wavelengths
Thick dashed line: synchrotron emission

# Active Galactic Nuclei

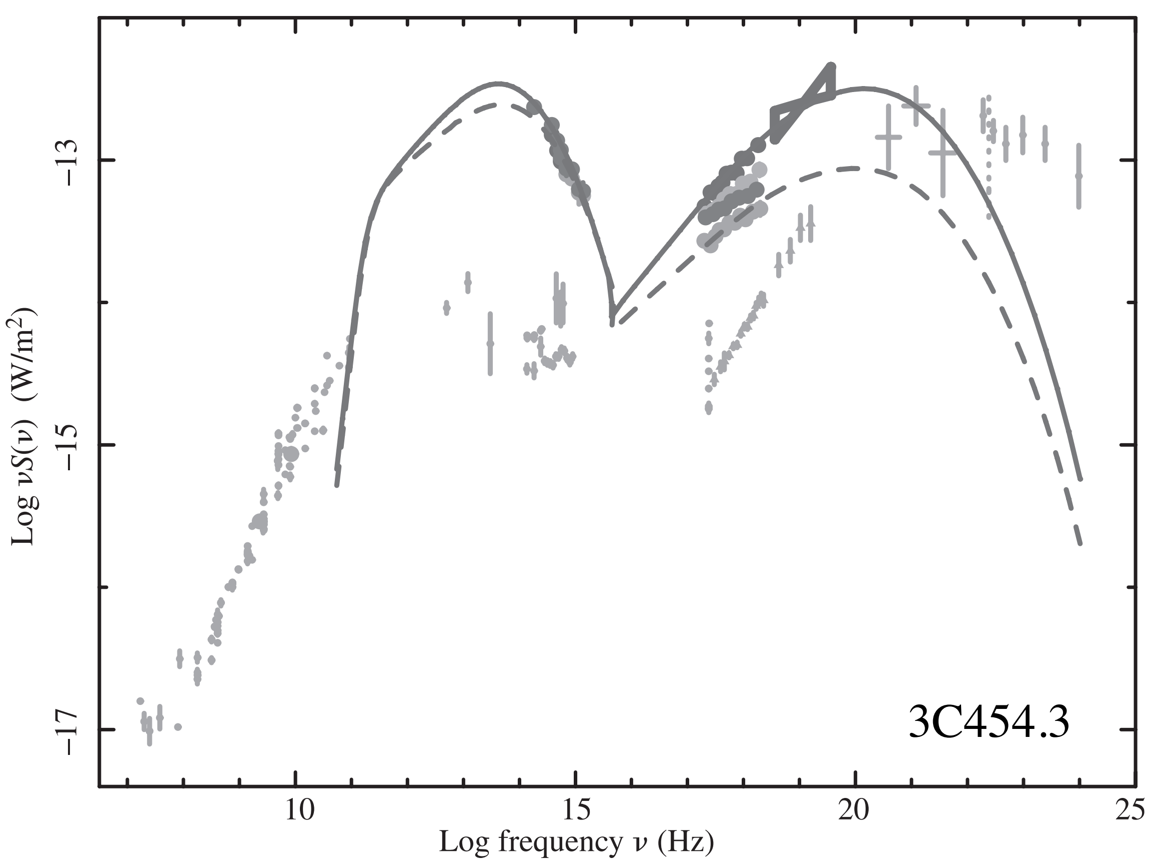
- A Unified Model for AGN

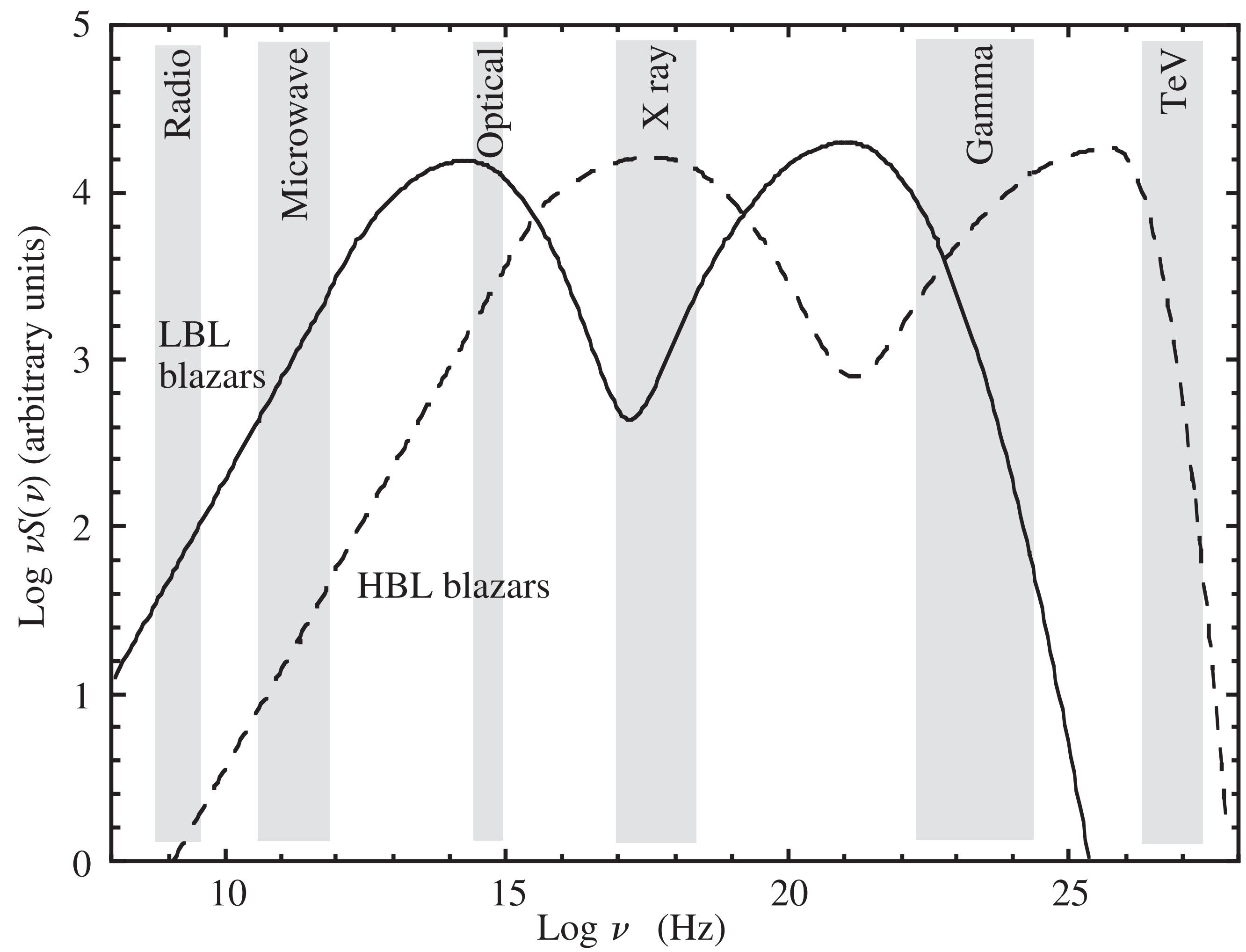


- **Blazars**: If the observer view is more or less normal to the accretion disk, the action close to the core becomes visible. The observer considered to lie within the jet beam. Such objects are known as blazars or as BL Lacertae objects.
- Blazars have SEDs that are typically two peaked. The peak at lower frequency is attributed to synchrotron radiation and the one at higher frequency to IC scattering.

