Modern Astronomy

Part 1. Interstellar Medium (ISM)

Week 4
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Virial Theorem

Virial Theorem

- For a collection of *N* point particles, the total moment of inertia is given by

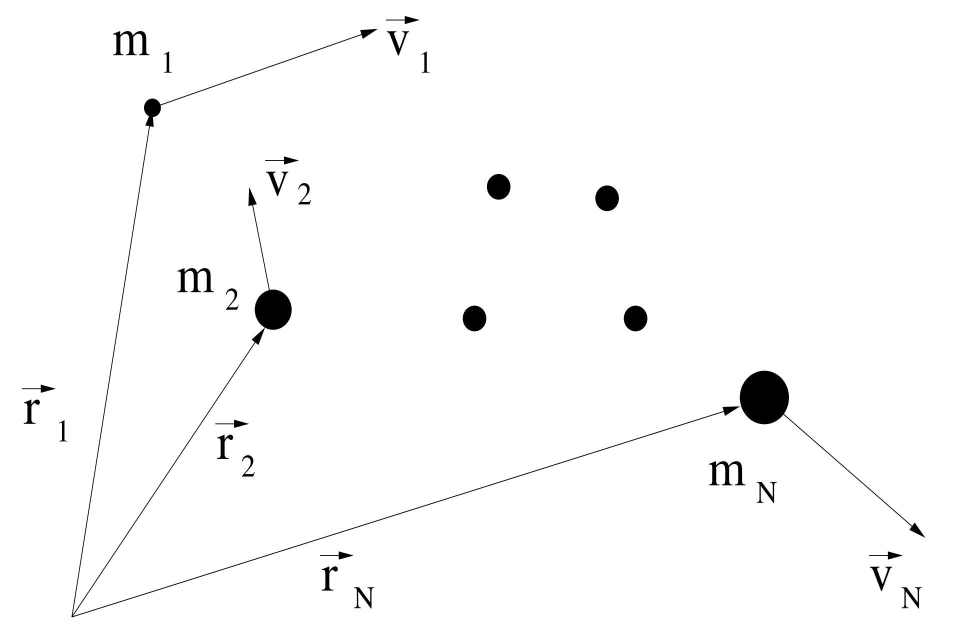
$$I = \sum_i m_i \left|\mathbf{r}_i\right|^2 = \sum_i m_i \mathbf{r}_i \cdot \mathbf{r}_i$$

- The time derivative of the moment of inertia is called the ***virial***:

$$\boxed{Q = \frac{1}{2}\frac{dI}{dt} = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i}$$

Here, momentum vector:

$$\mathbf{p}_i = m_i \mathbf{v}_i \quad \left(= m_i \frac{d\mathbf{r}_i}{dt}\right)$$



Now take the time derivative of the viral

$$\frac{dQ}{dt} = \sum \mathbf{p}_i \cdot \mathbf{v}_i + \sum_i \frac{d\mathbf{p}_i}{dt} \cdot \mathbf{r}_i$$

$$= \sum_i m_i \mathbf{v}_i \cdot \mathbf{v}_i + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i$$

Newton's second law:

$$\mathbf{F}_i = \frac{d\mathbf{p}_i}{dt} \quad \text{(the sum of all forces acting on particle } i\text{)}$$

The first term can be expressed in terms of the total kinetic energy of the system

$$\frac{dQ}{dt} = 2K + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \quad \left(\text{where } K = \sum_i \frac{1}{2} m_i \left|\mathbf{v}_i\right|^2\right)$$

Virial: from Latin “vis,” meaning “force” or “energy.”

- The total force on particle i is the sum of all the forces from the other particles j in the system:

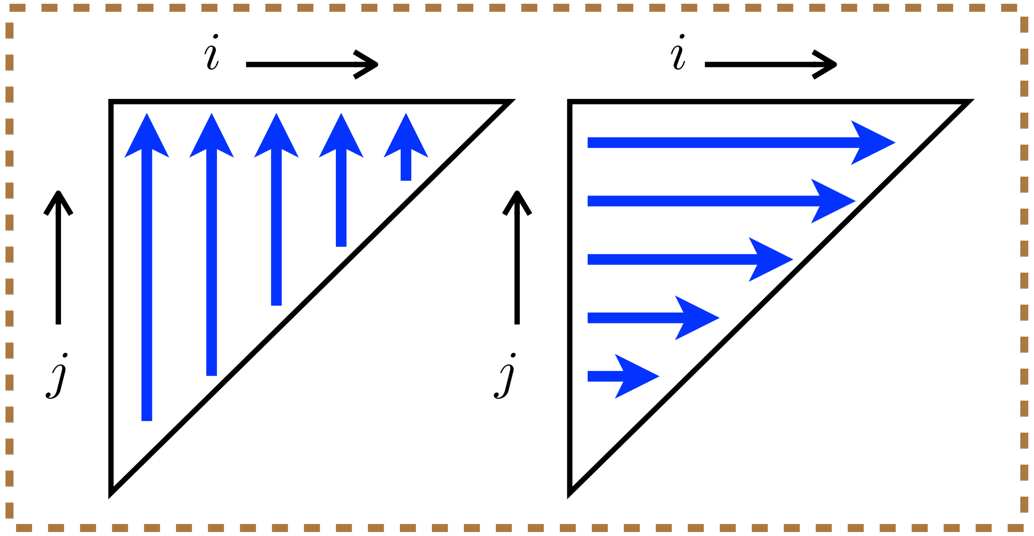
$$\mathbf{F}_i = \sum_{j \neq i} \mathbf{f}_{j \to i}$$

Here, $\mathbf{f}_{j \to i}$ is the force on particle i from particle j

Then,

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i = \sum_{i=1}^{N} \sum_{j \neq i} \mathbf{f}_{j \to i} \cdot \mathbf{r}_i$$

$$= \sum_{i=1}^{N} \sum_{j < i} \mathbf{f}_{j \to i} \cdot \mathbf{r}_i + \sum_{i=1}^{N} \sum_{j > i} \mathbf{f}_{j \to i} \cdot \mathbf{r}_i$$



The last term can be rewritten as follows:

$$\sum_{i=1}^{N} \sum_{j > i} \mathbf{f}_{j \to i} \cdot \mathbf{r}_i = \sum_{j=1}^{N} \sum_{i < j} \mathbf{f}_{j \to i} \cdot \mathbf{r}_i$$

interchange the ordering of the sums.

$$= \sum_{i=1}^{N} \sum_{j < i} \mathbf{f}_{i \to j} \cdot \mathbf{r}_j$$

interchange the name of the indices i and j.

$$= -\sum_{i=1}^{N} \sum_{j < i} \mathbf{f}_{j \to i} \cdot \mathbf{r}_j$$

Newton's third law: $\mathbf{f}_{i \to j} = -\mathbf{f}_{j \to i}$

Hence,

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i = \sum_{i=1}^{N} \sum_{j < i} \mathbf{f}_{j \to i} \cdot (\mathbf{r}_i - \mathbf{r}_j)$$

- Now, consider the gravitational force:

$$\mathbf{f}_{j\to i} = \frac{Gm_i m_j}{r_{ij}^3}(\mathbf{r}_j - \mathbf{r}_i) \longrightarrow \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i = -\sum_{i=1}^{N}\sum_{j<i} \frac{Gm_i m_j}{r_{ij}}$$

where $r_{ij} = |\mathbf{r}_j - \mathbf{r}_i|$

Note that the gravitational potential energy between particle i and j is given by $U_{ij} = -\frac{Gm_i m_j}{r_{ij}}$.

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i = \sum_{i=1}^{N}\sum_{j<i} U_{ij} = U$$ = total potential energy of the system

The total potential energy of the system is the sum of the potential energy between all possible pairs of particles (note that one pair of particle should be counted only once, this is why there is a $j < i$ in the second sum). We finally obtain:

- We take the mean value of the derivative of the virial over a long period of time:

$$\frac{dQ}{dt} = 2K + U \longrightarrow \left\langle \frac{dQ}{dt} \right\rangle = 2\langle K \rangle + \langle U \rangle$$

- If ***the system is bounded and the particles have finite momentum***, the average value will go to zero and we obtain the virial theorem:

$$\left\langle \frac{dQ}{dt} \right\rangle = \lim_{T\to\infty} \frac{Q(T) - Q(0)}{T} = 0 \longrightarrow 2\langle K \rangle + \langle U \rangle = 0$$

$\langle\rangle$ denotes both the ensemble average and time average.

Ergodic hypothesis: Averaging system variables over a long time period may be equal to averaging them over the ensemble.

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt \to \lim_{N\to} \frac{1}{N}\sum_{i=1}^{N}$$

If a bound system has a huge number of particles, it is equivalent to seeing the system over a long period of time.

Virial Mass Estimate

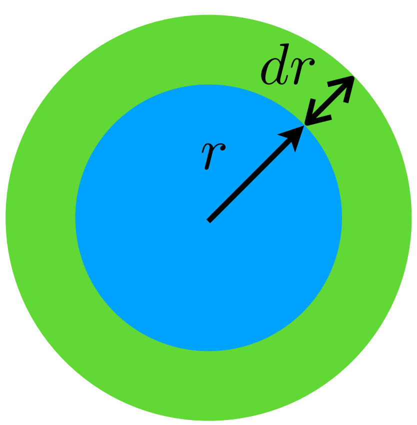
- A fundamental property is that GMCs are gravitationally bound and in viral equilibrium.
  - Their masses can then be estimated using line widths as a measure of the cloud velocities, following the arguments of Solomon et al.
  - The virial theorem provides a general equation that relates the average over time of the total kinetic energy of a stable, self-gravitating system of discrete particles, with the total potential energy of the system.

$$2\langle K\rangle + \langle U\rangle = 0$$

  - For a uniform density sphere with a mass M and radius R, the gravitational potential energy is

$$U = -\frac{3}{5}\frac{GM^2}{R}$$

$$\begin{aligned} U &= -\int_0^R \frac{GM_r dM_r}{r} \\ &= -\int_0^R \frac{G}{r}\left(\frac{4\pi}{3}r^3\rho\right)4\pi r^2 \rho dr \\ &= -\frac{(4\pi)^2}{3\times 5}G\rho^2 R^5 = -\frac{3}{5}\frac{GM^2}{R} \end{aligned}$$



Density $\rho = \frac{M}{(4\pi/3)R^3}$

Mass within a radius r

$$M_r = \int_0^r \rho(4\pi r'^2 dr') = \frac{4\pi}{3}r^3\rho$$

Mass between a shell (r, r+dr)

$$dM_r = \rho(4\pi r^2 dr)$$

- If the cloud is in equilibrium, with self-gravity being balanced by turbulent pressure, the virial theorem states that the turbulent velocity, in one dimension, to be

$$\langle K \rangle = \sum_i \frac{3}{2} m_i \langle v_i^2 \rangle = \frac{3}{2} \sum_i m_i \sigma_v^2 = \frac{3}{2} M \sigma_v^2$$

Here, σ_v is the rms velocity dispersion.

$$2\langle K \rangle + \langle U \rangle = 0 \quad \rightarrow \quad \sigma_v^2 = \frac{1}{5}\frac{GM}{R}$$

- Therefore, we obtain the total mass of the self-gravitating cloud in terms of the line width (broadening parameter):

virial mass: $$M = \frac{5b^2R}{2G} \approx 600\, M_\odot \left(\frac{b}{1\,\mathrm{km\,s^{-1}}}\right)^2 \left(\frac{R}{1\,\mathrm{pc}}\right)$$

Here, $b = \sqrt{2}\sigma_v$

Formation of Stars

Star-Gas-Star Cycle

- Stars (and their planetary systems) are formed out of the ISM material through gravitational contraction, making for a kind of star-gas-star cycle.

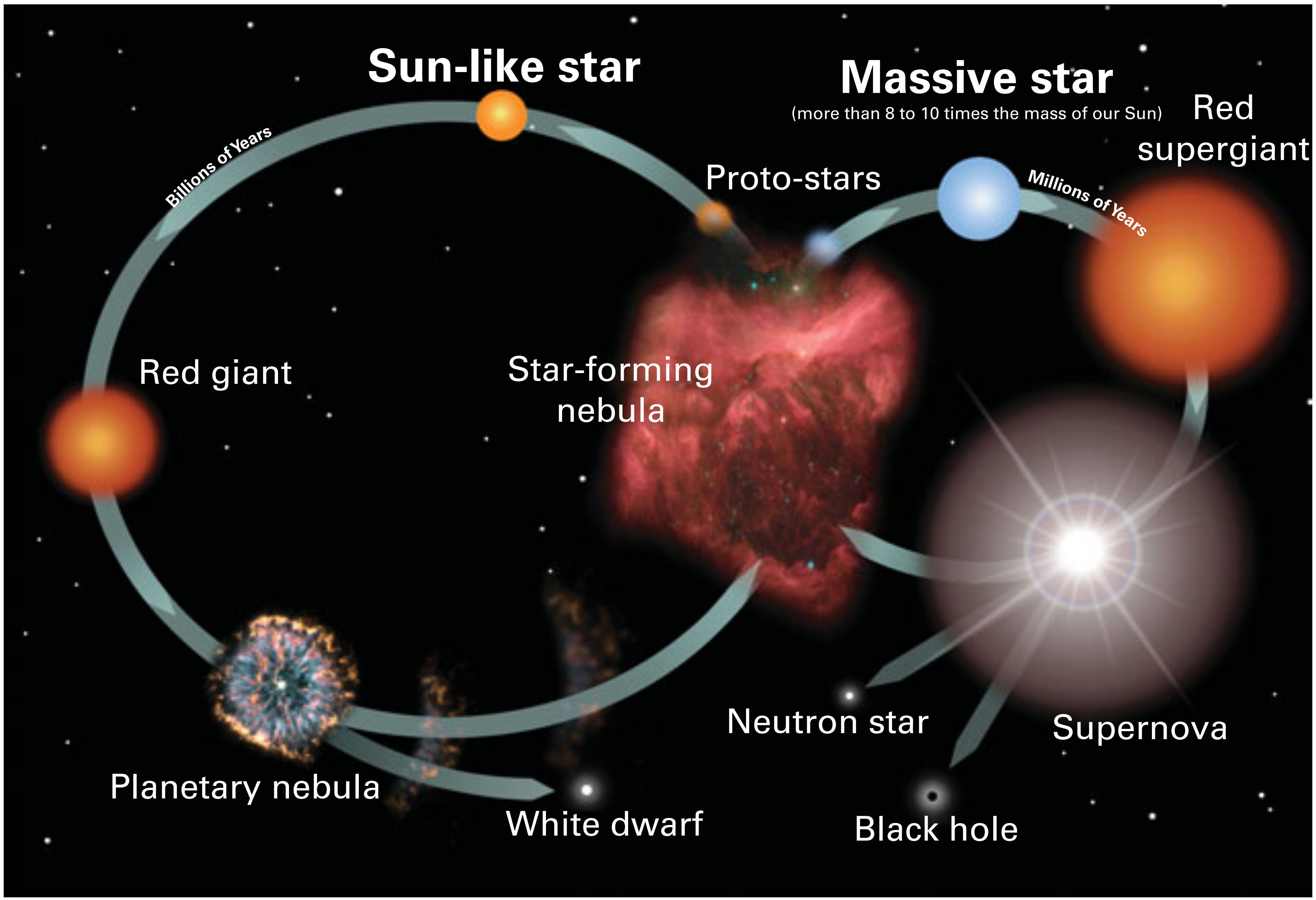


Illustration of formation of solar-type and massive stars from interstellar cloud [S. Owocki]

Star-Gas Cycle

- Assume that, on average, ***a typical atom spends roughly equal fractions of time in the star versus the ISM phase of this cycle***,
  - then the average density of gas in the ISM should be roughly equal to the mass of the stars spread out over the volume between them.
  - In the region of the Galaxy near the Sun (solar neighborhood), a typical separation between stars is

$$d \approx 2\,\mathrm{pc}$$

  - the mean number density of stars

$$n_* \approx 1/d^3 \approx 0.1\,\mathrm{pc}^{-3}$$

  - If we take the average mass of each star to be roughly that of the Sun, we obtain a mean mass density

$$\rho \approx M_\odot n_* \approx 7 \times 10^{-24}\,\mathrm{g/cm}^3$$

  - With a composition dominated by hydrogen, the associated ISM hydrogen-atom number density is

$$n \approx \rho/m_\mathrm{p} \approx 4\,\mathrm{cm}^{-3}$$

  - The characteristic ISM number $n \approx 1\,\mathrm{cm}^{-3}$ is comparable to this ‘‘very rough’’ estimate.

Jeans Criterion for Gravitational Contraction

- Stars generally form in clusters from the gravitational contraction of a dense, cold GMC.
  - The requirements for such gravitational contraction depend on the relative magnitudes of the total internal thermal (kinetic) energy K versus the gravitational binding energy U.
  - For a cloud of mass M, uniform temperature T, and mean mass per particle **μ**, the total number of particles N = M/**μ** have an associated total thermal energy,

$$K = \frac{3}{2} N k_{\mathrm{B}} T = \frac{3}{2} \frac{M k_{\mathrm{B}} T}{\mu}$$

  - For a spherical cloud with radius R and uniform density, the associated gravitational binding energy is

$$U = -\frac{3}{5} \frac{G M^2}{R}$$

  - Recall that $K = -U/2$ is the condition for stably bound systems in virial equilibrium.
  - Therefore, for a cloud with $K > -U/2$, the excess internal pressure would do work to expand the cloud against gravity, leading to it to be unbound.
  - Conversely, for $K < -U/2$, the too-low pressure would allow the cloud to gravitationally contract, leading to a more strongly bound cloud.

- The critical requirement, known as the ***Jeans criterion***, for gravitational contraction is

$$K < -U/2 \quad \Longrightarrow \quad \frac{3}{2}\frac{Mk_\mathrm{B}T}{\mu} < \frac{3}{10}\frac{GM^2}{R} \quad \Longrightarrow \quad \boxed{\frac{M}{R} > \frac{5k_\mathrm{B}T}{G\mu}}$$

- ***Jeans radius*** (in terms of the number density of atom $n = \rho/\mu$)

molecular hydrogen

$$M = \frac{4\pi}{3}R^3 n\mu \quad \Longrightarrow \quad \boxed{R > \left(\frac{15}{4\pi}\frac{k_\mathrm{B}T}{Gn\mu^2}\right)^{1/2} \equiv R_\mathrm{J}} \qquad R_\mathrm{J} \approx 15\,\mathrm{pc}\left(\frac{T/100\,\mathrm{K}}{n/10\,\mathrm{cm}^{-3}}\right)^{1/2}\left(\frac{2m_\mathrm{p}}{\mu}\right)$$

- ***Jeans mass***

$$M > \frac{4\pi}{3}R_\mathrm{J}^3 n\mu = \boxed{\frac{5}{\mu^2}\left(\frac{15}{4\pi n}\right)^{1/2}\left(\frac{k_\mathrm{B}T}{G}\right)^{3/2} \equiv M_\mathrm{J}} \qquad M_\mathrm{J} \approx 7300 M_\odot \frac{(T/100\,\mathrm{K})^{3/2}}{(n/10\,\mathrm{cm}^{-3})^{1/2}}\left(\frac{2m_\mathrm{p}}{\mu}\right)^2$$

- For typical ISM conditions, both the Jeans radius and mass are quite large, implying it can be actually quite difficult to initiate gravitational contraction.
- A general conclusion of such a large Jeans mass is that stars tend typically to be formed in large clusters, resulting from an initial contraction of a GMC, with mass of order 10^4 solar mass or more.
- ***Jeans fragmentation:*** The Jeans length and mass both scale inversely with the square root of the density. This suggests that ***a collapsing cloud may break into multiple smaller pieces*** as it becomes denser.

