Interstellar Medium (ISM)

Week 5
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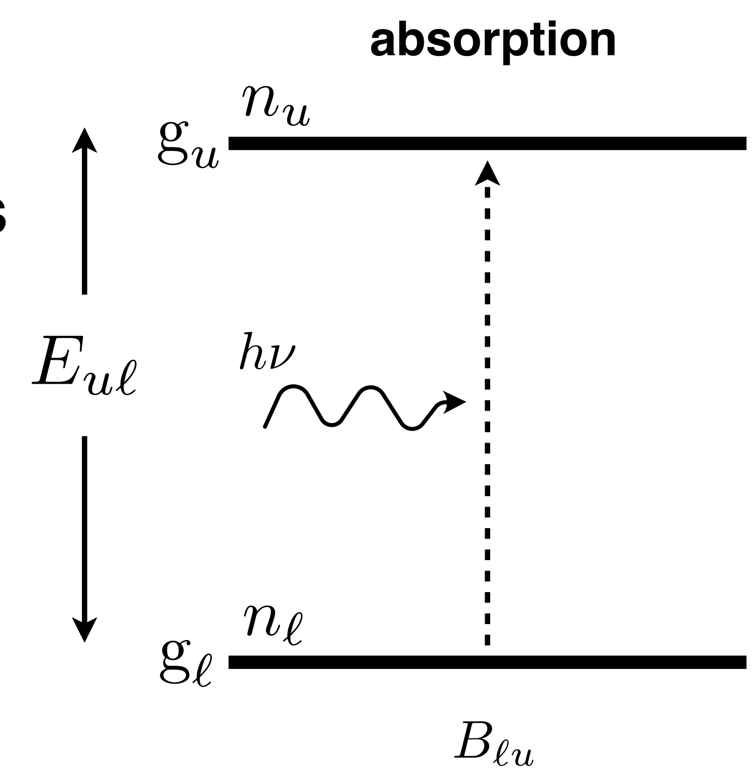
Atomic Processes

- **Excitation and de-excitation (Transition)**
  - Radiative excitation (photoexcitation; photoabsorption)
  - Radiative de-excitation (spontaneous emission and stimulated emission)
  - Collisional excitation
  - Collisional de-excitation
- **Emission Line**
  - Collisionally-excited emission lines
  - Recombination lines (recombination following photoionization or collisional ionization)
- **Ionization**
  - Photoionization and Auger-ionization
  - Collisional Ionization (Direct ionization and Excitation-autoionization)
- **Recombination**
  - Radiative recombination <=> Photoionization
  - Dielectronic Recombination (not dielectric!)
  - Three-body recombination <=> Direct collisional ionization
- **Charge exchange**

Radiative Excitation and De-excitation (Absorption and Emission)

- **Three Radiative Transitions and Einstein Coefficients**
  - ***Absorption***:



- If an absorber (atom, ion, molecule, or dust grain) X is in a lower level ℓ and there is radiation present with photons having an energy equal to $E_{u\ell}$. The absorber can absorb a photon and undergo an upward transition.

$$\text{absorption}: \quad X_\ell + h\nu \;\rightarrow\; X_u \; (h\nu = E_{u\ell})$$

- The rate per volume at which the absorbers absorb photons will be proportional to both the energy density u_ν of photons of the appropriate energy and the number density n_ℓ of absorbers in the lower level ℓ.

$$\left(\frac{dn_u}{dt}\right)_{\ell\rightarrow u} = -\left(\frac{dn_\ell}{dt}\right)_{\ell\rightarrow u} = n_\ell B_{\ell u} u_\nu$$

- The proportionality constant $B_{\ell u}$ is the ***Einstein B coefficient*** for the upward transition $\ell \rightarrow u$.