  The lower-energy case (LBL blazar) extends from the radio to the gamma-ray bands but is quiet in the TeV band. The higher-energy case (HBL blazar) reaches TeV energies but is quiet in the radio range.

LBL: Low-frequency peaked BL Lacs
HBL: High-frequency peaked BL Lacs



# [Sunyaev-Zeldovich effect]

This part taken from [Bradt, Astrophysical Processes]

- The **Sunyaev-Zeldovich effect** is the distortion of the blackbody spectrum ($T$ = 2.73 K) of the CMB owing to the IC scattering of the CMB photons by the energetic electrons in the galaxy clusters.

  Thermal SZ effects, where the CMB photons interact with electrons that have high energies due to their temperature.

  Kinematic SZ effects (Ostriker-Vishniac effect), a second-order effect where the CMB photons interact with electrons that have high energies due to their bulk motion (peculiar motion). The motions of galaxies and clusters of galaxies relative to the Hubble flow are called peculiar velocities. The plasma electrons in the cluster also have this velocity. The energies of the CMB photons that scattered by the electrons reflect this motion.

  Determinations of the peculiar velocities of clusters enable astronomers to map out the growth of large-scale structure in the universe. This topic is fundamental importance, and the kinetic SZ effect is a promising method for approaching it.

# - Thermal SZ effect

- The net effect of the IC scattering on the photon spectrum is obtained by multiplying the photon number spectrum by the kernel $K(\nu/\nu_0)$ and integrating over the spectrum.

$$N_{\rm scatt}(\nu) = \int_0^\infty N(\nu_0)K(\nu/\nu_0)d\nu_0$$

The net effect is that the BB spectrum is shifted to the right and distorted.

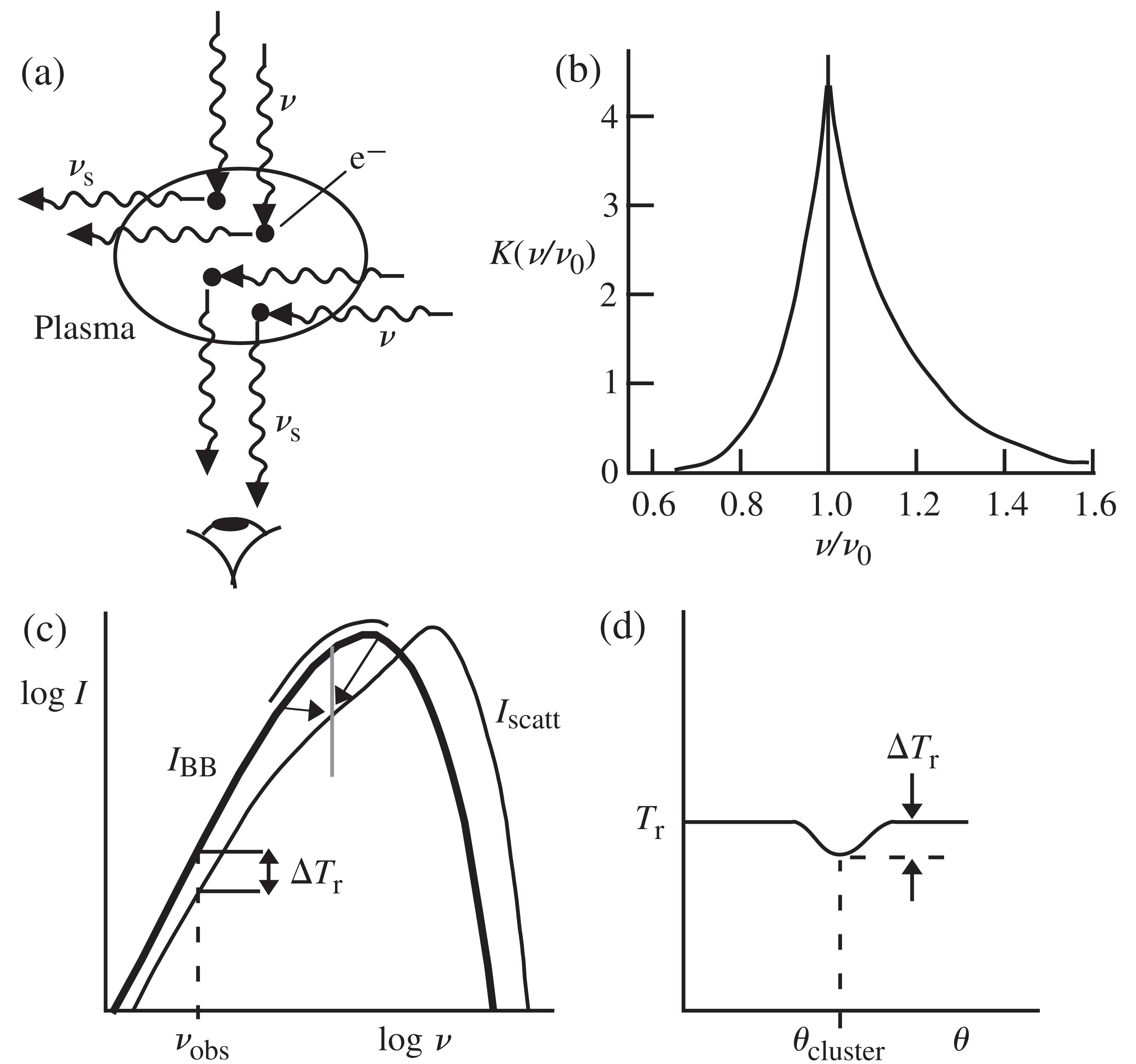
Observations of the CMB are most easily carried out in the low-frequency Rayleigh-Jeans region of the spectrum $(h\nu \ll kT_{\rm CMB})$.

Measurement of the CMB temperature as a function of position on the sky would thus exhibit antenna temperature dips in the directions of clusters that contain hot plasmas.

Note that the scattered spectrum is not a BB spectrum. The effect temperature increases. But, the total number of photons detected in a given time over the entire spectrum remains constant.

The result of such scatterings for an initial blackbody photon spectrum is shown in the following figure for the value:

$$\frac{kT_e}{mc^2}\tau = 0.5$$



- **Change of the BB temperature**

  In the Rayleigh-Jeans region,
$$I(\nu) = \frac{2\nu^2}{c^2} k_{\rm B} T_{\rm CMB}$$

  If the spectrum is shifted parallel to itself on a log-log plot, the fractional frequency change of a scattered photon is constant.

$$\varepsilon = \frac{\Delta\nu}{\nu} = \frac{\nu' - \nu}{\nu} = \text{ constant } \text{ or } \nu' = \nu(1+\varepsilon) \longrightarrow d\nu' = d\nu(1+\varepsilon)$$

  Total photon number is conserved: $N'(\nu')d\nu' = N(\nu)d\nu \rightarrow \dfrac{I'(\nu')}{h\nu'}d\nu' = \dfrac{I(\nu)}{h\nu}d\nu$

$$\therefore \quad I'(\nu') = I(\nu) \qquad I(\nu) = I\left(\frac{\nu'}{1+\varepsilon}\right) = \frac{2\nu'^2}{c^2(1+\varepsilon)^2} k_{\rm B} T_{\rm CMB}$$

$$\frac{\Delta I}{I} = \frac{I' - I}{I} = \frac{1}{(1+\varepsilon)^2} - 1 \approx -2\varepsilon = -2\frac{\Delta\nu}{\nu}$$

$$\frac{\Delta T_{\rm CMB}}{T_{\rm CMB}} = \frac{\Delta I}{I} \approx -2\epsilon = -2\frac{\Delta\nu}{\nu}$$

  The properly calculated result is $\varepsilon = \dfrac{\Delta\nu}{\nu} = \dfrac{k_{\rm B}T_{\rm CMB}}{mc^2}\tau$ . $\longrightarrow$ $\boxed{\dfrac{\Delta T_{\rm CMB}}{T_{\rm CMB}} \approx -2\dfrac{k_{\rm B}T_{\rm CMB}}{mc^2}\tau}$

A typical cluster have
an average electron density of $\sim 2.5 \times 10^{-3}\ \mathrm{cm}^{-3}$,
a core radius of $R_c \sim 10^{24}\ \mathrm{cm}\ (\sim 320\ \mathrm{pc})$,
and an electron temperature of $k_{\mathrm{B}} T_e \approx 5\ \mathrm{keV}$.