Free-fall time (Dynamical time scale)

- **Free-fall timescale**
  - In the absence of any support, the collapse can be described as a free-fall with acceleration determined by the gravitational force,

$$m\frac{d^2r}{dt^2} = -\frac{GMm}{r^2} \qquad (m = \text{test particle mass})$$

  - boundary condition: $v = \dfrac{dr}{dt} = 0$ at the outer radius $r = R$

$$\frac{d^2r}{dt^2} = \frac{dv}{dr}\frac{dr}{dt} = \frac{1}{2}\frac{dv^2}{dr} \longrightarrow \frac{dv^2}{dr} = -\frac{GM}{r^2} \implies v^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right) \implies \boxed{\frac{dr}{dt} = -\left[2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{1/2}}$$

the negative sign is chosen because the core is collapsing.

$$\int_0^{t_{\rm ff}} dt = -\int_R^0 \frac{dr}{\left[2GM\left(\frac{1}{R} - \frac{1}{r}\right)\right]^{1/2}} = \left(\frac{2R^3}{GM}\right)^{1/2}\frac{\pi}{4}$$

  - free-fall time (the times for the test particle to move from the outer radius to the center)

$$\boxed{t_{\rm ff} = \left(\frac{3\pi}{32G\rho}\right)^{1/2} = 3.6\left(\frac{100\,{\rm cm}^{-3}}{n}\right)^{1/2}\left(\frac{2m_{\rm p}}{\mu}\right)^{1/2}\ {\rm Myr}}$$

  - Note that we assumed no force against the gravity. But, there will be a significant source of internal pressure against the gravity while the material collapse.

- **Star Formation Efficiency**

  - In our Galaxy, the total mass in GMCs with density $n \gtrsim 100\,\mathrm{cm}^{-3}$ is $M_{\mathrm{GMC}} \approx 10^9 M_\odot$. Since this mass should collapse to stars over a free-fall time, it suggests an overall galactic star-formation rate is given by

$$\dot{M}_{\mathrm{SFR}} = \frac{M_{\mathrm{GMC}}}{t_{\mathrm{ff}}} \approx 280 M_\odot\,\mathrm{yr}^{-1}$$

  - But the observationally inferred star-formation rate is much smaller, only $\sim 1\,M_\odot\,\mathrm{yr}^{-1}$, implying an star-formation efficiency of only

$$\epsilon_{\mathrm{ff}} \lesssim 0.01$$

star formation efficiency =
the mass fraction of a cloud that ultimately turns into stars in a unit time interval

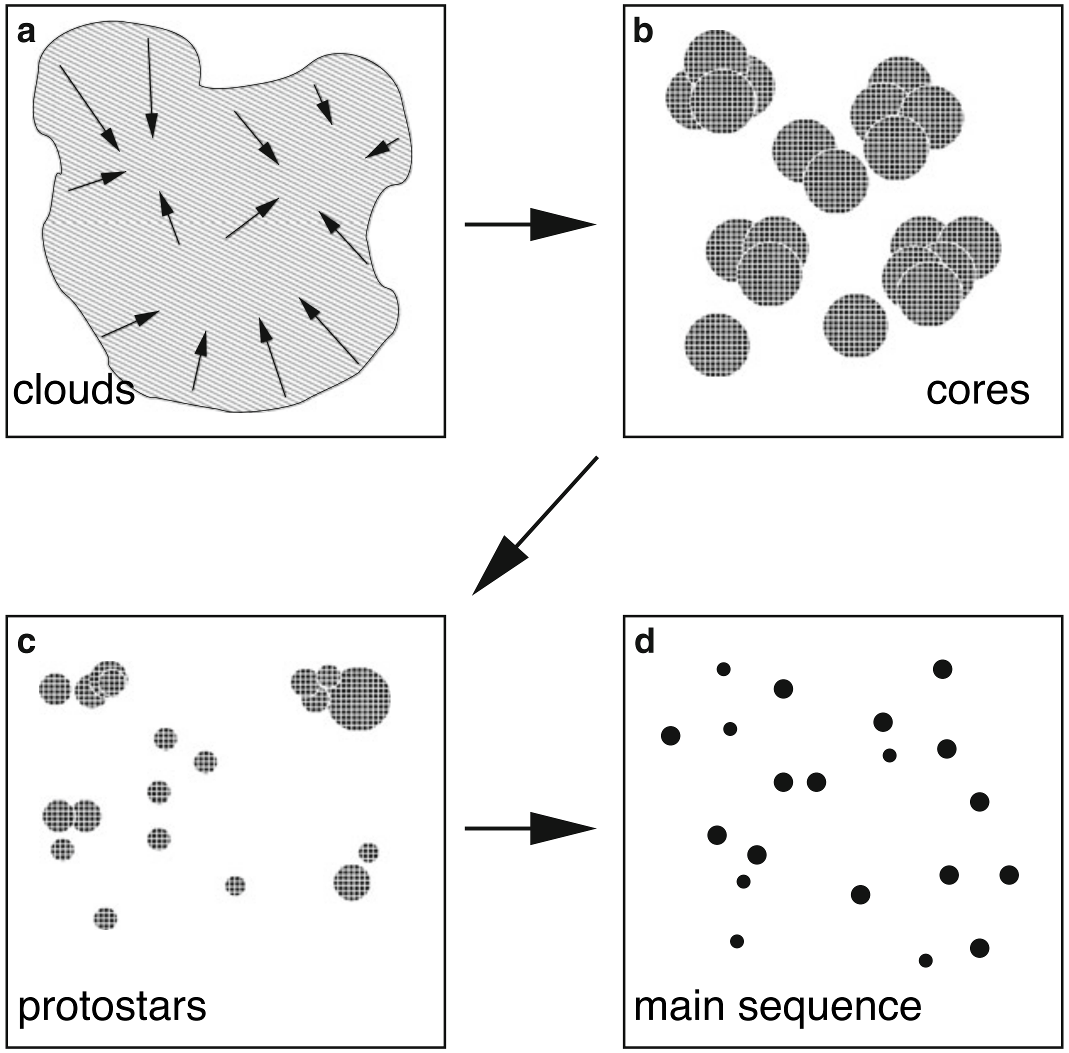
  - The reasons for this are not entirely clear, but may stem in part from inhibition of gravitational collapse by interstellar magnetic fields, and/or by interstellar turbulence.
  - Another likely factor is the feedback from hot, massive stars, which heat up and ionize the cloud out of which they form, thus preventing the further gravitational contraction of the cloud into more stars.
  - Modeling this feedback loop is a major challenge and a topic of current research.

Fragmentation into Cold Cores

- Fragmentation
  - In those portions of a GMC that do undergo gravitational collapse, the contraction soon leads to higher densities, and thus to smaller Jeans mass and Jeans radius, along with a shorter free-fall time.

$$M_{\rm J} \propto \rho^{-1/2}, \quad R_{\rm J} \propto \rho^{-1/2}, \quad \text{and} \quad t_{\rm ff} \propto \rho^{-1/2}$$

  - This tends to cause the overall cloud, with total mass, to fragment into much smaller, stellar-mass cloud “cores” that will form into individual stars.
  - The fragmentation process is a hierarchical process in which parent clouds break up into subclouds, which may themselves break into smaller structures.
    - ✦ 10 kpc - spiral arms of the Galaxy
    - ✦ 1 kpc - H I super clouds
    - ✦ 100 pc - giant molecular clouds
    - ✦ 10 pc - molecular clouds
    - ✦ 0.1 pc - molecular cloud cores
    - ✦ 100 AU - protostars
  - Stars are the final step of fragmentation.



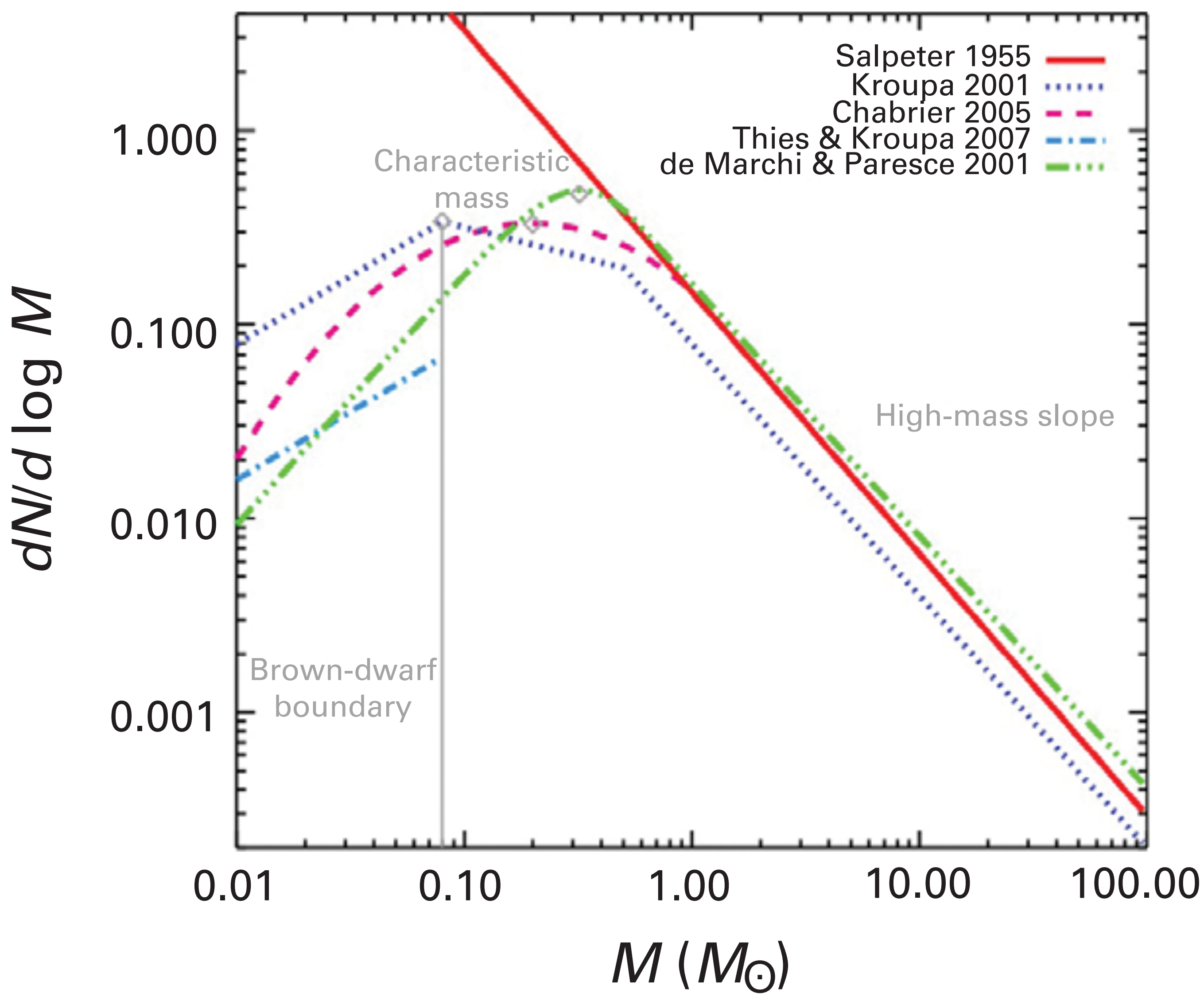
Initial Mass Function

- Initial Mass Function
  - A key, still-unsolved issue in star formation regards the physical processes and conditions that determine the mass distribution of these proto-stellar cores, leading then to what is known as the stellar initial mass function (IMF).
  - ***IMF = the number of stars within a mass range.***
  - Studies of the evolution of stellar clusters suggest that this can be roughly characterized by a power-law form:

$$\frac{dN}{dm} = Km^{-\alpha} \ (\alpha \approx 2.35 \text{ for } m > 1\,M_{\odot};\ \text{Salpeter IMF})$$

  - Here, K is a normalization factor that depends on the total number of stars. The large power-law index reflects the fact that higher-mass stars are much rarer than lower-mass stars.

- More modern models generally flatten the distribution at lower-mass, as illustrated in the following figure. This allows them to be normalized to a finite number when integrated over all masses.



Comparison of IMFs from various authors.

Except for the Salpeter pure-power-law form, the curves are normalized such that the integral over mass is unity.

[Offner et al. 2014]

- With a given form of the IMF for a collapsing GMC, one can model the evolution of the resulting stellar cluster, based on how each star with a given mass evolves through its various evolutionary phases, e.g., main sequence, red giant, etc.

The Bonnor-Ebert Sphere

- ***Resistance by thermal pressure***
  - The free-fall time is a useful guide to the scales involved in the collapse of a dense molecular core to form a star.
  - But, in practice, gravity is resisted by ***other forces. This can slow down collapse or allow stable cores to exist.***
- ***Bonnor-Ebert sphere*** = a spherical core in balance between self-gravity and thermal pressure
  - The balance can be described by the fluid equations for an ***isothermal cloud*** (T = constant), composed of molecular hydrogen, in spherical coordinates:

Poisson's equation $\nabla^2\phi = 4\pi G\rho$

hydrostatic equilibrium $\nabla P = -\rho\nabla\phi$

equation of state $P = nk_{\rm B}T$

Here, ϕ = gravitational potential
P = pressure
n = number density
ρ = mass density

$\longrightarrow$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi G\rho$$

$$\frac{dP}{dr} = -\rho\frac{d\phi}{dr}$$

$$P = \rho c_{\rm s}^2$$

$$c_{\rm s} = (k_{\rm B}T/m_{\rm H_2})^{1/2}, \quad \rho = nm_{\rm H_2}$$

$c_{\rm s}$ = sound speed

- ***singular isothermal sphere (SIS)***: a simple power law solution
  - The equations have a simple power law solution.

Assume:

$$\rho = \rho_0 r^\alpha \;\rightarrow\; P = \rho_0 c_{\rm s}^2 r^\alpha \;\longrightarrow\; \frac{d\phi}{dr} = -\frac{1}{\rho}\frac{dP}{dr} = -\alpha\frac{c_{\rm s}^2}{r}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi G\rho \;\rightarrow\; \frac{\alpha c_{\rm s}^2}{r^2} = -4\pi G\rho_0 r^\alpha$$

  - This implies $\alpha = -2$ and the solution is given by

$$\rho = \frac{c_{\rm s}^2}{2\pi G} r^{-\alpha}$$

  - This is called the singular isothermal sphere. It matched many of the early observations of dense molecular cores.
  - However, it is physically unrealistic in that the density and mass increase without limit on small and large scales, respectively.

$$\rho(r=0) \rightarrow \infty \quad \text{and} \quad \int_0^\infty \rho\,\left(4\pi r^2\right) dr \rightarrow \infty$$

  - Indeed, as technology allowed higher resolution observations at mm wavelengths, it became apparent that **core densities reached a plateau toward the center**.

- ***general solution*** (for the case of a finite central density)
  - For a finite central density $\rho_{\rm cen}$ at $r = 0$, we must numerically integrate the equations. These can be simplified by combining the 2nd (hydrodynamic equilibrium) and 3rd (equation of state) equations to

$$\frac{dP}{dr} = -\rho\frac{d\phi}{dr} \quad \rightarrow \quad c_{\rm s}^2 d\rho = -\rho d\phi$$

$\rho = \rho_{\rm cen}$ and $\phi = 0$ at $r = 0$

$$\rightarrow \quad \int \frac{1}{\rho} d\rho = -\frac{1}{c_{\rm s}^2}\int d\phi \quad \rightarrow \quad \boxed{\rho = \rho_{\rm cen} e^{-\phi/c_{\rm s}^2}}$$

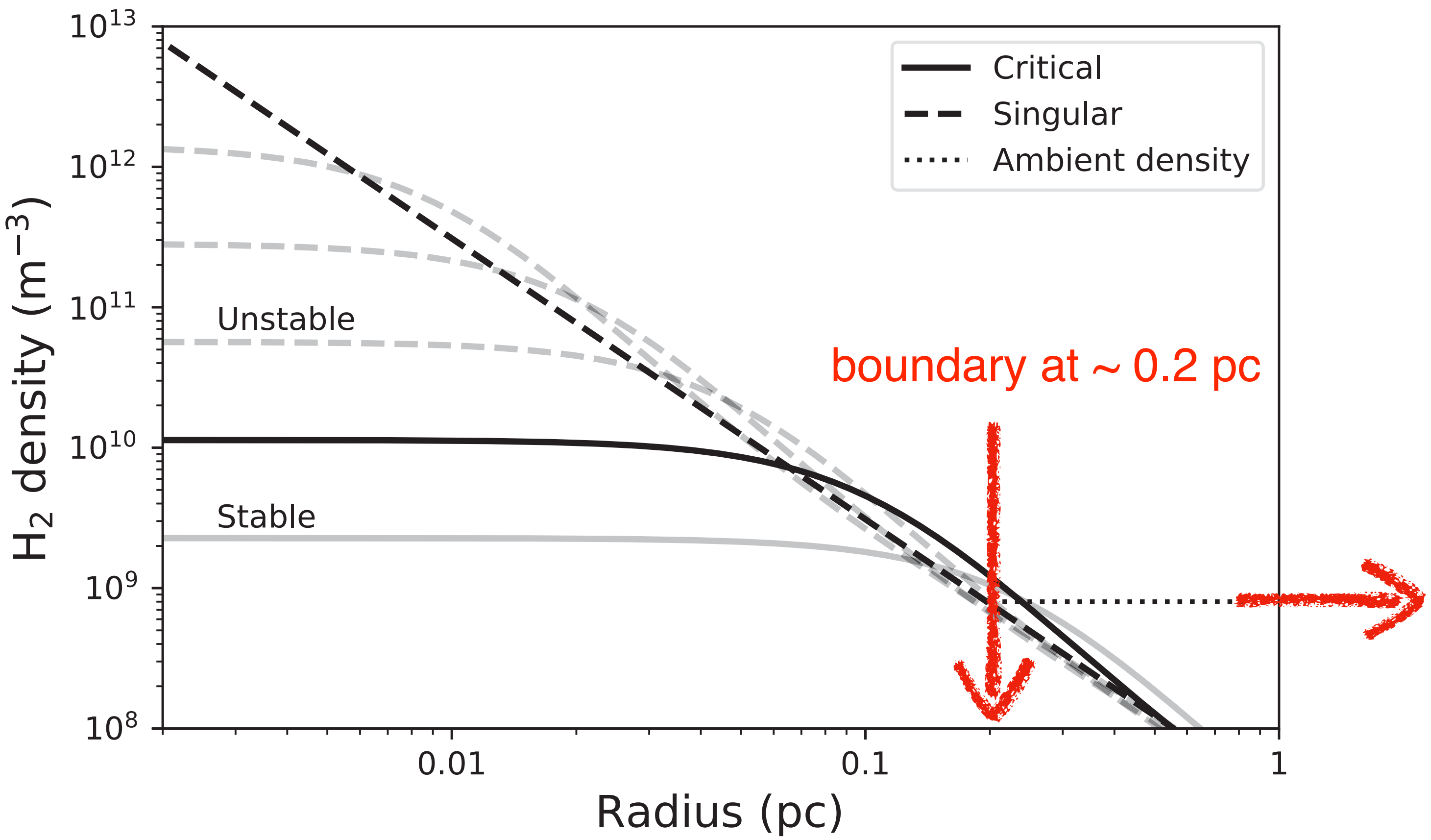
  - Then, the Poisson equation (1st eq) becomes the dimensionless equation:

$$\boxed{\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = e^{-y} \quad \text{where } x = \left(\frac{4\pi G\rho_{\rm cen}}{c_{\rm s}^2}\right)^{1/2} r, \; y = \frac{\phi}{c_{\rm s}^2}}$$

  - This is the Emden-Chandrasekhar equation and it can be numerically solved by integrating outwards in small steps, Δx, using Taylor expansions,

$$\begin{aligned} y''(x) &= e^{-y} - 2y'(x)/x \\ y(x+\Delta x) &= y(x) + y'(x)\Delta x + y''(x)\Delta x^2/2, \\ y'(x+\Delta x) &= y'(x) + y''(x)\Delta x \end{aligned}$$

    By symmetry, the boundary conditions are $\phi = d\phi/dr = 0$ at $r = 0$ ($y = y' = 0$ at $x = 0$). The limiting value of $y''(0) = 1/3$ can then be derived.



ambient pressure $P_{\rm amb} \simeq 10^{-13}\,{\rm Pa}$,
temperature $T = 10\,{\rm K}$
ambient density $n_{\rm amb} = P_{\rm amb}/k_{\rm B}T = 800\,{\rm cm}^{-3}$