- ***Emission***:
    - An absorber X in an excited level u can decay to a lower level ℓ with emission of a photon. There are two ways this can happen:

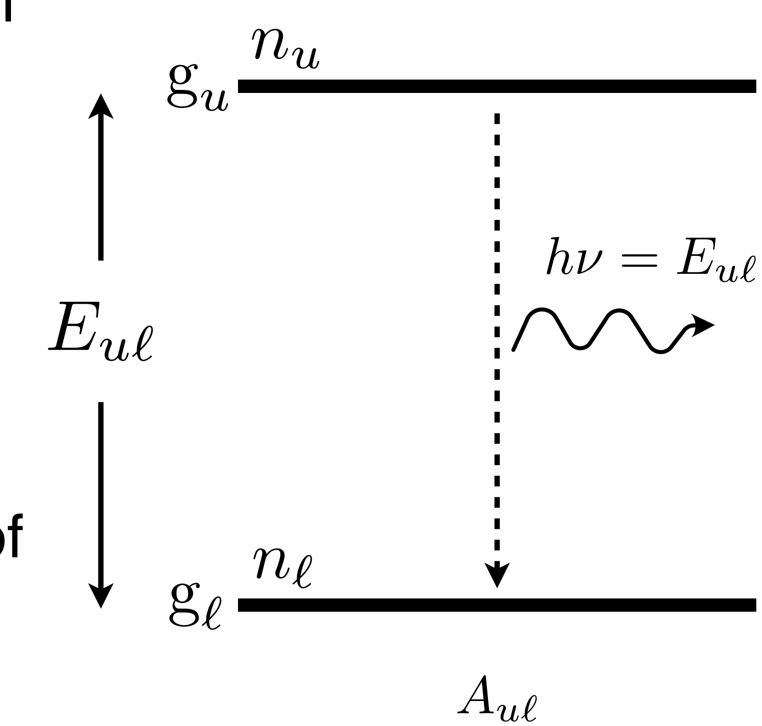
$$\text{spontaneous emission}: \quad X_u \;\rightarrow\; X_\ell + h\nu \quad (h\nu = E_{u\ell})$$
$$\text{stimulated emission}: \quad X_u + h\nu \;\rightarrow\; X_\ell + 2h\nu \quad (h\nu = E_{u\ell})$$

- ***Spontaneous emission*** is a random process, independent of the presence of a radiation field.
- ***Stimulated emission*** occurs if photons of the identical frequency, polarization, and direction of propagation are already present, and the rate of stimulated emission is proportional to the energy density u_ν of these photons.

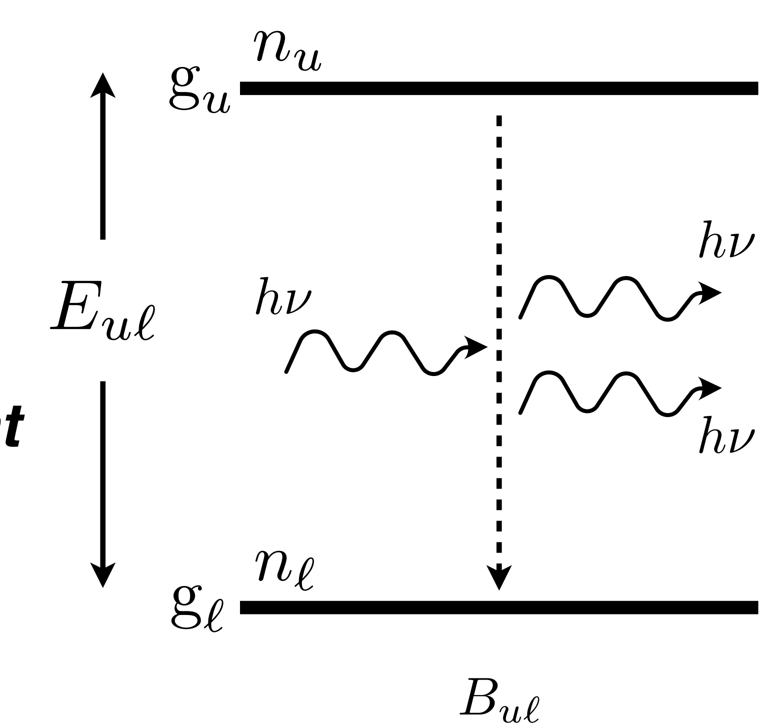
$$\left(\frac{dn_\ell}{dt}\right)_{u\rightarrow\ell} = -\left(\frac{dn_u}{dt}\right)_{u\rightarrow\ell} = n_u\left(A_{u\ell} + B_{u\ell}u_\nu\right)$$

