Interstellar Medium (ISM)

Week 8
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H II Regions and Strömgren Spheres

- ***Strömgren Sphere***:
  - Following Strömgren (1939), we consider the simple idealized problem of a fully ionized, spherical region of uniform medium plus a central source of ionizing photons.
  - The ionization is assumed to be maintained by absorption of the ionizing photons radiated by a central hot star. The central source produces ionizing photons, with energy $\nu > \nu_0 = I_{\rm H}/h$ at a constant rate Q_0 [photons s^{-1}].
  - At a distance r from the central star, the balance equation between ionization and recombination balance is

$$n_{\rm H^0}\int_{\nu_0}^{\infty}\frac{4\pi J_\nu}{h\nu}\sigma_\nu d\nu = n_p n_e \alpha_{\rm B,H}$$

L_ν = luminosity of the central star at frequency ν.

From the RT equation,

$$4\pi J_\nu = \underbrace{\frac{L_\nu}{4\pi r^2}e^{-\tau_\nu}}_{\text{geometrical attenuation + radiative absorption}}, \quad \text{where } \tau_\nu = \int_0^r n_{\rm H^0}\sigma_\nu dr$$

Integrating the balance equation over the whole volume:

$$\int_0^\infty\int_{\nu_0}^\infty \frac{L_\nu/h\nu}{4\pi r^2}e^{-\tau_\nu}n_{\rm H^0}\sigma_\nu d\nu(4\pi r^2)dr = \int_0^\infty n_p n_e \alpha_{\rm B,H}(4\pi r^2)dr$$

$$\int_{\nu_0}^\infty L_\nu/h\nu\left[\int_0^\infty e^{-\tau_\nu}n_{\rm H^0}\sigma_\nu dr\right]d\nu = \int_0^\infty n_p n_e \alpha_{\rm B,H}(4\pi r^2)dr$$

The square bracket term in the left side is

$$\int_0^\infty e^{-\tau_\nu} n_{\mathrm{H}^0} \sigma_\nu dr = \int_0^\infty e^{-\tau_\nu} d\tau_\nu = 1$$

Then, we obtain

total number of ionizing photons

$$Q_0 = \int_0^\infty n_p n_e \alpha_{\mathrm{B,H}} dV, \quad \text{where } Q_0 \equiv \int_{\nu_0}^\infty \frac{L_\nu}{h\nu} d\nu \text{ and } dV = 4\pi r^2 dr$$

- Assuming that ***the ionization is nearly complete (*** $n_p = n_e = n_{\mathrm{H}}$ ***) within*** R_s, and nearly zero ($n_{\mathrm{H}^0} = n_{\mathrm{H}},\ n_e = 0$) outside R_s, we obtain the size of the ionized region.

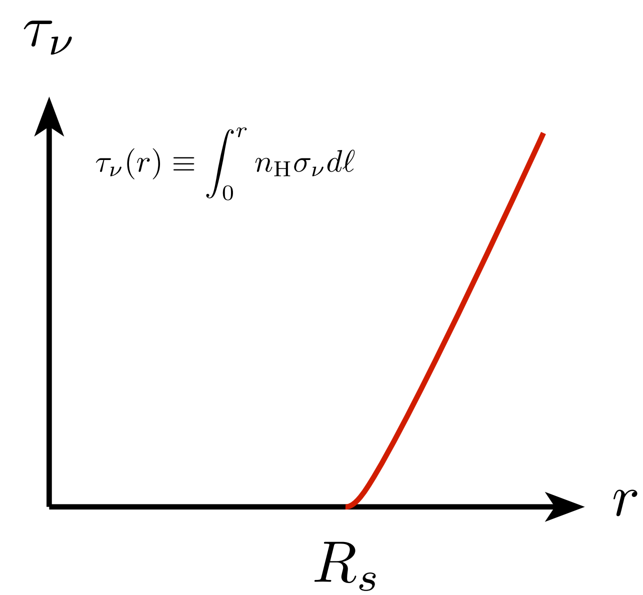
$$Q_0 = n_{\mathrm{H}}^2 \alpha_{\mathrm{B,H}} \frac{4\pi}{3} R_s^3$$

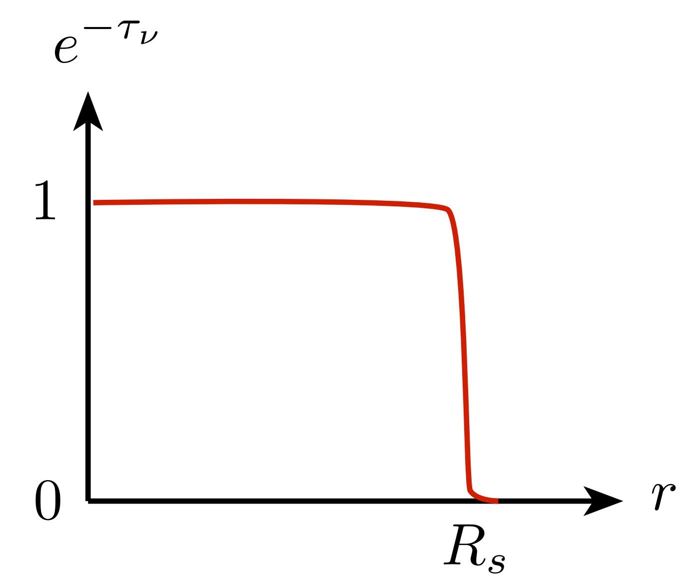
$$R_s = \left(\frac{3}{4\pi} \frac{Q_0}{\alpha_{\mathrm{B,H}} n_{\mathrm{H}}^2} \right)^{1/3}$$

$$= 3.17 \left(\frac{Q_0}{10^{49}\ \mathrm{photons\ s^{-1}}} \right)^{1/3} \left(\frac{n_{\mathrm{H}}}{10^2\ \mathrm{cm^{-3}}} \right)^{-2/3} \left(\frac{T}{10^4\ \mathrm{K}} \right)^{0.28} \quad [\mathrm{pc}]$$

The physical meaning of this is that ***the total number of ionizing photons emitted by the star balances the total number of recombinations within the ionized volume*** $(4\pi/3)R_s^3$, often called the Strömgren sphere. It's radius R_s is called the Strömgren radius.

- ***Opacity as a function of distance***
  - We note that the medium is fully ionized within the Strömgren sphere. Thus, within the Strömgren sphere, the opacity is nearly zero. The opacity suddenly increases at the boundary of the ionized region.





- ***Mean free path***
  - The mean free path of an ionizing photon is

$$\lambda_{\rm mfp} = \frac{1}{n_{\rm H}\sigma_{\rm pi}} \sim 5\times 10^{-4}\,{\rm pc}\left(\frac{n_{\rm H}}{10^2\,{\rm cm}^{-2}}\right)^{-1}\left(\frac{\sigma_{\rm pi}}{6.304\times 10^{-18}\,{\rm cm}^{-2}}\right)^{-1}$$

This tells us that the transition from ionized gas to neutral gas at the boundary of the H II region will occur over a distance that is very small compared to the Strömgren radius.

- Time Scales:
  - ***Ionization time scale***: The Strömgren sphere analysis assumes a steady state solution. What is the time scale for approach to the steady state? Suppose that we start with a neutral region, and the ionizing source is suddenly turned on.

$$t_{\rm ioniz.} = \frac{\text{total number of ions to be created}}{\text{number of photons supplied per unit time}}$$
$$= \frac{(4\pi/3)R_s^3 n_{\rm H}}{Q_0} = \frac{1}{\alpha_{\rm B,H} n_{\rm H}} = 1.22\times10^3\left(\frac{10^2\,{\rm cm}^{-3}}{n_{\rm H}}\right)\quad[\rm yr]$$

  - ***Recombination time scale***: Suppose that the ionizing source suddenly turns off. The ionized region will recombine on the recombination time scale:

$$t_{\rm rec} = \frac{1}{\alpha_{\rm B,H} n_{\rm H}} = 1.22\times10^3\left(\frac{10^2\,{\rm cm}^{-3}}{n_{\rm H}}\right)\quad[\rm yr]$$

Note that the recombination time scale is identical to the ionization time scale!

The ionization/recombination time scale is shorter than the main-sequence lifetime > 5 My for a massive star.

Ionization of Helium

- Now, what about helium?
  - Out of every 1000 atoms, there are on average 912 hydrogen atoms, 87 helium atoms and one heavy atom.
    - Looking at the photoionization cross sections for H^0, He^0, He^{+1}, we see that above the 24.6 eV threshold for ionizing He^0, the photoionization cross section for helium is larger than that for hydrogen.

$$\begin{aligned}\sigma_{\mathrm{pi,He^0}} &\approx 6.5\,\sigma_{\mathrm{pi,H^0}} \quad \text{at } h\nu \sim 24.6\,\mathrm{eV} \\ &\approx 14\,\sigma_{\mathrm{pi,H^0}} \quad \text{at } h\nu \sim 54.5\,\mathrm{eV}\end{aligned}$$

    - Thus, the photoionization cross section for He is ~ 10 times that of H, while the number density of He is ~ 0.1 times that of H.
    - This implies that if we suddenly turn on a hot star, ***the initial photons in the range*** $24.6\,\mathrm{eV} < h\nu < 54.4\,\mathrm{eV}$ ***will be about as likely to photoionize a helium atom as a hydrogen atom.***
    - ***In the range of*** $13.6\,\mathrm{eV} < h\nu < 24.6\,\mathrm{eV}$***, on the other hand, nearly all the photons go to ionize H***; scarcer atoms (metals like O and C) account for only a tiny fraction of the ionizations.

- ***Radiative Recombination of Helium***

$$\mathrm{He}^{++} + e^- \rightarrow \mathrm{He}^+$$

$$\alpha_\mathrm{A}(T) \approx 4.13 \times 10^{-13}\, Z (T_4/Z^2)^{-0.7131-0.0115\ln(T_4/Z^2)} \quad [\mathrm{cm^3\,s^{-1}}] \quad (30\,\mathrm{K} < T/Z^2 < 3\times10^4\,\mathrm{K})$$

$$\alpha_\mathrm{B}(T) \approx 2.54 \times 10^{-13}\, Z (T_4/Z^2)^{-0.8163-0.0208\ln(T_4/Z^2)} \quad [\mathrm{cm^3\,s^{-1}}]$$

$$\mathrm{He}^{+} + e^- \rightarrow \mathrm{He}^0$$

$$\alpha_{1\mathrm{s}^2,\mathrm{He}} = 1.54 \times 10^{-13}\, T_4^{-0.486} \quad [\mathrm{cm^3\,s^{-1}}] \qquad (0.5 < T_4 < 2)$$

$$\alpha_{\mathrm{B,He}} = 2.72 \times 10^{-13}\, T_4^{-0.789} \quad [\mathrm{cm^3\,s^{-1}}]$$

Here, $\alpha_{1\mathrm{s}^2,\mathrm{He}}$ is the recombination rate to the ground state $1\mathrm{s}^{2\,1}S_0$.,

and $\alpha_{\mathrm{B,He}}$ is the recombination rate coefficient to all states except the ground state.

Note: $\alpha_{\mathrm{B,H}} \approx \alpha_{\mathrm{B,He}}$ and $\alpha_{\mathrm{A,H}} \approx \alpha_{\mathrm{A,He}}$.

- ***Effective recombination rate coefficient for Helium***
  - Note that ***the stellar LyC photons with*** $h\nu > 24.6\,\mathrm{eV}$ ***are capable of photoionizing not only neutral helium atoms but also neutral hydrogen atoms***
  - The recombinations directly to the ***ground state*** $1\mathrm{s}^{2\ 1}S_0$ of neutral helium produce photons with $h\nu > 24.6\,\mathrm{eV}$. ***The recombination continuum photons are capable of photoionizing not only neutral helium atoms but also neutral hydrogen atoms***; ***the fraction of these that ionize hydrogen*** is

$$y = \frac{n_{\mathrm{H}^0}\sigma_{\mathrm{pi,H}^0}(E)}{n_{\mathrm{H}^0}\sigma_{\mathrm{pi,H}^0}(E) + n_{\mathrm{He}^0}\sigma_{\mathrm{pi,He}^0}(E)}$$

$$= \left[1 + \frac{n_{\mathrm{He}^0}}{n_{\mathrm{H}^0}}\frac{\sigma_{\mathrm{pi,He}^0}(E)}{\sigma_{\mathrm{pi,H}^0}(E)}\right]^{-1}, \quad \text{where } E \approx 24.6\,\mathrm{eV} + kT$$

$$\sigma_{\mathrm{pi,He}^0}/\sigma_{\mathrm{pi,H}^0} > 6.0 \quad \text{for } E > 24.6\,\mathrm{eV}$$

$$y < 0.5 \quad \text{if } n_{\mathrm{He}^0}/n_{\mathrm{H}^0} > 0.16$$

In an optically thick gas, the effective radiative recombination rate coefficient for $\mathrm{He}^+ \to \mathrm{He}^0$ is then

$$\alpha_{\mathrm{eff,He}} = \alpha_{\mathrm{B,He}} + y\alpha_{1\mathrm{s}^2,\mathrm{He}} = \alpha_{\mathrm{A,He}} - (1-y)\alpha_{1\mathrm{s}^2,\mathrm{He}}$$

At T = 10,000 K,

$$\alpha_{\mathrm{B,He}} = 2.72\times 10^{-13}\ [\mathrm{cm^3\,s^{-1}}] \quad \to \quad \alpha_{\mathrm{eff,He}} \approx 3.0\times 10^{-13}\ [\mathrm{cm^3\,s^{-1}}]$$

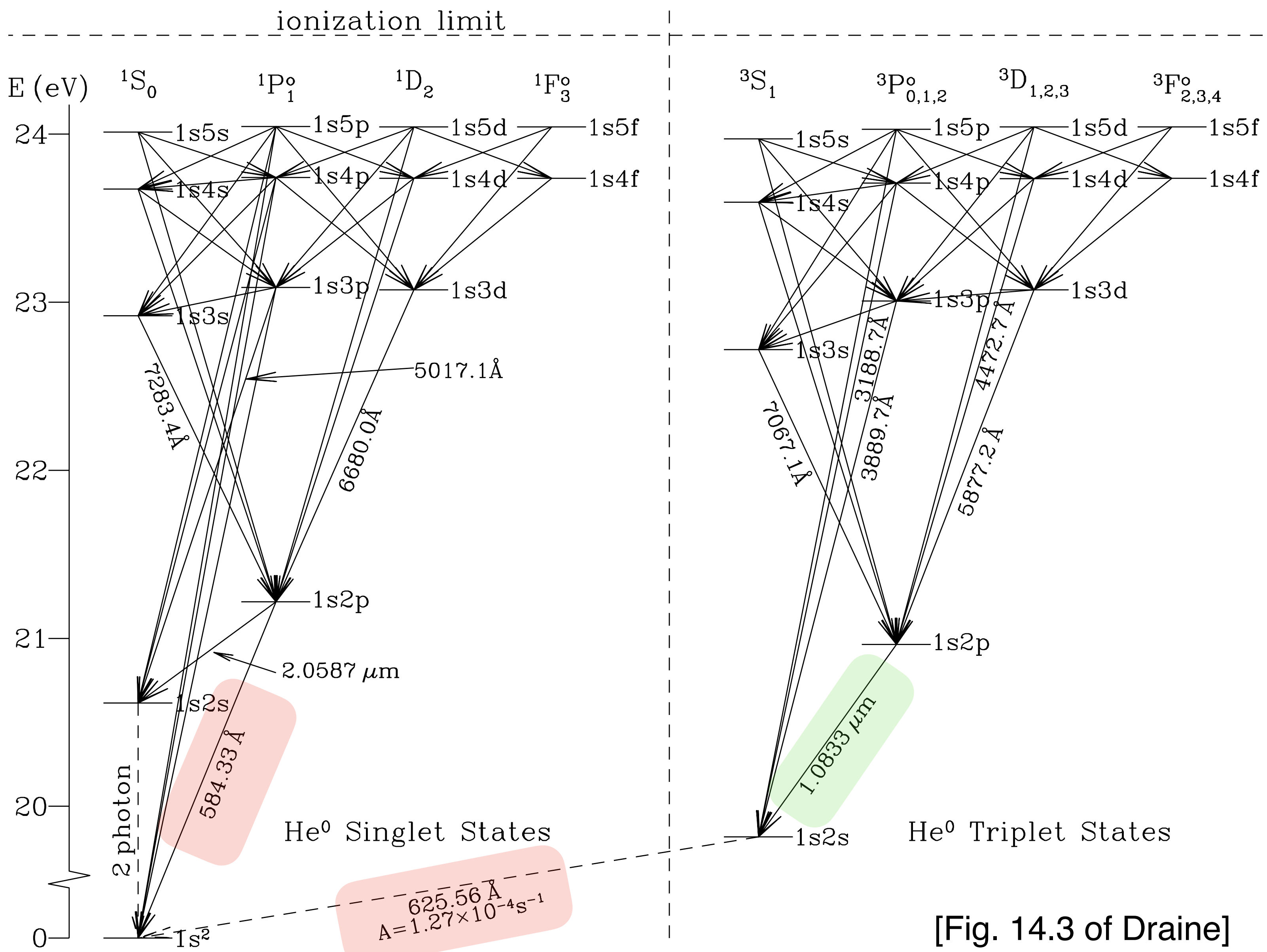
$$\alpha_{1\mathrm{s}^2,\mathrm{He}} = 1.54\times 10^{-13}\ [\mathrm{cm^3\,s^{-1}}] \qquad \approx 1.2\,\alpha_{\mathrm{B,H}}$$

$$y \approx 0.2$$

- This is not all. Consider now the recombination to ***excited levels*** of He⁰, which are followed by a radiative cascade down. Most of photons produced by the cascades have $h\nu > 13.6\,\mathrm{eV}$. ***A fraction of these photons are capable of photoionizing hydrogen. Let z be this fraction.*** However, note that ***this fraction is not relevant to the recombination of He, but contribute to the photoionization H.***

$$z \approx 0.96 \text{ at low densities}$$
$$\approx 0.67 \text{ at high densities}$$

We take an intermediate value $z \approx 0.8$.



[Fig. 14.3 of Draine]

- See Section 14.3.2 and 15.5 of [Draine] for details.
- [Ryden] assumes that $z = 1$.

Note that 1.08 μm is useful to probe the planetary atmosphere

- ***How many recombinations occur for He:*** Suppose that we have a Strömgren sphere with the cosmic abundance ratio of helium to hydrogen $f \equiv n_{\rm He}/n_{\rm H} \approx 0.096$. Now define:

$$Q_0 \equiv \int_{I_{\rm H}/h}^{\infty} \frac{L_\nu}{h\nu} d\nu, \quad Q_1 \equiv \int_{I_{\rm He}/h}^{\infty} \frac{L_\nu}{h\nu} d\nu \quad (Q_1 < Q_0)$$

  - In the very central region, the hydrogen would be fully ionized, and the helium would be all singly ionized. Even the hottest O stars don't produce a significant number of photons with $h\nu > 54.5\,{\rm eV}$; hence, there will be no significant amount of doubly ionized He^{+2}.
  - This will result in

$$\begin{aligned} n_p &= n_{\rm H} \\ n_{\rm He^+} &= n_{\rm He} = f n_{\rm H} \\ n_e &= n_p + n_{\rm He^+} = (1+f) n_{\rm H} \end{aligned}$$

inside the Strömgren sphere.

