# Astrophysics [Part I]

Lecture 2
Mar 14 (Thurs), 2019

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# Basic Theory of Radiation Fields

# Applicability of the Radiative Transfer Theory

- We defined specific intensity by the relation: $dE = I_\nu dA d\Omega d\nu dt$

  We should note that $dA$ and $d\Omega$ cannot both be made arbitrarily small because of the uncertainty principle:

$$dx dp_x dy dp_y = p^2 dA d\Omega \geq h^2 \quad \rightarrow \quad dA d\Omega \geq h^2/p^2 = \lambda^2$$

  There is another limitation because of the energy uncertainty principle:

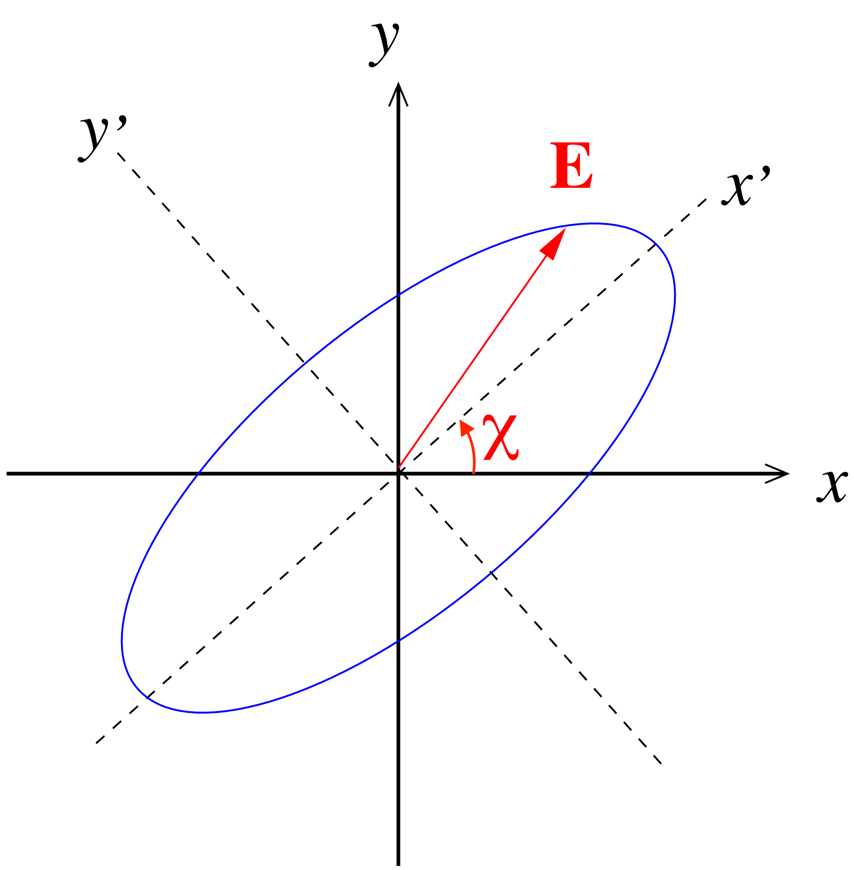
$$dE dt \geq h \quad \rightarrow \quad d\nu dt \geq 1$$

$a_0 = 0.53\text{Å}$

- Therefore, when the wavelength of light is larger than atomic dimensions (Bohr radius), as in the optical, we cannot describe the interaction of light on the atomic scale in terms of specific intensity.
- However, we may still regard transfer theory as a valid macroscopic theory, <u>provided the absorption and emission properties are correctly calculated from microscopic theories</u> (electromagnetic or quantum theory).
- A more precise, classical treatment of the validity of rays is known as the eikonal approximation. (from German "eikonal", which is from Greek word meaning "image")

# Polarization

- Let us consider a plane EM wave propagating in the +*z* direction, and examine the electric vector at z = 0. Because the electric field is transverse, the electric field can be expressed as




$$\mathbf{E} = \hat{\mathbf{x}} E_1 e^{i(kz-\omega t)} + \hat{\mathbf{y}} E_2 e^{i(kz-\omega t)}$$
$$= \hat{\mathbf{x}} E_1 e^{-i\omega t} + \hat{\mathbf{y}} E_2 e^{-i\omega t} \quad \text{at z} = 0$$

Complex amplitudes can be expressed as

$$E_1 = \mathcal{E}_1 e^{i\phi_1} \qquad E_2 = \mathcal{E}_2 e^{i\phi_2} \quad \text{where } \mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2 \text{ are real.}$$

Then, the real part of **E** is

$$\mathbf{E} = \hat{\mathbf{x}} \mathcal{E}_1 \cos(\omega t - \phi_1) + \hat{\mathbf{y}} \mathcal{E}_2 \cos(\omega t - \phi_2)$$

<u>As a function of time, the tip of **E** will trace out an ellipse, meaning that the general wave is **elliptically polarized**.</u>

- In general, the principal axes of this ellipse will have a tilt angle $\chi$ w.r.t. to *x*-*y* axes. We define the zero of time so that **E** lies along the *x*' direction at *t* = 0.

$$\mathbf{E} = \hat{\mathbf{x}}' \mathcal{E}_1' \cos \omega t + \hat{\mathbf{y}}' \mathcal{E}_2' \sin \omega t$$

- We can satisfy the late part of the above equation by defining an **ellipticity angle**:

$$\mathcal{E}_1' = \mathcal{E}_0 \cos\beta \quad \mathcal{E}_2' = -\mathcal{E}_0 \sin\beta \;\; \text{where} \;\; -\pi/2 \le \beta \le \pi/2 \;\; (\text{or} \;\; \mathcal{E}_2' = \mathcal{E}_0 \sin\beta', \;\; \beta' = -\beta)$$

- With the relations

$$\begin{aligned} &\hat{\mathbf{x}}\mathcal{E}_1 \cos(\omega t - \phi_1) + \hat{\mathbf{y}}\mathcal{E}_2 \cos(\omega t - \phi_2) \\ &= \hat{\mathbf{x}}'\mathcal{E}_0 \cos\beta \cos\omega t - \hat{\mathbf{y}}'\mathcal{E}_0 \sin\beta \sin\omega t \end{aligned} \quad + \quad \begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \end{pmatrix} = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix}$$

we obtain the relations:

Given $\mathcal{E}_1, \phi_1, \mathcal{E}_2, \phi_2,$ we can solve for $\mathcal{E}_0, \beta, \chi,$

$$\begin{aligned} \mathcal{E}_1 \cos\phi_1 &= \mathcal{E}_0 \cos\beta \cos\chi \\ \mathcal{E}_1 \sin\phi_1 &= \mathcal{E}_0 \sin\beta \sin\chi \\ \mathcal{E}_2 \cos\phi_2 &= \mathcal{E}_0 \cos\beta \sin\chi \\ \mathcal{E}_2 \sin\phi_2 &= -\mathcal{E}_0 \sin\beta \cos\chi \end{aligned} \quad \longrightarrow \quad \begin{aligned} \mathcal{E}_1^2 + \mathcal{E}_2^2 &= \mathcal{E}_0^2 \\ \mathcal{E}_1^2 - \mathcal{E}_2^2 &= \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \\ 2\mathcal{E}_1\mathcal{E}_2 \cos\delta\phi &= \mathcal{E}_0^2 \cos 2\beta \sin 2\chi \\ 2\mathcal{E}_1\mathcal{E}_2 \sin\delta\phi &= \mathcal{E}_0^2 \sin 2\beta \end{aligned}$$

(where $\delta\phi \equiv \phi_1 - \phi_2$)

# Polarization

- Taking time average of the $|\mathbf{E}|^2$, we obtain:

$$\langle |\mathbf{E}|^2 \rangle = \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_1'^2 + \mathcal{E}_2'^2 = \text{constant} \equiv \mathcal{E}_0^2$$

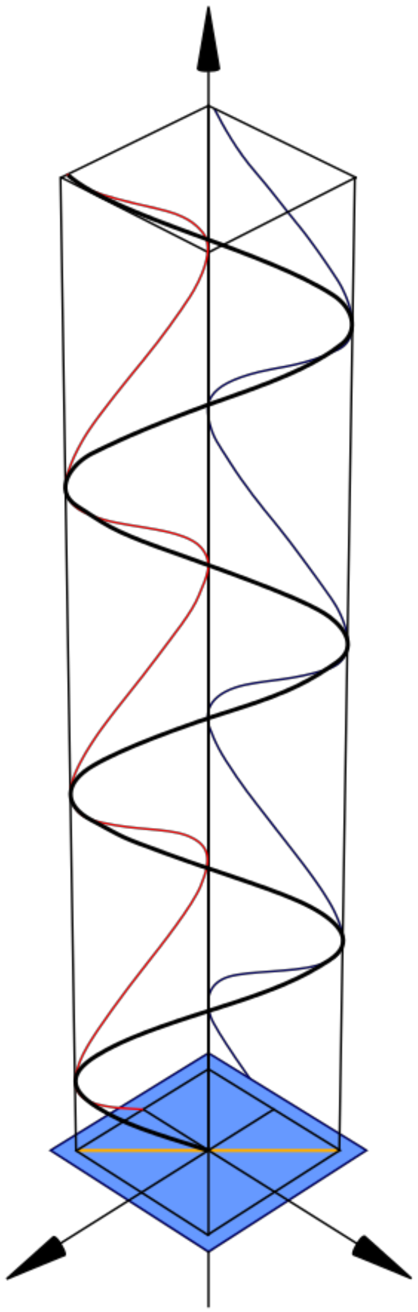
- **Polarization:**

$\beta = \pm\pi/4$ : circularly polarized

$\beta = 0$ or $\pm\pi/2$ : linearly polarized

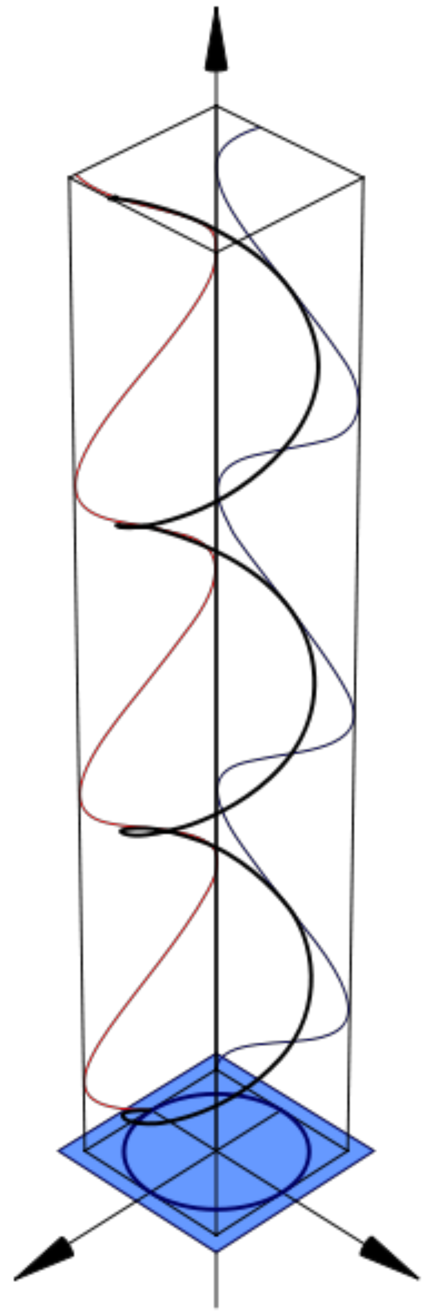
| | Right-Handed Polarization | Left-Handed Polarization |
|---|---|---|
| Helicity | + (positive) | - (negative) |
| Rotation at a fixed position | Counterclockwise | Clockwise |
| Screw at a fixed time | Left-Handed Screw | Right-Handed Screw |
| $\beta$ | $-\pi/2 < \beta < 0$ | $0 < \beta < \pi/2$ |
| $\delta\phi$ | $-\pi/2 < \delta\phi < 0$ | $0 < \delta\phi < \pi/2$ |
| Stokes V | V > 0 | V < 0 |

Figures from Wikipedia



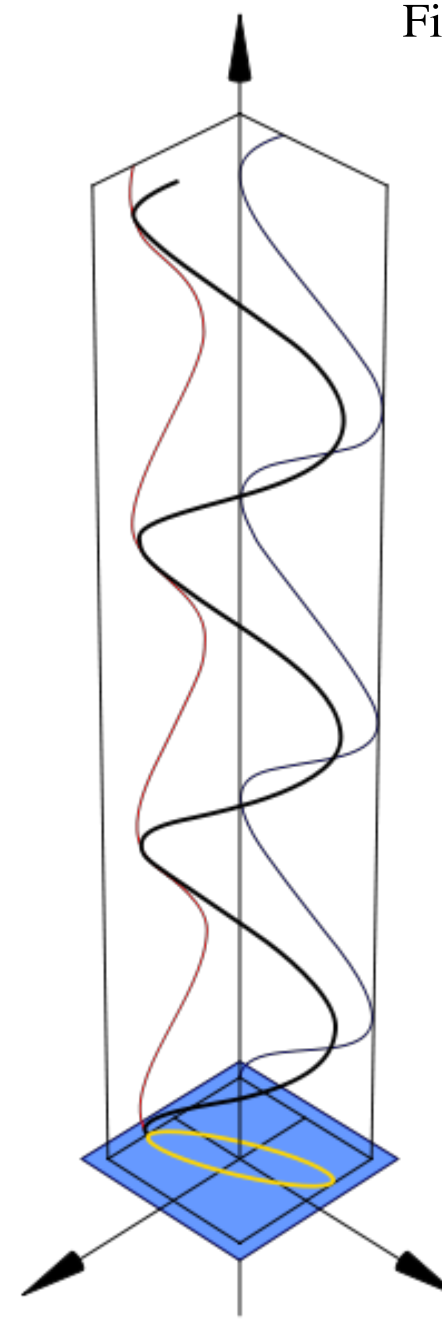
Linear

$$|\phi_1 - \phi_2| = 0$$
$$|\beta| = 0, \pi/2$$
$$\mathcal{E}_1/\mathcal{E}_2 = \text{const.}$$



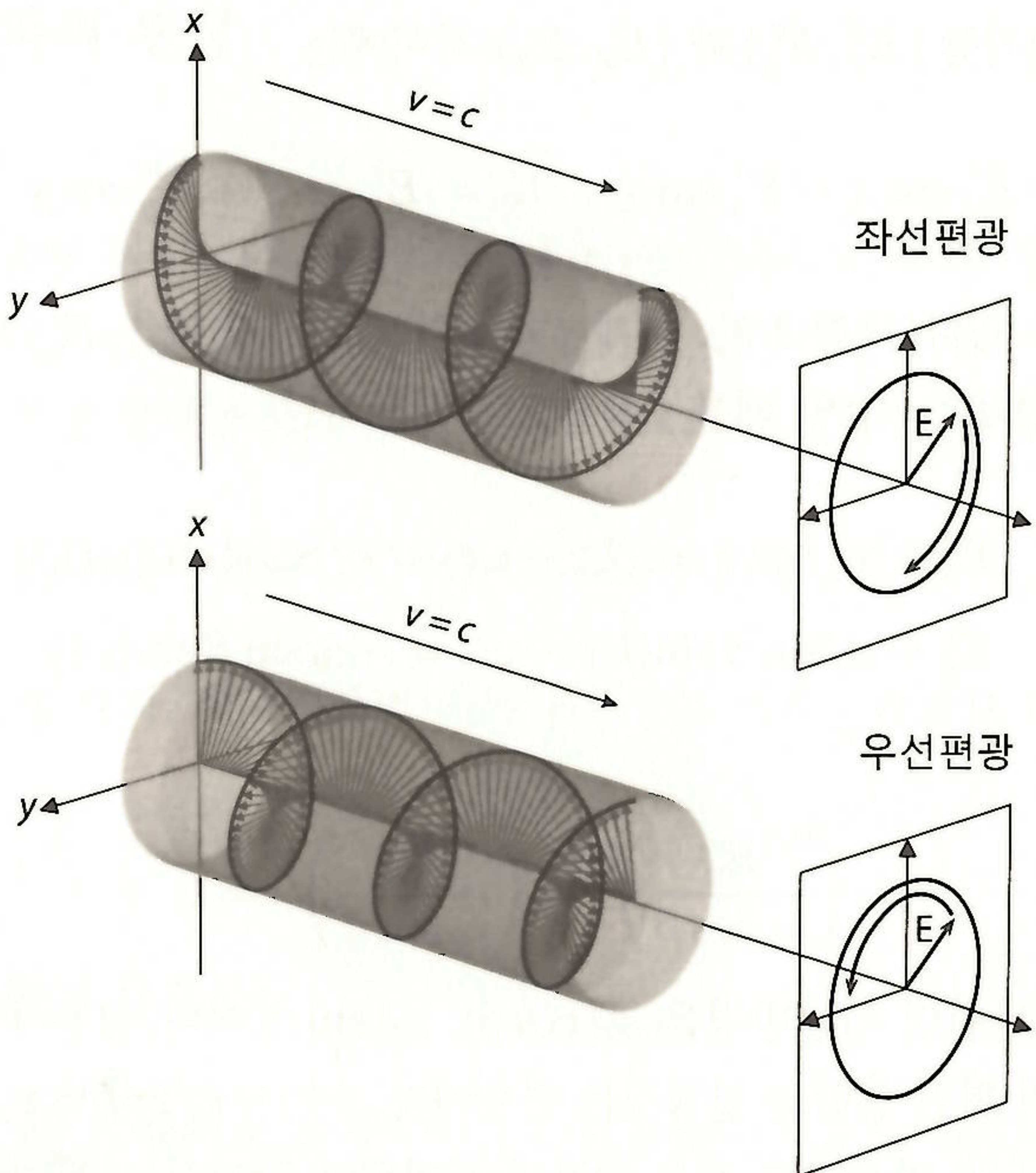
Circular

$$|\phi_1 - \phi_2| = \pi/2$$
$$|\beta| = \pi/4$$
$$|\mathcal{E}_1/\mathcal{E}_2| = 1$$



Elliptical

$$|\phi_1 - \phi_2| \neq 0, \pi/2$$
$$|\beta| \neq 0, \pi/4, \pi/4$$
$$\mathcal{E}_1/\mathcal{E}_2 \neq \pm 1$$




from 천체물리학 (구본철, 김웅태)

IEEE (1969) standard
IAU (1974) recommandation

| | RHP | LHP |
|---|---|---|
| Helicity | + (positive) | - (negative) |
| Rotation at a fixed | Counterclockwise | Clockwise |
| Screw at a fixed time | Left-Handed Screw | Right-Handed Screw |
| $\beta$ | $-\pi/2 < \beta < 0$ | $0 < \beta < \pi/2$ |
| $\delta\phi$ | $-\pi/2 < \delta\phi < 0$ | $0 < \delta\phi < \pi/2$ |
| Stokes V | V > 0 | V < 0 |

# Stokes Parameters (for monochromatic waves)

- A convenient way to solve these equations is by means of the **Stokes parameters for monochromatic waves**.

$$
\begin{aligned}
I &\equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2 \\
Q &\equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \\
U &\equiv 2\mathcal{E}_1\mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi \\
V &\equiv -2\mathcal{E}_1\mathcal{E}_2 \sin(\phi_1 - \phi_2) = -\mathcal{E}_0^2 \sin 2\beta
\end{aligned}
\quad \longrightarrow \quad I^2 = Q^2 + U^2 + V^2
$$

for a monochromatic wave (pure polarization)

Then, we have

$$\mathcal{E}_0 = \sqrt{I}, \;\; \sin 2\phi = -\frac{V}{I}, \;\; \tan 2\chi = \frac{U}{Q}$$

Pure elliptical polarization is determined sole by three parameters $(\mathcal{E}_0, \; \beta, \; \chi)$.