- The probability per unit time $A_{u\ell}$ is the ***Einstein A coefficient*** for spontaneous transition. The coefficient $B_{u\ell}$ is the ***Einstein B coefficient*** for the downward transition $u \rightarrow \ell$.

spontaneous emission



stimulated emission



Relations between the Einstein coefficients

- The three Einstein coefficients are not mutually independent.
- ***In thermal equilibrium***, the radiation field becomes the “blackbody” radiation field and the two levels must be populated according to the Boltzmann distribution.

$$(u_\nu)_{\rm TE} = \frac{4\pi}{c} B_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_{\rm B}T} - 1}$$

$$\left(\frac{n_u}{n_\ell}\right)_{\rm TE} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} e^{-E_{u\ell}/k_{\rm B}T} \quad \text{Here, } E_{u\ell} = h\nu.$$

- The net rate of change of level u should be equal to zero, in TE.

$$\begin{aligned} \frac{dn_u}{dt} &= \left(\frac{dn_u}{dt}\right)_{\ell\to u} + \left(\frac{dn_u}{dt}\right)_{u\to\ell} \\ &= n_\ell B_{\ell u} u_\nu - n_u \left(A_{u\ell} + B_{u\ell} u_\nu\right) \\ &= 0 \end{aligned}$$

$$n_\ell B_{\ell u} u_\nu - n_u \left(A_{u\ell} + B_{u\ell} u_\nu\right) = 0$$

$$\left(n_\ell B_{\ell u} - n_u B_{u\ell}\right) u_\nu = n_u A_{u\ell}$$

$$\begin{aligned} u_\nu &= \frac{n_u A_{u\ell}}{n_\ell B_{\ell u} - n_u B_{u\ell}} \\ &= \frac{\left(n_u A_{u\ell}\right) / \left(n_\ell B_{\ell u}\right)}{1 - \left(n_u B_{u\ell}\right) / \left(n_\ell B_{\ell u}\right)} \\ &= \frac{\left(\mathrm{g}_u/\mathrm{g}_\ell\right) e^{-h\nu/kT} \left(A_{u\ell}/B_{\ell u}\right)}{1 - \left(\mathrm{g}_u/\mathrm{g}_\ell\right) e^{-h\nu/kT} \left(B_{u\ell}/B_{\ell u}\right)} \quad \leftarrow \quad \frac{n_u}{n_\ell} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} e^{-h\nu_{u\ell}/kT_{\rm exc}} \\ &= \frac{\left(\mathrm{g}_u/\mathrm{g}_\ell\right) \left(A_{u\ell}/B_{\ell u}\right)}{e^{h\nu/kT} - \left(\mathrm{g}_u/\mathrm{g}_\ell\right) \left(B_{u\ell}/B_{\ell u}\right)} \end{aligned}$$

[Note] If there is no stimulated emission $(B_{u\ell} = 0)$, the only way to make the left eq. consistent with the Planck function is to assume $h\nu/kT \gg 1$ (Wien's regime). Therefore, the stimulated emission is negligible in the Wien's regime. In other words, the stimulated emission term is required in the Rayleigh-Jean regime.

Comparing the above eq. with Planck function,

$$u_\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

we can immediately recognize that the following relations should be satisfied.

$$\left(\mathrm{g}_u/\mathrm{g}_\ell\right) \left(A_{u\ell}/B_{\ell u}\right) = \frac{8\pi h\nu^3}{c^3}$$

$$\left(\mathrm{g}_u/\mathrm{g}_\ell\right) \left(B_{u\ell}/B_{\ell u}\right) = 1$$

Therefore, only one coefficient is independent.

