Astrophysics [Part I]

Lecture 3
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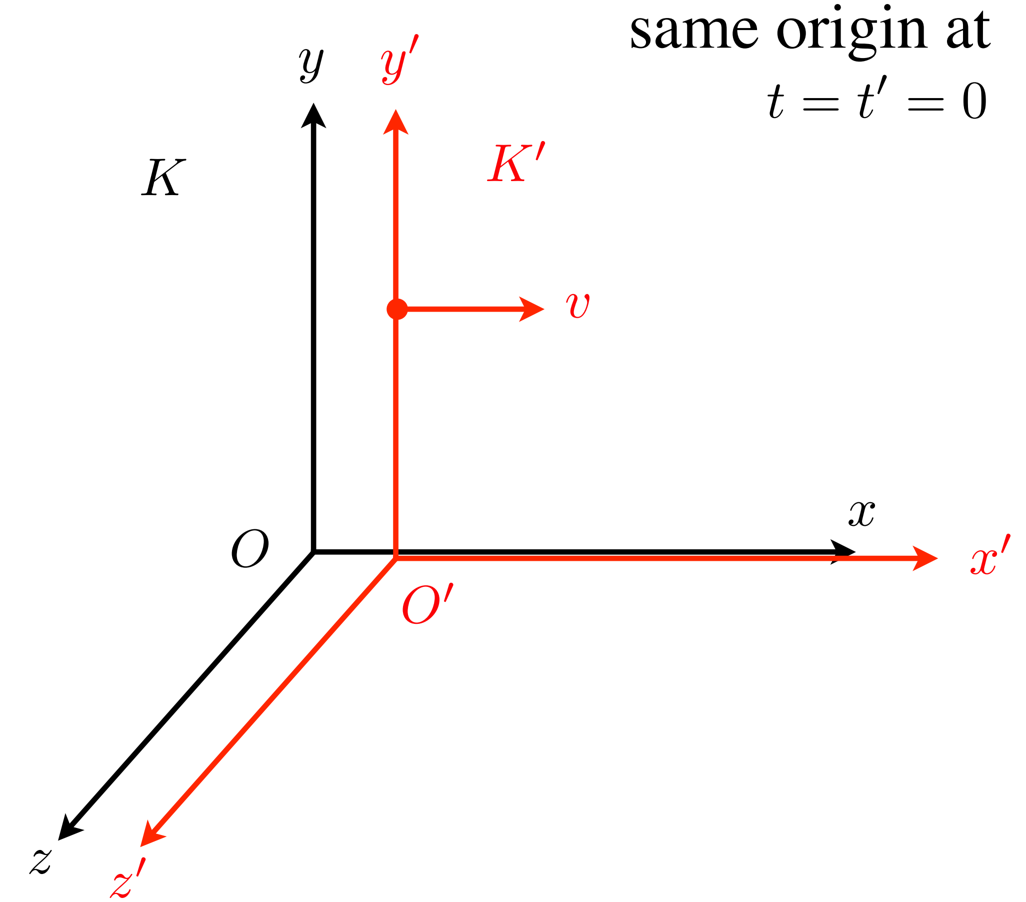
Kwang-il Seon
KASI / UST

Relativistic Theory and Kinematics

Galilean Transformation/Relativity

- Galilean transformation is used to transform between the coordinates of two **inertial frames of reference** which differ only by constant relative motion within the constructs of Newtonian physics.



$$
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

Newton's law is invariant under the Galilean transformation.

However, Maxwell's equations are not invariant under the Galilean transformation.

- **Lorentz transformation is the result of attempts by Lorentz and others to explain how the speed of light was observed to be independent of the reference frame, and to understand the symmetries of the Maxwell's equations.**

* Review of Lorentz Transformations *

- **Postulates in the special theory of relativity**

  (1) The laws of nature are the same in two frames of reference in uniform relative motion with no rotation.

  (2) The speed of light is c in all such frames.

- **Space-time event**: an event that takes place at a location in space and time.

- **Lorentz transforms**:

$$
\begin{aligned}
x' &= \gamma(x - vt) & \qquad x &= \gamma(x' + vt') \\
y' &= y & y &= y \\
z' &= z & z &= z \\
t' &= \gamma\left(t - \frac{v}{c}x\right) & t &= \gamma\left(t' + \frac{v}{c}x'\right)
\end{aligned}
$$

$$\text{where } \gamma \equiv \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} = \left(1-\beta^2\right)^{-1/2}; \quad \beta \equiv \frac{v}{c}$$

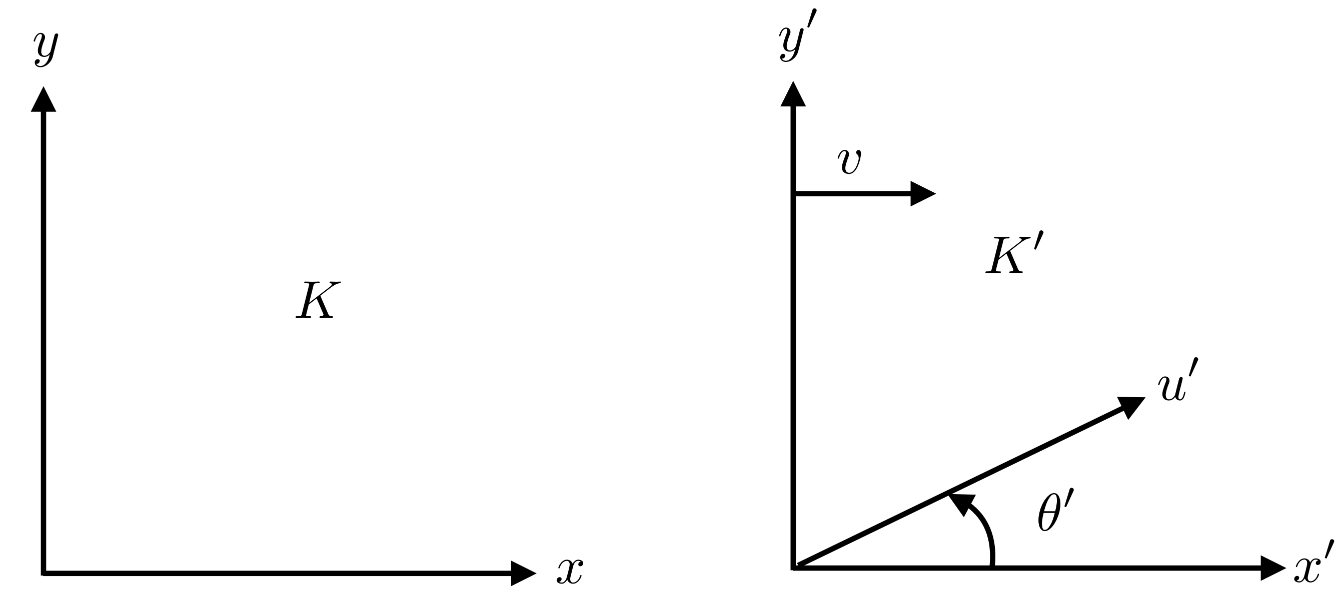
$$\text{Lorentz factor } 1 \le \gamma \le \infty; \quad 0 \le \beta \le 1$$

Transformation of Velocities

- If a point has a velocity $\mathbf{u}'$ in frame K', what is its velocity $\mathbf{u}$ in frame K?

$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + v u'_{\parallel}/c^2}$$

$$u_{\perp} = \frac{u'_{\perp}}{\gamma\left(1 + v u'_{\parallel}/c^2\right)}$$



- Aberration formula: the directions of the velocities in the two frames are related by

$$\tan\theta = \frac{u_{\perp}}{u_{\parallel}} = \frac{u'_{\perp}}{\gamma\left(u'_{\parallel} + v\right)} = \frac{u'\sin\theta'}{\gamma\left(u'\cos\theta' + v\right)}$$

- **Aberration of light** $(u' = c)$

$$\tan\theta = \frac{\sin\theta'}{\gamma\left(\cos\theta' + \beta\right)}$$

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$\sin\theta = \frac{\sin\theta'}{\gamma\left(1 + \beta\cos\theta'\right)}$$

$$\longrightarrow \quad \boxed{\tan\left(\frac{\theta}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^2 \tan\left(\frac{\theta'}{2}\right) \quad \rightarrow \quad \theta < \theta'}$$

- **Beaming ("headlight") effect**:

If photons are emitted isotropically in K', then half will have $\theta' < \pi/2$ and half $\theta' > \pi/2$.

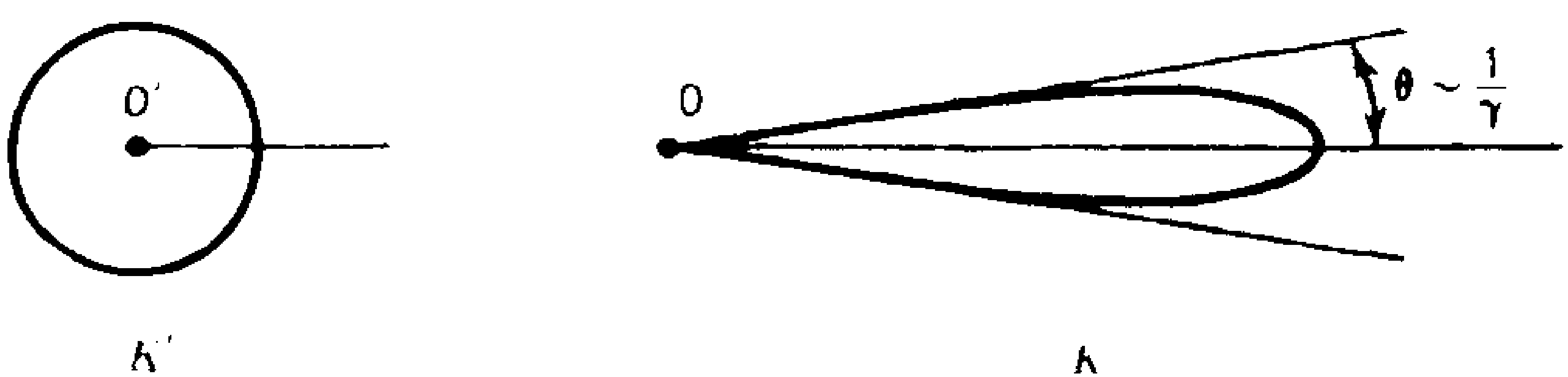
Consider a photon emitted at right angles to v in K'. Then we have

beam half-angle: $$\sin\theta_b = \frac{1}{\gamma}, \quad \cos\theta = \beta, \quad \text{or} \quad \tan\left(\frac{\theta_b}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}$$

For highly relativistic speeds, $\gamma \gg 1$, θ_b becomes small:

$$\boxed{\theta_b \approx \frac{1}{\gamma}}$$

Therefore, in frame K, photons are concentrated in the forward direction, with half of them lying within a cone of half-angle $1/\gamma$. Very few photons will be emitted $\theta \gg 1/\gamma$.

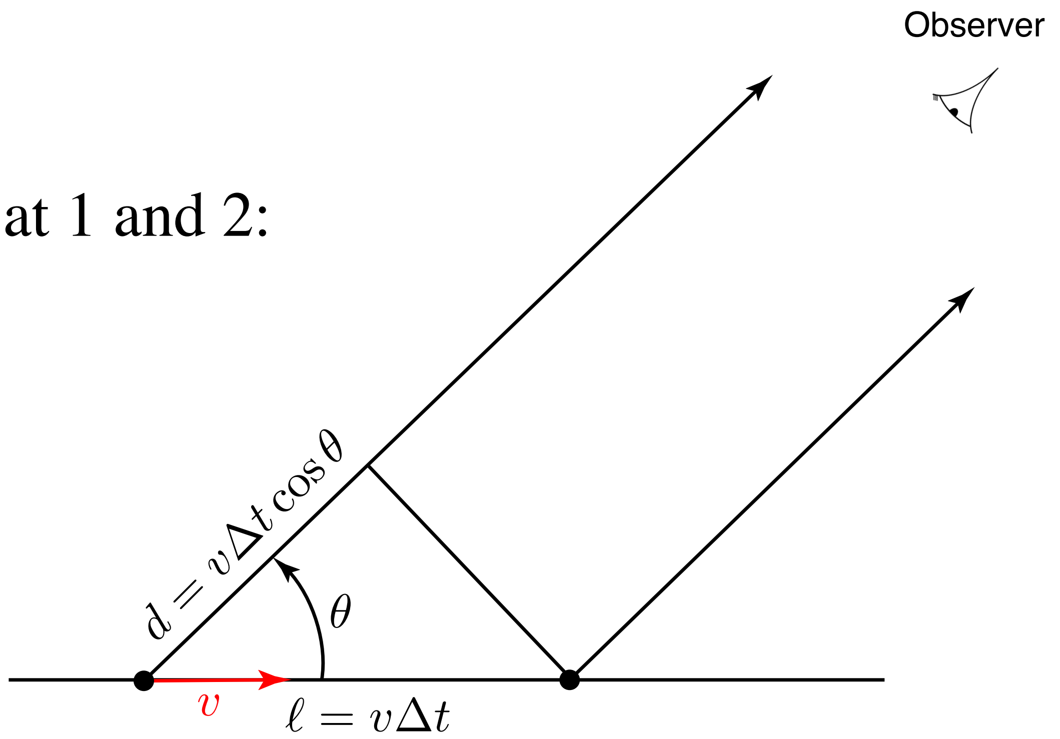


Doppler Effect

- In the rest frame of the observer K, imagine that the moving source emits one period of radiation as it moves from point 1 to point 2 at velocity v.

  Let frequency of the radiation in the rest frame K' of the source = ω'. Then the time taken to move from point 1 to point 2 in the observer's frame is given by the time-dilation effect:

$$\Delta t = \Delta t' \gamma = \frac{2\pi}{\omega'}\gamma$$



Difference in arrival times Δt_A of the radiation emitted at 1 and 2:

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t\left(1 - \frac{v}{c}\cos\theta\right)$$

Therefore, the observed frequency ω will be

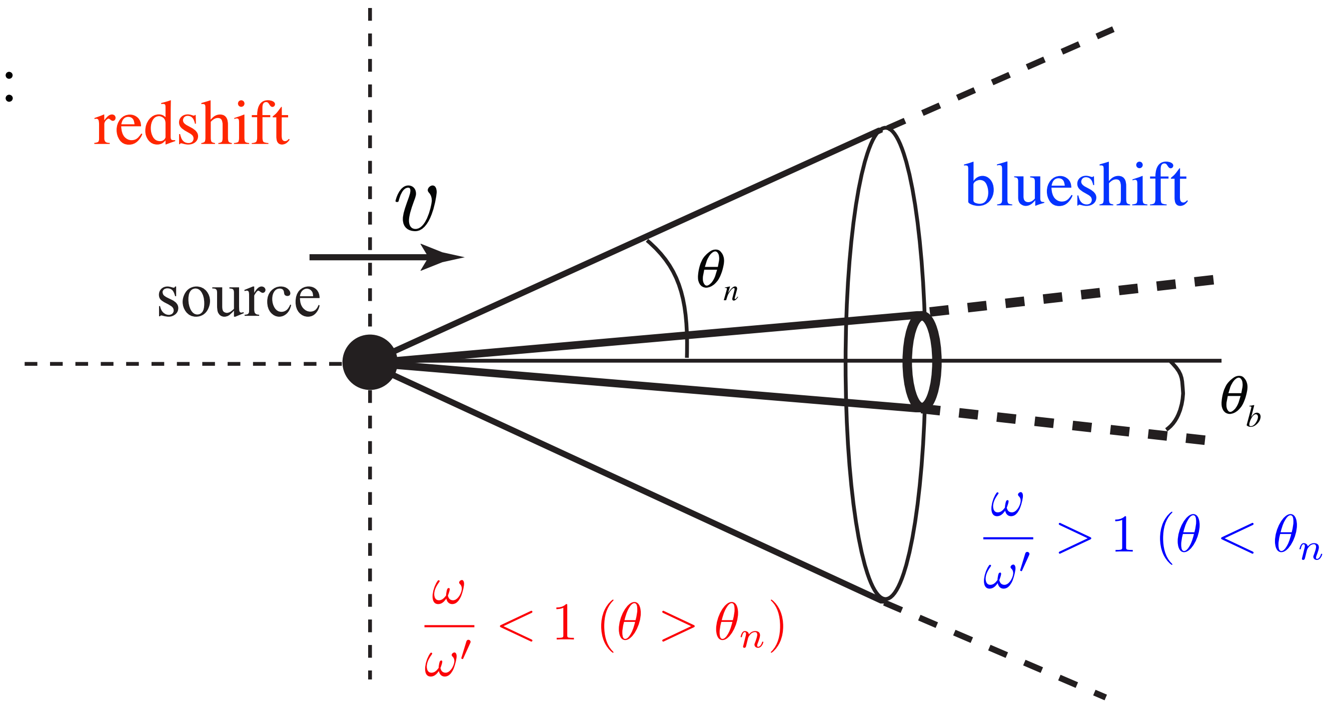
$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma\,(1-\beta\cos\theta)} \qquad \boxed{\frac{\omega}{\omega'} = \frac{1}{\gamma\,(1-\beta\cos\theta)}}$$

Note $1-\beta\cos\theta$ appears even classically. The factor γ^{-1} is purely a relativistic effect.

- Recall beam half-angle = $\theta_b = \sin^{-1}\gamma^{-1}$
- Angle for null Doppler shift is defined by:

$$\frac{\omega}{\omega'} = \frac{1}{\gamma\left(1 - \beta\cos\theta_n\right)} = 1$$

$$\rightarrow \quad \cos\theta_n = \frac{1-\gamma^{-1}}{\beta} = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{-1/2}$$

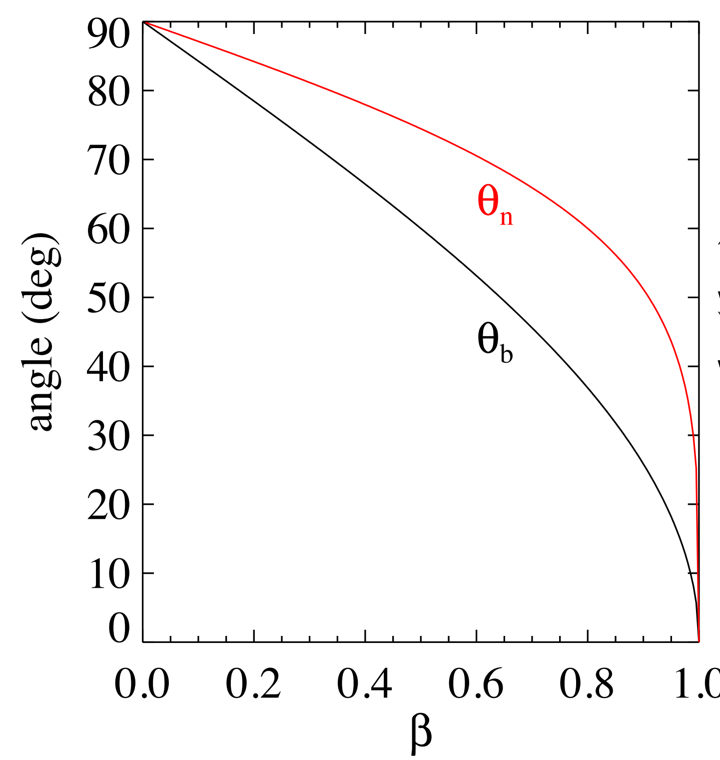


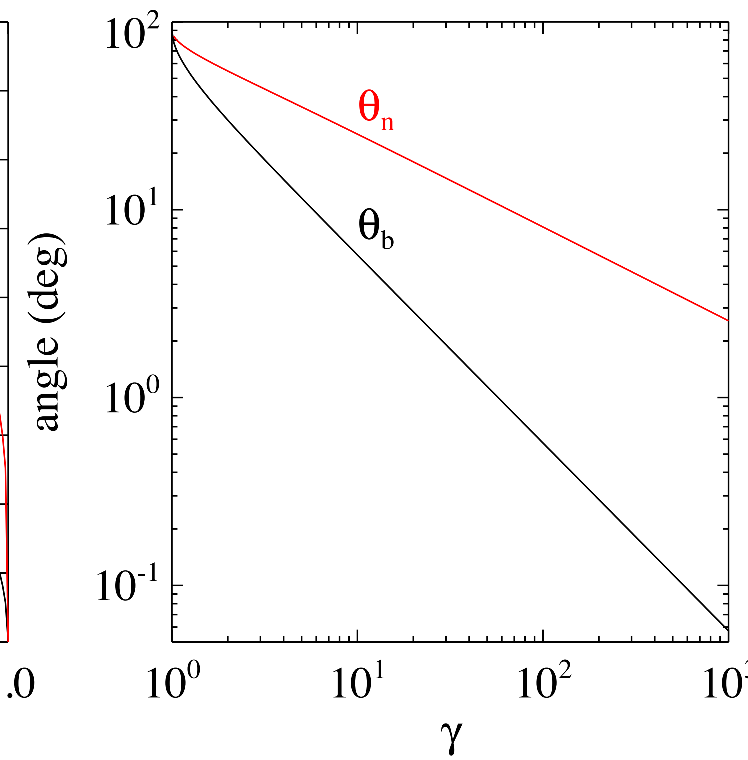
Relativistic Doppler effect can yield redshift even as a source approaches.

$$\cos\theta_n = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{-1/2} \approx 1 - \frac{1}{\gamma} \quad \text{for} \quad \gamma \gg 1$$

$$1 - \frac{\theta_n^2}{2} \approx 1 - \frac{1}{\gamma}$$

$$\boxed{\theta_n \approx \sqrt{\frac{2}{\gamma}} \approx \sqrt{2\theta_b}}$$





- Note $\theta_b < \theta_n$

Momentum and Energy

- Four-momentum of a particle with a mass m_0 is defined by

$$P^\mu = (P^0,\ \mathbf{p}) = (\gamma m_0 c,\ \gamma m_0 \mathbf{v})$$

- In the nonrelativistic limit, $P^0 c = m_0 c^2 \gamma \approx m_0 c^2 + \frac{1}{2} m_0 v^2 + \cdots$

  Therefore, we interpret $E = P^0 c = \gamma m_0 c^2$ as the total energy of the particle.

- Photons are massless, but we can still define

$$P^\mu = (E/c,\ \mathbf{p}), \quad E = |\mathbf{p}|c$$

- From Quantum relations: $E = h/nu = \hbar\omega$

$$p = E/c = \hbar k$$

  we can define four wavenumber vector: $\vec{k} \equiv \frac{1}{\hbar}\vec{P} = (\omega/c,\ \mathbf{k})$

  Note that it's a null vector:

$$\vec{k}\cdot\vec{k} = |\mathbf{k}|^2 - \omega^2/c^2 = 0$$

  From the property that scalar product of two four vectors is an invariant, we obtain an invariant:

$$\vec{k}\cdot\vec{x} = k_\mu x^\mu = \mathbf{k}\cdot\mathbf{x} - \omega t \qquad \boxed{\mathbf{k}'\cdot\mathbf{x}' - \omega' t' = \mathbf{k}\cdot\mathbf{x} - \omega t}$$

  Therefore, the phase of the plane wave is an invariant.

- **Lorentz invariant**: A quantity (scalar) that remains unchanged by a Lorentz transform is said to be a "Lorentz invariant."

Transformation of Electromagnetic Fields & Emission Power

- In general,

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \qquad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$$
$$\mathbf{E}_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \beta \times \mathbf{B}\right) \qquad \mathbf{B}_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \beta \times \mathbf{E}\right)$$

The concept of a pure electric or pure magnetic is not Lorentz invariant.