  - The volumetric rate of the hydrogen recombination is

$$\frac{dn_p}{dt} = -\alpha_{\rm B,H} n_e n_p = -\alpha_{\rm B,H}(1+f) n_{\rm H}^2$$

  - The volumetric rate of He recombination is

$$\frac{dn_{\rm He^+}}{dt} = -\alpha_{\rm eff,He} n_e n_{\rm He^+} = -\alpha_{\rm eff,He} f (1+f) n_{\rm H}^2$$

- Comparing the two equations, we see that

$$\begin{aligned}\frac{dn_{\mathrm{He}^+}}{dt} &= \left(\frac{\alpha_{\mathrm{eff,He}}}{\alpha_{\mathrm{B,H}}}\right) f \frac{dn_p}{dt} \\ &\approx (1.2)(0.096)\frac{dn_p}{dt} \\ &\approx 0.11\frac{dn_p}{dt}\end{aligned}$$

- Thus, for every helium recombination, we expect about 9 hydrogen recombinations.

- Remember the recombination paths, under the Case B condition:
  - $13.6\,\mathrm{eV} < h\nu < 24.6\,\mathrm{eV}$: A stellar photon will ionize one H atom.
  - $h\nu > 24.6\,\mathrm{eV}$: For a fraction of y of the photoionization followed by the direct recombinations to the ground state, a stellar photon will ionize one H atom. For the remaining fraction $(1-y)$ of these, a stellar photon will ionize one He atom.
  - $h\nu > 24.6\,\mathrm{eV}$: For the photoionization followed by the recombinations to excited states, a stellar photon will ionize one H atom for a fraction of z of the recombination events.

- ***Number of ionized atoms***: The number of ionized helium and hydrogen, $N(\mathrm{He}^+)$ and $N(\mathrm{H}^+)$, within the ionized regions can be estimated by balancing recombinations and photoionizations:

$$\boxed{\begin{aligned} &N(\mathrm{He}^+)n_e\left(\alpha_{\mathrm{B,He}} + y\alpha_{1\mathrm{s}^2,\mathrm{He}}\right) = (1-y)Q_1 \\ &N(\mathrm{H}^+)n_e\alpha_{\mathrm{B,H}} = (Q_0 - Q_1) + yQ_1 + N(\mathrm{He}^+)n_e\left(z\alpha_{\mathrm{B,He}} + y\alpha_{1\mathrm{s}^2,\mathrm{He}}\right) \end{aligned}}$$

Contribution by the recombination to the ground state.

stellar LyC with $h\nu > 24.5\,\mathrm{eV}$ that ionize H

Contribution by the recombination to the excited state.

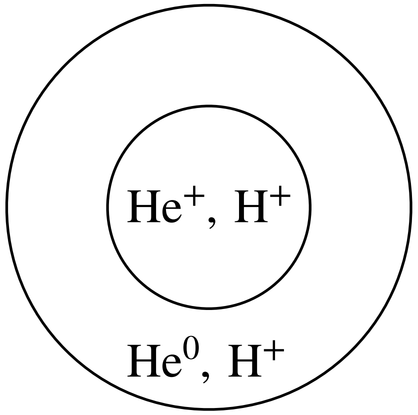
$$\rightarrow \quad N(\mathrm{H}^+)n_e\alpha_{\mathrm{B,H}} = Q_0 - N(\mathrm{He}^+)n_e\left(1-z\right)\alpha_{\mathrm{B,He}}$$

$$\longrightarrow \quad \boxed{\frac{N(\mathrm{He}^+)}{N(\mathrm{H}^+)} = \frac{(1-y)\alpha_{\mathrm{B,H}}(Q_1/Q_0)}{\alpha_{\mathrm{B,He}} + y\alpha_{1\mathrm{s}^2,\mathrm{He}} - (1-y)(1-z)(Q_1/Q_0)\alpha_{\mathrm{B,He}}}}$$

$$\frac{N(\mathrm{He}^+)}{N(\mathrm{H}^+)} \approx \frac{0.68(Q_1/Q_0)}{1-0.17(Q_1/Q_0)} \quad \text{for } z \approx 0.8,\ T = 8000\,\mathrm{K},\ \text{and } y = 0.2$$

- Condition for full ionization of the He in the H+ Strömgren sphere:

$$\frac{N(\mathrm{He}^+)}{N(\mathrm{H}^+)} = \frac{n_{\mathrm{He}}}{n_{\mathrm{H}}} = 0.096 \quad \rightarrow \quad \frac{Q_1}{Q_0} \approx 0.15$$



- ***Radius of the He+ zone***:

$$N(\mathrm{He}^+) = \frac{4\pi}{3} R_{\mathrm{He}}^3 n_{\mathrm{He}}$$

$$N(\mathrm{H}^+) = \frac{4\pi}{3} R_{\mathrm{H}}^3 n_{\mathrm{H}}$$

$$\frac{R_{\mathrm{He}}}{R_{\mathrm{H}}} = \left[\frac{n_{\mathrm{H}}}{n_{\mathrm{He}}} \frac{N(\mathrm{He}^+)}{N(\mathrm{H}^+)}\right]^{1/3} = \left[\frac{7.08(Q_1/Q_0)}{1-0.17(Q_1/Q_0)}\right]^{1/3}$$

$$R_{\mathrm{He}} < R_{\mathrm{H}} \quad \text{if} \quad Q_1/Q_0 \lesssim 0.15$$

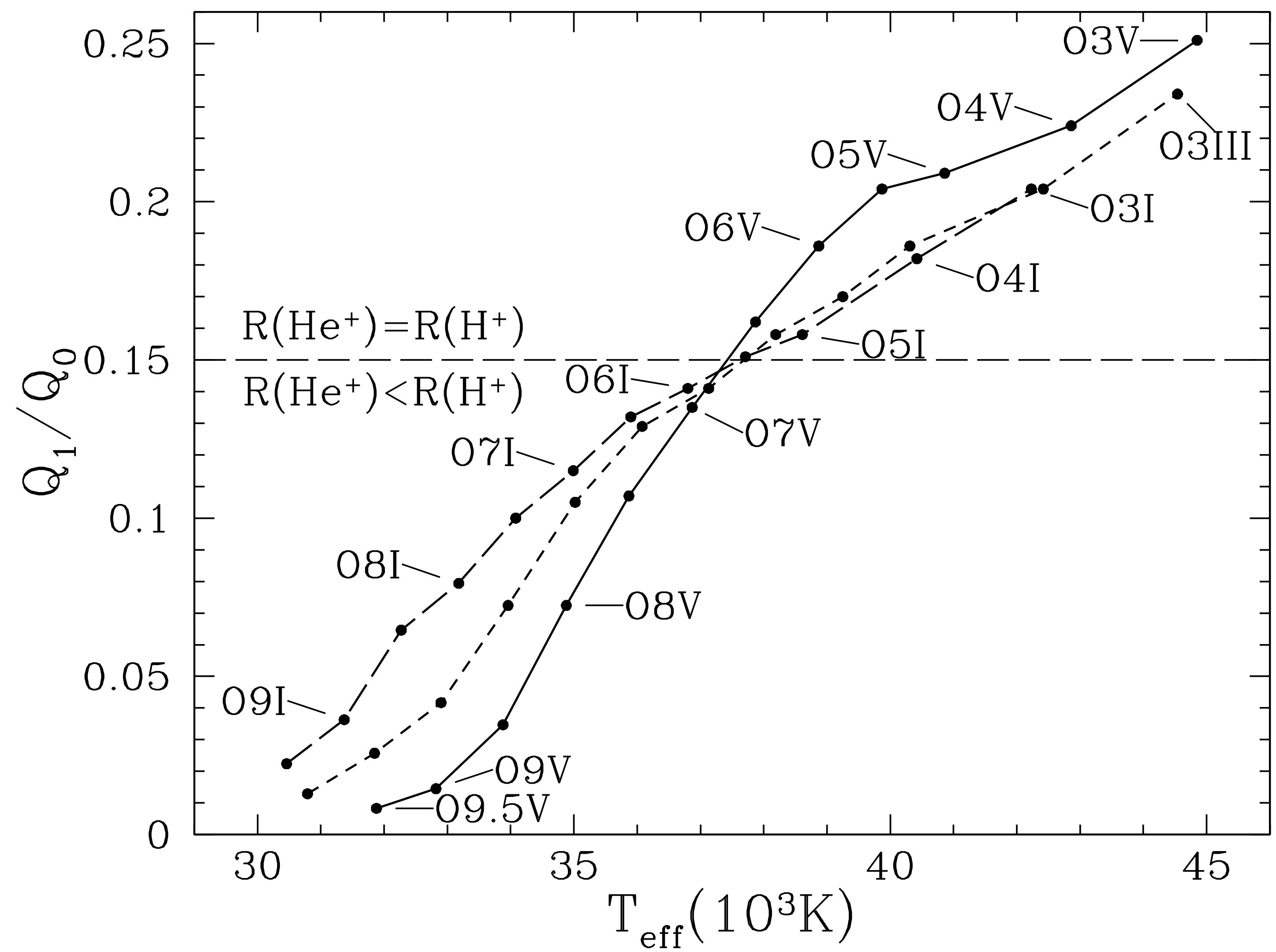
- On the main sequence, a star with spectral class O7, corresponding to effective temperature T_{eff} = 37,000 K, will have a critical ratio $Q_1/Q_0 \sim 0.14$.
  - For cooler ionizing stars, the ionized helium sphere will have a radius that is smaller than the radius of the ionized hydrogen sphere.
  - For stellar temperature T_{eff} > 37,000 K, the ionized helium sphere has the same size as the ionized hydrogen sphere, because of the limit on the abundance. The photons with $h\nu > 24.6\,\mathrm{eV}$ will be used up to ionize H.

Table 15.1 [Draine]

SpTp	$M/M_\odot$	$T_{\rm eff}$(K)	$\log_{10}(Q_0/{\rm s}^{-1})$ [b]	Q_1/Q_0 [c]	$\log_{10}(L/L_\odot)$ [d]
O3V	58.0	44850	49.64	0.251	5.84
O4V	46.9	42860	49.44	0.224	5.67
O5V	38.1	40860	49.22	0.209	5.49
O5.5V	34.4	39870	49.10	0.204	5.41
O6V	31.0	38870	48.99	0.186	5.32
O6.5V	28.0	37870	48.88	0.162	5.23
O7V	25.3	36870	48.75	0.135	5.14
O7.5V	22.9	35870	48.61	0.107	5.05
O8V	20.8	34880	48.44	0.072	4.96
O8.5V	18.8	33880	48.27	0.0347	4.86
O9V	17.1	32830	48.06	0.0145	4.77
O9.5V	15.6	31880	47.88	0.0083	4.68
O3III	56.0	44540	49.77	0.234	5.96
O4III	47.4	42420	49.64	0.204	5.85
O5III	40.4	40310	49.48	0.186	5.73
O5.5III	37.4	39250	49.40	0.170	5.67
O6III	34.5	38190	49.32	0.158	5.61
O6.5III	32.0	37130	49.23	0.141	5.54
O7III	29.6	36080	49.13	0.129	5.48
O7.5III	27.5	35020	49.01	0.105	5.42
O8III	25.5	33960	48.88	0.072	5.35
O8.5III	23.7	32900	48.75	0.0417	5.28
O9III	22.0	31850	48.65	0.0257	5.21
O9.5III	20.6	30790	48.42	0.0129	5.15
O3I	67.5	42230	49.78	0.204	5.99
O4I	58.5	40420	49.70	0.182	5.93
O5I	50.7	38610	49.62	0.158	5.87
O5.5I	47.3	37710	49.58	0.151	5.84
O6I	44.1	36800	49.52	0.141	5.81
O6.5I	41.2	35900	49.46	0.132	5.78
O7I	38.4	34990	49.41	0.115	5.75
O7.5I	36.0	34080	49.31	0.100	5.72
O8I	33.7	33180	49.25	0.079	5.68
O8.5I	31.5	32270	49.19	0.065	5.65
O9I	29.6	31370	49.11	0.0363	5.61
O9.5I	27.8	30460	49.00	0.0224	5.57

[a] After Martins et al. (2005).
[b] Q_0 = rate of emission of $h\nu > 13.6$ eV photons.
[c] Q_1 = rate of emission of $h\nu > 24.6$ eV photons.
[d] L = total electromagnetic luminosity.

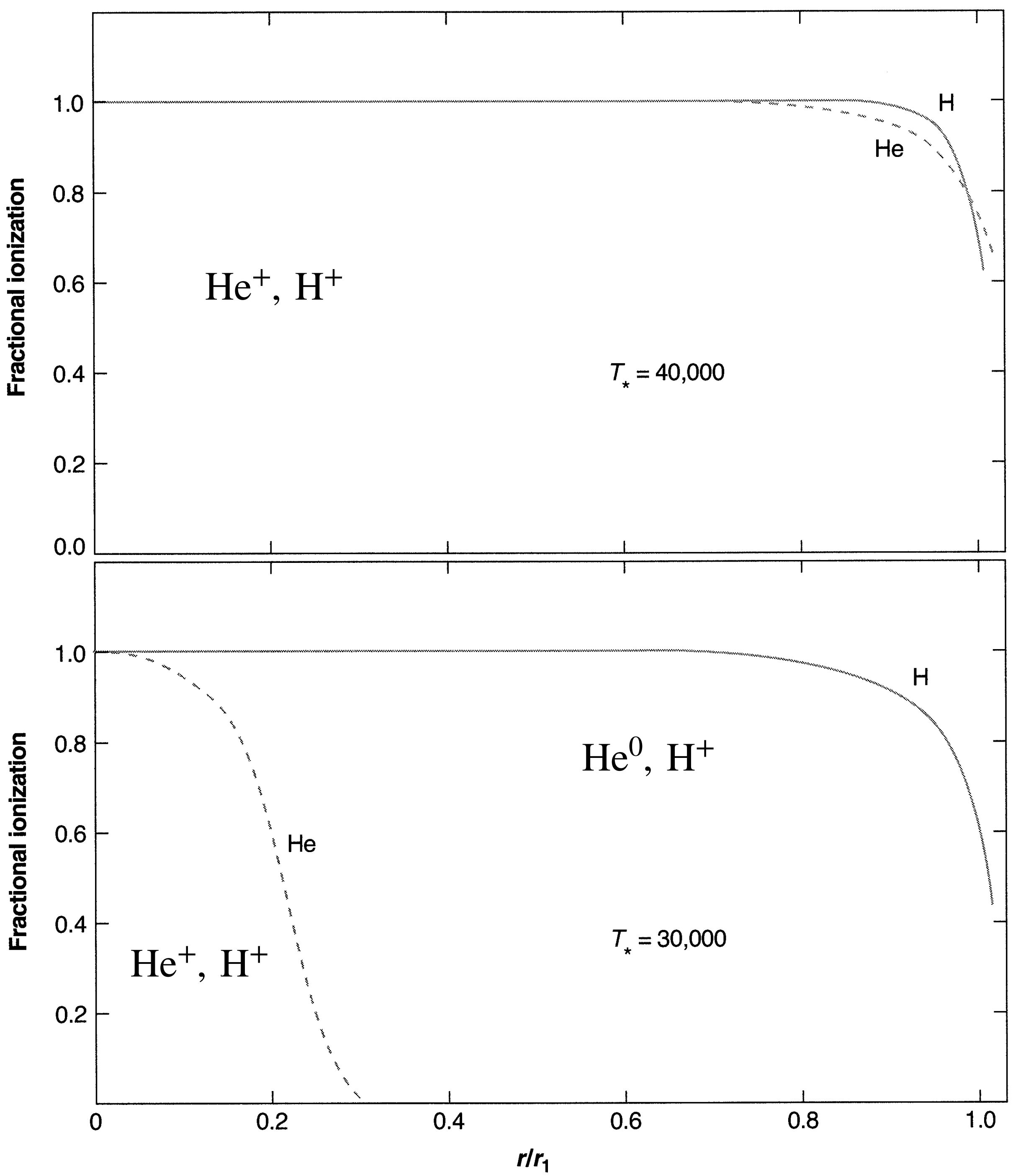
Figure 15.5 [Draine]



Q_1/Q_0 vs. stellar temperature

$Q_1/Q_0 > 0.15$ is required for He to be ionized throughout the H II region, corresponding to $T_{\rm eff} > 37{,}000$ K.

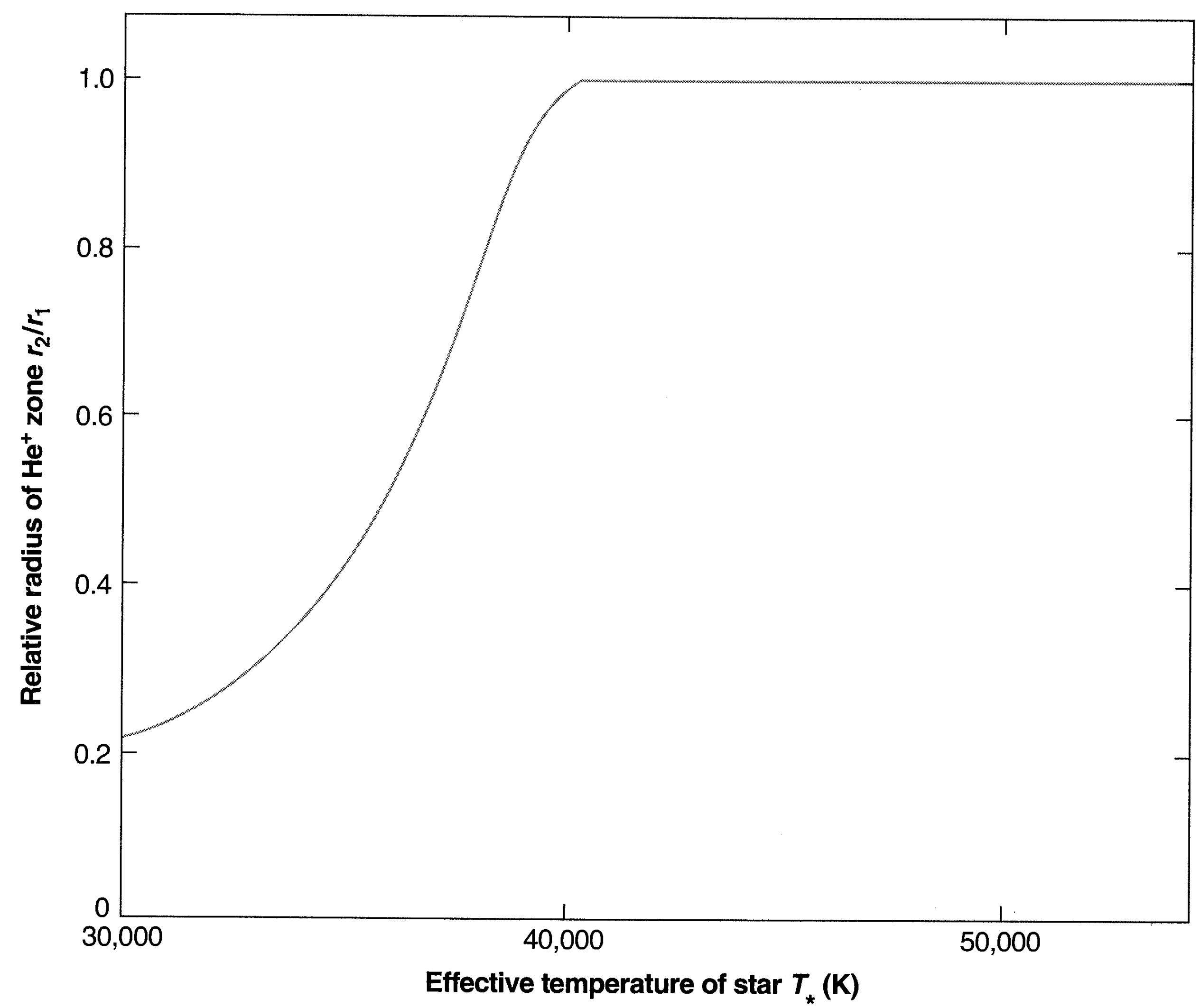
Figure 2.4 [Osterbrock]



Ionization structure of two homogeneous H + He models for H II regions.