In summary, we obtained the following relations between the Einstein coefficients.

$$A_{u\ell} = \frac{8\pi h\nu^3}{c^3} B_{u\ell} \qquad B_{u\ell} = \frac{c^3}{8\pi h\nu^3} A_{u\ell}$$

$$B_{\ell u} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} B_{u\ell} \qquad B_{\ell u} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} \frac{c^3}{8\pi h\nu^3} A_{u\ell}$$

- We note the Einstein coefficients are intrinsic properties of the absorbing material, irrelevant to the assumption of TE. Hence, **the relations between the Einstein coefficients should hold in any condition.**
- Using the relation, we can rewrite the downward and upward transition rates:

$$\left(\frac{dn_u}{dt}\right)_{\ell \to u} = n_\ell \frac{g_u}{g_\ell} \frac{c^3}{8\pi h \nu^3} A_{u\ell} u_\nu \qquad \left(\frac{dn_\ell}{dt}\right)_{u \to \ell} = n_u A_{u\ell} \left(1 + \frac{c^3}{8\pi h \nu^3} u_\nu\right)$$

- It is helpful to use a dimensionless quantity, the photon occupation number:

$$n_\gamma \equiv \frac{c^2}{2h\nu^3} I_\nu \quad \xrightarrow{\text{averaging over directions}} \quad \langle n_\gamma \rangle = \frac{c^2}{2h\nu^3} J_\nu = \frac{c^3}{8\pi h \nu^3} u_\nu$$

- Then, the above transition rates are simplified:

$$\left(\frac{dn_u}{dt}\right)_{\ell \to u} = n_\ell \frac{g_u}{g_\ell} A_{u\ell} \langle n_\gamma \rangle \qquad \left(\frac{dn_\ell}{dt}\right)_{u \to \ell} = n_u A_{u\ell} \left(1 + \langle n_\gamma \rangle\right)$$

- The photon occupation number determines the relative importance of stimulated and spontaneous emission: stimulated emission is important only when $\langle n_\gamma \rangle \gg 1$.

Absorption and Emission Coefficients in terms of Einstein coefficients

- The Einstein coefficients are useful means of analyzing absorption and emission processes. However, we often find it even more useful to use cross section because **the cross section has a natural geometric meaning**.
- ***(pure) Absorption cross section:***
  - The number density of photons per unit frequency interval is $u_\nu/h\nu$. Let $\sigma_{\ell u}(\nu)$ be the cross section for absorption of photons for the transition $\ell \to u$. Then, the absorption rate is

$$\left(\frac{dn_u}{dt}\right)_{\ell\to u} = n_\ell \int d\nu \sigma_{\ell u}(\nu) c \frac{u_\nu}{h\nu} \approx n_\ell u_\nu \frac{c}{h\nu_{u\ell}} \int d\nu \sigma_{\ell u}(\nu)$$

  - Here, we assumed that $u_\nu/h\nu$ do not vary appreciably over the profile of the cross section. Therefore, we derive a simple relation between the absorption cross section and the Einstein B coefficient:

$$\int d\nu \sigma_{\ell u}(\nu) = \frac{h\nu_{u\ell}}{c} B_{\ell u} = \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell}$$

  - If the cross section has a normalized profile of ϕ_ν, we can write the absorption cross section as follows:

$$\boxed{\sigma_{\ell u}(\nu) = \frac{h\nu_{u\ell}}{c} B_{\ell u}\phi_\nu = \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell}\phi_\nu \qquad \text{with} \quad \int \phi_\nu d\nu = 1}$$

• *(effective) Absorption Coefficient*

- We note that the stimulated emission is proportional to the energy density of ambient radiation field. In the radiative transfer equation, it is convenient to include the stimulated emission term in the absorption coefficient as a negative absorption.

$$\left(\frac{dn_u}{dt}\right)_{\ell\rightarrow u} - \left(\frac{dn_\ell}{dt}\right)_{u\rightarrow\ell}^{\text{stimulated}} = n_\ell B_{\ell u} u_\nu - n_u B_{u\ell} u_\nu$$
$$= n_\ell B_{\ell u} u_\nu - n_u \left(\frac{g_\ell}{g_u} B_{\ell u}\right) u_\nu$$