- There is no singularity at the center but ***they still have arbitrarily large masses at large scales*** since they have the same behavior as the SIS, $\rho \propto 1/r^2$.
- The density does not go to zero, but the core will merge into the ambient ISM. The outer radius will be set by the ambient density. This sets a finite boundary at ~ 0.2 pc and therefore a finite mass.
- For the SIS, the outer boundary and the mass contained within this radius are

$$R_{\rm SIS} = \frac{c_{\rm s}}{(2\pi G\rho_{\rm amb})^{1/2}} \quad \text{and} \quad M_{\rm SIS} = \int_0^{R_{\rm SIS}} 4\pi r^2 \rho dr = \left(\frac{2}{\pi G^3 \rho_{\rm amb}}\right)^{1/2} c_{\rm s}^3$$

- In general, the total enclosed mass is given by the dimensionless variables,

$$M_{\rm enclosed} = \left(\frac{c_{\rm s}^3}{4\pi G^3 \rho_{\rm amb}}\right)^{1/2} \left[\left(\frac{\rho_{\rm cen}}{\rho_{\rm amb}}\right)^{-1/2} \int_0^{x_{\rm amb}} x^2 e^{-y} dx\right]$$

Here, $x_{\rm amb}$ is determined by the ambient density constraint,

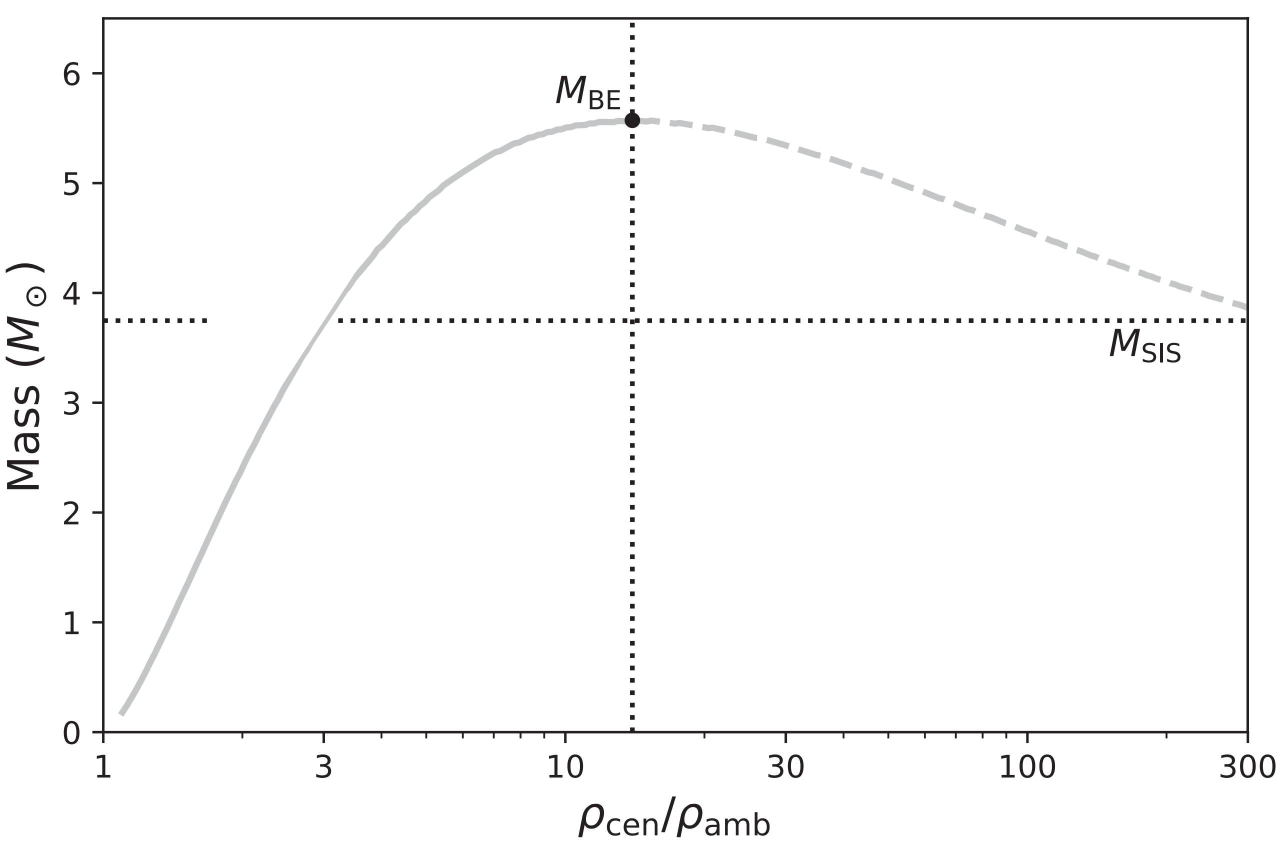
$$y(x_{\rm amb}) = c_{\rm s}^2 \ln\left(\rho_{\rm cen}/\rho_{\rm amb}\right)$$

- As the central density increase, the mass initially increases but eventually the core shrinks, the volume decreases, and the enclosed mass drops.
- The maximum value of the enclosed mass is known as the ***Bonnor-Ebert mass***, which is given by

$$M_{\rm BE} = 1.5 M_{\rm SIS} = 1.5 \left(\frac{2}{\pi G^3 \rho_{\rm amb}}\right)^{1/2} c_{\rm s}^3$$

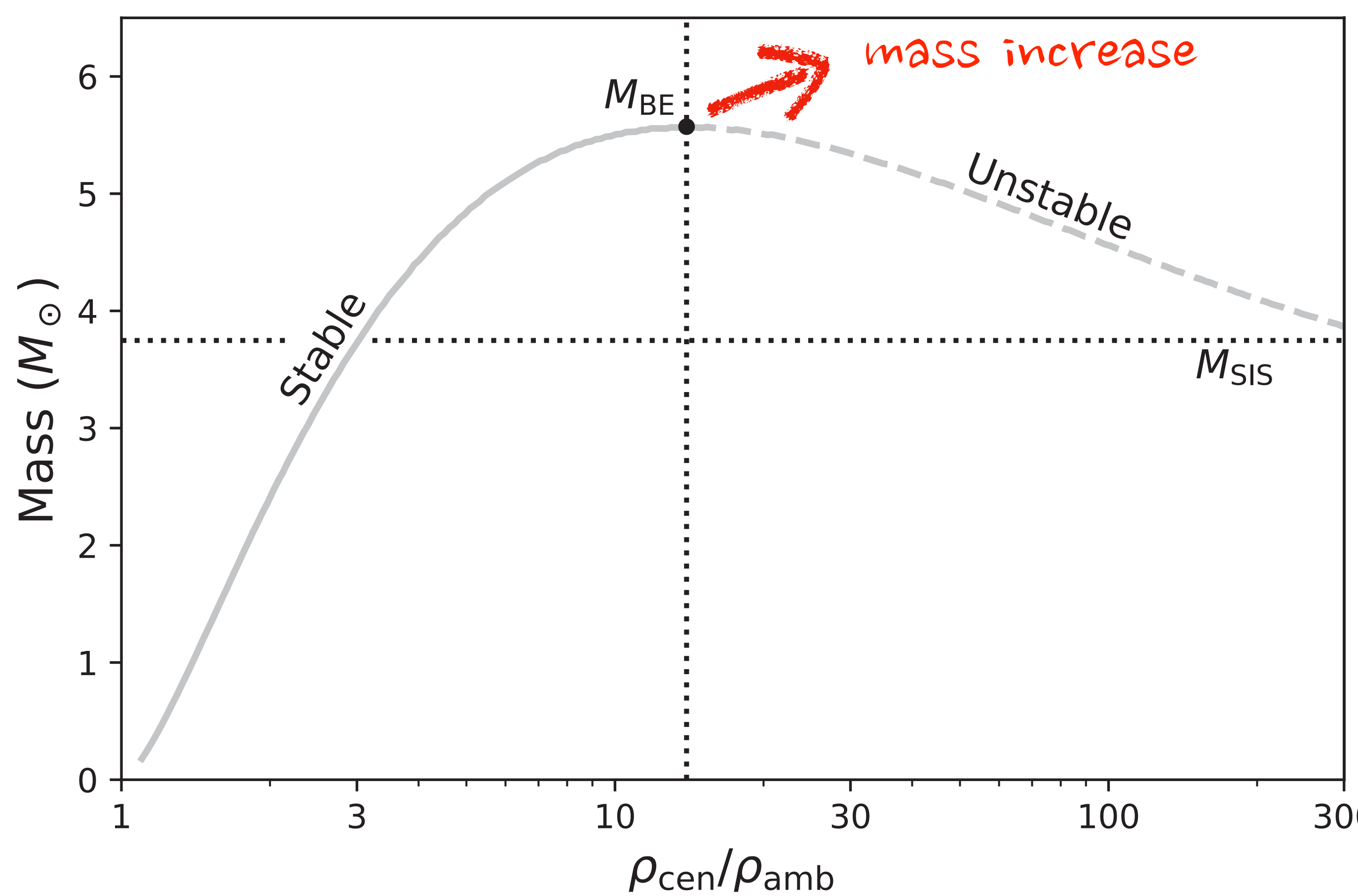
occurring at $\rho_{\rm cen}/\rho_{\rm amb} = 14$

Total enclosed mass vs. the ambient density



- ***Stability of the core***
  - The core will start from zero mass at $\rho_{\rm cen}/\rho_{\rm amb} = 1$.
  - As its mass increases, the central density will increase until it reaches the maximum at $\rho_{\rm cen}/\rho_{\rm amb} = 14$.
  - ***Beyond this point ($\rho_{\rm cen}/\rho_{\rm amb} > 14$), as it gains additional mass***, it cannot remain in equilibrium because it will be so dense that self-gravity overcomes thermal pressure. ***Eventually, the core collapse.***
  - This implies that the right side ($\rho_{\rm cen}/\rho_{\rm amb} > 14$) is unstable.
- ***Triggered star formation***
  - We can imagine a core with the maximum stable mass.
  - If external forces change the pressure and thus the ambient density, then the core will become unstable and collapse.
  - This is the triggered star formation, by which the impulsive force of an expanding H II region or supernova remnant can induce the collapse of marginally stable cores in neighboring molecular clouds.

$M_{\rm BE}$ = the maximum stable mass



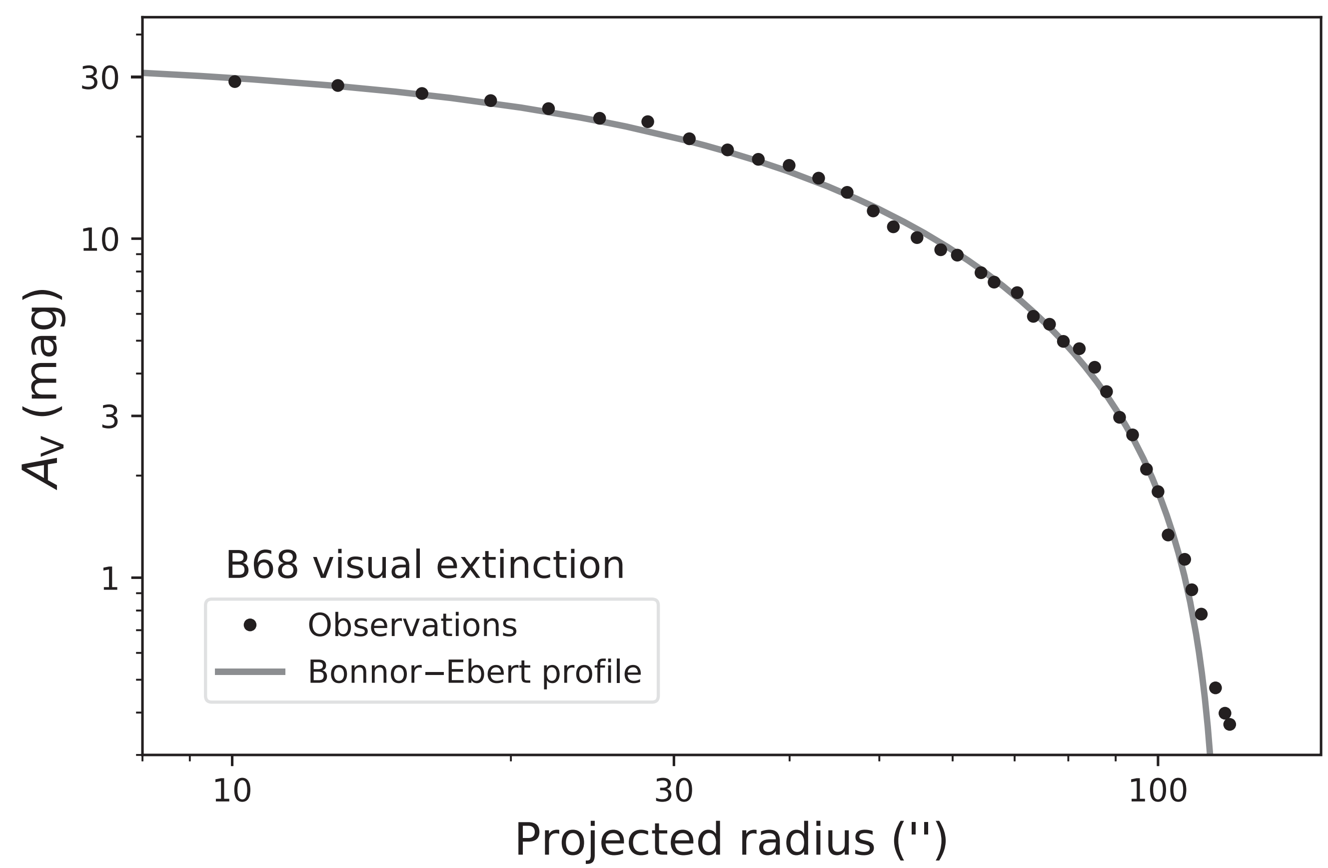
- Jeans mass vs. Bonnor-Ebert mass

$$M_{\rm J} = \frac{5}{\mu^2}\left(\frac{15}{4\pi n}\right)^{1/2}\left(\frac{k_{\rm B}T}{G}\right)^{3/2}$$
$$= 5\left(\frac{15}{4\pi}\right)^{1/2}\left(\frac{1}{G^3\rho}\right)^{1/2}\left(\frac{k_{\rm B}T}{m_{\rm H_2}}\right)^{3/2}$$

$$M_{\rm BE} = 1.5\left(\frac{2}{\pi G^3\rho_{\rm amb}}\right)^{1/2} c_{\rm s}^3$$
$$= 1.5\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{1}{G^3\rho_{\rm amb}}\right)^{1/2}\left(\frac{k_{\rm B}T}{m_{\rm H_2}}\right)^{3/2}$$

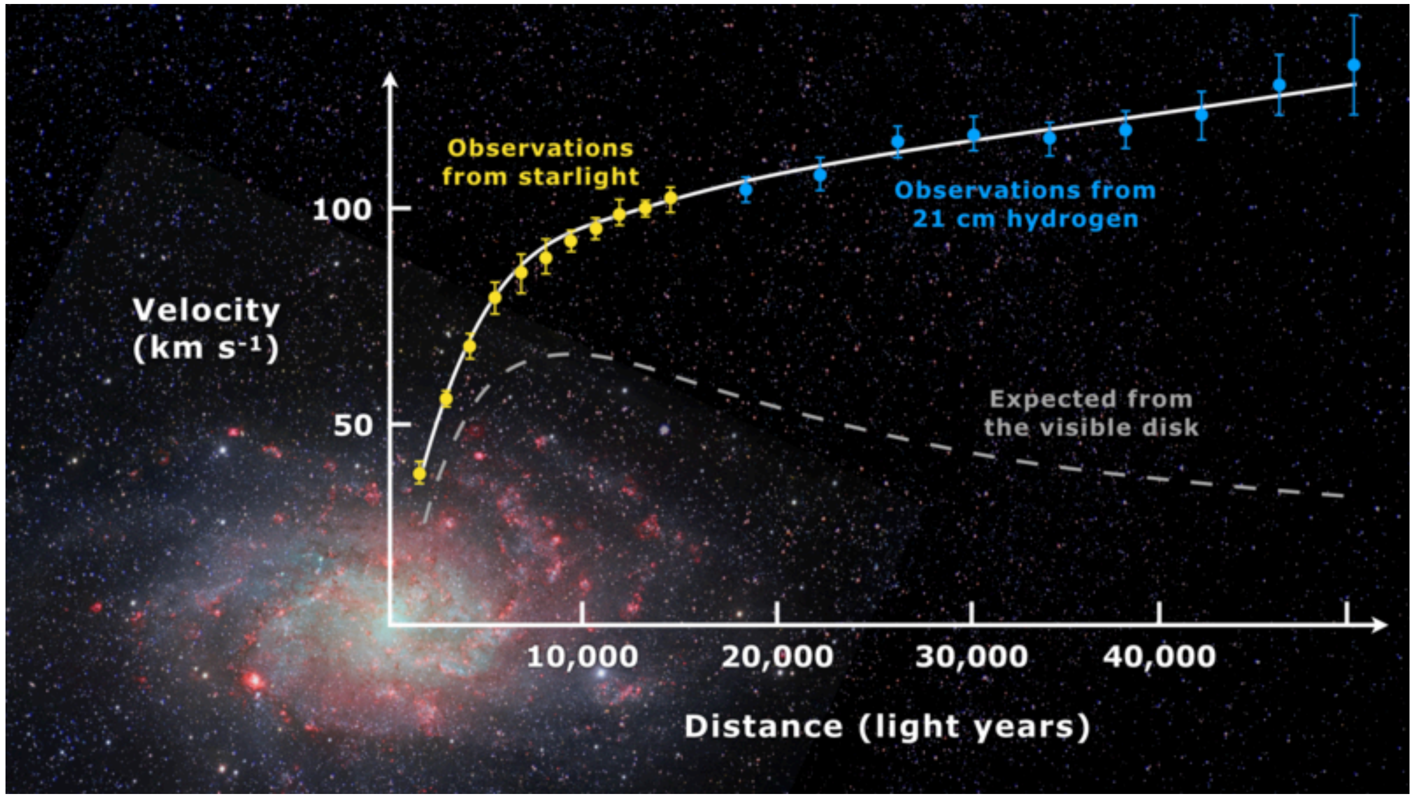
  - They have the same dependence on density and temperature, except for a numerical factor.

- Barnard 68 dark cloud
  - A_V was derived from star counts in the optical and Near-IR.
  - The inferred central density is close to the critical value of 14. Therefore, if it were to gain additional mass or externally perturbed, it would undergo collapse.