Figure 2.5 [Osterbrock]

$r_1 = R(\mathrm{H})$
$r_2 = R(\mathrm{He})$



Relative radius of He+ zone as a function of effective temperature of exciting star.

- ***Metals***: Ions that requires $E > 24.6$ eV for their formation will present only in the He^{+} zone.

[Draine] **Table 15.2** Abundant Ions in H II Regions[a]

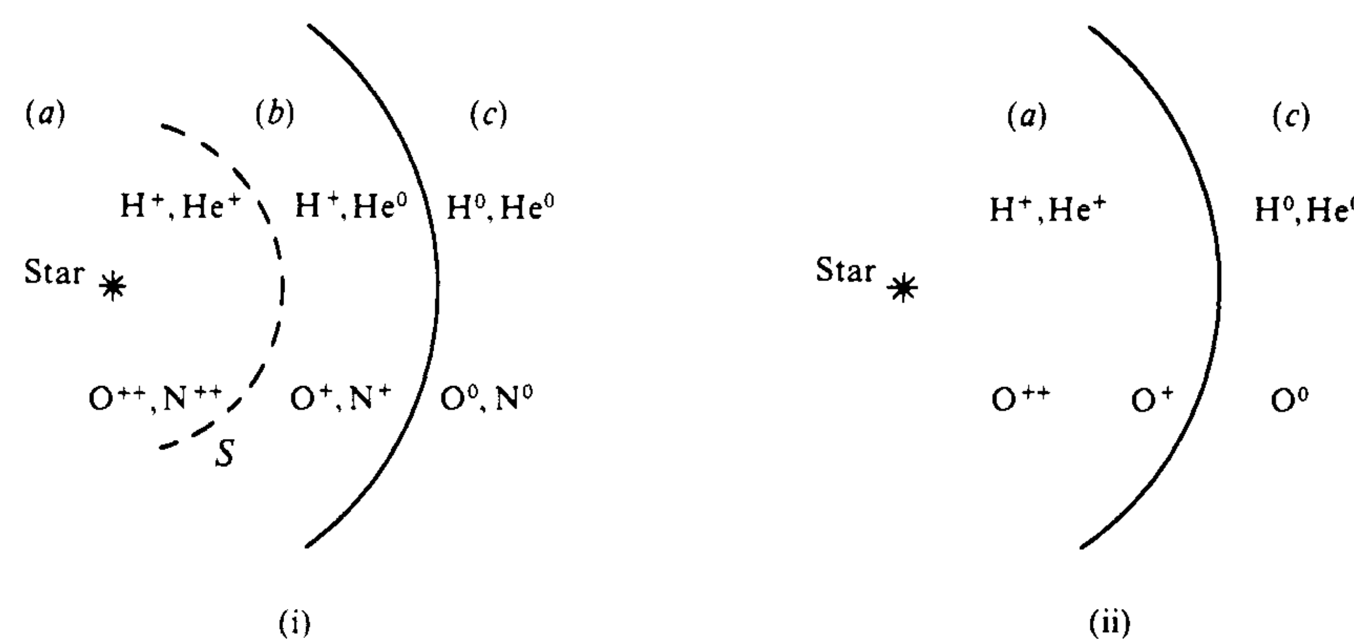
	H II and He I zone[b]		H II and He II zone[c]	
Element	Ion	$h\nu$ (eV)[d]	Ion	$h\nu$ (eV)[d]
H	H II	13.60	H II	13.60
He	He I	0	He II	24.59
C	C II	11.26	C III[e]	24.38
			C IV	47.88
N	N II	14.53	N III	29.60
			N IV	47.45
O	O II	13.62	O III	35.12
Ne	Ne II	21.56	Ne III	40.96
Na	(Na II)[f]	5.14	(Na II)[f]	5.14
			Na III	47.29
Mg	Mg II	7.65	(Mg III)[f]	15.04
	(Mg III)[f]	15.04		
Al	Al III	18.83	(Al IV)[f]	28.45
Si	Si III	16.35	Si IV	33.49
			(Si V)[f]	45.14
S	S II	10.36	S III	23.33
	S III	23.33	S IV	34.83
Ar	Ar II	15.76	Ar III	27.63
			Ar IV	40.74
Ca	Ca III	11.87	Ca IV	50.91
Fe	Fe III	16.16	Fe IV	30.65
Ni	Ni III	18.17	Ni IV	35.17

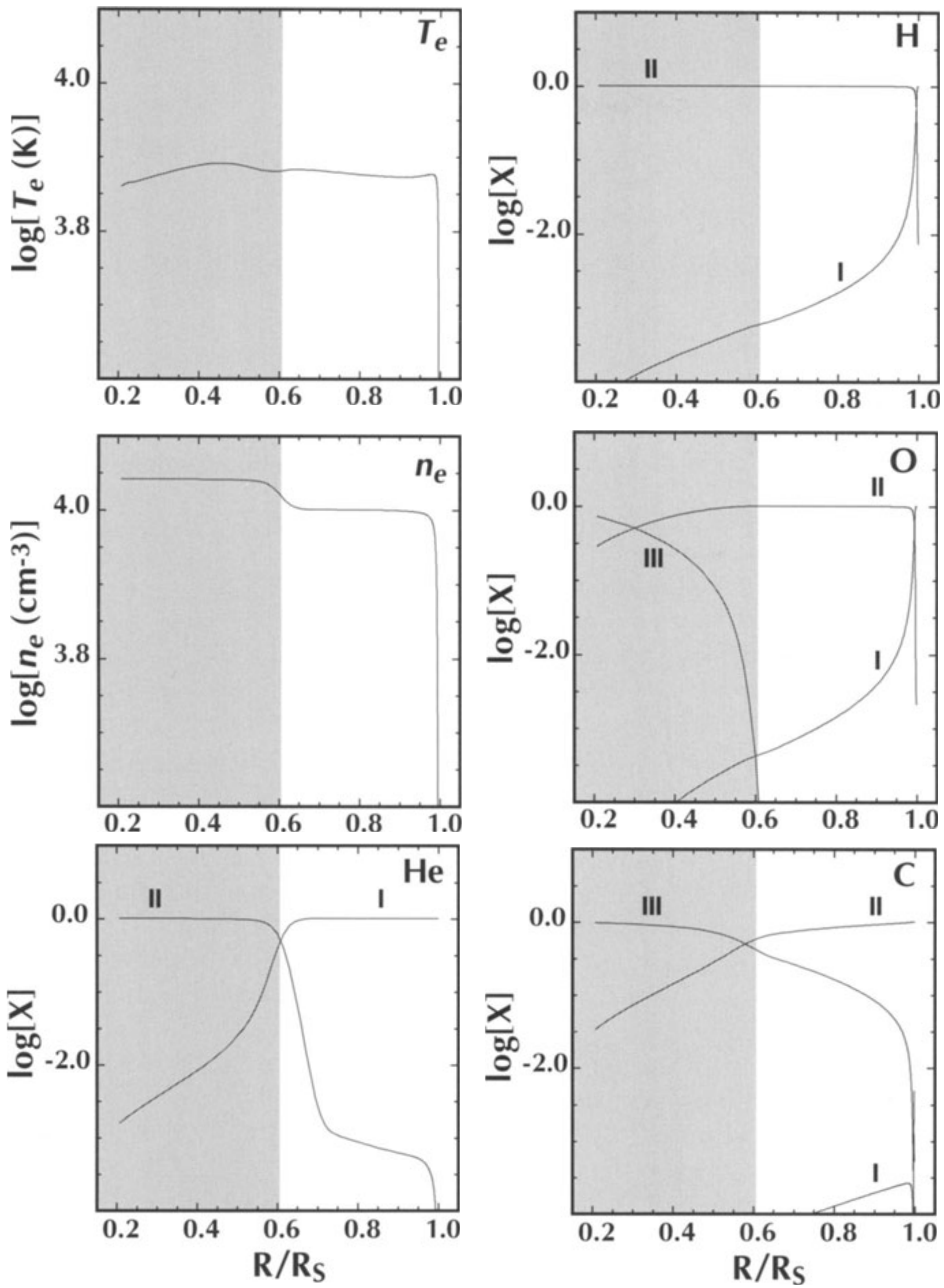
[a] Limited to elements X with $N_X/N_H > 10^{-6}$.
[b] Ions that can be created by radiation with $13.60 < h\nu < 24.59$ eV.
[c] Ions that can be created by radiation with $24.59 < h\nu < 54.42$ eV.
[d] Photon energy required to create ion.
[e] Ionization potential is just below 24.59 eV.
[f] Closed shell, with no excited states below 13.6 eV.

[Page 238, Dopita]
The dominant ionization zones of the nebula for the most abundant elements and important coolants are as follows:

H I, He I	:	C II, N I, O I, Ne I, S II,
H II, He I	:	C II, (C III), N II, O II, Ne II, S II, (S III),
H II, He II	:	C III, (C IV), N III, O III, Ne III, S III, (S IV, S V),
H II, He III	:	C IV, N IV, O IV, Ne III, S V, and higher,

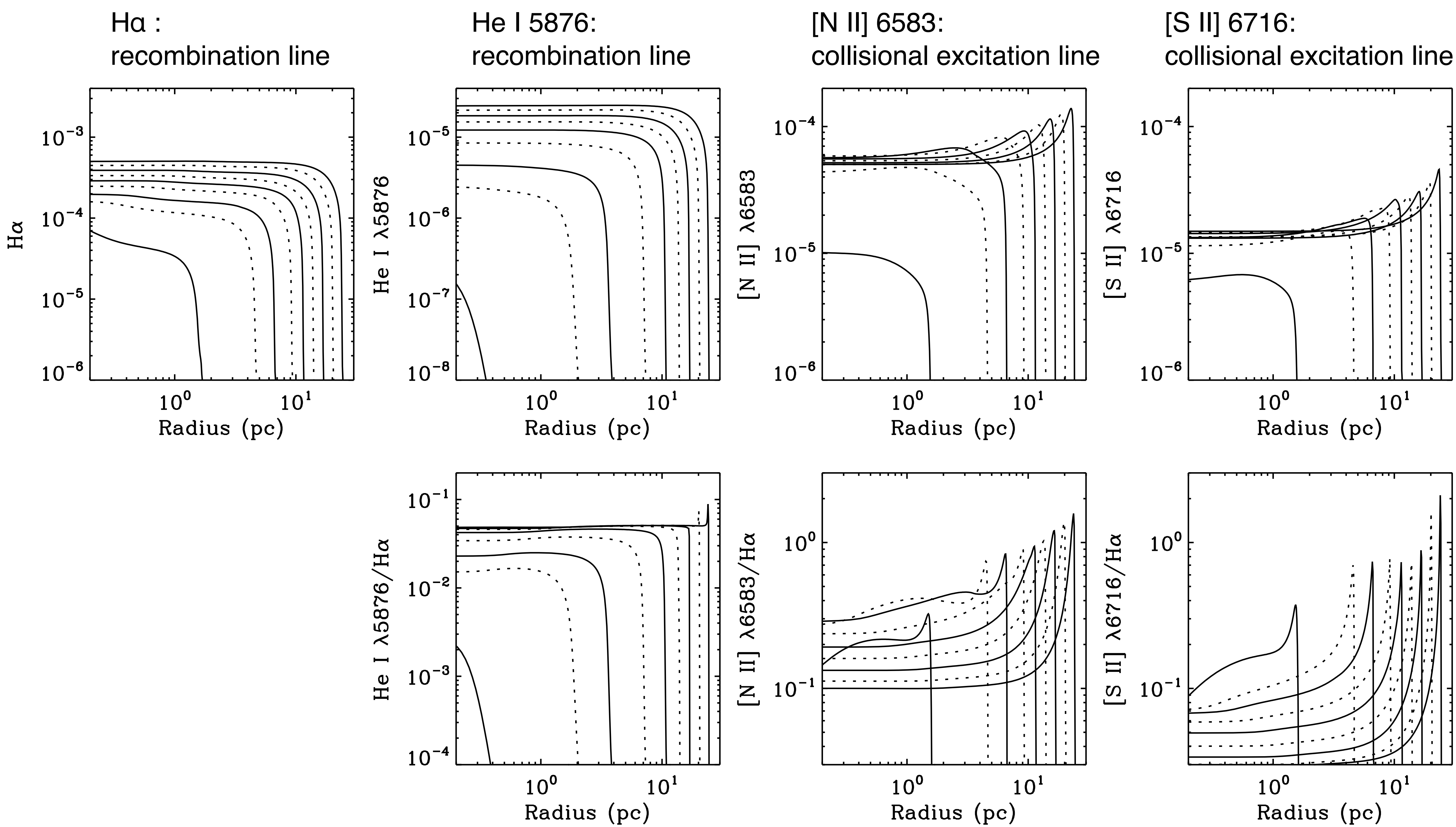
[Figure 5.3, Dyson]
Ionization stratification in a nebula. (i) Low stellar temperature, (ii) High stellar temperature





[Figure 9.4, Dopita, Astrophysics of the Diffuse Universe]

The temperature, density, and ionization structure of a model H II region illuminated by a star with an effective temperature of 53,000 K. Note how the ionization structure in the heavy elements follows that of hydrogen and helium.



[Seon & Witt, 2012, ApJ, 758, 19]

Figure 4. Top: brightness profiles of Hα, He I λ5867, [N II] λ6583, and [S II] λ6716 lines (in units of erg cm^{-2} s^{-1} sr^{-1}) for various central ionization sources. Bottom: brightness profiles of line ratios He I/Hα, [N II]/Hα, and [S II]/Hα. Elemental abundances for WNM and hydrogen density of $n_{\rm H} = 10$ cm^{-3} were assumed for the photoionization models. The curves from the outermost to innermost correspond to O3V to B1V stars progressively. Solid and dashed lines were alternatively used for clarification.

Recombination lines

- Recombination Radiation = Recombination Lines + Recombination Continuum
- Diagnostics using the recombination lines
  - ***Temperature***: The hydrogen recombination spectrum depends on temperature *T*, and therefore measured line ratios can be used to estimate *T*.
  - ***Reddening***: Measurements of the relative intensities of recombination lines with different wavelengths can be used to estimate the reddening by dust between us and the emitting region.
- ***Case A Recombination Spectrum***
  - In the optically thin limit, the power radiated per volume in the transition $n\ell \to n'\ell'$ is

$$4\pi j(n\ell \to n'\ell') = n_e n_p \frac{A(n\ell \to n'\ell') h\nu_{n\ell \to n'\ell'}}{\sum_{n''\ell''} A(n\ell \to n''\ell')} \times \left[\underbrace{\alpha(n\ell)}_{\text{direct recombination}} + \underbrace{\sum_{n''\ell''\ (n''>n)} \alpha(n''\ell'') P_A(n''\ell', n\ell)}_{\text{cascade}} \right]$$

Note a typo in Eq (14.7) of Draine

$P_A(n''\ell'', n\ell)$ is the Case A probability that an atom in level $n''\ell''$ will follow a decay path that takes it through level $n\ell$. It can be readily calculated from the known transition probabilities $A(n\ell \to n'\ell')$ using straightforward branching probability arguments.

- ***Case B Recombination Spectrum***
  - The resonant absorption cross-sections for Lyα, Lyβ,… are much larger than photoionization cross sections.

$$\tau_0(\mathrm{Ly}\alpha) = 8.02 \times 10^4 \left(\frac{15\,\mathrm{km\,s^{-1}}}{b} \right) \tau(\mathrm{Ly\ cont})$$

$$\tau(\mathrm{Ly\ cont}) = 6.30 \times 10^{-18}\,\mathrm{cm}^2 N(\mathrm{H})$$

  - ***Any nebula that is optically thick to Lyman continuum (E > 13.6 eV) will be very optical thick to all of the Lyman series ($n \rightarrow 1$) transitions.***
  - (Note that the cross sections for resonant absorption in the $1 \rightarrow n$ transitions becomes equal to the photoionization cross section as $n \rightarrow \infty$.)

$$\tau_{\mathrm{reson.}}(1 \rightarrow n) \geq \tau_{\mathrm{reson.}}(1 \rightarrow \infty) = \tau_{\mathrm{photo.}}$$

- **On-the-spot approximation**:
    - **Therefore, under Case B condition, Lyman series photons will (immediately) be resonantly absorbed by other hydrogen atoms in the ground state.** They will travel only a short distance before being reabsorbed.
    - It is helpful to think about the radiative decay and resonant reabsorption as through the photon were reabsorbed by the same atom as emitted.
    - Consider a hydrogen atom in level $n \geq 3$ (for instance, $n = 3$) . Then, Lyβ, Lyɣ,… will immediately be resonantly absorbed, returning back to the initial state $n \geq 3$. After returning to the initial state, the atom will again decay one of its allowed decay paths (for instance, $3 \to 2 \to 1$ and $4 \to 2 \to 1$). The atom may emit another Lyman series photon, which will be again be absorbed.
    - This process will repeat until eventually **“non-Lyman transitions” + a “Lyα transition” (or “non-Lyman transitions” + 2-photon transition)** occur.

      For instance,

      Hα(3-2) + Lyα(2-1) for n = 3

      Pα(4-3) + Hα(3-2) + Lyα(2-1) or Hβ(4-2) + Lyα(2-1) for n = 4.

      Two-photon continuum emission can also occur, if the repeated process eventually populates 2s state, instead of 2p.
    - Under this condition, no Lyman series (except for Lyα) lines will be produced.

- ***Balmer lines***:
  - Under Case B condition, the rate coefficients for recombinations that result in emission of Hα, Hβ can be approximated by

$$\alpha_{\mathrm{eff,H}\alpha} \approx 1.17 \times 10^{-13}\, T_4^{-0.942-0.031\ln T_4} \quad [\mathrm{cm^3\,s^{-1}}] \qquad (T_4 \equiv T/10^4\,\mathrm{K})$$
$$\alpha_{\mathrm{eff,H}\beta} \approx 3.03 \times 10^{-14}\, T_4^{-0.874-0.058\ln T_4} \quad [\mathrm{cm^3\,s^{-1}}]$$

  - Emissivities of Balmer lines:

    Using the statistical balance for the level population, we can obtain the emissivity. (Note that, ***in the case of hydrogen and helium, the population caused collisional excitation is negligible.***)

Population of u state by recombination = Depopulation by radiative decay.

$$4\pi j_{u\ell} = n_u A_{u\ell}\,(h\nu_{u\ell}) = n_e n_p \alpha_{\mathrm{eff},u}\,(h\nu_{u\ell})$$

$$4\pi j_{\mathrm{H}\alpha} = n_e n_p \alpha_{\mathrm{eff,H}\alpha} h\nu_{\mathrm{H}\alpha}$$
$$4\pi j_{\mathrm{H}\beta} = n_e n_p \alpha_{\mathrm{eff,H}\beta} h\nu_{\mathrm{H}\beta}$$

  - ***Balmer Decrement*** : The ratio between Balmer lines can be used **to estimate the dust reddening.**

$$\frac{j_{\mathrm{H}\alpha}}{j_{\mathrm{H}\beta}} = \frac{\alpha_{\mathrm{eff,H}\alpha}}{\alpha_{\mathrm{eff,H}\beta}} \frac{\nu_{\mathrm{H}\alpha}}{\nu_{\mathrm{H}\beta}} = 2.86\, T_4^{-0.068+0.027\ln T_4}$$

Note: $\lambda_{\mathrm{H}\alpha} = 6563\text{Å}$

$\lambda_{\mathrm{H}\beta} = 4861\text{Å}$

See Table 14.2 of Draine for other lines.