- Therefore, we may define the cross section for stimulated emission and the net (effective) absorption coefficient as follows:

pure absorption coefficient

$$\sigma_{u\ell} = \frac{g_\ell}{g_u}\sigma_{\ell u}$$

$$\kappa_\nu = \boxed{n_\ell \sigma_{\ell u}} - n_u \sigma_{u\ell}$$
$$= n_\ell \sigma_{\ell u}\left(1 - \frac{n_u/n_\ell}{g_u/g_\ell}\right)$$

- Using the definition of the excitation temperature, we can rewrite them:

$$\kappa_\nu = n_\ell \sigma_{\ell u}\left[1 - \exp\left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}}\right)\right] \quad \text{or} \quad \sigma_\nu^{\text{eff}} = \sigma_{\ell u}\left[1 - \exp\left(-\frac{h\nu_{u\ell}}{k_B T_{\text{exc}}}\right)\right]$$

• *Emission coefficient (Emissivity)*

- The emissivity is defined as the power radiated per unit frequency per unit solid angle per unit volume.
- The line emissivity can be expressed in terms of the spontaneous downward transition rate:

$$4\pi \int d\nu j_\nu = h\nu_{u\ell} \left(\frac{dn_\ell}{dt} \right)_{u\to\ell}^{\text{spontaneous}}$$

- Comparing with the definition of the Einstein A coefficient, we obtain:

$$\int d\nu j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell}$$

- If the emission line has a normalized profile of ϕ_ν, we can write the emissivity as follows:

$$j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell}\phi_\nu \qquad \text{with} \quad \int d\nu \phi_\nu = 1$$

- The correction factor for the stimulated emission in absorption coefficient:
  - For Lyα line,

$$h\nu_{u\ell} = 10.2\,\mathrm{eV} \quad \rightarrow \quad 1 - \exp\left(-\frac{h\nu_{u\ell}}{k_\mathrm{B}T_\mathrm{exc}}\right) = 1 - \exp\left(-\frac{1.1837\times 10^5\,\mathrm{K}}{T_\mathrm{exc}}\right)$$
$$\simeq 1 \quad \text{for } T_\mathrm{exc} \approx T_\mathrm{gas} < 1\times 10^5\,\mathrm{K}$$

    - The stimulated emission is negligible.
  - For 21 cm line,

$$h\nu_{u\ell} = 6\,\mu\mathrm{eV} \quad \rightarrow \quad 1 - \exp\left(-\frac{h\nu_{u\ell}}{k_\mathrm{B}T_\mathrm{exc}}\right) = 1 - \exp\left(-\frac{0.068\,\mathrm{K}}{T_\mathrm{exc}}\right)$$
$$\simeq \frac{0.068\,\mathrm{K}}{T_\mathrm{exc}} \ll 1 \quad \text{for } T_\mathrm{exc} \approx T_\mathrm{gas} \sim 100\,\mathrm{K}$$

    - The correction for stimulated emission is very important. ***We, therefore, need to take into account the stimulated emission in dealing with the 21 cm line.***
- Two limiting cases:
  - At radio and sub-mm frequencies, the upper levels are often appreciably populated, and it is important to include both spontaneous and stimulated emission.
  - When we consider propagation of optical, UV, or X-ray radiation in cold ISM, the upper levels of atoms and ions usually have negligible populations, and stimulated emission can be neglected.

- ***Source Function***:

$$
\begin{aligned}
S_\nu &= \frac{j_\nu}{\kappa_\nu} \\
&= \frac{n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell}\phi_\nu^{\rm emiss}}{n_\ell \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell}\phi_\nu^{\rm abs}\left[1 - \exp\left(-h\nu_{u\ell}/k_{\rm B}T_{\rm exc}\right)\right]} & \leftarrow \quad \frac{n_u}{n_\ell} = \frac{g_u}{g_\ell}\exp\left(-h\nu_{u\ell}/k_{\rm B}T_{\rm exc}\right) \\
& & \leftarrow \quad \phi_\nu^{\rm emiss} = \phi_\nu^{\rm abs} \\
&= \frac{2h\nu_{u\ell}^3}{c^2}\frac{1}{\exp\left(h\nu_{u\ell}/k_{\rm B}T_{\rm exc}\right) - 1}
\end{aligned}
$$

- This is called the ***generalized Kirchhoff's law***.
- **The intrinsic profiles for absorption and emission are the same**.
    - The source function should approach the Planck function in LTE ($T_{\rm exc} = T_{\rm kinetic}$). For this to be true, the intrinsic profile of emission line should be the same as that of absorption line.
    - We can show that the intrinsic emission and absorption profiles are, indeed, the same, using a semi-classical model for an atom.