- Total emitted power:

Imagine **an instantaneous rest frame** *K'*, such that the particle has zero velocity at a certain time. We can then calculate the radiation emitted by use of the dipole (Larmor) formula.

$$dW = \gamma dW'$$

The time interval dt is simply $dt = \gamma dt'$

The total power emitted in frames *K* and *K'* are given by

$$P = \frac{dW}{dt}, \quad P' = \frac{dW'}{dt'}$$

Thus **the total emitted power is a Lorentz invariant** for any emitter that emits with front-back symmetry in its instantaneous rest frame.

$$\boxed{P = P'}$$

Larmor formula

- **The Larmor formula:**

Recall that $P' = \frac{2q^2}{3c^3}|\mathbf{a}'|^2$ in an instantaneous rest frame of the particle.

$$t = \gamma\left(t' + \frac{v}{c^2}x'_{\parallel}\right)$$

$$u_{\parallel}\left(\equiv \frac{dx_{\parallel}}{dt}\right) = \frac{u'_{\parallel} + v}{1 + vu'_{\parallel}/c^2}$$

$$u_{\perp}\left(\equiv \frac{dx_{\perp}}{dt}\right) = \frac{u'_{\perp}}{\gamma\left(1 + vu'_{\parallel}/c^2\right)}$$

$$\longrightarrow$$

$$a_{\parallel}\left(\equiv \frac{du_{\parallel}}{dt}\right) = \frac{1}{\gamma^3\sigma^3}a'_{\parallel}$$

$$a_{\perp}\left(\equiv \frac{du_{\perp}}{dt}\right) = \frac{1}{\gamma^2\sigma^3}\left(\sigma a'_{\perp} - \frac{vu'_{\perp}}{c^2}a_{\parallel}\right)$$

$$\text{where} \quad \sigma \equiv \left(1 + \frac{vu'_{\parallel}}{c^2}\right)$$

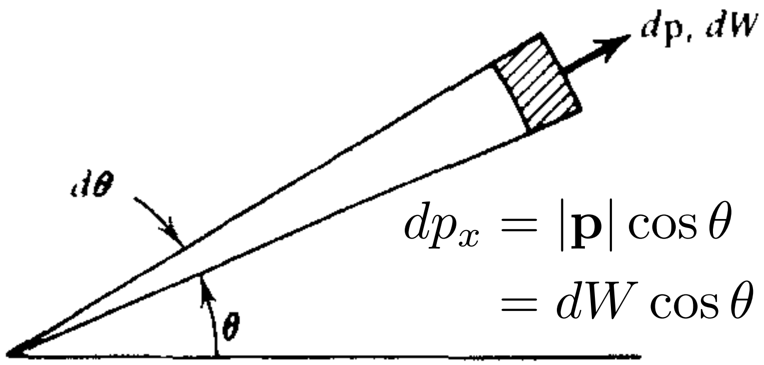
In an instantaneous rest frame of the particle.

$$u'_{\parallel} = u'_{\perp} = 0, \quad \sigma = 1 \longrightarrow \boxed{\begin{aligned} a'_{\parallel} &= \gamma^3 a_{\parallel} \\ a'_{\perp} &= \gamma^2 a_{\perp} \end{aligned}}$$

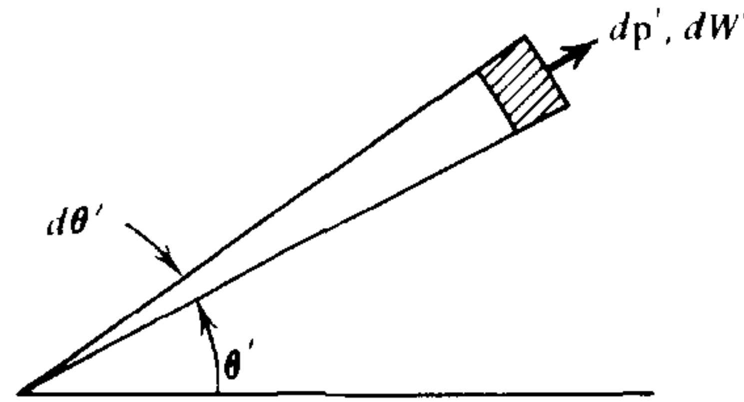
$$\boxed{\begin{aligned} P' &= \frac{2q^2}{3c^3}\left(a'^2_{\parallel} + a'^2_{\perp}\right) \\ P &= \frac{2q^2}{3c^3}\gamma^4\left(\gamma^2 a^2_{\parallel} + a^2_{\perp}\right) \end{aligned}}$$

Differential Power

- <u>Angular Distribution of Emitted and Received Power</u>



$$dp_x = |\mathbf{p}|\cos\theta = dW\cos\theta$$



Note:

$d\phi' = d\phi$

In the instantaneous rest frame of the particle, let us consider an amount of energy *dW'* that is emitted into the solid angle $d\Omega' = \sin\theta' d\theta' d\phi'$ (see the above figure).

$$\mu \equiv \cos\theta \quad \rightarrow \quad d\Omega = d\mu d\phi$$
$$\mu' \equiv \cos\theta' \quad \rightarrow \quad d\Omega' = d\mu' d\phi'$$

Recall the aberration formula:

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \quad \rightarrow \quad \mu = \frac{\mu' + \beta}{1 + \beta\mu'} \quad \rightarrow \quad \mu' = \frac{\mu - \beta}{1 - \beta\mu}$$

$$\boxed{\begin{array}{lcl} d\mu = \dfrac{d\mu'}{\gamma^2 (1 + \beta\mu')^2} & & d\Omega = \dfrac{d\Omega'}{\gamma^2 (1 + \beta\mu')^2} \\ & \longrightarrow & \\ d\mu' = \dfrac{d\mu}{\gamma^2 (1 - \beta\mu)^2} & & d\Omega' = \dfrac{d\Omega}{\gamma^2 (1 - \beta\mu)^2} \end{array}}$$

- <u>Power</u>

Recall that energy and momentum form a four-vector

$$\vec{P} = (E/c,\ \mathbf{p}) \quad \text{and} \quad |\mathbf{p}| = E/c \quad \longrightarrow \quad dW = \gamma(dW' + v dp'_x) = \gamma(1+\beta\mu')dW'$$

$$\therefore\ dW = \gamma(1+\beta\mu')dW'$$
$$dW' = \gamma(1-\beta\mu)dW$$

$$\boxed{\frac{dW}{d\Omega} = \gamma^3 (1+\beta\mu')^3 \frac{dW'}{d\Omega'}, \quad \frac{dW'}{d\Omega'} = \gamma^3 (1-\beta\mu)^3 \frac{dW}{d\Omega}}$$

In the rest frame, **the power emitted in a unit time interval** is $\dfrac{dP'}{d\Omega'} \equiv \dfrac{dW'}{dt' d\Omega'}$

However, in the observer's frame, there are two possible choices for the time interval to calculate the power.

(1) <u>$dt = \gamma dt'$:</u>

This is the time interval during which the emission occurs. With this choice we obtain **the emitted power**.

(2) <u>$dt_A = \gamma(1-\beta\mu)dt'$ or $dt_A = \gamma^{-1}(1+\beta\mu')^{-1}dt'$:</u>

This is the time interval of the radiation as received by a stationary observer in *K*. With this choice we obtain **the received power**.

- Thus we obtain the two results:

$$\boxed{\begin{aligned}\frac{dP_e}{d\Omega} &= \gamma^2\left(1+\beta\mu'\right)^3\frac{dP'}{d\Omega'} = \frac{1}{\gamma^4\left(1-\beta\mu\right)^3}\frac{dP'}{d\Omega'}\\ \frac{dP_r}{d\Omega} &= \gamma^4\left(1+\beta\mu'\right)^4\frac{dP'}{d\Omega'} = \frac{1}{\gamma^4\left(1-\beta\mu\right)^4}\frac{dP'}{d\Omega'}\end{aligned}}$$

P_r is the power actually measured by an observer. It has the expected symmetry property of yielding the inverse transformation by interchanging primed and unprimed variables, along with a change of sign of β.

P_e is used in the discussion of emission coefficient.

In practice, **the distinction between emitted and received power is often not important, since they are equal in an average sense for stationary distributions of particles.**

- Beaming effect:

If the radiation if isotropic in the particle's frame, then the angular distribution in the observer's frame will be highly peaked in the forward direction for highly relativistic velocities.

The factor $\gamma^{-4}\left(1-\beta\mu\right)^{-4}$ is sharply peaked near $\theta \approx 0$ with an angular scale of order $1/\gamma$.

$$\gamma^{-4}\left(1-\beta\mu\right)^{-4} \approx \gamma^{-4}\left[1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right)\right]^{-4} = \gamma^{-4}\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^{-4} = \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^{-4}$$

- Dipole emission from a slowly moving particle

$$\frac{dP'}{d\Omega'} = \frac{q^2 a'^2}{4\pi c^3} \sin^2 \Theta'$$

Θ' = the angle between the acceleration and the direction of emission.

Using $a'_{\parallel} = \gamma^3 a_{\parallel},\ a'_{\perp} = \gamma^2 a_{\perp}$ and $\dfrac{dP_r}{d\Omega} = \gamma^{-4}(1-\beta\mu)^{-4}\dfrac{dP'}{d\Omega'}$, we obtain

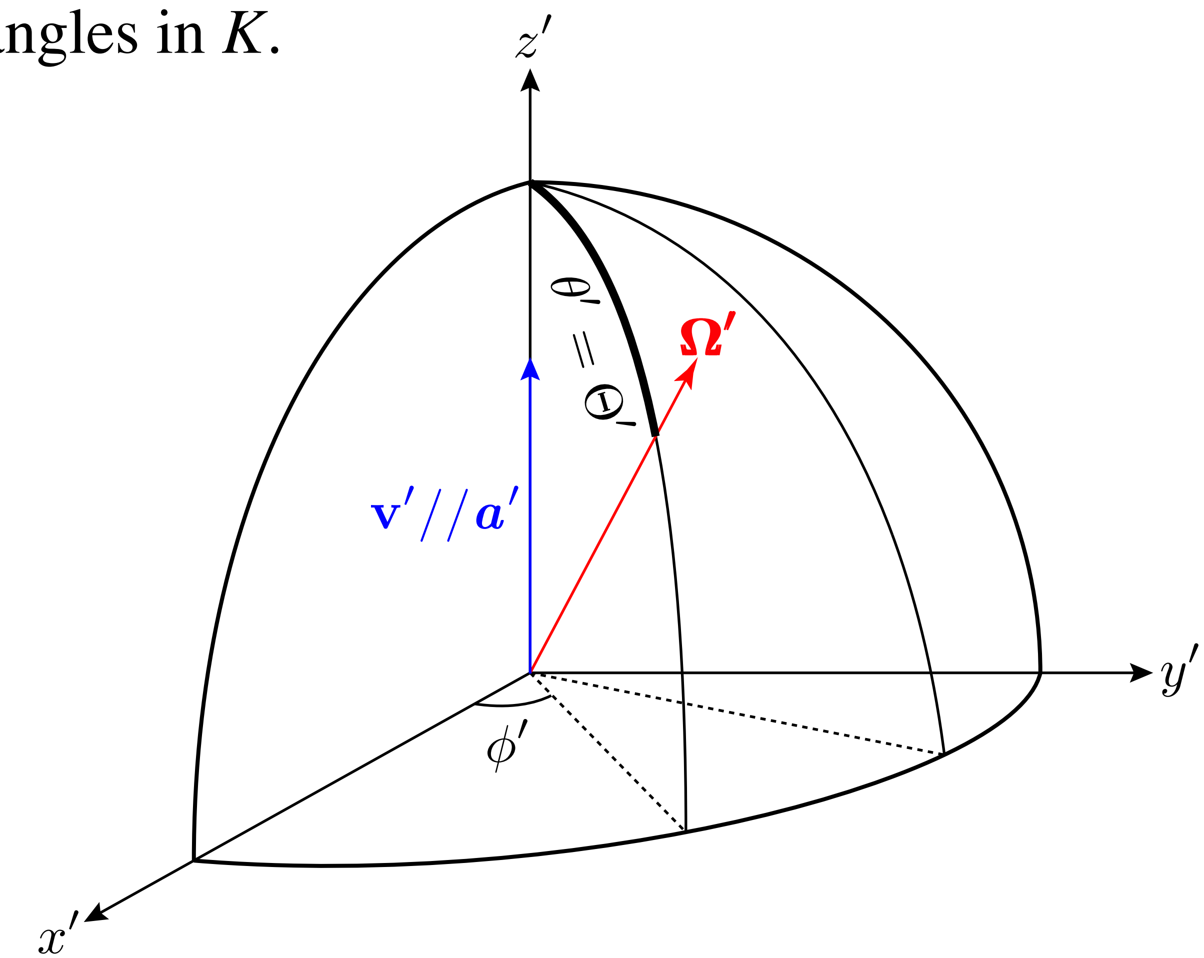
$$\boxed{\frac{dP_r}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{\left(\gamma^2 a_{\parallel}^2 + a_{\perp}^2\right)}{(1-\beta\mu)^4}\sin^2\Theta'}$$

To use this formula, we must relate Θ' to the angles in K.

(1) Acceleration parallel to velocity: $\Theta' = \theta',\ a_{\perp} = 0$

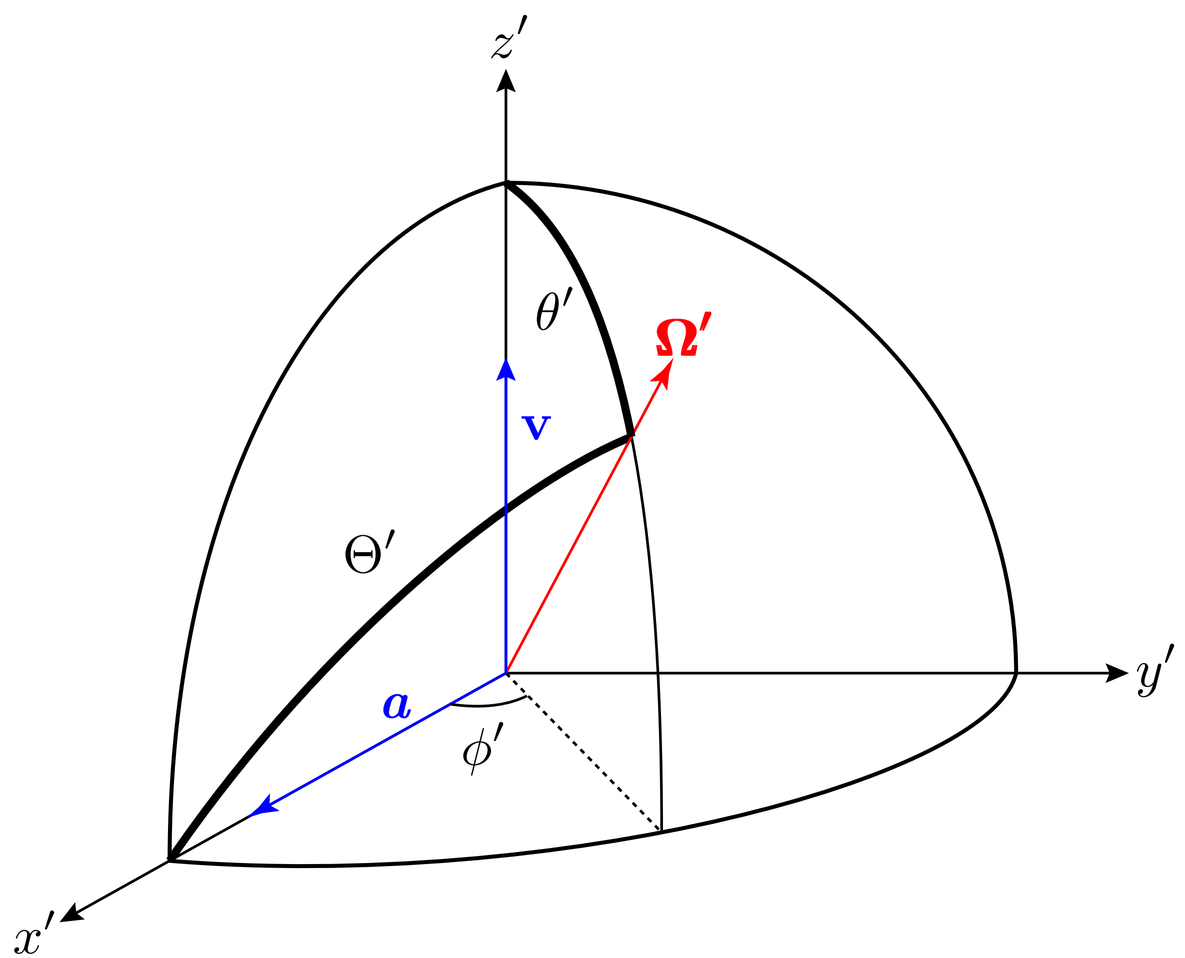
$$\sin^2\Theta' = 1 - \mu'^2 = 1 - \left(\frac{\mu-\beta}{1-\beta\mu}\right)^2 = \frac{1-\mu^2}{\gamma^2(1-\beta\mu)^2}$$

$$\longrightarrow \quad \frac{dP_{r\parallel}}{d\Omega} = \frac{q^2 a_{\parallel}^2}{4\pi c^3}\frac{1-\mu^2}{(1-\beta\mu)^6}$$



(2) Acceleration perpendicular to velocity: $\cos\Theta' = \sin\theta' \cos\phi'$ (when *a* is in *x*-direction in the figure)

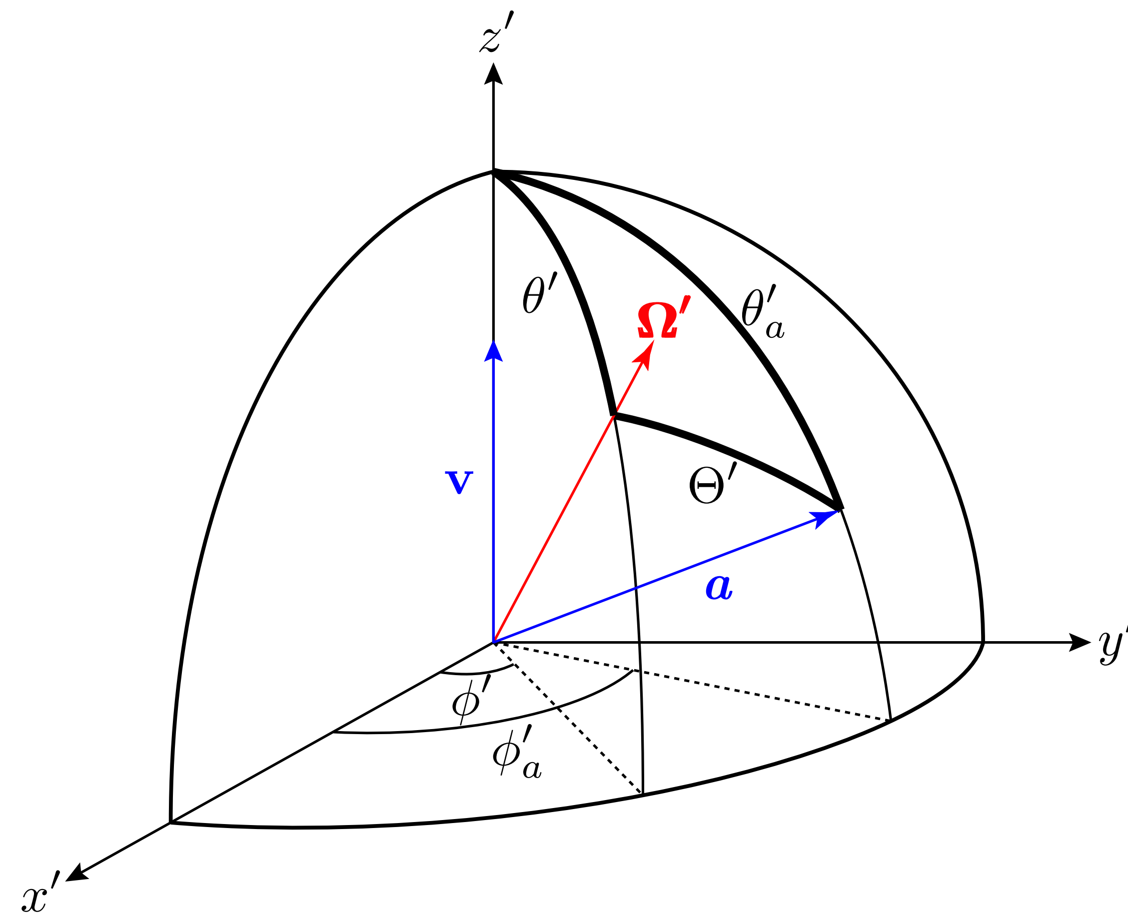
$$\sin^2\Theta' = 1 - \frac{\left(1-\mu^2\right)\cos^2\phi}{\gamma^2\left(1-\beta\mu\right)^2} \longrightarrow \frac{dP_{r\perp}}{d\Omega} = \frac{q^2 a_\perp^2}{4\pi c^3}\frac{1}{\left(1-\beta\mu\right)^4}\left[1 - \frac{\left(1-\mu^2\right)\cos^2\phi}{\gamma^2\left(1-\beta\mu\right)^2}\right]$$



(3) In general

$$\cos\Theta' = \mu'\mu'_a + \left(1 - \mu'^2\right)^{1/2}\left(1 - \mu_a'^2\right)^{1/2}\cos\left(\phi' - \phi'_a\right)$$

See Eq. (219) in Chadrasekhar (1960)

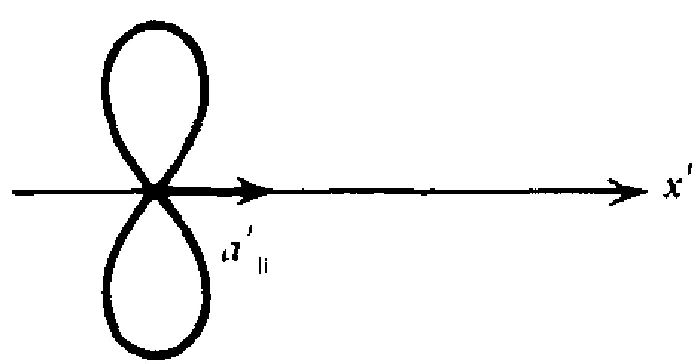


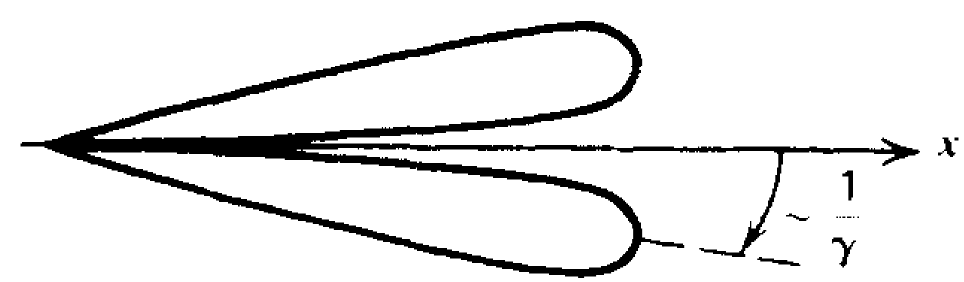
- In the extreme relativistic limit, the radiation becomes strongly peaked in the forward direction.

particle's rest frame:

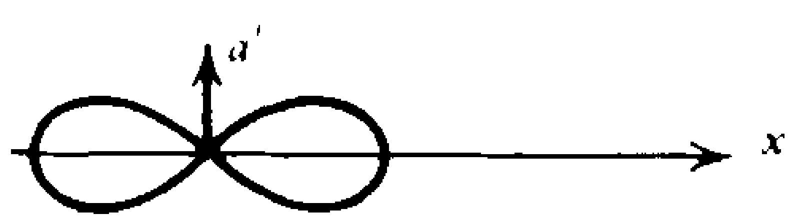
observer's frame:

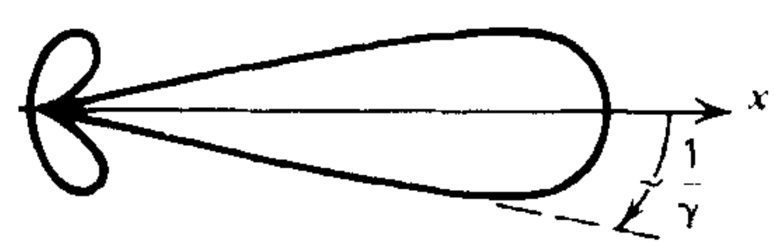
parallel acceleration:





perpendicular acceleration:





[Invariant Phase Volumes and Specific Intensity]

- **Phase volume**

Consider a group of particles that occupy a slight spread in position and in momentum at a particular time. In a rest frame comoving with the particles, they occupy a spatial volume element and a momentum volume element.

$$d^3\mathbf{x}' = dx'dy'dz'$$
$$d^3\mathbf{p}' = dp'_x dp'_y dp'_z$$

$\longrightarrow$ phase volume in the comoving frame:

$$d\mathcal{V}' \equiv d^3\mathbf{x}' d^3\mathbf{p}' = dx'dy'dz'dp'_x dp'_y dp'_z$$

In the observer's frame, $dx = \dfrac{1}{\gamma}dx'$ (length contraction), $dy = dy'$, $dz = dz'$

$$dp_x = \gamma\left(dp'_x + \beta dP'_0\right), \quad dp_y = dp'_y, \quad dp_z = dp'_z$$

We note that $dP'_0 = 0 + \mathcal{O}(dp'^2_x)$ because the velocities are near zero in the comoving frame and the energy is quadratic in velocity. Therefore, we have $dp_x = \gamma dp'_x$.

$$\boxed{d\mathcal{V}' \equiv d^3\mathbf{x}' d^3\mathbf{p}' = d^3\mathbf{x} d^3\mathbf{p} \equiv d\mathcal{V}}$$: Lorentz invariant

This contains no reference to particle mass, and therefore it has applicability to photons.

The phase space density $\boxed{f \equiv \dfrac{dN}{d\mathcal{V}} = \dfrac{dN}{d^3\mathbf{x} d^3\mathbf{p}}}$

is an invariant, since the number of particles within the phase volume element is a countable quantity and itself invariant.