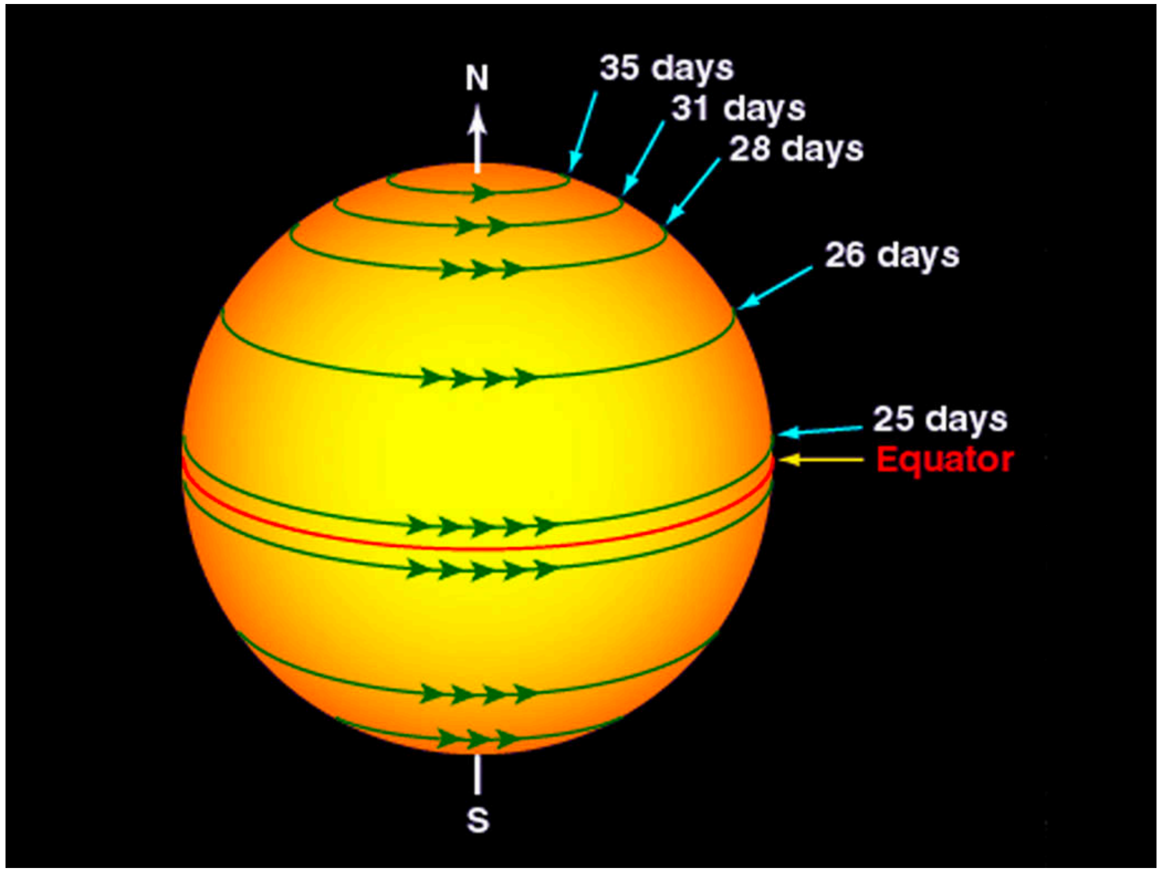


Angular Momentum: Rotation is ubiquitous

Rotation of spiral galaxy M33



Rotation of the Sun

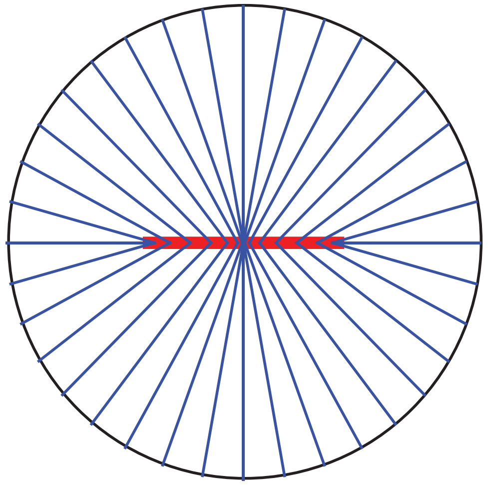


Core Collapse - Angular Momentum Conservation

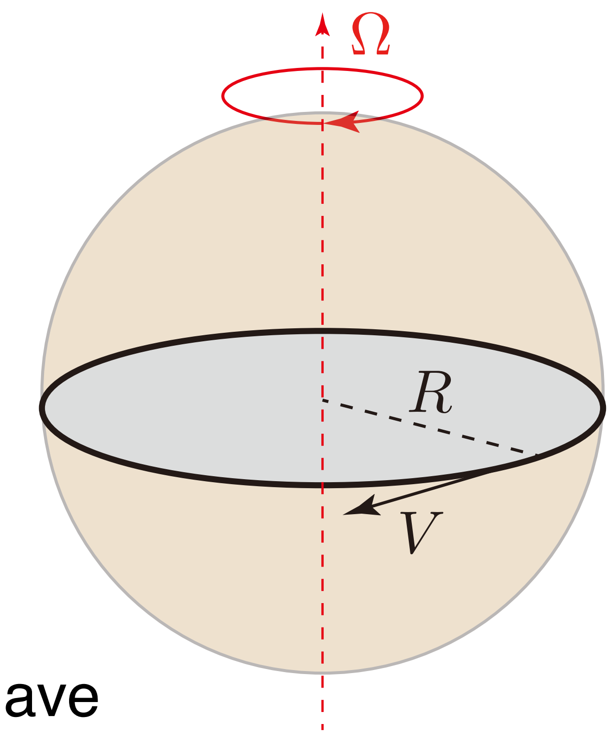
- The gravitational collapse of a cloud core to a star is accompanied by an enormous change in size scale, about six to seven orders of magnitude.

$$\frac{R_{\rm core}}{R_{\rm star}} \approx \frac{0.2\,{\rm pc}}{3 \times 6.9 \times 10^5\,{\rm km}} \approx 3 \times 10^6 \qquad \leftarrow \qquad R_{\rm star} \approx 3 R_{\rm sun}$$

- Rotation is a ubiquitous phenomenon from the largest to the smallest scales in our Galaxy; the Galaxy itself rotates as a whole, its individual stars spin too.
- In the same way as ice skaters spin faster as they pull in their arms, if all the angular momentum of the core were to be transferred to the star, it would rotate very rapidly.
- It is observed that molecular clouds do have a small amount of rotation, with velocities of order ***a few hundred meters per second*** (~ 0.15 km/s).
- ***The formation of planets around stars would not be possible if a certain amount of angular momentum was not present at the beginning of the star formation process.*** The angular momentum thus plays a significant role during the process of star and planet formation.



- Consider the gravitational collapse of a uniform spherical core with radius R_{core} and mass M to a star with radius R_{star}.



rotational speed (angular frequency): $\Omega = V/R$

moment of inertia: $I = (2/5)MR^2$

angular momentum: $J = I\Omega$

rotation period: $P = 2\pi/\Omega = 2\pi R/V$

- From the conservation of mass and angular momentum, we have

$$I_{\rm core}\Omega_{\rm core} = I_{\rm star}\Omega_{\rm star} \implies MR^2_{\rm core}\Omega_{\rm core} = MR^2_{\rm star}\Omega_{\rm star} \longrightarrow$$

$$\therefore \quad \Omega_{\rm star} = \left(\frac{R_{\rm core}}{R_{\rm star}}\right)^2 \Omega_{\rm core}$$

$$P_{\rm star} = \left(\frac{R_{\rm star}}{R_{\rm core}}\right)^2 P_{\rm core}$$

For $R_{\rm core} = 0.2\,{\rm pc}$,
$R_{\rm star} = 3R_\odot = 3\times(7\times10^8)\,{\rm m}$,
$V_{\rm core} = 150\,{\rm m\,s^{-1}}$

We obtain $P_{\rm star} \simeq 10^{-13} P_{\rm core}$, $P_{\rm core} \simeq 0.9\,{\rm Myr}$, $P_{\rm star} \simeq 0.5\,{\rm mins}$

$P_{\rm sun} \simeq 27$ days for reference

- ***Stars cannot spin that fast because they would tear apart.*** Instead, the core may fragment as its density increases and form more than one star. The bulk of the core rotation then goes into orbital motion of multiple stars (and/or planets) rather than the rotation of a single star.

Core Collapse - Disk Formation

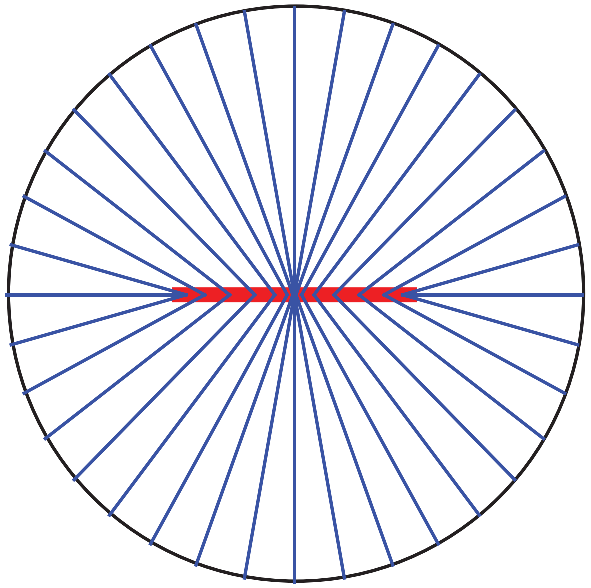
- Ultimately, a core that collapses to a single object will contract until gravity is balanced by centrifugal force. This only acts perpendicular to the rotational axis so the spherical core flattens to a disk.
- If we consider a test particle with mass m at radius R rotating at speed V, the ratio of the two forces is

$$\frac{F_{\rm grav}}{F_{\rm cen}} = \frac{GMm/R^2}{mv^2/R} = \frac{GM}{R^3\Omega^2}$$

This condition is called "Keplerian motion."

- At the ***outer boundary of the disk***, the ratio should be equal to 1.

$$R_{\rm disk}^3\Omega_{\rm disk}^2 = GM$$



- Assuming the conservation of angular momentum,

$$R_{\rm disk}^2\Omega_{\rm disk} = R_{\rm core}^2\Omega_{\rm core} \quad \Longrightarrow \quad R_{\rm disk}^4\Omega_{\rm disk}^2 = R_{\rm core}^4\Omega_{\rm core}^2$$

- Therefore, we have

$$R_{\rm disk} = \frac{R_{\rm core}^4\Omega_{\rm core}^2}{GM} = 2\beta_{\rm eq}R_{\rm core} \quad \Longleftarrow \quad \beta_{\rm eq} \equiv \frac{\Omega_{\rm core}^2 R_{\rm core}^2/2}{GM/R_{\rm core}}$$

Core Collapse - Disk Formation

- Here, β is ***the ratio of rotational to gravitational energy*** at the equator of the core.

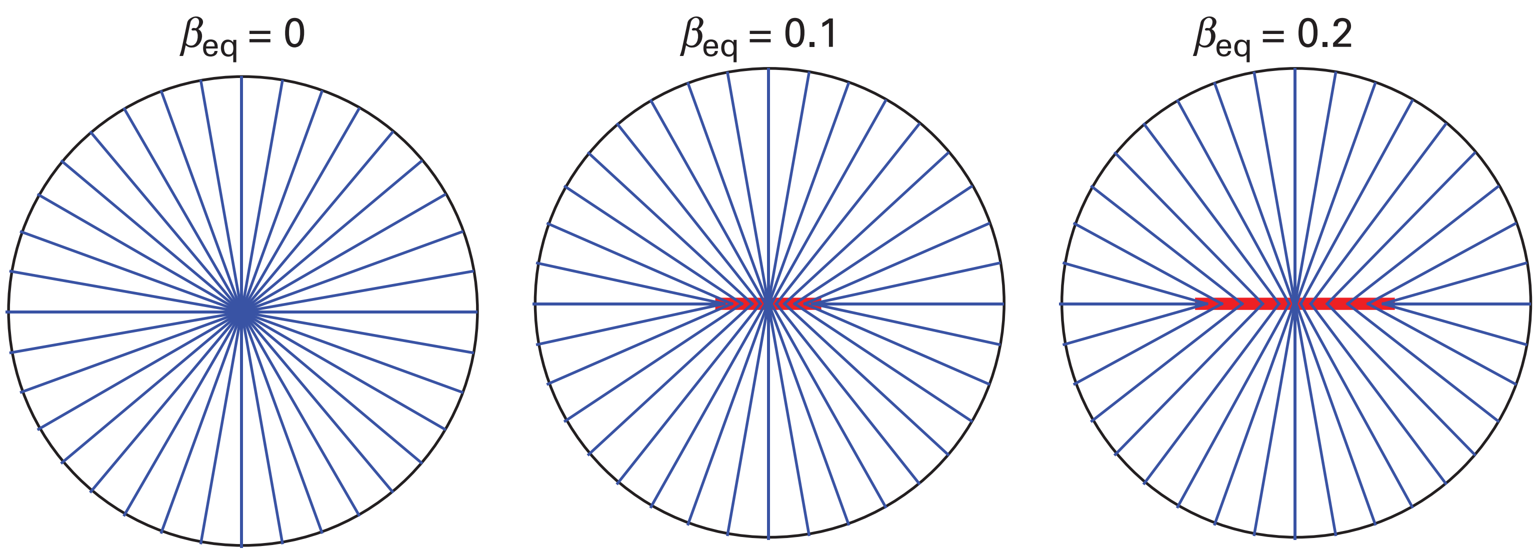
$$\beta_{\rm eq} \equiv \frac{\Omega_{\rm core}^2 R_{\rm core}^2/2}{GM/R_{\rm core}}$$

- Observations of cloud cores gives a typical value of $\beta_{\rm eq} \approx 0.02$ (Goodman et al. 1993).
- For a cloud core size $R \approx 0.05\,{\rm pc}$, the expected disk radius is a few hundred AU, comparable to the inferred sized of proto-stellar disks.

$$R_{\rm disk} \approx 410\,{\rm AU}\left(\frac{\beta_{\rm eq}}{0.02}\right)\left(\frac{R_{\rm core}}{0.05\,{\rm pc}}\right)$$

$R_{\rm heliosphere} \approx 100\,{\rm AU}$ for reference

- The following figure illustrates, for various values of $\beta_{\rm eq}$, how material from various latitudes in the original clouds track toward the center star or disk under gravitational collapse.

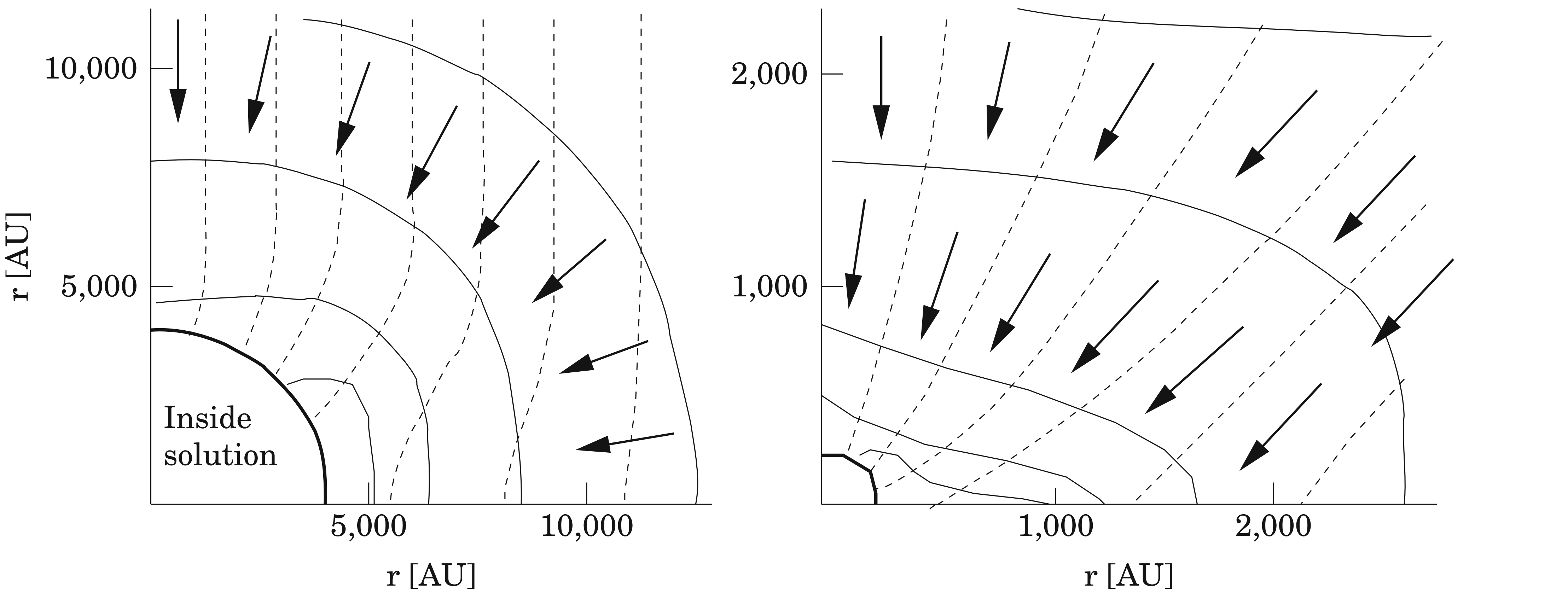


black circle: rigidly rotating spherical cloud

red: orbiting disk

- In disks with Keplerian orbits, the orbital frequency increase inward with radius $\Omega \sim r^{-3/2}$, meaning that between two neighboring rings there is an overall shear in orbital speed.
  - ✦ Any frictional interaction - due to viscosity - between such neighboring rings will tend to transport angular momentum from the faster inner ring to the slower outer ring, allowing the inner mass to fall further inward, while the angular momentum receiving material moves further outward.
  - ✦ This outward viscous diffusion of angular momentum allows, over time, for most of the mass to accrete onto the star, with just a small mass fraction retaining the original angular momentum.

- Eventually, this remnant disk-mass can fragment into its own gravitationally collapsing cores to form planets.
  - ✦ In our solar system, the most massive planet Jupiter has only 0.1% the mass of the Sun, but 99% of the solar system's angular momentum.
  - ✦ Earth too originated from the evolving proto-solar disk.



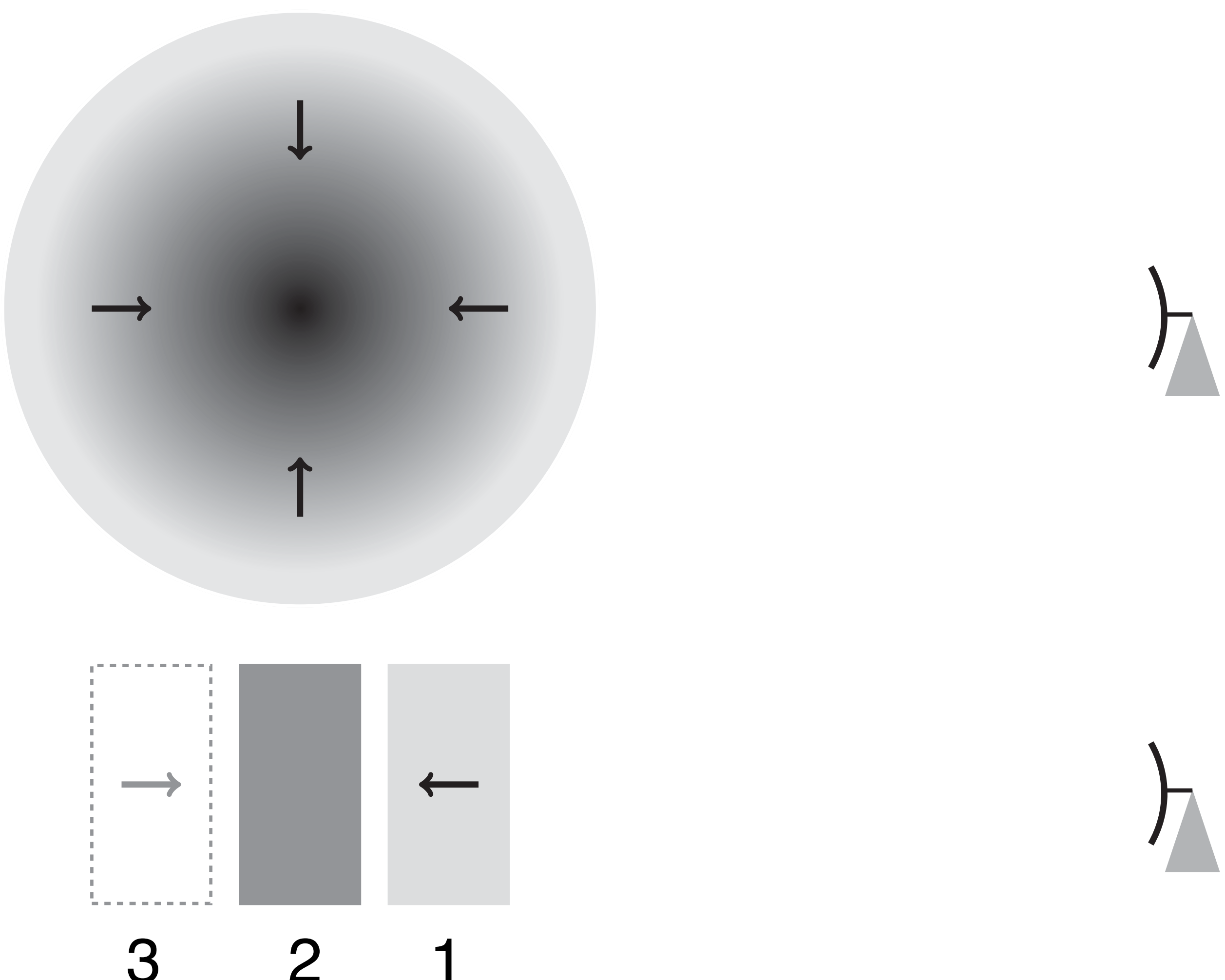
- Illustration of a collapse of a magnetic spherical, isothermal sphere fixed at about 2×10^5 yr after the start of the collapse and at the end of the free-fall phase.
- The solid lines are contours of equal density, the dashed lines are magnetic field lines, the thick arrows show the velocity field.
- Adapted from Galli & Shu

Observations of Core Collapse: Molecular Emission

- Cooling by Molecular Emission
  - At low temperature $T < 100$ K, cooling by molecular radiation is dominated by CO (carbon monoxide).
  - Such CO molecular cooling is a key factor in initiating and maintaining cloud contraction, by allowing the cloud to shed the increased internal energy gained from the tighter gravitational binding.
  - In virial equilibrium only half this energy is lost, and so the interior would still heat up in proportion to the stronger gravitational binding.
  - But, in practice, CO emission is often so efficient that the could interior can stay cool, or even become cooler, as it contracts.
  - The resulting dramatic reduction in interior pressure support then leads to a full gravitational collapse.