A typical optical depth is thus

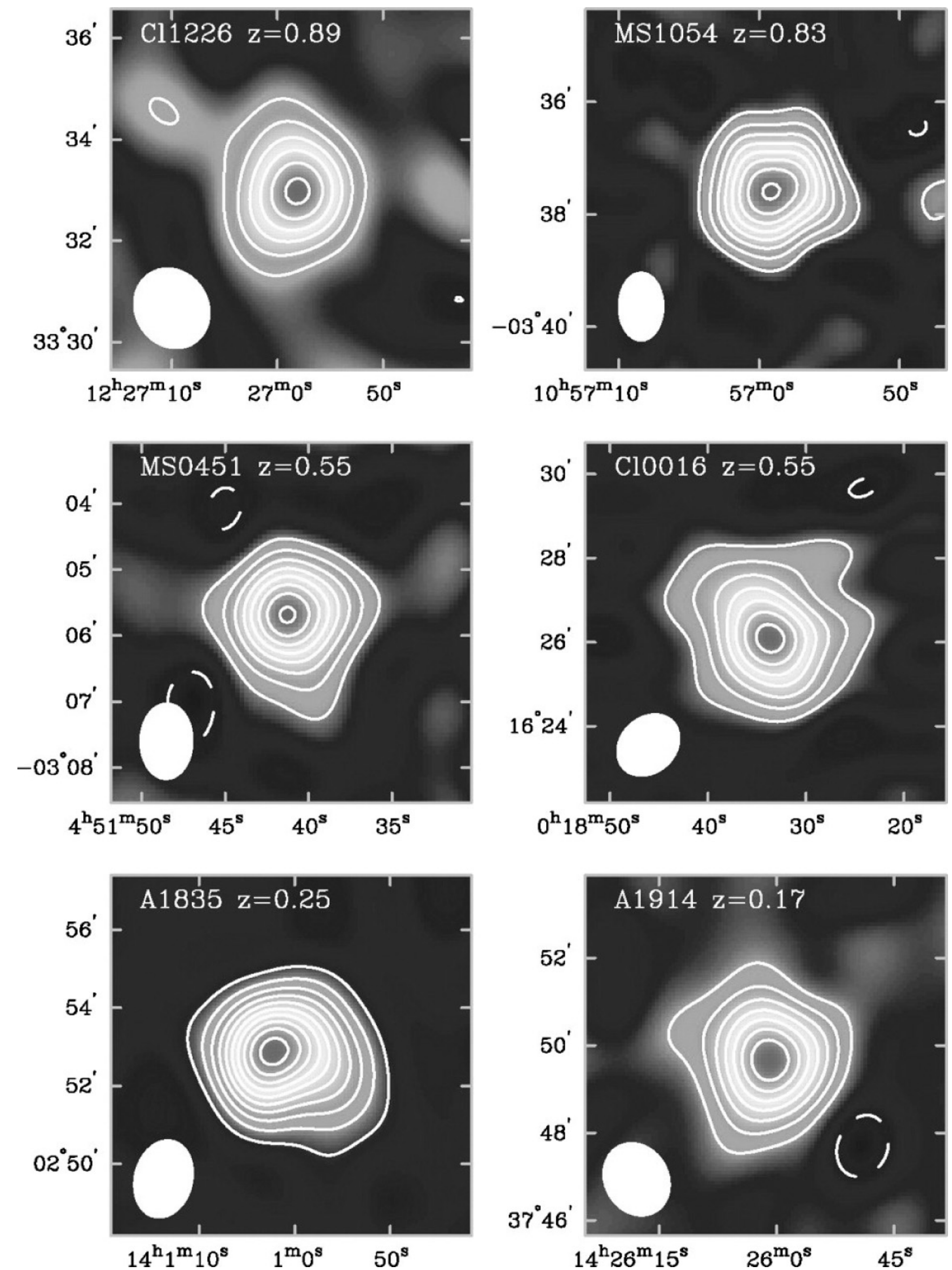
$$\tau \approx 3\sigma_T n_e R_e \approx 0.005$$

The expected antenna temperature change is

$$\frac{\Delta T_{\mathrm{CMB}}}{T_{\mathrm{CMB}}} \approx -1 \times 10^{-4}$$

$$\Delta T_{\mathrm{CMB}} \approx -0.3\ \mathrm{mK\ for}\ T_{\mathrm{CMB}} = 2.7\ \mathrm{K}$$

This effect has been measured in dozens of clusters.




Interferometric images at 30 GHz of six clusters of galaxies. The solid white contours indicate negative decrements to the CMB. (Carlstrom et al. 2002, ARAA, 40, 643)

# - Hubble Constant

- A value of the Hubble constant is obtained for a given galaxy only if one has independent measures of a recession speed v and a distance *d* of a galaxy.

$$H_0 = \frac{v}{d}$$

Recession speed is readily obtained from the spectral redshift

Distance:

X-ray observations: $I(\nu, T_e) = C\frac{g(\nu, T_e)}{T_e^{1/2}}\exp(-h\nu/kT_e)n_e^2(2R)$

S-Z CMB decrement: $\frac{\Delta T_{\rm CMB}}{T_{\rm CMB}} = -2\frac{kT_e}{mc^2}\tau = -2\frac{kT_e}{mc^2}\left(\sigma_T n_e 2R\right)$

The radio and X-ray measurements yield absolute values of the electron density $n_e$ and cluster radius $R$ without a priori knowledge of the cluster distance.

Imaging of the cluster in the radio or X-ray band yields the angular size of the cluster $\theta$. Then the distance $d$ to the cluster is obtained by

$$d = \frac{R}{\theta}$$

The SZ effect (at radio frequencies) in conjunction with X-ray measurements can give distances to clusters of galaxies. This can be used to derive the Hubble constant.

# [Kompaneets Equation]

- The Kompaneets equation describes the time evolution of the distribution of photon occupancies in the case where photons and electrons are interacting through Compton scattering.
- Boltzmann transport equation

$$\frac{\partial n(\omega)}{\partial t} = c\int d^3p \int d\Omega \frac{d\sigma}{d\Omega}\left[f_e(\mathbf{p})' n(\omega')(1+n(\omega)) - f_e(\mathbf{p})n(\omega)(1+n(\omega'))\right]$$

In $1 + n(\omega)$, the "1" for spontaneous Compton scattering, and the $n(\omega)$ for stimulated Compton scattering.