- **Specific Intensity and Source Function**

Definition of the energy density per unit solid angle per frequency range.

$$h\nu \frac{dN}{d\mathcal{V}} p^2 dp d\Omega d^3\mathbf{x} = u_\nu(\Omega) d\Omega d\nu d^3\mathbf{x} \longrightarrow \begin{aligned} u_\nu(\Omega) &= h\nu \frac{dN}{d\mathcal{V}} p^2 \frac{dp}{d\nu} \qquad \leftarrow \quad p = \frac{h\nu}{c} \\ &= h\nu \left(\frac{h\nu}{c}\right)^2 \left(\frac{h}{c}\right) \frac{dN}{d\mathcal{V}} \end{aligned}$$

Since $u_\nu(\Omega) = I_\nu / c$, we find that $\boxed{\frac{I_\nu}{\nu^3} = \text{ Lorentz invariant}}$

Because the source function occurs in the transfer equation as the difference $I_\nu - S_\nu$, the source function must have the same transformation properties as the intensity.

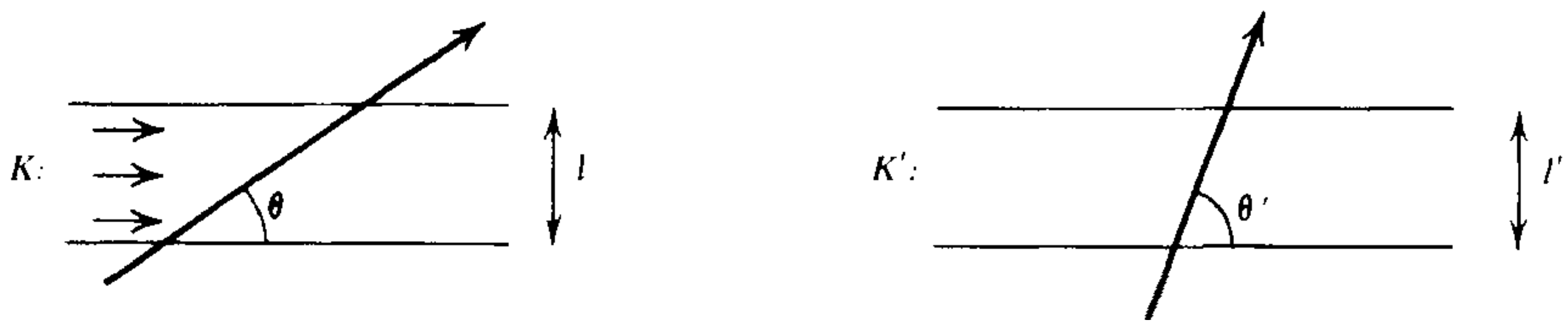
- **Optical Depth, Absorption Coefficient and Emission Coefficient**

The optical depth must be an invariant, since $\exp(-\tau)$ gives the fraction of photons passing through the material, and this involves simple counting.

$$\boxed{\tau = \text{ Lorentz invariant}}$$

- **Absorption Coefficient and Emission Coefficient**

Consider the optical depth in two frames:



Then, the optical depth is

$$\tau_\nu = \frac{\ell\alpha_\nu}{\sin\theta} = \frac{\ell}{\nu\sin\theta}\nu\alpha_\nu = \text{ Lorentz invariant}$$

Note that $\nu\sin\theta$ is proportional to the *y* component of the photon four-momentum $\vec{k} = (\omega/c,\ \mathbf{k})$

Both k_y and ℓ are the same in both frames, being perpendicular to the motion. Therefore, we have

$$\nu\sin\theta \propto k_y, \quad k_y = k'_y, \quad \ell' = \ell \qquad\qquad \boxed{\nu\alpha_\nu = \text{ Lorentz invariant}}$$

Finally, we obtain the transformation of the emission coefficient from the definition of the source function: $S_\nu \equiv \dfrac{j_\nu}{\alpha_\nu}$

$$\boxed{\frac{j_\nu}{\nu^2} = \text{ Lorentz invariant}}$$

Bremstrahlung

[Bremsstrahlung]

- **Bremsstrahlung (= “breaking radiation”) (or free-free emission)**: radiation due to the acceleration of a charge in the Coulomb field of another charge.

  Consider bremsstrahlung radiated from a plasma of temperature *T* and densities $n_e\ (\mathrm{cm}^{-3})$ electrons with charge -*e* and $n_i\ (\mathrm{cm}^{-3})$ ions with charge *Ze*.

  We calculate an important ratio:

$$
\begin{aligned}
\frac{\text{Coulomb potential energy}}{\text{thermal kinetic energy}} &\approx \frac{Ze^2/\langle r\rangle}{kT} \approx \frac{Ze^2/n_e^{1/3}}{kT} \\
&= 1.670 \times 10^{-7}\ Z \left(\frac{1\ \mathrm{cm}^{-3}}{n_e}\right)^{1/3} \frac{10^4\ K}{T} \\
&\ll 1
\end{aligned}
$$

  for typical $n_e < 1\ \mathrm{cm}^{-3}$ and $T \sim 10^4 - 10^8\ \mathrm{K}$.

  Therefore, **Coulomb interaction is only a perturbation on the thermal motions of the electrons.**

A full understanding of this process requires a quantum treatment. However, a classical treatment is justified in some regimes, and the formulas so obtained have the correct functional dependence for most of the physical parameters.

- **Bremsstrahlung due to the collision of identical particles (electron-electron, proton-proton) is zero in the dipole approximation (in absence of external forces)**, because the dipole moment is simply proportional to the center of mass (a constant of motion).

$$\sum e_i \mathbf{r}_i = e \sum \mathbf{r}_i \propto m \sum \mathbf{r}_i = \sum m_i \mathbf{r}_i$$

- Approximations:

(1) In electron-ion bremsstrahlung, we can treat the electron as moving in a fixed Coulomb field of the ion, since the relative accelerations are inversely proportional to the masses.

$$\frac{a_i}{a_e} = \frac{m_e}{m_i} \sim (1800)^{-1} < 10^{-3}$$

(2) A series of small-angle scatterings

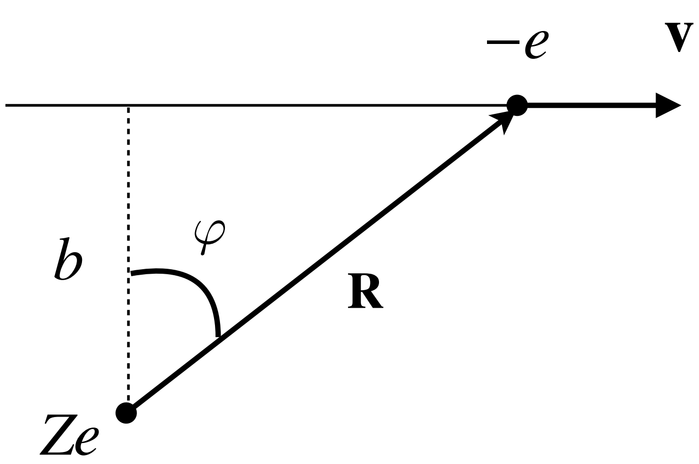
(3) Classical calculation => Quantum correction

(4) Non-relativistic => Relativistic

[Emission from single-speed Electrons]

- **Small-angle scattering** approximation:

The electron moves rapidly enough so that the deviation of its path from a straight line is negligible.



dipole moment: $\mathbf{d} = -e\mathbf{R}$

$\ddot{\mathbf{d}} = -e\dot{\mathbf{v}}$

Take Fourier transform of the second derivative of the dipole moment.

$$-\omega^2 \bar{\mathbf{b}}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\mathbf{v}} e^{i\omega t} dt \approx -\frac{e}{2\pi}\int_{-\tau}^{\tau} \dot{\mathbf{v}} e^{i\omega t} dt$$

Collision time: the electron would be in close interaction with the ion over a time interval.

$$\tau = \frac{b}{v}$$

For $\omega\tau \gg 1$, the exponential in the integral oscillates rapidly.

For $\omega\tau \ll 1$, the exponential is essentially unity, so we may write

$$\bar{\mathbf{d}}(\omega) \approx \begin{cases} \frac{e}{2\pi\omega^2}\Delta\mathbf{v} & \text{if } \omega\tau \ll 1 \\ 0 & \text{if } \omega\tau \gg 1 \end{cases}$$

where $\Delta\mathbf{v}$ is the change of velocity during the collision.

- **Spectrum of the emitted radiation by a single electron**:

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\left|\bar{d}(\omega)\right|^2 = \begin{cases} \dfrac{2e^2}{3\pi c^3}|\Delta\mathbf{v}|^2 & \text{if } \ \omega\tau \ll 1 \\ 0 & \text{if } \ \omega\tau \gg 1 \end{cases}$$

Let us now estimate $\Delta\mathbf{v}$. Since the path is almost linear, the change in velocity is predominantly normal to the path.

$$\left(F_\perp = F\cos\varphi, \quad F = Ze^2/R^2, \quad \cos\varphi = b/R, R = (b^2+v^2t^2)^{1/2}\right)$$

$$\begin{aligned}\Delta v \approx \Delta v_\perp &= \frac{1}{m_e}\int F_\perp dt \\ &= \frac{Ze^2}{m_e}\int_{-\infty}^{\infty}\frac{b}{(b^2+v^2t^2)^{3/2}}dt = \frac{2Ze^2}{m_e b v}\end{aligned} \longleftarrow \begin{aligned}\int_{-\infty}^{\infty}\frac{dx}{(b^2+x^2)^{3/2}} &= \left.\frac{x}{b^2(b^2+x^2)^{1/2}}\right|_{-\infty}^{\infty} \\ &= \frac{2}{b^2}\end{aligned}$$

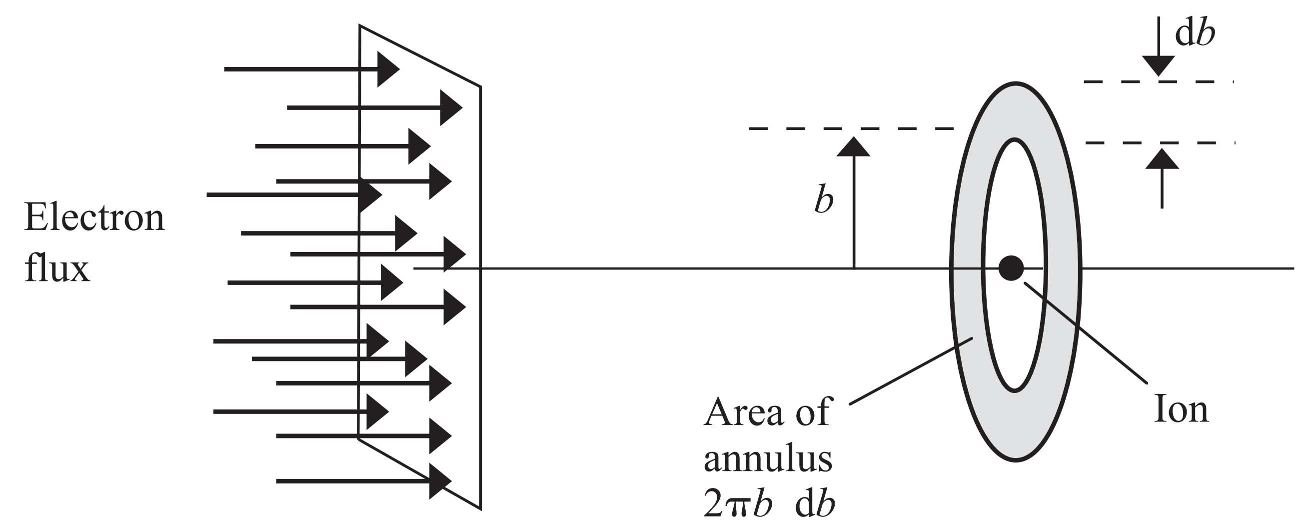
Thus for small angle scatterings, **the emission from a single collision** is

$$\frac{dW(b)}{d\omega} = \begin{cases} \dfrac{8Z^2e^6}{3\pi c^3 m_e^2 v^2 b^2} & \text{if } \ b \ll v/\omega \\ 0 & \text{if } \ b \gg v/\omega \end{cases}$$

- **Total spectrum for a medium** with ion density n_i, electron density n_e and for a fixed electron speed.

  - flux of electrons (per unit area per unit time) incident on one ion = $n_e v$
  - element of area = $2\pi b db$
  - a good approximation is obtained in low-frequency regimes:

$$\begin{aligned}\frac{dW}{d\omega dV dt} &= n_e n_i v \int_{b_{\min}}^{\infty} \frac{dW(b)}{d\omega} 2\pi b db \\ &= \frac{16e^6}{3c^3 m_e^2 v} n_e n_i Z^2 \int_{b_{\min}}^{b_{\max}} \frac{db}{b} \\ &= \frac{16e^6}{3c^3 m_e^2 v} n_e n_i Z^2 \ln\left(\frac{b_{\max}}{b_{\min}}\right)\end{aligned}$$



[Bradt (2008) Astrophysics Processes]

- Upper limit: $b_{\max} \sim v/\omega$

  The integral is negligible for $b \gg b_{\max} \sim v/\omega$

- Lower limits

  by the small-angle approximation: $\Delta v/v \sim (Ze^2/b)/(m_e v^2/2) < 1 \quad \rightarrow \quad b_{\min} > b_{\min}^{(1)} \equiv Ze^2/m_e v^2$

  by the uncertainty principle: $\Delta x \Delta p \geq \hbar \quad \rightarrow \quad b_{\min} > b_{\min}^{(2)} \equiv \hbar/mv$ (de Broglie wavelength)

  When $b_{\min}^{(1)} \gg b_{\min}^{(2)}$ or $\frac{1}{2}m_e v^2 \ll Z^2\mathrm{Ry}$ $\left(\mathrm{Ry} \equiv \frac{m_e c^4}{2\hbar^2} = 13.6\ \mathrm{eV} = \text{Rydberg energy for H atom}\right)$

  a classical description of the scattering process is valid. Then, $b_{\min} = b_{\min}^{(1)}$

  When $b_{\min}^{(1)} \ll b_{\min}^{(2)}$ or $\frac{1}{2}m_e v^2 \gg Z^2\mathrm{Ry}$

  the uncertainty principle plays an important role. Then, $b_{\min} = b_{\min}^{(2)}$

$$\therefore\ b_{\min} = \max\left(b_{\min}^{(1)}, b_{\min}^{(2)}\right)$$

- For any regime the exact results are conveniently stated in terms of correction factor or **Gaunt factor**. Precise expression of the Gaunt factor comes from QED (Quantum Electrodynamics) computation.

$$4\pi j_\omega(v,\omega) = \frac{dW}{d\omega dV dt} = \frac{16\pi e^6}{3\sqrt{3}c^3 m_e^2 v} n_e n_i Z^2 g_{\mathrm{ff}}(v,\omega) \qquad g_{\mathrm{ff}}(v,\omega) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{b_{\max}}{b_{\min}}\right)$$

Typically $g_{\mathrm{ff}} \approx 1$ to a few.

Tables and plots are available by Bressaard and van de Hulst (1962) and Karzas and Latter (1961).

[Thermal Bremsstrahlung Emission]

- We now average the above single-speed expression over a thermal distribution of electron speeds.

$$f(\mathbf{v})d^3\mathbf{v} = \left(\frac{m_e}{2\pi kT}\right)^{3/2} e^{-m_e v^2/2kT} d^3\mathbf{v} = \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} e^{-m_e v^2/2kT} v^2 dv$$

At frequency ν, the incident velocity must be at least such that $\frac{1}{2}m_e v^2 \geq h\nu$, because otherwise a photon of energy $h\nu$ could not be created.

This cutoff in the lower limit of the integration over electron velocities is called a ***photon discreteness effect***.

$$\frac{dW}{dV\,dt\,d\omega} = \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} \int_{v_{\rm min}}^{\infty} \frac{dW(v,\omega)}{dV\,dt\,d\omega} v^2 e^{-m_e v^2/2kT} dv \qquad \left(\text{where} \quad v_{\rm min} \equiv \sqrt{\frac{2h\nu}{m_e}}\right)$$

$$= \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} \frac{16\pi e^6}{3^{3/2}c^3 m_e^2} n_i n_e Z^2 \int_{v_{\rm min}}^{\infty} \frac{g_{\rm ff}(v,\omega)}{v} v^2 e^{-m_e v^2/2kT} dv$$

$$= \left(\frac{m_e}{kT}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} \frac{16\pi e^6}{3^{3/2}c^3 m_e^2} n_i n_e Z^2 \int_{v_{\rm min}}^{\infty} g_{\rm ff}(v,\omega) e^{-m_e v^2/2kT} d(v^2/2)$$

The exponential factor can be written as

$$\exp\left(-\frac{m_e v^2}{2kT}\right) = \exp\left(-\frac{m_e v_{\rm min}^2}{2kT}\right) \exp\left(-\frac{m_e (v^2 - v_{\rm min}^2)}{2kT}\right) = \exp\left(-\frac{h\nu}{kT}\right) \exp\left(-\frac{m_e (v^2 - v_{\rm min}^2)}{2kT}\right)$$

$$\frac{dW}{dV\,dt\,d\omega} = \left(\frac{m_e}{kT}\right)^{3/2}\left(\frac{2}{\pi}\right)^{1/2}\frac{16\pi e^6}{3^{3/2}c^3 m_e^2} n_i n_e Z^2 e^{-h\nu/kT}\left(\frac{m_e}{kT}\right)^{-1}\int_0^\infty g_{\rm ff}(v,\omega)e^{-u}du$$

$$\left(\text{where} \quad u \equiv \frac{m_e(v^2 - v_{\rm min}^2)}{2kT}\right)$$

In terms of $\nu = \omega/(2\pi)$, the volume emissivity is

$$\begin{aligned}
\varepsilon_\nu^{\rm ff} \equiv \frac{dW}{dV\,dt\,d\omega} &= 2\pi\left(\frac{m_e}{kT}\right)^{1/2}\left(\frac{2}{\pi}\right)^{1/2}\frac{16\pi e^6}{3^{3/2}c^3 m_e^2} n_i n_e Z^2 e^{-h\nu/kT}\int_0^\infty g_{\rm ff}(v,\omega)e^{-u}du \\
&= \left(\frac{2}{kT}\right)^{1/2}\frac{32\pi^{3/2}e^6}{3^{3/2}c^3 m_e^{3/2}} n_i n_e Z^2 e^{-h\nu/kT}\int_0^\infty g_{\rm ff}(v,\omega)e^{-u}du \\
&= \frac{2^5\pi e^6}{3m_e c^3}\left(\frac{2\pi}{3km_e}\right)^{1/2} n_i n_e Z^2 T^{-1/2} e^{-h\nu/kT}\overline{g_{\rm ff}}
\end{aligned}$$

$$\boxed{\varepsilon_\nu^{\rm ff} = 6.8\times 10^{-38} n_i n_e Z^2 T^{-1/2} e^{-h\nu/kT}\overline{g_{\rm ff}}}$$

$$\overline{g_{\rm ff}} \equiv \int_0^\infty g_{\rm ff}(v,\omega)e^{-u}du$$

$$g_{\rm ff}(v,\omega) = (\sqrt{3}/\pi)\ln(b_{\rm max}/b_{\rm min})$$

where $\overline{g_{\rm ff}}$ is the velocity-averaged free-free Gaunt factor.

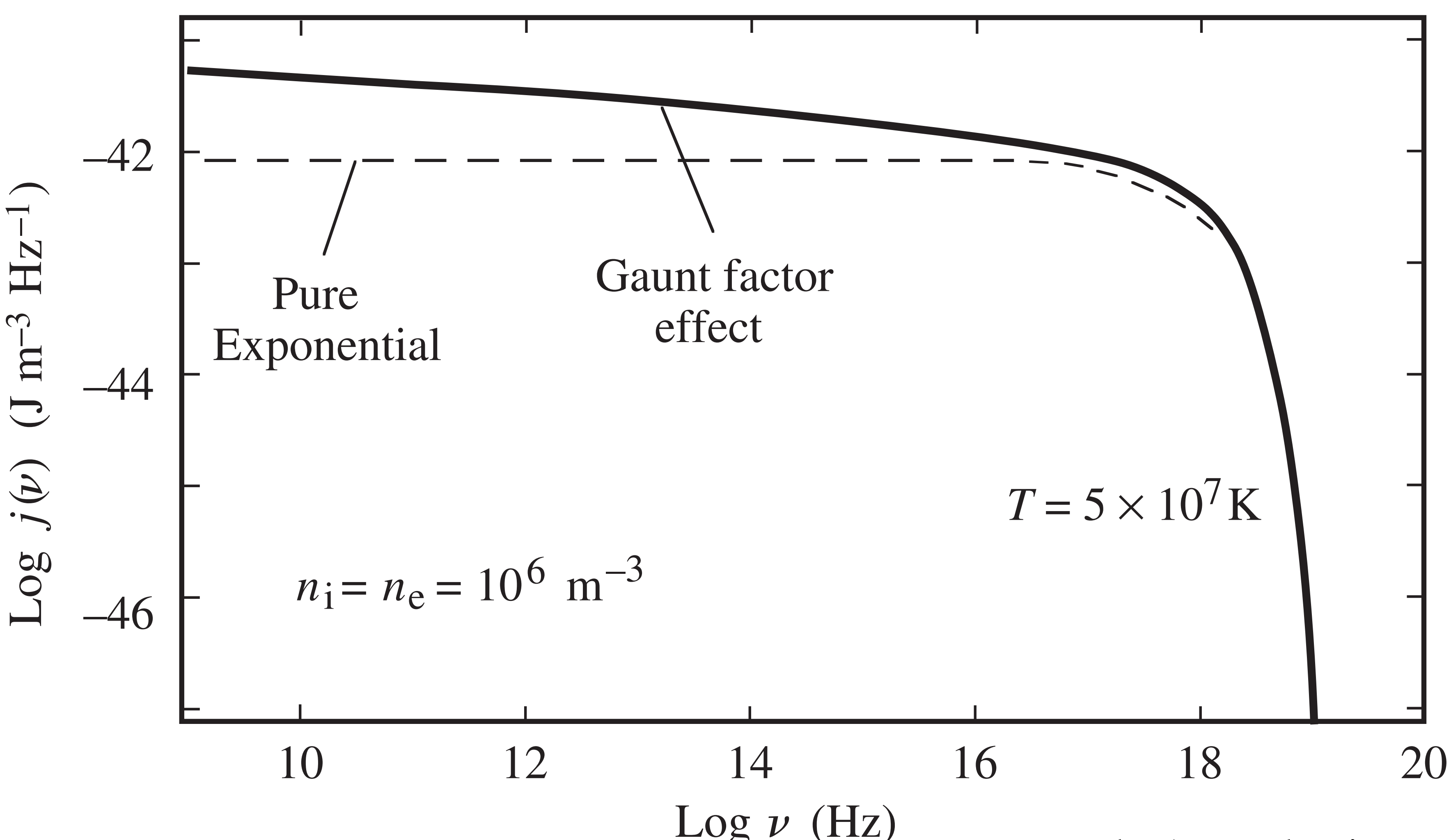
Summing over all ion species gives the emissivity:

$$\varepsilon_\nu^{\rm ff} = 6.8\times 10^{-38}\sum_i n_i n_e Z^2 T^{-1/2} e^{-h\nu/kT}\overline{g_{\rm ff}} \quad (\text{erg s}^{-1}\ \text{cm}^{-3}\ \text{Hz}^{-1})$$

Note that main frequency dependence is $\varepsilon_\nu^{\rm ff} \propto e^{-h\nu/kT}$, which shows a "flat spectrum" with a cut off at $\nu \sim kT/h$. The spectrum can be used to determine temperature of hot plasma.