- *Lyman α*

- Let $\alpha_{\rm eff,2s}$ and $\alpha_{\rm eff,2p}$ be the effective rate coefficients for populating the 2s and 2p states. By definition, it is clear that the case B radiative recombination process must eventually take the atom to either the 2s level or the 2p level. Thus,

$$\alpha_{\rm eff,2s} + \alpha_{\rm eff,2p} = \alpha_{\rm B}$$

- The fractions $f(2s) \equiv \dfrac{\alpha_{\rm eff,2s}}{\alpha_{\rm B}} \approx \dfrac{1}{3}$ and $f(2p) \equiv \dfrac{\alpha_{\rm eff,2p}}{\alpha_{\rm B}} \approx \dfrac{2}{3}$ are given in the following table.

T(K)	f(2s)	f(2p)
4000	0.285	0.715
5000	0.305	0.695
10000	0.325	0.675
20000	0.356	0.644

Tables 14.2 and 14.3 of [Draine]

A minor discrepancy between this and Cantalupo et al. (2008, ApJ, 672, 48):

$$\begin{aligned} f(\mathrm{Ly}\alpha) =& 0.686 - 0.106 \log(T/10^4\,\mathrm{K}) \\ & - 0.009\,(T/10^4\,\mathrm{K})^{-0.44} \end{aligned}$$

- Then, the emissivity for Lyα is

$$\begin{aligned} 4\pi j_{\mathrm{Ly}\alpha} &= n_e n_p \alpha_{\rm eff,2p} h\nu_{\mathrm{Ly}\alpha} \\ &\approx \frac{2}{3} n_e n_p \alpha_{\rm B} h\nu_{\mathrm{Ly}\alpha} \end{aligned}$$

In a high density medium ($n_e \gtrsim 1.55 \times 10^4\,\mathrm{cm}^{-3}$), the Lyα emissivity will be increased by the collisional transition from 2s to 2p state (see 14.2.4 of [Draine]).

How many Lyα, Hα, and Hβ photons are produced for each recombination event:

$$f(\mathrm{Ly}\alpha) = \frac{\alpha_{\rm eff,2p}}{\alpha_{\rm B}} \approx \frac{2}{3} \quad \Rightarrow f(2p)$$

$$f(\mathrm{H}\alpha) = \frac{\alpha_{\mathrm{eff,H}\alpha}}{\alpha_{\rm B}} = 0.452\, T_4^{-0.109-0.003\ln T_4}$$

$$f(\mathrm{H}\beta) = \frac{\alpha_{\mathrm{eff,H}\beta}}{\alpha_{\rm B}} = 0.117\, T_4^{-0.041-0.02\ln T_4}$$

- ***Radiative Recombination: Heavy Elements***
  - We do not concern ourselves with the possibility that photons emitted from recombination to the ground state could be reabsorbed locally by another atom.
  - That is, *we assume Case A condition when studying the recombination of heavy elements.*

  - Radiative recombination of elements such as O and Ne is accompanied by emission of characteristic lines - the recombining electrons are captured into excited states, which then emit a cascade of line radiation.
  - For example, radiative recombination of O III sometimes populates an excited state, resulting in O II 4462.8Å and O II 4073.79Å emission (allowed lines).
  - In H II regions and planetary nebulae, these recombination lines are faint compared to the recombination lines of H, simply because of the greatly reduced abundance of heavy elements, but can nevertheless be measured.

  - The abundances obtained from **recombination lines** should, in principle, agree with the abundances derived from the much stronger **collisionally excited lines**. However, it is known that recombination lines give abundances that are larger than that estimated from collisionally excited lines. ***This is a puzzle that is yet to be resolved.***

Appendix: Ionization Fraction within an H II region

- Let's consider a shell between radii r and $r + dr$.
  - Number of ionizing photons within the volume = Number of Recombinations within in the volume

$$|Q(r+\Delta r) - Q(r)| = n_p n_e \alpha_{\rm B} \Delta V$$

$$\frac{dQ}{dr} = -n_p n_e \alpha_{\rm B} 4\pi r^2$$

$$\begin{aligned} Q(r) &= Q_0 - \int_0^r n_p n_e \alpha_{\rm B} 4\pi r'^2 dr' \\ &= Q_0 \left[1 - 3\int_0^{r/R_s} x^2 y^2 dy \right] \end{aligned}$$

where $Q_0 \equiv Q(r=0)$

$x \equiv n_p/n_{\rm H} = n_e/n_{\rm H}$

$y \equiv r/R_s$

$$R_s = \left(\frac{3}{4\pi} \frac{Q_0}{\alpha_{\rm B,H} n_{\rm H}^2} \right)^{1/3}$$

- At each point,
  - The rate of Case B recombinations per volume must be balanced by the rate of photoionization per volume:

$$\frac{Q(r)}{4\pi r^2} n_{\rm H^0} \sigma_{\rm pi} = n_p n_e \alpha_{\rm B}$$

- This can be rewritten as

$$\frac{Q(r)}{4\pi r^2}(1-x)n_{\rm H}\sigma_{\rm pi} = x^2 n_{\rm H}^2 \alpha_{\rm B}$$

$$\frac{Q(r)}{Q_0}\frac{(4\pi/3)R_s^3\alpha_{\rm B}n_{\rm H}^2}{4\pi r^2}(1-x)n_{\rm H}\sigma_{\rm pi} = x^2 n_{\rm H}^2 \alpha_{\rm B}$$

$$\frac{x^2}{1-x} = \frac{Q(r)}{Q_0}\frac{\tau_s}{3y^2}$$

$$\text{where} \quad \tau_s \equiv n_{\rm H}\sigma_{\rm pi}R_s$$

$$= 2880\left(\frac{Q_0}{10^{49}\, s^{-1}}\right)^{1/3}\left(\frac{n_{\rm H}}{10^2\,{\rm cm}^{-3}}\right)^{1/3}\left(\frac{T}{10^4\,{\rm K}}\right)^{0.28}\left(\frac{\sigma_{\rm pi}}{2.95\times 10^{-18}\,{\rm cm}^2}\right)$$

- Now, we can estimate the ionization degree x at each point r, by simultaneously solving the following equations:

$$\frac{x^2}{1-x} = \frac{Q(y)}{Q_0}\frac{\tau_s}{3y^2}$$

$$\frac{Q(y)}{Q_0} = \left[1 - 3\int_0^y x^2 y'^2 dy'\right] \qquad (0 \le y = r/R_s \le 1)$$

Heading and Cooling in H II Regions: Heating

- ***Temperature***
  - $T_{\rm HII}$ ~ 10,000 K. Observations indicate that the temperatures of H II regions are remarkably independent of the effective temperature of the central star.
  - The temperature is not determined by the central star. It is ***the result of a balance between heating and cooling mechanisms*** in the ionized gas of the H II region.
  - The main source of heating in an ionized nebula is photoionization.
- ***Photoionization Heating***
  - When hydrogen is photoionized from its ground state, the photoelectron that is emitted carries away a kinetic energy:

$$E = h\nu - I_{\rm H} \quad (h\nu = \text{energy of incident photon})$$

  The mean energy of the ejected electrons, averaged over the all photoionization, is

$$\langle E \rangle = \langle h\nu \rangle - I_{\rm H}$$

  - The average energy $\langle h\nu \rangle$ of an ionizing photon must be weighted by the photoionization cross-section.

$$\langle h\nu \rangle = \frac{\int_{\nu_0}^{\infty} (4\pi J_\nu / h\nu)(h\nu)\sigma_{\rm pi} d\nu}{\int_{\nu_0}^{\infty} (4\pi J_\nu / h\nu)\sigma_{\rm pi} d\nu}$$

- Although stars are not blackbodies, we will use the Planck function. Because the energy of ionizing photons is $h\nu > 13.6\,\mathrm{eV}$, we use the high-energy Wien tail with an effective temperature T_{eff}.

$$J_\nu \propto \nu^3 \exp\left(-\frac{h\nu}{kT_{\mathrm{eff}}}\right) \qquad \text{and} \qquad \sigma_{\mathrm{pi}} \propto \nu^{-3}$$

$$\begin{aligned}
\langle h\nu \rangle &= \frac{h\int_{\nu_0}^{\infty}\left(\nu^2 e^{-h\nu/kT_{\mathrm{eff}}}\right)\nu\cdot\nu^{-3}d\nu}{\int_{\nu_0}^{\infty}\left(\nu^2 e^{-h\nu/kT_{\mathrm{eff}}}\right)\nu^{-3}d\nu} \\
&= kT_{\mathrm{eff}}\frac{\int_{x_0}^{\infty} e^{-x}dx}{\int_{x_0}^{\infty} e^{-x}x^{-1}dx} \qquad \text{Here, } x \equiv h\nu/kT_{\mathrm{eff}} \text{ and } x_0 \equiv h\nu_0/kT_{\mathrm{eff}} \\
&= kT_{\mathrm{eff}}\frac{e^{-x_0}}{\int_{x_0}^{\infty} e^{-x}x^{-1}dx} = kT_{\mathrm{eff}}\frac{e^{-x_0}}{E_1(x_0)}
\end{aligned}$$

The integral in the denominator is the first exponential integral $E_1(x_0)$. Then, we obtain

$$E_1(x_0) \simeq \frac{e^{-x_0}}{x_0}\left[1 - \frac{1}{x_0} + \mathcal{O}\left(x_0^{-2}\right)\right] \quad \text{for} \quad x_0 \gg 1$$

$$\langle h\nu \rangle \approx kT_{\mathrm{eff}}\,x_0\left(1+\frac{1}{x_0}\right) = h\nu_0 + kT_{\mathrm{eff}} \longrightarrow$$

Mean kinetic energy of the ejected electrons:

$$\langle E \rangle = \langle h\nu \rangle - I_{\mathrm{H}} \approx kT_{\mathrm{eff}}$$

- ***Volumetric heating rate***: In photoionization equilibrium,

$$n_{\mathrm{H}^0}\zeta_{\mathrm{pi}} = n_e n_p \alpha_{\mathrm{B,H}}$$

Hence, the volumetric heating rate is

$$\begin{aligned}\mathcal{G}_{\mathrm{pi}} &= n_{\mathrm{H}^0}\zeta_{\mathrm{pi}}\,\langle E\rangle & &\longleftarrow\quad n_{\mathrm{H}^0}\zeta_{\mathrm{pi}} = n_e n_p \alpha_{\mathrm{B,H}} \quad \text{and} \quad \langle E\rangle = kT_{\mathrm{eff}} \\ &= n_{\mathrm{H}}^2 \alpha_{\mathrm{B,H}}\, kT_{\mathrm{eff}} & &\longleftarrow\quad \alpha_{\mathrm{B,H}} \approx 2.59\times10^{-13}\left(T_{\mathrm{gas}}/10^4\,\mathrm{K}\right)^{-0.833} \quad [\mathrm{cm}^3\,\mathrm{s}^{-1}] \\ &\propto T_{\mathrm{gas}}^{-0.83} T_{\mathrm{eff}} & & \end{aligned}$$

Notice that *the volumetric heating rate decreases with increasing gas temperature*.

- ***Necessity of the cooling mechanisms***
  - An O3 main sequence star has an effective temperature $T_{\mathrm{eff}} \sim 44{,}850$ K ($kT_{\mathrm{eff}} \sim 3.9$ eV), and thus the photoelectrons will have a mean energy of 3.9 eV.
  - However, the free electrons in a 10,000 K nebula have a mean energy (3/2) $kT_{\mathrm{gas}} \sim 1.3$ eV.
  - Therefore, some cooling mechanism must be reducing the average kinetic energy of the photoelectrons.

Heading and Cooling in H II Regions: Cooling

- ***Main cooling sources in H II regions***:
  - Recombination Continuum and Line Emission (free-bound)
  - Thermal Bremsstrahlung (free-free)
  - Collisionally Excited Line Emission
- ***Recombination Cooling***:
  - Recombination cooling occurs when electrons undergo radiative recombination with protons to form neutral hydrogen atoms. The volumetric cooling rate is then

$$\mathcal{L}_{\rm rr} = n_e n_p \alpha_{\rm B,H} \langle E_{\rm rr} \rangle$$

where $\langle E_{\rm rr} \rangle$ is the mean kinetic energy of the recombining electrons. The mean kinetic energy is obtained by weighting by cross section and integrating over the Maxwell distribution

$$\langle E_{\rm rr} \rangle = \frac{\langle E \sigma_{\rm rr} v \rangle_{\rm Maxwell}}{\langle \sigma_{\rm rr} v \rangle_{\rm Maxwell}} = \frac{\int v^2 dv e^{-E/kT_{\rm gas}} \sigma_{\rm rr} v E}{\int v^2 dv e^{-E/kT_{\rm gas}} \sigma_{\rm rr} v} = \frac{\int E^2 \sigma_{\rm rr} e^{-E/kT_{\rm gas}} dE}{\int E \sigma_{\rm rr} e^{-E/kT_{\rm gas}} dE}$$

Note that $\langle E_{\rm rr} \rangle \neq (3/2) kT_{\rm gas}$. This is because the radiative recombination cross-section is a decreasing function of electron kinetic energy.

We will perform the integration by approximating that the radiative recombination cross-section, at about $T \sim 10^4$ K, by a power-law:

$$\sigma_{\rm rr}(E) = \sigma_0 \, (E/E_0)^{\gamma} \quad \text{where } \gamma \approx -1.316 \text{ for Case B}$$

See Section 27.3.1 of [Draine] for the derivation of the power-law index.

Then, the mean energy per recombining electron (for Case B) is

$$\begin{aligned}\langle E_{\rm rr}\rangle &= \frac{\Gamma(3+\gamma)}{\Gamma(2+\gamma)} kT_{\rm gas} = (2+\gamma) kT_{\rm gas} \\ &= 0.684 \, kT_{\rm gas}\end{aligned} \quad \longleftarrow \quad \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

The cooling rate from the recombination is

$$\mathcal{L}_{\rm rr} = n_e n_p \alpha_{\rm B,H} \, (\gamma + 2) \, kT_{\rm gas}$$

- Gas temperature:

If radiative recombination were the only cooling mechanism, then the gas temperature would be found by equating the photoionization heating with the recombination cooling.

$$\mathcal{G}_{\rm pi} = \mathcal{L}_{\rm rr} \quad \longrightarrow \quad n_e n_p \alpha_{\rm B,H} \, kT_{\rm eff} = n_e n_p \alpha_{\rm B,H} \, (\gamma + 2) \, kT_{\rm gas}$$

$$T_{\rm gas} = \frac{T_{\rm eff}}{2+\gamma} = \frac{T_{\rm eff}}{0.684} = 1.462\,T_{\rm eff}$$

The resulting temperature would be ~46% higher than the effective temperature of the central star. *For an O3 main sequence star with T_{eff} = 44,900 K, the nebula temperature will be T_{gas} = 66,000 K*

This is because radiative recombination selectively removes the lower-energy free electrons (because of the higher cross section at lower energy), and thus increases the mean kinetic energy of electrons that are left without being captured.

- Hence, we need an additional cooling mechanism.

- ***Free-free cooling***:
  - Bremsstrahlung cooling occurs when free electrons are accelerated by close encounters with protons or other ions, and thus emit radiation.
  - The emissivity is

$$4\pi j_\nu^{\rm ff} = \frac{32\pi}{3}\left(\frac{2\pi}{3}\right)^{1/2} \frac{Z_i^2 e^6}{m_e^2 c^3}\left(\frac{m_e}{kT_{\rm gas}}\right)^{1/2} n_i n_e\, g_{\rm ff}\, e^{-h\nu/kT_{\rm gas}} \qquad (Z_i = 1,\ n_i = n_p \text{ for H})$$

where $g_{\rm ff}$ is the Quantum mechanical Gaunt factor.

- The volumetric cooling rate for a pure hydrogen gas is

$$\mathcal{L}_{\rm ff} = \int_0^\infty 4\pi j_\nu^{\rm ff} d\nu$$
$$= \frac{32\pi}{3}\left(\frac{2\pi}{3}\right)^{1/2} \frac{e^6}{m_e^2 h c^3} \left(m_e k T_{\rm gas}\right)^{1/2} n_p n_e \, \bar{g}_{\rm ff}$$

where $\bar{g}_{\rm ff}$ is the frequency-averaged Gaunt factor. For temperature near $T_{\rm gas} = 10^4$ K, a Quantum-mechanical calculation yields

$$\bar{g}_{\rm ff} \approx 1.34 \left(T/10^4\,{\rm K}\right)^{0.05}$$

- The ratio between the RR cooling and free-free cooling rates is

$$\frac{\mathcal{L}_{\rm ff}}{\mathcal{L}_{\rm rr}} = \frac{32\pi}{3}\left(\frac{2\pi}{3}\right)^{1/2}\frac{e^6}{m_e^2 h c^3}\left(\frac{m_e}{kT_{\rm gas}}\right)^{1/2}\frac{\bar{g}_{\rm ff}}{(2+\gamma)\alpha_{\rm B,H}}$$

$$\frac{\mathcal{L}_{\rm ff}}{\mathcal{L}_{\rm rr}} \approx 0.79\left(T_{\rm gas}/10^4\,{\rm K}\right)^{0.37}$$

Note that both cooling mechanisms are two-body processes and thus the factors $n_e n_{\rm H^+}$ cancel.

- Adding the free-free cooling, we can estimate the gas temperature, as follows:

$$\mathcal{G}_{\rm pi} = \mathcal{L}_{\rm rr} + \mathcal{L}_{\rm ff} \longrightarrow T_{\rm eff} = (\gamma+2)\,T_{\rm gas}\left[1+0.79\left(T_{\rm gas}/10^4\,{\rm K}\right)^{0.37}\right]$$

$$\gamma + 2 = 0.684$$

Example: for an O3 main sequence star with $T_{\rm eff}$ = 44,900 K, the nebula temperature will be $T_{\rm gas}$ = 30,000 K if both the radiative recombination and free-free coolings are taken into account. This temperature is still higher than that is actually observed in H II regions.

- ***Collisional excited line cooling***
  - If a free electron collisionally excites an atom or ion from a lower energy level to an excited level, the energy difference between the levels is taken from the free electron's kinetic energy. If the excited atom or ion then undergoes radiative de-excitation, and if the emitted photon escapes from the nebula, then there is a net cooling of the gas.

- To cool from T ~ 30,000 K to ~ 10,000 K, the energy levels of the excited system must be separated by a difference $\Delta E \approx 1 - 3\,\mathrm{eV}$ $\left[T \approx (1.2 - 3.5) \times 10^4\,\mathrm{K}\right]$.
    - If ΔE is much lower than this value, then the photons emitted by radiative de-excitation will carry away only a small amount of energy.
    - If ΔE is much higher than this value, then only a small fraction of free electrons will have high enough energies to excite the ions or atoms.
    - In H II regions, most of the hydrogen will be ionized. Even if some He or He^+ is present, the energy of the first excited state is so far above the ground state that the rate for collisional excitation is negligible. Lyα ($\Delta E = 10.2\,\mathrm{eV}$) emission from neutral hydrogen atoms is not effective at cooling H II regions. Similarly, the first excited state of neutral helium is far too energetic ($\Delta E = 20.6\,\mathrm{eV}$) to be collisional excited.
- ***This is where the heavy atoms such as oxygen and nitrogen play a key role in cooling H II regions.***
    - In particular, O II, N II, and O III have forbidden transitions in the 1 - 3 eV range.
    - If the collisional excitation is followed by a collisional de-excitation, the kinetic energy of the gas will be unchanged.
    - Therefore, if a collisional excitation is to result in cooling, it must be followed by a radiative de-excitation. For radiative de-excitation to dominate over collisional de-excitation, the number density of electrons must be lower than the critical density n_{crit}.
    - The critical density for these forbidden lines are indeed high compared to typical densities in an H II region.