- Meaning of the Stokes parameters:

$I$ : total energy flux or intensity

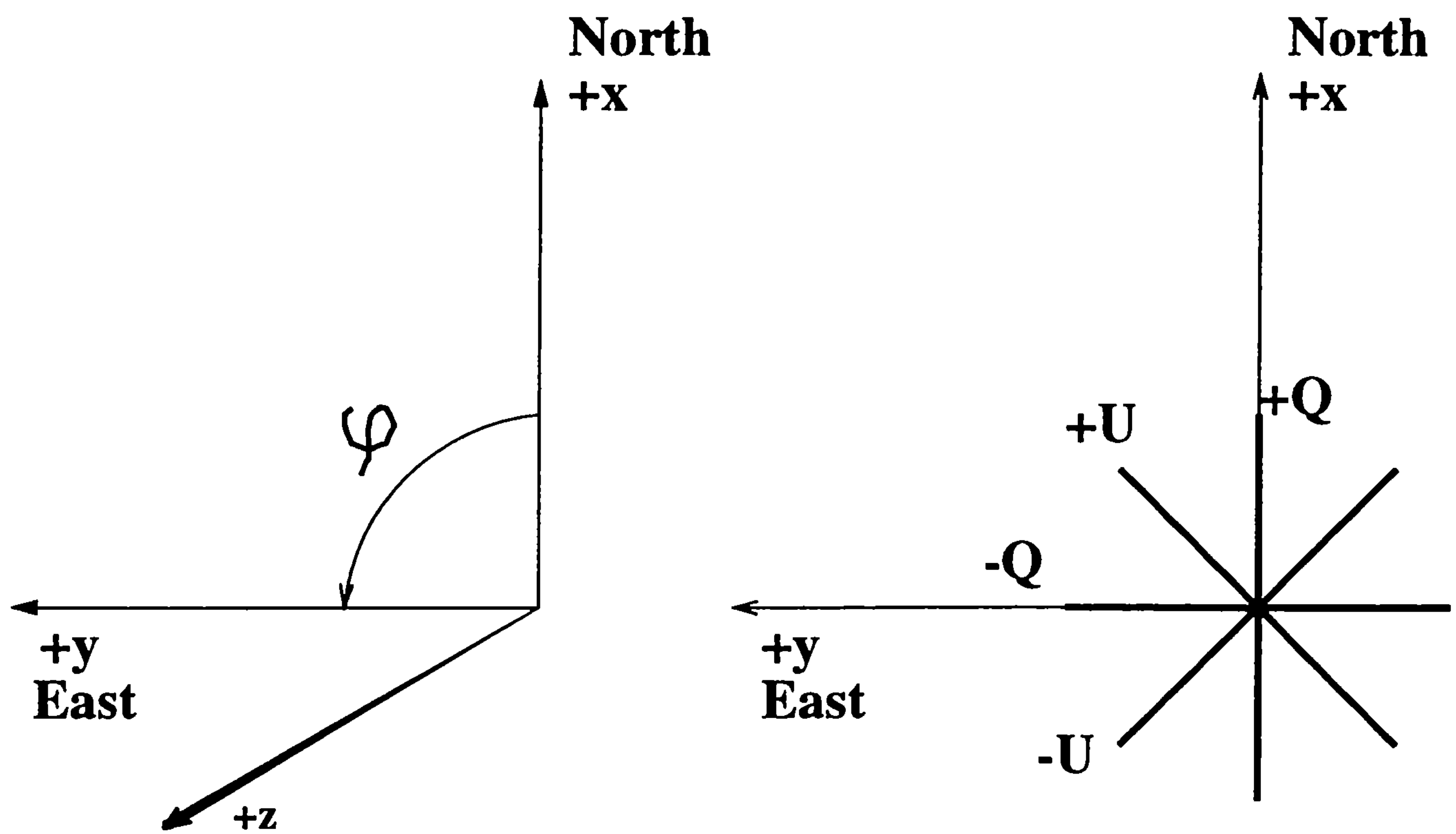
$V$ : circularity parameter ($V > 0$ : right-handed, $V < 0$ : left-handed)

$Q$, $U$ : orientation of the ellipse (or line) relative to the $x$-axis

| | |
|---|---|
| $Q \times U \neq 0, V = 0$ | : linear polarization |
| $Q = U = 0, V \neq 0$ | : circular polarization |
| $Q \times U \neq 0, V \neq 0$ | : elliptical polarization |

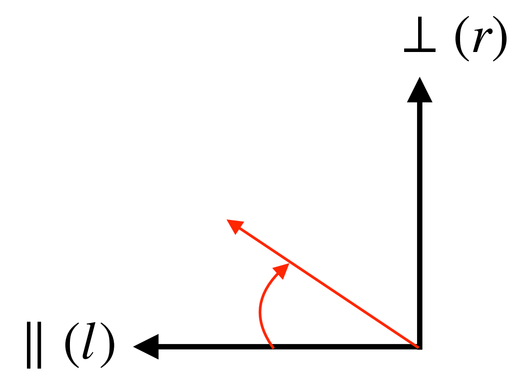
# The IAU definition of coordinate system

from Hamaker & Bregman (1996, A&AS)



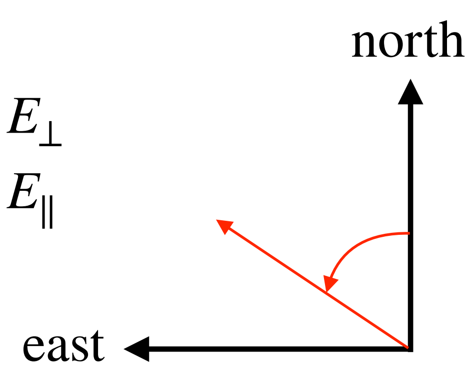
# Definition - Stokes vector (a)

- Bohren & Huffman (Absorption and Scattering of Light by Small Particles)
- Chandrasekhar (Radiative Transfer)
- IAU recommendation
- IEEE standard




$$I_{\mathrm{BH}} = E_{\parallel}E_{\parallel}^{*} + E_{\perp}E_{\perp}^{*}$$
$$Q_{\mathrm{BH}} = E_{\parallel}E_{\parallel}^{*} - E_{\perp}E_{\perp}^{*}$$
$$U_{\mathrm{BH}} = E_{\parallel}E_{\perp}^{*} + E_{\perp}E_{\parallel}^{*}$$
$$V_{\mathrm{BH}} = i\left(E_{\parallel}E_{\perp}^{*} - E_{\perp}E_{\parallel}^{*}\right)$$
$$V_{\mathrm{C}} = -i\left(E_{\parallel}E_{\perp}^{*} - E_{\perp}E_{\parallel}^{*}\right)$$

$$E_n = E_{\perp}$$
$$E_e = E_{\parallel}$$




$$I_{\mathrm{IAU}} = E_nE_n^{*} + E_eE_e^{*}$$
$$Q_{\mathrm{IAU}} = E_nE_n^{*} - E_eE_e^{*}$$
$$U_{\mathrm{IAU}} = E_nE_e^{*} + E_eE_n^{*}$$
$$V_{\mathrm{IAU}} = i\left(E_nE_e^{*} - E_eE_n^{*}\right)$$

$$\therefore \begin{pmatrix} I_{\mathrm{IAU}} \\ Q_{\mathrm{IAU}} \\ U_{\mathrm{IAU}} \\ V_{\mathrm{IAU}} \end{pmatrix} = \begin{pmatrix} I_{\mathrm{BH}} \\ -Q_{\mathrm{BH}} \\ U_{\mathrm{BH}} \\ -V_{\mathrm{BH}} \end{pmatrix}$$

# Conventions adopted by various authors

Peest et al. (2017, A&A, 601, A92) + α
(Typo: +/-U should read as +/-Q)

| | +Q | -Q |
|---|---|---|
| +V | IAU (1974)<br>Martin (1974)<br>Tsang et al. (1985)<br>Trippe (2014) | Chandrasekhar (1950)<br>van de Hulst (1957)<br>Hovenier & van der Mee (1983)<br>Fischer et al. (1994)<br>Code & Whitney (1995)<br>Mishchenko et al. (1999)<br>Gordon et al. (2001)<br>Lucas (2003)<br>Gorski et al. (2005) |
| -V | Shurcliff (1962)<br>Bianchi et al. (1996) | Bohren & Huffman (1998)<br>Rybicki & Lightman (1979)<br>Mishchenko et al. (2002) |

# Stokes Parameters (for quasi-monochromatic waves)

- In general, EM waves vary over time and with wavenumber. Clearly, then, the practical measurement of EM waves involves taking a time average over a time interval.
- Consider EM wave with **slowly varying** amplitudes and phases:

$$E_1(t) = \mathcal{E}_1(t)e^{i\phi_1(t)} \qquad E_2(t) = \mathcal{E}_2(t)e^{i\phi_2(t)}$$

- How slow is slow? **Quasi-monochromatic wave**:

  Assumption: over a time interval $\Delta t > \Delta t_c \equiv 1/\omega$ , the amplitudes and phases do not change significantly. By the uncertainty relation, its frequency spread $\Delta\omega$ about the value $\omega$ can be estimated as $\Delta\omega/\omega \approx \Delta t_c/\Delta t < 1$ .

  For this reason, the wave slowly varying over a time interval $\Delta t > \Delta t_c = 1/\omega$ is called **quasi-monochromatic**, and the time $\Delta t_c$ is called the **coherence time**.

- The **Stokes parameters for quasi-monochromatic waves** are defined by the following average over time, to be consistent with the definition for monochromatic waves:

$$\boxed{\begin{aligned} I &\equiv \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle = \left\langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \right\rangle \\ Q &\equiv \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle = \left\langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \right\rangle \\ U &\equiv \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle = 2 \left\langle \mathcal{E}_1 \mathcal{E}_2 \cos\left(\phi_1 - \phi_2\right) \right\rangle \\ V &\equiv i \left( \langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle \right) = -2 \left\langle \mathcal{E}_1 \mathcal{E}_2 \sin\left(\phi_1 - \phi_2\right) \right\rangle \end{aligned}}$$

- With the Schwartz inequality $\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle \geq \langle E_1 E_2^* \rangle \langle E_2 E_1^* \rangle$

  we can easily verify that

$$\boxed{I^2 \geq Q^2 + U^2 + V^2}$$

  The equality holds only for a completely polarized wave.

- Most sources of EM radiation a large number of atoms or molecules that emit light. The orientation of the electric fields produced by these emitters may not be correlated, in which case the light is said to be **unpolarized**. For completely **unpolarized** wave, where the phase difference $\phi_1 - \phi_2$ between $E_1$ and $E_2$ maintain no permanent relation and where there is no preferred orientation in the *x*-*y* plane, so that $\langle \mathcal{E}_1^2 \rangle = \langle \mathcal{E}_2^2 \rangle$.

$$Q = U = V = 0$$

Proof of the inequality:
Homework:
(1) Derive the Schwartz inequality.
(2) Show that $I^2 \geq Q^2 + U^2 + V^2$

# Superposition of independent waves

- Radiation will generally originate from a variety of regions different polarizations and different wave phases. Consider therefore a beam consisting of a mixture of many independent waves:

$$E_1 = \sum_k E_1^{(k)} \qquad E_2 = \sum_k E_2^{(k)} \qquad \text{where } k = 1, 2, 3, \cdots .$$

$$\left\langle E_i E_j^* \right\rangle = \sum_k \sum_l \left\langle E_i^{(k)} E_j^{(l)*} \right\rangle = \sum_k \left\langle E_i^{(k)} E_j^{(k)*} \right\rangle \qquad (i, j = 1 \text{ or } 2)$$

Because the relative phases are random, only term $k = l$ survive the averaging. Therefore, the **Stokes parameters have additive properties**:

$$\boxed{I = \sum_k I^{(k)}, \quad Q = \sum_k Q^{(k)}, \quad U = \sum_k U^{(k)}, \quad V = \sum_k V^{(k)}}$$

- By the superposition principle, an arbitrary wave can be decomposed of a completely unpolarized wave and a completely polarized wave.

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I - \sqrt{Q^2 + U^2 + V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{Q^2 + U^2 + V^2} \\ Q \\ U \\ V \end{pmatrix}$$

- Proof of the inequality: $I^2 = (I_{\rm pol} + I_{\rm unpol})^2 \geq I_{\rm pol}^2 = Q^2 + U^2 + V^2$

- **Degree of polarization** for a partially polarized wave = ratio of the intensity of the polarized part to the total intensity

$$\boxed{\Pi \equiv \frac{I_{\rm pol}}{I} = \frac{\sqrt{Q^2+U^2+V^2}}{I}}$$

- In the case of partial **linear polarization (*V* = 0)**, the measurement consists of rotating a linear polarization filter until the maximum values of intensity are found. The maximum value will occur when the filter is aligned with the plane of polarization, and the minimum value will occur along in the direction perpendicular to it.

  Total value of the unpolarized intensity is shared equally between any two perpendicular directions. Therefore,

$$I_{\rm max} = \frac{1}{2}I_{\rm unpol} + I_{\rm pol} \qquad \text{where} \quad I_{\rm unpol} = I - \sqrt{Q^2+U^2}$$

$$I_{\rm min} = \frac{1}{2}I_{\rm unpol} \qquad\qquad I_{\rm pol} = \sqrt{Q^2+U^2}$$

$$\boxed{\therefore \quad \Pi_{\rm linear} = \frac{I_{\rm max}-I_{\rm min}}{I_{\rm max}+I_{\rm min}}}$$

  This equation will underestimate the true degree of polarization if circular or elliptical polarization is present.

$$\begin{aligned} I_{\rm max} &= \tfrac{1}{2}(I_{\rm unpol}+I_{\rm cir}) + I_{\rm lin} \\ I_{\rm min} &= \tfrac{1}{2}(I_{\rm unpol}+I_{\rm cir}) \end{aligned} \quad \rightarrow \quad \frac{I_{\rm max}-I_{\rm min}}{I_{\rm max}+I_{\rm min}} = \frac{I_{\rm lin}}{I} < \frac{I_{\rm pol}}{I} = \frac{I_{\rm lin}+I_{\rm cir}}{I_{\rm unpol}+I_{\rm lin}+I_{\rm cir}}$$

# Useful Mathematical Formulae

- Dirac delta function:

$$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega-\omega')t}dt$$

- Fourier Transform:

Rybicki

$$\bar{a}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(t)e^{-i\omega t}dt$$

$$a(t) = \int_{-\infty}^{\infty} \bar{a}(\omega)e^{i\omega t}d\omega$$

Parseval's Theorem

$$\int_{-\infty}^{\infty} a(t)b^*(t)dt = 2\pi\int_{-\infty}^{\infty} \bar{a}(\omega)\bar{b}^*(\omega)d\omega$$

$$\int_{-\infty}^{\infty} |a(t)|^2dt = 2\pi\int_{-\infty}^{\infty} |\bar{a}(\omega)|^2d\omega$$

Afken (Mathematical Methods for Physicists)

$$\bar{a}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(t)e^{i\omega t}dt$$

$$a(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{a}(\omega)e^{-i\omega t}d\omega$$

$$\int_{-\infty}^{\infty} a(t)b^*(t)dt = \int_{-\infty}^{\infty} \bar{a}(\omega)\bar{b}^*(\omega)d\omega$$

$$\int_{-\infty}^{\infty} |a(t)|^2dt = \int_{-\infty}^{\infty} |\bar{a}(\omega)|^2d\omega$$

- Vector identities:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\nabla\times(\nabla\phi) = 0$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

# Electromagnetic force on a single charged particle

- **Lorentz force**: If a particle of charge $q$ and mass $m$ moves with velocity $\mathbf{v}$ in the presence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$, then it will experience a force:

$$\boxed{\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)}$$

(in Gaussian units, or cgs units)

- **Power** supplied by the EM fields (the rate of work done by the fields) on a particle is

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{F} &= q\mathbf{v} \cdot \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \\
\mathbf{v} \cdot m\frac{d\mathbf{v}}{dt} &= q\mathbf{v} \cdot \mathbf{E} \\
\therefore \frac{dU_{\rm mech}}{dt} &\equiv \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\mathbf{v} \cdot \mathbf{E}
\end{aligned}$$

- Note $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$, meaning that the magnetic fields do not work.

# Electromagnetic force on a continuous medium

- Consider a medium with **charge density** and **current density**:

$$\rho \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i$$

$$\mathbf{j} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i$$

- **Force density** (force per unit volume):

$$\mathbf{f} = \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B}$$

- **Power density** supplied by the field (the rate of work done by the field per unit volume):

$$\frac{du_{\mathrm{mech}}}{dt} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i \cdot \mathbf{E} = \mathbf{j} \cdot \mathbf{E}$$

Note typos in the Rybicki & Lightman's book. They use the same symbol to denote the energy density $u$ and the total energy $U$ within a volume.

# Maxwell's equations

- **Maxwell's eqs.** (in macroscopic forms) relates fields to charge and current densities.

| | |
|---|---|
| $\nabla \cdot \mathbf{D} = 4\pi\rho$ | Gauss's law |
| $\nabla \cdot \mathbf{B} = 0$ | Gauss's law for magnetism (no magnetic monopoles) |
| $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ | Maxwell-Faraday equation |
| $\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$ | Ampere-Maxwell equation |

$\mathbf{D}$,$\mathbf{H}$ : macroscopic fields
$\mathbf{B}$,$\mathbf{E}$ : microscopic fields

$\mathbf{D} = \epsilon \mathbf{E}$

$\mathbf{B} = \mu \mathbf{H}$

$\epsilon$ : dielectric constant

$\mu$ : magnetic permeability

**Dielectric material (절연체)**: an electrical insulator that can be polarized by an applied electric field. Electric charges do not flow through the material as they do in a conductor, but only slightly shift from their average equilibrium positions causing dielectric polarization.

**Permeability (투자율)**: the degree of magnetization of a material in response to a magnetic field.

Note $\epsilon = \mu = 1$ in the absence of dielectric or permeability media.

- **Conservation of charge**

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \left( \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \right)$$

$$0 = \frac{4\pi}{c}\nabla \cdot \mathbf{j} + \frac{1}{c}\frac{\partial}{\partial t}(4\pi\rho)$$

$$\rightarrow \boxed{\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0}$$

# Poynting's Theorem: Electromagnetic Field Energy

- Use the Ampere's law to obtain the mechanical energy density

$$\frac{du_{\text{mech}}}{dt} = \mathbf{j}\cdot\mathbf{E} = \frac{1}{4\pi}\mathbf{E}\cdot\left(c\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)$$

- Use a vector identity and Faraday's law:

$$\begin{aligned}\mathbf{E}\cdot(\nabla\times\mathbf{H}) &= \mathbf{H}\cdot(\nabla\times\mathbf{E}) - \nabla\cdot(\mathbf{E}\times\mathbf{H}) \\ &= -\frac{1}{c}\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{H})\end{aligned}$$

Then,

$$\mathbf{j}\cdot\mathbf{E} = \frac{1}{4\pi}\left(-c\nabla\cdot(\mathbf{E}\times\mathbf{H}) - \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right)$$

- **Poynting's theorem** in differential form.

$$\mathbf{j}\cdot\mathbf{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2 + \frac{B^2}{\mu}\right) = -\nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\right)$$

- **Electromagnetic field density** (field energy per unit volume) and **Poynting vector** (electromagnetic flux vector) are identified:

$$\boxed{u_{\text{field}} = \frac{1}{8\pi}\left(\epsilon E^2 + \frac{B^2}{\mu}\right) = u_E + u_B \qquad \mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}}$$

- The Poynting's theorem becomes an expression of the **local conservation of energy**.

$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{field}}) + \nabla \cdot \mathbf{S} = 0$$

- Integrating the equation over a volume element and using the divergence theorem, we obtain the **conservation of energy**:

$$\boxed{\frac{d}{dt}(U_{\text{mech}} + U_{\text{field}}) = -\int_\Sigma \mathbf{S} \cdot d\mathbf{A}}$$

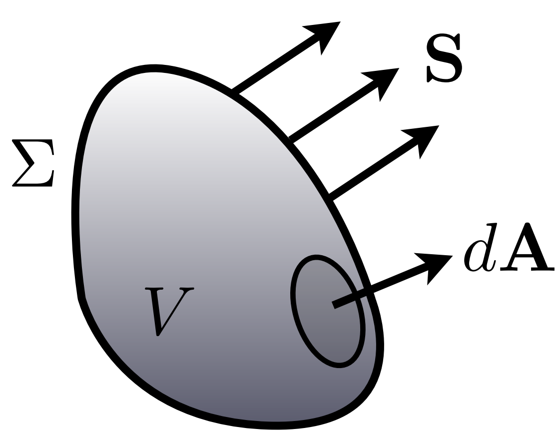
Here,

$$U_{\text{mech}} \equiv \int_V u_{\text{mech}} dV \quad \text{and} \quad U_{\text{field}} \equiv \int_V u_{\text{field}} dV$$

or

$$\int_V (\mathbf{j} \cdot \mathbf{E}) dV + \frac{d}{dt} \int_V \left( \frac{\epsilon E^2 + B^2/\mu}{8\pi} \right) dV = -\int_\Sigma \mathbf{S} \cdot d\mathbf{A}$$

Meaning: <u>the rate of change of total (mechanical + field) energy within the volume *V* is equal to the net inward flow of energy through the bounding surface $\Sigma$.</u>




divergence theorem:

$$\int_V \nabla \cdot \mathbf{S} dV = \int_\Sigma \mathbf{S} \cdot d\mathbf{A}$$

- In electrostatics and magnetostatics, we recall that

$$\mathbf{E} \propto r^{-2} \quad \text{and} \quad \mathbf{B} \propto r^{-2} \quad \text{as} \quad r \to \infty \qquad \to \quad \mathbf{S} \propto r^{-4}$$

$$\therefore \quad \int_{\Sigma} \mathbf{S} \cdot d\mathbf{A} = 0 \quad \text{as} \quad r \to \infty$$

- However, for time-varying fields, we will find that

$$\mathbf{E} \propto r^{-1} \quad \text{and} \quad \mathbf{B} \propto r^{-1} \quad \text{as} \quad r \to \infty$$

$$\therefore \quad \int_{\Sigma} \mathbf{S} \cdot d\mathbf{A} \neq 0 \quad \text{as} \quad r \to \infty$$

- This finite energy flowing outward (or inward) at large distances is called **radiation**. Those parts of **E** and **B** that decreases as $r^{-1}$ at large distances are said to constitute the **radiation field**.