For a hydrogen plasma ($Z = 1$) with $T > 3 \times 10^5$ K at low frequencies $(h\nu \ll kT)$ Gaunt factor is given by

$$\overline{g_{\rm ff}} = \frac{\sqrt{3}}{\pi} \ln\left(\frac{2.25kT}{h\nu}\right)$$



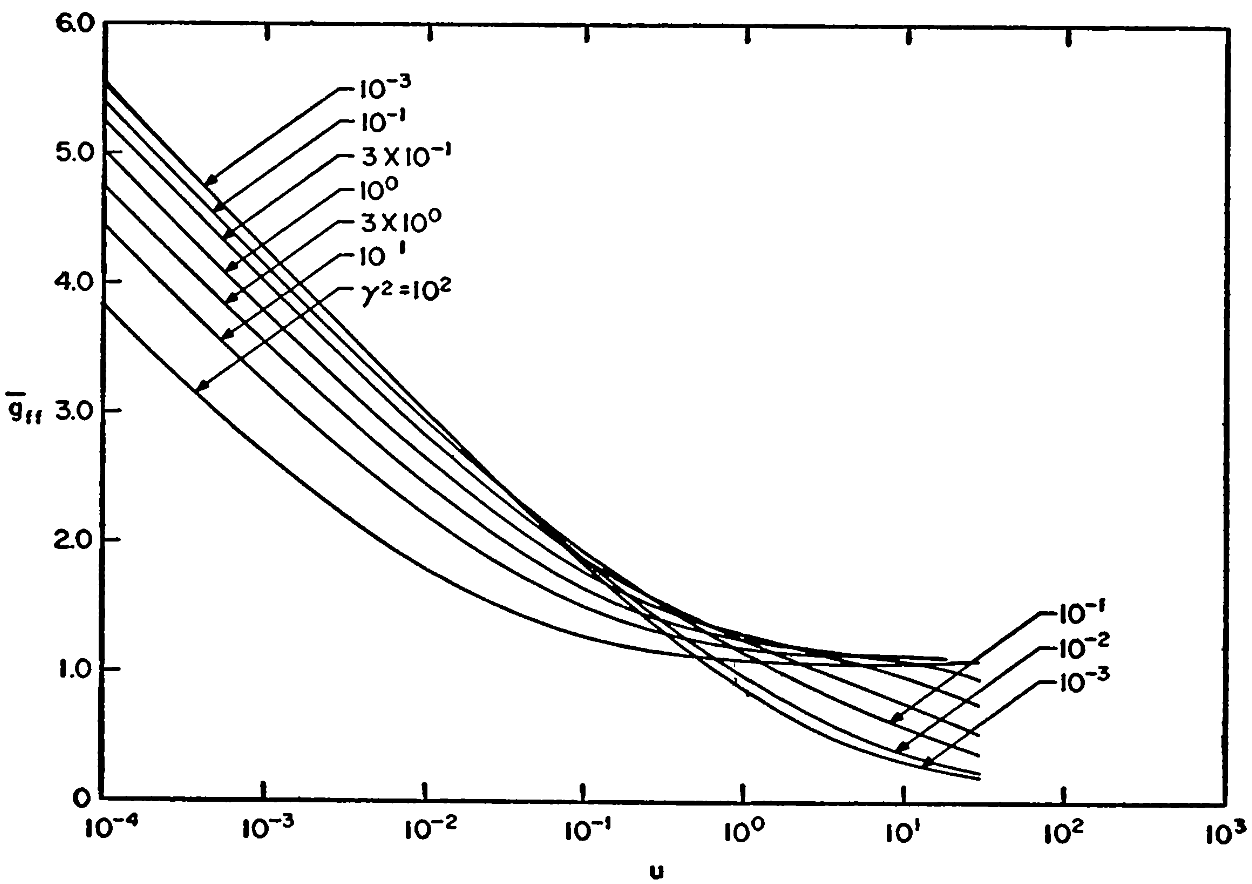
[Bradt, Astrophysics Processes]

- Gaunt Factor

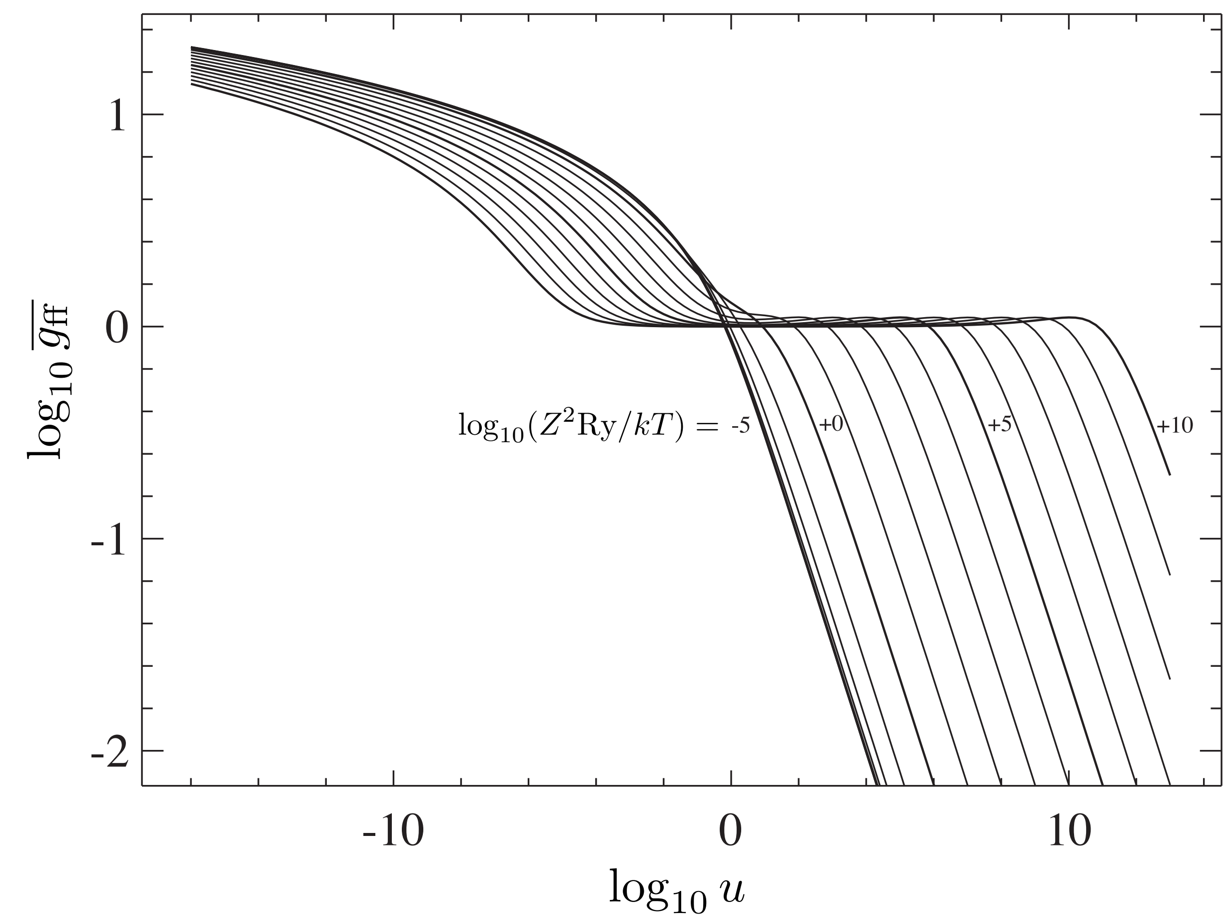
- Note that the values of Gaunt factor for $u = h\nu/kT \gg 1$ are not important, since the spectrum cuts off for these values.

$$\overline{g_{\rm ff}} \sim \begin{cases} 1 & \text{for } u \sim 1 \\ 1-5 & \text{for } 10^{-4} < u < 1 \end{cases}$$

Karzas & Latter (1961, ApJS, 6, 167)



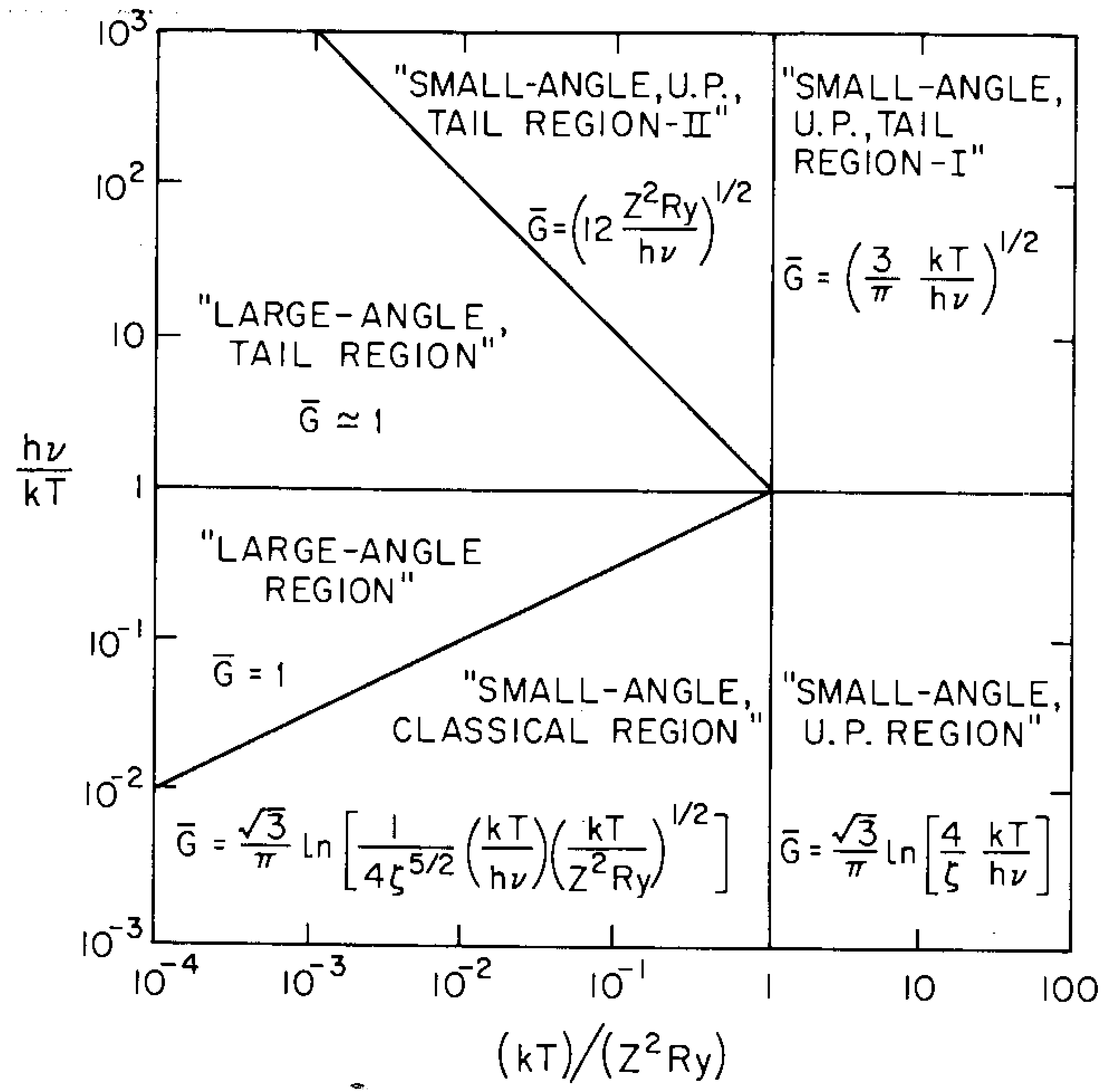
van Hoof et al. (2014, MNRAS, 444, 420)



$$u = h\nu/kT = 4.8\times10^{11}\nu/T$$

$$\gamma^2 = Z^2{\rm Ry}/kT = 1.58\times10^5 Z^2/T$$

- Novikov & Thorne (1973, in Black Holes, Les Houches)



U.P. = Uncertainty principle

- To obtain the formulas for non thermal bremsstrahlung, one needs to know the actual distributions of velocities, and the formula for emission from a single-speed electron must be averaged over that distribution. One also must have the appropriate Gaunt factors.
- Integrated Bremsstrahlung emission per unit volume:

$$
\begin{aligned}
\varepsilon^{\rm ff} \equiv \int \varepsilon^{\rm ff}(\nu)d\nu &= \frac{2^5\pi e^6}{3m_e c^3}\left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-1/2} n_i n_e Z^2 \int e^{-h\nu/kT}\overline{g_{\rm ff}}d\nu \\
&= \frac{2^5\pi e^6}{3m_e c^3}\left(\frac{2\pi}{3km_e}\right)^{1/2}\left(\frac{kT^{1/2}}{h}\right) n_i n_e Z^2 \int_0^\infty e^{-u}\overline{g_{\rm ff}}du \\
&= \left(\frac{2\pi kT}{3m_e}\right)^{1/2}\frac{2^5\pi e^6}{3hm_e c^3} n_i n_e Z^2 \overline{g_{\rm B}}
\end{aligned}
$$

$$
\varepsilon^{\rm ff}\left(\equiv \frac{dW}{dtdV}\right) = 1.42\times 10^{-27} n_i n_e Z^2 T^{1/2}\overline{g_{\rm B}}\ \text{erg cm}^{-3}\ \text{s}^{-1} \longrightarrow \varepsilon_{\rm ff} \propto T^{1/2}
$$

where frequency average of the velocity averaged Gaunt factor:

$$
\begin{aligned}
\overline{g_{\rm B}} &= \int_0^\infty e^{-u}\overline{g_{\rm ff}}du \quad (u = h\nu/kT) \\
&= 1.3 \pm 0.2
\end{aligned}
$$

$$
\overline{g_{\rm B}} \approx 1 + \frac{0.44}{1+0.058\left[\ln(T/10^{5.4}Z^2K)\right]^2}
$$

$$
\left(\text{for } 10^{4.2}\ \text{K} \le T/Z^2 \le 10^{8.2}\ \text{K},\ \text{Draine (2011)}\right)
$$

[Thermal Bremsstrahlung (free-free) Absorption]

- Absorption of radiation by free electrons moving in the field of ions:

For thermal system, Kirchoff's law says:

$$\frac{1}{4\pi}\frac{dW}{dV\,dt\,d\nu} = j_\nu^{\mathrm{ff}} = \alpha_\nu^{\mathrm{ff}} B_\nu(T) \qquad\qquad B_\nu(T) = \left(2h\nu^3/c^2\right)\left[\exp(h/nu/kT) - 1\right]^{-1}$$

We have then

$$\begin{aligned}\alpha_\nu^{\mathrm{ff}} &= \frac{4e^6}{3m_e hc}\left(\frac{2\pi}{3km_e}\right)^{1/2} n_i n_e Z^2 T^{-1/2}\nu^{-3}\left(1 - e^{-h\nu/kT}\right)\overline{g_{\mathrm{ff}}} \\ &= 3.7\times 10^8\ n_i n_e Z^2 T^{-1/2}\nu^{-3}\left(1 - e^{-h\nu/kT}\right)\overline{g_{\mathrm{ff}}}\ \ (\mathrm{cm}^{-1})\end{aligned}$$

For $h\nu \gg kT$, $\alpha_\nu^{\mathrm{ff}} = 3.7\times 10^8 n_i n_e Z^2 T^{-1/2}\nu^{-3}\overline{g_{\mathrm{ff}}}\ \ (\mathrm{cm}^{-1}) \longrightarrow \tau_\nu \propto \alpha_\nu^{\mathrm{ff}} \propto \nu^{-3}$

For $h\nu \ll kT$, $\alpha_\nu^{\mathrm{ff}} = \dfrac{4e^6}{3m_e kc}\left(\dfrac{2\pi}{3km_e}\right)^{1/2} n_i n_e Z^2 T^{-3/2}\nu^{-2}\overline{g_{\mathrm{ff}}} \longrightarrow \tau_\nu \propto \alpha_\nu^{\mathrm{ff}} \propto \nu^{-2}$

$$= 0.018 n_i n_e Z^2 T^{-3/2}\nu^{-2}\overline{g_{\mathrm{ff}}}$$

Bremsstrahlung self-absorption: The medium becomes always optically thick at sufficiently small frequency. Therefore, the free-free emission is absorbed inside plasma.

[Overall Spectral Shape]

- An approximate formula for the free-free Gaunt factor is given by Draine (2011).

$$\overline{g_{\rm ff}} \approx 6.155(Z\nu_9)^{-0.118}T_4^{0.177} \quad (0.14 < Z\nu_9/T_4^{3/2} < 250) \quad \text{where } \nu_9 = \nu/10^9 \text{ Hz}, \quad T_4 = T/10^4 \text{ K}$$

- Emission and absorption coefficients:

$$j_\nu = \frac{1}{4\pi}\varepsilon_\nu \approx 3.35\times 10^{-40} n_i n_e Z^{1.882} T_4^{-0.323} \nu_9^{-0.118} \text{ erg cm}^{-3}\text{ s}^{-1}\text{ Hz}^{-1}\text{ sr}^{-1}$$

$$\alpha_\nu = \frac{j_\nu}{B_\nu} \approx 3.37\times 10^{-7} n_i n_e Z^{1.882} T_4^{-1.323} \nu_9^{-2.118} \text{ pc}^{-1}$$

- Optical depth:

$$\tau_\nu = \int \alpha_\nu ds \approx 3.37\times 10^{-7} Z^{1.882} T_4^{-1.323}\nu_9^{-2.118}\left(\frac{n_i}{n_p}\right)\left[\frac{\rm EM}{\rm cm^{-6}pc}\right] \quad \text{where EM } \equiv \int n_e n_p ds$$

- SED (Spectral Energy Density) from a uniform sphere

$$\text{for } \tau_\nu \gg 1,\ h\nu \ll kT \longrightarrow I_\nu = S_\nu = B_\nu \qquad F_\nu = \pi B_\nu \left(\frac{R}{d}\right)^2 \propto \nu^2 \quad \text{(Rayleigh-Jeans Law)}$$

$$\text{for } \tau_\nu \ll 1,\ h\nu \ll kT \longrightarrow I_\nu = \int j_\nu ds \qquad F_\nu = 4\pi j_\nu \left(\frac{4\pi R^2}{3}\right)\frac{1}{4\pi d^2} \propto \nu^{-0.1}$$

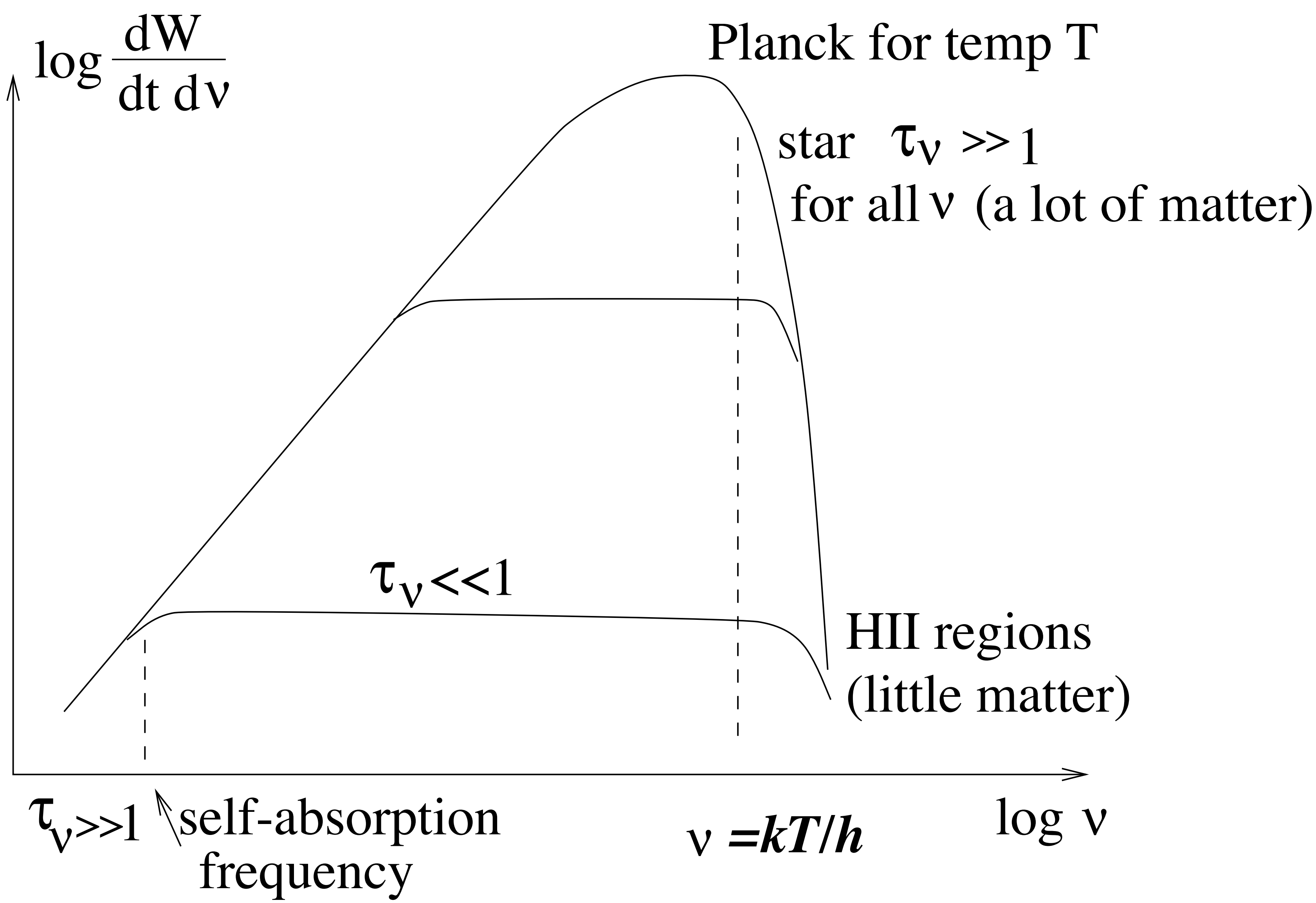


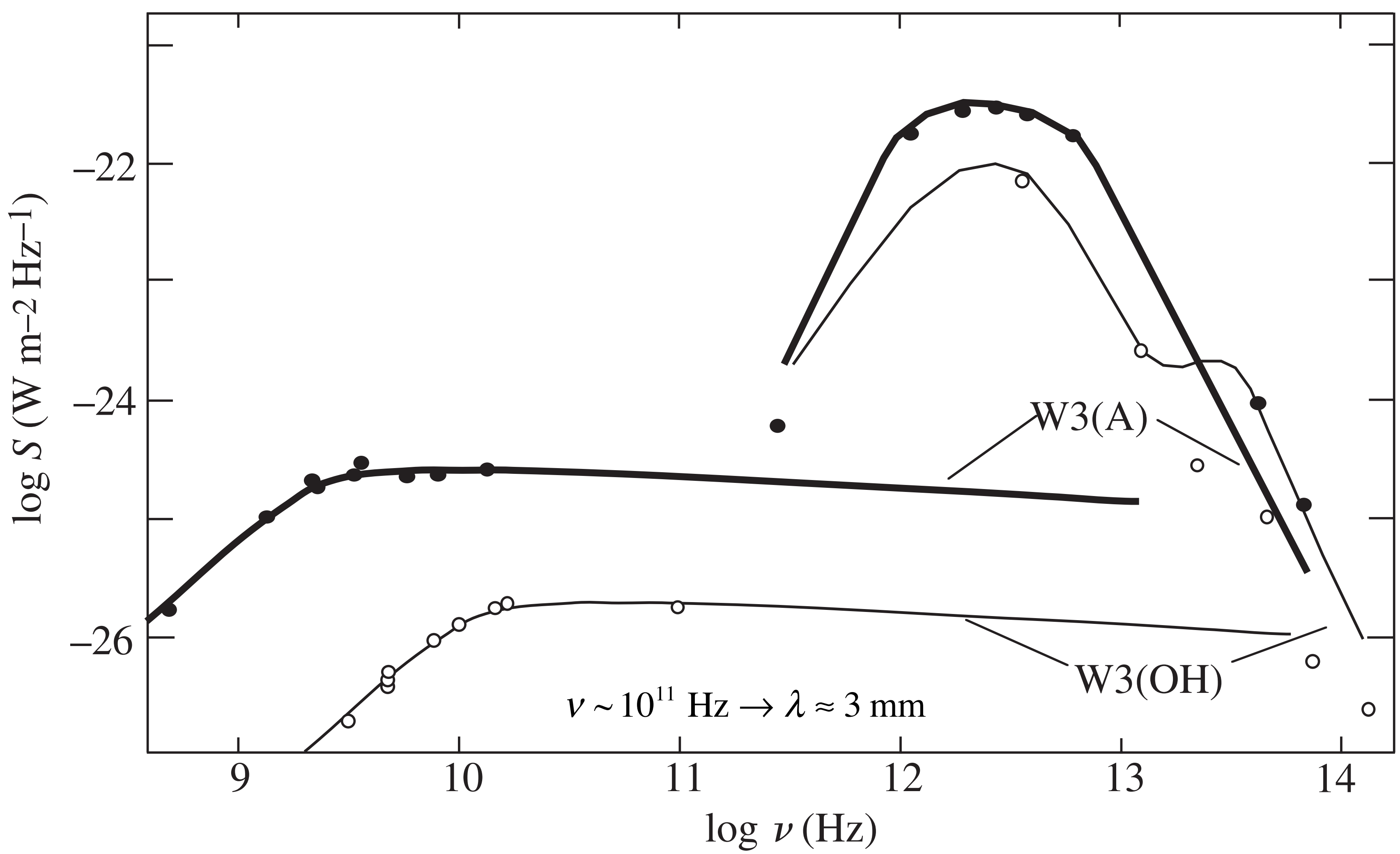
Figure from the Lecture Note of J. Poutanen

Astronomical Examples - H II regions

- The radio spectra of H II regions clearly show the flat spectrum of an optically thin thermal source. The bright stars in the H II regions emit copiously in the UV and thus ionize the hydrogen gas.
- Continuum spectra of two H II regions, W3(A) and W3(OH):

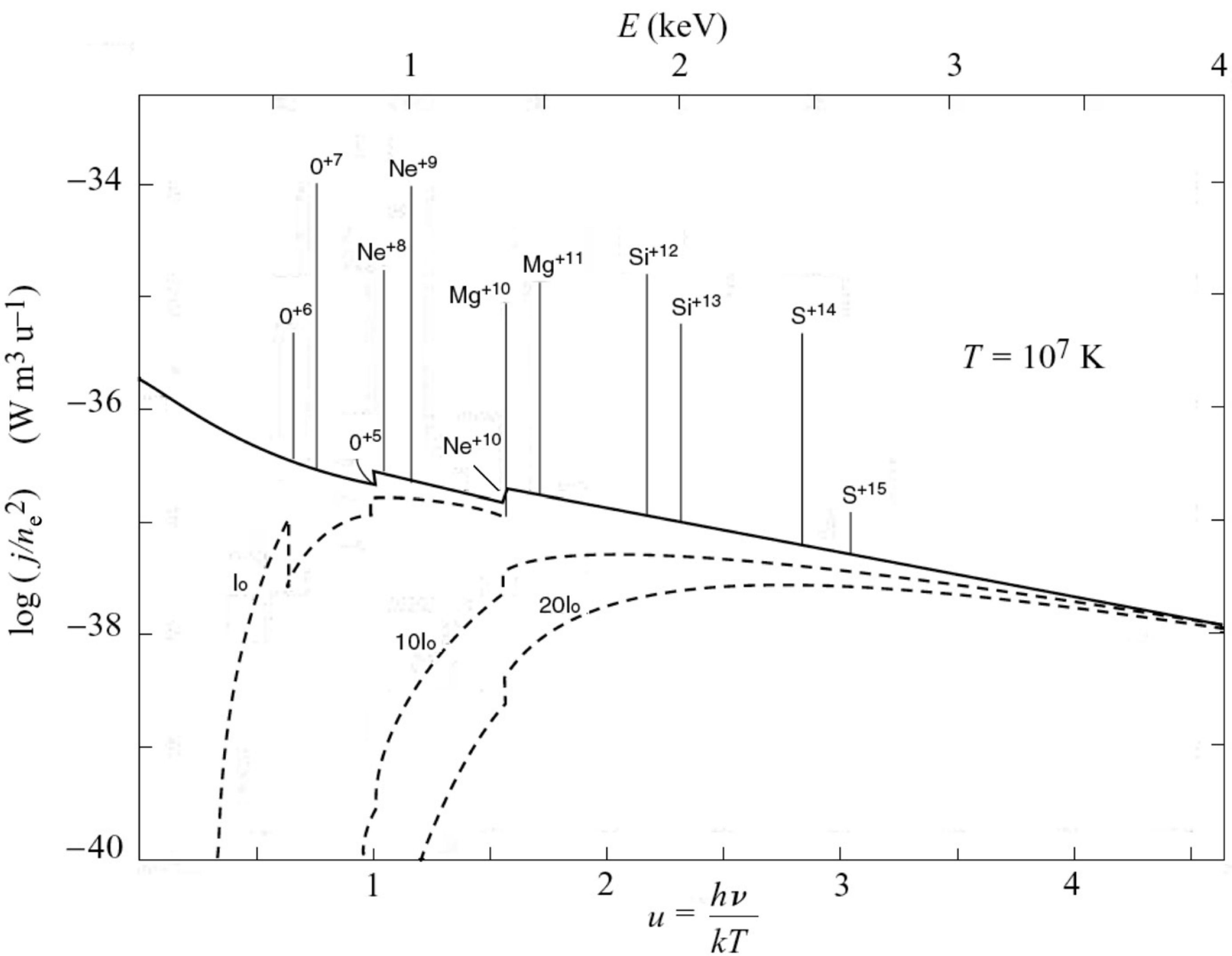
  Note a flat thermal bremsstrahlung (radio), a low-frequency cutoff (radio, self absorption), and a large peak at high frequency (infrared, $10^{12} - 10^{13}$ Hz) due to heated, but still "cold" dust grains in the nebula.