The Boltzmann equation may be expanded to second order in the small energy transfer, yielding an approximation called the *Fokker-Plank equation*. For photons scattering off a nonrelativistic, thermal distribution of electrons, the Fokker-Plank equation was first derived by A. S. Kompaneets (1957) and is known as the Kompaneets equation.

$$\frac{\partial n(\omega)}{\partial t} = \left(\frac{k_{\mathrm{B}}T}{mc^2}\right)\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n + n^2\right)\right] \quad \text{where} \quad x \equiv \frac{\hbar\omega}{k_{\mathrm{B}}T}, \quad \text{and} \quad t_c \equiv (n_e\sigma_T c)\,t$$

- For the complete derivation, see the books "X-ray spectroscopy in Astrophysics (eds. van Paradijs)", pages 213-218, and the book "High Energy Astrophysics (Katz)", pages 103-110.
- Monte Carlo Simulation of Compton scattering: see "Pozdnyakov, Sobol, and Suyaev (1983, Soviet Scientific Reviews, vol. 2, 189-331)" (1983ASPRv...2..189P)

# 8. Plasma Effects

# [Dispersion in Cold, Isotropic Plasma]

- Roughly speaking a **plasma** is a

  globally neutral

  partially or completely ionized gas

- **<u>Plasma Dispersion Relation</u>**

  Assume a space and time variation of all quantities of the form $\exp\left[i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right]$ .

$$\begin{aligned} \nabla\cdot\mathbf{E} &= 4\pi\rho \\ \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\times\mathbf{B} &= \frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \end{aligned} \longrightarrow \begin{aligned} i\mathbf{k}\cdot\mathbf{E} &= 4\pi\rho \\ i\mathbf{k}\cdot\mathbf{B} &= 0 \\ i\mathbf{k}\times\mathbf{E} &= i\frac{\omega}{c}\mathbf{B} \\ i\mathbf{k}\times\mathbf{B} &= \frac{4\pi}{c}\mathbf{j}-i\frac{\omega}{c}\mathbf{E} \end{aligned}$$

Equation of motion of electrons when there is no external magnetic field

$$m\dot{\mathbf{v}} = -e\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right) = -e\mathbf{E} \longrightarrow \begin{aligned} -im\omega\mathbf{v} &= -e\mathbf{E} \\ \mathbf{v} &= \frac{e\mathbf{E}}{i\omega m} \end{aligned}$$

<u>Current density:</u>

$$\mathbf{j} = -ne\mathbf{v} = \sigma\mathbf{E}$$

**conductivity:** $\sigma \equiv \dfrac{ine^2}{\omega m}$

## Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \qquad -i\omega\rho + i\mathbf{k}\cdot\mathbf{j} = 0 \;\rightarrow\; \rho = \frac{1}{\omega}\mathbf{k}\cdot\mathbf{j} = \frac{\sigma}{\omega}\mathbf{k}\cdot\mathbf{E}$$

Using

$$\mathbf{j} = \sigma\mathbf{E}$$
$$\rho = \frac{\sigma}{\omega}\mathbf{k}\cdot\mathbf{E}$$

$$\begin{aligned} i\mathbf{k}\cdot\mathbf{E} &= 4\pi\rho \\ i\mathbf{k}\cdot\mathbf{B} &= 0 \\ i\mathbf{k}\times\mathbf{E} &= i\frac{\omega}{c}\mathbf{B} \\ i\mathbf{k}\times\mathbf{B} &= \frac{4\pi}{c}\mathbf{j} - i\frac{\omega}{c}\mathbf{E} \end{aligned} \longrightarrow \begin{aligned} i\mathbf{k}\cdot\epsilon\mathbf{E} &= 0 \\ i\mathbf{k}\cdot\mathbf{B} &= 0 \\ i\mathbf{k}\times\mathbf{E} &= i\frac{\omega}{c}\mathbf{B} \\ i\mathbf{k}\times\mathbf{B} &= -i\frac{\omega}{c}\epsilon\mathbf{E} \end{aligned}$$

These equations are now “source-free.”

$\mathbf{k}$, $\mathbf{E}$, $\mathbf{B}$ form a mutually orthogonal right-hand vector triad.

The dispersion relation is

$$\begin{aligned} c^2k^2 &= \epsilon\omega^2 \\ &= \omega^2 - \omega_p^2 \end{aligned}$$

$$\boxed{\omega^2 = \omega_p^2 + c^2k^2}$$

where **dielectric constant**:

$$\begin{aligned} \epsilon \equiv 1 - \frac{4\pi\sigma}{i\omega} &= 1 - \frac{4\pi ne^2}{m\omega^2} \\ &= 1 - \left(\frac{\omega_p}{\omega}\right)^2 \end{aligned}$$

**plasma frequency:**

$$\begin{aligned} \omega_p &\equiv \sqrt{\frac{4\pi ne^2}{m}} \\ &= 5.63\times10^4\,\left(n/\mathrm{cm}^{-3}\right)^{1/2}\ \mathrm{s}^{-1} \end{aligned}$$

- If $\omega < \omega_p$, then $k^2 < 0$. Thus, wavenumber is imaginary!

$$k = \frac{i}{c}\sqrt{\omega_p^2 - \omega^2}$$

The wave amplitude decreases as $e^{-|k|r}$ on a scale of the order of $\approx 2\pi c/\omega_p$.

Thus $\omega_p$ defines a plasma cutoff frequency below which there is no electromagnetic propagation.

For instance, Earth ionosphere prevents extraterrestrial radiation at frequencies less than ~ 1 MHz from being observed at the earth's surface ($n \sim 10^4$ cm$^{-3}$).

<u>Method of probing the ionosphere:</u>

Let a pulse of radiation in a narrow range about $\omega$ be directed straight upward from the earth's surface.

When there is a layer at which $n$ is large enough to make $\omega < \omega_p$, the pulse will be totally reflected from the layer.

The time delay of the pulse provides information on the height of the layer.

By making such measurements at many different frequencies, the electron density can be determined as a function of height.

- **Group and Phase Velocity**

  When $\omega > \omega_p$ , waves do propagate without damping.

  Phase velocity

$$v_{\mathrm{ph}} \equiv \frac{\omega}{k} = \frac{\omega}{\sqrt{\omega^2 - \omega_p^2}/c} = \frac{c}{n_r}$$

  where the index of refraction $n_r \equiv \sqrt{\epsilon} = \sqrt{1 - \frac{\omega_p^2}{\omega}} \quad \rightarrow \quad v_{\mathrm{ph}} > c$

  Group velocity

$$\omega^2 = \omega_p^2 + c^2 k^2 \quad \longrightarrow \quad v_{\mathrm{g}} \equiv \frac{\partial \omega}{\partial k} = \frac{c^2 k}{\omega} = c\sqrt{1 - \frac{\omega_p^2}{\omega}} \quad \rightarrow \quad v_{\mathrm{g}} < c$$

- The wave energy travels at the group velocity, as does any modulation of the wave (information coding).

- **Pulsar**

  Each small range of frequencies travels at a slightly different group velocity and will reach earth at a slightly different time.

  Let $d$ = pulsar distance.

  The time required for a pulse to reach earth at frequency $\omega$ is $t_p = \int_0^d \frac{ds}{v_\mathrm{g}}$.