# Plane Electromagnetic Waves

- In vacuum $(\rho = 0 = \mathbf{j},\ \epsilon = 1 = \mu)$, Maxwell's equations give the vector wave equations:

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$\longrightarrow$ These equations are invariant under
$$\begin{aligned} \mathbf{E} &\to \mathbf{B} \\ \mathbf{B} &\to -\mathbf{E} \end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times (\nabla \times \mathbf{E}) &= -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \\
\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \\
\therefore\ \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \\
\nabla \times (\nabla \times \mathbf{B}) &= \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \\
\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} \\
\therefore\ \nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} &= 0
\end{aligned}$$

$\longrightarrow$ We obtain the vector wave equations:

$$\boxed{\begin{aligned} \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0 \\ \nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} &= 0 \end{aligned}}$$

# Properties of a single Fourier mode

- Consider an arbitrary Fourier mode in vacuum:

$$\mathbf{E} = E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{e}}, \qquad \mathbf{B} = B_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{b}}$$

wave vector $\mathbf{k}$

angular frequency $\omega = 2\pi\nu$

( $E_0$ and $B_0$ are complex constants.)

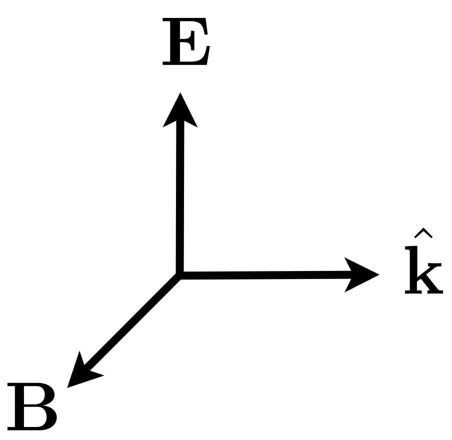
- Substituting into Maxwell's equations yields:

$$\nabla\cdot\mathbf{E} = 0 \quad\rightarrow\quad \mathbf{k}\cdot\mathbf{E} = 0$$

$$\nabla\cdot\mathbf{B} = 0 \quad\rightarrow\quad \mathbf{k}\cdot\mathbf{B} = 0 \qquad \left(\mathbf{k} = \hat{\mathbf{k}}\frac{\omega}{c}\right)$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \quad\rightarrow\quad \mathbf{k}\times\mathbf{E} = \frac{\omega}{c}\mathbf{B} \qquad \text{or} \quad \hat{\mathbf{k}}\times\mathbf{E} = \mathbf{B}$$

$$\nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \quad\rightarrow\quad \mathbf{k}\times\mathbf{B} = -\frac{\omega}{c}\mathbf{E} \qquad \text{or} \quad \hat{\mathbf{k}}\times\mathbf{B} = -\mathbf{E}$$




$$\hat{\mathbf{k}}\times\mathbf{E} = \mathbf{B} \quad\rightarrow\quad \hat{\mathbf{k}}\times\hat{\mathbf{e}}E_0 = \hat{\mathbf{b}}B_0$$

$$E_0\left(\hat{\mathbf{k}}\times\hat{\mathbf{e}}\right)\cdot\hat{\mathbf{b}} = B_0$$

$$E_0 = B_0$$

$$\begin{aligned} E_0 &= |\mathbf{E}|e^{i\phi_E} \\ B_0 &= |\mathbf{B}|e^{i\phi_B} \end{aligned} \quad\rightarrow\quad \phi_E = \phi_B$$

(1) EM waves are **transverse** (perpendicular to the direction of propagation).

(2) $\mathbf{E}$ and $\mathbf{B}$ are **orthogonal** to each other.

(3) $\left(\hat{\mathbf{k}},\ \hat{\mathbf{e}},\ \hat{\mathbf{b}}\right)$ form an orthogonal basis (triad).

(4) **Field amplitudes and phases are equal**: $|\mathbf{B}| = |\mathbf{E}|, \quad B_0 = E_0 \quad \text{and} \quad \phi_B = \phi_E$

- Fourier transform of fields:

$$\bar{\mathbf{E}}(\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int d^3\mathbf{r}\int dt\mathbf{E}(\mathbf{r},t)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

- Inverse transformation:

$$\mathbf{E}(\mathbf{r},t) = \int d^3\mathbf{k}\int d\omega\bar{\mathbf{E}}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

- Apply the wave equation to Fourier expansion:

$$\begin{aligned}\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t^2} &= -\int d^3\mathbf{k}\int d\omega\left(k^2 - \frac{\omega^2}{c^2}\right)\mathbf{E}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \\ &= 0\end{aligned}$$

$$\therefore k = \frac{\omega}{c}$$

- Phase velocity

$$v_{\mathrm{ph}} \equiv \frac{\omega}{k} = c$$

# Dispersion relation

- We obtain the vacuum dispersion relation, phase velocity, and group velocity:

$$\omega = ck \qquad v_{\mathrm{ph}} \equiv \frac{\omega}{k} = c \qquad v_{\mathrm{g}} \equiv \frac{\partial \omega}{\partial k} = c$$

dispersion relation = a function which gives $\omega$ as a function of $k$.

phase velocity = the rate at which the phase of the wave propagates in space.

group velocity = the velocity with which the overall shape of the waves' amplitudes (modulation or envelope of the wave) propagates through space.

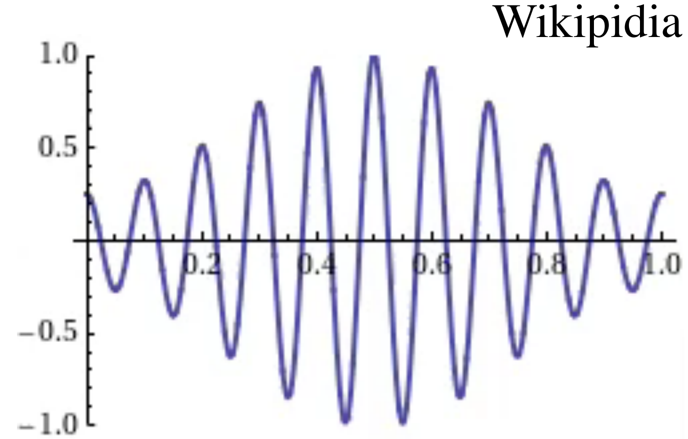
- Assume the wave packet $E$ is almost monochromatic, so that its Fourier component is nonzero only in the vicinity of a central wavenumber $k_0$. Then, linearization gives:

$$\begin{aligned}\omega(k) &\approx \omega_0 + (k - k_0)\frac{\partial \omega(k)}{\partial k}|_{k=k_0} \\ &= \omega_0 + (k - k_0)\omega_0'\end{aligned}$$

$$\begin{aligned}E(x,t) &= \int dk \int d\omega \bar{E}(k,\omega) e^{i(kx - \omega t)} \\ &\approx e^{it(\omega_0' k_0 - \omega_0)} \int dk \bar{E}(k,\omega_0) e^{ik(x - \omega_0' t)}\end{aligned}$$

$$|E(x,t)| = |E(x - \omega_0' t, 0)|$$

Wikipidia




$\leftarrow v_{\mathrm{ph}} \quad \rightarrow v_{\mathrm{g}}$

- The envelope of the wavepacket travels at velocity $\omega_0' = (\partial \omega / \partial k)_{k=k_0}$.

- If *A(t)* and *B(t)* are two complex quantities with the same sinusoidal time dependence,

$$A(t) = \mathcal{A}e^{i\omega t} \qquad B(t) = \mathcal{B}e^{i\omega t}$$

then the time average of the product of their real parts is

$$\begin{aligned}
\langle \mathrm{Re}A(t) \cdot \mathrm{Re}B(t) \rangle &= \frac{1}{4} \left\langle \left( \mathcal{A}e^{i\omega t} + \mathcal{A}^* e^{-i\omega t} \right) \left( \mathcal{B}e^{i\omega t} + \mathcal{B}^* e^{-i\omega t} \right) \right\rangle \\
&= \frac{1}{4} \langle \mathcal{A}\mathcal{B}^* + \mathcal{A}^*\mathcal{B} \rangle \\
&= \frac{1}{2}\mathrm{Re}(\mathcal{A}\mathcal{B}^*) = \frac{1}{2}\mathrm{Re}(\mathcal{A}^*\mathcal{B})
\end{aligned}$$

- Time-averaged Poynting vector amplitude:

$$\begin{aligned}
\langle S \rangle &= \frac{c}{4\pi} \left\langle \mathrm{Re}\left( E_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \right) \mathrm{Re}\left( B_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \right) \right\rangle \\
&= \frac{c}{8\pi} \mathrm{Re}\left( E_0 B_0^* \right) \\
&= \frac{c}{8\pi} |E_0|^2 = \frac{c}{8\pi} |B_0|^2 \qquad \leftarrow E_0 = B_0
\end{aligned}$$

Energy per unit area per unit time:

$$\frac{dW}{dAdt} = \frac{c}{8\pi} |E_0|^2$$

- Time-averaged field energy density:

$$\langle U_{\text{field}} \rangle = \frac{1}{8\pi} \left\langle |\mathbf{E}|^2 + |\mathbf{B}|^2 \right\rangle = \frac{1}{16\pi} \mathrm{Re}(E_0 E_0^* + B_0 B_0^*) = \frac{1}{8\pi} |E_0|^2 = \frac{1}{8\pi} |B_0|^2$$

- Velocity of energy flow:

$$\langle S \rangle / \langle U_{\text{field}} \rangle = c$$

# Power Spectrum

- A common property of any wave theory:

  If we have a time record of the radiation field of length $\Delta t$, we can only define the spectrum to within a frequency resolution $\Delta\omega$ where

$$\boxed{\Delta\omega\Delta t > 1. \quad \text{(uncertainty relation)}}$$

- Let us consider only a component of the transverse electric field: $E(t) \equiv \hat{\mathbf{a}} \cdot \mathbf{E}(t)$
- Fourier transform and its inverse are:

$$\bar{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt, \qquad E(t) = \int_{-\infty}^{\infty} \bar{E}(\omega)e^{-i\omega t}d\omega$$

  Since *E*(t) is real, the negative frequencies are redundant, i.e., $\bar{E}(-\omega) = \bar{E}^*(\omega)$.

- Total energy per unit area per unit time: $\dfrac{dW}{dAdt} = \dfrac{c}{4\pi}E^2(t)$ (Poynting vector)
- Total energy per unit area:

$$\frac{dW}{dA} = \int_{-\infty}^{\infty} \frac{dW}{dAdt}dt = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)dt$$
$$= \frac{c}{2}\int_{-\infty}^{\infty} |\bar{E}(\omega)|^2 d\omega = c\int_{0}^{\infty} |\bar{E}(\omega)|^2 d\omega$$

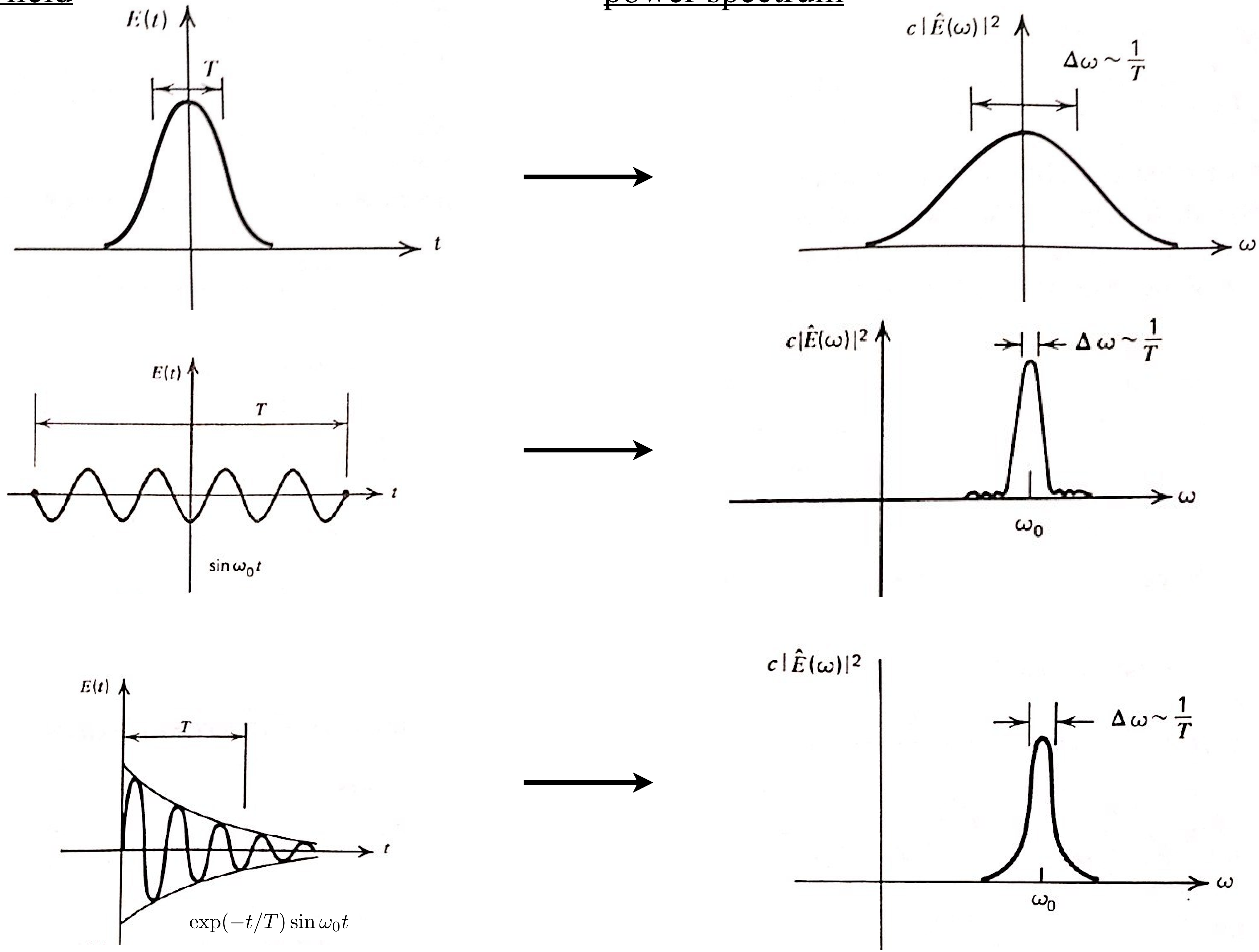
Here, we used Parseval's theorem: $\int_{-\infty}^{\infty} E^2(t)dt = 2\pi\int_{-\infty}^{\infty} |\bar{E}(\omega)|^2 d\omega$

Energy per unit area per unit frequency:

$$\frac{dW}{dAd\omega} = c|\bar{E}(\omega)|^2$$

electric field

power spectrum



# Electromagnetic Potentials

(Gauss' law for magnetism)

- **Vector potential**: from the vector identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, the equation $\nabla \cdot \mathbf{B} = 0$ yields

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \qquad \mathbf{A}: \text{ vector potential}$$

(Maxwell-Faraday equation)

Then, $\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 0$

- **Scalar potential**: from the vector identity $\nabla \times (\nabla\phi) = 0$, this equation can be satisfied if we define a potential such as

$$\boxed{\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}} \qquad \phi: \text{ scalar potential}$$

- **Gauge invariance**:

**B** will be unchanged for any transformation

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\psi \qquad \text{since } \nabla \times (\nabla\psi) = 0$$

**E** will also be unchanged if at the same time the scalar potential is changed by

$$\phi \rightarrow \phi' = \phi - \frac{1}{c}\frac{\partial \psi}{\partial t}$$

EM field is invariant under the **Gauge transform** $\boxed{(\phi,\ \mathbf{A}) \rightarrow \left(\phi - \frac{1}{c}\frac{\partial \psi}{\partial t},\ \mathbf{A} + \nabla\psi\right)}$

# Lorentz Gauge (Lorentz condition)

- Using the potential, the inhomogeneous Maxwell's equations can be written as

$$\nabla \cdot \mathbf{E} = 4\pi\rho \;\rightarrow\; \nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -4\pi\rho$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} \;\rightarrow\; \nabla\times(\nabla\times\mathbf{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = \frac{4\pi}{c}\mathbf{j}$$

$$\rightarrow\; -\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} + \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = \frac{4\pi}{c}\mathbf{j}$$

$$\text{Note}: \nabla\times(\nabla\times\mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A})$$

- The Lorentz gauge is the most important gauge in the EM theory, defined by:

$$\boxed{\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0}$$

Note that we can alway choose a function $\psi$ such as:

$$\nabla\cdot\mathbf{A}' + \frac{1}{c}\frac{\partial\phi'}{\partial t} = \nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} + \left(\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}\right) = 0$$

- Then, with the Lorentz gauge, the above equations become:

$$\boxed{\begin{aligned}\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -4\pi\rho\\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{j}\end{aligned}}$$

Note: Column gauge is $\nabla\cdot\mathbf{A} = 0$

# Retarded potentials

- The solutions to the above equations are (called the **retarded potentials**, see Jackson):

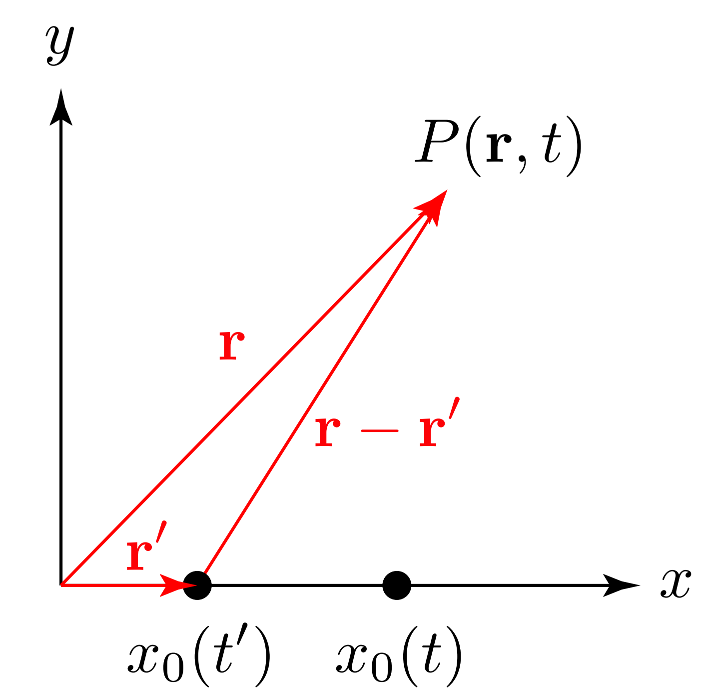
$$\phi(\mathbf{r},t) = \int_V d^3\mathbf{r}' \int dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int_V d^3\mathbf{r}' \int dt' \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)$$

$$\longrightarrow \phi(\mathbf{r},t) = \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}, \qquad \mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}$$

where

$$t' \equiv t_{\rm ret} = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$




$$x_0(t) - x_0(t') = c(t-t')$$

- The **retarded time** refers to conditions at the point $\mathbf{r}'$ that existed at a time earlier than $t$ by just the time required for light to travel from $\mathbf{r}'$ to $\mathbf{r}$.
- The potentials responds to the changes after "retarded time" delay.

# Radiation from Moving Charges

# A single moving charge: Potentials

- Recall the retarded potentials:

$$\phi(\mathbf{r},t) = \int d^3\mathbf{r}' \int dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$
$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}' \int dt' \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

- Consider a particle of charge $q$ that moves along a trajectory $\mathbf{r}=\mathbf{r}_0(t)$.
- Its velocity is then $\mathbf{u}(t)=\dot{\mathbf{r}}_0(t)$. The charge and current densities are given by

$$\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{r}_0(t)), \qquad \mathbf{j}(\mathbf{r},t) = q\mathbf{u}(t)\delta(\mathbf{r}-\mathbf{r}_0(t))$$

- Then, the potentials become

$$\phi(\mathbf{r},t) = q\int \frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|}\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)$$
$$\mathbf{A}(\mathbf{r},t) = \frac{q}{c}\int \frac{dt'\mathbf{u}(t')}{|\mathbf{r}-\mathbf{r}_0(t')|}\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)$$

# Note on the Dirac delta function.

- $$\delta(\mathbf{r}-\mathbf{r}_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

- $\int f(x)\delta(x-x_0)dx = f(x_0)$ if $x_0$ is not a function of $x$.