Figure from [Bradt, Astrophysics Processes]
Data from Kruegel & Mezger (1975, A&A, 42, 441)

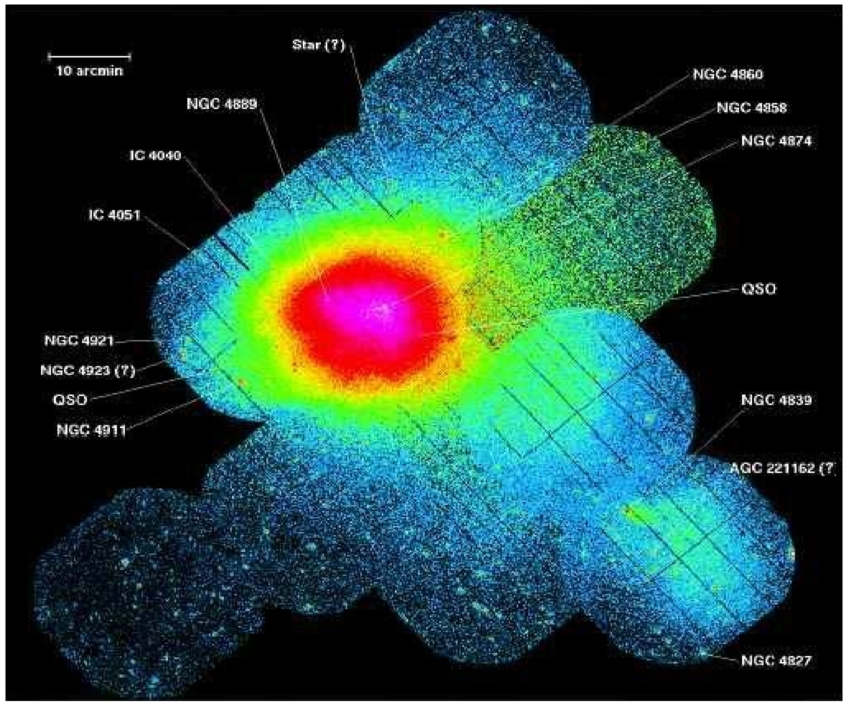


Astronomical Examples - X-ray emission

- Theoretical spectrum for a plasma of temperature 10^7 K that takes into account quantum effects. Comparison with real spectra from clusters of galaxies allows one to deduce the actual amounts of different elements and ionized species in the plasma as well as its temperature. It is only in the present millennium that X-ray spectra taken from satellites (e.g., Chandra and the XMM Newton satellite) have had sufficient resolution to distinguish these narrow lines The dashed lines show the effect of X-ray absorption by interstellar gas [Bradt, Astrophysics Processes].



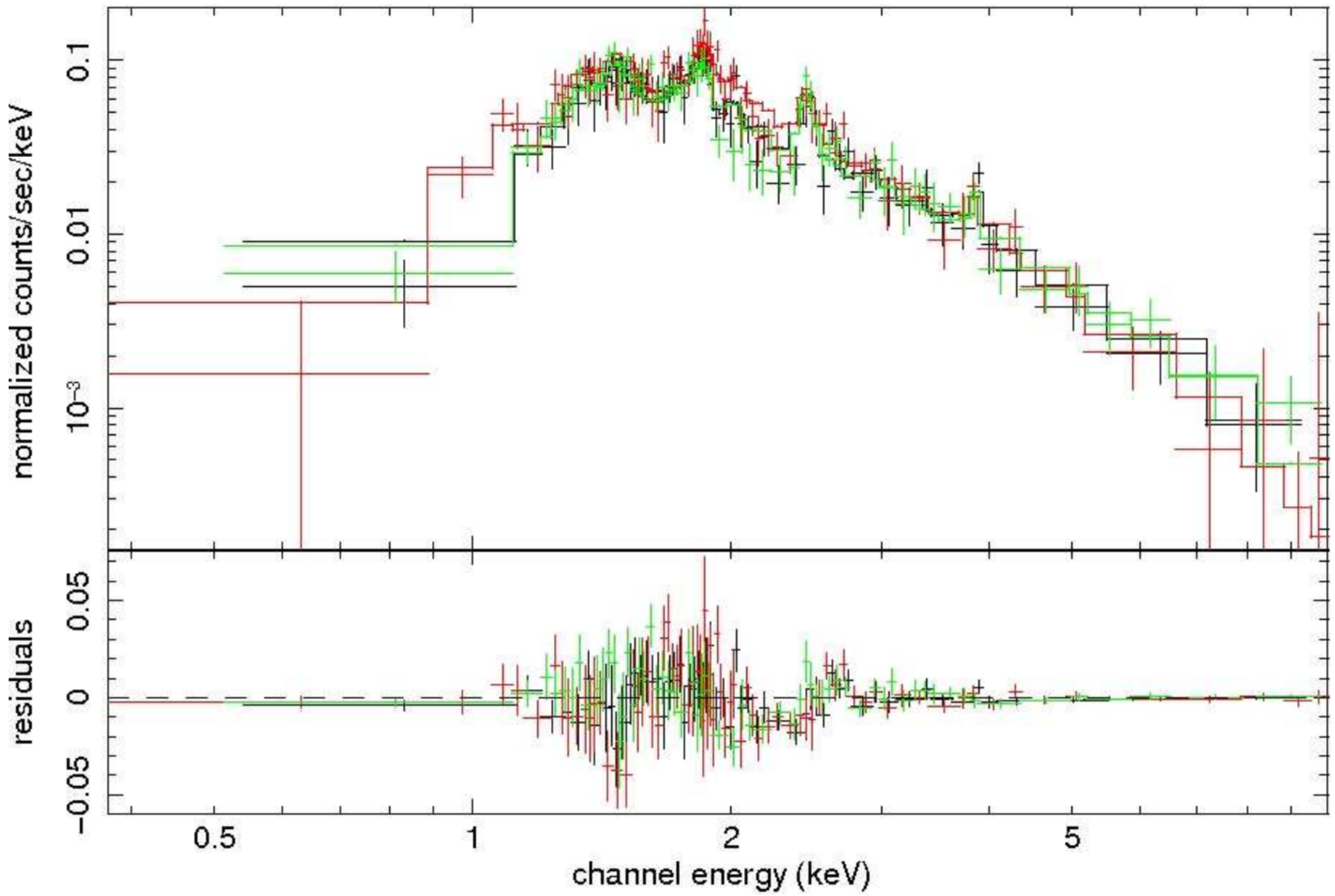
Coma cluster (z = 0.0232), size ~ 1 Mpc



Astronomical Examples - Supernova Remnants

- SNR G346.6-0.2

  X-ray spectra of the SNR from three of the four telescopes on-board Suzaku (represented by green, red and black). The underlying continuum is thermal bremsstrahlung, while the spectral features are due to elements such as Mg, S, Si, Ca and Fe. The roll over in the spectrum at low and high energies is due to a fall in the detector response, which is forward-modeled together with the spectrum.



Sezer et al. (2011, MNRAS, 415, 301)

Cooling function

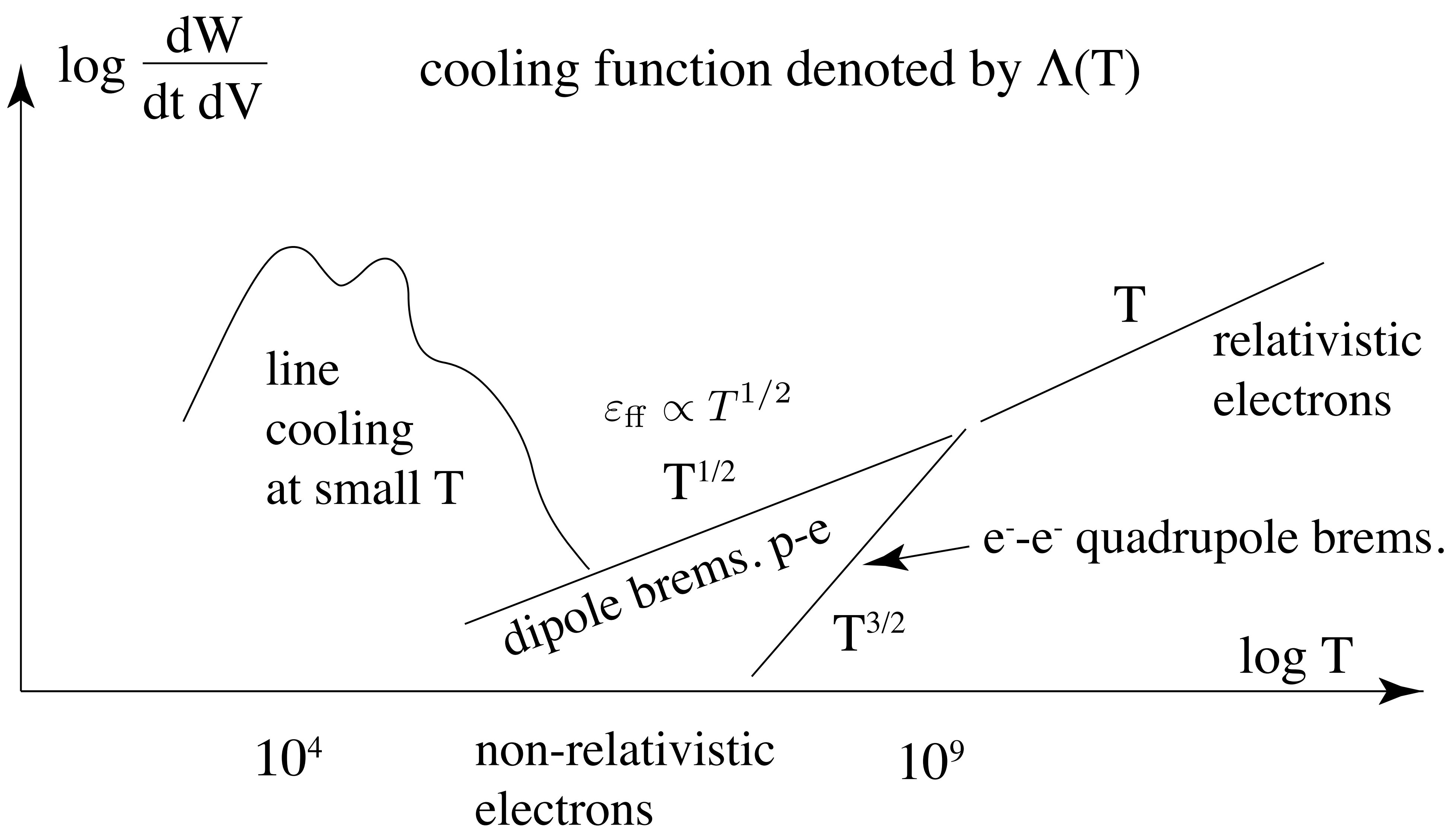


Figure from the Lecture Note of J. Poutanen

[Relativistic Bremsstrahlung]*

- Normally, the ions move rather slowly in comparison to the electrons.

  However, in a frame of reference in which electron is initially at rest, the ion appears to move rapidly toward the electron. **The electrostatic field of the ion appears to the electron to be a pulse of electromagnetic radiation. This radiation then Compton (or Thompson) scatters off the electron to produce emitted radiation.** Transforming back to the rest frame of the ion (or lab frame) we obtain the bremsstrahlung emission of the electron. **Relativistic bremsstrahlung can be regarded as the Compton scattering of the virtual quanta of the ion's electrostatic field as seen in the electron's frame.**

- In the (primed) electron rest frame, the spectrum of the pulse of the virtual quanta:

$$\frac{dW'}{dA'd\omega'} = \frac{q^2}{\pi^2 b'^2 v^2}\left(\frac{b'\omega'}{\gamma v}\right)^2 K_1^2\left(\frac{b'\omega'}{\gamma v}\right) = \frac{(Ze)^2}{\pi^2 b'^2 v^2}\left(\frac{b'\omega'}{\gamma v}\right)^2 K_1^2\left(\frac{b'\omega'}{\gamma c}\right) \quad \leftarrow \; v \approx c$$

(in the ultrarelativistic limit)

In the low-frequency limit, the scattered radiation is

$$\frac{dW}{d\omega} = \sigma_T \frac{dW'}{dA'd\omega'} \qquad \left(\sigma_T = \frac{2\pi}{3}\frac{e^4}{m_e^2 c^4}\right)$$

Transverse lengths are unchanged, $b = b'$, and $\omega = \gamma\omega'(1+\beta\cos\theta')$. The scattering is forward-backward symmetric, we therefore have the averaged relation $\omega = \gamma\omega'$.

- For a plasma with a single-speeds

$$
\begin{aligned}
\frac{dW}{dV\,dt\,d\omega} &= n_e n_i v \int_{b_{\min}}^{\infty} \frac{dW(b)}{d\omega} 2\pi b db \\
&= \frac{16 Z^2 e^6}{3 c^3 m_e^2} n_e n_i \int_{b_{\min}}^{b_{\max}} \left(\frac{b\omega}{\gamma^2 c}\right) K_1 \left(\frac{b\omega}{\gamma^2 c}\right) db \\
&= \frac{16 Z^2 e^6}{3 c^3 m_e^2} n_e n_i \ln\left(\frac{0.68\gamma^2 c}{\omega b_{\min}}\right)
\end{aligned}
$$

- For a Maxwell distribution of electrons, a useful approximate expression for the frequency integrated power is given by Novikov & Thorne (1973).

$$
\varepsilon_\nu^{\mathrm{ff}} = 1.4 \times 10^{-27} n_i n_e Z^2 T^{1/2} \overline{g_{\mathrm{B}}} \left(1 + 4.4 \times 10^{-10} T\right) \quad (\mathrm{erg\ s^{-1}\ cm^{-3}})
$$

See also Itoh et al. (2000, ApJS, 128, 125), Zekovic (2013, arXiv:1310.5639v1)

- At higher frequencies Klein-Nishina corrections must be used.

Synchrotron Radiation

[Synchrotron Radiation]

- Particles accelerated by a magnetic field will radiate.

- **Cyclotron radiation**: For nonrelativistic velocities, the radiation is called cyclotron radiation. The frequency of emission is simply the frequency of gyration in the magnetic field.

- **Synchrotron radiation**: For extreme relativistic particles, the frequency spectrum is much more complex and can extend to many times the gyration frequency. This radiation is known as synchrotron radiation.

[Total Emitted Power]

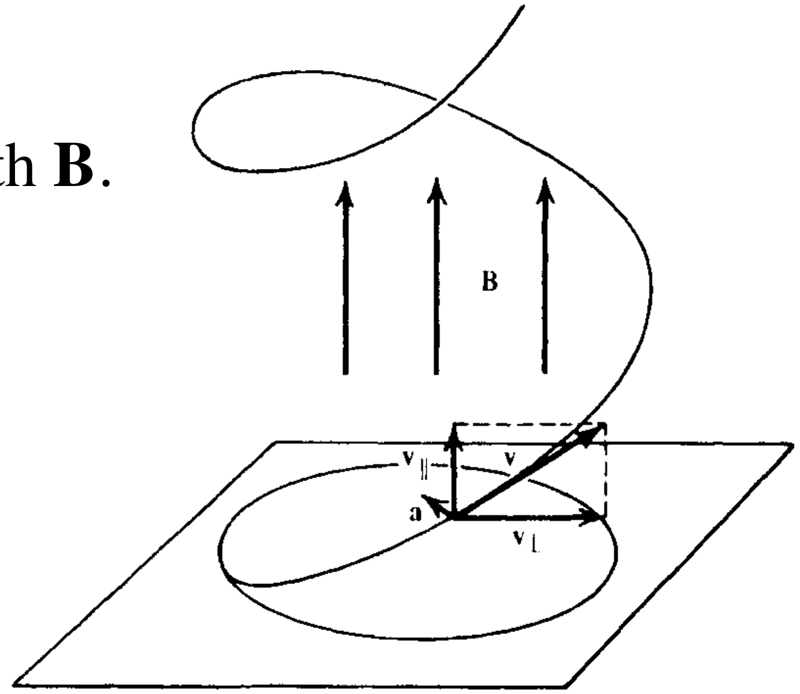
- Consider a particle of mass m and charge q moving in **a uniform magnetic field, with no electric field.**
- Equations of motion:

$$\frac{dE}{dt} = \frac{d(\gamma mc^2)}{dt} = q\mathbf{v}\cdot\mathbf{E} = 0$$
$$\frac{d\mathbf{p}}{dt} = \frac{d(\gamma m\mathbf{v})}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}$$

The first equation implies that $\underline{\gamma = \text{constant (or equivalently } |\mathbf{v}| = \text{constant)}}$. Therefore, it follows that

$$\gamma m\frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}$$

Decompose the velocity into $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$, and take dot product with $\mathbf{B}$.



$$\mathbf{B}\cdot\left(\gamma m\frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}\right) \quad\rightarrow\quad \begin{cases} \dfrac{d\mathbf{v}_{\parallel}}{dt} = 0 \\[2ex] \dfrac{d\mathbf{v}_{\perp}}{dt} = \dfrac{q}{\gamma mc}\mathbf{v}_{\perp}\times\mathbf{B} \end{cases}$$

Therefore,

$$\mathbf{v}_{\parallel} = \text{constant}$$
$$|\mathbf{v}_{\perp}| = \text{constant (since } |\mathbf{v}| = \text{constant)}$$

Helical motion: The perpendicular velocity component processes around $\mathbf{B}$. Therefore, the motion is a combination of the uniform circular motion and the uniform motion along the field.

Equation of Motion

- Equation of motion of an electron in a uniform magnetic field:

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \longrightarrow \frac{d\mathbf{v}_{\parallel}}{dt} = 0$$

$$\frac{d\mathbf{v}_{\perp}}{dt} = \frac{-e}{\gamma m_e c}\mathbf{v}_{\perp} \times \mathbf{B} = -\frac{\omega_{\mathrm{B}}}{B}\mathbf{v}_{\perp} \times \mathbf{B} \qquad \left(\text{Here, } \omega_{\mathrm{B}} \equiv \frac{eB}{\gamma m_e c}\right)$$

$$\longrightarrow \frac{d^2\mathbf{v}_{\perp}}{dt^2} = -\omega_{\mathrm{B}}\frac{d\mathbf{v}_{\perp}}{dt} \times \mathbf{B} = \omega_{\mathrm{B}}^2(\mathbf{v}_{\perp} \times \mathbf{B}) \times \mathbf{B} = \omega_{\mathrm{B}}^2\left[-\mathbf{v}_{\perp}(\mathbf{B}\cdot\mathbf{B}) + \mathbf{B}(\mathbf{B}\cdot\mathbf{v}_{\perp})\right]$$

$$\therefore \frac{d^2\mathbf{v}_{\perp}}{dt^2} = -\omega_{\mathrm{B}}^2\mathbf{v}_{\perp}$$

- Solution:

$$\mathbf{v}(t) = v_{\perp}\left(-\hat{\mathbf{x}}\sin\omega_{\mathrm{B}}t + \hat{\mathbf{y}}\cos\omega_{\mathrm{B}}t\right) + \hat{\mathbf{z}}v_{\parallel} \qquad \text{where } v_{\parallel} = v\cos\alpha, \quad v_{\perp} = v\sin\alpha$$

$$\mathbf{r}(t) = \frac{v_{\perp}}{\omega_{\mathrm{B}}}\left(\hat{\mathbf{x}}\cos\omega_{\mathrm{B}}t + \hat{\mathbf{y}}\sin\omega_{\mathrm{B}}t\right) + \hat{\mathbf{z}}v_{\parallel}t \qquad (\text{assuming } \mathbf{v}_{\perp}(t=0) \parallel \mathbf{y} \text{ and } \mathbf{x}_{\perp}(t=0) \parallel \mathbf{x})$$

Larmor frequency:
(Cychrotron frequency,
non-relativistic gyrofrequency)

$$\omega_{\mathrm{L}} = \frac{eB}{m_e c}$$

gyrofrequency:

$$\omega_{\mathrm{B}} = \frac{eB}{\gamma m_e c} = \frac{\omega_{\mathrm{L}}}{\gamma} = \frac{17.6}{\gamma}\frac{B}{\mu\mathrm{G}}\ (\mathrm{Hz})$$

gyroradius:

$$r_{\mathrm{B}} = \frac{v_{\perp}}{\omega_B} = 1.7 \times 10^9 \gamma\beta_{\perp}\left(\frac{B}{\mu\mathrm{G}}\right)\ (\mathrm{cm})$$

- Total emitted power:

Since $a_\perp = \omega_B v_\perp$, and $a_\parallel = 0$,

$$P = \frac{2q^2}{3c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right)$$
$$= \frac{2}{3}\gamma^2\frac{q^4B^2}{m^2c^5}v_\perp^2 = \frac{2}{3}r_e^2 c\beta^2\gamma^2 B^2\sin^2\alpha \qquad \leftarrow (v_\perp = v\sin\alpha)$$
$$= 2\sigma_T c(\gamma\beta)^2 U_B \sin^2\alpha$$

where α is the pitch angle, the angle between magnetic field and velocity.

$$\cos\alpha \equiv \frac{\mathbf{v}\cdot\mathbf{B}}{vB},\ r_e \equiv \frac{e^2}{mc^2},\ \sigma_T = \frac{8\pi}{3}r_e^2,\ U_B = \frac{B^2}{8\pi}$$

For an isotropic distribution of velocities, it is necessary to average the formula over all angles.

$$\boxed{P = \frac{4}{3}\sigma_T c(\gamma^2 - 1)U_B} \quad \leftarrow \quad \langle\sin^2\alpha\rangle = \frac{1}{4\pi}\int\sin^2\alpha d\Omega = \frac{2}{3}$$
$$\beta^2 = 1 - \frac{1}{\gamma^2}$$

Note that $P \propto \begin{cases} m^{-2} & \text{for a fixed velocity } (\gamma) \\ m^{-4} & \text{for a fixed energy } (\gamma mc^2) \end{cases} \longrightarrow$ This indicates that the synchrotron radiation is mostly due to electrons.