- Calculation of the cooling rate for the collisionally excitation lines (electron impact emission lines)
  - If the collisionally excited levels are radiatively de-excited, the rate of energy loss by the gas is

$$\mathcal{L}_{\mathrm{ce}} = \sum_X \sum_u n(X,u) \sum_{\ell<u} A_{u\ell} E_{u\ell}$$

where $E_{u\ell} \equiv E_u - E_\ell$

where the sum is over species X and excited states u.

Recall:

[population balance for two level atoms], ignoring the stimulated emission

$$n_\ell n_e k_{\ell u} = n_u \left(n_e k_{u\ell} + A_{u\ell}\right)$$

$$\longrightarrow \quad \frac{n_u}{n_\ell} = \frac{n_e k_{\ell u}}{n_e k_{u\ell} + A_{u\ell}} \quad \rightarrow \quad \frac{n_u}{n_\ell} \simeq n_e \frac{k_{\ell u}}{A_{u\ell}} \quad \text{for low density.}$$

[collisional excitation & de-excitation rate coefficients]

$$k_{u\ell} = \langle \sigma_{u\ell} v \rangle = \frac{\beta}{T^{1/2}} \frac{\langle \Omega_{u\ell} \rangle}{g_u} \quad [\mathrm{cm^3\ s^{-1}}],$$

$$k_{\ell u} = \langle \sigma_{\ell u} v \rangle = \frac{\beta}{T^{1/2}} \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} e^{-(E_u - E_\ell)/kT_{\mathrm{gas}}} \quad [\mathrm{cm^3\ s^{-1}}]$$

$$\left(\beta = 8.62942 \times 10^{-6}\right)$$

[emissivity] $4\pi j_\nu = n_u A_{u\ell} \left(E_u - E_\ell\right)$

[principle of detailed balance]

$$\frac{k_{\ell u}}{k_{u\ell}} = \frac{g_u}{g_\ell} e^{-(E_u - E_\ell)/kT_{\mathrm{gas}}}$$

We need (1) $A_{u\ell}$ and (2) $\langle \Omega_{u\ell} \rangle$.
For three or more levels, the balance equation becomes more complicated.
See Appendix F of Draine, Table 4.1 of Lequeux, Table 9.3 & 9.4 in Draine

Density and Metallicity Effects

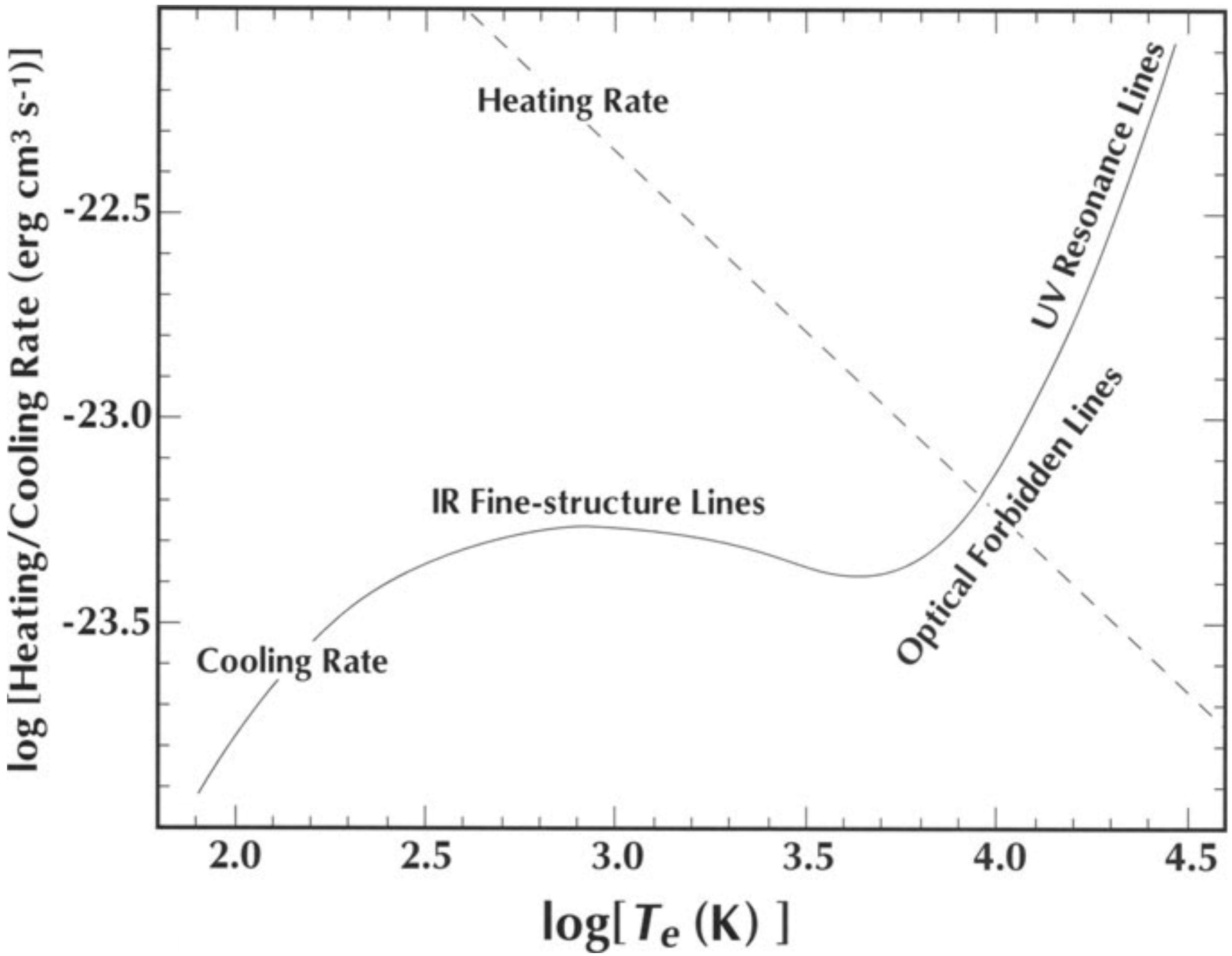
- ***Density Effect***: If the density is high, fewer of the possible cooling lines are above the critical density.
  - Thus, cooling becomes less effective at higher densities, and **the equilibrium temperature of the nebula goes up.**
  - For instance, the temperature of an Orion-like nebula increases from $T_{\rm gas}$ = 6600 K at n_H = 100 cm^{-3} to T = 9050 K at n_H = 10^6 cm^{-3}.

Main contributors to line cooling
in H II regions [Table 4.1 in Ryden]

Name	λ [Å]	$A_{u\ell}$ $[10^{-3}\,{\rm s}^{-1}]$	$n_{\rm crit}$ $[10^4\,{\rm cm}^{-3}]$
[O II]^{4}S - ^{2}D	3726	0.16	1.5
	3729	0.036	0.34
[N II]^{3}P - ^{1}D	6548	0.98	6.6
	6583	3.0	6.6
[O III]^{3}P - ^{1}D	4959	6.8	68
	5007	20.	68

- ***Metallicity Effect***:
  - The equilibrium temperature of a nebula also depends on its metallicity. If the metallicity is lowered, its temperature raises.
  - An Orion-like nebula (around a star with $T_{\rm eff}$ = 35,000 K) has a gas temperature of $T_{\rm gas}$ ~ 8050 K.
  - If the metallicity is lowered to $Z = Z_\odot/10$, its temperature rises to 15,600 K.
  - If the metallicity were zero, the gas temperature would be $T_{\rm gas}$ ~ 250,000 K.
  - If the metallicity were 3 times that of the Orion Nebula, its temperature would be $T_{\rm gas}$ ~ 5400 K.

Heading and Cooling Function



[Figure 9.5, Dopita]

The cooling function for a fixed ionization state produced by an O star with $T_{\rm eff}$ = 40,000 K as a function of electron temperature.
The heating rate is related to the recombination rate. The equilibrium temperature is defined by the point at which these cross.

At different electron temperatures, different collisionally excited emission lines dominate the cooling rate.

A $T \sim 1000$ K (kT ~ 0.1 eV), the colling is dominated by infrared fine-structure lines, such as [S II] 18.7 μm line and, at lower temperatures, the [O I] 63 μm and [C II] 158 μm lines, which cool the CNM.

At $T \sim 25{,}000$ K, the cooling is dominated by ultraviolet lines such are Lyα.

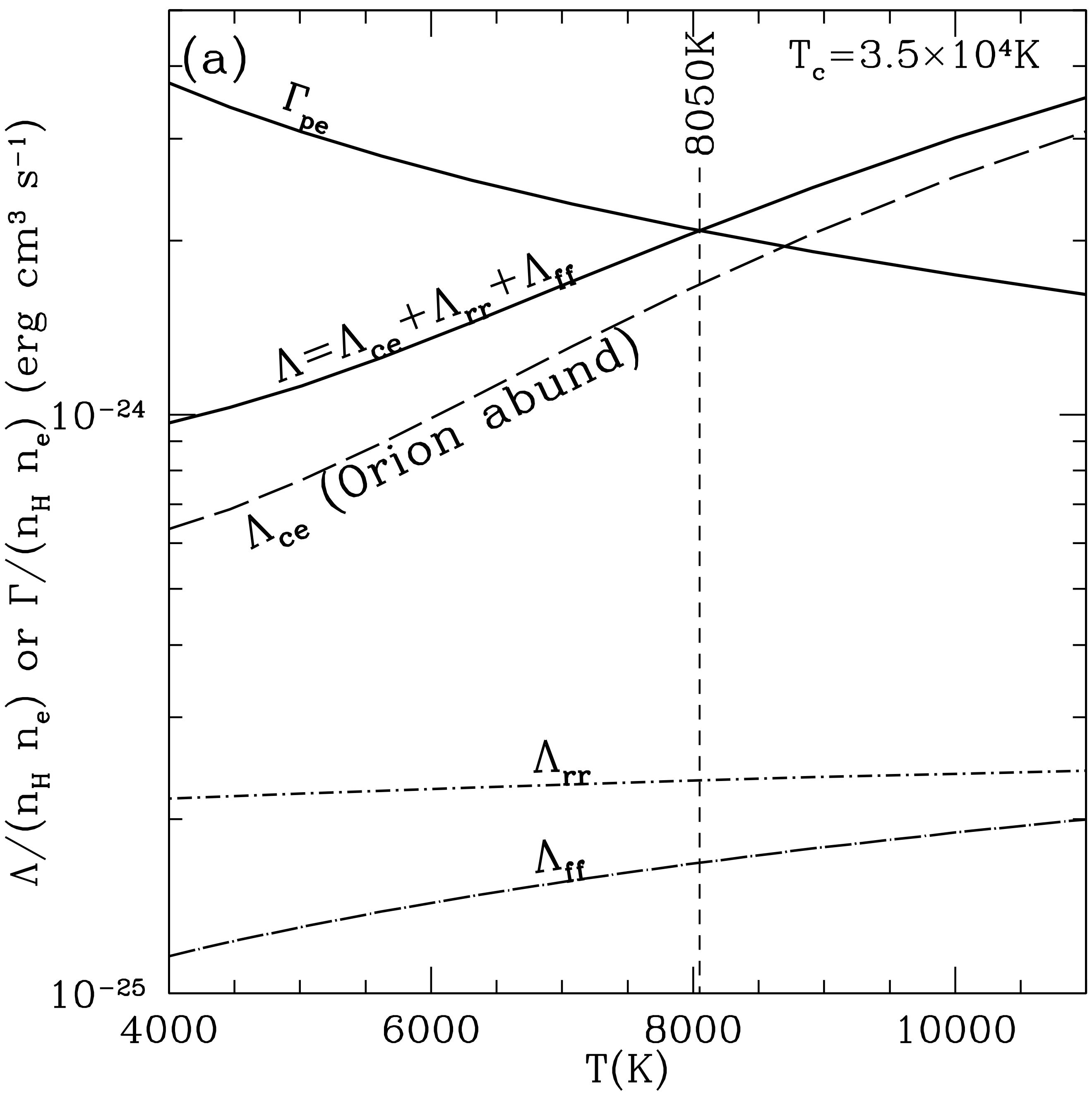
At intermediate temperatures, $T \sim 8000$ K, optical forbidden lines from O II, N II, and O III dominate the cooling rate.

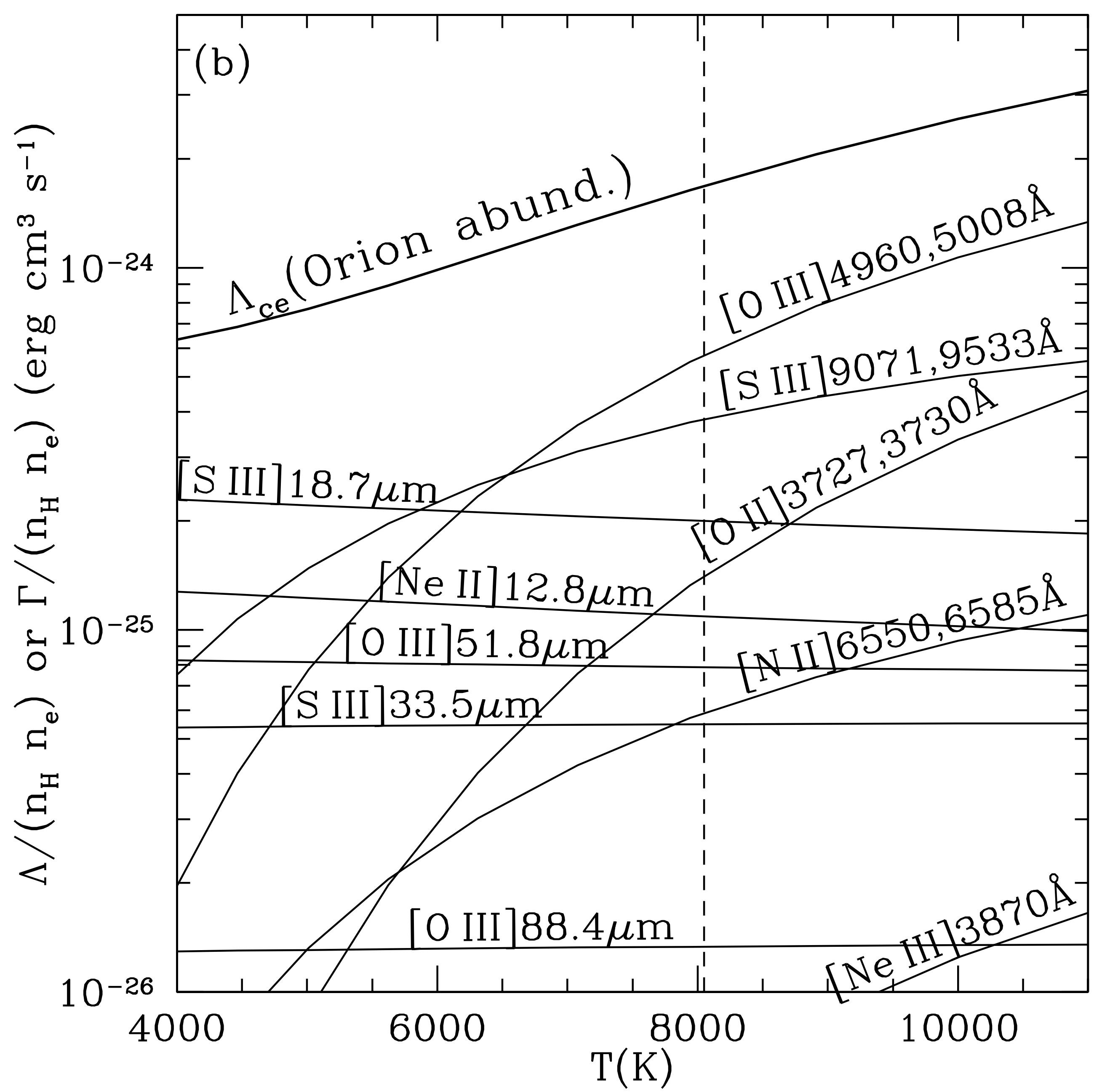
Heating and Cooling function as a function of gas temperature in an H II region with Orion-like abundances and density $n_{\mathrm{H}} = 4000\,\mathrm{cm}^{-3}$.

Heating and cooling balance at $T_{\mathrm{gas}} \sim 8050$ K.

Contributions of individual collisionally-excited lines to the cooling function.

[Figure 27.1 in Draine]





Heating and Cooling - Dependence on Metallicity

Heating and Cooling function for different metal abundances

(a) For an abundance of 10% of that of the Orion Nebula
(b) For 3 times higher abundance

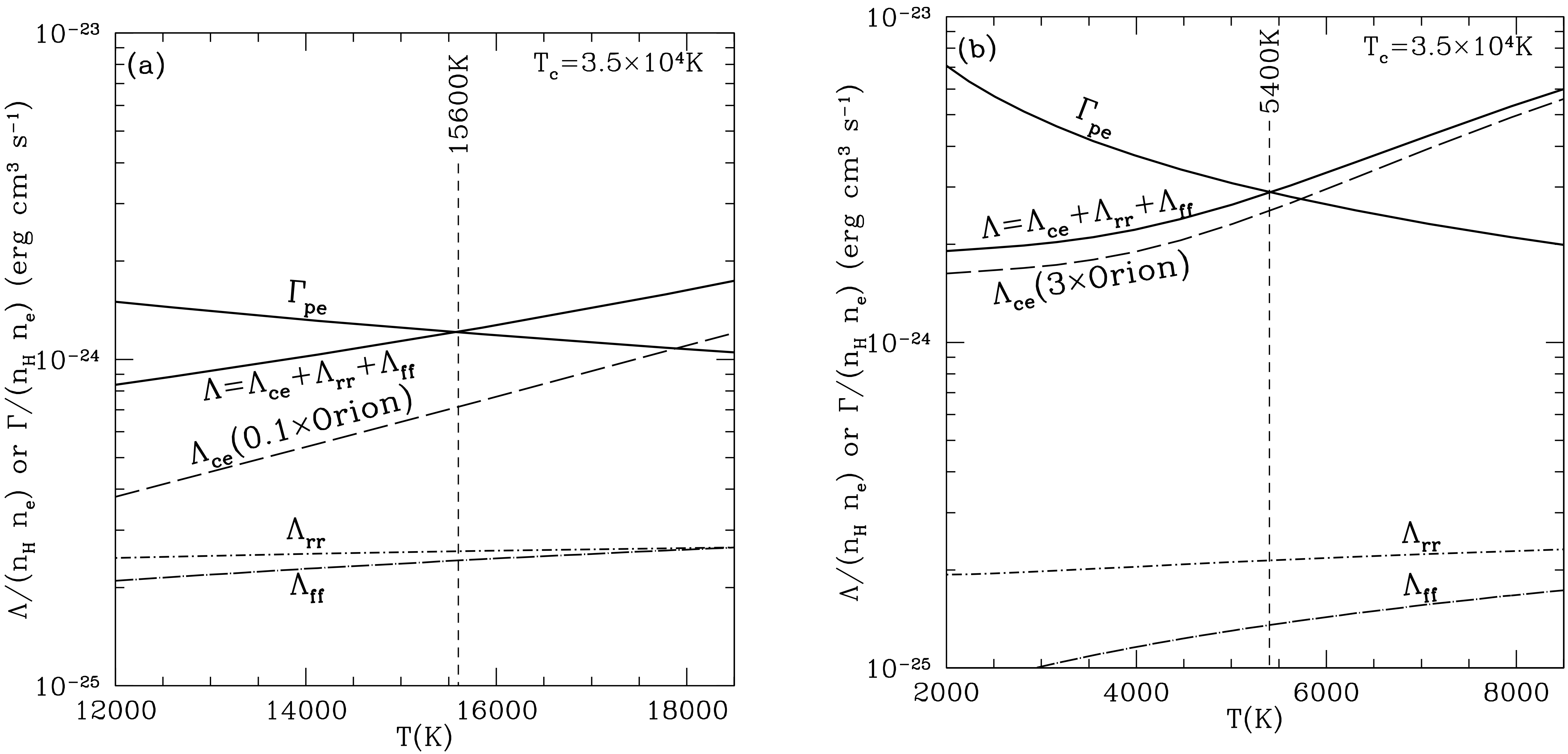


Figure 27.2 in Draine

Heading and Cooling - Dependence on Density

Cooling function for different densities.