- $$\int f(x)\delta(g(x))dx = \int f(x)\delta(y)\frac{dy}{(dg/dx)} \qquad \leftarrow \qquad \begin{aligned} y &\equiv g(x') \\ dy &= (dg/dx')dx' \\ dx' &= \frac{dy}{(dg/dx')} \end{aligned}$$
$$= \sum_{x_j} \frac{f(x_j)}{dg/dx|_{x_j}}$$

  where $x_j$ are roots of the equation $y = g(x) = 0$

- Let's define

$$\mathbf{R}(t') \equiv \mathbf{r} - \mathbf{r}_0(t') \quad \rightarrow \quad R(t') = |\mathbf{r} - \mathbf{r}_0(t')|$$

- We then have

$$\phi(\mathbf{r}, t) = q \int \frac{dt'}{R(t')} \delta(t' - t + R(t')/c)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{c} \int \frac{\mathbf{u}(t') dt'}{R(t')} \delta(t' - t + R(t')/c)$$

- Change of variables:

(1) $$t'' = t' - t + R(t')/c \quad \rightarrow \quad dt'' = \left[1 + \frac{1}{c}\dot{R}(t')\right] dt'$$

(2)
$$R^2(t') = \mathbf{R}(t') \cdot \mathbf{R}(t')$$
$$2R(t')\dot{R}(t') = -2\mathbf{R}(t') \cdot \mathbf{u}(t') \qquad \leftarrow \dot{\mathbf{R}}(t') = -\mathbf{u}(t')$$
$$\dot{R}(t') = -\frac{\mathbf{R}(t')}{R(t')} \cdot \mathbf{u}(t')$$

(3)
$$\dot{R}(t') = -\mathbf{n}(t') \cdot \mathbf{u}(t')$$
$$dt'' = [1 - \mathbf{n}(t') \cdot \boldsymbol{\beta}(t')]\, dt'$$

where $\mathbf{n}(t') \equiv \dfrac{\mathbf{R}(t')}{R(t')}$ and $\boldsymbol{\beta} \equiv \dfrac{\mathbf{u}}{c}$

(4) $$dt'' = \kappa(t') dt'$$

where $\kappa(t') = 1 - \mathbf{n}(t') \cdot \boldsymbol{\beta}(t')$

# A single moving charge: The Lienard-Wiechart Potential

$$\phi(\mathbf{r}, t) = q \int \frac{dt''}{\kappa(t')R(t')} \delta(t'')$$

$$\mathbf{A}(\mathbf{r}, t) = q \int \frac{dt''\boldsymbol{\beta}(t')}{\kappa(t')R(t')} \delta(t'')$$

The equations then becomes by setting $t'' = 0 \quad \rightarrow \quad t' = t_{\mathrm{ret}} \equiv t - R(t_{\mathrm{ret}})/c$

$$\boxed{\begin{aligned} \phi(\mathbf{r}, t) &= \frac{q}{\kappa(t_{\mathrm{ret}})R(t_{\mathrm{ret}})} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{q\boldsymbol{\beta}(t_{\mathrm{ret}})}{\kappa(t_{\mathrm{ret}})R(t_{\mathrm{ret}})} \end{aligned}}$$

$Li\acute{e}nard - Wiechart\ potentials$

(리에나르-비헤르트)

(French-German)

- **Beaming effect**: The factor $\kappa(t_{\mathrm{ret}}) = 1 - \mathbf{n}(t_{\mathrm{ret}}) \cdot \boldsymbol{\beta}(t_{\mathrm{ret}})$ becomes very important at velocities close to the speed of light, where, it tends to concentrate the potentials into a narrow cone about the particle velocity.
- **Retardation makes it possible for a particle to radiate**:

- In static case, differentiation of the $1/r$ potential to find the fields gives a $1/r^2$ decrease.
- However, the implicit dependence of the retarded time on position gives $1/r$ behavior of the fields. This allows radiation energy to flow to infinite distances.

# A single moving charge: Electromagnetic Fields

- Then, the electromagnetic fields is obtained (see Jackson 14.1):
- Note $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{n}$ form a right-handed triad of mutually perpendicular vectors, and that . These properties are consistent with the solutions of the source-free Maxwell equations.

$$|\mathbf{E}| = |\mathbf{B}|$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

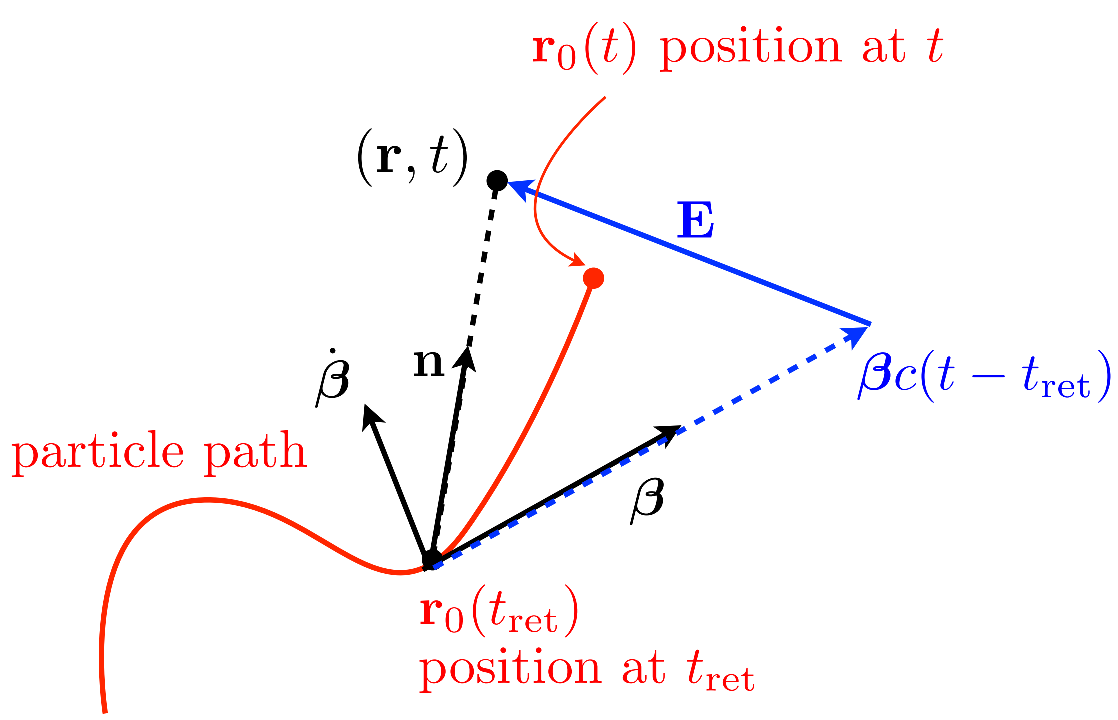
$$\mathbf{B} = \nabla \times \mathbf{A}$$

$\longrightarrow$

velocity field — acceleration field

$$\mathbf{E}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right] + \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R} \times \{(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}\right]$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{n} \times \mathbf{E}(\mathbf{r},t)$$




Geometry for calculation of the radiation field at a point $(\mathbf{r},t)$ in spacetime.

where

$$\mathbf{u} \equiv \dot{\mathbf{r}}_0(t_{\mathrm{ret}})$$

$$\boldsymbol{\beta} \equiv \frac{\mathbf{u}(t_{\mathrm{ret}})}{c} = \frac{\dot{\mathbf{r}}_0(t_{\mathrm{ret}})}{c}$$

$$\dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{u}}(t_{\mathrm{ret}})}{c} = \frac{\ddot{\mathbf{r}}_0(t_{\mathrm{ret}})}{c}$$

$$\mathbf{R} \equiv \mathbf{r} - \mathbf{r}_0(t_{\mathrm{ret}})$$

$$\mathbf{n} \equiv \frac{\mathbf{R}}{R} = \frac{\mathbf{r}-\mathbf{r}_0(t_{\mathrm{ret}})}{|\mathbf{r}-\mathbf{r}_0(t_{\mathrm{ret}})|}$$

$$\kappa \equiv 1 - \mathbf{n}\cdot\boldsymbol{\beta}$$

# “Velocity” Field

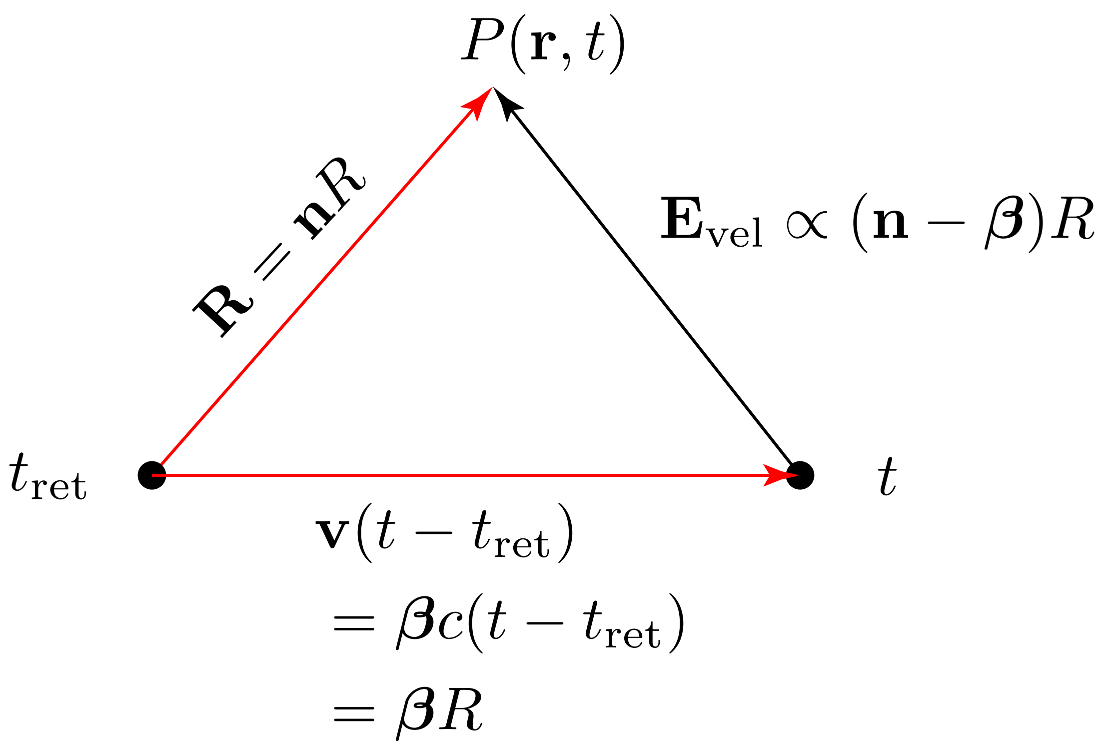
- The first term depends only on position and velocity. When the particle moves with constant velocity it is only this term that contributes to the fields.

$$\mathbf{E}_{\rm vel}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]$$

Displacement (of the photon) from the retarded point $\mathbf{r}_0(t_{\rm ret})$ (point at $t_{\rm ret}$ ) to the field point $\mathbf{r}$ during the light travel time = $\mathbf{n}c(t-t_{\rm ret})$ .

In the same time, the particle undergoes a displacement $\boldsymbol{\beta}c(t-t_{\rm ret})$ .

The displacement between the field point and the current position of the particle is given by $(\mathbf{n}-\boldsymbol{\beta})c(t-t_{\rm ret})$ which is the direction of the velocity field.




- Therefore, **the “velocity” electric field always points along the line toward the “current” position of the particle.**

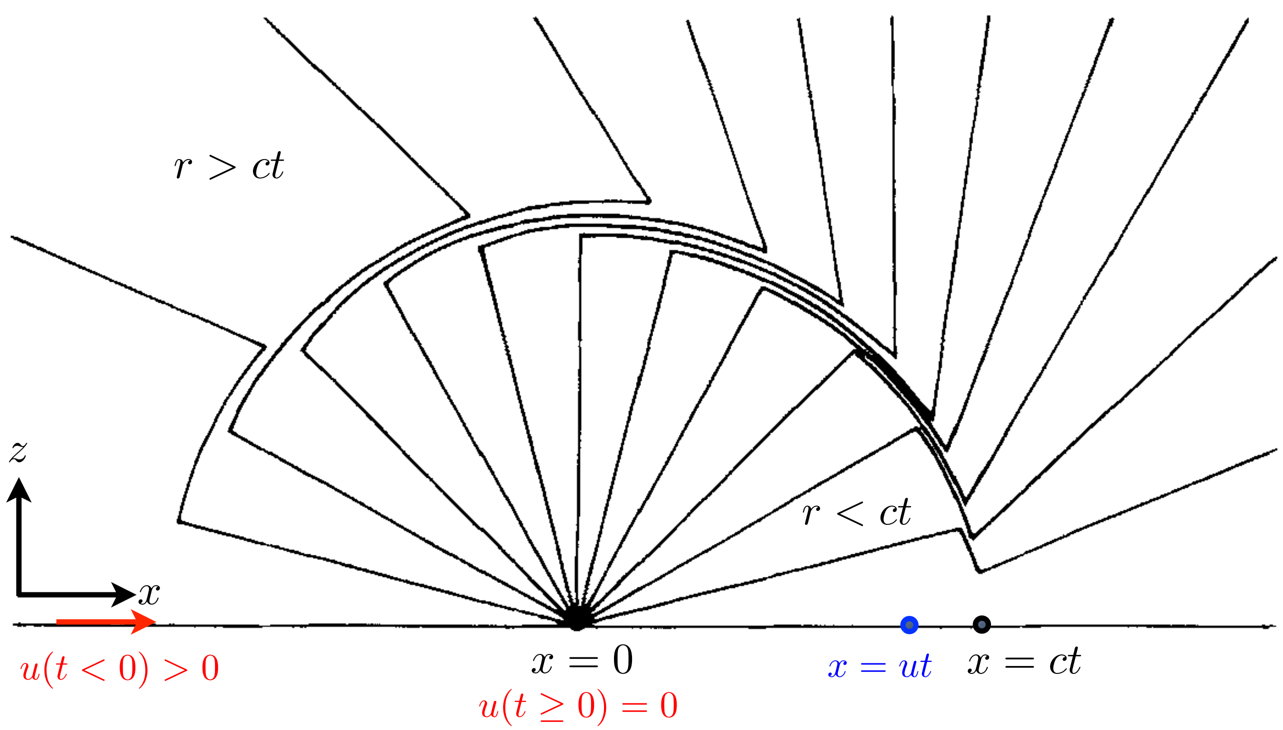
- As $u \ll c$ $(\beta \ll 1)$, the “velocity” field becomes precisely Coulomb’s law.

# “Acceleration” (or “radiation”) Field

- The second term (a) falls off as $1/R$, (b) is proportional to the particle’s acceleration, and (c) is perpendicular to $\mathbf{n}$.

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) = \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$
$$\mathbf{B}_{\rm rad}(\mathbf{r},t) = \mathbf{n}\times\mathbf{E}_{\rm rad}(\mathbf{r},t)$$

- Let’s consider a particle, which originally moved with a constant velocity along the $x$-axis and stopped at $x = 0$ at time $t = 0$. At time $t$, the field outside radius $ct$ is radial and points to the position where the particle would have been if there had been no deceleration, since no information of the deceleration has yet propagated. On the other hand, the field inside radius $ct$ is “informed” and is radially directed to the true position of the particle.




The figure demonstrates how an acceleration can give rise to a transverse field that decreases as $1/R$.

- The electric field in the transition (shell) zone is transverse.
- The radial thickness of the shell corresponds to the deceleration time scale, and will be constant.
- However, the radius of the shell increases as $R$.
- Since the total number of flux lines (in $xy$-plane) is conserved, we have

$$E(\delta x)(2\pi R) = \text{constant} \;\rightarrow\; E \propto \frac{1}{R}$$

# A single moving charge: Radiation Power

- Power per unit frequency per unit solid angle

$$\frac{dW}{d\omega d\Omega} = \frac{R^2 dW}{d\omega dA} = R^2 c|\bar{\mathbf{E}}(\omega)|^2$$

$$= \frac{c}{4\pi^2}\left|\int [R\mathbf{E}(t)]\, e^{i\omega t} dt\right|^2$$

$$= \frac{q^2}{4\pi^2 c}\left|\int \left[\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-3}\right] e^{i\omega t} dt\right|^2$$

Note: the expression in the brackets is evaluated at the retarded time $t' = t - R(t')/c$.

change of variables: $t' = t - R(t')/c, \quad dt = \kappa(t')dt' = (1 - \mathbf{n}(t')\cdot\boldsymbol{\beta}(t'))dt'$

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-2} e^{i\omega(t'+R(t')/c)} dt'\right|^2$$

If $|\mathbf{r}_0| \ll |\mathbf{r}| = r$ , (1) $R(t') = |\mathbf{r} - \mathbf{r}_0(t')| = [(\mathbf{r}-\mathbf{r}_0)\cdot(\mathbf{r}-\mathbf{r}_0)]^{1/2}$

$$= \left[r^2 - 2(\mathbf{r}\cdot\mathbf{r}_0) + r_0^2\right]^{1/2} = r\left[1 - \frac{2(\mathbf{r}\cdot\mathbf{r}_0)}{r^2} + \frac{r_0^2}{r^2}\right]^{1/2}$$

$$\approx r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}_0}{r^2}\right)$$

$$= r - \mathbf{n}\cdot\mathbf{r}_0 \qquad \leftarrow \quad \mathbf{n} \equiv \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|} \approx \frac{\mathbf{r}}{r}$$

(2) $\kappa(t') = 1 - \mathbf{n}(t')\cdot\boldsymbol{\beta}(t') \approx 1 - \mathbf{n}\cdot\boldsymbol{\beta}(t')$ where $\mathbf{n}$ is independent of $t'$.

(3) $e^{i\omega(t'+R(t')/c)} = e^{i\omega r/c} e^{i\omega(t'-\mathbf{n}\cdot r_0(t')/c)}, \quad |e^{i\omega r/c}| = 1 \quad \rightarrow$

$$\therefore \ \frac{dW}{d\omega d\Omega} \approx \frac{q^2}{4\pi^2 c} \left| \int \mathbf{n} \times \{(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\} \kappa^{-2} e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)} dt' \right|^2$$

We can integrate the above equation by parts to obtain an expression without $\dot{\boldsymbol{\beta}}$.

How?

We first need to show that: $\kappa^{-2}\mathbf{n} \times \{(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\} = \frac{d}{dt'}\left[\kappa^{-1}\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})\right]$

With the rule: $\displaystyle\int f' g dt = fg - \int f g' dt$

we obtain

This term vanishes under the assumption of a finite wave train.

$$\begin{aligned}
&\int \mathbf{n} \times \{(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\} \kappa^{-2} e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)} dt' \\
&= \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) \kappa^{-1} e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)} \Big|_{-\infty}^{\infty} - \int \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) \kappa^{-1} \{i\omega(1-\mathbf{n}\cdot\dot{\mathbf{r}}_0(t')/c)\} e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)} dt' \\
&= -i\omega \int \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) \kappa^{-1} (1-\mathbf{n}\cdot\boldsymbol{\beta}) e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)} dt' \\
&= -i\omega \int \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)} dt' \qquad \leftarrow \quad \kappa = 1 - \mathbf{n}\cdot\boldsymbol{\beta}
\end{aligned}$$

$$\boxed{\frac{dW}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c} \left| \int \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) \exp\left[i\omega\left(t' - \frac{\mathbf{n}\cdot\mathbf{r}_0(t')}{c}\right)\right] dt' \right|^2}$$

This formula will be used later.