- Cooling Time

- The energy balance equation becomes:

$$mc^2 \frac{d\gamma}{dt} = -P$$

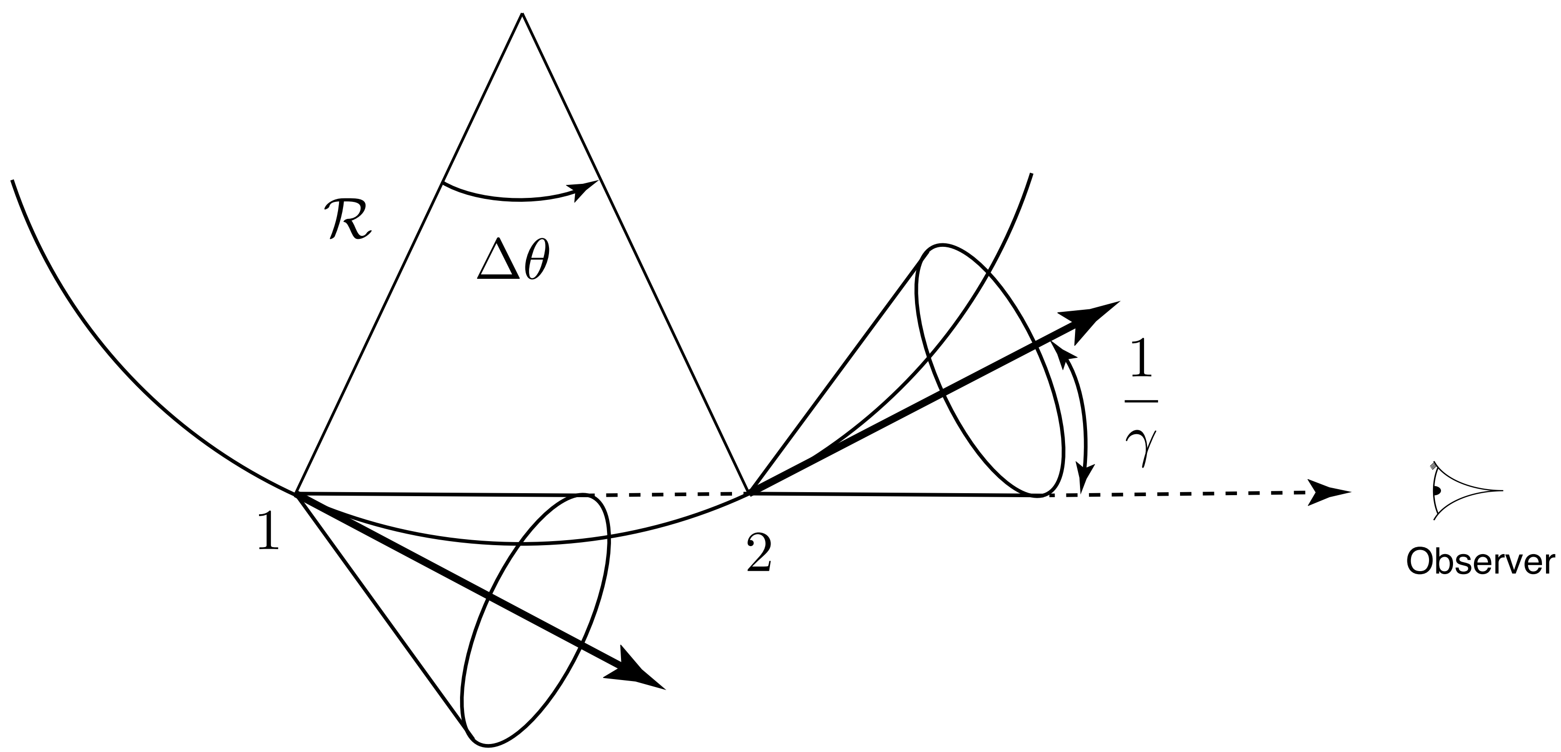
- **cooling time**: the typical timescale for the electron to loose about of its energy is approximately

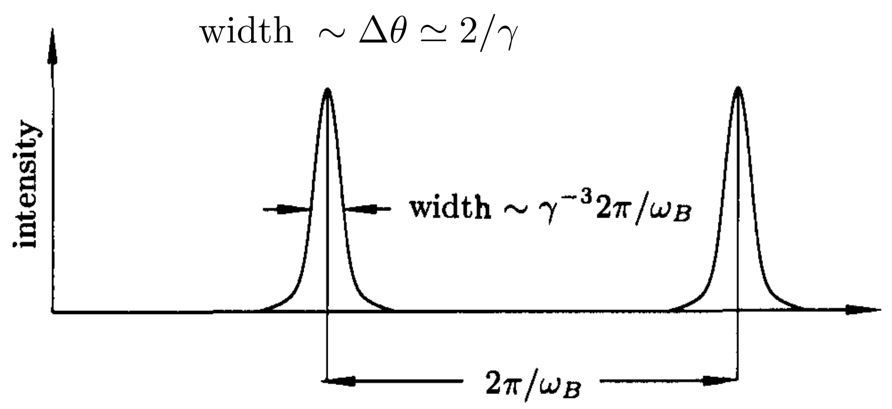
$$t_{\rm cool} = \frac{\rm energy}{\rm cooling\ rate} = \frac{\gamma mc^2}{P} = \frac{4\pi mc}{\sigma_T} \frac{1}{\gamma B^2 \sin^2\alpha} = \frac{15\ \rm years}{\gamma B^2 \sin^2\alpha}$$

- for $\gamma = 10^3$

Location	Typical B	$t_{\rm cool}$	cooling length $\approx ct_{\rm cool}$	size of object
Interstellar medium	10^{-6} G	10^{10} years	10^{28} cm	10^{22} cm
Stellar atmosphere	1 G	5 days	10^{15} cm	10^{11} cm
Supermassive black hole	10^4 G	10^{-3} sec	3 10^7 cm	10^{14} cm
White dwarf	10^8 G	10^{-11} sec	3 mm	1000 km
Neutron star	10^{12} G	10^{-19} sec	3 10^{-9} cm	10 km

[Spectrum of Synchrotron Radiation: A Qualitative Discussion]





- Because of beaming effects the emitted radiation fields appear to be concentrated in a narrow set of directions about the particle's velocity.

  The observer will see a pulses of radiation confined to a time interval much smaller than the gyration period. The spectrum will thus be spread over a much broader frequency range than on of order $\omega_{\rm B}$.

  The cone of emission has an angular width $\sim 1/\gamma$. Therefore, the observer will sees emission over the angular range of $\Delta\theta \simeq 2/\gamma$.

- The radiation appears beamed toward the direction of the observer in a series of pulses spaced in time (period) $2\pi/\omega_{\rm B}$ apart, but with each pulse lasting only $\Delta\theta \simeq 2/\gamma$.

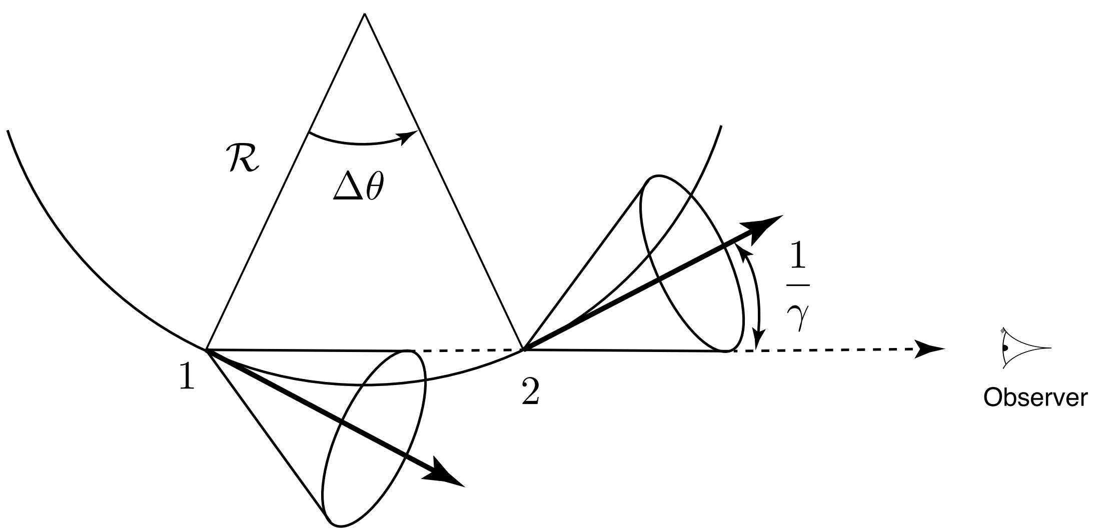
- To Fourier analyze the pulse shape, we need to **calculate the interval of the arrival times of the pulse**. Let's consider an instantaneous rest frame of the electron.

  The path length from point 1 to 2 is $\Delta s = \mathcal{R}\Delta\theta$, where $\mathcal{R}$ is the radius of curvature of the path.

  The equation of motion:
$$\gamma m_e \frac{\Delta \mathbf{v}}{\Delta t} = \frac{e}{c}\mathbf{v}\times\mathbf{B}$$

  Since $|\Delta \mathbf{v}| = v\Delta\theta$ and $\Delta s = v\Delta t$, we find
$$\gamma m_e \frac{v\Delta\theta}{\Delta s/v} = \frac{e}{c} vB\sin\alpha \;\rightarrow\; \mathcal{R} = \frac{\Delta s}{\Delta\theta} = \frac{v}{\omega_{\mathrm{B}}\sin\alpha}$$

  Note that the curvature is different from the gyroradius. Therefore the path length is given by
$$\Delta s = \mathcal{R}(2/\gamma) = \frac{2v}{\gamma\omega_{\mathrm{B}}\sin\alpha} = \frac{2v}{\omega_{\mathrm{L}}\sin\alpha}$$



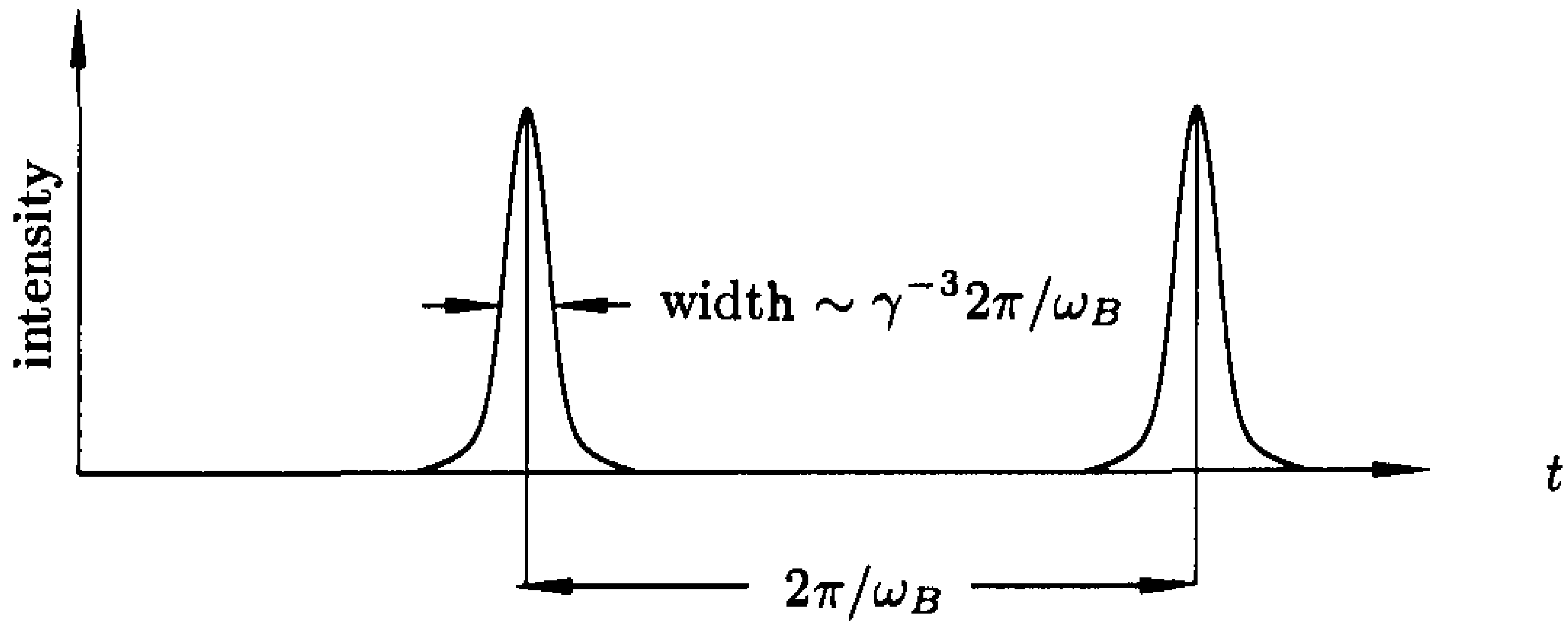
  Time interval that the particle passes from point 1 to 2:
$$\Delta t = t_2 - t_1 = \frac{\Delta s}{v} \simeq \frac{2}{\omega_{\mathrm{L}}\sin\alpha}$$

  Note that point 2 is closer than point 1 by $\Delta s/c$. The difference of the arrival times of the pulse is
$$\Delta t^A = t_2^A - t_1^A = \Delta t - \frac{\Delta s}{c} = \Delta t\left(1 - \frac{v}{c}\right) \approx \frac{1}{\gamma^2\omega_{\mathrm{L}}\sin\alpha} \qquad \leftarrow \quad 1 - \frac{v}{c} \approx \frac{1}{2\gamma^2}$$

$$\Delta t^A = t_2^A - t_1^A \approx \frac{1}{\gamma^2 \omega_{\rm L} \sin\alpha} = \frac{1}{\gamma^3 \omega_{\rm B} \sin\alpha}$$

Therefore, the width of the observed pulses is smaller than the gyration by a factor γ^3.

- Temporal pattern of received pulses:



- We define a critical frequency: $\omega_c \equiv \frac{3}{2}\gamma^2 \omega_{\rm L} \sin\alpha = \frac{3}{2}\gamma^3 \omega_{\rm B} \sin\alpha$

From the properties of Fourier transformation, we expect that the spectrum will be fairly broad, within the frequency range of $\omega_{\rm B} \lesssim \omega \lesssim \omega_{\rm c}$.

- We can derive an important property of the spectrum for the synchrotron radiation.

  Remember that the electric field is a function of $\gamma\theta$, where θ is a polar angle about the direction of motion, because of the beaming effect. Then we can write

$$E(t) \propto F(\gamma\theta)$$

  Let time $t = 0$ and the path length $s = 0$ when the pulse is centered on the observer. Then, we find

$$\theta \approx \frac{s}{\mathcal{R}} \quad \text{and} \quad t \approx \frac{s}{v}\left(1 - \frac{v}{c}\right) \approx \frac{s}{v}\frac{1}{2\gamma^2}$$

  Then we have

$$\gamma\theta \approx \gamma\frac{s}{\mathcal{R}} = \gamma\left(\frac{s}{v}\omega_{\mathrm{B}}\sin\alpha\right) = \gamma\left(2\gamma^2 t\omega_{\mathrm{B}}\sin\alpha\right) \propto \omega_c t$$

  The time dependence of the electric field can be written as

$$E(t) \propto g(\omega_c t)$$

  The Fourier transform of the electric field is

$$\begin{aligned}\bar{E}(\omega) &\propto \int_{-\infty}^{\infty} g(\omega_c t)e^{i\omega t}dt \qquad \leftarrow \quad \xi \equiv \omega_c t \\ &= \int_{-\infty}^{\infty} g(\xi)e^{i(\omega/\omega_c)\xi}d\xi\end{aligned}$$

  Therefore, the power per unit frequency is a function of ω/ω_c: $\quad P(\omega) \propto \left|\bar{E}(\omega)\right|^2 = C_1 F\left(\frac{\omega}{\omega_c}\right)$

[Spectral Index for Power-law Electron Distribution]

- Often the number density of particles with energies between E and $E + dE$ can be approximately expressed in the form:

$$N(\gamma)d\gamma = C\gamma^{-p}d\gamma \;\; (\gamma_1 < \gamma < \gamma_2)$$
$$N(E)dE = CE^{-p}dE \;\; (E_1 < E < E_2)$$

- The total power radiated per unit volume per unit frequency by such a distribution is given by

$$\begin{aligned} P_{\rm tot}(\omega) &= \int_{\gamma_1}^{\gamma_2} N(\gamma)P(\omega)d\gamma \\ &\propto \int_{\gamma_1}^{\gamma_2} \gamma^{-p} F\left(\frac{\omega}{\omega_c}\right) d\gamma \qquad\qquad \leftarrow \quad x \equiv \frac{\omega}{\omega_c} \propto \gamma^{-2}\omega \\ &\propto \omega^{-(p-1)/2} \int_{x_1}^{x_2} x^{-(p-3)/2} F(x) dx \end{aligned}$$

- Then, the spectrum is also a power law and the power-law spectral index s is related to the particle distribution index p by

$$\boxed{\begin{aligned} P_{\rm tot}(\omega) &\propto \omega^{-s} \\ &\propto \omega^{-(p-1)/2} \quad \longrightarrow \quad s = \frac{p-1}{2} \end{aligned}}$$

[Spectrum of Synchrotron Radiation: A Detailed Discussion]

- We will use the formula derived in Chapter 3.

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int \mathbf{n}\times(\mathbf{n}\times\mathbf{B})\exp\left[i\omega\left(t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c}\right)\right]dt'\right|^2$$

- The coordinate system is chosen so that the particle has velocity **v** along the *x'* axis at time *t'* = 0.

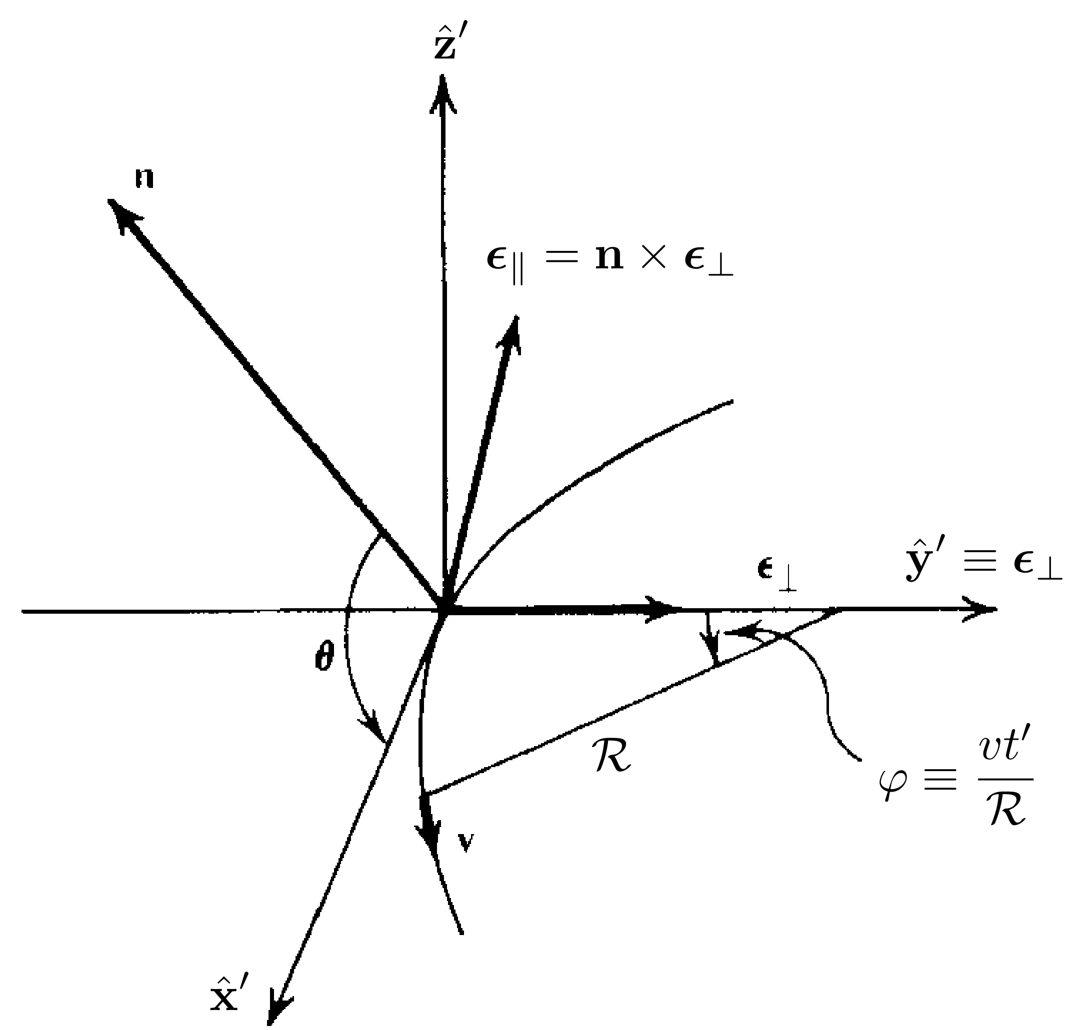
  $\boldsymbol{\epsilon}_\perp$ is a unit vector along the *y'* axis in the orbital (*x'*-*y'*) plane.

  Let θ represent the angle between the observing direction (**n**) and the velocity vector **v** at *t'* = 0.

  Then, an equivalent circular orbit at *t'* is given by

$$\mathbf{v}(t') = v(\hat{\mathbf{x}}'\cos\varphi + \hat{\mathbf{y}}'\sin\varphi), \quad \text{where } \varphi \equiv \frac{vt'}{\mathcal{R}} = (\omega_{\rm B}\sin\alpha)t'$$

$$\mathbf{r}(t') = \mathcal{R}(\hat{\mathbf{x}}'\sin\varphi - \hat{\mathbf{y}}'\cos\varphi)$$



Note that (1) $\mathbf{n}\times\hat{\mathbf{x}}' = \sin\theta\hat{\mathbf{y}}'$, (2) $\mathbf{n}\times\hat{\mathbf{y}}' = \mathbf{n}\times\boldsymbol{\epsilon}_\perp = \boldsymbol{\epsilon}_\parallel$
(3) $\mathbf{n}\cdot\hat{\mathbf{y}}' = 0$, (4) $\mathbf{n}\cdot\hat{\mathbf{x}}' = \cos\theta$

$$\begin{aligned}\mathbf{n}\times(\mathbf{n}\times\mathbf{B}) &= \beta\left[\mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{x}}')\cos\varphi + \mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{y}}')\sin\varphi\right]\\ &= \beta\mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{x}}')\cos\varphi + \beta\left(\mathbf{n}(\mathbf{n}\cdot\hat{\mathbf{y}}') - \hat{\mathbf{y}}'\right)\sin\varphi\\ &= \boldsymbol{\epsilon}_\parallel\beta\sin\theta\cos\varphi - \boldsymbol{\epsilon}_\perp\beta\sin\varphi\end{aligned}$$

- We note that

$$
\begin{aligned}
t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c} &= t' - \frac{\mathcal{R}}{c}\cos\theta\sin\varphi && \leftarrow \quad \mathbf{n}\cdot\hat{\mathbf{x}}' = \cos\theta \\
&= t' - \frac{\mathcal{R}}{c}\left(1-\frac{\theta^2}{2}\right)\left(\varphi - \frac{\varphi^3}{6}\right) && \leftarrow \quad \varphi = \frac{vt'}{\mathcal{R}} \\
&= t'\left[1-\frac{v}{c}\left(1-\frac{\theta^2}{2}\right)\left(1-\frac{(vt')^2}{6\mathcal{R}^2}\right)\right] && \leftarrow \quad 1-\frac{v}{c}\approx\frac{1}{2\gamma^2} \\
&\approx t'\left[1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right)\left(1-\left(1-\frac{1}{2\gamma^2}\right)^2\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] \\
&\approx \frac{t'}{2\gamma^2}\left[2\gamma^2-(2\gamma^2-1)\left(1-\frac{\theta^2}{2}\right)\left(1-\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] && \leftarrow \quad ct' \ll \mathcal{R},\ \theta \ll 1 \\
&\approx \frac{t'}{2\gamma^2}\left[2\gamma^2-(2\gamma^2-1)\left(1-\frac{\theta^2}{2}-\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] \\
&\approx \frac{t'}{2\gamma^2}\left[2\gamma^2-(2\gamma^2-1)+2\gamma^2\left(\frac{\theta^2}{2}+\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] \\
&= \frac{1}{2\gamma^2}\left[\left(1+\gamma^2\theta^2\right)t' + \frac{c^2\gamma^2t'^3}{3\mathcal{R}^2}\right]
\end{aligned}
$$

- We also note that

$$
\begin{aligned}
\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) &= \boldsymbol{\epsilon}_{\parallel}\beta\sin\theta\cos\varphi - \boldsymbol{\epsilon}_{\perp}\beta\sin\varphi \qquad \leftarrow \quad \beta\approx 1\\
&\approx \boldsymbol{\epsilon}_{\parallel}\sin\theta\cos\varphi - \boldsymbol{\epsilon}_{\perp}\sin\varphi\\
&\approx \boldsymbol{\epsilon}_{\parallel}\theta - \boldsymbol{\epsilon}_{\perp}\varphi\\
&= \boldsymbol{\epsilon}_{\parallel}\theta - \boldsymbol{\epsilon}_{\perp}\frac{vt'}{\mathcal{R}}\\
&\approx \boldsymbol{\epsilon}_{\parallel}\theta - \boldsymbol{\epsilon}_{\perp}\frac{ct'}{\mathcal{R}}
\end{aligned}
$$

$$
t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c} \approx \frac{1}{2\gamma^2}\left[\left(1+\gamma^2\theta^2\right)t' + \frac{c^2\gamma^2 t'^3}{3\mathcal{R}^2}\right]
$$

- We can identify the contribution to the received power in the two orthogonal polarized directions.

$$
\frac{dW}{d\omega d\Omega} \equiv \frac{dW_{\parallel}}{d\omega d\Omega} + \frac{dW_{\perp}}{d\omega d\Omega}
$$

$$
\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int \mathbf{n}\times(\mathbf{n}\times\mathbf{B})\exp\left[i\omega\left(t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c}\right)\right]dt'\right|^2
$$

$$
\frac{dW_{\parallel}}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int \frac{ct'}{\mathcal{R}}\exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3\mathcal{R}^2}\right)\right]dt'\right|^2
$$

$$
\frac{dW_{\perp}}{d\omega d\Omega} = \frac{e^2\omega^2\theta^2}{4\pi^2 c}\left|\int \exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3\mathcal{R}^2}\right)\right]dt'\right|^2
$$

$$
\theta_\gamma \equiv 1 + \gamma^2\theta^2
$$

Define the following variables

$$
y \equiv \gamma\frac{ct'}{\mathcal{R}\theta_\gamma} \text{ and } \eta \equiv \frac{\omega\mathcal{R}\theta_\gamma^3}{3c\gamma^3}
$$

$$
\frac{dW_{\parallel}}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left(\frac{\mathcal{R}\theta_\gamma^2}{\gamma^2 c}\right)^2\left|\int_{-\infty}^{\infty} y\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dt'\right|^2
$$

$$
\frac{dW_{\perp}}{d\omega d\Omega} = \frac{e^2\omega^2\theta^2}{4\pi^2 c}\left(\frac{\mathcal{R}\theta_\gamma}{\gamma c}\right)^2\left|\int_{-\infty}^{\infty} \exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dt'\right|^2
$$

- The integrals are functions only of the parameter η. Since most of the radiation occurs at angle $\theta \approx 0$, η can be written as

$$\eta \approx \eta(\theta = 0) = \frac{\omega \mathcal{R}}{3c\gamma^3} = \frac{\omega v}{3c\gamma^3 \omega_\mathrm{B} \sin\alpha} \approx \frac{\omega}{2\omega_c} \qquad \text{where} \quad \omega_c \equiv \frac{3}{2}\frac{\gamma^2 eB\sin\alpha}{m_e c} = \frac{3}{2}\gamma^3\omega_\mathrm{B}\sin\alpha$$

The frequency dependence of the spectrum depends on ω only through ω/ω_c.