Observations of Core Collapse: Spectral Signature of Core Collapse

- The gravitational collapse of a cloud core, though fast by astronomical standards, can be observed ***through Doppler shifts of molecular spectral lines***.
- A collapsing core core will have a greater velocity dispersion than a static core. However, because we only measure projected motions along the line of sight, we cannot distinguish collapse from expansion via the dispersion alone.
- The trick is to consider observations of an ***optically thick line*** such that we effectively only see the motion of the front side of the core relative to its center.

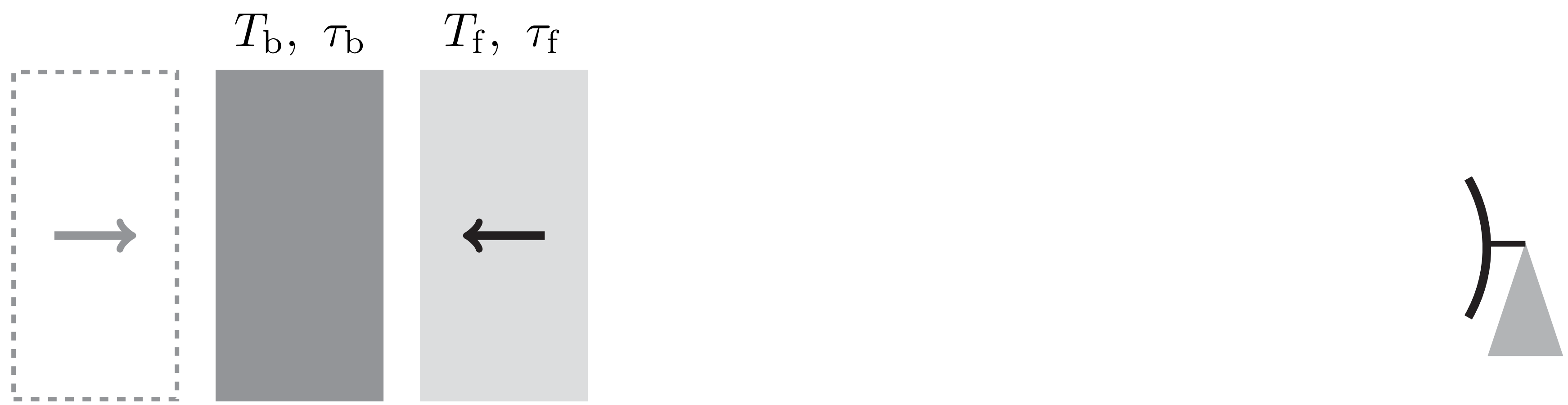


[J. P. Williams]

A collapsing core with a density gradient observed by a radio telescope can be approximated by a slab model for an optically thick line.

The observed emission comes predominantly from the central region (2) and the front layer (1) closest to the observer.

The third layer (3) is ignored as its emission is mostly absorbed by the central region.



- The observed intensity is then the superposition of (1) the front layer and (2) the back, attenuated by the front.

$$I_\nu = B_\nu(T_{\mathrm{f}})\left(1 - e^{-\tau_{\mathrm{f}}}\right) + B_\nu(T_{\mathrm{b}})\left(1 - e^{-\tau_{\mathrm{b}}}\right) e^{-\tau_{\mathrm{f}}}$$

✦ Here, the front layer absorb some of the emission from the back, a phenomenon known as ***self-absorption***.

✦ Now, in velocity space relative to the core center (back layer), we assume that the optical depth in each layer is distributed as a Gaussian with uniform velocity dispersion σ and peak values τ_{f0} and τ_{b0} such that

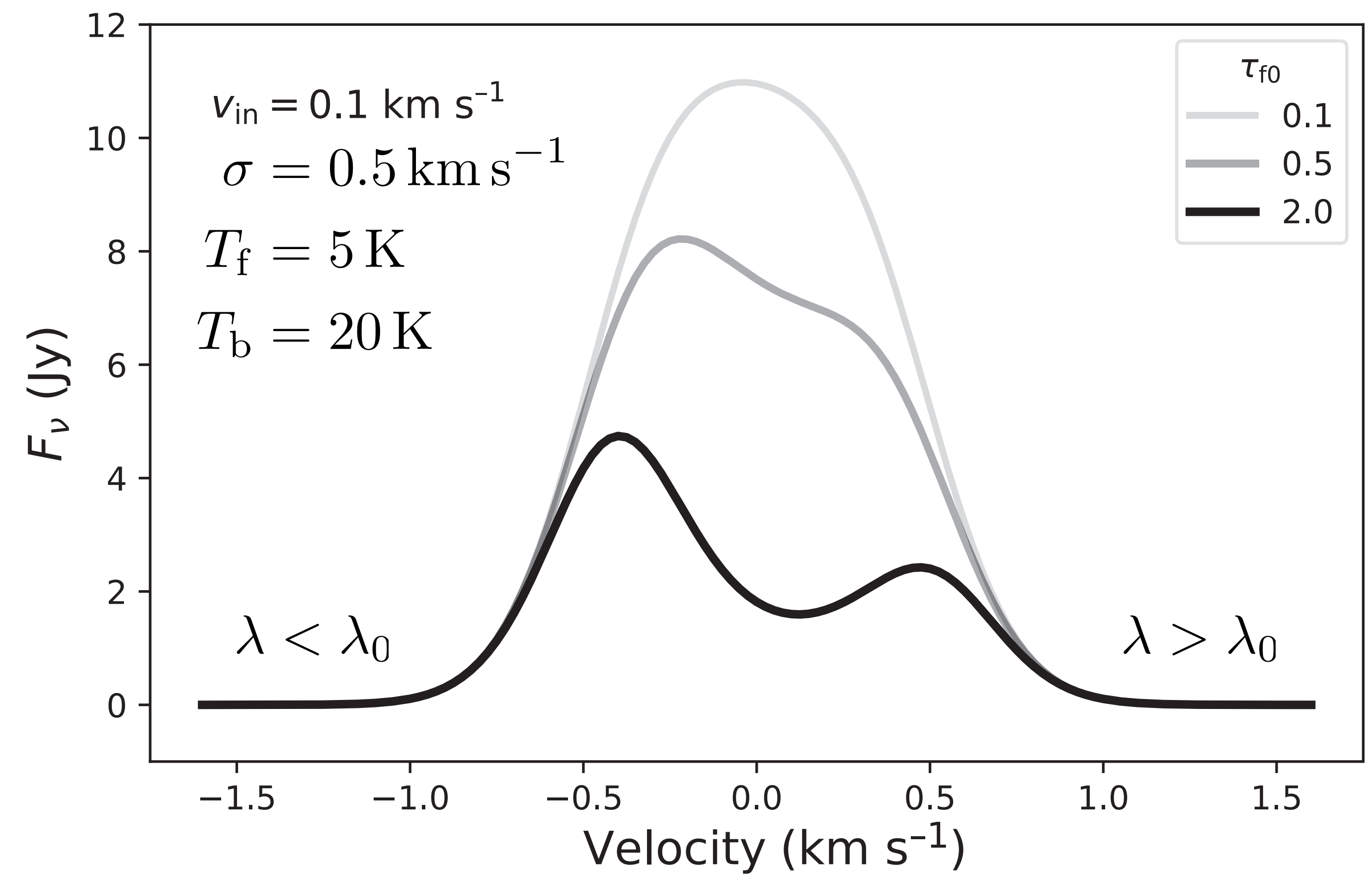
$$\tau_{\mathrm{f}} = \tau_{\mathrm{f0}} e^{-(v - v_{\mathrm{in}})^2/2\sigma^2} \qquad \text{where } \frac{v}{c} = \frac{\lambda - \lambda_0}{\lambda_0}$$
$$\tau_{\mathrm{b}} = \tau_{\mathrm{b0}} e^{-v^2/2\sigma^2}$$

✦ Here, v_{in} (>0) is the infall speed of the collaping core.

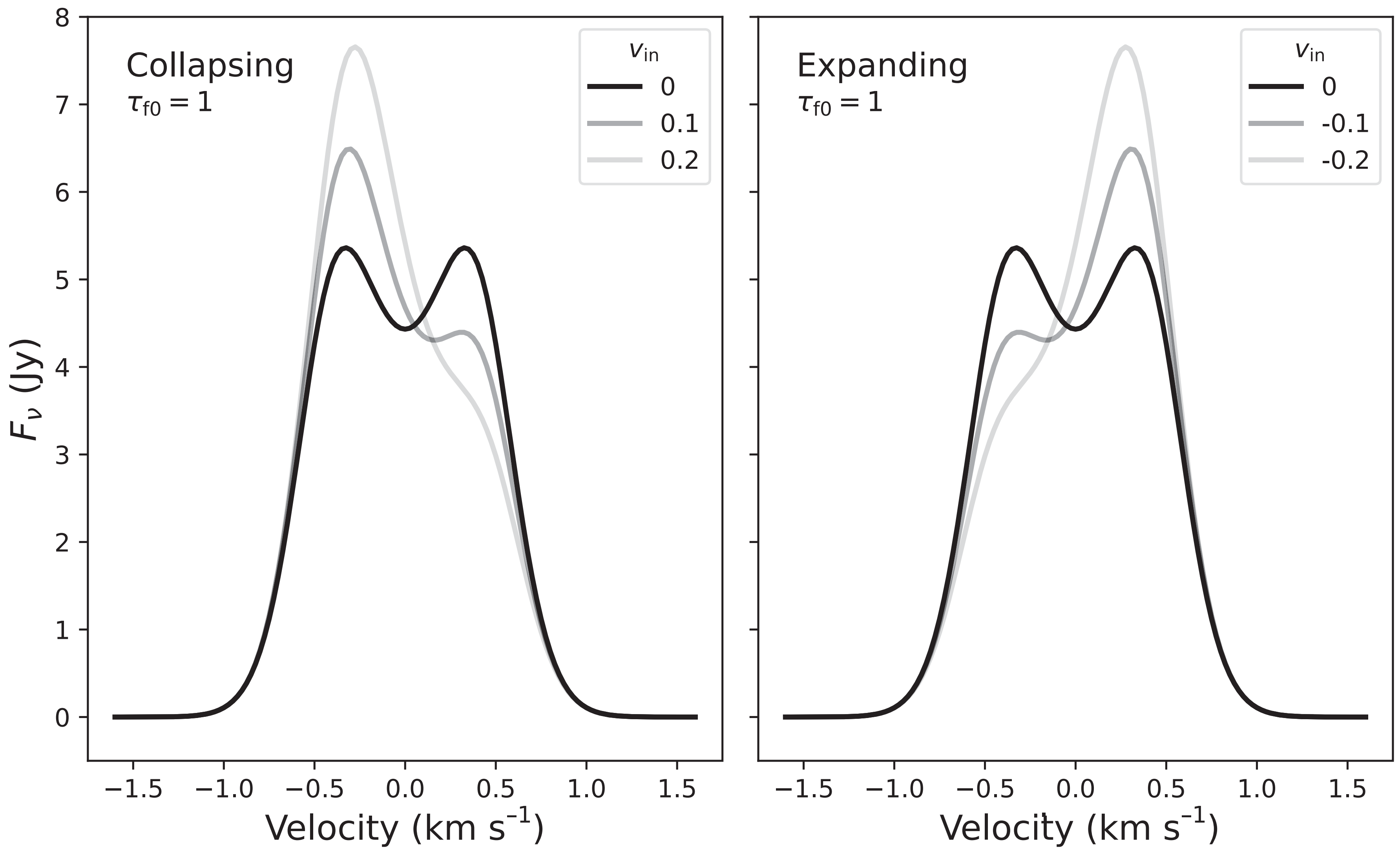
- The model has six parameters: two temperatures, a velocity dispersion, two optical depths, and an infall velocity.
  - ✦ The two temperatures determine the flux scale, and the dispersion sets the velocity scale.
  - ✦ The general spectral shape is primarily determined by the front layer optical depth and ratio of infall speed to dispersion.

- ***Variation of optical depth:*** The spectral profile changes from near-Gaussian for $\tau_{f0} \ll 1$ to double-peaked for $\tau_{f0} > 1$ as the front layer absorbs an increasing amount of the emission from the back layer.

- The low-velocity peak is greater than the high-velocity peak because ***the front layer is moving away from the observer, and therefore predominantly absorbs the redshifted side*** of the emission from the back layer.



- ***Variation of infall velocity*** (and expansion velocity): The sensitivity of the line profile to $v_{\rm in}$ shows that we can distinguish relative motions in a core and determine how fast they are.



- ***P-Cygni profile:*** Blue-shifted absorption, corresponding to expansion, is often seen in the spectra of massive stars due to a wind blowing away from the photosphere.
- Thus the signature of a collapsing core is referred to as an ***inverse P-Cygni profile***.

Young Stellar Objects (YSOs)

- Young Stellar Object denotes a star in its early stage of evolution.
  - This class consists of two groups of objects: protostars and pre-main-sequence stars.
  - A ***protostar*** is a very young star that is still gathering mass from its parent molecular cloud.
    - A dense core is initially in balance between self-gravity and gas/magnetic pressure. As the dense core acquares mass from its larger, surrounding cloud, self-gravity begins to overwhelm pressure, and collapse begins. The protostellar phase begins when the molecular cloud fragment first collapses under the force of self-gravity and an opaque, pressure supported core forms inside the collapsing fragment.
    - The gas that collapses toward the center of the dense core first builds up a low-mass protostar, and then a protoplanetary disk orbiting the object. As the collapse continues, an increasing amount of gas impacts the disk rather than the star, a consequence of angular momentum conservation.
  - The protostellar phase ends when the infalling gas is depleted, leaving a ***pre-main-sequence star***.
    - After the protostar blows away the envelope, it is optically visible, and appears on the Hayashi track in the Hertzsprung-Russell diagram. A pre-main-sequence star contracts to later become a main-sequence star at the onset of hydrogen fusion producing helium.
    - The energy source of PMS objects is gravitational contraction, as opposed to hydrogen burning in main-sequence stars.

- ***Protostar***
  - **Maximum infall rate:** Combining the free-fall timescale and maximum stable core mass, we can derive an upper limit to the mass infall rate.

$$\dot{M}_{\rm in} = \frac{M_{\rm BE}}{t_{\rm ff}} \approx 2.2 \frac{c_{\rm s}^3}{G} \approx 4 \times 10^{-6} M_\odot/{\rm yr} \longleftarrow \quad M_{\rm BE} = 1.5 \left(\frac{2}{\pi G^3 \rho}\right)^{1/2} c_{\rm s}^3, \quad t_{\rm ff} = \left(\frac{3\pi}{32 G\rho}\right)^{1/2}$$

  - **Accretion luminosity:** This is high enough that the release of gravitational energy, as gas falls from core to stellar scales, provides significant accretion luminosity.

$$L_{\rm acc} = -GM\dot{M}_{\rm in}\left(\frac{1}{R_{\rm core}} - \frac{1}{R_*}\right)$$
$$\approx 9.3 \left(\frac{M}{1\,M_\odot}\right)\left(\frac{\dot{M}_{\rm in}}{10^{-6}\,M_\odot\,{\rm yr}^{-1}}\right)\left(\frac{R_*}{3R_\odot}\right)^{-1} L_\odot \longleftarrow$$

core radius $R_{\rm core}$ >> stellar radius R_*
typical size of solar-mass stars at early times $R_* \approx 3R_\odot$

  - This calculation shows that a protostar is detectable well before it begins nuclear fusion.
  - Due to the high dust extinction of the surrounding core, the very earliest phase of star formation is only visible at $\lambda > 100\,\mu{\rm m}$.

- But, within $\sim 10^5$ yr after collapse, protostars become detectable in the near-IR and mid-IR.
- As the core is used up, both the dust emission an extinction decrease. The protostar becomes increasingly visible at shorter wavelengths and the longer wavelength emission decreases.
- The evolutionary state of a protostar is generally classified based on its IR SED, but mm wavelength imaging of its surrounding envelope and disk is increasingly used in addition (e.g., ALMA).

• *Pre-main sequence star*

- Once the collapse phase ends and the protostar had reached its final mass, it is known as a pre-main sequence star.
- It slowly contracts through the loss of gravitational energy by radiation on a ***Kelvin-Helmholtz timescale (thermal timescale)***, which determines how quickly a star contracts before nuclear fusion starts - i.e. sets roughly the pre-main sequence lifetime

$$t_{\rm KH} = \frac{U}{L_*} = \frac{GM_*^2/R_*}{L_*} \simeq 31.1 \left(\frac{M_*}{M_\odot}\right)^2 \left(\frac{R_*}{R_\odot}\right)^{-1} \left(\frac{L_*}{L_\odot}\right)^{-1} \rm Myr$$

Here, U = internal thermal energy from the Virial theorem

- Pre-main sequence stars with $0.1\,M_\odot < M_* < 2\,M_\odot$ is known as T Tauri stars. These stars were first knwon in the 1940s and this name is remains a common alternative name for pre-main sequence stars for historical reasons
- The intermediate mass counterpart, $2\,M_\odot < M_* < 8\,M_\odot$, are known as Herbig Ae/Be stars, where A and B refer to the stellar spectral types. The “e” denotes the photospheric emission lines that were used to identify their youth.
- More massive stars are much more luminous and they contract too quickly. Hence, they have no pre-main-sequence stage. By the time they become visible, the hydrogen in their centers is already fusing and they are main-sequence objects.

- The first stages of their evolution on the Hertzsprung-Russell diagram is a near vertical line known as the Hayashi track. Ultimately their central temperatures increase to $\sim 10^7\,\mathrm{K}$ where hydrogen fusion begins and they become main sequence stars.