- Proof of the relation:

$$\kappa^{-2}\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\} = \frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right]$$

note the vector identity: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{C}\cdot\mathbf{A}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$$\frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right] = \kappa^{-2}\left[-\frac{d\kappa}{dt'}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) + \kappa\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right]$$

$$\text{Here, use the relations : } \kappa = 1-\mathbf{n}\cdot\boldsymbol{\beta}, \quad \frac{d\kappa}{dt'} = -\mathbf{n}\cdot\dot{\boldsymbol{\beta}}$$

$$\begin{aligned}
\frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right] &= \kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\} + (1-\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\}\right] \\
&= \kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\} + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}) - (\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\}\right] \\
&= \kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta}) - \boldsymbol{\beta}\} + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}) - (\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}}) - \dot{\boldsymbol{\beta}}\}\right] \\
&= \kappa^{-2}\left[-(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta} + (\mathbf{n}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}} + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right] \\
&= \kappa^{-2}\left[-\mathbf{n}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}) + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right] \\
&= \kappa^{-2}\left[\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]
\end{aligned}$$

# Radiation from Nonrelativistic Particles

- The previous formulae is fully relativistic. However, for the moment, we will discuss nonrelativistic particles:

$$\beta = \frac{u}{c} \ll 1$$

- Order of magnitude comparison of the two fields:

$$E_{\rm rad} \approx \frac{q}{c} \frac{\dot{\beta}}{\kappa^3 R}, \quad E_{\rm vel} \approx \frac{q}{\kappa^3 R^2} \quad \rightarrow \quad \frac{E_{\rm rad}}{E_{\rm vel}} \approx \frac{R\dot{u}}{c^2}$$

If the particle has a characteristic frequency of oscillation $\nu \sim 1/T$, then $\dot{u} \sim u\nu$, and the above equation becomes:

$$\frac{E_{\rm rad}}{E_{\rm vel}} \sim \frac{Ru\nu}{c^2} = \frac{u}{c}\frac{R}{\lambda}$$

**For field points inside the "near zone", $R \lesssim \lambda$, the velocity field is stronger than the radiation field by a factor $c/u = 1/\beta$.**

**For field points sufficiently far in the "far zone", $R \gg \lambda(c/u)$, the radiation field dominates.**

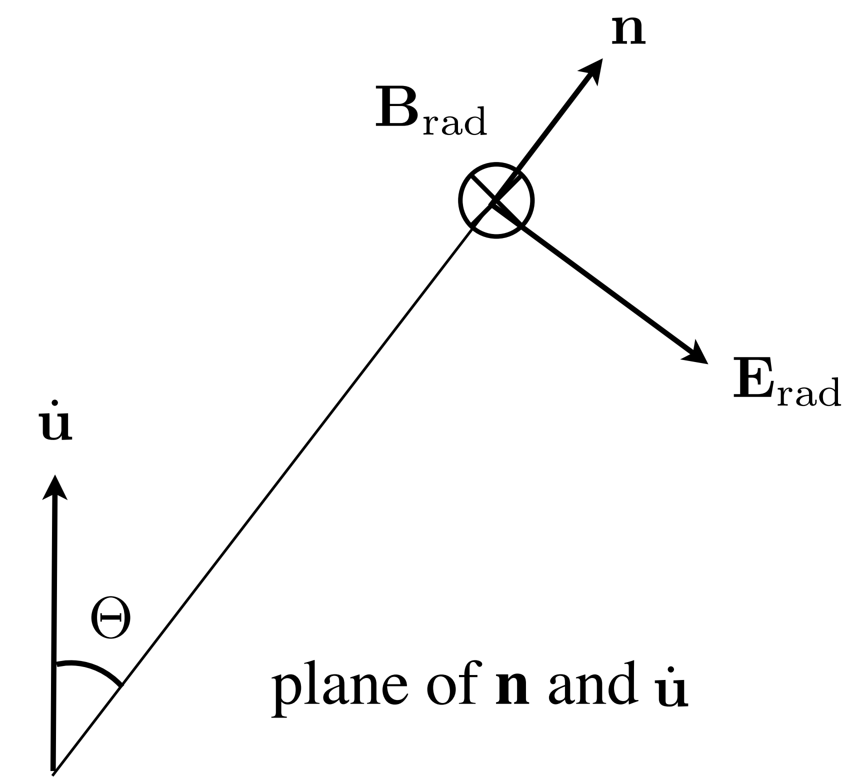
In astronomy, we are only interested in the "far zone".

Therefore, let's consider only the radiation field.

# Larmor's Formula

- When $\beta \ll 1$,

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) = \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$
$$\approx \left[\frac{q}{Rc^2}\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}})\right]$$
$$\mathbf{B}_{\rm rad}(\mathbf{r},t) = \mathbf{n}\times\mathbf{E}_{\rm rad}(\mathbf{r},t)$$




Note: $\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}}) = \mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{u}}) - \dot{\mathbf{u}}$

$$\begin{aligned}\{\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}})\}^2 &= (\mathbf{n}\cdot\dot{\mathbf{u}})^2 + (\dot{\mathbf{u}})^2 - 2(\mathbf{n}\cdot\dot{\mathbf{u}})^2 \\ &= \dot{u}^2\cos^2\Theta + \dot{u}^2 - 2\dot{u}^2\cos^2\Theta \\ &= \dot{u}^2(1-\cos^2\Theta) \\ &= \dot{u}^2\sin^2\Theta\end{aligned}$$

$$\therefore\ |\mathbf{E}_{\rm rad}| = |\mathbf{B}_{\rm rad}| = \frac{q\dot{u}}{Rc^2}\sin\Theta$$

The $E_{\rm rad}$ field is in the plane of $(\mathbf{n},\dot{\mathbf{u}})$.

- The Poynting vector is in direction of $\mathbf{n}$ and has a magnitude.

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}E_{\rm rad}^2\mathbf{n}$$
$$S = \frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^4}\sin^2\Theta \equiv \frac{dW}{dtdA} \quad ({\rm erg\ s^{-1}\ cm^{-2}})$$

- Radiation pattern: energy emitted per unit time into solid angle about **n**:

$$\frac{dW}{dtd\Omega} = R^2\frac{dW}{dtdA} = R^2 S$$
$$= \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta$$

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta}$$

- Total power emitted into all angles:

$$P = \frac{dW}{dt} = \int d\Omega\frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta = \frac{q^2\dot{u}^2}{2c^3}\int_{-1}^{1}(1-\mu^2)d\mu$$

$$\boxed{P = \frac{2q^2\dot{u}^2}{3c^3}}$$ Larmor's Formula

1. The Power emitted is proportional to the square of the charge and the square of the acceleration.
2. Characteristic dipole pattern $\propto \sin^2\Theta$ : no radiation is emitted along the direction of acceleration, and the maximum is emitted perpendicular to acceleration.
3. The instantaneous direction of $\mathbf{E}_{\rm rad}$ is determined by $\dot{\mathbf{u}}$ and **n**. If the particle accelerates along a line, the radiation will be 100% linearly polarized in the plane of $\dot{\mathbf{u}}$ and **n**.

# Dipole Approximation

- Consider many particles with positions $\mathbf{r}_i$, velocities $\mathbf{u}_i$, and charges $q_i (i = 1, 2, \cdots N)$. The radiation field at large distances can be found by adding together the $\mathbf{E}_{\rm rad}$ from each particle.
- However, the radiation field equations refer to conditions at retarded time, and the retarded times will differ for each particle. Therefore, we must keep track of the phase relations between the particles.
- There are situations in which it is possible to ignore this difficulty:

Let $L$ = typical size of the system
$\tau$ = typical time scale for variations within the system

If $\tau \gg L/c$ (light-travel-time), the differences in retarded time across the source are negligible.

Note that $\tau$ can represent the time scale over which significant changes in the radiation field, and this in turn determines typical characteristic frequency of the emitted radiation.

The above condition is equivalent to the condition for the characteristic frequency (or characteristic wavelength) :

$$\nu \approx \frac{1}{\tau} \ll \frac{c}{L} \quad \text{or} \quad \lambda = \frac{c}{\nu} \gg L$$

Therefore, **the differences in retarded times can be ignored when the system size is smaller than the characteristic wavelength.**

We may also characterize $\tau$ as the time a particle takes to change its motion substantially.

Let $\ell$ be a characteristic scale of the particle's orbit and $u$ be a typical velocity, then $\tau \sim \ell/u$ .

The above condition $\tau \gg L/c$ then imply $u/c = \ell/(\tau c) \ll \ell/L$

But since $\ell < L$, **the condition for dipole approximation is simply equivalent to the nonrelativistic condition:**

$$u \ll c$$

- With the above conditions met we can write:

$$\boxed{\mathbf{E}_{\rm rad} = \sum_i \frac{q_i}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}_i)}{R_i}}$$

- Let $R_0$ be the distance from some point in the system to the field point.
- If $R_i = R_0 + \ell_i \approx R_0$ as $R_0 \gg \ell_i$, we have

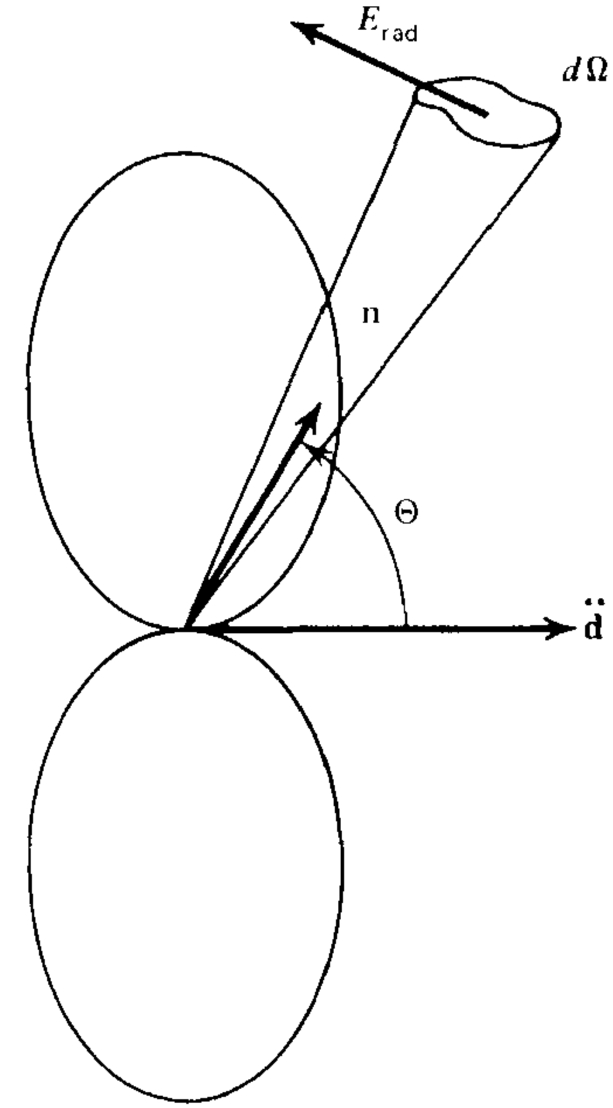
$$\begin{aligned} \mathbf{E}_{\rm rad} &= \frac{1}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \sum_i q_i \dot{\mathbf{u}}_i)}{R_0} \\ &= \frac{\mathbf{n} \times (\mathbf{n} \times \ddot{\mathbf{d}})}{c^2 R_0} \end{aligned}$$

where the electric dipole moment is $$\mathbf{d} \equiv \sum_i q_i \mathbf{r}_i = \int_V \rho(\mathbf{r}', t) d^3\mathbf{r}'$$

Note that the right-hand side of the above equations must still be evaluated at a retarded time, but using any point within the region, say, $R_0$.

- Dipole approximation: $$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\sin^2\Theta, \quad P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}$$

  The instantaneous polarization of E lines in the plane of $\ddot{\mathbf{d}}$ and **n**.




- Power spectrum of radiation in the dipole approximation:

  For simplicity we assume that d always lies in a single direction.

$$E(t) = \ddot{d}(t)\frac{\sin\Theta}{c^2 R_0}, \qquad d(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\bar{d}(\omega)d\omega$$

$$\ddot{d}(t) = -\int_{-\infty}^{\infty} \omega^2 e^{-i\omega t}\bar{d}(\omega)d\omega$$

$$\therefore \;\; \bar{E}(\omega) = -\frac{1}{c^2 R_0}\omega^2\bar{d}(\omega)\sin\Theta$$

$$\frac{dW}{d\omega d\Omega} = R_0^2\frac{dW}{d\omega dA} \quad \rightarrow \quad \begin{aligned}\frac{dW}{d\omega d\Omega} &= \frac{1}{c^3}\omega^4|\bar{d}(\omega)|^2\sin^2\Theta \\ \frac{dW}{d\omega} &= \frac{8\pi\omega^4}{3c^3}|\bar{d}(\omega)|^2\end{aligned}$$

Note the $\omega^4 \propto \lambda^{-4}$ dependence.

the spectrum of the emitted radiation is related directly to the frequencies of oscillation of the dipole moment. However, this property is not true for relativistic particles.

# Multipole Expansion

- Vector potential:

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}'\int dt'\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

- Consider a Fourier analysis of the sources and fields:

$$\mathbf{j}_\omega(\mathbf{r}) = \int \mathbf{j}(\mathbf{r},t)e^{i\omega t}dt$$

$$\mathbf{A}_\omega(\mathbf{r}) = \int \mathbf{A}(\mathbf{r},t)e^{i\omega t}dt = \frac{1}{c}\int d^3\mathbf{r}'\int dt'\int dt\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}e^{i\omega t}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

$$= \frac{1}{c}\int d^3\mathbf{r}'\int dt'\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}e^{i\omega t'}e^{i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$= \frac{1}{c}\int d^3\mathbf{r}'\frac{\mathbf{j}_\omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|} \qquad \leftarrow \qquad k\equiv\omega/c$$

Note this equation relate single Fourier components of source and potential.

- Let’s choose an origin of coordinates inside the source of size $L$. At field points such that $r \ll L$.

$$|\mathbf{r}-\mathbf{r}'| = \left[(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')\right]^{1/2}$$

$$= \left[r^2 - 2(\mathbf{r}\cdot\mathbf{r}') + r'^2\right]^{1/2} = r\left[1 - \frac{2(\mathbf{r}\cdot\mathbf{r}')}{r^2} + \frac{r'^2}{r^2}\right]^{1/2}$$

$$\approx r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)$$

$$= r - \mathbf{n}\cdot\mathbf{r}' \qquad \leftarrow \quad \mathbf{n}\equiv\frac{\mathbf{r}}{r}$$

- The Fourier component becomes:

$$\mathbf{A}_\omega(\mathbf{r}) \approx (e^{ikr}/cr) \int \mathbf{j}_\omega(\mathbf{r}')e^{-ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}'$$

- The factor $\exp(ikr)$ outside the integral expresses the effect of retardation from the source as a whole. The factor $\exp(-ik\mathbf{n}\cdot\mathbf{r}')$ inside the integral expresses the relative retardation of each element of the source.

- In our slow-motion approximation, $kL = 2\pi L/\lambda \ll 1$. Expanding the exponential in the integral:

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{e^{ikr}}{cr} \int \mathbf{j}_\omega(\mathbf{r}') \sum_{n=0}^{\infty} \frac{(-ik\mathbf{n}\cdot\mathbf{r}')^n}{n!} d^3\mathbf{r}'$$

  Dipole approximation results from taking just the first term ($n = 0$):

$$\mathbf{A}_\omega(\mathbf{r})|_{\mathrm{dipole}} = \frac{e^{ikr}}{cr} \int \mathbf{j}_\omega(\mathbf{r}') d^3\mathbf{r}'$$

  Quadrupole term is the second term:

$$\mathbf{A}_\omega(\mathbf{r})|_{\mathrm{quad}} = \frac{-ike^{ikr}}{cr} \int \mathbf{j}_\omega(\mathbf{r}')(\mathbf{n}\cdot\mathbf{r}') d^3\mathbf{r}'$$

# Thomson Scattering ((free) Electron Scattering)

- Recall the dipole formula
$$\frac{dP}{d\Omega} = \frac{dW}{dt d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2\Theta, \quad P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}$$

- Let us consider the **process in which a free charged particle (electron) radiates in response to an incident electromagnetic wave**.

- In non-relativistic case, we may neglect magnetic force.

  magnetic/electric force ratio in Lorentz force: $F_B/F_E \sim (v/c)B/E = v/c \ll 1$

- Consider a monochromatic wave with frequency $\omega_0$ and linearly polarized in direction $\hat{\boldsymbol{\epsilon}}$ :

$$\mathbf{E} = \hat{\boldsymbol{\epsilon}} E_0 \sin\omega_0 t$$

Thus the force on a particle with the charge *e* is

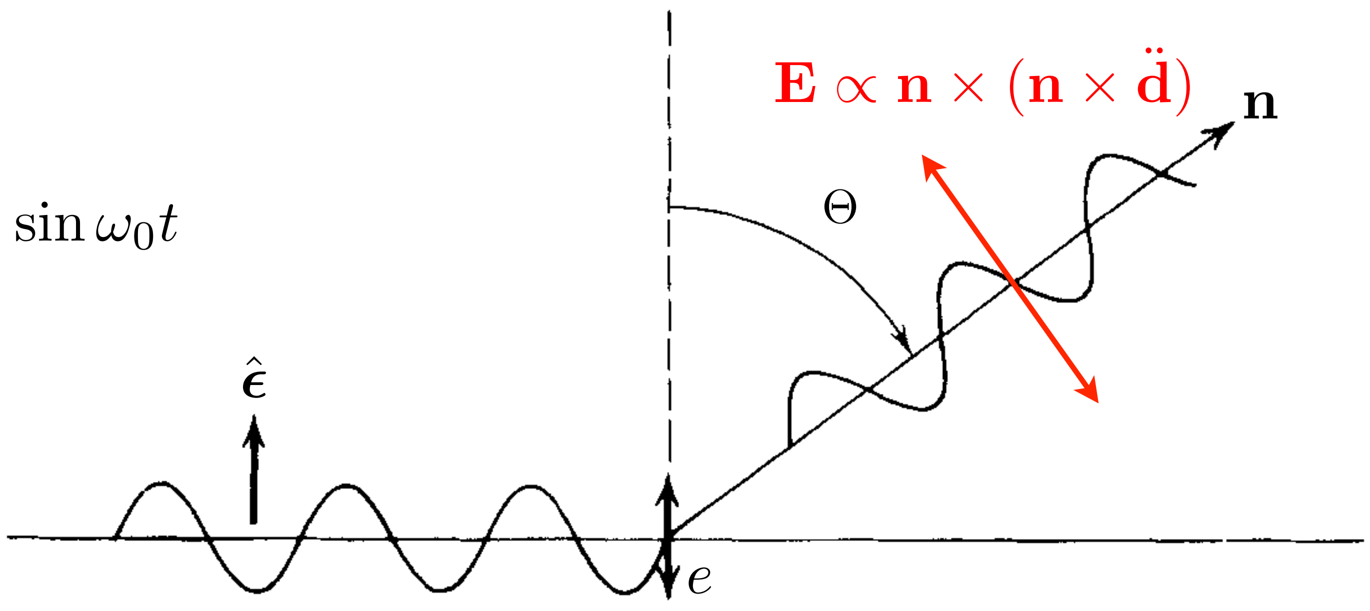
$$\mathbf{F} = e\mathbf{E} = \hat{\boldsymbol{\epsilon}} e E_0 \sin\omega_0 t$$

the acceleration of the electron is

$$\ddot{\mathbf{r}} = \hat{\boldsymbol{\epsilon}} \frac{eE_0}{m} \sin\omega_0 t, \quad \ddot{\mathbf{d}} = e\ddot{\mathbf{r}} = \hat{\boldsymbol{\epsilon}} \frac{e^2 E_0}{m} \sin\omega_0 t$$

the dipole moment is

$$\mathbf{d} = -\hat{\boldsymbol{\epsilon}} \left( \frac{e^2 E_0}{m\omega_0^2} \right) \sin\omega_0 t$$



- We obtain the time-averaged power per solid angle $\left(\langle \sin^2 \omega_0 t \rangle = 1/2\right)$:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{< \ddot{\mathbf{d}}^2 >}{4\pi c^3} \sin^2 \Theta = \frac{e^4 E_0^2}{8\pi m^2 c^3} \sin^2 \Theta, \quad \langle P \rangle = \frac{e^4 E_0^2}{3 m^2 c^3}$$

Note that the time-averaged incident flux is

$$\langle S \rangle = \frac{c}{8\pi} E_0^2$$

The **differential cross section**, $\dfrac{d\sigma}{d\Omega}$, for linearly polarized radiation is obtained by

$$\frac{d\sigma}{d\Omega} \equiv \left\langle \frac{dP}{d\Omega} \right\rangle / \langle S \rangle, \quad \boxed{\therefore \frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \sin^2 \Theta = r_0^2 \sin^2 \Theta, \quad r_0 \equiv \frac{e^2}{mc^2}}$$

where the quantity $r_0$ gives a measure of the "size" of the point charge. (Note electrostatic potential energy $e\phi = e^2/r_0$).

For an electron, the classical electron radius has a value $r_0 = 2.82 \times 10^{-13}$ cm.

The total cross section is found by integrating over solid angle.

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi r_0^2 \int_{-1}^{1} (1 - \mu^2) d\mu = \frac{8\pi}{3} r_0^2$$

For an electron, the scattering process is then called Thomson scattering or electron scattering, and the **Thomson cross section** is

$$\boxed{\sigma_{\rm T} = \frac{8\pi}{3} r_0^2 = 6.652 \times 10^{-25} \ {\rm cm}^2}$$

- Note:

  The total and differential cross sections are frequency independent.

  The scattered radiation is linearly polarized in the plane of the incident polarization vector $\hat{\epsilon}$ and the direction of scattering $\mathbf{n}$.

  $\sigma \propto 1/m^2$ : electron scattering is larger than ions by a factor of $(m_\mathrm{p}/m_e)^2 = (1836)^2 \approx 3.4 \times 10^6$.

  We have implicitly assumed that electron recoil is negligible. This is only valid for nonrelativistic energies. For higher energies, the (quantum-mechanical) Klein-Nishina cross section has to be used.

- What is **the cross section for scattering of unpolarized radiation**?

  An unpolarized beam can be regarded as the independent superposition of two linear-polarized beams with perpendicular axes.