The angular dependence uses θ only through the combination $\gamma\theta$.

- The integrals can be expressed in terms of the modified Bessel functions of 1/3 and 2/3 order.

From 10.4.26, 10.4.31, and 10.4.32 of Abramovitz & Stegun (1965)
See Westfold 1959, ApJ, 130, 241

$$\frac{dW_\parallel}{d\omega d\Omega} = \frac{e^2\omega^2}{3\pi^2 c}\left(\frac{\mathcal{R}\theta_\gamma^2}{\gamma^2 c}\right)^2 K_{2/3}^2(\eta)$$

$$\frac{dW_\perp}{d\omega d\Omega} = \frac{e^2\omega^2\theta^2}{3\pi^2 c}\left(\frac{\mathcal{R}\theta_\gamma}{\gamma c}\right)^2 K_{1/3}^2(\eta)$$

- The energy per frequency range radiated by the particle per complete orbit in the projected normal plane can be obtained by integrating over solid angle.

- We note that the emitted radiation is almost completely confined to the solid angle shown shaded in the following figure, which lies within an angle $1/\gamma$ of a cone of half-angle α. Therefore, the integral over the solid angle can be approximated by

$$\frac{W_{\|}}{d\omega} = \int_0^{\pi} \frac{dW_{\|}}{d\omega d\Omega} 2\pi \sin\theta d\theta \approx \int_{-\infty}^{\infty} \frac{dW_{\|}}{d\omega d\Omega} 2\pi \sin\alpha d\theta$$

- Therefore,

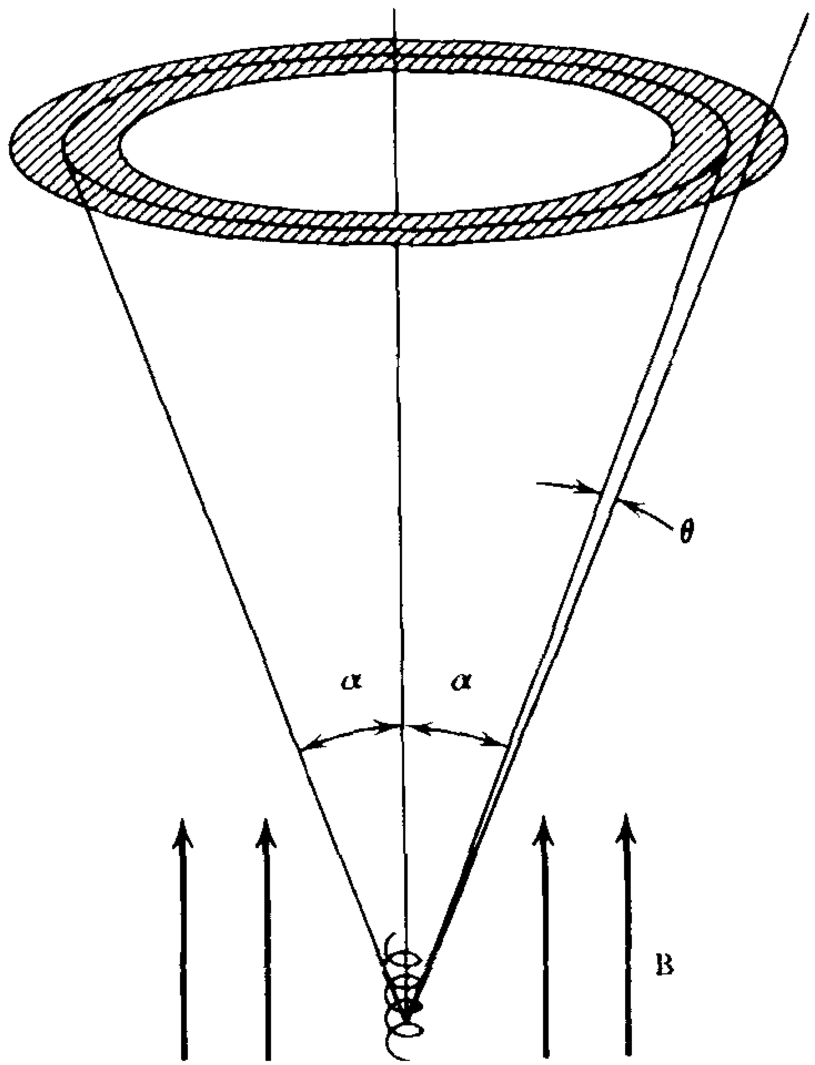
$$\frac{dW_{\|}}{d\omega} = \frac{2e^2\omega^2\mathcal{R}^2 \sin\alpha}{3\pi c^3\gamma^4} \int_{-\infty}^{\infty} \theta_\gamma^4 K_{2/3}^2(\eta) d\theta$$

$$\frac{dW_{\perp}}{d\omega} = \frac{2e^2\omega^2\mathcal{R}^2 \sin\alpha}{3\pi c^3\gamma^4} \int_{-\infty}^{\infty} \theta_\gamma^2 K_{1/3}^2(\eta) d\theta$$

- The emitted power per frequency is obtained by dividing the orbital period of the charge $T = 2\pi/\omega_{\rm B}$:

$$P_{\|}(\omega) \equiv \frac{1}{T}\frac{dW_{\|}}{d\omega}$$

$$P_{\perp}(\omega) \equiv \frac{1}{T}\frac{dW_{\perp}}{d\omega}$$



- Emitted power:

$$P_{\parallel}(\omega) \equiv \frac{\sqrt{3}e^3 B \sin\alpha}{4\pi m_e c^2}\left[F(x) - G(x)\right]$$

$$P_{\perp}(\omega) \equiv \frac{\sqrt{3}e^3 B \sin\alpha}{4\pi m_e c^2}\left[F(x) + G(x)\right]$$

where

$$F(x) \equiv x \int_x^{\infty} K_{5/3}(\xi)d\xi$$

$$G(x) \equiv x K_{2/3}(x)$$

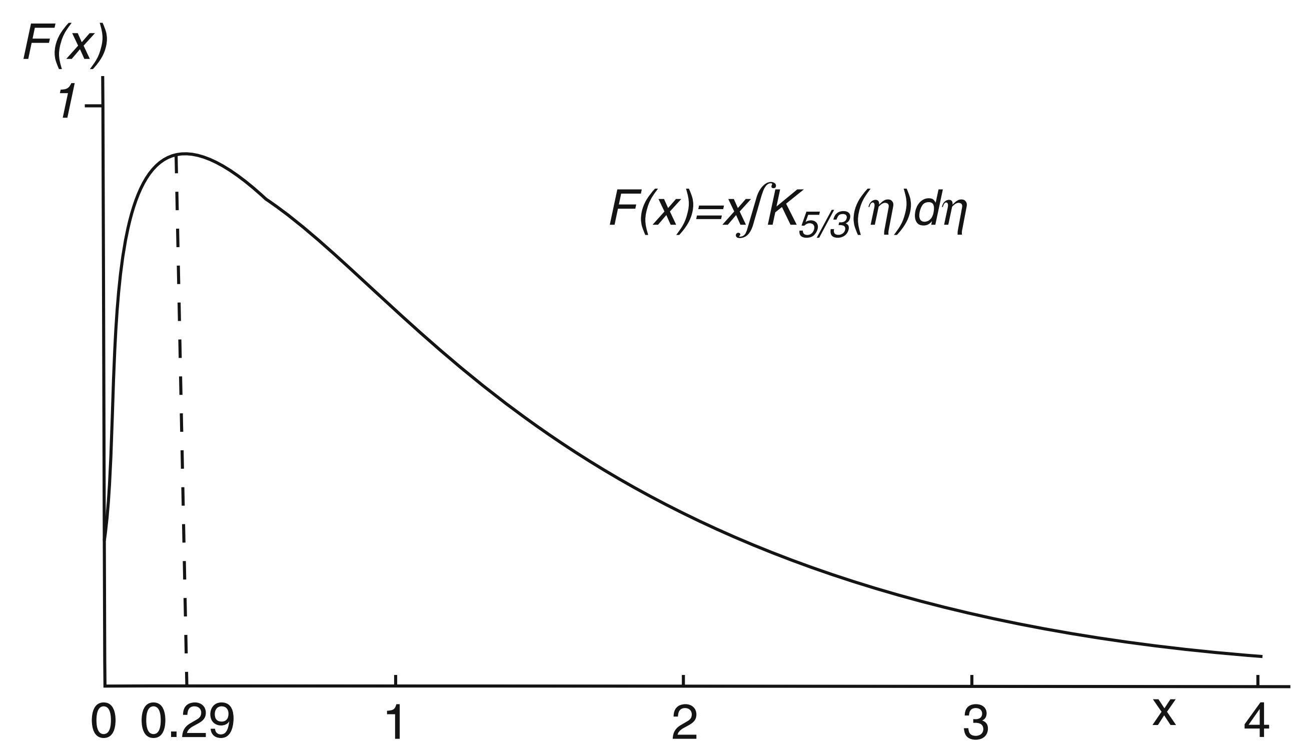
$$x \equiv \omega/\omega_c$$

- Total emitted power per frequency:

$$P(\omega) \equiv P_{\parallel}(\omega) + P_{\perp}(\omega) = \frac{\sqrt{3}e^3 B \sin\alpha}{2\pi m_e c^2} F(x)$$

$$F(x) \sim \frac{4\pi}{\sqrt{3}\Gamma(1/3)}\left(\frac{x}{2}\right)^{1/3} \quad \text{if} \;\; x \ll 1$$

$$F(x) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-x} x^{1/2} \qquad \text{if} \;\; x \gg 1$$



- For a power-law distribution of electrons $N(\gamma)d\gamma = C\gamma^{-p}d\gamma \quad (\gamma_1 < \gamma < \gamma_2)$, we obtain the total power per unit volume per unit frequency:

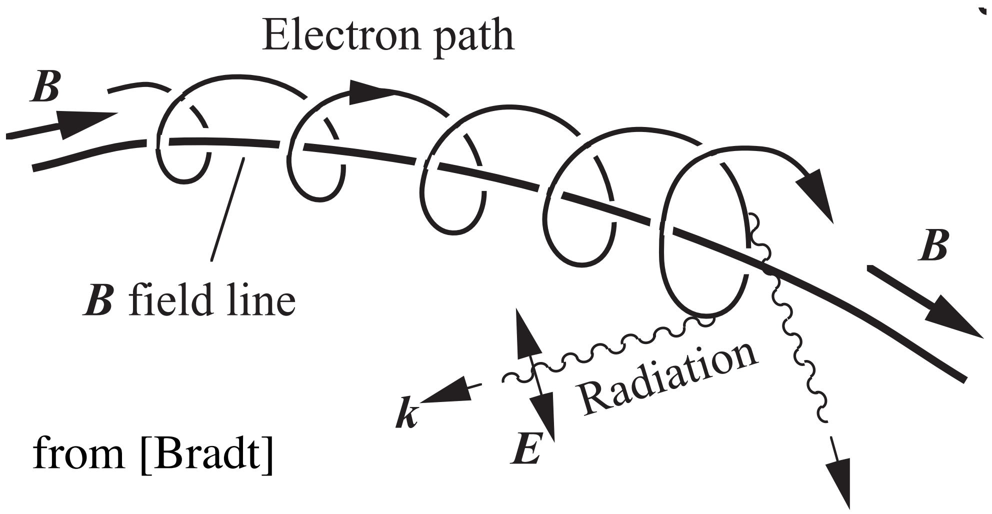
$$
\begin{aligned}
P_{\rm tot} &= \int N(\gamma)P(\omega)d\gamma \\
&\equiv \frac{\sqrt{3}e^3 CB\sin\alpha}{2\pi m_e c^2 (p+1)}\Gamma\left(\frac{p}{4}+\frac{19}{12}\right)\Gamma\left(\frac{p}{4}-\frac{1}{12}\right)\left(\frac{m_e c\omega}{3eB\sin\alpha}\right)^{-(p-1)/2}
\end{aligned}
$$

$$P_{\rm tot} \propto \omega^{-(p-1)/2}$$

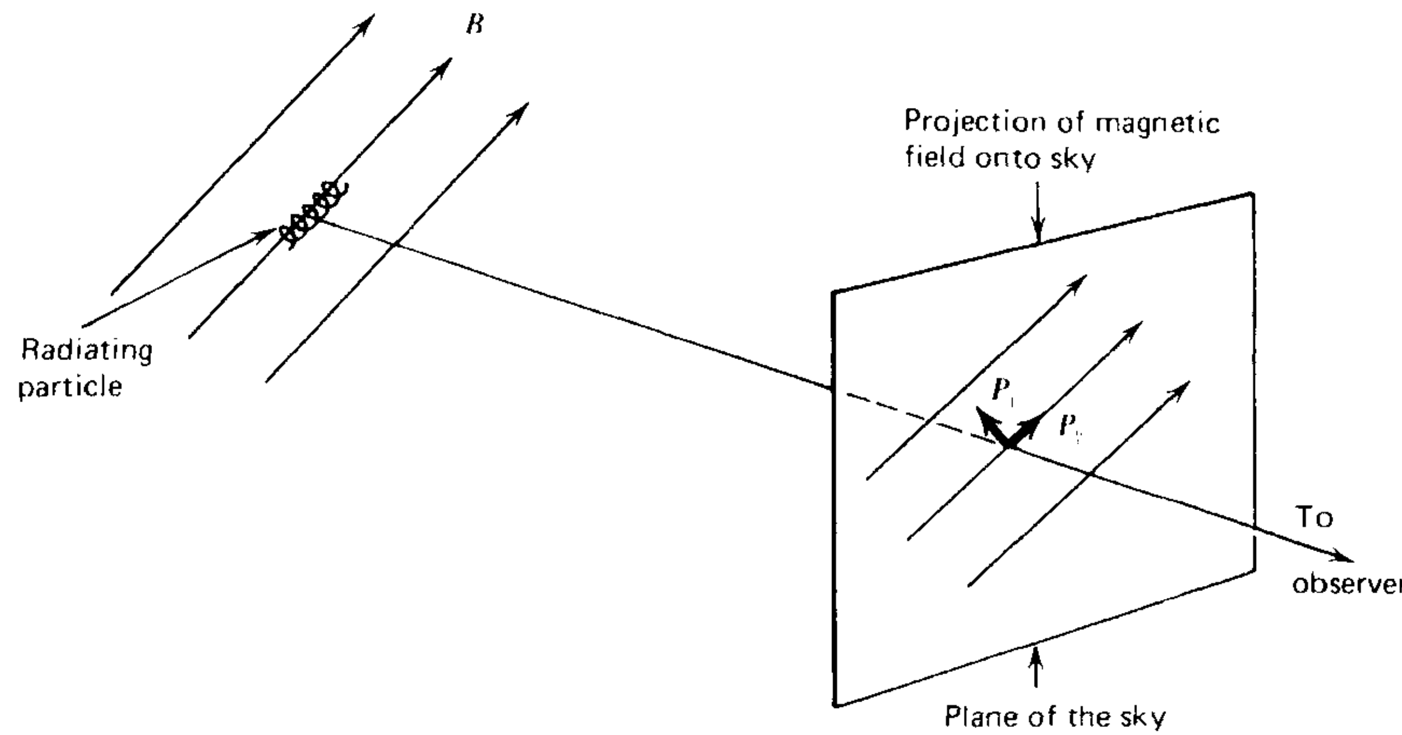
- For the complete derivation of the formula, see Westfold (1959).

[Polarization of Synchrotron Radiation]

- In general, the radiation from a single charge will be elliptically polarized. For any reasonable distribution of particles that varies smoothly with pitch angle, the elliptical component will cancel out as emission cones will contribute equally from both sides of the line of sight. Thus **the radiation will be partially linearly polarized.**



from [Bradt]



Recall $\mathbf{E}_{\rm rad} \propto \mathbf{n} \times (\mathbf{n} \times \boldsymbol{a})$
On average, the radiation will be polarized perpendicular to the magnetic field.
(See 천체물리학(구본철, 김웅태) for detailed description)

- Degree of linear polarization of a single energy:

$$\Pi(\omega) \equiv \frac{P_\perp(\omega) - P_\parallel(\omega)}{P_\perp(\omega) + P_\parallel(\omega)} = \frac{G(x)}{F(x)}$$

- For particles with a power law distribution of energies:

$$\begin{aligned}\Pi(\omega) &= \frac{\int G(x)\gamma^{-p}d\gamma}{\int F(x)\gamma^{-p}d\gamma} \qquad \leftarrow \;\; \gamma \propto x^{-1/2} \\ &= \frac{\int G(x)x^{(p-3)/2}dx}{\int F(x)x^{(p-3)/2}dx} \\ &= \frac{(p+1)/2}{2}\frac{1}{\frac{p-3}{4}+\frac{4}{3}} \\ &= \frac{p+1}{p+\frac{7}{3}}\end{aligned}$$

$\longleftarrow$

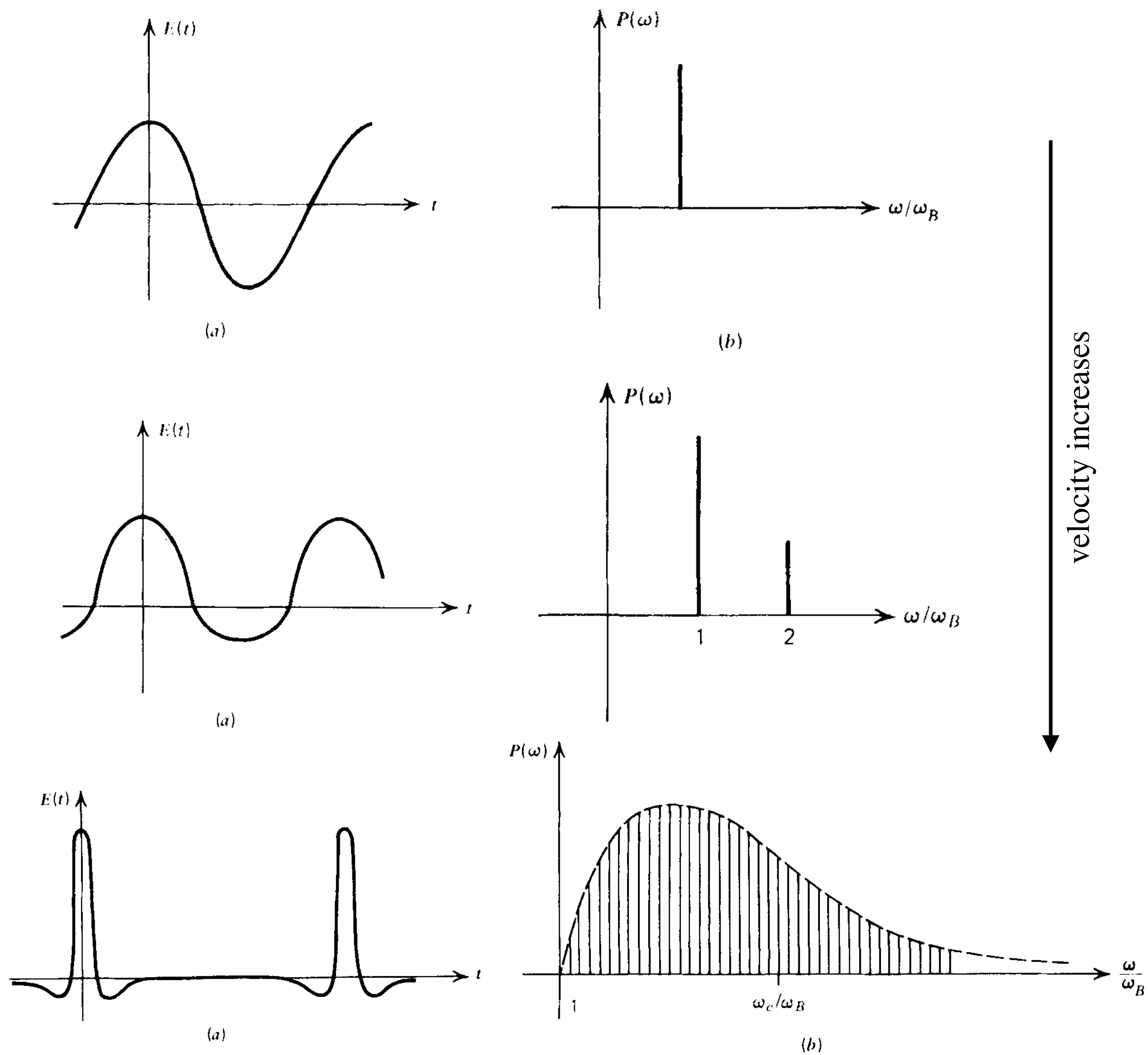
$$\begin{aligned}\int_0^\infty x^\mu F(x)dx &= \frac{2^{\mu+1}}{\mu+2}\Gamma\left(\frac{\mu}{2}+\frac{7}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right) \\ \int_0^\infty x^\mu G(x)dx &= 2^\mu\Gamma\left(\frac{\mu}{2}+\frac{4}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right)\end{aligned}$$

- For particles of a single energy, the polarization degree of the frequency integrated radiation is

$$\begin{aligned}\Pi(\omega) &= \frac{p+1}{p+\frac{7}{3}} \qquad \leftarrow \;\; p=3 \\ &= \frac{3}{4} \\ &= 75\%\end{aligned}$$

[Transition from Cyclotron to Synchrotron Emission]

- For low energies, the electric field components vary sinusoidally with the same frequency as the gyration in the magnetic field. The spectrum consists of a single line.
- When v/c increases, higher harmonics of the fundamental frequency begin to contribute.
- For very relativistic velocities, the originally sinusoidal form of *E(t)* has now become a series of sharp pulses, which is repeated at time intervals $2\pi/\omega_B$. The spectrum now involves a great number of harmonics, the envelope of which approaches the form of the function *F(x)*.