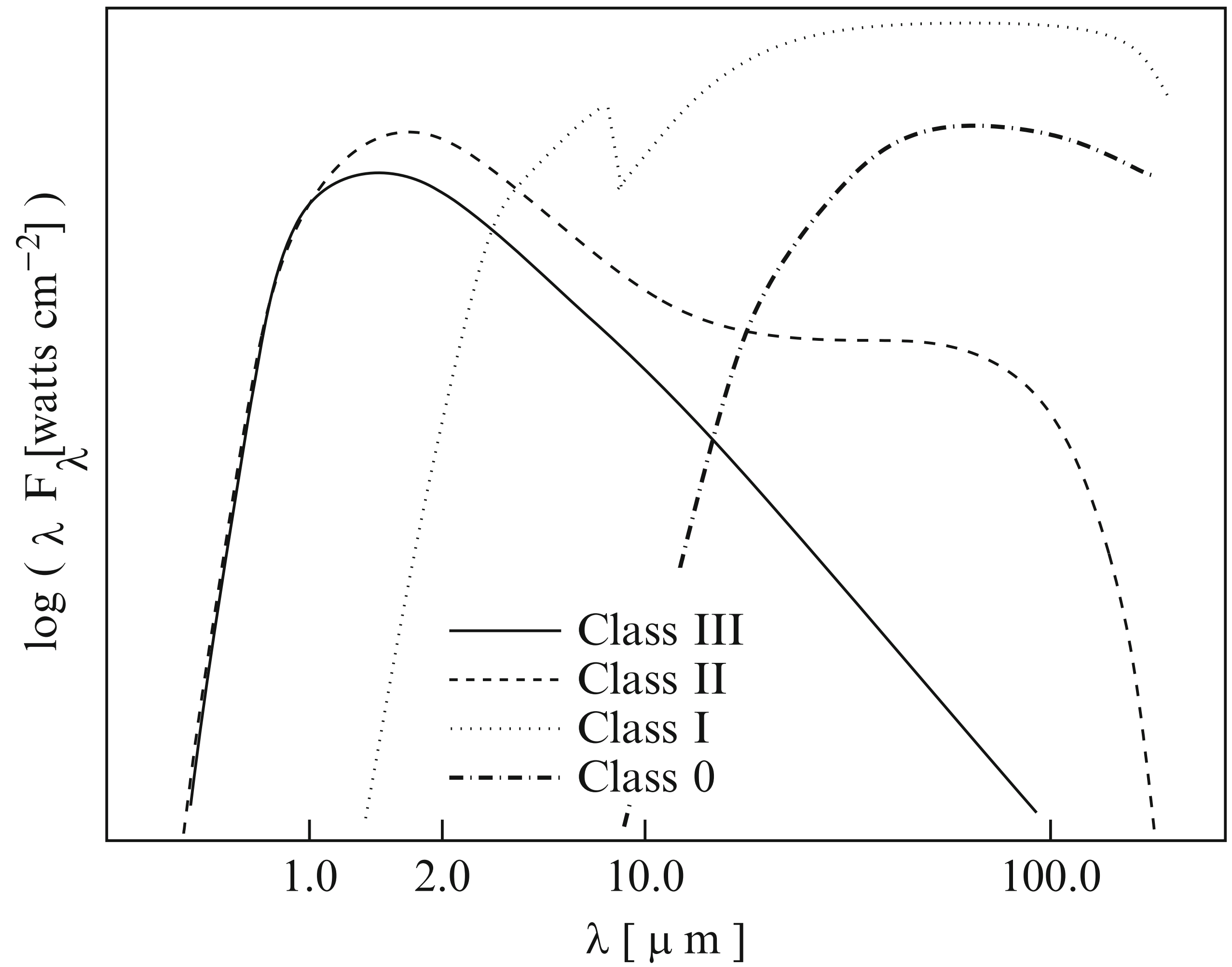
- Timescales in stellar evolution
  - Free-fall timescale = dynamic timescale
  - Kelvin-Helmholtz timescale = thermal timescale
  - Nuclear time scale
    - ✦ Time scale on which the star will exhaust its supply of nuclear fuel if it keeps burning it at the current rate
    - ✦ Energy release from fusing one gram of H to He is $6 \times 10^{18}\,\mathrm{erg}$. Therefore, the nuclear timescale is

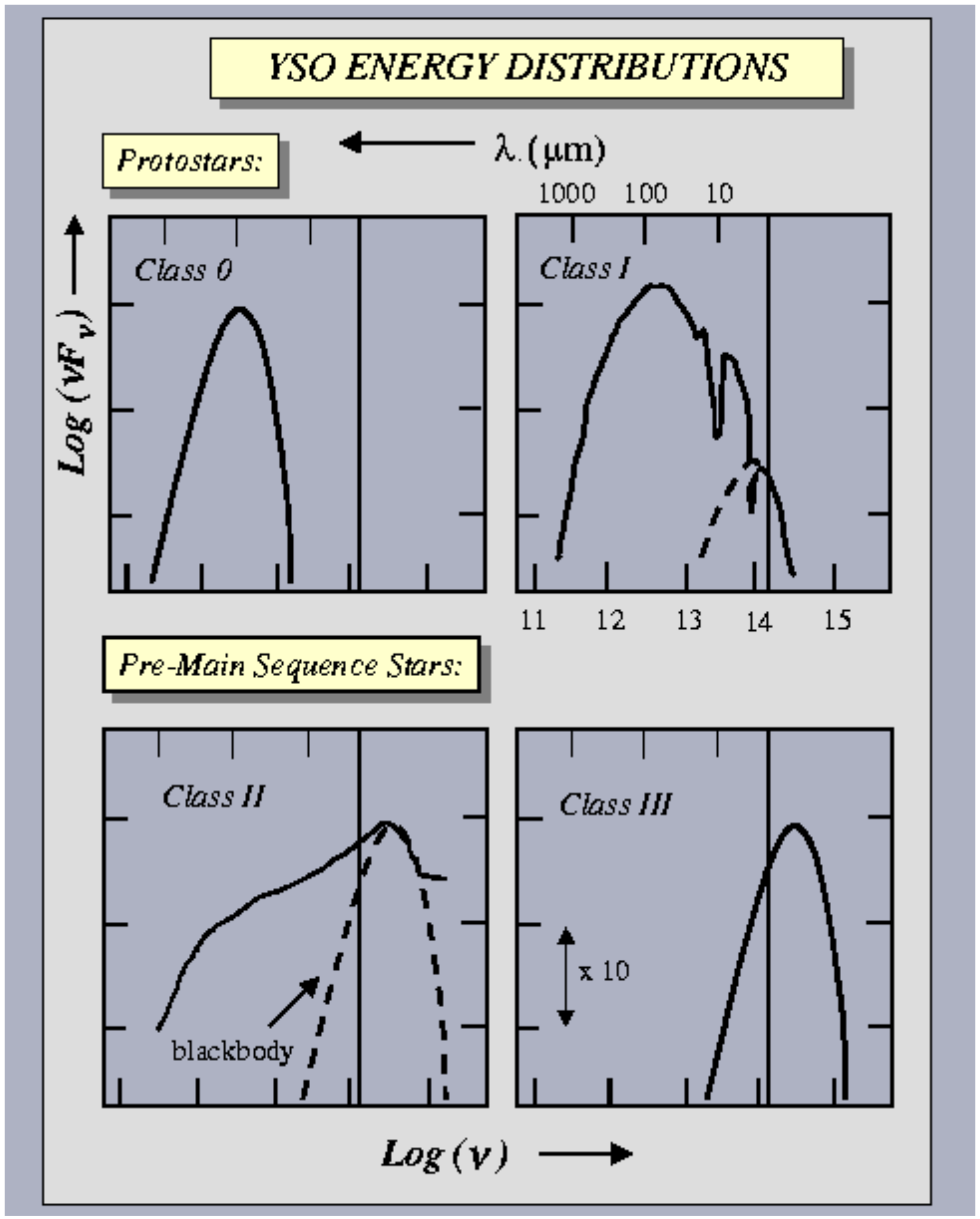
$$
\begin{aligned}
t_{\mathrm{nuc}} &= \frac{qXM \times (6 \times 10^{18}\,\mathrm{erg\,g^{-1}})}{L_*} \\
&\approx 7 \left(\frac{X}{0.7}\right) \left(\frac{q}{0.1}\right) \left(\frac{M}{M_\odot}\right) \left(\frac{L_\odot}{L_*}\right) \mathrm{Gyr}
\end{aligned}
$$

Here, X is the mass fraction of H initially present and q is the fraction of fuel available to burn in the core.

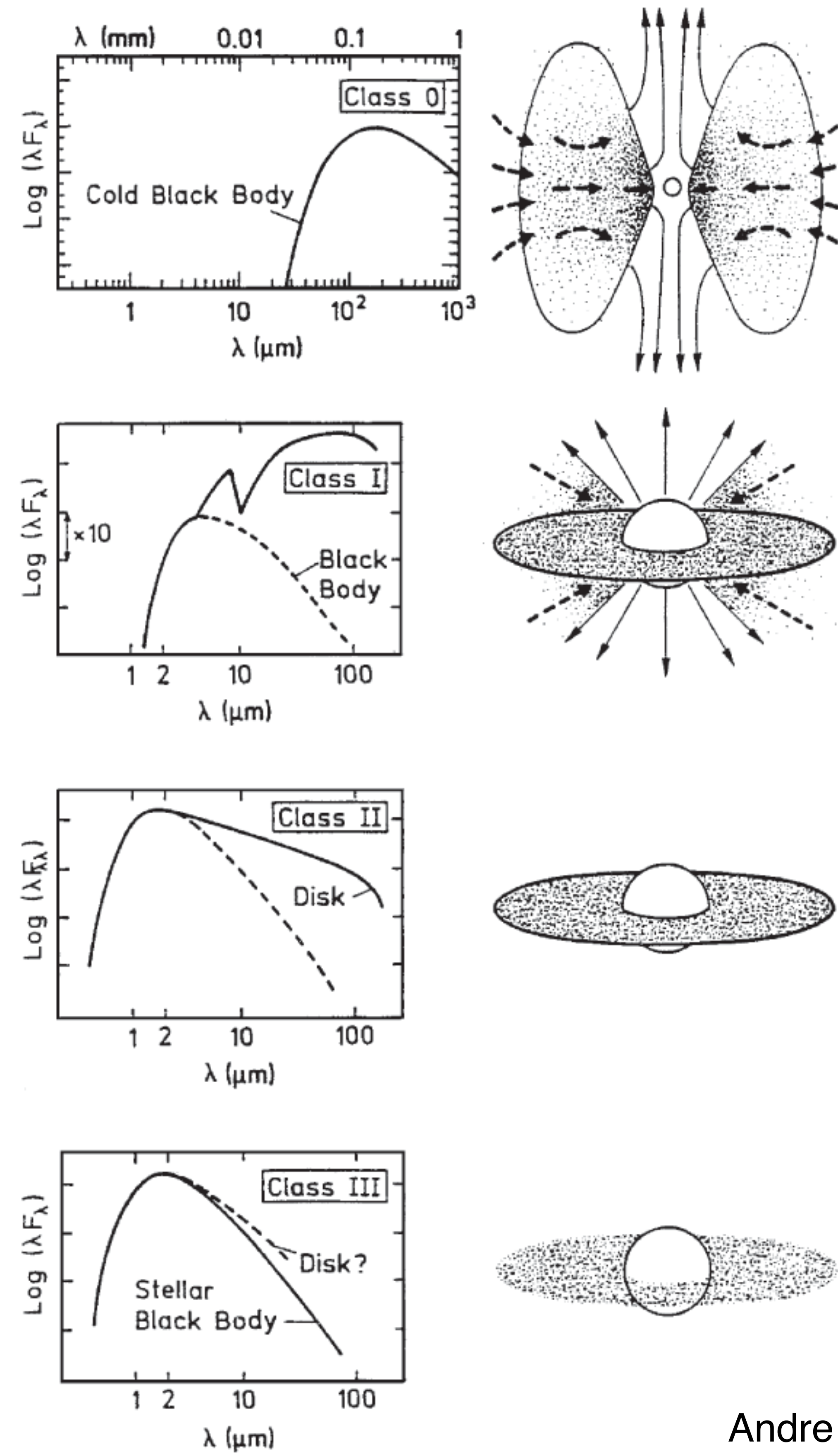
Spectral Energy Distributions (SEDs) of YSOs

- Lada & Wilking (1984) and Lada (1987) investigated SEDs of IR sources observed in the 1-100 μm wavelength band in the core of the Ophiuchi dark cloud, and proposed a general IR classification scheme.
  - ✦ A spectral index is defined between 2.2 and 25 μm: $\alpha_{\rm IR} = \frac{d\log(\lambda F_\lambda)}{d\log\lambda}$ $\lambda F_\lambda \propto \lambda^\alpha$
  - ✦ Class I sources have very broad SEDs with $\alpha_{\rm IR} > 0$.
  - ✦ Class 2 sources have $-2 < \alpha_{\rm IR} < 0$.
  - ✦ Class 3 sources have $\alpha_{\rm IR} < -2$.
  - ✦ The large IR excesses are attributed to thermal emission from dust in large circumstellar envelopes and Class I sources are likely to be evolved protostars.
- Andre et al. (1993) discovered embedded sources that remained undetected below 25 μm indicating significantly larger amounts of circumstellar material than in Class I sources and the proposed a younger Class 0 of YSOs.



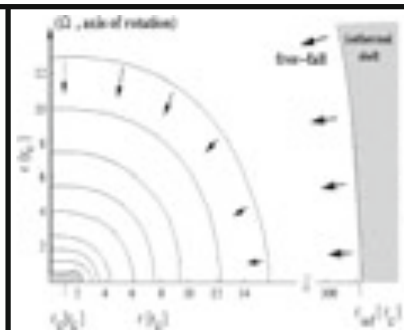
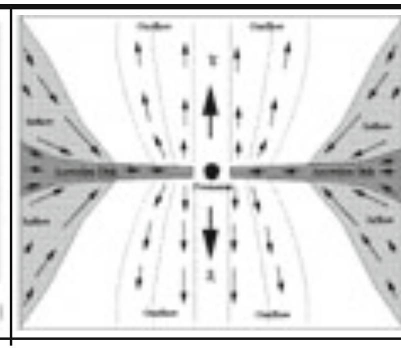
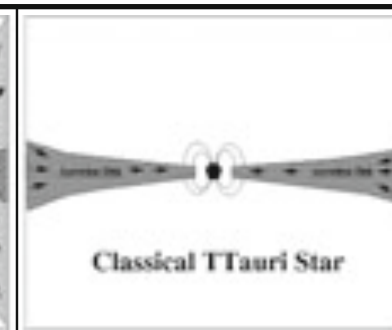
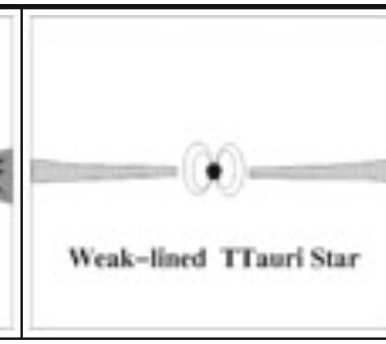
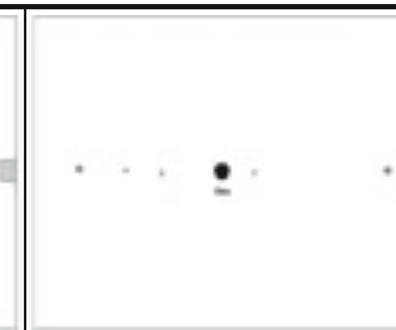


Lada (1999)



Andre (1994)

YSO classification

	Infalling protostar	Accreting protostar	Contracting PMS star		MS star
YSO properties					
Phase	adiabatic (A,B,C)	accretion (D) deuterium burning onset of convection	convective radiative onset of nuclear burning		convective radiative full nuclear burning
Matter flows	mostly infall disk & outflows form	some infall mostly accretion outflows, jets	low accretion	?	—
Envelope/ disk size	< 10000 AU	< 1000 AU	< 400 AU	~ 100 AU	—
Infall/ accretion rate	10^{-4}	10^{-5}	10^{-6} -- 10^{-7}	?	—
Age	$10^4 - 10^5$ yr	10^5 yr	10^6 -- 10^7 yr	10^6 -- 10^7 yr	—
Emission bands (except IR)	thermal radio X–ray?	radio X–ray	radio optical strong X–ray	non–therm. radio optical strong X–ray	non–therm. radio optical X–ray
Classes	Class 0	Class I	Class II	Class III	ZAMS

Angular Momentum Problem & Outflow

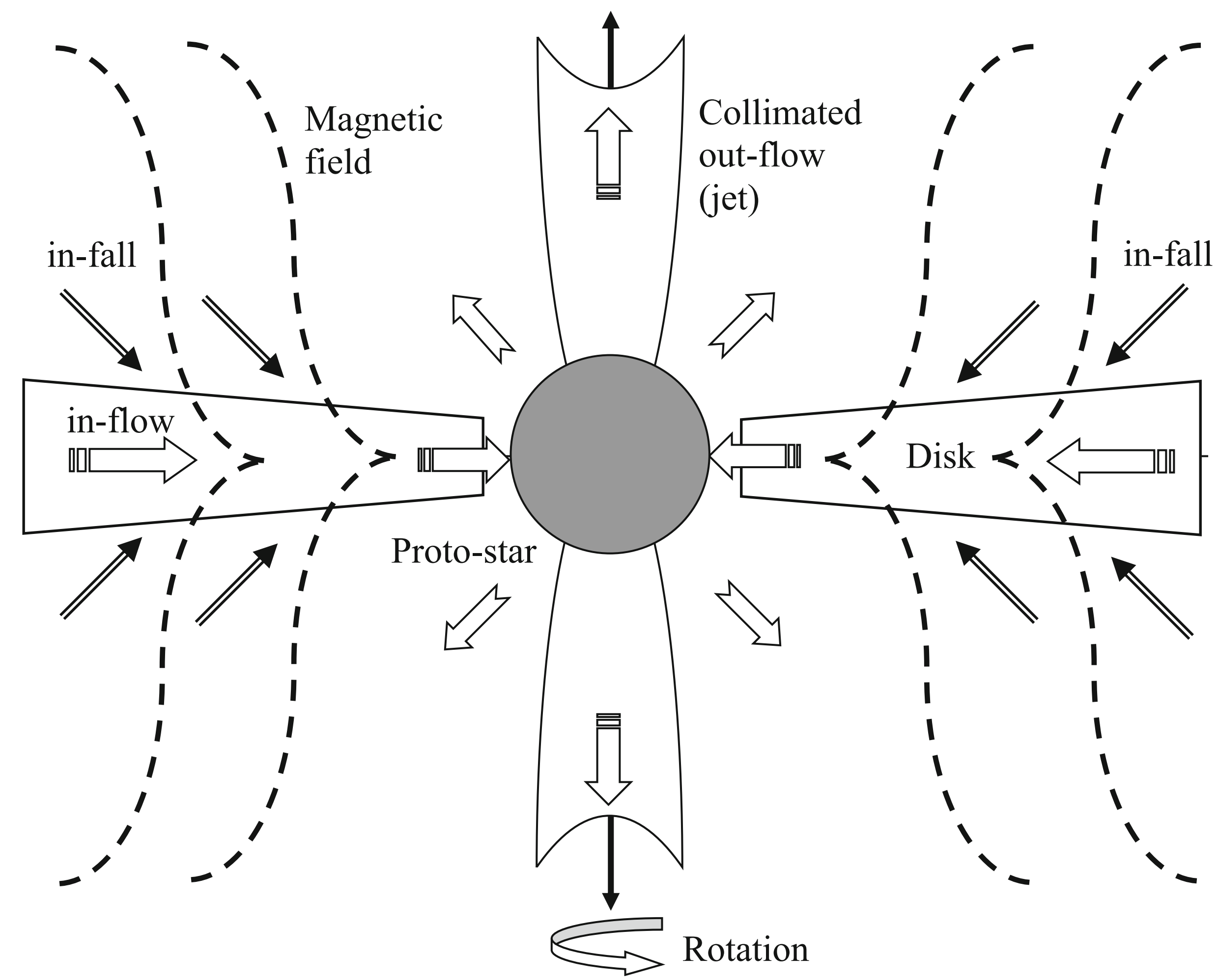
- Angular Momentum Problem
  - The typical specific angular momenta measured on different scales and at different evolutionary stages, from dense cores in molecular clouds down to the Sun is listed in the following table.
  - A parcel of gas initially located in a dense core has to reduce its angular momentum by 6 to 7 orders of magnitude in order to participate in the building of a typical star like out Sun. This puzzle has long been regarded as the “angular momentum problem”.
  - Since angular momentum is a conserved quantity, this loss of angular momentum has to occur by transfer to other particles that will not be incorporated into the star.

Table 1. Typical specific angular momenta from dense cores to the Sun

Object	J/M (cm^2 s^{-1})	References
Dense cores in molecular clouds	10^{21-22}	1
Protoplanetary disks	10^{19-21}	2
Pre-main-sequence binaries	10^{19-20}	3
Pre-main-sequence stars	10^{16-17}	4
Extrasolar planetary systems (exoplanet(s) + star)	10^{16-18}	5, 6
Solar system (planets + Sun)	10^{17}	7
Sun	10^{15}	8

References: 1: Goodman et al. (1993), 2: Williams & Cieza (2011), 3: Chen et al. (2007), 4: Mathieu (2004), 5: Armstrong et al. (2007), 6: Berget & Durrance (2010), 7: Allen (1973), 8: Pinto et al. (2011).