  The plasma frequency in ISM is usually quite low (~ $10^3$ Hz), so we can assume that $\omega \gg \omega_p$.

$$\frac{1}{v_\mathrm{g}} = \frac{1}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} \approx \frac{1}{c}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right) \longrightarrow \quad t_p \approx \frac{d}{c} + \frac{2\pi e^2}{cm\omega^2}\int_0^d n ds$$

= transit time for a vacuum + plasma correction

- **Dispersion measure**:

  What is usually measured is **the rate of change of arrival time with respect to frequency**.

$$\frac{dt_p}{d\omega} = -\frac{4\pi e^2}{cm\omega^3}\mathcal{DM} \qquad \text{where } \mathcal{DM} \equiv \int_0^d n ds$$

  The arrival interval of pulsar signal between two frequencies $\nu_1$ and $\nu_2$:

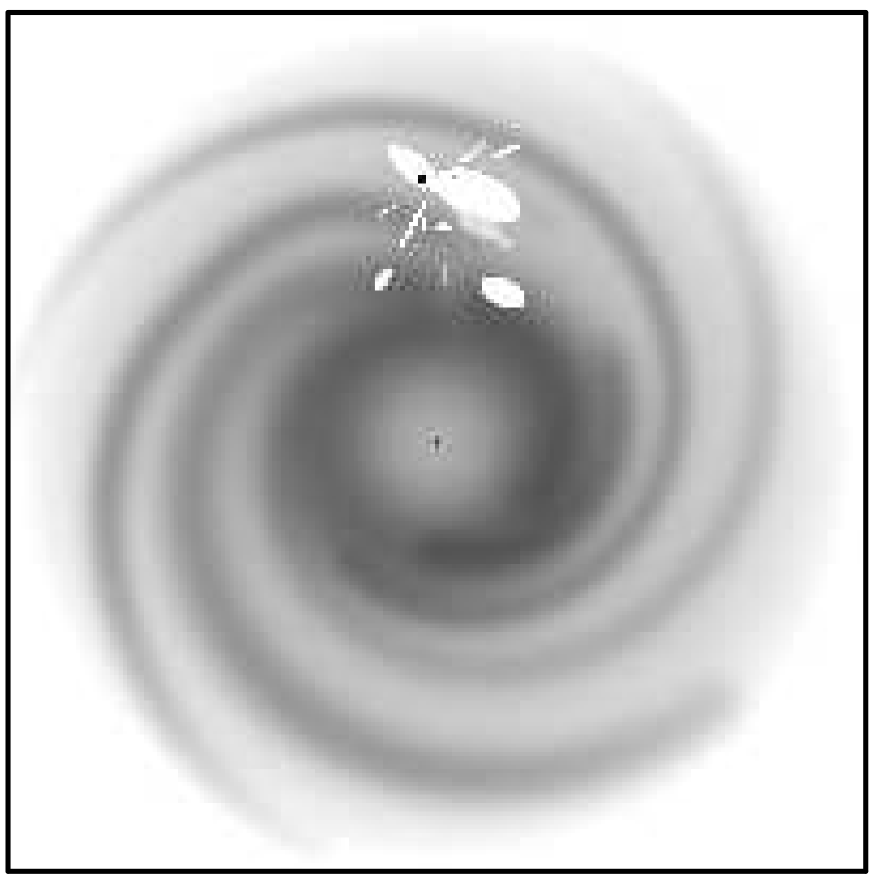
$$\Delta t_p = \frac{e^2}{2\pi c m_e}\left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2}^{2}\right)\mathcal{DM} = 4.15 \times 10^{15}\left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2}^{2}\right)\frac{\mathcal{DM}}{\mathrm{cm}^{-3}\ \mathrm{pc}}\ \mathrm{sec}$$

If one has idea of pulsar distance, one can use pulsar data to map free electron density.

Taylor & Cordes (1993, ApJ, 411, 674)

Cordes & Lazio (2003, arXiv:astro-ph/0207156)

Schnitzeler (2012, MNRAS, 427, 664)



Cordes & Lazio (2003)

FIG. 2.— Electron density corresponding to the best fit model plotted as a grayscale with logarithmic levels on a 30×30 kpc x-y plane at z=0 and centered on the Galactic center. The most prominent large-scale features are the spiral arms, a thick, tenuous disk, a molecular ring component. A Galactic center component appears as a small dot. The small-scale, lighter features represent the local ISM and underdense regions required for by some lines of sight with independent distance measurements. The small dark region embedded in one of the underdense, ellipsoidal regions is the Gum Nebula and Vela supernova remnant.

# [Propagation along a Magnetic Field; Faraday Rotation]

- Now we consider the effect of an external, fixed magnetic field $\mathbf{B}_0$.

  The properties of the waves will then depend on the direction of propagation relative to the magnetic field direction.

$$\text{cyclotron frequency} \quad \omega_{\mathrm{B}} = \frac{eB_0}{mc} = 1.67 \times 10^7 \ (B_0/\mathrm{G}) \ \mathrm{s}^{-1}$$
$$\hbar\omega_{\mathrm{B}} = 1.16 \times 10^{-8} \ (B_0/\mathrm{G}) \ \mathrm{eV}$$

  If the fixed magnetic field $\mathbf{B}_0$ is much stronger than the field strengths of the propagating wave, then the equation of motion of an electron is approximately

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - \frac{e}{c}\mathbf{v} \times \mathbf{B}_0$$

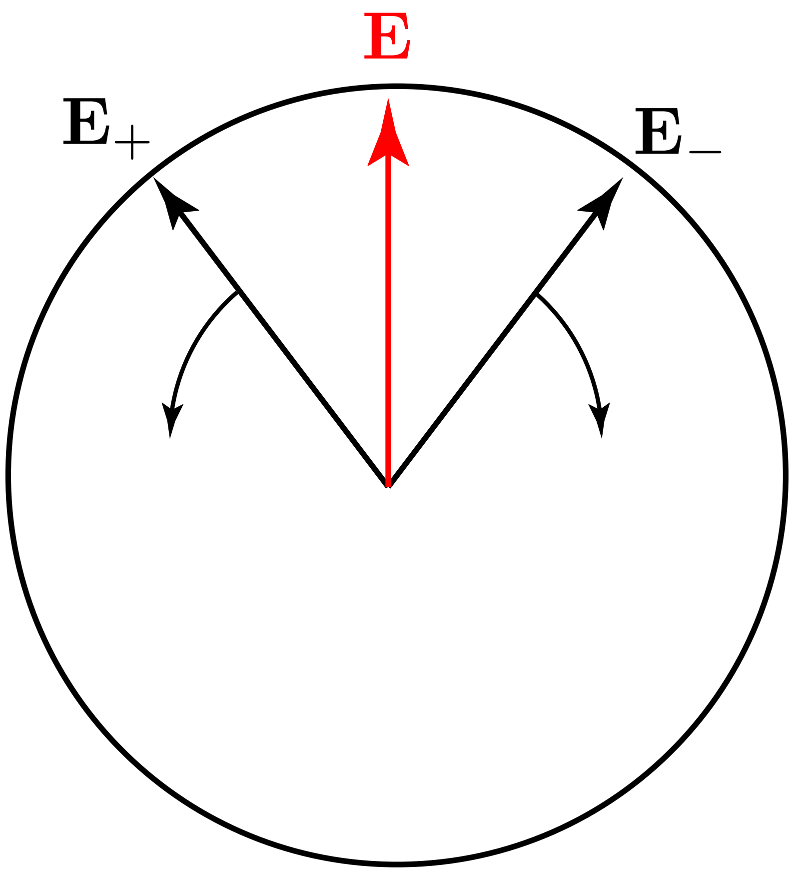
  For simplicity, assume that the wave propagates along the fixed field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, and assume that the wave is circularly polarized and sinusoidal.

$$\mathbf{E}_{\pm}(t) = Ee^{-i\omega t}\left(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}\right)$$

  Here, $\pm$ denotes right and left circular polarization.