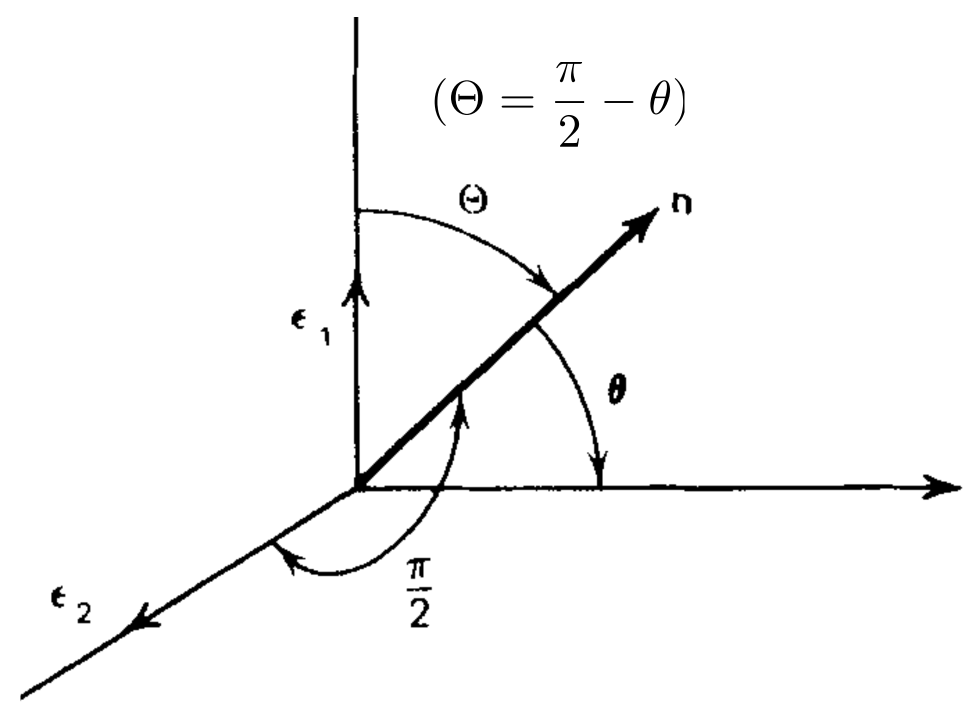
  Let us assume that $\mathbf{n}$ = direction of scattered radiation

  $\mathbf{k}$= direction of incident radiation

  Choose

  the first electric field along $\hat{\epsilon}_1$, which is in the $\mathbf{n} - \mathbf{k}$ plane

  the second one along $\hat{\epsilon}_2$ orthogonal to this plane and to $\mathbf{n}$



- Let $\Theta$ = angle between $\boldsymbol{\epsilon}_1$ and $\mathbf{n}$, and note that angle between $\boldsymbol{\epsilon}_2$ and $\mathbf{n} = \pi/2$.

  $\theta = \pi/2 - \Theta$ = angle between the scattered wave and incident wave

  Then, the differential cross section for unpolarized radiation

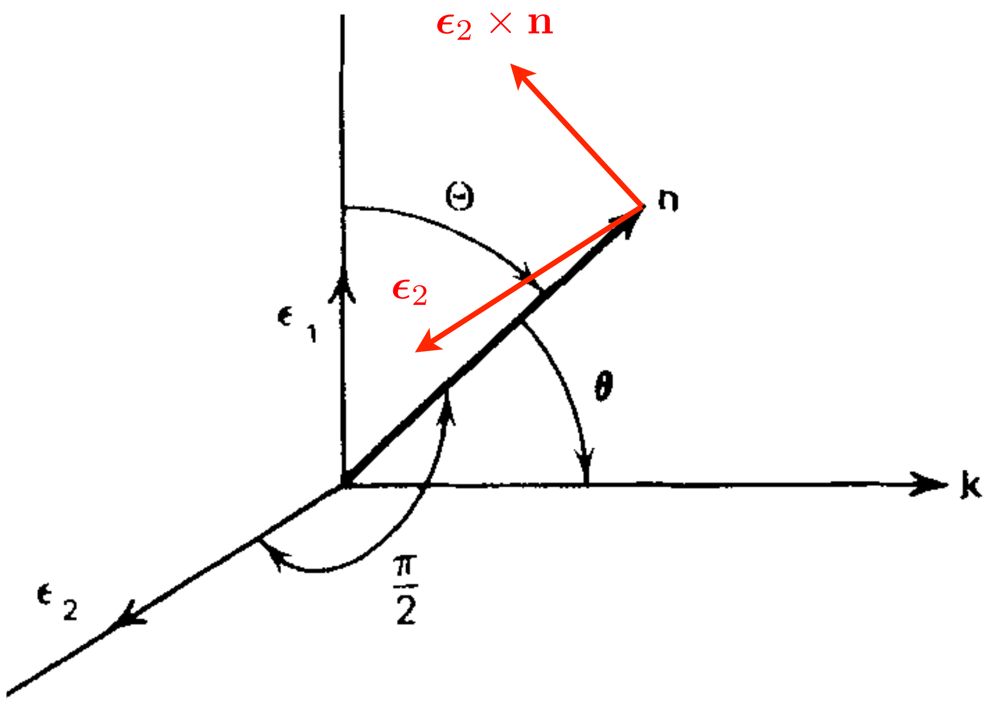
  is the average of the cross sections for scattering of two electric fields.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2}\left[\left(\frac{d\sigma}{d\Omega}\right)_{\epsilon_2} + \left(\frac{d\sigma}{d\Omega}\right)_{\epsilon_1}\right]$$

$$= \frac{1}{2}\left[\left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\text{pol}} + \left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\text{pol}}\right]$$

$$= \frac{1}{2}r_0^2\left(1 + \sin^2\Theta\right)$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2}r_0^2\left(1 + \cos^2\theta\right)}$$




This depends only on the angle between the incident and scattered directions, as it should for unpolarized radiation.

Total cross section:

$$\sigma_{\text{unpol}} = \int \left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} d\Omega = \pi r_0^2 \int_{-1}^{1} (1+\mu^2)\,d\mu$$

$$= \frac{8\pi}{3}r_0^2$$

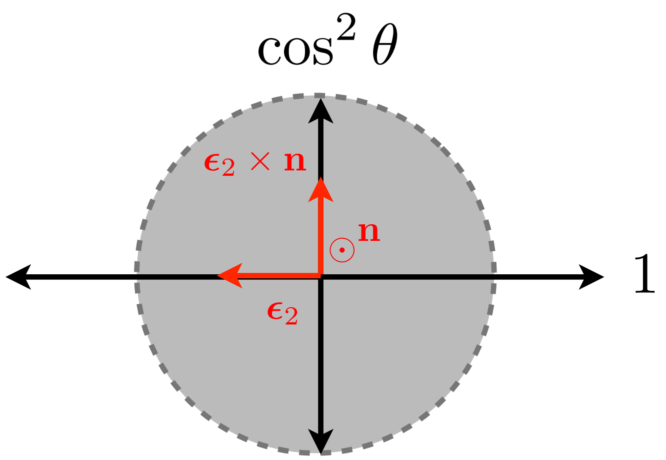
$$= \sigma_{\text{pol}}$$

# Properties of Thomson Scattering

- Forward-backward symmetry: differential cross section is symmetric under $\theta \to -\theta$.
- Total cross section of unpolarized incident radiation = total cross section for polarized incident radiation. This is because the electron at rest has no preferred direction defined.
- **Scattering creates polarization**

The scattered intensity is proportional to $1 + \cos^2\theta$, of which $1$ arises from the incident electric field along $\boldsymbol{\epsilon}_2$ and $\cos^2\theta$ from the incident electric field along $\boldsymbol{\epsilon}_1$.

"$\cos^2\theta$" of the polarization along $\boldsymbol{\epsilon}_2$ will be cancelled out by the independent polarization along $\boldsymbol{\epsilon}_2 \times \mathbf{n}$.




Therefore, the degree of polarization of the scattered wave:

$$\Pi = \frac{1 - \cos^2\theta}{1 + \cos^2\theta}$$

**Electron scattering of a completely unpolarized incident wave produces a scattered wave with some degree of polarization.**

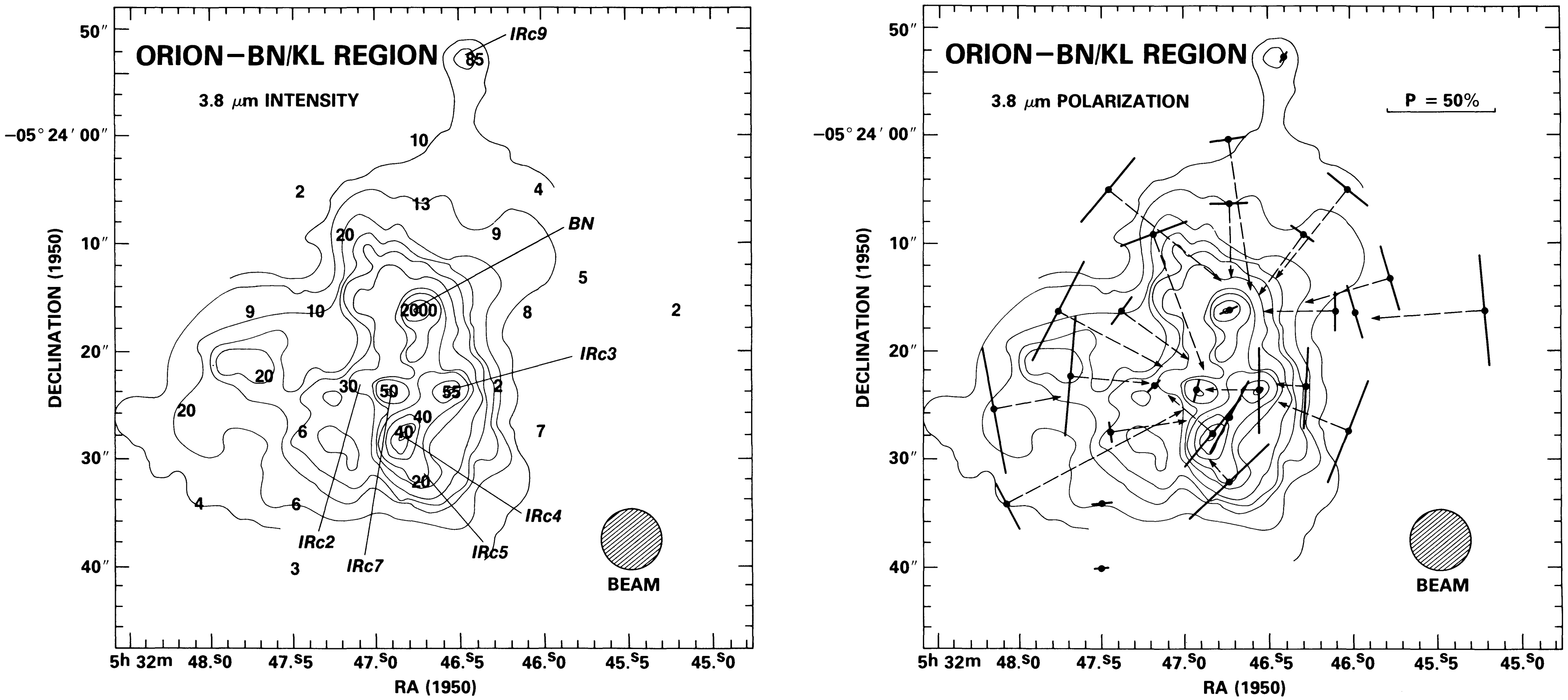
<u>No net polarization along the incident direction</u> $(\theta = 0)$, since, by symmetry, all directions are equivalent.

<u>100% polarization perpendicular to the incident direction</u> $(\theta = \pi/2)$, since the electron's motion is confined to a plane normal to the incident direction.

# Astrophysical Applications of Polarization by Scattering

- Detection of a concentric pattern of polarization vectors in an extended region indicates that the light comes via scattering from a central point source.

Werner et al. (1983, ApJL, 265, L13)




- Left map shows the IR intensity map at 3.8 um of the Becklin-Neugebauer/Kleinmann-Low region of Orion. It is not easy to identify which bright spots correspond to locations of possible protostars.
- However, the polarization map singles out only two positions of intrinsic luminosity: IRc2 (now known to be an intense protostellar wind) and BN (suspected to be a relatively high-mass star)
- All the other bright spots (IRc3 through 7) correspond to IR reflection nebulae.

# Harmonic Oscillator: classical model to the motion of an electron in an atom

- **<u>Lorentz Oscillator Model</u> to describe the interaction between atoms and electric fields**: The electron (with small mass) is bound to the nucleus of the atom (with a much larger mass) by a force that behaves according to Hooke's Law (a spring-like force). An applied electric field would then interact with the charge of the electron, causing "stretching" or "compression" of the spring.
- **The electron's equation of motion**:

$$\boxed{m\ddot{\mathbf{x}} = -k\mathbf{x} + \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{rad}}}$$

$$k = m\omega_0^2 \text{ , where } k = \text{spring constant}$$

$$\omega_0 = \text{natural (fundamental or resonant) freqency}$$

$\mathbf{F}_{\text{ext}} =$ external force, driving force, or external electric field

$\mathbf{F}_{\text{rad}} =$ radiation reaction force (radiation damping)

the damping of a charge's motion which arises because of the emission of radiation)

# Free Oscillator: radiation damping

- **Undriven Harmonically Bound Particles** (free oscillator)

Since an oscillating electron represents a continuously accelerating charge, the electron will radiate energy. The energy radiated away must come from the particle's own energy (energy conservation). In other words, **there must be a force acting on a particle by virtue of the radiation it produces. This is called the radiation reaction force.**

Let's derive the formula for the radiation reaction force from the fact that the energy radiated must be compensated for by the work done against the radiation reaction force.

(1) On one hand, the radiative loss rate of energy, averaged over one cycle of the oscillating dipole, can be represented by the radiative reaction force:

$$\frac{dW}{dt} = \langle \mathbf{F}_{\rm rad} \cdot \dot{\mathbf{x}} \rangle$$

(2) On the other hand, from the Larmor's formula for a dipole, the radiative loss will be:

$$\frac{dW}{dt} = -\frac{2e^2 \left\langle |\ddot{\mathbf{x}}|^2 \right\rangle}{3c^3}$$

# Free Oscillator: Abraham-Lorentz formula

$$\therefore \quad \langle \mathbf{F}_{\rm rad} \cdot \dot{\mathbf{x}} \rangle = -\frac{2e^2 \left\langle |\ddot{\mathbf{x}}|^2 \right\rangle}{3c^3}$$

Here, $$\left\langle |\ddot{\mathbf{x}}|^2 \right\rangle \equiv \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}} dt$$
$$= \frac{1}{\tau} \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} |_{-\tau/2}^{\tau/2} - \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt$$

We assume that the initial and final states are the same: $\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}(-\tau/2) = \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}(\tau/2)$

Then,

$$\left\langle |\ddot{\mathbf{x}}|^2 \right\rangle = -\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \dddot{\mathbf{x}} \cdot \ddot{\mathbf{x}} dt = -\left\langle \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \right\rangle \quad \rightarrow \quad \langle \mathbf{F}_{\rm rad} \cdot \dot{\mathbf{x}} \rangle = \frac{2e^2 \left\langle \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \right\rangle}{3c^3}$$

Therefore, we can obtain $\boxed{\mathbf{F}_{\rm rad} = \frac{2e^2 \dddot{\mathbf{x}}}{3c^3} \quad : \text{ Abraham-Lorentz formula}}$

- **Abraham-Lorentz formula**:

$$\boxed{\mathbf{F}_{\mathrm{rad}} = \frac{2e^2 \dddot{\mathbf{x}}}{3c^3}}$$

This formula depends on the derivative of acceleration. This increases the degree of the equation of motion of a particle and can lead to some nonphysical behavior if not used properly and consistently.

For a simple harmonic oscillator with a frequency $\omega_0$, we can avoid the difficulty by using

$$\dddot{\mathbf{x}} = -\omega_0^2 \dot{\mathbf{x}}$$

***This is a good assumption as long as the energy is to be radiated on a time scale that is long compared to the period of oscillation.*** In this regime, radiation reaction may be considered as a *perturbation* on the particle's motion. We then rewrite the radiation reaction force as

$$\mathbf{F}_{\mathrm{rad}} = -\frac{2e^2\omega_0^2}{3c^3}\dot{\mathbf{x}} = -m\gamma\dot{\mathbf{x}}, \qquad \gamma \equiv \frac{2e^2\omega_0^2}{3mc^3} \quad : \quad \text{damping constant}$$

Note $\gamma = A_{21}$

Condition for this approximation:

$T$ = the time interval over which the kinetic energy of the particle is changed substantially by the emission of radiation:

$$T \sim \frac{mv^2}{dW/dt} \sim \frac{3mc^3}{2e^2}\left(\frac{v}{a}\right)^2$$

$t_p$ = the typical orbital time scale for the particle: $t_p \sim \frac{v}{a}$ or $t_p = \frac{2\pi}{\omega_0}$

Then, the condition is

$$\left(\text{electron radius, } r_e = \frac{e^2}{mc^2}\right)$$

$$\frac{T}{t_p} \gg 1 \;\rightarrow\; \frac{3mc^3}{2e^2}t_p = \frac{t_p}{\tau_c} \gg 1 \;\rightarrow\; t_p \gg \tau_c \equiv \frac{2}{3}\frac{e^2}{mc^3} = \frac{2}{3}\frac{r_e}{c}(\sim 10^{-23}\ \mathrm{s})$$

where $\tau_c$ is the time for radiation to cross a distance comparable to the classical electron radius.

In terms of frequency of the oscillator, this condition is equivalent to:

$$\frac{2\pi}{\tau_c} = 3\pi\frac{c}{r_e} \equiv \omega_c \gg \omega_0$$

In terms of wavelength of the oscillator,

$$\lambda_0 = \frac{2\pi c}{\omega_0} \gg \lambda_c \equiv \frac{2\pi c}{\omega_c} = \frac{2}{3}r_e \;(\sim\ 2\times 10^{-13}\ \mathrm{cm} = 2\times 10^{-5}\mathrm{\AA})$$

Therefore, **in most cases, the approximation is valid.**

At this limit:

$$\frac{\gamma}{\omega_0} = \frac{2e^2}{3mc^2}\frac{\omega_0}{c}$$

$$= \frac{2}{3}\frac{r_e}{\lambda_0}2\pi$$

$$\therefore \frac{\gamma}{\omega_0} \ll 1 \text{ for } \lambda_0 \gg r_e = 2.82 \times 10^{-13} \text{ cm}$$

- Equation of motion of the electron in a Lorentz atom:

$$\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x} = 0$$

This equation may be solved by assuming that $x(t) \propto e^{\alpha t}$.

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \;\rightarrow\; \alpha = -(\gamma/2) \pm \sqrt{(\gamma/2)^2 - \omega_0^2}$$
$$= -\gamma/2 \pm i\omega_0 + \mathcal{O}(\gamma^2/\omega_0^2)$$

Assuming initial conditions

$$x(0) = x_0,\; \dot{x}(0) = 0 \text{ at } t = 0$$

we have

$$x(t) = \frac{1}{2}x_0\left[e^{-(\gamma/2 - i\omega_0)t} + e^{-(\gamma/2 + i\omega_0)}\right] = x_0 e^{-\gamma/2}\cos\omega_0 t \longrightarrow \text{Damping oscillator}$$

- Power spectrum:

$$\bar{x}(\omega) = \frac{1}{2\pi}\int_0^\infty x(t)e^{i\omega t}dt = \frac{x_0}{4\pi}\left[\frac{1}{\gamma/2 - i(\omega + \omega_0)} + \frac{1}{\gamma/2 - i(\omega - \omega_0)}\right]$$

This becomes large in the vicinity of $\omega = \omega_0$ and $\omega = -\omega_0$.

We are ultimately interested only in positive frequencies, and only in regions in which the values become large. Therefore, we obtain

$$\bar{x}(\omega) \approx \frac{x_0}{4\pi}\frac{1}{\gamma/2 - i(\omega - \omega_0)}, \quad |\bar{x}(\omega)|^2 = \left(\frac{x_0}{4\pi}\right)^2\frac{1}{(\omega - \omega_0)^2 + (\gamma/2)^2}$$

# Emission Line profile

Recall
$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} e^2 |\bar{x}(\omega)|^2$$

Energy radiated per unit frequency:

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} \frac{e^2 x_0^2}{(4\pi)^2} \frac{1}{(\omega-\omega_0)^2 + (\gamma/2)^2} = \frac{1}{2} m \left(\frac{\omega^4}{\omega_0^2}\right) x_0^2 \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$
$$\approx \frac{1}{2} m\omega_0^2 x_0^2 \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

For a harmonic oscillator, note that the equation of motion is $\mathbf{F} = -k\mathbf{x} = -m\omega_0^2\mathbf{x}$ , spring constant is $k = m\omega_0^2$, and the potential energy (energy stored in spring) is $(1/2)kx_0^2$.