The gas is assumed to have Orion-like abundances and ionization conditions.
As the gas density is varied from 10^2 to 10^5 cm^{-3}, the equilibrium temperature varies from 6600 K to 9050 K, *because of the contribution of collisional de-excitation.*

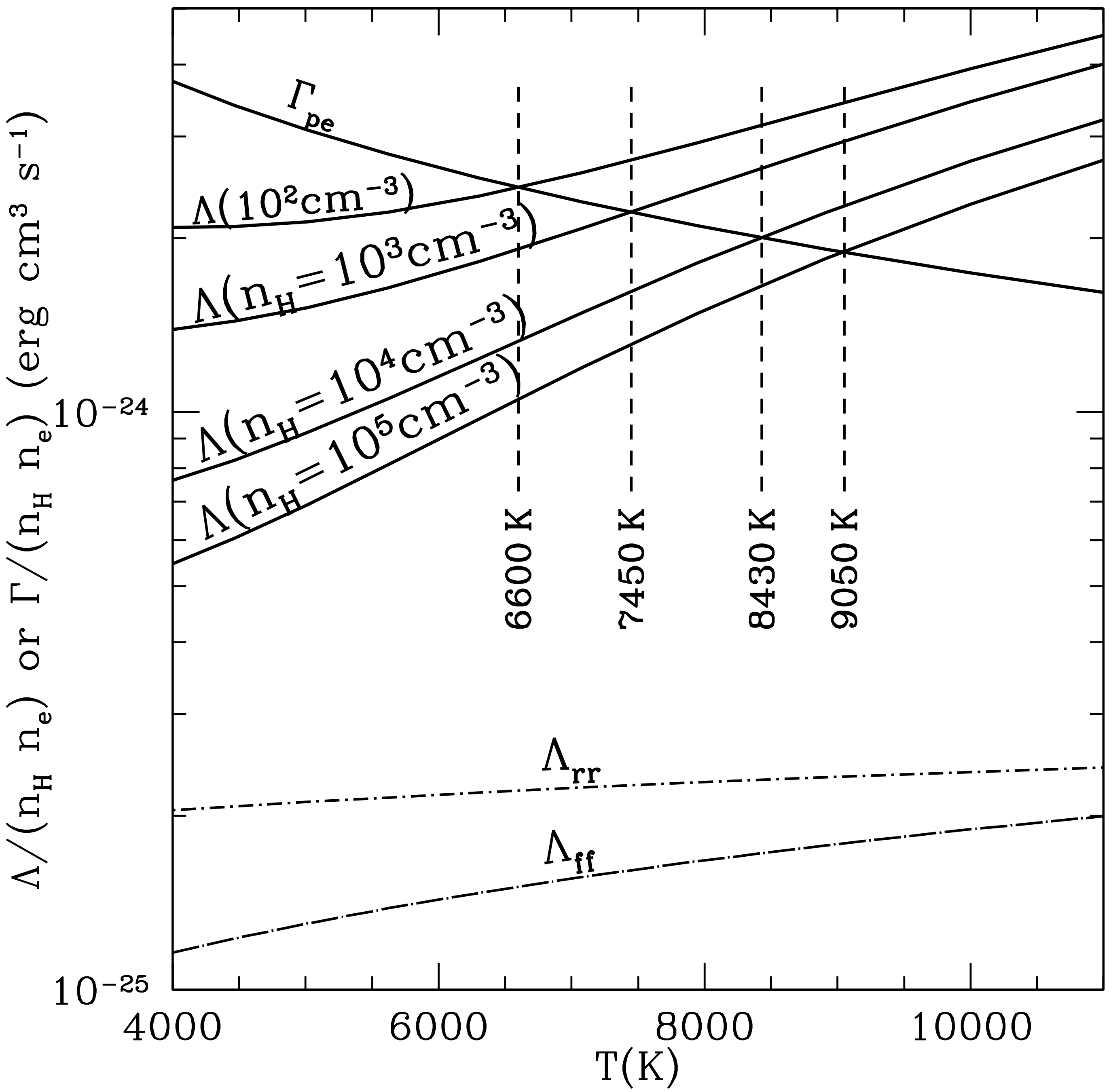
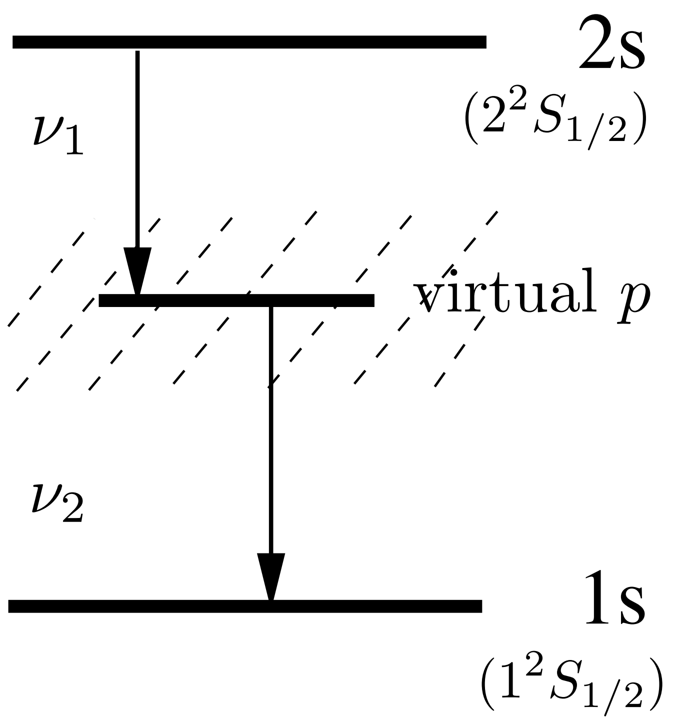


Figure 27.3 in Draine

Additional Cooling Mechanisms

- Thermal Emission of Dust
  - H II regions contain dust which scatters the light of the exciting stars. *Dust grains also absorbs some of the photons emitted by the stars and some of the Lyman α emission that fills the H II region. They re-emit the absorbed energy in the mid- and far-infrared, producing thermal continuum.*

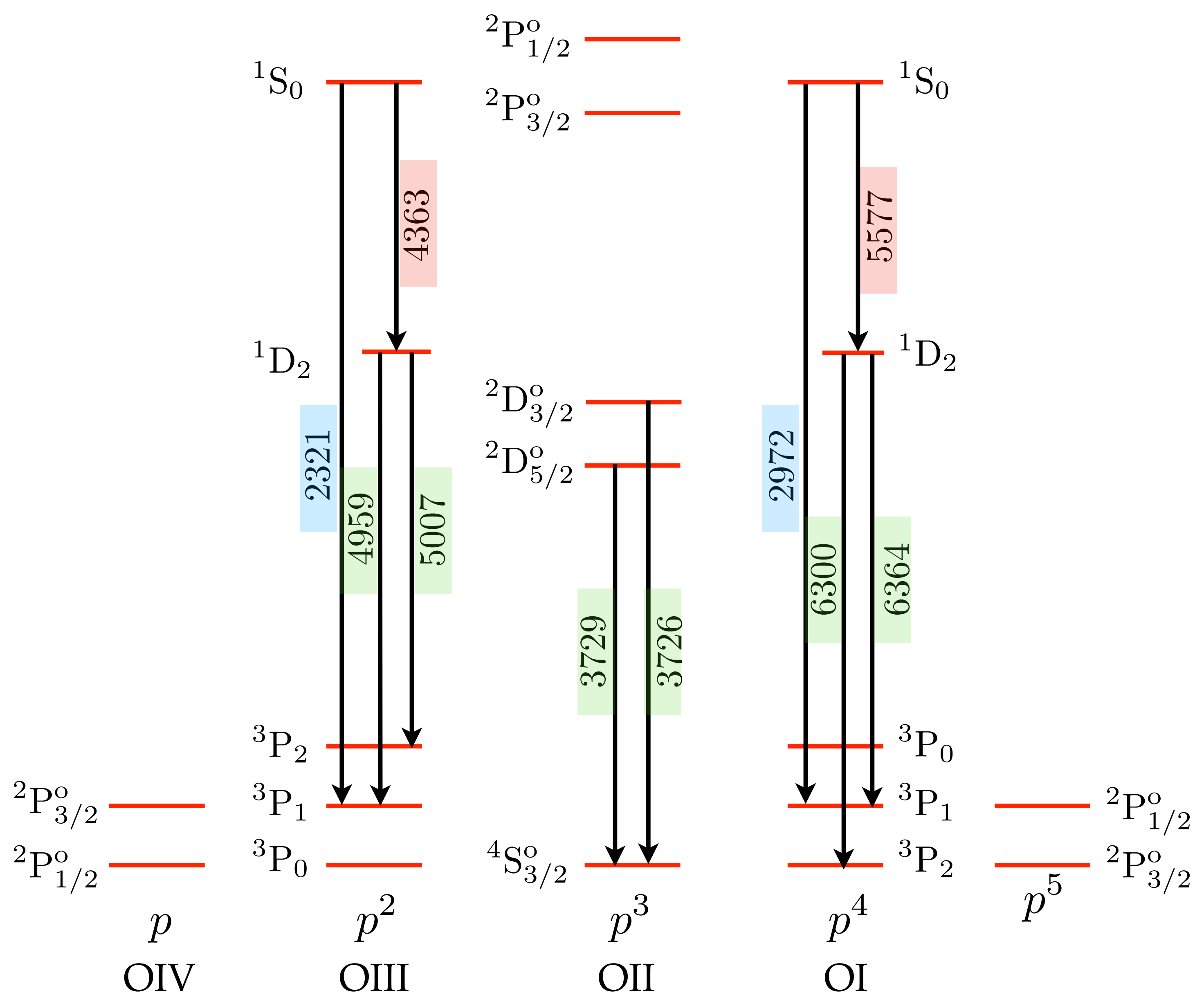
- Two-Photon (Continuum) Emission
  - The emission of radiation from an atomic level can arise through the intermediate of a virtual state. In this case, two photons are emitted, the sum of their energies being equal to the energy of the transition.
  - The probability of this 2-photon emission is small, but it can become the main channel for the de-excitation of a metastable level if collisions are negligible.
  - *This is the case for neutral hydrogen and helium.*



Nebular, Auroral Lines

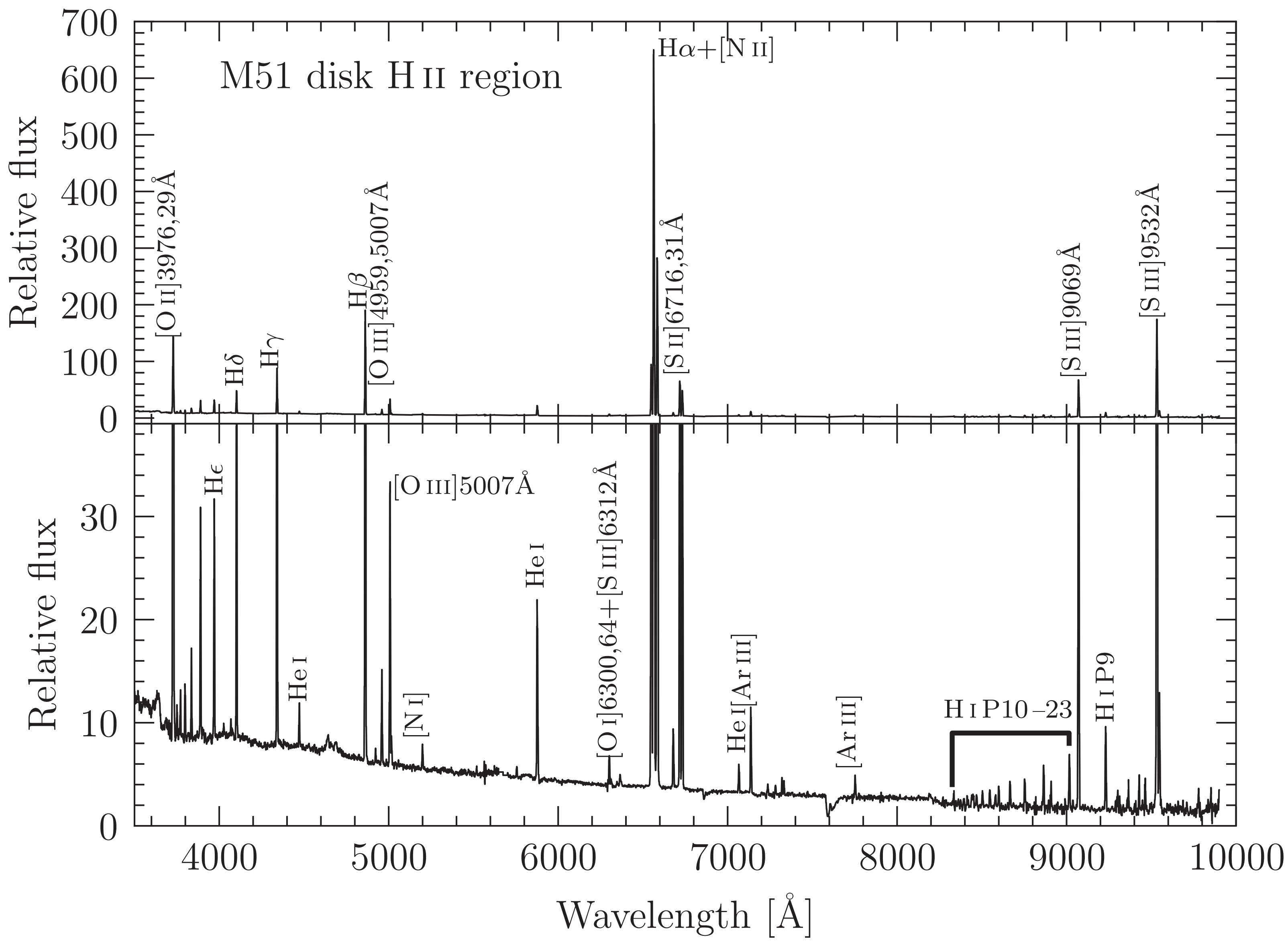
- Definitions:
  - **Auroral**: the transitions between ***two higher terms*** of configurations p^2, p^3, and p^4 are named auroral.
  - **Nebular**: the transitions between ***the middle and the lowest terms*** give nebular lines.
  - **Transauroral**: the transitions between ***the highest and the lowest terms*** give the transauroral lines.

The term structure for the ground configurations with p, p^2 p^3, p^4, and p^5 outermost shells. (not to scale)



Temperature, Density & Abundance Diagnostics

- In the figure, the continuum is a mixture of free-bound continuum (from radiative recombination), free-free emission (thermal bremsstrahlung), and two-photon emission.
- If we know enough about the temperature dependence of these continuum radiation, we can estimate the nebula temperature. However, the ***collisionally excited emission lines*** are much stronger than the continuum spectrum.



Spectrum of a disk HII region in the Whirlpool galaxy (M51).

(top) bright lines
(bottom) scaled to show faint lines.

Figure 4.5 [Ryden]
Data from Croxall et al. (2015)

Temperature Determination

- The key to using emission lines to estimate temperature is finding ***two excited states of the same ion whose energy differs by ~ kT.*** For nebulae with T ~ 10^4 K, this implies energy differences of order of 1 eV or so.

- Atoms and ions with six (or 14) electrons have $2p^2$ (or $3p^2$) as their lowest configuration, and have 3 terms, as shown in the figure. Because the energy difference between the terms are very different. *The relative strengths of the emission lines will be very sensitive to temperature.*

- The measured intensity ratio can be used to determine the nebula temperature.

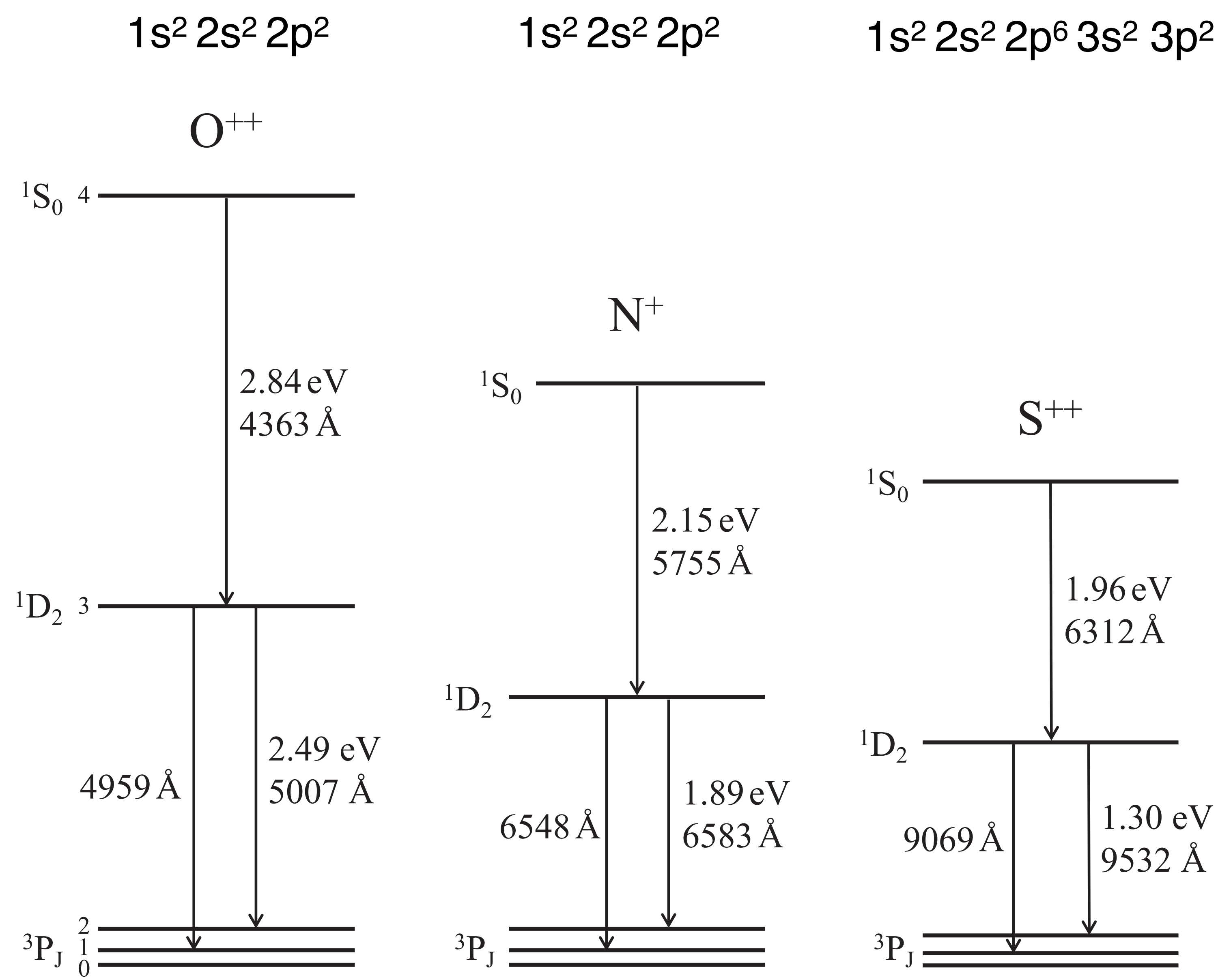


Figure 4.6 [Ryden]

- Candidate $2p^2$ ions are C I, N II, O III, Fe IV, Ne V, and so on.
  - C I is easily photoionized, and will have very low abundance in an H II region.
  - Fe IV, Ne VI,… have an ionization potential exceeding 54.8 eV, and we do not expect such high ionization states to be abundant in H II regions.
  - This leaves N II and O III as the only $2p^2$ ions that will be available in H II regions.
- The lowest excited states of singly ionized nitrogen (N II) and doubly ionized oxygen (O III) are useful tools for estimating the temperatures of H II regions and planetary nebulae.
- N II and O III have six bound electrons, and thus their fine structure energy levels in their lowest configuration are very similar in structure.