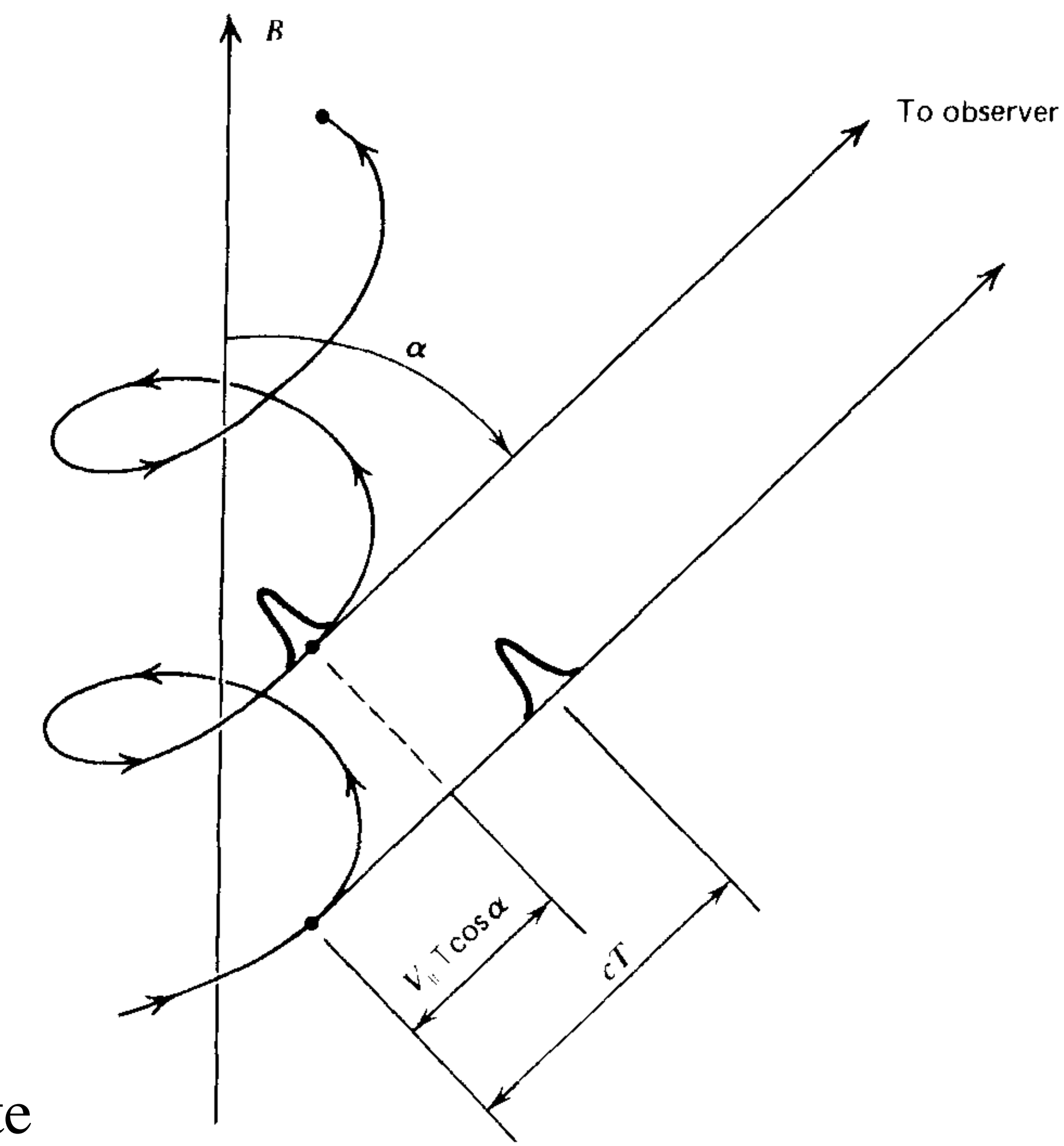


[Distinction between Received and Emitted Power]

- If $T = 2\pi/\omega_B$ is the orbital period of the projected motion, then time-delay effects will give a period between the arrival of pulses T_A satisfying

$$T_A = T\left(1 - \frac{v_\parallel}{c}\cos\alpha\right) = T\left(1 - \frac{v}{c}\cos^2\alpha\right)$$
$$\approx T\left(1 - \cos^2\alpha\right) = \frac{2\pi}{\omega_B}\sin^2\alpha$$

Therefore, the fundamental observed frequency is $\omega_B/\sin^2\alpha$ rather than ω_B .



- Two modifications to the preceding results:

(1) Spacing of the harmonics is $\omega_B/\sin^2\alpha$. For extreme relativistic particles this is not important, because one sees a continuum rather than the harmonic structure. Note that we did take the Doppler effects in deriving the pulse width Δt_A and consequently for the critical frequency ω_c. The continuum radiation is still a function of ω/ω_c.

(2) The emitted power was found by dividing the energy by the period T. But the received power must be obtained by dividing by T_A. Thus we have $P_r = P_e/\sin^2\alpha$.

The average power emitted and received will be the same, because the total number of emitted and received pulses but be the same in the long run. These corrections are not important for most cases of interest.

[Synchrotron Self-Absorption]

- **Opacity**

In order to calculate the opacity for non-thermal velocity distribution of electrons. We first need to generalize the Einstein coefficients to **include continuum states.**

For a given energy of a photon $h\nu$ there are many possible transitions, meaning that the absorption coefficient should be obtained by summing over all upper states 2 and lower stats 1:

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \phi_{21}(\nu)$$

The profile function $\phi_{21}(\nu)$ is essentially a Dirac delta-function: $\phi_{21}(\nu) = \delta\left(\nu - (E_2 - E_1)/h\right)$

In terms of the Einstein coefficients, the emitted power is given by

$$P(\nu, E_2) = h\nu \sum_{E_1} A_{21} \phi_{21}(\nu) = (2h\nu^3/c^2) h\nu \sum_{E_1} B_{21} \phi_{21}(\nu) \longleftarrow A_{21} = \left(2h\nu^3/c^2\right) B_{21}$$

- Using this, the part due to stimulated emission of α_ν can be represented in terms of P:

$$-\frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} n(E_2) B_{21} \phi_{21}(\nu) = -\frac{c^2}{8\pi h\nu^3} \sum_{E_2} n(E_2) P(\nu, E_2)$$

- The true absorption coefficient (first part) is:

$$\frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} n(E_1) B_{12} \phi_{21}(\nu) = \frac{c^2}{8\pi h\nu^3} \sum_{E_2} n(E_2 - h\nu) P(\nu, E_2) \longleftarrow \begin{array}{l} B_{12} = B_{21} \\ E_1 = E_2 - h\nu \end{array}$$

Therefore, we have
$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\sum_{E_2}\left[n(E_2 - h\nu) - n(E_2)\right] P(\nu, E_2)$$

We need to convert the discrete summation to an integral over continuum energy (or momentum).

(1) Let $f(p)d^3p \equiv$ number of electrons per volume with momenta in d^3p about p.

(2) Number of quantum states in d^3p = $g\frac{d^3p}{h^3}$ $\quad (g = 2 \text{ for spin } 1/2 \text{ paticles})$

(3) Electron density per quantum state = $\frac{f(p)d^3p}{gd^3p/h^3} = \frac{h^3}{g}f(p)$

Therefore, we can make the replacements
$$\sum_2 \;\rightarrow\; \frac{g}{h^3}\int d^3p, \qquad n(E_2) \;\rightarrow\; \frac{h^3}{g}f(p)$$

Then, the absorption coefficient becomes
$$\boxed{\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int d^3p_2\left[f(p_2^*) - f(p_2)\right] P(\nu, E_2)}$$

where p_2^* is the momentum corresponding to energy $E_2 - h\nu$.

- For a thermal distribution of particles, we can derive the **Kirchhoff's Law for continuum states**.

$$f(p) = K \exp\left[-\frac{E(p)}{kT}\right]$$

$$\begin{aligned} f(p_2^*) - f(p) &= K \exp\left(-\frac{E_2 - h\nu}{kT}\right) - K \exp\left(-\frac{E_2}{kT}\right) \\ &= f(p_2)\left(e^{h\nu/kT} - 1\right) \end{aligned}$$

Thus, the absorption coefficient is

$$\begin{aligned} \alpha_\nu &= \frac{c^2}{8\pi h\nu^3}\left(e^{h\nu/kT} - 1\right) \int d^3p_2 f(p_2) P(\nu, E_2) \\ &= \frac{1}{4\pi}\frac{c^2}{2h\nu^3}\left(e^{h\nu/kT} - 1\right) 4\pi j_\nu \end{aligned}$$

This integral is the total power per volume per frequency range. i.e., $4\pi j_\nu$

Therefore, we obtained the Kirchhoff's Law for thermal emission.

$$\alpha_\nu = \frac{j_\nu}{B_\nu(T)}$$

- **<u>For an isotropic, and extremely relativistic electron distribution</u>**, we can use energy instead of momentum:

$$E = \sqrt{(pc)^2 + (mc^2)^2} \approx pc \longrightarrow d^3p = 4\pi p^2 dp = \frac{4\pi}{c^3} E^2 dE$$

$$f(p) 4\pi p^2 dp = N(E) dE \longrightarrow f(p) = \frac{N(E)dE}{4\pi p^2 dp} = \frac{N(E)}{(4\pi/c^3)E^2}$$

$$\Rightarrow \begin{cases} d^3p f(p) = dE E^2 \dfrac{N(E)}{E^2} \\ d^3p f(p^*) = dE E^2 \dfrac{N(E^*)}{E^{*2}} \end{cases}$$

where $E^* = E - h\nu$

Then,

$$\alpha_\nu = \frac{c^2}{8\pi h \nu^3} \int dE P(\nu, E) E^2 \left[\frac{N(E - h\nu)}{(E - h\nu)^2} - \frac{N(E)}{E^2} \right]$$

Assume that $h\nu \ll E$ (in fact, a necessary condition for the application of classical electrodynamics) and expand to first order in $h\nu$.

$$\frac{N(E-h\nu)}{(E-h\nu)^2} \approx \frac{N(E)}{E^2} - h\nu \frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right] + \mathcal{O}\left((h\nu)^2\right)$$

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2} \int dE P(\nu, E) E^2 \frac{\partial}{\partial E} \left[\frac{N(E)}{E^2} \right]$$

- <u>For a power law distribution of particles:</u>

$$N(E) = CE^{-p}$$

$$-E^2 \frac{d}{dE}\left[\frac{N(E)}{E^2}\right] = (p+2)CE^{-(p+1)} = \frac{(p+2)N(E)}{E}$$

$$\Rightarrow$$

$$\alpha_\nu = \frac{(p+2)c^2}{8\pi\nu^2} \int dE P(\nu, E) \frac{N(E)}{E}$$

$$\propto \nu^{-2} \int dE F(x) \frac{E^{-p}}{E} \quad \leftarrow \quad \text{set } x = \frac{\omega}{\omega_c} \propto \nu \gamma^{-2} \propto \nu E^{-2}$$

$$\propto \nu^{-2} \int \nu^{1/2} x^{-3/2} dx F(x) \nu^{-(p+1)/2} x^{(p+1)/2}$$

$$\boxed{\alpha_\nu \propto \nu^{-(p+4)/2}}$$

Note $\alpha_\nu \propto \nu^{-(p+4)/2}$ indicates that **the synchrotron emission is optically thick at low frequencies and optically thin at high frequencies.**

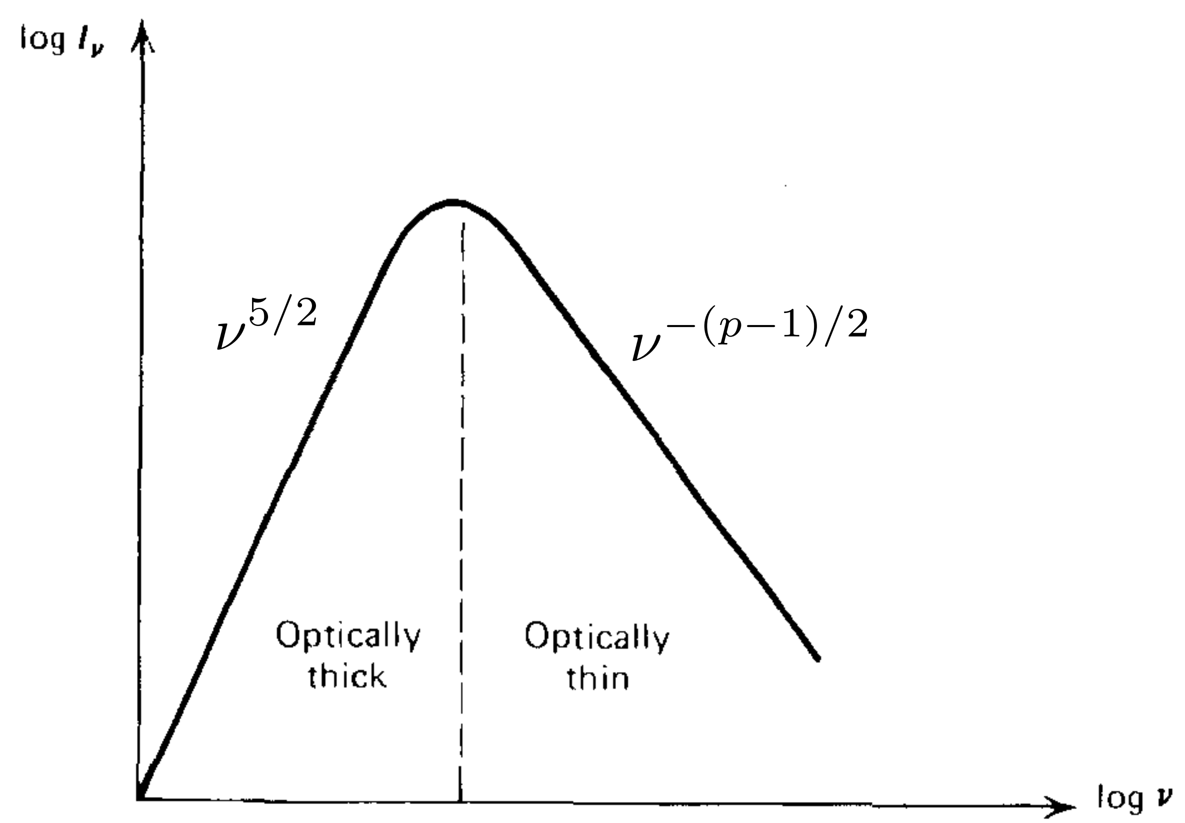
The source function is

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{P(\nu)}{4\pi\alpha_\nu} \propto \nu^{5/2} \longleftarrow P(\nu) \propto \nu^{-(p-1)/2}, \quad \alpha_\nu \propto \nu^{-(p+4)/2}$$

For optically thin synchrotron emission, $I_\nu = \int j_\nu ds \propto \nu^{-(p-1)/2}$

For optically thick emission, $I_\nu = S_\nu \propto \nu^{5/2}$

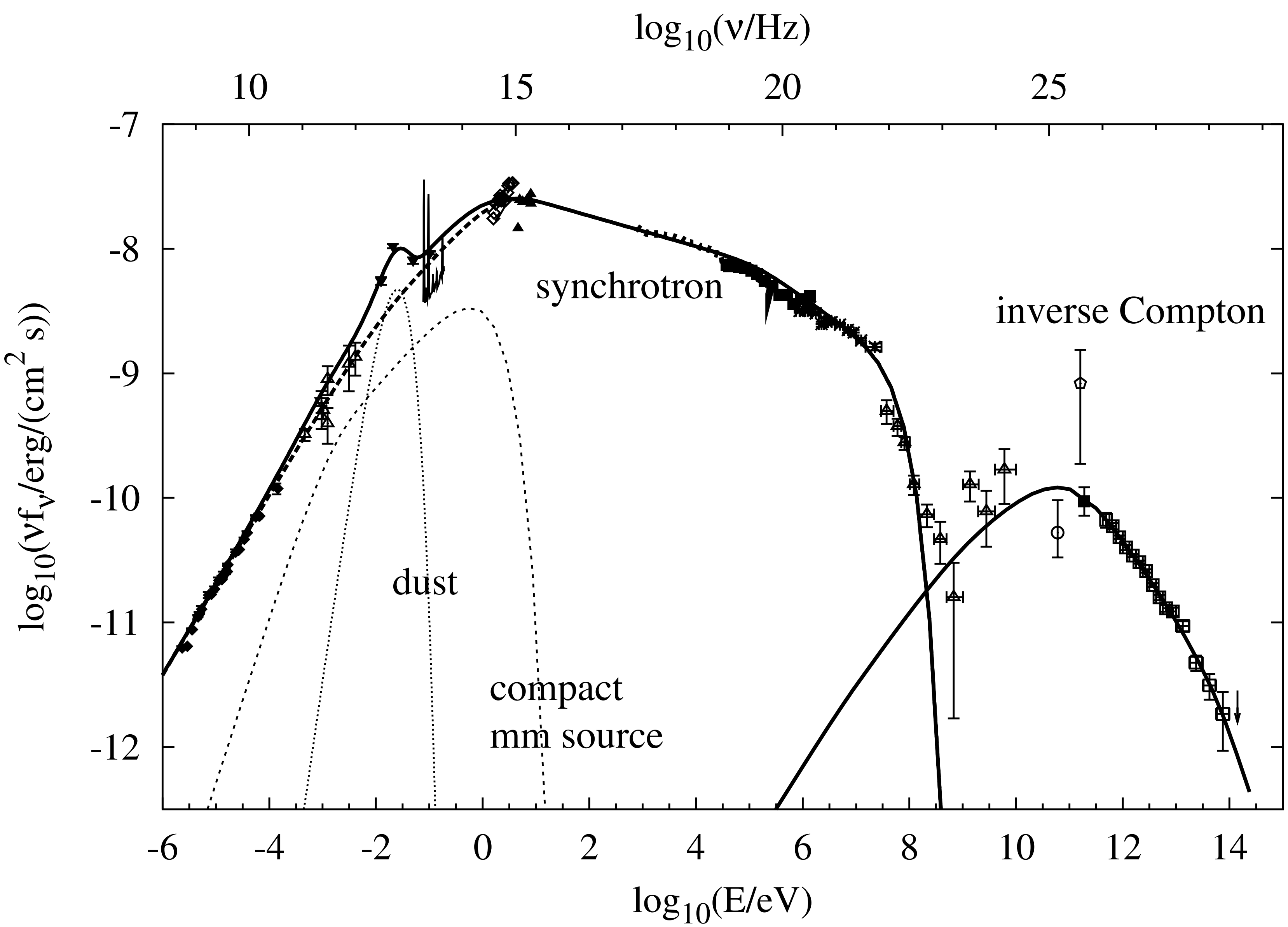
Therefore, the synchrotron spectrum from a power-law distribution of electrons is

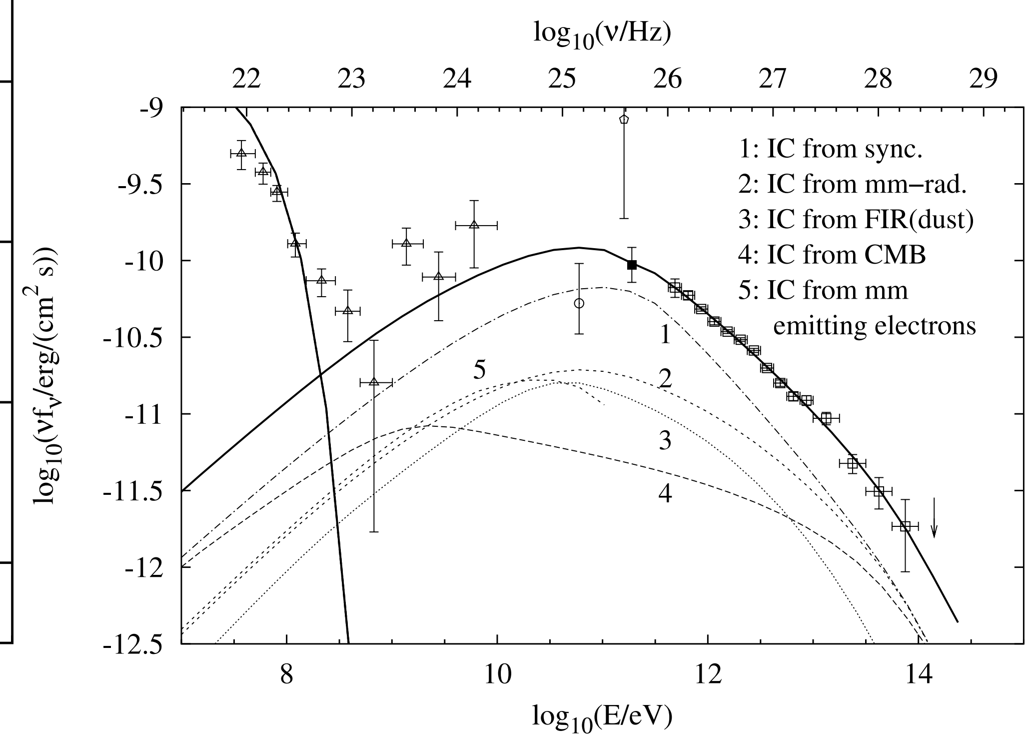


Astrophysical Example

- Crab nebula

Dots: modified blackbody with $T = 46$ K.
Thin dashed line: emission at mm wavelengths
Thick dashed line: synchrotron emission





Aharonian et al. (2004)

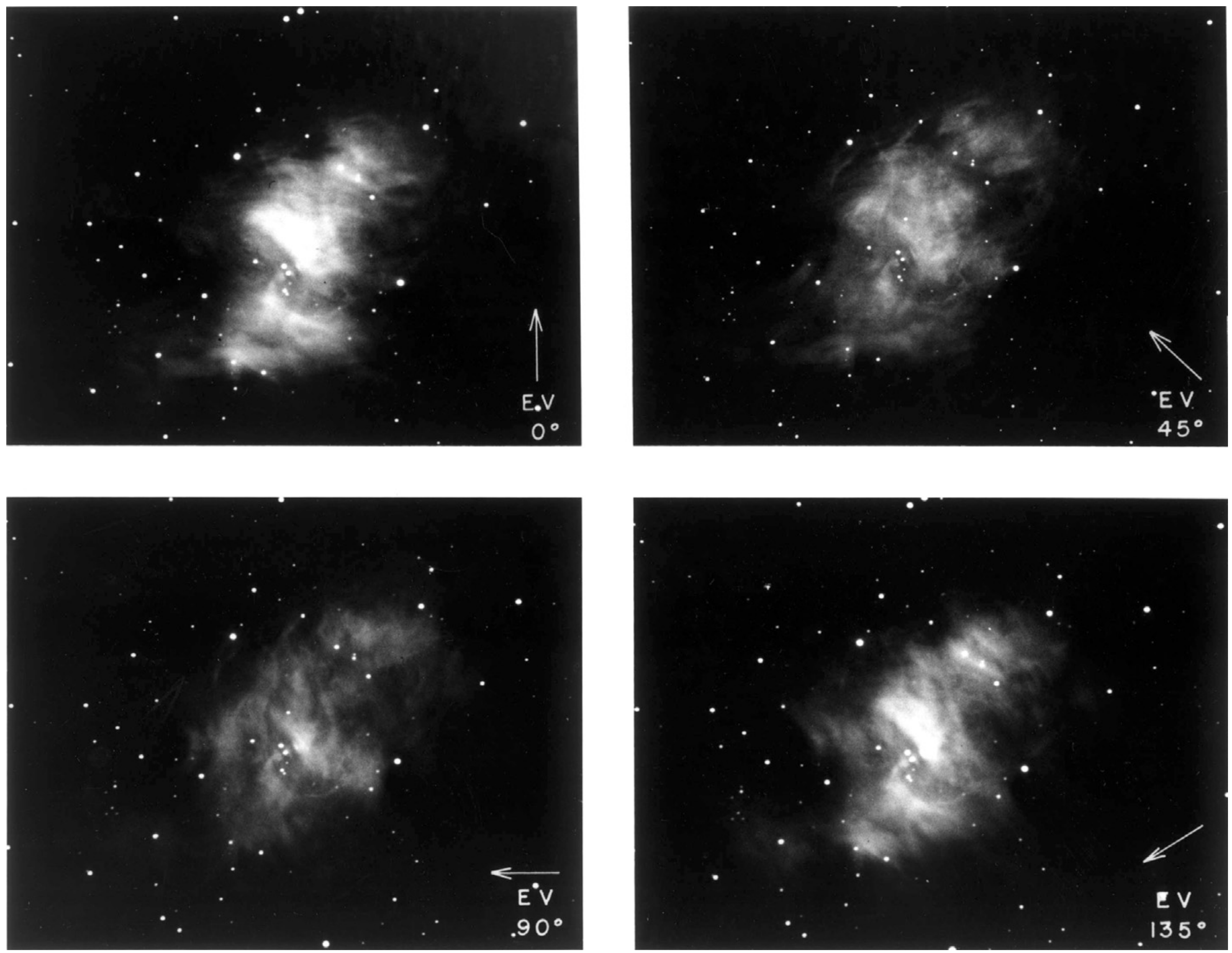


Fig. 8.3: Photographs of the Crab nebula in polarized light with the polarizer at different orientations. The arrows show the directions or planes of the transmitted transverse electric vector. Note the changing brightness pattern from photo to photo. The nebula has angular size $4' \times 6'$ and is $\sim$6 000 LY distant from the solar system. North is up and east to the left. The pulsar is the southwest (lower right) partner of the doublet at the center of the nebula. [Palomar Observatory/CalTech]