Oscillator Strength

- In the previous slides, we characterized the absorption cross section by the Einstein A coefficient. Equivalently, we can express the cross section in terms of the oscillator strength for the absorption transition $\ell \rightarrow u$, defined by the relation:

$$\int \sigma_{\ell u}(\nu) d\nu = \frac{\pi e^2}{m_e c} f_{\ell u} \quad \rightarrow \quad \boxed{\sigma_{\ell u}(\nu) = \frac{\pi e^2}{m_e c} f_{\ell u} \phi_\nu}$$

  - Here, the factor $\frac{\pi e^2}{m_e c} = 0.02654 \ \mathrm{cm^2 \, Hz}$ is the cross-section, integrated over the line profile, for a classical oscillator model.
  - The oscillator strength is the factor which corrects the classical result. The quantum mechanical process can be interpreted as being due to a (fractional) number f of equivalent classical electron oscillators of the same frequency.

- The Einstein A coefficient is related to the absorption oscillator strength of the upward transition by

$$\boxed{A_{u\ell} = \frac{8\pi^2 e^2 \nu_{u\ell}^2}{m_e c^3} \frac{g_\ell}{g_u} f_{\ell u} = \left(\frac{0.8167\,\mathrm{cm}}{\lambda_{u\ell}} \right)^2 \frac{g_\ell}{g_u} f_{\ell u} \ [\mathrm{s}^{-1}]}$$

  - For 21.1 cm line, $g_u = 3, \ g_\ell = 1 \ (g_F = 2F + 1)$

$$A_{u\ell} = 2.88 \times 10^{-15} \ [\mathrm{s}^{-1}] = (11\,\mathrm{Myr})^{-1} \qquad f_{\ell u} = 5.75 \times 10^{-12}$$

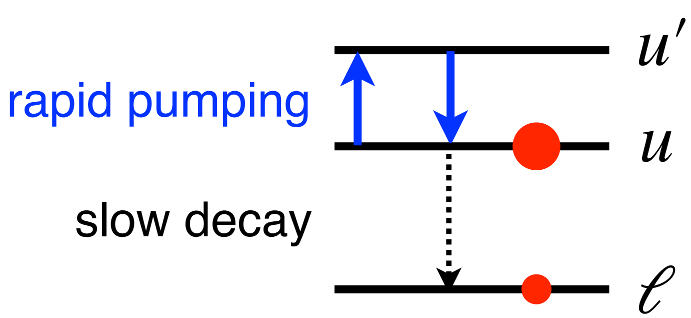
  - For Lyα (1215.67Å) line, $g_u = 3, \ g_\ell = 1 \ (g_L = 2L + 1)$

$$A_{u\ell} = 6.265 \times 10^8 \ [\mathrm{s}^{-1}] \qquad f_{\ell u} = 0.4164 \ \text{for} \ 1^2S_{1/2} \rightarrow 2^2P$$

$$\begin{aligned} f_{\ell u} &= 0.27760 \ \text{for} \ {}^2S_{1/2} \rightarrow {}^2P_{3/2} \\ &= 0.13881 \ \text{for} \ {}^2S_{1/2} \rightarrow {}^2P_{1/2} \end{aligned}$$

Maser Lines

- Population inversion
  - Under some conditions, a process may act to “pump” an excited state u by either collisional or radiative excitation of a higher level u' that then decays to populate level u. If this pumping process is rapid enough (relative to the processes that depopulate u), it may be possible for the relative level populations between u and ℓ to satisfy the inequality (also to have a negative excitation temperature).