Temperature-sensitive nebular lines (Å).

p^2 Ions	[N II]	[O III]	[Ne V]	[S III]
$^1S_0 \rightarrow {}^1D_2$	5755	4363	2974	6312
$^1D_2 \rightarrow {}^3P_2$	6583	5007	3426	9532
$^1D_2 \rightarrow {}^3P_1$	6548	4959	3346	9069
p^4 Ions	[O I]	[Ne III]	[Ar III]	
$^1S_0 \rightarrow {}^1D_2$	5577	3343	5192	
$^1D_2 \rightarrow {}^3P_2$	6300	3869	7136	
$^1D_2 \rightarrow {}^3P_1$	6363	3968	7751	

Element		I→II	II→III	III→IV	IV→V	V→VI	VI→VII	VII→VIII
1	H	13.5984						
2	He	24.5874	54.416					
3	Li	5.3917	75.640	122.454				
4	Be	9.3227	18.211	153.894	217.719			
5	B	8.2980	25.155	37.931	259.375	340.226		
6	C	11.2603	24.383	47.888	64.494	392.089	489.993	
7	N	14.5341	29.601	47.449	77.474	97.890	552.072	667.046
8	O	13.6181	35.121	54.936	77.414	113.899	138.120	739.293
9	F	17.4228	34.971	62.708	87.140	114.243	147.163	185.189
10	Ne	21.5645	40.963	63.423	97.117	126.247	154.214	207.271
11	Na	5.1391	47.286	71.620	98.91	138.40	172.183	208.50
12	Mg	7.6462	15.035	80.144	109.265	141.270	186.76	225.02
13	Al	5.9858	18.829	28.448	119.992	153.825	190.477	241.76
14	Si	8.1517	16.346	33.493	45.142	166.767	205.267	246.481
15	P	10.4867	19.769	30.203	51.444	65.025	220.422	263.57
16	S	10.3600	23.338	34.790	47.222	72.594	88.053	280.948
17	Cl	12.9676	23.814	39.911	53.465	67.819	97.030	114.201
18	Ar	15.7596	27.630	40.735	59.686	75.134	91.00	124.328
19	K	4.3407	31.628	45.806	60.913	82.66	99.4	117.6
20	Ca	6.1132	11.872	50.913	67.27	84.51	108.8	127.2
21	Sc	6.5615	12.800	24.757	73.489	91.69	110.7	138.0
22	Ti	6.8281	13.576	24.492	43.267	123.7	119.533	140.846
23	V	6.7462	14.655	29.311	46.709	65.282	128.125	150.641
24	Cr	6.7665	16.486	30.959	49.160	69.456	90.635	160.175
25	Mn	7.4340	15.640	33.668	51.2	72.4	95.60	119.203
26	Fe	7.9024	16.188	30.651	54.801	75.010	99.063	124.976
27	Co	7.8810	17.084	33.50	51.27	79.5	102.	129.
28	Ni	7.6398	18.169	35.187	54.925	76.06	107.87	133.
29	Cu	7.7264	20.292	36.841	57.380	79.846	103.031	138.862
30	Zn	9.3492	17.964	39.723	59.573	82.574	133.903	133.903

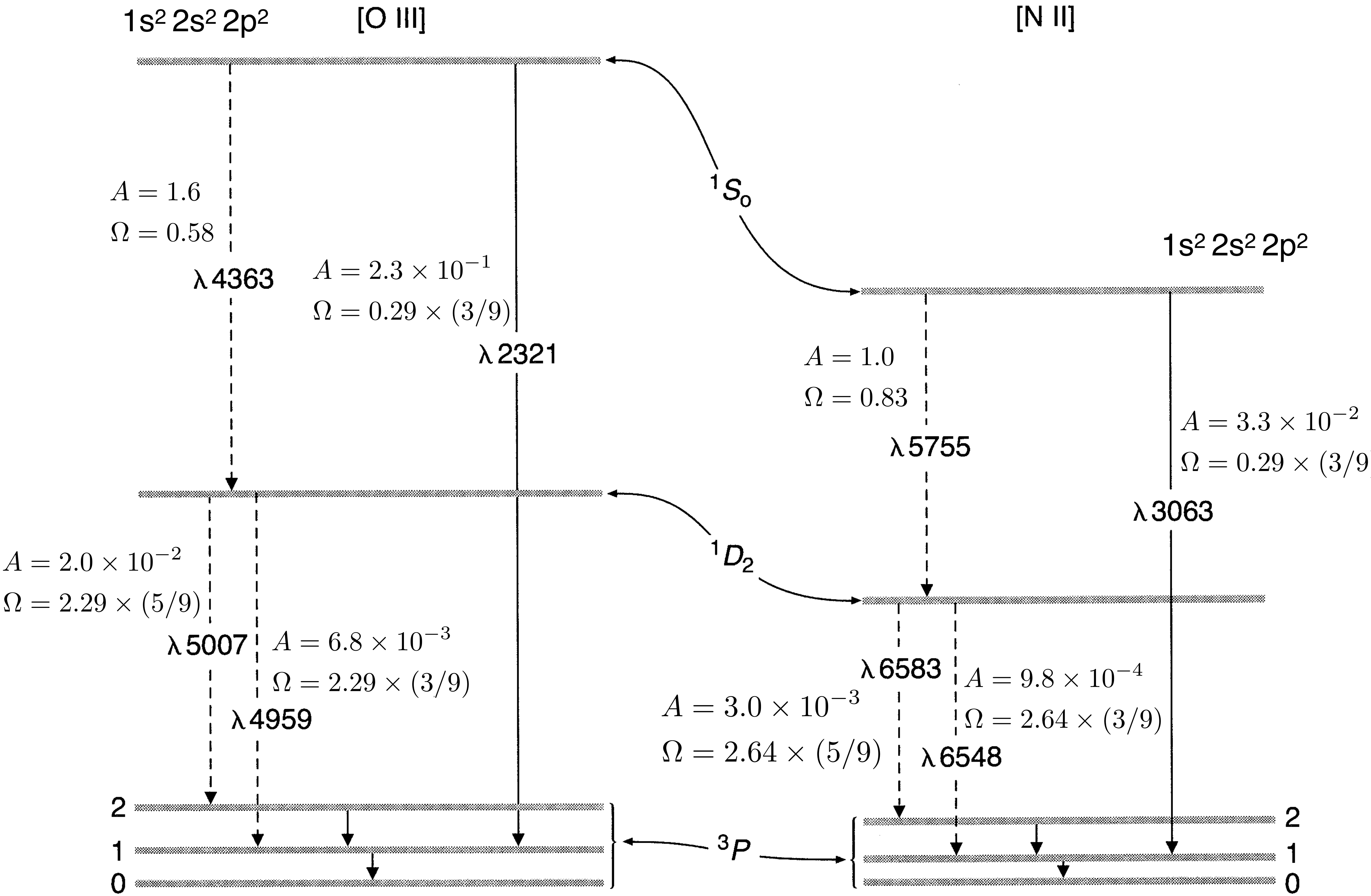


Figure 3.1 [Osterbrock]
Astrophysics of Gaseous Nebulae
and Active Galactic Nuclei

See Table 3.12 for A and
Table 3.6 for Collision Strength

Ion	$^3P,\ ^1D$	$^3P,\ ^1S$	$^1D,\ ^1S$
N^+	2.64	0.29	0.83
O^{+2}	2.29	0.29	0.58

Table 3.6
Collision Strength

- Let's suppose that we are in ***the low-density limit***, so that the free electron density is less than the critical density for collisional de-excitation of each line.
  - In this case, every collisional excitation will be followed by radiative decays returning the ion to the ground state, with branching ratios that are determined by the Einstein coefficients.
  - $4 \rightarrow 3$ transition:
    - The emissivity of the $4 \rightarrow 3$ transition, integrated over the entire line width is:

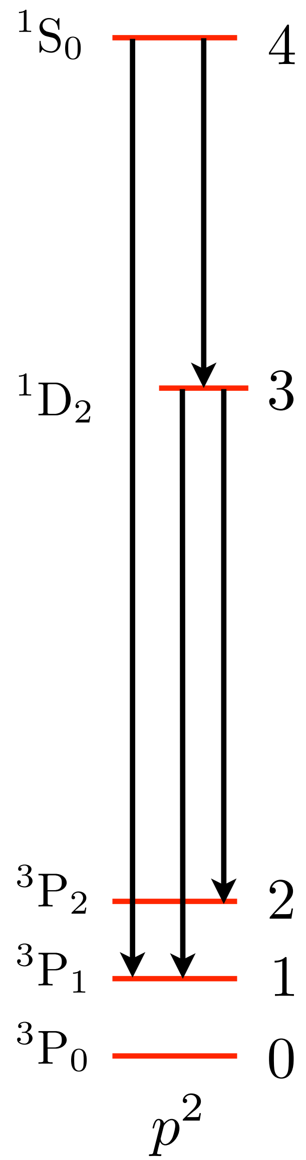
$$4\pi j(4 \rightarrow 3) = n_4 A_{43} h\nu_{43}$$

    - The rate of collisional excitation from "0" to level "4" is balanced by radiative de-excitation from "4" to "3" and "1":

$$n_e n_0 k_{04} = n_4 \left(A_{43} + A_{41}\right)$$

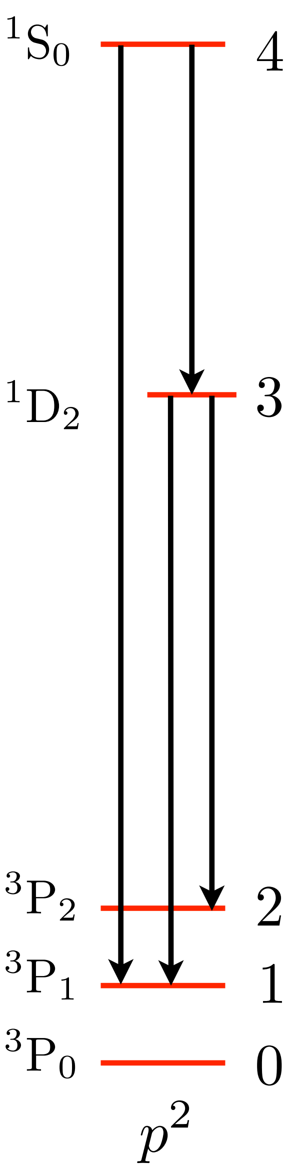
    - Therefore,

$$4\pi j(4 \rightarrow 3) = n_e n_0 k_{04} \frac{A_{43}}{A_{43} + A_{41}} h\nu_{43}$$



- $3 \to 2$ transition:
    - The level "3" can be populated in two ways: (1) by collisional excitation directly from the ground state, and (2) by a collisional excitation from the ground to "4", followed by radiative de-excitation to "3".

$$4\pi j(3 \to 2) = n_e n_0 \left(k_{03} + k_{04} \frac{A_{43}}{A_{43} + A_{41}} \right) \frac{A_{32}}{A_{32} + A_{31}} h\nu_{32}$$



- The relative strength between $4 \to 3$ and $3 \to 2$ emission line:

$$\frac{j(4 \to 3)}{j(3 \to 2)} = \frac{A_{43}}{A_{32}} \frac{\nu_{43}}{\nu_{32}} \frac{(A_{32} + A_{31}) k_{04}}{(A_{43} + A_{41}) k_{03} + A_{43} k_{04}}$$

- We notice that the temperature dependence in the above equation enters solely through the collisional rate coefficients k_{04} and k_{03}. Using the relation between the collisional excitation and de-excitation rate coefficients,

$$\begin{aligned} k_{0u} &= k_{u0} \frac{g_u}{g_0} e^{-h\nu_{u0}/kT} \\ &= \frac{\beta}{g_0} \frac{\langle \Omega_{u0} \rangle}{T^{1/2}} e^{-h\nu_{u0}/kT} \end{aligned}$$

- We obtain

$$\frac{j(4 \to 3)}{j(3 \to 2)} = \frac{A_{43}}{A_{32}} \frac{\nu_{43}}{\nu_{32}} \frac{(A_{32} + A_{31}) \langle \Omega_{40} \rangle \, e^{-h\nu_{43}/kT}}{(A_{43} + A_{41}) \langle \Omega_{30} \rangle + A_{43} \langle \Omega_{40} \rangle \, e^{-h\nu_{43}/kT}} \quad \text{where} \quad h\nu_{43} = h\nu_{40} - h\nu_{30}$$

- Notice that all the temperature dependence, aside from the weak dependence of collision strengths on temperature, is contained in the exponential factor. Thus, the line ratio is sensitive to the temperature $kT \sim h\nu_{43}$ (2.15 eV for N II, 2.84 eV for O III).

- At the high and low temperatures, the ratio can be expressed as

$$\begin{aligned} \frac{j(4 \to 3)}{j(3 \to 2)} &\approx \frac{A_{43}}{A_{32}} \frac{\nu_{43}}{\nu_{32}} \frac{(A_{32} + A_{31}) \langle \Omega_{40} \rangle}{(A_{43} + A_{41}) \langle \Omega_{30} \rangle + A_{43} \langle \Omega_{40} \rangle} && \text{for} \quad kT \gg h\nu_{43} \\ &\approx \frac{A_{43}}{A_{32}} \frac{\nu_{43}}{\nu_{32}} \frac{(A_{32} + A_{31}) \langle \Omega_{40} \rangle}{(A_{43} + A_{41}) \langle \Omega_{30} \rangle} e^{-h\nu_{43}/kT} && \text{for} \quad kT \ll h\nu_{43} \end{aligned}$$

At high temperatures, the line ratio becomes more or less independent of temperature.

$$\frac{j(4\rightarrow 3)}{j(3\rightarrow 2)} = \frac{A_{43}}{A_{32}}\frac{\nu_{43}}{\nu_{32}}\frac{(A_{32}+A_{31})\langle\Omega_{40}\rangle\, e^{-h\nu_{43}/kT}}{(A_{43}+A_{41})\langle\Omega_{30}\rangle + A_{43}\langle\Omega_{40}\rangle\, e^{-h\nu_{43}/kT}}$$

Dependence of collision strength on temperature is very weak.
So, we will adopt a typical value.

$$T_4 \equiv T/10^4\,\mathrm{K}$$

[O III]

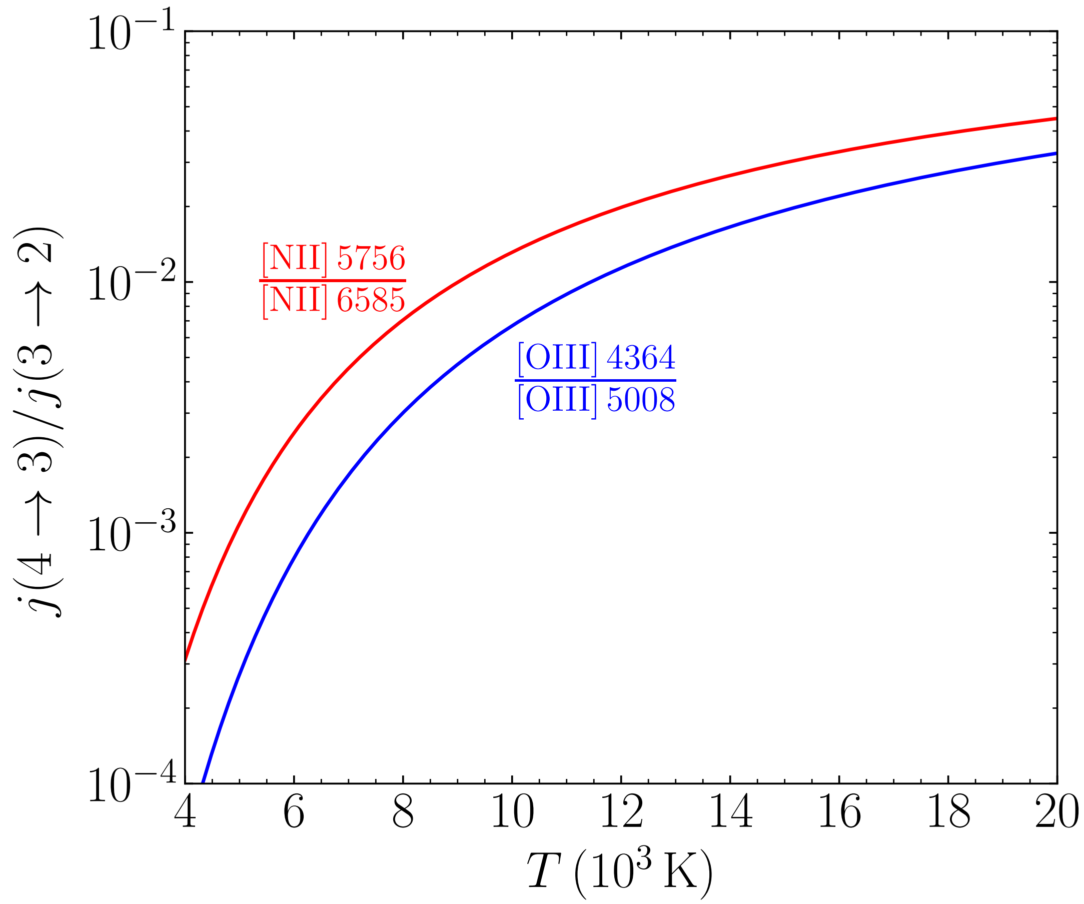
$$\langle\Omega_{30}\rangle = 2.29\times(1/9) \qquad A_{32} = 2.0\times10^{-2}\ [\mathrm{s}^{-1}]$$
$$\langle\Omega_{40}\rangle = 0.29\times(1/9) \qquad A_{31} = 6.8\times10^{-3}\ [\mathrm{s}^{-1}]$$
$$E_{40}/k = 61207\,[\mathrm{K}] \qquad A_{43} = 1.6\ [\mathrm{s}^{-1}]$$
$$E_{30}/k = 29169\,[\mathrm{K}] \qquad A_{41} = 2.3\times10^{-1}\ [\mathrm{s}^{-1}]$$
$$E_{20}/k = 441\,[\mathrm{K}]$$
$$E_{10}/k = 163\,[\mathrm{K}]$$

$$\frac{j(4364)}{j(5008)} = 0.1655\frac{e^{-3.2038/T_4}}{1+0.1107e^{-3.2038/T_4}}$$

[N II]