# Bases vectors for circularly rotating electric field

$$
\begin{aligned}
Re\left(\mathbf{E}_{\pm}\right) &= Re\left(\left(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}\right) E e^{-i\omega t}\right) \\
&= Re\left(\left(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}\right) E \left(\cos\omega t - i\sin\omega t\right)\right) \\
&= (\cos\omega t)\hat{\mathbf{x}} \pm (\sin\omega t)\hat{\mathbf{y}}
\end{aligned}
$$



$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - \frac{e}{c}\mathbf{v}\times\mathbf{B}_0 \qquad \frac{d\mathbf{v}_\parallel}{dt} = 0 \qquad \mathbf{v}_\parallel = \text{constant}$$

Set $\quad \mathbf{v}_\perp = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$

$$(-i\omega)e^{-i\omega t}\left(v_x\hat{\mathbf{v}} + v_y\hat{\mathbf{y}}\right) = -\frac{e}{m}Ee^{-i\omega t}\left(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}\right) - \frac{e}{mc}e^{-i\omega t}\left(v_yB_0\hat{\mathbf{x}} - v_xB_0\hat{\mathbf{y}}\right)$$

$$\longrightarrow \quad v_x = -\frac{ie}{m\omega}E - \frac{ieB_0}{mc\omega}v_y, \quad v_y = \pm\frac{e}{m\omega}E + \frac{ieB_0}{mc\omega}v_x$$

$$(1)\ v_x = -\frac{ie}{m\omega}E - \frac{ieB_0}{mc\omega}\left(\pm\frac{e}{m\omega}E + \frac{ieB_0}{mc\omega}v_x\right)$$

$$\to \quad \left(1 - \frac{e^2B_0^2}{m^2c^2\omega^2}\right)v_x = -\frac{ie}{m\omega}E\left(1 \pm \frac{eB_0}{mc\omega}\right)$$

$$\to \quad \left(\omega^2 - \omega_{\mathrm{B}}^2\right)v_x = -\frac{ie}{m}E\left(\omega \pm \omega_{\mathrm{B}}\right)$$

$$\to \quad v_x = -\frac{ie}{m}E\frac{1}{\omega \mp \omega_{\mathrm{B}}}$$

$$(2)\ v_y = \pm\frac{e}{m\omega}E + \frac{ieB_0}{mc\omega}\left(-\frac{ie}{m\omega}E - \frac{ieB_0}{mc\omega}v_y\right)$$

$$\to \quad \left(1 - \frac{e^2B_0^2}{m^2c^2\omega^2}\right)v_y = \pm\frac{e}{m\omega}E\left(1 \pm \frac{eB_0}{mc\omega}\right)$$

$$\to \quad \left(\omega^2 - \omega_{\mathrm{B}}^2\right)v_y = \pm\frac{e}{m}E\left(\omega \pm \omega_{\mathrm{B}}\right)$$

$$\to \quad v_y = \pm\frac{e}{m}E\frac{1}{\omega \mp \omega_{\mathrm{B}}}$$

Then, the current density and conductivity are given by

$$\therefore \quad \mathbf{v} = \frac{-ie}{m\left(\omega \mp \omega_{\mathrm{B}}\right)}\mathbf{E}(t)$$

$$\boxed{\begin{array}{l} \mathbf{j} \equiv -ne\mathbf{v} = \sigma\mathbf{E} \\ \text{where } \sigma_{\mathrm{R,L}} \equiv \dfrac{ine^2}{m\left(\omega \mp \omega_{\mathrm{B}}\right)} \end{array}}$$

## Dielectric constant

$$\epsilon \equiv 1 - \frac{4\pi\sigma}{i\omega} = 1 - \frac{4\pi n e^2}{m\omega\left(\omega \mp \omega_{\mathrm{B}}\right)}$$

$$\epsilon_{\mathrm{R,L}} = 1 - \frac{\omega_p^2}{\omega\left(\omega \mp \omega_{\mathrm{B}}\right)}$$

Dispersion relation

$$\omega^2 = k^2 c^2 + \frac{\omega_p^2 \omega}{\omega \mp \omega_{\mathrm{B}}}$$

Phase velocity

$$v_p \equiv \frac{\omega}{k} = \frac{c}{n_r}$$

$$n_r = \sqrt{1 - \frac{\omega_p^2}{\omega\left(\omega \mp \omega_{\mathrm{B}}\right)}}$$

Right (+) and left(-) circularly polarized waves travel with different speeds.

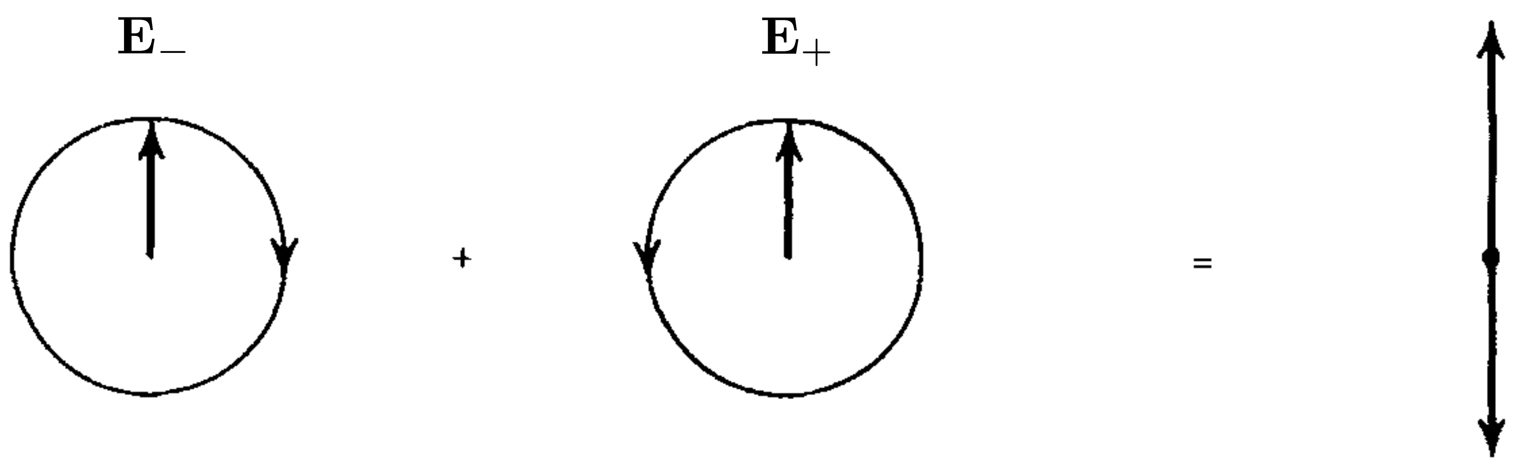
Speed difference sense is $v_{\mathrm{R}} > v_{\mathrm{L}}$.

Note that Rybiki & Lightman define oppositely the RCP and LCP.

- **<u>Faraday Rotation</u>**

If the incident radiation is ***circularly polarized*** (either R or L), then the radiation will encounter different dispersion than unmagnetized case. But, the radiation will still remain circularly polarized.

If the incident radiation is ***linearly polarized***, i.e., a linear superposition of a right-hand and a left-hand polarized wave, then ***<u>the line of polarization will rotate as it propagates</u>***. This effect is called Faraday rotation.