- Schematic cross section through an accretion disk located about a newly forming protostar.
  - Material falls onto the disk surface from the surrounding (and collapsing) natal cloud, and with appropriate angular momentum sorting moves through the disk to be accreted by the central protostar.
  - As a consequence of the high surface temperature of the protostar, however, material at the inner edge of the disk will become ionized and therefore begin to strongly interact with any disk-threaded magnetic field lines.
  - The combined effects of rotation and the winding up of the magnetic field lines result in the formation of bipolar jets along the spin-axis of the system, with material being carried away from the central object.

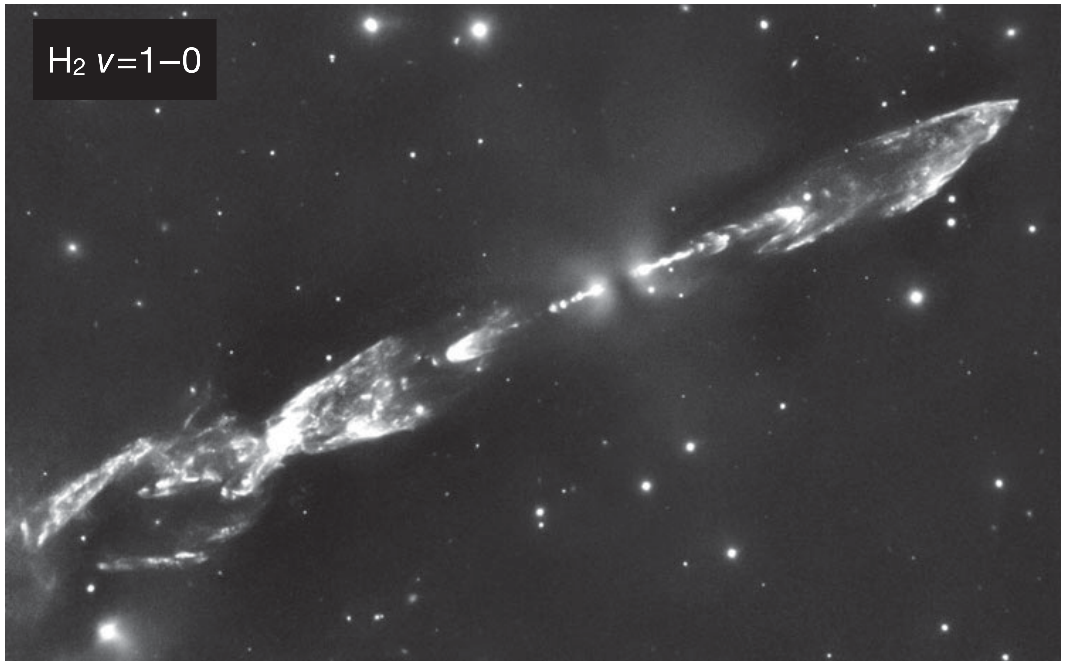


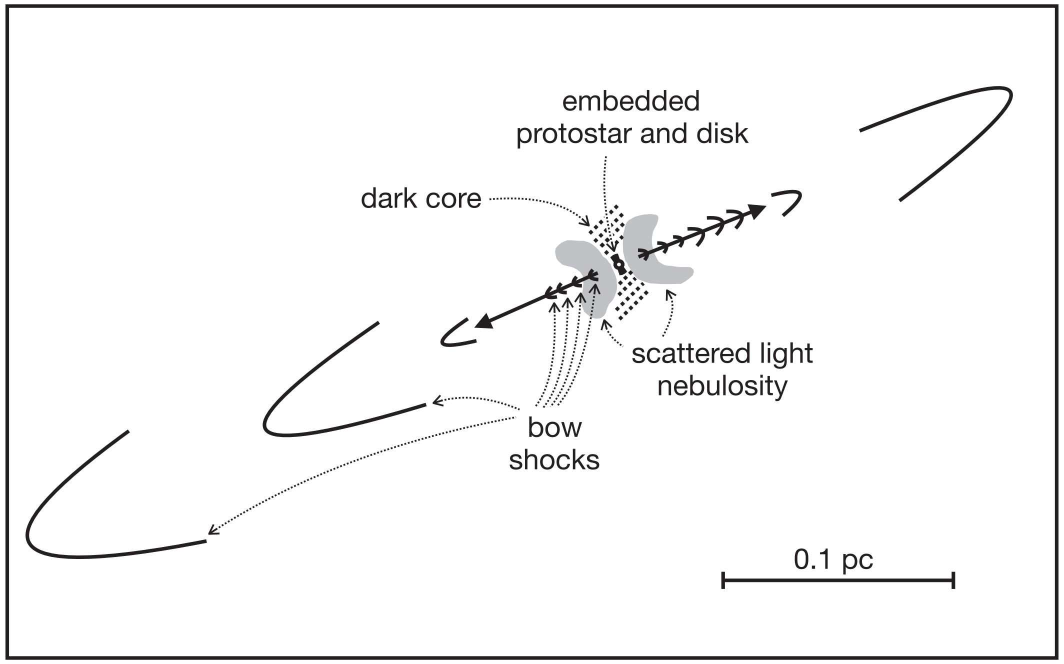
[Martin Beech]
Introducing the Stars: Formation, Structures and Evolution

Protostellar outflow

- The disk is rotationally connected to the protostar through strong magnetic fields (which are also concentrated during the collapse).
- Their mutual interaction produces an outflow that can carry away angular momentum. These are high-velocity, unbound, flows of gas that originate at the star-disk interface or within the inner few AU of the disk.
- Observations show highly collimated, fast jets from very young, highly embedded protostars and broader flows from later evolutionary classes.
- Magnetic field exerts a torque on the disk and slows it down.
- Outflows also entrain and sweep away the surrounding molecular core. This removes much of the surrounding mass reservoir and helps advance the evolutionary sequence along from embedded protostar to optically visible pre-main sequence star.
- Eventually, then mass infall rate decreases and the outflow stops. Young, rapidly rotating, pre-main sequence stars slow down as they lose angular momentum through stellar winds.

IR image of v = 1-0 line of H2 at 2.12 um produced by a protostellar outflow in Orion (named HH212 because of its prominence in this line) observed by the Very Large Telescope in Chile.

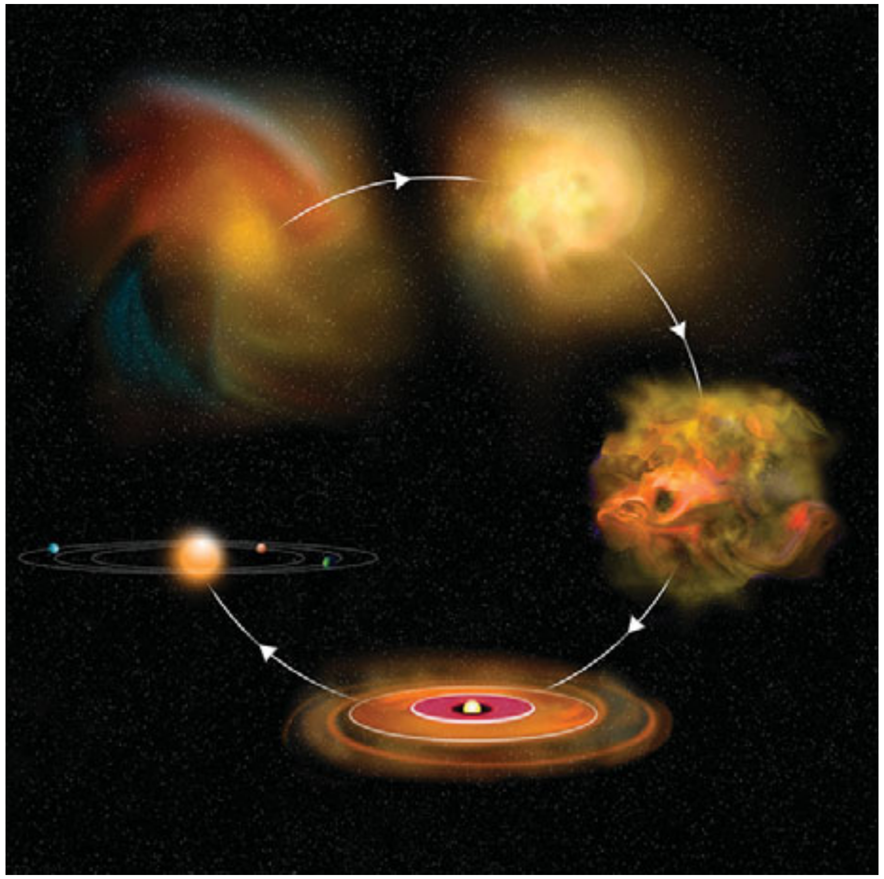




Planets

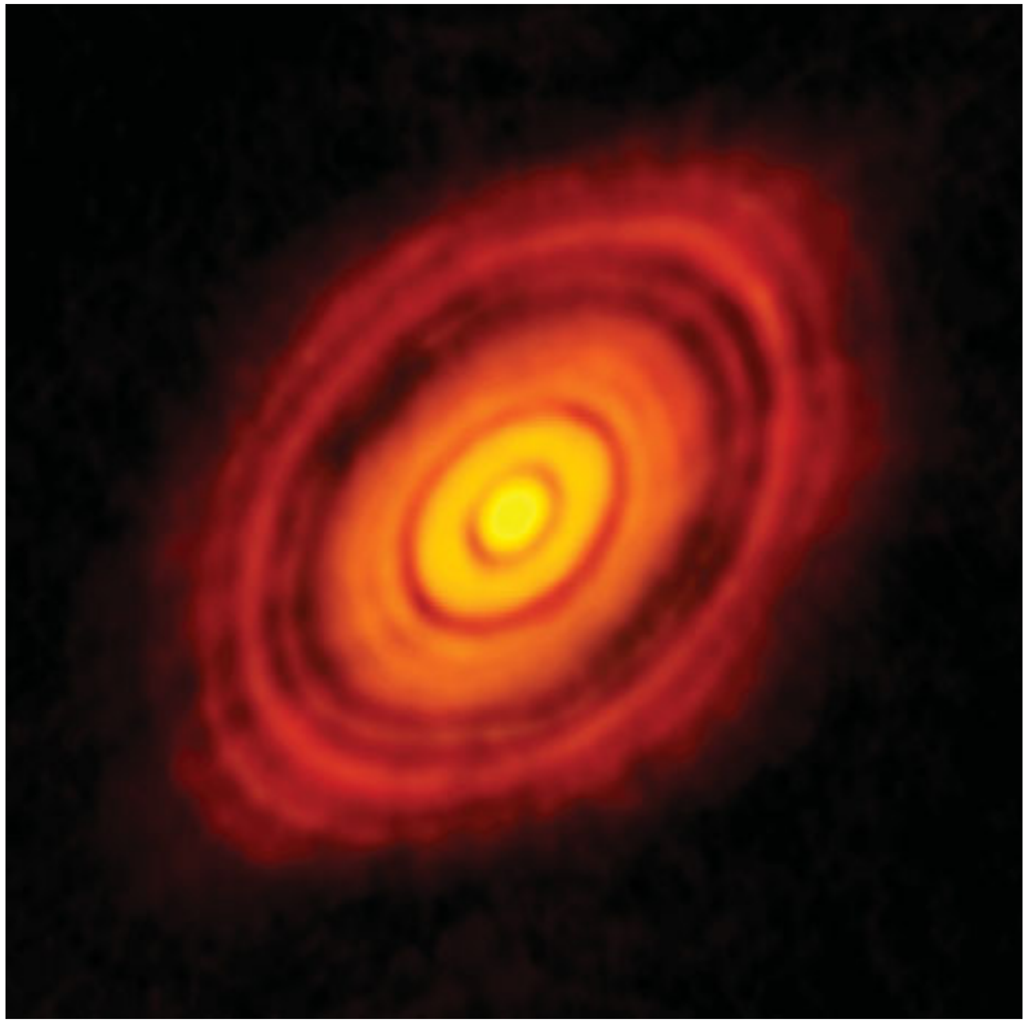
Origin of Planetary Systems

- The Nebular Model (basic ideas by Kant and Laplace)
  - The disk-formation process forms the basis for the "nebular model" for the formation of planetary systems, including our own solar system.
  - As a proto-stellar cloud collapses under the pull of its own gravity, conservation of its initial angular momentum leads naturally to form of an orbiting disk, which surrounds the central core mass.
  - This disk is initially gaseous, held **in a vertical hydrostatic equilibrium** about the disk mid-plane, **with radial support against gravity provided by the centrifugal force**.
  - This stops the rapid, dynamical infall, but as the viscous coupling between differentially rotating rings (and the entrainment of disk material by an outflowing stellar wind) transports angular momentum outward, there remains relatively slow inward diffuse of material that causes much of the initial disk mass to gradually accrete onto the young star.
  - This gradually depletes H and He gas (over a few million years). During this period, the heavier elements can gradually bond together to make molecules.
  - These in turn nucleate into grains of dust, and eventually into rocks.
  - Collisions among these rocks leads to a combination of fragmentation and accumulation, with the latter eventually forming asteroid-size (m to km) bodies. These then make planetoids, and eventually planets.



Observations of Proto-Planetary Disks

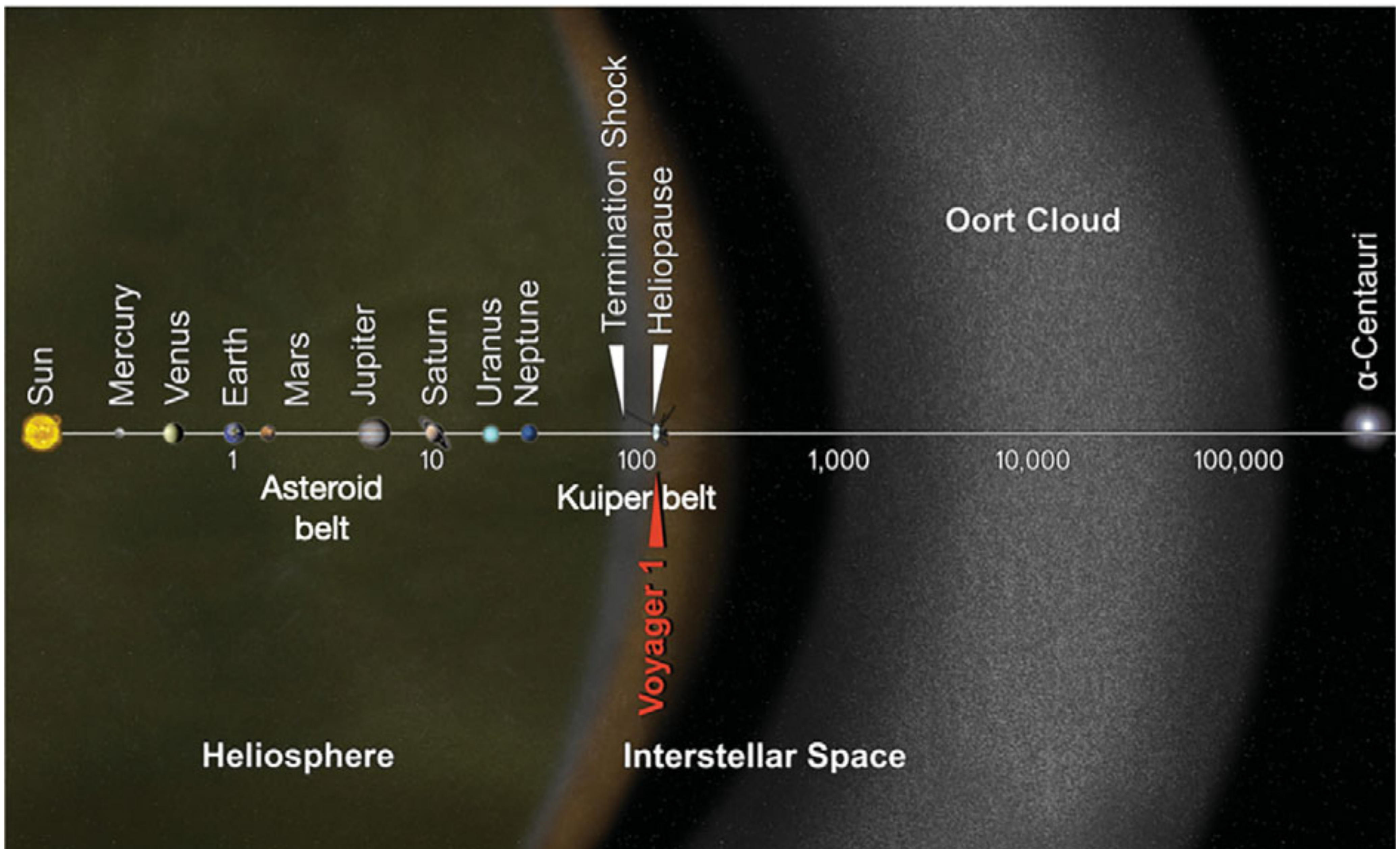
- Young stellar objects (YSOs) often show clear evidence of proto-planetary disks.
- With advent of telescope array (e.g., ALMA) observing in the far-IR and submm spectral regions, it is now becoming possible to image such disks directly.
- The figure shows an ALMA image of a proto-planetary disk in the T Tauri star HL Tauri, made in mm wavelengths.



- Interferometry from the array allows spatial resolution ranging down to 0.025 arcsec. At HL Tauri's distance of 140 pc, this corresponds to 3.5 AU, with the visible disk extending over a diameter of ~ 200 AU.
- The disk gaps likely represent regions where planet formation is clearing out disk debris, though there is so far no direct evidence of fully formed planets in this system.
- A key issue in planet formation is whether this can occur quickly enough to compete with disk depletion by various processes, like accretion onto the star, dissociation by stellar UV radiation, and entrainment in a outflowing stellar wind.

Our Solar System

- Planets
  - Rocky planets (rocky dwarfs; terrestrial planets): Mercury, Venus, Earth, Mars
  - Gas giants: Jupiter and Saturn
  - Ice giants: Uranus and Neptune



- Snow line (also known as the ice line or frost line)
  - the particular distance in the solar nebula from the central protostar where it is cold enough for volatile compounds such as water, ammonia, methane, carbon dioxide, and carbon monoxide to condense into solid ice grains.
  - In the colder outer regions, these condensed to form ice, which gradually collected into ever larger solid cores, eventually growing massive enough to gravitationally attract and retain the even more abundant but lighter gases of hydrogen and helium.
    - ✦ This is the basis for formation of the outer gas and ice giant planets, with an overall composition similar to the solar nebula, and the present day Sun.
  - In the inner nebula, where it was too warm to form ice, such light atoms of H and He escaped from the weaker gravity of the smaller, rocky planets, effectively preventing their growth and so keeping them relatively small.

- Hot Jupiter
    - ✦ a planet with a mass comparable to (actually even larger than) Jupiter, but orbiting at such a close distance that the stellar heating would make it quite hot.
    - ✦ Such gas giants had been supposed to form only beyond the ice line. The detection of hot Jupiters around several stars was a real surprise.
    - ✦ They are though to have formed outside the snow line, and later migrated inwards to their current positions. Gravitational interaction with the proto-stellar disk and/or other giant planets out there is supposed to lead some to be flung into an inward migration, so that they finally ended up very close to their star.

Equilibrium Temperature

- For an absorbing sphere with radius r at a distance d from the star, the intercepted flux of the stellar luminosity L_* is

$$\frac{\pi r^2 L_*}{4\pi d^2} = \pi r^2 \sigma_{\mathrm{SB}} T_*^4 \left(\frac{R_*}{d}\right)^2$$

- where R_* and T_* are the star's radius and effective temperature, and σ_{SB} is the Stefan-Boltzmann constant.