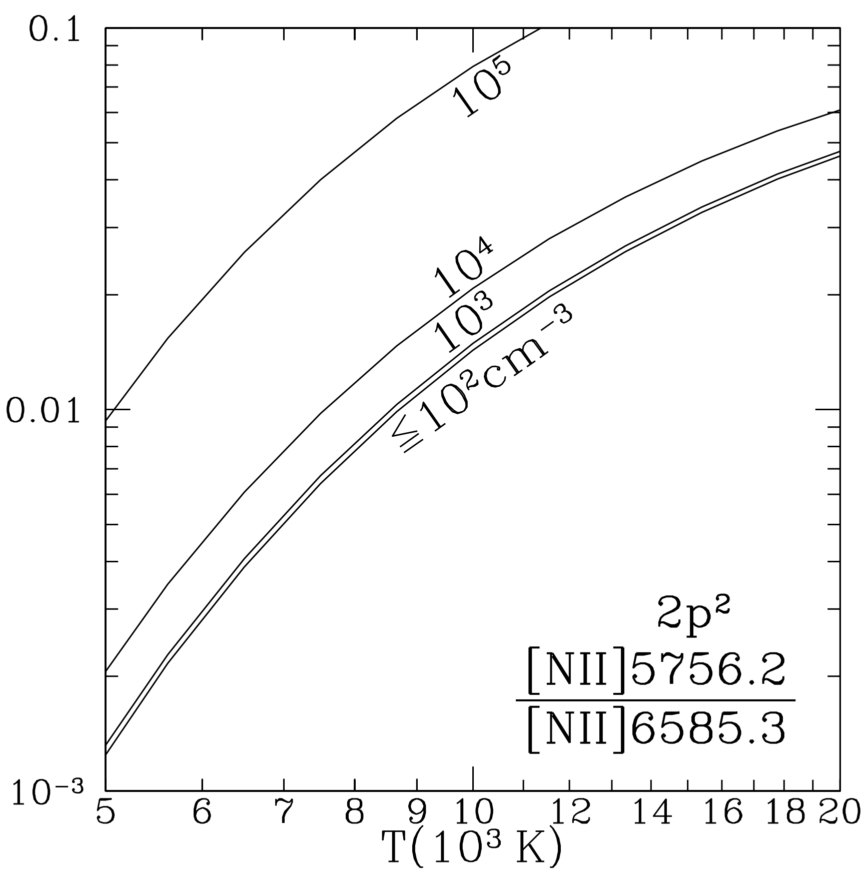
$$\langle\Omega_{30}\rangle = 2.64\times(1/9) \qquad A_{32} = 3.0\times10^{-3}\ [\mathrm{s}^{-1}]$$
$$\langle\Omega_{40}\rangle = 0.29\times(1/9) \qquad A_{31} = 9.8\times10^{-4}\ [\mathrm{s}^{-1}]$$
$$E_{40}/k = 47033\,[\mathrm{K}] \qquad A_{43} = 1.0\ [\mathrm{s}^{-1}]$$
$$E_{30}/k = 22037\,[\mathrm{K}] \qquad A_{41} = 3.3\times10^{-2}\ [\mathrm{s}^{-1}]$$
$$E_{20}/k = 188\,[\mathrm{K}]$$
$$E_{10}/k = 70\,[\mathrm{K}]$$

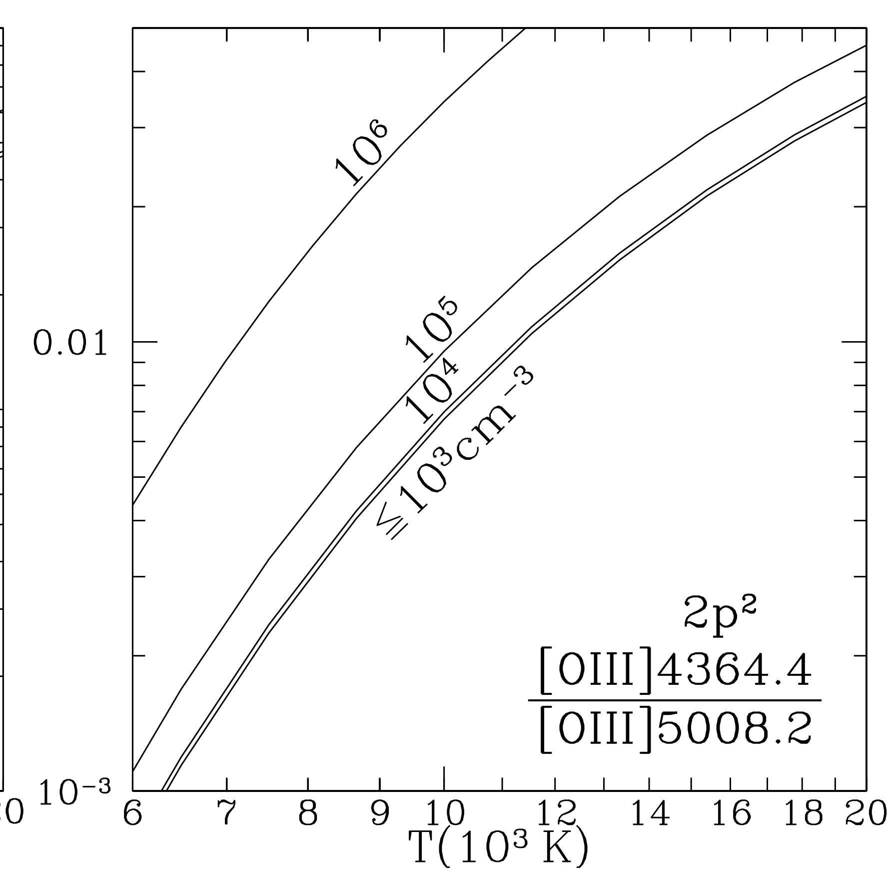
$$\frac{j(5756)}{j(6585)} = 0.1614\frac{e^{-2.4996/T_4}}{1+0.1063e^{-2.4996/T_4}}$$



Line ratios as a function of temperature, which is obtained using the equations in the previous slide.

Unfortunately, the auroral line at [O III] 4364Å can only be observed when the temperature is sufficiently high.





Line ratios for temperature diagnostics.

Curves indicate electron density. For each ion, the low density limit is shown, as well as results for higher densities.

Figure 18.2 [Draine]

- Sometimes, it would be better to combine $3 \to 2$ and $3 \to 1$ transitions:
    - $3 \to 1$ transition:

$$4\pi j(3 \to 2) = n_e n_0 \left(k_{03} + k_{04} \frac{A_{43}}{A_{43} + A_{41}} \right) \frac{A_{32}}{A_{32} + A_{31}} h\nu_{32}$$

$$4\pi j(3 \to 1) = n_e n_0 \left(k_{03} + k_{04} \frac{A_{43}}{A_{43} + A_{41}} \right) \frac{A_{31}}{A_{32} + A_{31}} h\nu_{31}$$

$$4\pi \left[j(3 \to 1) + j(3 \to 2) \right] = n_e n_0 \left(k_{03} + k_{04} \frac{A_{43}}{A_{43} + A_{41}} \right) h\overline{\nu} \qquad \text{where } \; \overline{\nu} \equiv \frac{A_{32}\nu_{32} + A_{31}\nu_{31}}{A_{32} + A_{31}}$$

- Recall that

$$4\pi j(4 \to 3) = n_e n_0 k_{04} \frac{A_{43}}{A_{43} + A_{41}} h\nu_{43}$$

Combining these equations, we obtain

$$\frac{j(3 \to 1) + j(3 \to 2)}{j(4 \to 3)} = \frac{\overline{\nu}}{\nu_{43}} \frac{A_{43} + A_{41}}{A_{43}} \frac{k_{03}}{k_{04}} \left(1 + \frac{k_{04}}{k_{03}} \frac{A_{43}}{A_{43} + A_{41}} \right)$$

$$k_{0u} = \frac{\beta}{T^{1/2}} \frac{\langle \Omega_{u0} \rangle}{g_0} e^{-E_{u0}/kT_{\rm gas}}$$

$$\frac{k_{03}}{k_{04}} = \frac{\langle \Omega_{30} \rangle}{\langle \Omega_{40} \rangle} \frac{e^{-h\nu_{30}/kT}}{e^{-h\nu_{40}/kT}} = \frac{\langle \Omega_{30} \rangle}{\langle \Omega_{40} \rangle} e^{h\nu_{43}/kT} \quad (\text{where } \nu_{43} = \nu_{40} - \nu_{30})$$

$$\begin{aligned} \frac{j(3 \to 1) + j(3 \to 2)}{j(4 \to 3)} &= \frac{\overline{\nu}}{\nu_{43}} \frac{A_{43} + A_{41}}{A_{43}} \frac{\langle \Omega_{30} \rangle}{\langle \Omega_{40} \rangle} e^{h\nu_{43/kT}} \left(1 + \frac{\langle \Omega_{40} \rangle}{\langle \Omega_{30} \rangle} \frac{A_{43}}{A_{43} + A_{41}} e^{-h\nu_{43}/kT} \right) \\ &\simeq \frac{\overline{\nu}}{\nu_{43}} \frac{A_{43} + A_{41}}{A_{43}} \frac{\langle \Omega_{30} \rangle}{\langle \Omega_{40} \rangle} e^{h\nu_{43/kT}} \end{aligned}$$

Note $\langle \Omega_{40} \rangle < \langle \Omega_{30} \rangle$ and $e^{-h\nu_{43}/kT} \ll 1$.

Thus, the second term inside the parenthesis is negligible.

$$\frac{j(3\to 1)+j(3\to 2)}{j(4\to 3)} \simeq \frac{\overline{\nu}}{\nu_{43}}\frac{A_{43}+A_{41}}{A_{43}}\frac{\langle\Omega_{30}\rangle}{\langle\Omega_{40}\rangle}e^{h\nu_{43/kT}}$$

Use the following data:

[O III]

$$\langle\Omega_{30}\rangle = 2.29\times(1/9) \qquad A_{32} = 2.0\times10^{-2}\ [\mathrm{s}^{-1}]$$
$$\langle\Omega_{40}\rangle = 0.29\times(1/9) \qquad A_{31} = 6.8\times10^{-3}\ [\mathrm{s}^{-1}]$$
$$E_{40}/k = 61207\,[\mathrm{K}] \qquad A_{43} = 1.6\ [\mathrm{s}^{-1}]$$
$$E_{30}/k = 29169\,[\mathrm{K}] \qquad A_{41} = 2.3\times10^{-1}\ [\mathrm{s}^{-1}]$$
$$E_{20}/k = 441\,[\mathrm{K}]$$
$$E_{10}/k = 163\,[\mathrm{K}]$$

[N II]

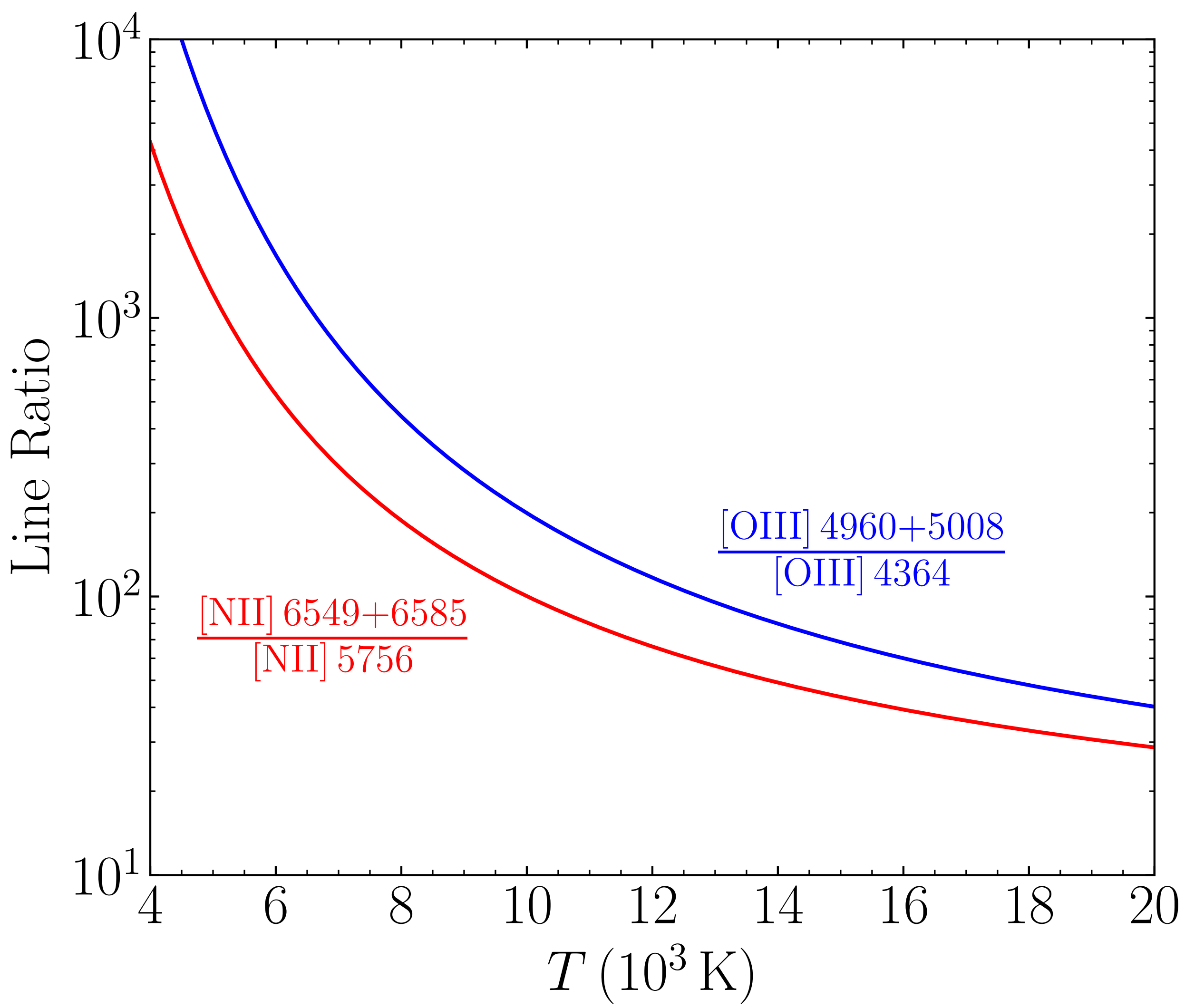
$$\langle\Omega_{30}\rangle = 2.64\times(1/9) \qquad A_{32} = 3.0\times10^{-3}\ [\mathrm{s}^{-1}]$$
$$\langle\Omega_{40}\rangle = 0.29\times(1/9) \qquad A_{31} = 9.8\times10^{-4}\ [\mathrm{s}^{-1}]$$
$$E_{40}/k = 47033\,[\mathrm{K}] \qquad A_{43} = 1.0\ [\mathrm{s}^{-1}]$$
$$E_{30}/k = 22037\,[\mathrm{K}] \qquad A_{41} = 3.3\times10^{-2}\ [\mathrm{s}^{-1}]$$
$$E_{20}/k = 188\,[\mathrm{K}]$$
$$E_{10}/k = 70\,[\mathrm{K}]$$

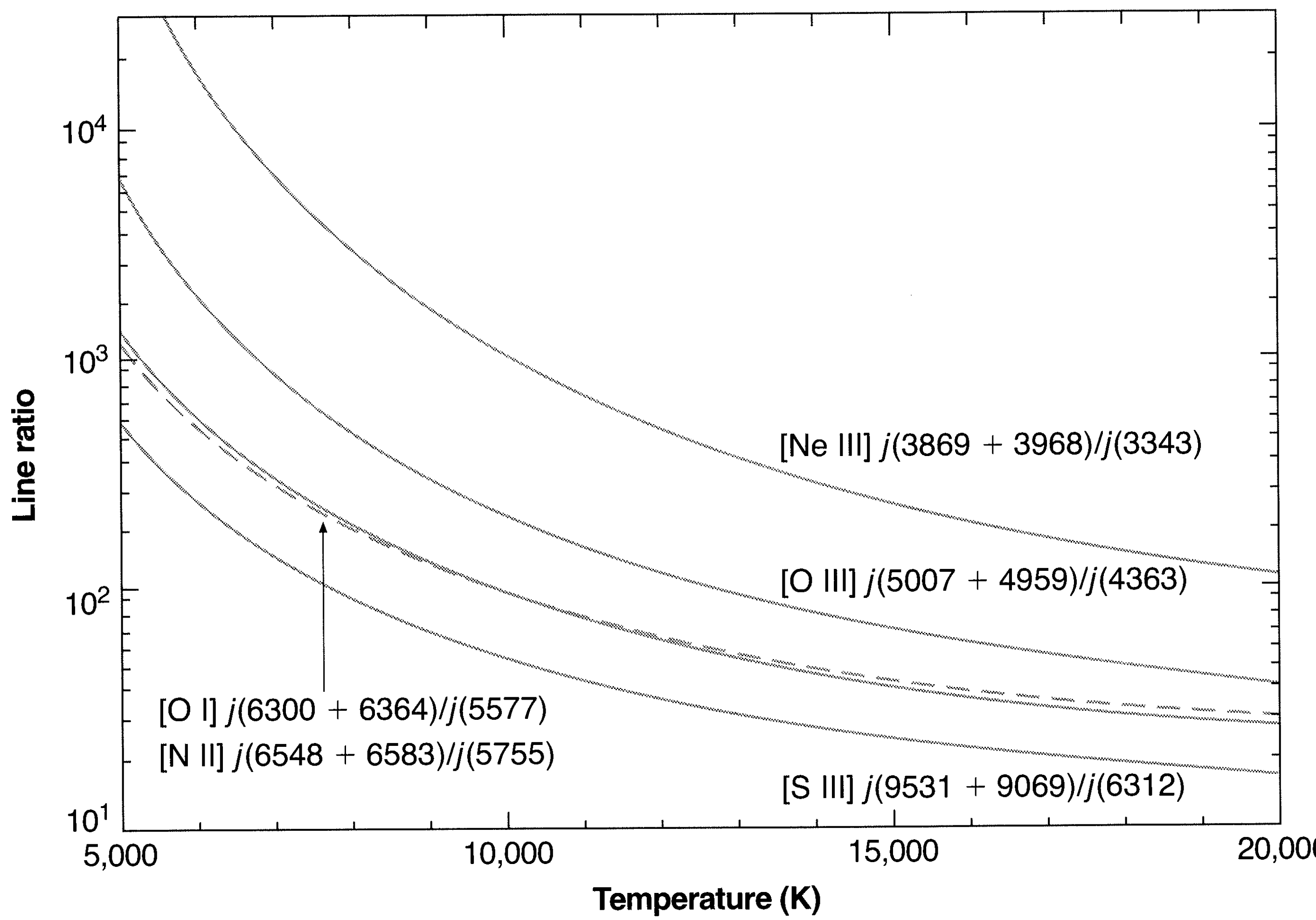
We obtain the line ratio as a function of temperature.

$$\frac{[\mathrm{O\,III}]\,4960+5008}{[\mathrm{O\,III}]\,4364} = 8.12\,e^{3.20/T_4}$$

$$\frac{[\mathrm{N\,II}]\,6549+6585}{[\mathrm{N\,II}]\,5756} = 8.23\,e^{2.50/T_4}$$

Figure 5.1 [Osterbrock]





See Equations (5.4)-(5.7) for a correction factor for the density effect.

$$\text{[O III]}\frac{j_{\lambda4959}+j_{\lambda5007}}{j_{\lambda4363}}=\frac{7.90\exp(3.29\times10^4/T)}{1+4.5\times10^{-4}n_e/T^{1/2}}$$

$$\text{[N II]}\frac{j_{\lambda6548}+j_{\lambda6583}}{j_{\lambda5755}}=\frac{8.23\exp(2.50\times10^4/T)}{1+4.4\times10^{-3}n_e/T^{1/2}}$$

$$\text{[Ne III]}\frac{j_{\lambda3869}+j_{\lambda3968}}{j_{\lambda3343}}=\frac{13.7\exp(4.30\times10^4/T)}{1+3.8\times10^{-5}n_e/T^{1/2}}$$

$$\text{[S III]}\frac{j_{\lambda9532}+j_{\lambda9069}}{j_{\lambda6312}}=\frac{5.44\exp(2.28\times10^4/T)}{1+3.5\times10^{-4}n_e/T^{1/2}}$$