$$n_u > \frac{g_u}{g_\ell} n_\ell \quad \rightarrow \quad T_{\mathrm{exc},u\ell} < 0.$$

  - When this population inversion occurs, stimulated emission is stronger than pure absorption, and ***the radiation is amplified as it propagates***. Then, the effective absorption coefficient, optical depth, and attenuation factor are

$$\kappa_\nu = \sigma_{\ell u}\left(1 - \frac{n_u/g_u}{n_\ell/g_\ell}\right) < 0, \quad \tau_\nu = \int \kappa_\nu ds < 0, \quad e^{-\tau_\nu} > 1$$

- Maser
  - Such population inversion have been observed for microwave transitions of H I, OH, H_2O, and SiO, and hence we speak of ***maser (microwave amplification by stimulated emission of radiation)*** emission.

- Observational properties
  - If $|k_{\mathrm{B}} T_{\mathrm{exc,u}\ell}| \gg h\nu$, the RT equation becomes

$$\begin{aligned} T_A &= T_A(0) e^{-\tau_\nu} + T_{\mathrm{exc}} \left(1 - e^{-\tau_\nu}\right) \\ &= \left(T_A(0) + |T_{\mathrm{exc}}|\right) e^{|\tau_\nu|} - |T_{\mathrm{exc}}| \end{aligned}$$

  - The factor $e^{|\tau_\nu|}$ is in some cases very large - some OH and H_2O masers have been observed to have $T_A > 10^{11}\,\mathrm{K}$.

  - We note that
    - $e^{|\tau_\nu|}$ is more strongly peaked on the sky than $|\tau_\nu|$ - the angular size of the maser is less than the actual transverse dimension of the masing region.
    - $e^{|\tau_\nu|}$ is more strongly in ν than $|\tau_\nu|$ - the maser line is narrower than the actual velocity distribution of the masing species.

  - Some maser can be very bright, allowing the use of interferometry, as well as observations of sources at large distances.
    - This has enabled measurements of proper motion of maser spots in star-forming regions of the Milky Way, as well as in material orbiting a supermassive black hole in the spiral galaxy NGC 4258 (Hernstein et al. 1999).

Collisional Excitation & De-excitation

- **Collisional Rate (Two Level Atom)**
  - The cross section $\sigma_{\ell u}$ for collisional excitation from a lower level ℓ to an upper level u is, in general, inversely proportional to the impact energy (or v^2) above the energy threshold $E_{u\ell}$ and is zero below.
  - The collisional cross section can be expressed in the following form using a dimensionless quantity called the ***collision strength*** $\Omega_{\ell u}$:

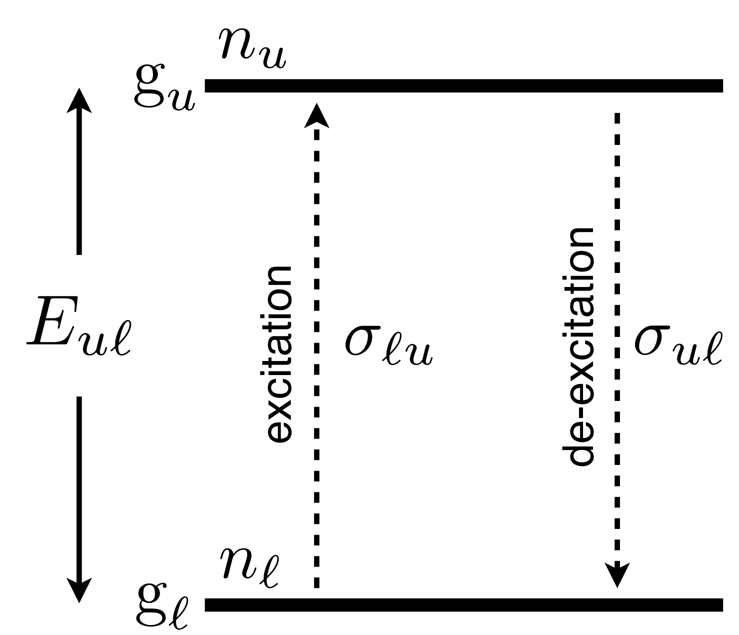
$$\sigma_{\ell u}(v) = (\pi a_0^2)\left(\frac{hR_{\rm H}}{\frac{1}{2}m_e v^2}\right)\frac{\Omega_{\ell u}}{g_\ell}\ \ \mathrm{cm}^2 \quad \mathrm{for}\quad \frac{1}{2}m_e v^2 > E_{u\ell}$$