- Phase after traveling a distance $d$:

$$\phi_{\mathrm{R,L}} = \int_0^d k_{\mathrm{R,L}} dz$$

Assume that $\omega \gg \omega_p,\ \omega \gg \omega_{\mathrm{B}}$

$$k_{\mathrm{R,L}} = \frac{\omega}{c}\sqrt{\epsilon_{\mathrm{R,L}}} = \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{\omega^2\left(1 \mp \omega_{\mathrm{B}}/\omega\right)}\right]^{1/2} \approx \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{2\omega^2}\left(1 \pm \frac{\omega_{\mathrm{B}}}{\omega}\right)\right]$$
$$= \frac{\omega}{c} - \frac{\omega_p^2}{2c\omega} \mp \frac{\omega_p^2 \omega_{\mathrm{B}}}{2c\omega^2} = k_0 \mp \Delta k \quad (\Delta k \equiv k_{\mathrm{L}} - k_{\mathrm{R}})$$

Consider an electromagnetic wave that starts off linearly polarized in the $x$-direction at the source.

$$\mathbf{E}(t) = Ee^{-i\omega t}\hat{\mathbf{x}} = \frac{1}{2}\left[(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + (\hat{\mathbf{x}} - i\hat{\mathbf{y}})\right] Ee^{-i\omega t}$$

Let's define

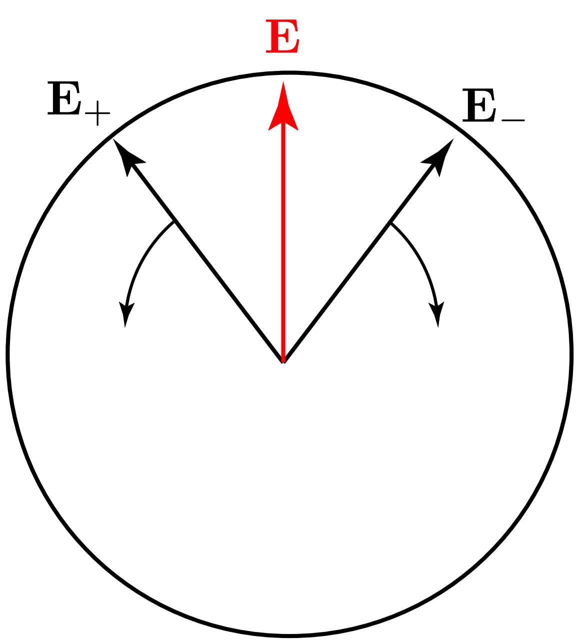
$$\int_0^d k_{\mathrm{R,L}} = \int_0^d k_0 dz \mp \int_0^d \Delta k dz \equiv \phi \mp \varphi$$

Then, after propagating a distance d through a magnetized plasma toward the observer, the electric field will be

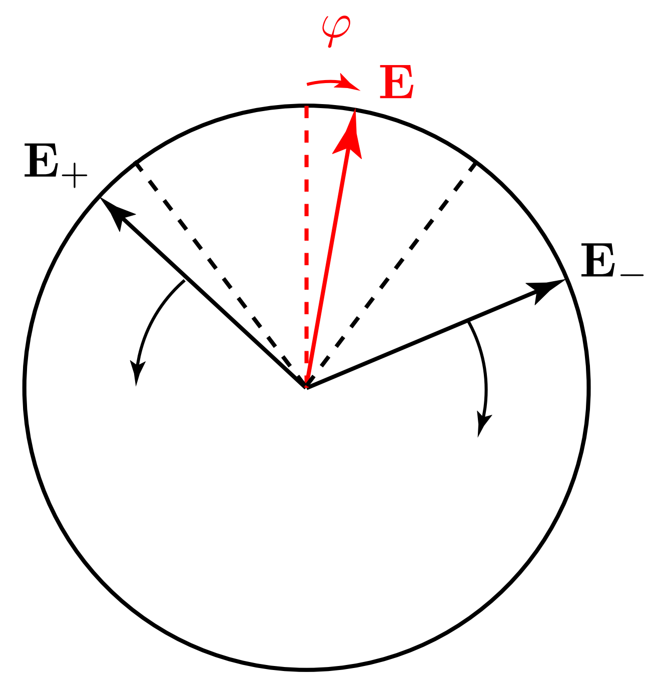
$$\mathbf{E}(t) = \frac{1}{2}\left[(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\, e^{i(\phi-\varphi)} + (\hat{\mathbf{x}} - i\hat{\mathbf{y}})\, e^{i(\phi+\varphi)}\right] Ee^{-i\omega t}$$
$$= (\hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\sin\varphi)\, Ee^{i(\phi-\omega t)}$$

- Radiation that starts linearly polarized in a certain direction is rotated by the *Faraday effect* through an angle $\varphi$ after propagating a distance *d* through a magnetized plasma.

$$\varphi = \int_0^d \Delta k dz = \frac{1}{2}\int_0^d \frac{\omega_p^2 \omega_{\mathrm{B}}}{c\omega^2} ds = \frac{2\pi e^3}{m^2 c^2 \omega^2}\int_0^d n B_{\parallel} ds$$




Before passing through the medium




After passing through the medium

- We cannot, of course, generally measure the absolute rotation angle, since we do not know the intrinsic polarization direction of the radiation when it started from the source.

  However, since $\varphi$ varies with frequency (as $\omega^{-2}$), we can determine the value of integral $\int nB_{\parallel}ds$ by making measurements at several frequencies. This can give information about the interstellar magnetic field.

  **Rotation measure** is defined by $$\varphi = \frac{2\pi e^3}{m^2c^2\omega^2}\mathcal{RM} = \frac{e^3\lambda^2}{2\pi m^2 c^4}\mathcal{RM}, \text{ where } \mathcal{RM} \equiv \int_0^d nB_{\parallel}ds$$

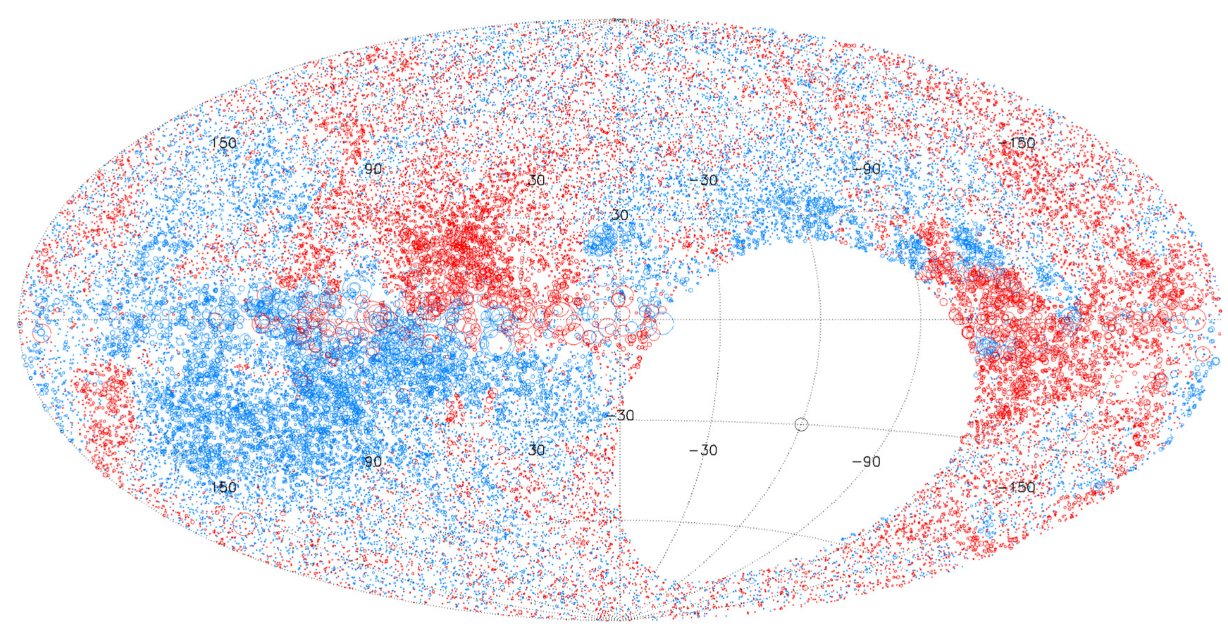
  However, the field changes direction often along the line of sight and this method gives only a lower limit to actual field magnitudes.