From
$$\int_{-\infty}^{\infty} \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2} d\omega = \frac{1}{\pi} \tan^{-1}\{2(\omega-\omega_0)/\gamma\}|_{-\infty}^{\infty} = 1$$

Total emitted energy = initial potential energy of the oscillator:

$$W = \int_0^{\infty} \frac{dW}{d\omega} d\omega = \frac{1}{2} k\omega_0^2$$

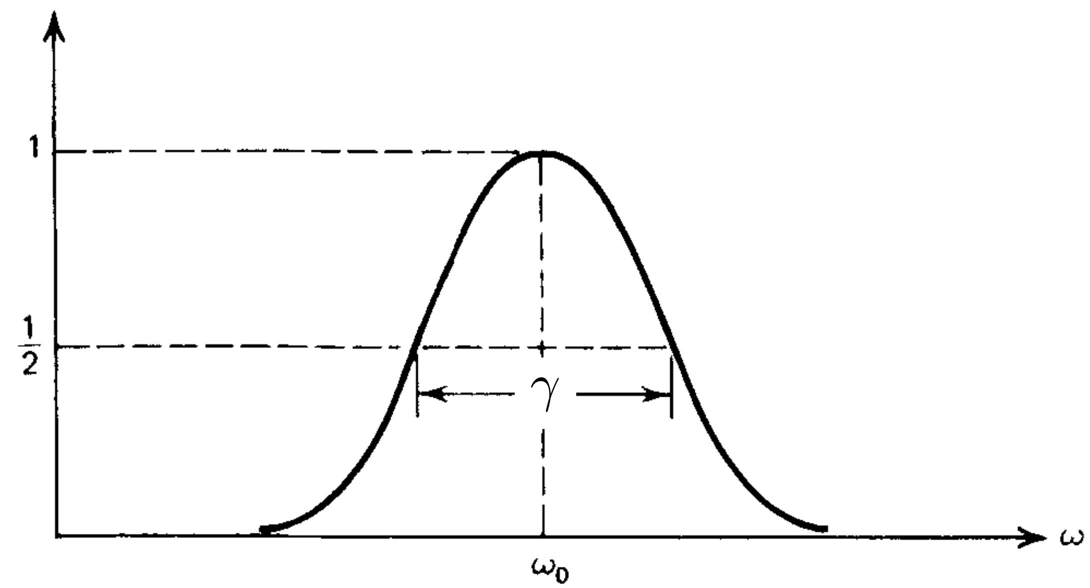
Profile of the emitted spectrum:
$$\boxed{\phi(\omega) = \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}}$$
**Lorentz (natural) profile**

Damping constant is the full width at half maximum (FWHM).

$$\phi(\omega) = \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

$$\phi(\nu) = \frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2}$$

Note $\phi(\omega)d\omega = \phi(\nu)d\nu$




The line width $\Delta\omega = \gamma$ is a universal constant when expressed in terms of wavelength:

$$\lambda = \frac{2\pi c}{\omega}$$

$$\Delta\lambda = 2\pi c\frac{\Delta\omega}{\omega^2} = 2\pi c\frac{2}{3}\frac{r_e}{c} \longleftarrow \left(\Delta\omega = \gamma = \frac{2}{3}r_e\frac{\omega_0^2}{c}\right)$$

$$= \frac{4}{3}\pi r_e$$

$$= 1.2\times 10^{-4}\,\text{Å}$$

# Driven Oscillator (scattering)

- **Driven Harmonically Bound Particles** (forced oscillators)

Electron's equation of motion (electric charge = -e):

$$\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x} = -\frac{e\mathbf{E}_0}{m}e^{i\omega t}$$

Rybicki & Lightman use the following equation.

$$\ddot{\mathbf{x}} - (\gamma/\omega_0^2)\dddot{\mathbf{x}} + \omega_0^2\mathbf{x} = -\frac{e\mathbf{E}_0}{m}e^{i\omega t}$$

Steady-state solution of this equation:

$$\mathbf{x} = \mathbf{x}_0 e^{i\omega t} \equiv |\mathbf{x}_0|e^{i(\omega t+\delta)} \quad \rightarrow \quad (-\omega^2 + i\omega\gamma + \omega_0^2)\mathbf{x}_0 e^{i\omega t} = -\frac{e\mathbf{E}_0}{m}e^{i\omega t}$$

$$\mathbf{x}_0 = \frac{(e/m)\mathbf{E}_0}{(\omega^2 - \omega_0^2) - i\omega\gamma}$$

$$\mathbf{x}_0 = |\mathbf{x}_0|e^{i\delta} \propto (\omega^2 - \omega_0^2) + i\omega\gamma \quad \rightarrow \quad \delta = \tan^{-1}\left(\frac{\omega\gamma}{\omega^2 - \omega_0^2}\right)$$

The response is slightly out of phase with respect to the imposed field.

For $\omega > \omega_0$, the particle "leads" the driving force and for $\omega < \omega_0$ it "lags."

Time-averaged total power radiated:

$$P = \left\langle \frac{dW}{dt} \right\rangle = \frac{2e^2\left\langle |\ddot{\mathbf{x}}|^2 \right\rangle}{3c^3} = \frac{e^2\omega^4|\mathbf{x}_0|^2}{3c^3}$$

$$= \frac{e^4E_0^2}{3m^2c^3}\frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}$$

- Scattering cross section:

$$\sigma_{\rm sca} \equiv \frac{\langle P\rangle}{\langle S\rangle}, \quad \langle S\rangle = \frac{c}{8\pi}E_0^2 \quad \longrightarrow \quad \sigma_{\rm sca}(\omega) = \frac{8\pi e^4}{3m^2c^4}\frac{\omega^4}{(\omega^2-\omega_0^2)^2+(\omega\gamma)^2}$$

$$= \sigma_T \frac{\omega^4}{(\omega^2-\omega_0^2)^2+(\omega\gamma)^2}$$

- Limiting Cases of Interest

(a) $\omega \gg \omega_0$ (Thomson scattering by free electron)

$$\boxed{\sigma_{\rm sca} = \sigma_T = \frac{8\pi}{3}r_e^2}$$

At high incident energies, the binding becomes negligible.

(b) $\omega \ll \omega_0$ (Rayleigh scattering by bound electron)

$$\boxed{\sigma_{\rm sca} = \sigma_T\left(\frac{\omega}{\omega_0}\right)^4}$$

The electric filed appears nearly static and produces a nearly static force.

Blue color of the sky at sunrise:

Red color of the sun at sunset: when the path through the atmosphere is longer, the blue and green components are removed almost completely leaving the longer wavelength orange and red.

(c) $\omega \approx \omega_0$ (Resonance scattering of line radiation)

$$\sigma_{\rm sca}(\omega) \approx \sigma_T \frac{\omega_0^4}{(\omega-\omega_0)^2(2\omega_0)^2+(\omega_0\gamma)^2}$$
$$= \sigma_T \frac{\omega_0^2/4}{(\omega-\omega_0)^2+(\gamma/2)^2}$$

Note $\sigma_{\rm scat}(\omega) = \sigma_\nu(\nu)$

$$\sigma_T \frac{\omega_0^2}{4} = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \times \frac{1}{4} \times \left(\gamma \frac{3}{2}\frac{mc^3}{e^2\omega_0^2}\right) = 2\pi^2 \frac{e^2}{mc}(\gamma/2\pi) \longrightarrow$$

$$\boxed{\begin{aligned} \sigma_{\rm sca}(\omega) &= \frac{2\pi^2 e^2}{mc}\frac{\gamma/2\pi}{(\omega-\omega_0)^2+(\gamma/2)^2} \\ \sigma_{\rm sca}(\nu) &= \frac{\pi e^2}{mc}\frac{\gamma/4\pi^2}{(\nu-\nu_0)^2+(\gamma/4\pi)^2} \end{aligned}}$$

In the neighborhood of the resonance, the shape of the scattering cross section is the same as the emission line profile from the free oscillator.

Total scattering cross section: $$\int_0^\infty \sigma(\omega)d\omega = \frac{2\pi^2 e^2}{mc}, \quad \int_0^\infty \sigma(\nu)d\nu = \frac{\pi e^2}{mc}$$

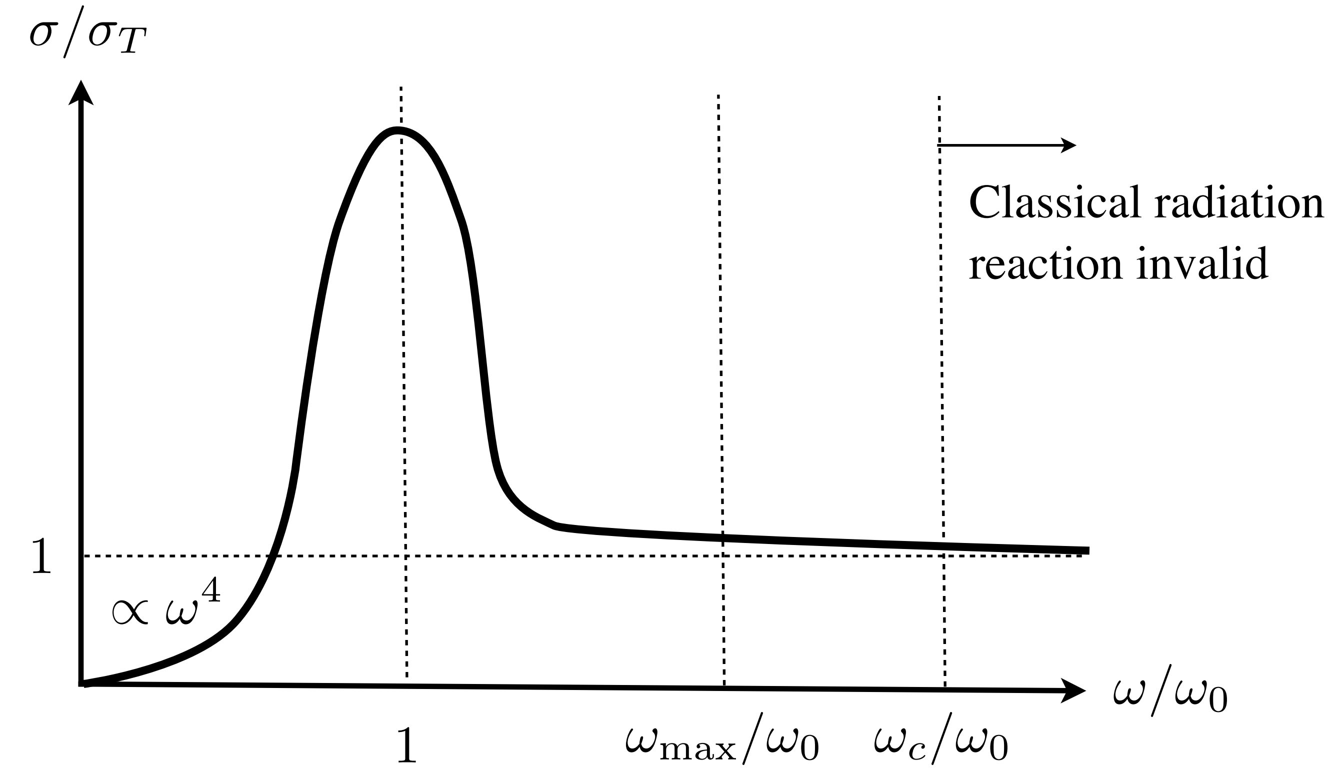
In evaluating this integral, we have apparently neglected a divergence, since the cross section approaches $\sigma_T$ for large $\omega$ .

However, note that the approximate formula for radiation reaction is only valid for $\omega_0 \ll \omega_c$ . Therefore, we must cut off the integral at a $\omega_{\rm max}$ such that $\omega_0 \ll \omega_{\rm max} \ll \omega_c$ .

We also note that the contribution to the integral from the constant Thomson limit is less than

$$\int_0^{\omega_{\max}} \sigma_T d\omega = \sigma_T \omega_{\max} \ll \sigma_T \omega_c = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \times 3\pi\left(\frac{mc^3}{e^2}\right) = \frac{8\pi^2 e^2}{mc} \approx \int_0^{\infty} \sigma_{\mathrm{sca}}(\omega) d\omega$$

The contribution is therefore negligible.




In the quantum theory of spectral lines, we obtain similar formulas, which are conveniently stated in terms of the classical results as

$$\boxed{\int_0^{\infty} \sigma(\nu) d\nu = \frac{\pi e^2}{mc} f_{nn'}}$$

where $f_{nn'}$ is called the ***oscillator strength*** or *f-value* for the transition between states *n* and *n'*.

# Resonance Lines

Draine, Physics of the interstellar and intergalactic medium

**Table 9.4** Selected Resonance Lines[a] with $\lambda < 3000$ Å

| | Configurations | $\ell$ | $u$ | $E_\ell/hc(\mathrm{cm}^{-1})$ | $\lambda_{\mathrm{vac}}(\text{Å})$ | $f_{\ell u}$ |
|---|---|---|---|---|---|---|
| C IV | $1s^22s - 1s^22p$ | $^2\mathrm{S}_{1/2}$ | $^2\mathrm{P}^o_{1/2}$ | 0 | 1550.772 | 0.0962 |
| | | $^2\mathrm{S}_{1/2}$ | $^2\mathrm{P}^o_{3/2}$ | 0 | 1548.202 | 0.190 |
| N V | $1s^22s - 1s^22p$ | $^2\mathrm{S}_{1/2}$ | $^2\mathrm{P}^o_{1/2}$ | 0 | 1242.804 | 0.0780 |
| | | $^2\mathrm{S}_{1/2}$ | $^2\mathrm{P}^o_{3/2}$ | 0 | 1242.821 | 0.156 |
| O VI | $1s^22s - 1s^22p$ | $^2\mathrm{S}_{1/2}$ | $^2\mathrm{P}^o_{1/2}$ | 0 | 1037.613 | 0.066 |
| | | $^2\mathrm{S}_{1/2}$ | $^2\mathrm{P}^o_{3/2}$ | 0 | 1037.921 | 0.133 |
| C III | $2s^2 - 2s2p$ | $^1\mathrm{S}_0$ | $^1\mathrm{P}^o_1$ | 0 | 977.02 | 0.7586 |
| C II | $2s^22p - 2s2p^2$ | $^2\mathrm{P}^o_{1/2}$ | $^2\mathrm{D}^o_{3/2}$ | 0 | 1334.532 | 0.127 |
| | | $^2\mathrm{P}^o_{3/2}$ | $^2\mathrm{D}^o_{5/2}$ | 63.42 | 1335.708 | 0.114 |
| N III | $2s^22p - 2s2p^2$ | $^2\mathrm{P}^o_{1/2}$ | $^2\mathrm{D}^o_{3/2}$ | 0 | 989.790 | 0.123 |
| | | $^2\mathrm{P}^o_{3/2}$ | $^2\mathrm{D}^o_{5/2}$ | 174.4 | 991.577 | 0.110 |
| C I | $2s^22p^2 - 2s^22p3s$ | $^3\mathrm{P}_0$ | $^3\mathrm{P}^o_1$ | 0 | 1656.928 | 0.140 |
| | | $^3\mathrm{P}_1$ | $^3\mathrm{P}^o_2$ | 16.40 | 1656.267 | 0.0588 |
| | | $^3\mathrm{P}_2$ | $^3\mathrm{P}^o_2$ | 43.40 | 1657.008 | 0.104 |
| N II | $2s^22p^2 - 2s2p^3$ | $^3\mathrm{P}_0$ | $^3\mathrm{D}^o_1$ | 0 | 1083.990 | 0.115 |
| | | $^3\mathrm{P}_1$ | $^3\mathrm{D}^o_2$ | 48.7 | 1084.580 | 0.0861 |
| | | $^3\mathrm{P}_2$ | $^3\mathrm{D}^o_3$ | 130.8 | 1085.701 | 0.0957 |
| N I | $2s^22p^3 - 2s^22p^23s$ | $^4\mathrm{S}^o_{3/2}$ | $^4\mathrm{P}_{5/2}$ | 0 | 1199.550 | 0.130 |
| | | $^4\mathrm{S}^o_{3/2}$ | $^4\mathrm{P}_{3/2}$ | 0 | 1200.223 | 0.0862 |
| O I | $2s^22p^4 - 2s^22p^33s$ | $^3\mathrm{P}_2$ | $^3\mathrm{S}^o_1$ | 0 | 1302.168 | 0.0520 |
| | | $^3\mathrm{P}_1$ | $^3\mathrm{S}^o_1$ | 158.265 | 1304.858 | 0.0518 |
| | | $^3\mathrm{P}_0$ | $^3\mathrm{S}^o_1$ | 226.977 | 1306.029 | 0.0519 |
| Mg II | $2p^63s - 2p^63p$ | $^2\mathrm{S}_{1/2}$ | $^2\mathrm{P}^o_{1/2}$ | 0 | 2803.531 | 0.303 |
| | | $^2\mathrm{S}_{1/2}$ | $^2\mathrm{P}^o_{3/2}$ | 0 | 2796.352 | 0.608 |
| Al III | $2p^63s - 2p^63p$ | $^2\mathrm{S}_{1/2}$ | $^2\mathrm{P}^o_{1/2}$ | 0 | 1862.790 | 0.277 |
| | | $^2\mathrm{S}_{1/2}$ | $^2\mathrm{P}^o_{3/2}$ | 0 | 1854.716 | 0.557 |

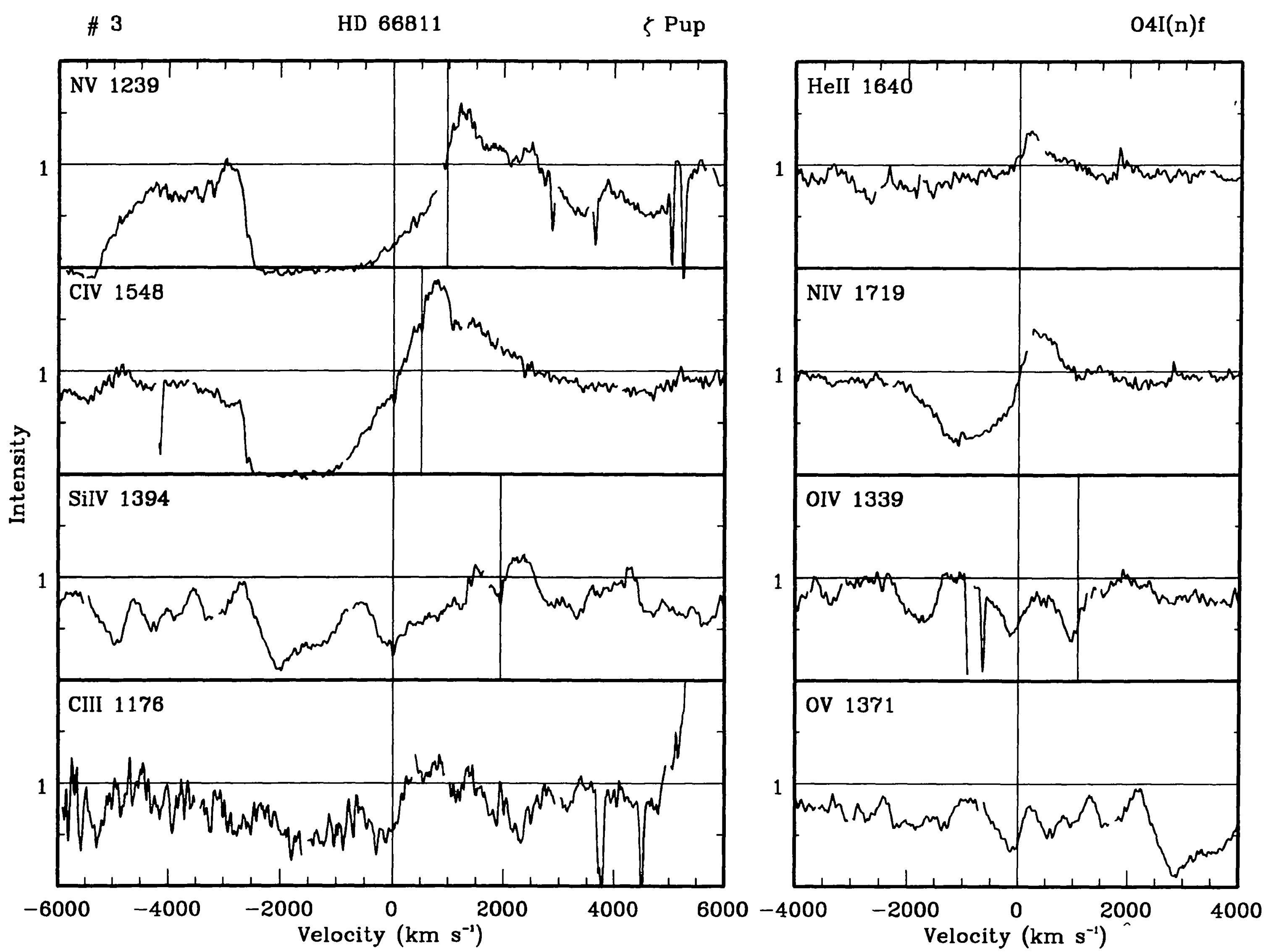
**Table 9.4** contd.