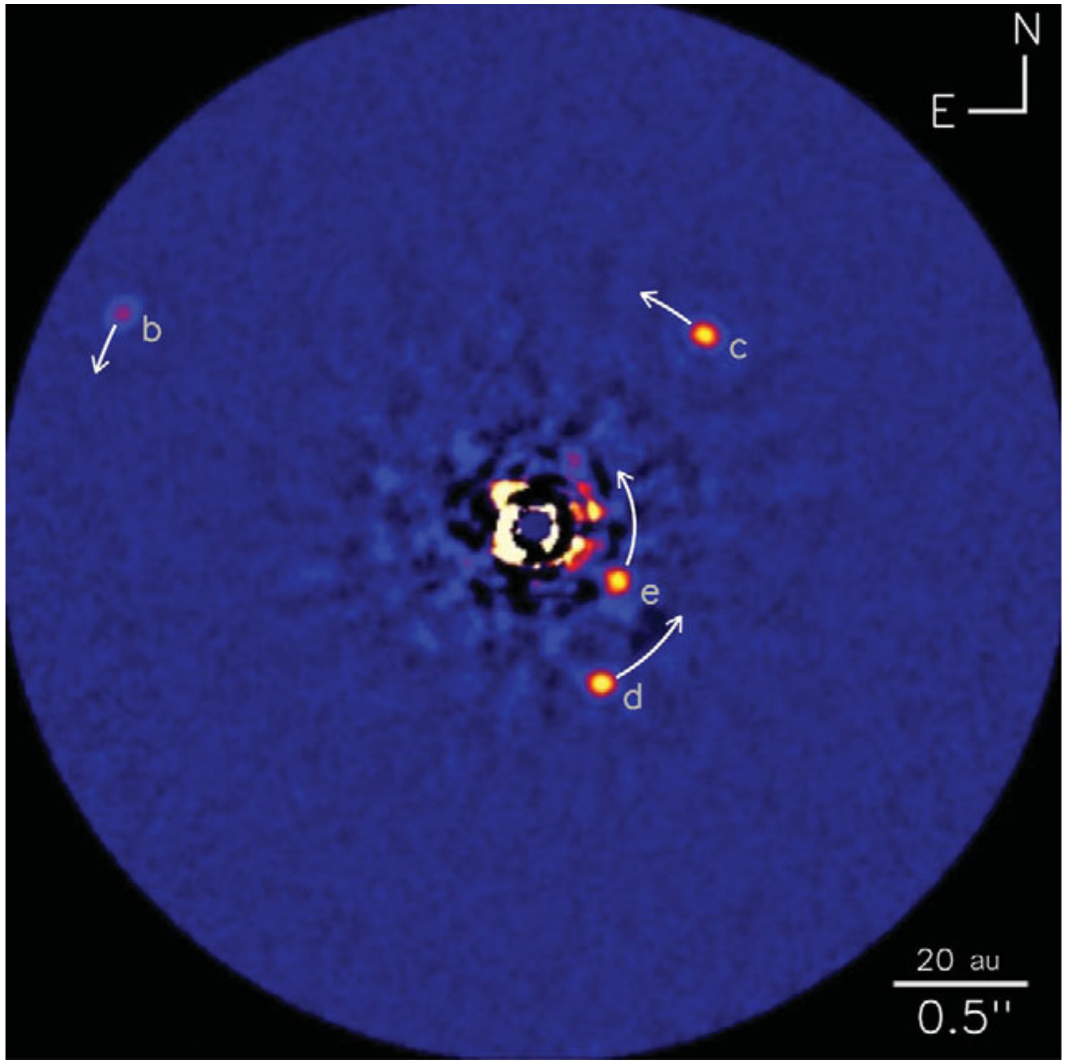
- If we assume this sphere then radiates this energy as a blackbody over its surface area, then solving for its ***equilibrium temperature*** gives

$$\pi r^2 \sigma_{\mathrm{SB}} T_*^4 \left(\frac{R_*}{d}\right)^2 = 4\pi r^2 \sigma_{\mathrm{SB}} T_{\mathrm{planet}}^4$$

$$T_{\mathrm{planet}}(d) = T_* \sqrt{\frac{R_*}{2d}} \approx 290\,\mathrm{K} \sqrt{\frac{1\,\mathrm{AU}}{d}}$$

Detection of Exoplanets (Extra-solar planets)

- Direct-Imaging Method
  - Because they are much cooler than stars, the thermal emission from planets is mostly in the IR.
  - Their appearance at visible wavelengths comes by reflected light from their host stars. This greatly complicates direct detection of extra-solar planets, since this reflected light is generally overwhelmed by the direct light from the stars.
  - Nowadays, there are ~ 20 such detect imaging detections of exoplanets.



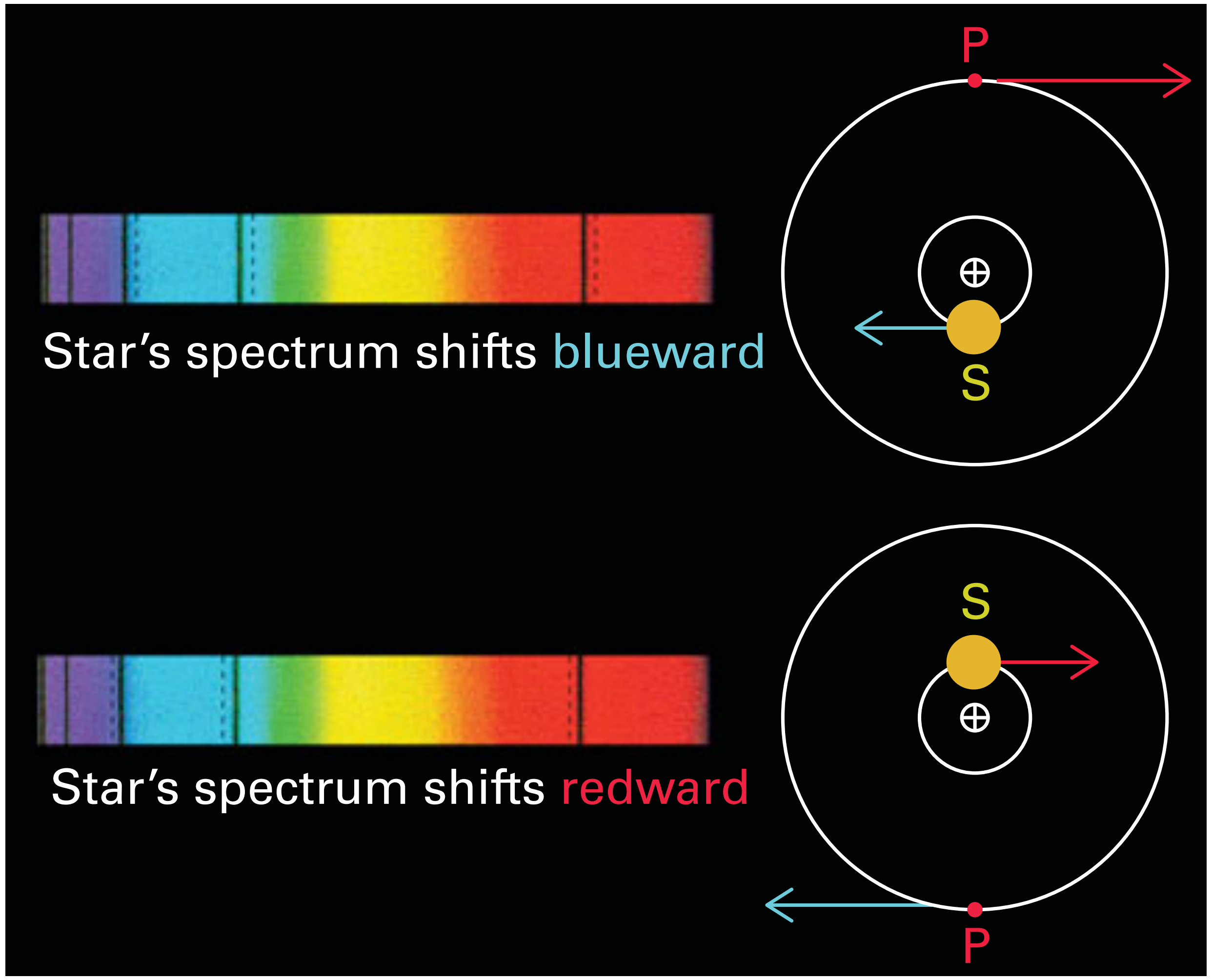
Direct image by the Keck Observatory of 4 exoplanets orbiting HR8799.

The arrows indicates their orbital motion from monitoring their positions over more than a decade. The orbital periods are 49, 100, 189, and 474 years for planets e, d, c, and b, respectively.

For reference, the orbital period of Neptune is 165 years.

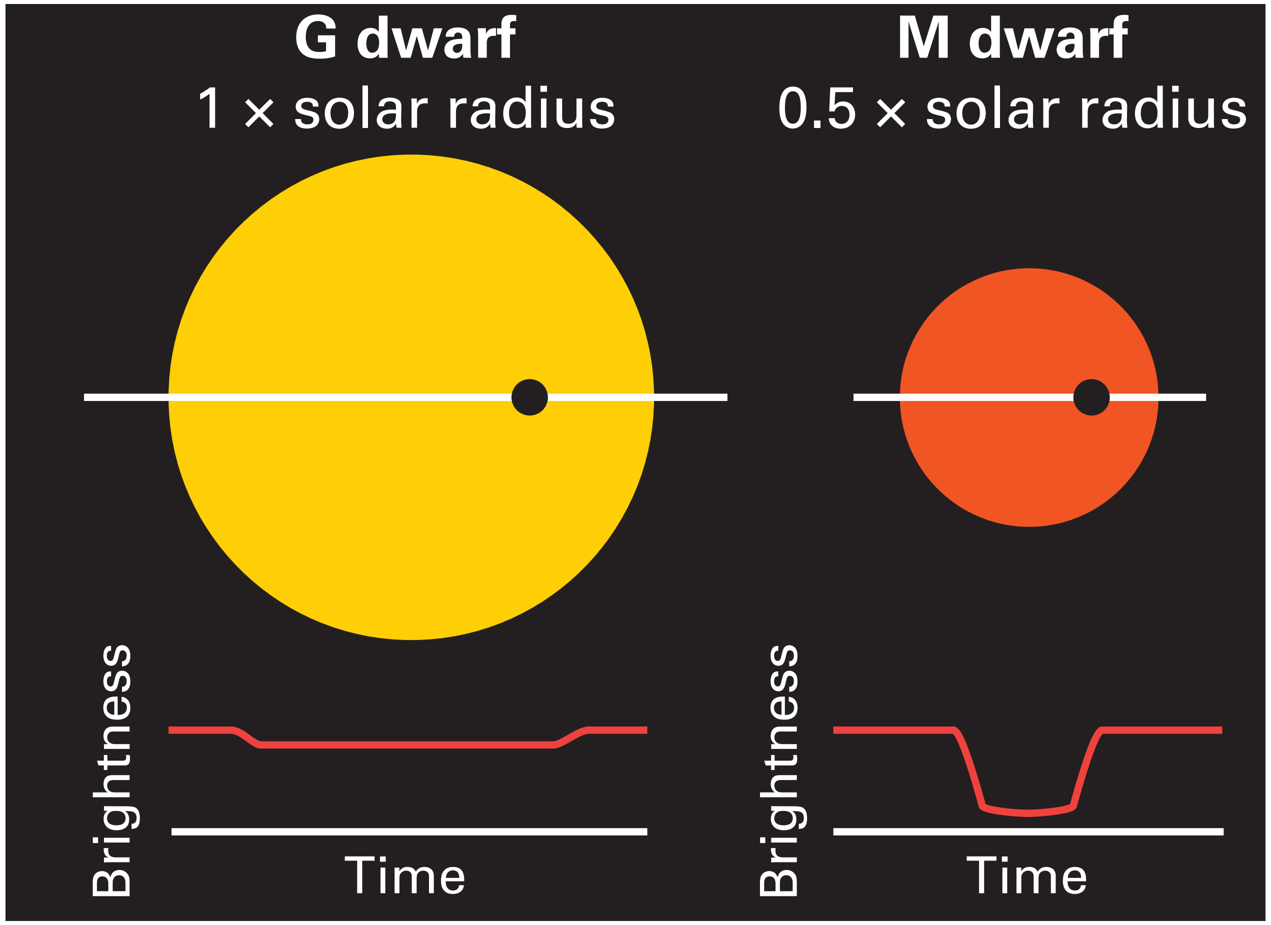
[NTV-HIS/C. Marois/W. M. Keck Observatory]

- Radial-Velocity Method
  - The periodic movement of the host star due to the gravitational pull of the planet causes spatial “wobble.”
  - This wobble is not directly detectable, but, its associated motion toward and away from the observer can be detected via very precise spectroscopic measurements of the systematic Doppler shift from multiple absorption lines in the star’s spectrum.
  - This is the same method as used in spectroscopic binaries.



- Transit Method
  - This method simply looks for the slight dimming of the star's apparent brightness whenever a planet "transits" in front of it.
  - Instead of elaborate spectroscopic measurement of the slight Doppler shift, this merely requires precise photometric measurements of changes in the star's total apparent brightness. (This is analogous to eclipsing binaries)
  - The fractional drop in the star's brightness will be

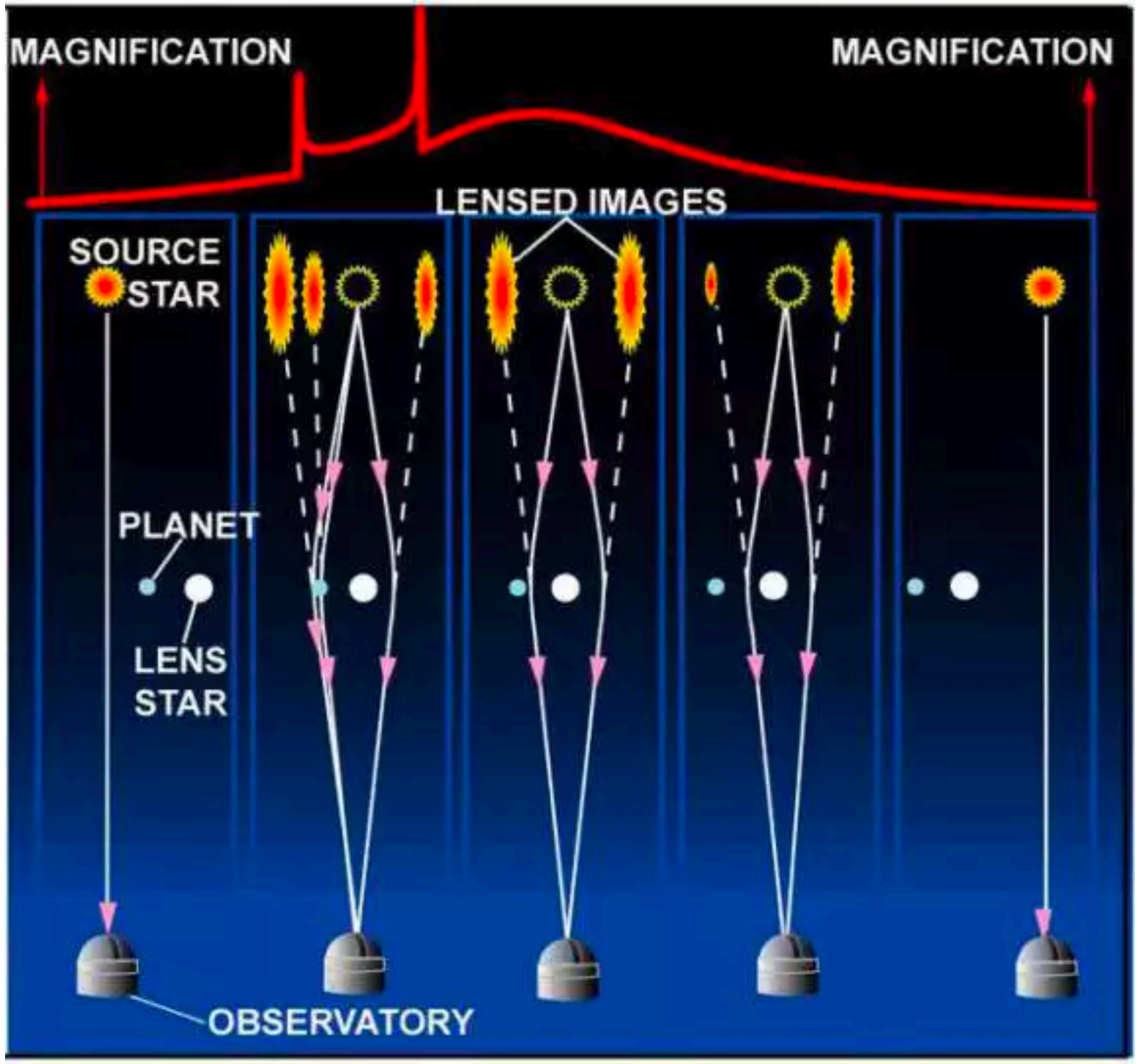
$$\frac{\Delta F}{F} = \left(\frac{R_{\rm p}}{R_*}\right)^2$$



- Microlensing Method
  - Microlensing is a form of gravitational lensing in which the light from a background source is bent by the gravitational field of a foreground lens to create distorted, multiple and/or brightened images.

[planetary.org]

The lensing star (white) moves in front of the source star (yellow) magnifying its image and creating a microlensing event.
In the second image (from left), the planet adds its own microlensing effect, creating the two characteristic spikes in the light curve.



Homework (due date: 10/07)

Problem 1 - Virial Theorem

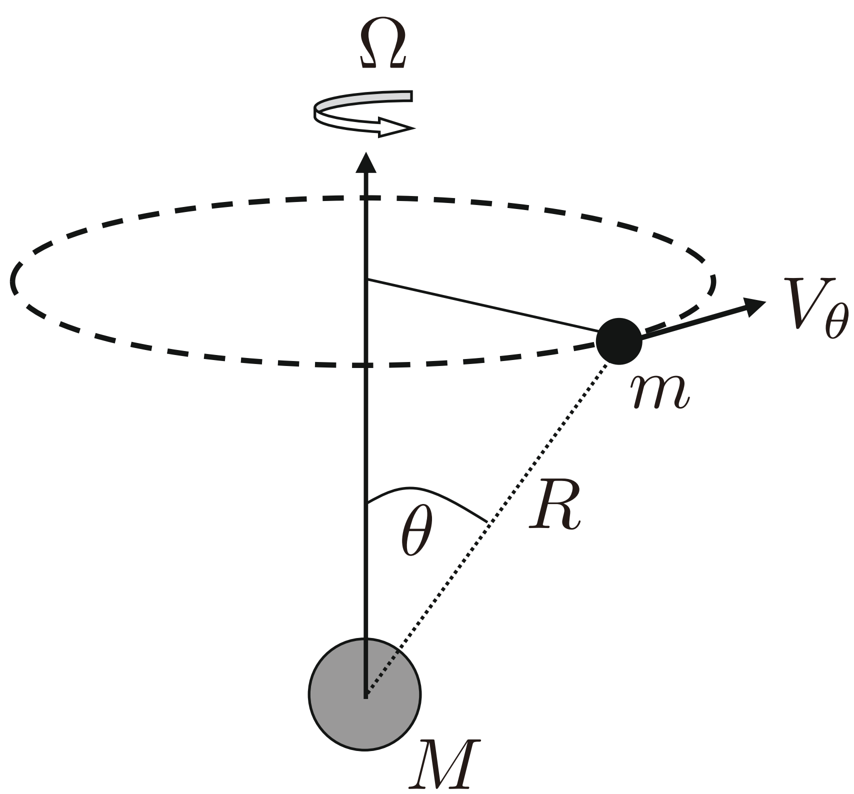
Consider a spherical cloud with a total mass *M* and a radius *R*. Assume that the cloud has a radial density profile of $\rho = \rho_0 (r/R)^{-\beta}$. The gravitational potential energy can be expressed as follows:

$$U = -\alpha \frac{GM^2}{R}$$

(a) For the above equation to be satisfied, what is the condition for β ?
(b) Express α in terms of β.
(c) What is the virial mass? Express the mass in terms of the cloud radius *R*.

Problem 2 - Disk collapse from various latitudes

- For a spherical cloud of radius R and rotational speed Ω, consider locations away from the equator, with colatitude θ measured from the polar axis. Derive an expression for the associated ratio $\beta(\theta)$ of the local rotational energy to gravitational energy, writing this in terms of the equatorial ratio $\beta_{\mathrm{eq}} \equiv \beta(\theta = \pi/2)$ derived in this lecture note.



- Use this to derive an expression for the associated disk radius $r(\theta)$ to which material contracts from various colatitudes on the initial spherical surface of radius R. (You may assume that throughout the contraction, the gravitational attraction is that from a point source of mass M at the cloud center.) The blue lines in the lower figure draw connections between this disk radius and its source location at various colatitudes on the cloud surface, for a choice of the parameter β_{eq}.