  For measurements toward sources (pulsars) where the dispersion measure (DM) is also known, we can derive an estimate of the mean field strength along the line of sight.

  $$\langle B_{\parallel}\rangle = \frac{\mathcal{RM}}{\mathcal{DM}}$$

  Radio astronomers have concluded that

  $$\langle n_e\rangle \approx 0.03 \text{ cm}^{-3}$$
  $$\langle B_{\parallel}\rangle \approx 3\mu\text{G}$$



(Taylor, Stil, & Sunstrum 2009, ApJ, 702, 1230)
Red circles are positive RM and blue circles are negative.
The size of the circle scales linearly with magnitude of RM.

# [Plasma Effects in High-Energy Emission Processes]

- <u>Maxwell equations in dielectric medium:</u>

$$
\begin{aligned}
\nabla \cdot (\epsilon \mathbf{E}) &= 4\pi\rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial(\epsilon \mathbf{E})}{\partial t}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} \\
\mathbf{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}
\end{aligned}
$$

These equations formally result from Maxwell's equation in vacuum by the substitutions.

$$
\begin{aligned}
\mathbf{E} &\rightarrow \sqrt{\epsilon}\mathbf{E} \\
c &\rightarrow c/\sqrt{\epsilon} \\
\mathbf{B} &\rightarrow \mathbf{B} \\
e &\rightarrow e/\sqrt{\epsilon} \\
\phi &\rightarrow \sqrt{\epsilon}\phi \\
\mathbf{A} &\rightarrow \mathbf{A}
\end{aligned}
$$

These equations may be solved in the same manner as before for the retarded and Lienard-Wiechert potentials.

# - Cherenkov Radiation

- Radiation from relativistic charges moving in a plasma with $n_r \equiv \sqrt{\epsilon} > 1$.

  In this case, the velocity of the charges can exceed the phase velocity:

$$v_p = \frac{c}{n_r} < v < c \quad \rightarrow \quad \beta n_r > 1$$

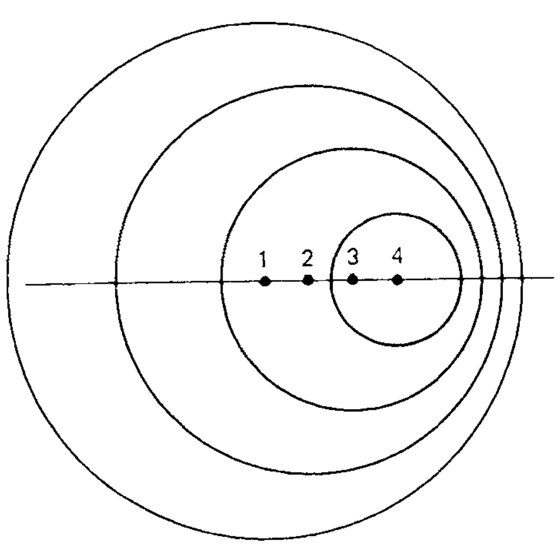
  The beaming term of the Lienard-Wiechert potentials can vanish for an angle $\theta$ such that $\cos\theta = (n_r\beta)^{-1}$.

$$\kappa = 1 - (v/c)\cos\theta \quad \rightarrow \quad \kappa = 1 - \beta n_r \cos\theta$$

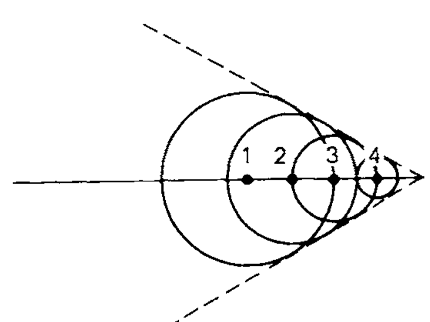
  The potentials become infinite at certain places. In consequence, the uniformly moving particle can now radiate.

  Cherenkov cone: Outside the cone, points feel no potentials yet. Inside the cone, each point is intersected by two spheres. The resulting radiation is called Cherenkov radiation.

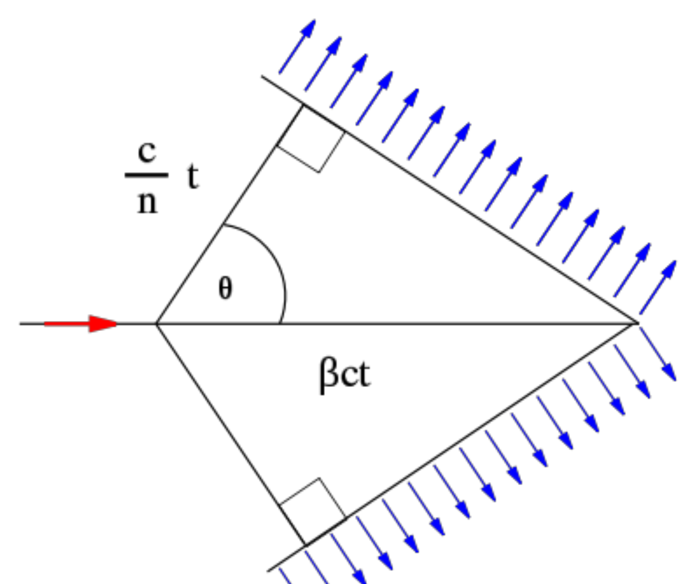
  A common analogy is the sonic boom of a supersonic aircraft or bullet.




$v < v_p = \dfrac{c}{n_r}$




$v > v_p$



# - Razin-Tsytovich Effect

- When $n_r < 1$, Cherenkov radiation cannot occur.

  The critical angle defining the beaming effect in a vacuum was shown to be $\theta_b \sim 1/\gamma = \sqrt{1-\beta^2}$. But in a plasma we have instead

$$\theta_b \sim \sqrt{1 - n_r^2 \beta^2}$$

  If $n_r \ll 1,$ and $\beta \sim 1$,

$$\theta_b \sim \sqrt{1 - n_r^2} = \sqrt{1 - \left(1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_\mathrm{B})}\right)} \approx \frac{\omega_p}{\omega}$$

  If $\omega < \gamma\omega_p$ , $\theta_p > 1/\gamma$ and the beaming effect is suppressed.

  Below the frequency $\gamma\omega_p$ , the synchrotron spectrum will be cut off because of the suppression of beaming. This is called the Razin-Tsytovich effect.

  As frequencies increase, $\theta_b$ decreases until it becomes of order of the vacuum value $1/\gamma$ , and therefore the vacuum results apply.

  Therefore, the plasma medium effect is unimportant when $\omega \gg \gamma\omega_p$ .