| | Configurations | $\ell$ | $u$ | $E_\ell/hc\,(\mathrm{cm}^{-1})$ | $\lambda_{\mathrm{vac}}(\text{Å})$ | $f_{\ell u}$ |
|---|---|---|---|---|---|---|
| Mg I | $2p^63s^2 - 2p^63s3p$ | $^1\mathrm{S}_0$ | $^1\mathrm{P}^o_1$ | 0 | 2852.964 | 1.80 |
| Al II | $2p^63s^2 - 2p^63s3p$ | $^1\mathrm{S}_0$ | $^1\mathrm{P}^o_1$ | 0 | 1670.787 | 1.83 |
| Si III | $2p^63s^2 - 2p^63s3p$ | $^1\mathrm{S}_0$ | $^1\mathrm{P}^o_1$ | 0 | 1206.51 | 1.67 |
| P IV | $2p^63s^2 - 2p^63s3p$ | $^1\mathrm{S}_0$ | $^1\mathrm{P}^o_1$ | 0 | 950.655 | 1.60 |
| Si II | $3s^23p - 3s^24s$ | $^2\mathrm{P}^o_{1/2}$ | $^2\mathrm{S}_{1/2}$ | 0 | 1526.72 | 0.133 |
| | | $^2\mathrm{P}^o_{3/2}$ | $^2\mathrm{S}_{1/2}$ | 287.24 | 1533.45 | 0.133 |
| P III | $3s^23p - 3s3p^2$ | $^2\mathrm{P}^o_{1/2}$ | $^2\mathrm{D}_{3/2}$ | 0 | 1334.808 | 0.029 |
| | | $^2\mathrm{P}^o_{3/2}$ | $^2\mathrm{D}_{5/2}$ | 559.14 | 1344.327 | 0.026 |
| Si I | $3s^23p^2 - 3s^23p4s$ | $^3\mathrm{P}_0$ | $^3\mathrm{P}^o_1$ | 0 | 2515.08 | 0.17 |
| | | $^3\mathrm{P}_1$ | $^3\mathrm{P}^o_2$ | 77.115 | 2507.652 | 0.0732 |
| | | $^3\mathrm{P}_2$ | $^3\mathrm{P}^o_2$ | 223.157 | 2516.870 | 0.115 |
| P II | $3s^23p^2 - 3s3p^3$ | $^3\mathrm{P}_0$ | $^3\mathrm{P}^o_1$ | 0 | 1301.87 | 0.038 |
| | | $^3\mathrm{P}_1$ | $^3\mathrm{P}^o_2$ | 164.9 | 1305.48 | 0.016 |
| | | $^3\mathrm{P}_2$ | $^3\mathrm{P}^o_2$ | 469.12 | 1310.70 | 0.115 |
| S III | $3s^23p^2 - 3s3p^3$ | $^3\mathrm{P}_0$ | $^3\mathrm{D}^o_1$ | 0 | 1190.206 | 0.61 |
| | | $^3\mathrm{P}_1$ | $^3\mathrm{D}^o_2$ | 298.69 | 1194.061 | 0.46 |
| | | $^3\mathrm{P}_2$ | $^3\mathrm{D}^o_3$ | 833.08 | 1200.07 | 0.51 |
| Cl IV | $3s^23p^2 - 3s3p^3$ | $^3\mathrm{P}_0$ | $^3\mathrm{D}^o_1$ | 0 | 973.21 | 0.55 |
| | | $^3\mathrm{P}_1$ | $^3\mathrm{D}^o_2$ | 492.0 | 977.56 | 0.41 |
| | | $^3\mathrm{P}_2$ | $^3\mathrm{D}^o_3$ | 1341.9 | 984.95 | 0.47 |
| P I | $3s^23p^3 - 3s^23p^24s$ | $^4\mathrm{S}^o_{3/2}$ | $^4\mathrm{P}_{5/2}$ | 0 | 1774.951 | 0.154 |
| S II | $3s^23p^3 - 3s^23p^24s$ | $^4\mathrm{S}^o_{3/2}$ | $^4\mathrm{P}_{5/2}$ | 0 | 1259.518 | 0.12 |
| Cl III | $3s^23p^3 - 3s^23p^24s$ | $^4\mathrm{S}^o_{3/2}$ | $^4\mathrm{P}_{5/2}$ | 0 | 1015.019 | 0.58 |
| S I | $3s^23p^4 - 3s^23p^34s$ | $^3\mathrm{P}_2$ | $^3\mathrm{S}^o_1$ | 0 | 1807.311 | 0.11 |
| | | $^3\mathrm{P}_1$ | $^3\mathrm{S}^o_1$ | 396.055 | 1820.343 | 0.11 |
| | | $^3\mathrm{P}_0$ | $^3\mathrm{S}^o_1$ | 573.640 | 1826.245 | 0.11 |
| Cl II | $3s^23p^4 - 3s3p^5$ | $^3\mathrm{P}_2$ | $^3\mathrm{P}^o_2$ | 0 | 1071.036 | 0.014 |
| | | $^3\mathrm{P}_1$ | $^3\mathrm{P}^o_2$ | 696.00 | 1079.080 | 0.00793 |
| | | $^3\mathrm{P}_0$ | $^3\mathrm{P}^o_1$ | 996.47 | 1075.230 | 0.019 |
| Cl I | $3s^23p^5 - 3s^23p^44s$ | $^2\mathrm{P}^o_{3/2}$ | $^2\mathrm{P}_{3/2}$ | 0 | 1347.240 | 0.114 |
| | | $^2\mathrm{P}^o_{1/2}$ | $^2\mathrm{P}_{3/2}$ | 882.352 | 1351.657 | 0.0885 |
| Ar II | $3s^23p^5 - 3s3p^6$ | $^2\mathrm{P}^o_{3/2}$ | $^2\mathrm{S}_{1/2}$ | 0 | 919.781 | 0.0089 |
| | | $^2\mathrm{P}^o_{1/2}$ | $^2\mathrm{S}_{1/2}$ | 1431.583 | 932.054 | 0.0087 |
| Ar I | $3p^6 - 3p^54s$ | $^1\mathrm{S}_0$ | $^2[1/2]^o$ | 0 | 1048.220 | 0.25 |

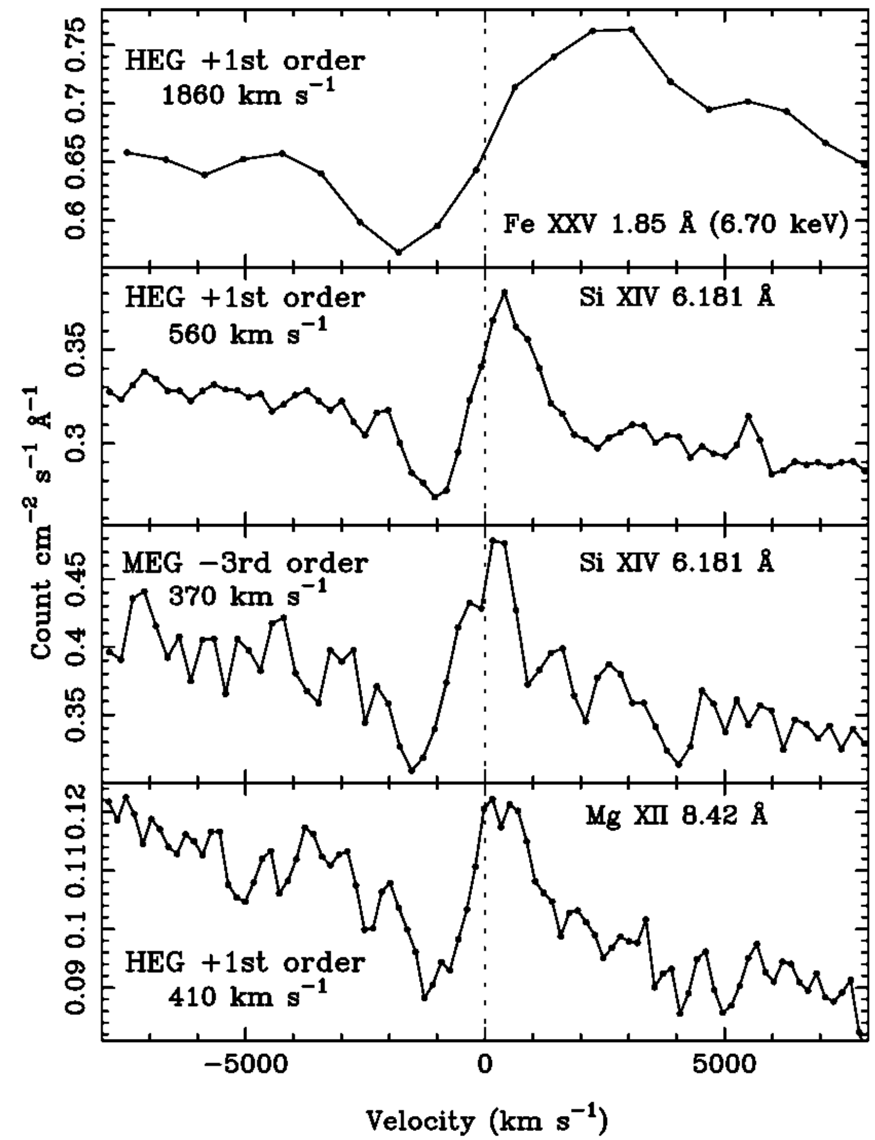
[a] Transition data from NIST Atomic Spectra Database v4.0.0 (Ralchenko et al. 2010)

# P Cygni Profile

- The PCygni profile is characterized by strong emission lines with corresponding blueshifted absorption line.



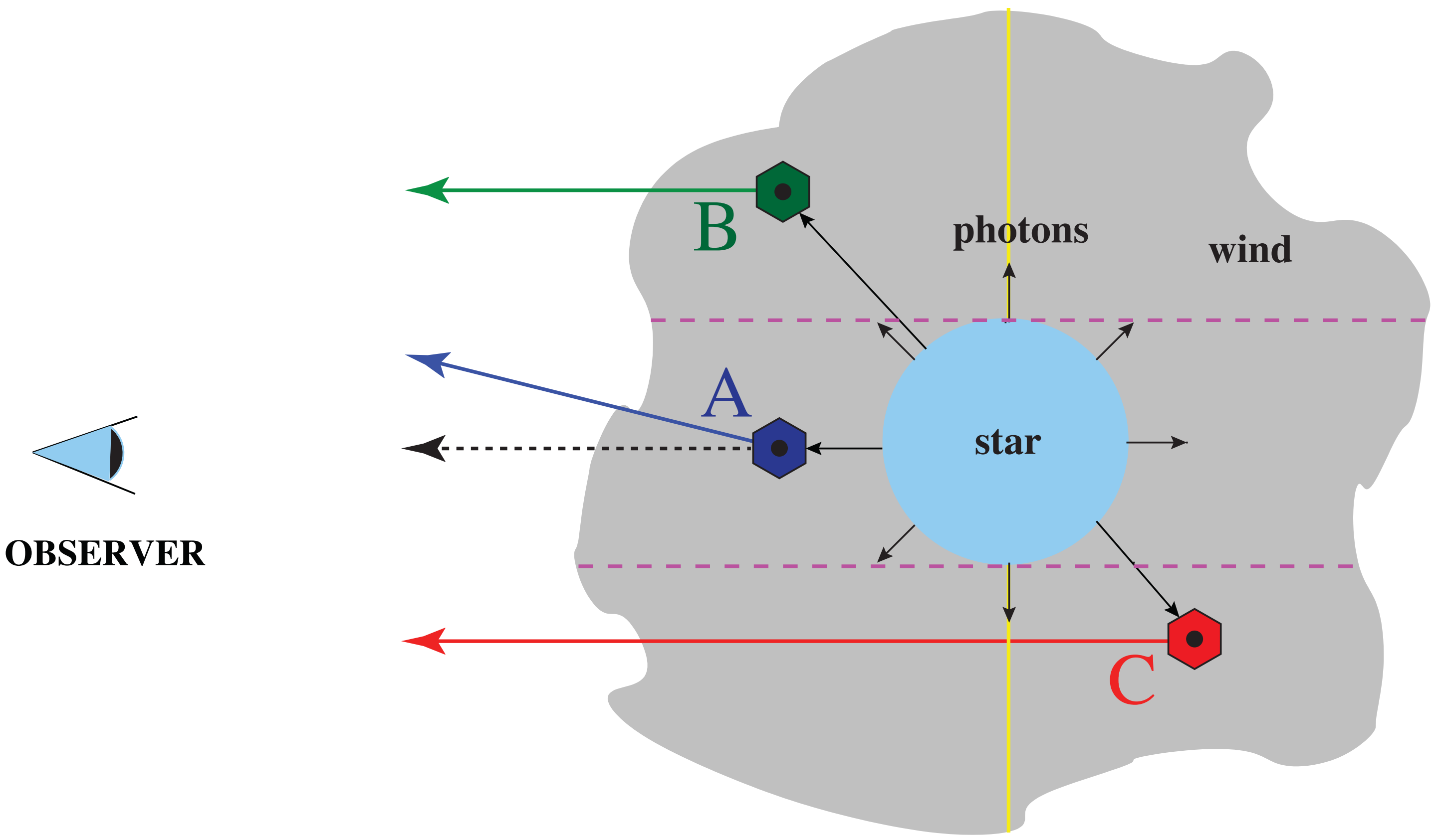

zeta Puppis (Snow et al., 1994, ApJS, 95, 163)



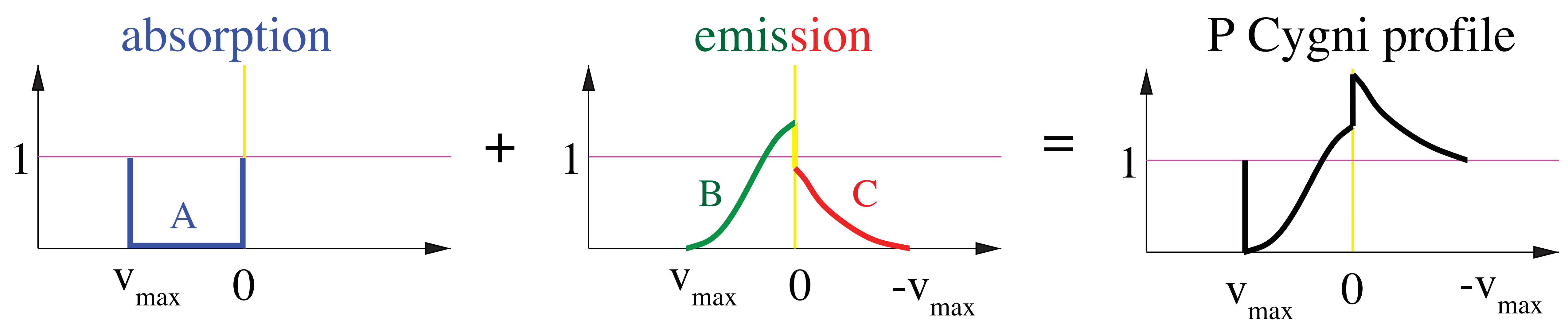

Circinus X-1
(Brandt & Schulz, 2000, ApJ, 544, L123)

# P Cygni profile formation

- The blueshifted absorption line is produced by material moving away from the star and toward us, whereas the emission come from other parts of the expanding shell.




Figures from Joachim Puls slightly modified



# Lyα Resonance Scattering

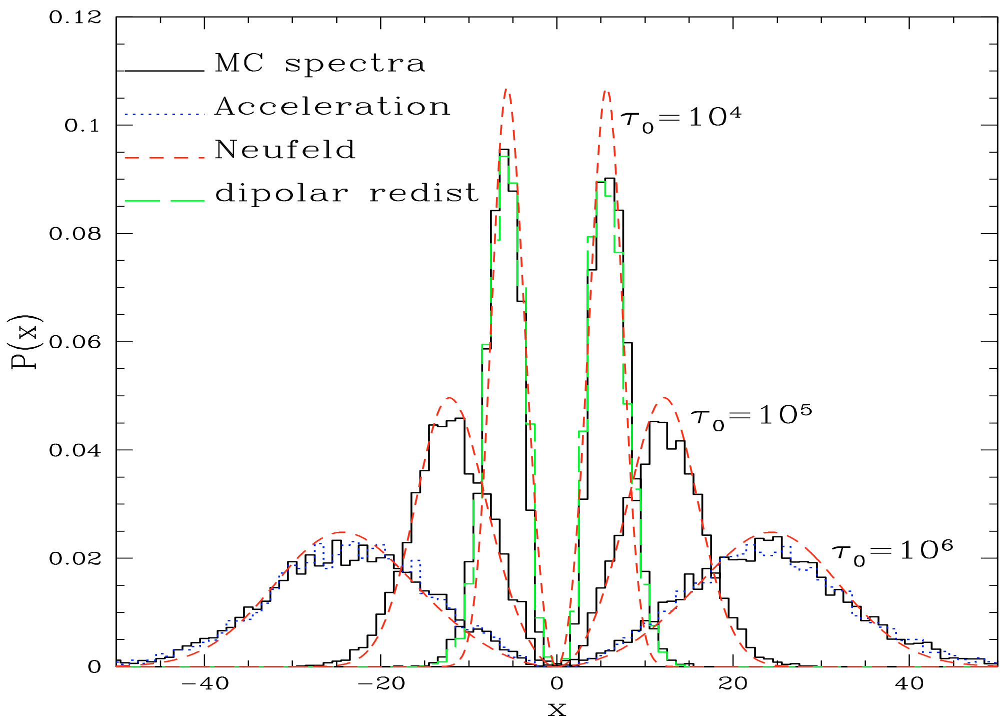
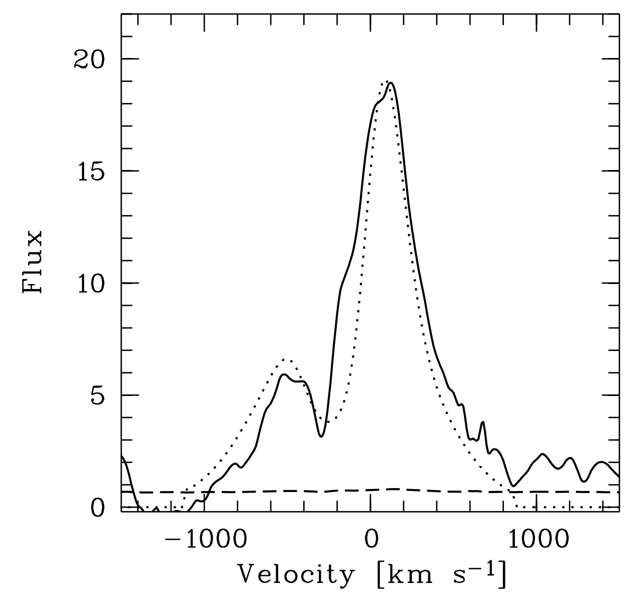
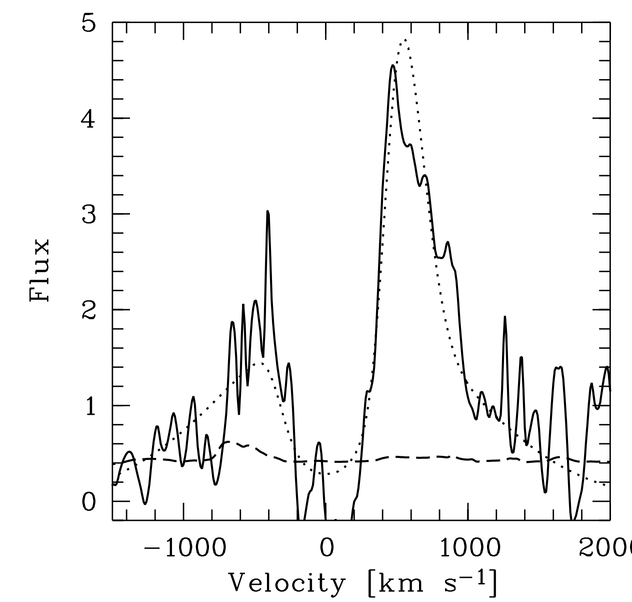



**Fig. 1.** Predicted emergent Ly$\alpha$ profiles for monochromatic line radiation emitted in a dust-free slab of different optical depths (solid lines) compared with analytic solutions from Neufeld (1990, dashed). The dotted blue curve shows the line profile obtained using a frequency redistribution function, which skips a large number of resonant core scatterings. The adopted conditions of the medium are: $T = 10$ K (i.e. $a = 1.5 \times 10^{-2}$) and $\tau_0 = 10^4, 10^5, 10^6$ *from top to bottom*. The green long-dashed curve, obtained with a dipolar angular redistribution, overlaps perfectly the black solid line obtained with the isotropic angular redistribution function, illustrating the fact that in static media, isotropy is a very good approximation.

Verhamme et al. (2006, A&A, 460, 397)




(a) FDF-5215




(b) FDF-7539

| ID | $v_{dis}$(core) [km s$^{-1}$] | $v_{dis}$(shell) [km s$^{-1}$] | $N_{HI}$ [cm$^{-2}$] | $v_{outflow}$ [km s$^{-1}$] | $z$ |
|---|---|---|---|---|---|
| 4691 | 600 | 60 | $4 \times 10^{17}$ | 12 | 3.30 |
| 5215 | 500 | 125 | $<2 \times 10^{16}$ | 125 | 3.15 |
| 7539 | 1140 | 190 | $2.5 \times 10^{16}$ | 190 | 3.29 |

Comparizon of the observed Lyα lines (solid lines) and the best-fit theoretical models. The dashed line indicates the noise level of the observed spectrum.
Tapken et al. (2007, A&A, 467, 63)