$$= \frac{h^2}{4\pi m_e^2 v^2}\frac{\Omega_{\ell u}}{g_\ell}$$

$$\mathrm{or}\quad \sigma_{\ell u}(E) = \frac{h^2}{8\pi m_e E}\frac{\Omega_{\ell u}}{g_\ell} \qquad \left(E = \frac{1}{2}m_e v^2\right)$$

$$\mathrm{where,}\quad a_0 = \frac{\hbar^2}{m_e e^2} = 5.12\times 10^{13}\ \ \mathrm{cm,\ Bohr\ radius}$$

$$R_{\rm H} = \frac{m_e e^4}{4\pi\hbar^3} = 109,737\ \ \mathrm{cm}^{-1},\quad \mathrm{Rydberg\ constant}\qquad \left(\hbar = \frac{h}{2\pi}\right)$$



  - The collision strength $\Omega_{\ell u}$ is a function of electron velocity (or energy) but is often approximately constant near the threshold. Here, g_ℓ and g_u are the statistical weights of the lower and upper levels, respectively.

- Advantage of using the collision strength is that (1) it removes the primary energy dependence for most atomic transitions and (2) they have the symmetry between the upper and the lower states.

The principle of detailed balance states that ***in thermodynamic equilibrium each microscopic process is balanced by its inverse.***

$$n_e n_\ell v_\ell \sigma_{\ell u}(v_\ell) f(v_\ell) dv_\ell = n_e n_u v_u \sigma_{u\ell}(v_u) f(v_u) dv_u$$

Here, v_ℓ and v_u are related by $\frac{1}{2}m_e v_\ell^2 = \frac{1}{2}m_e v_u^2 + E_{u\ell}$, and $f(v)$ is a Maxwell velocity distribution of electrons. Using the Boltzmann equation of thermodynamic equilibrium,

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right)$$

we derive the following relation between the cross-sections for excitation and de-excitation are

$$\boxed{g_\ell v_\ell^2 \sigma_{\ell u}(v_\ell) = g_u v_u^2 \sigma_{u\ell}(v_u) \ \ \text{Here,} \ \frac{1}{2}m_e v_\ell^2 = \frac{1}{2}m_e v_u^2 + E_{u\ell}} \longrightarrow g_\ell \cdot (E + E_{u\ell}) \cdot \sigma_{\ell u}(E + E_{u\ell}) = g_u \cdot E \cdot \sigma_{u\ell}(E)$$

where $E = \frac{1}{2}m_e v_u^2$

and the symmetry of the collision strength between levels.

$$\boxed{\Omega_{\ell u} = \Omega_{u\ell}} \qquad \text{more precisely} \ \boxed{\Omega_{\ell u}\left(E + E_{u\ell}\right) = \Omega_{u\ell}\left(E\right)}$$

These two relations were derived in the TE condition. However, ***the cross-sections are independent on the assumptions, and thus the above relations should be always satisfied.***

- Collisional excitation and de-excitation rates

The ***collisional de-excitation rate per unit volume per unit time, which is thermally averaged,*** is

$$\left(\frac{dn_\ell}{dt}\right)_{u\rightarrow\ell} = n_e n_u \int_0^\infty v\sigma_{u\ell}(v)f(v)dv$$
$$= n_e n_u k_{u\ell} \quad [\mathrm{cm}^{-3}\ \mathrm{s}^{-1}]$$

$$k_{u\ell} \equiv \langle\sigma v\rangle_{u\rightarrow\ell}$$

$$\begin{aligned} k_{u\ell} &= \int_0^\infty v\sigma_{u\ell}(v)f(v)dv \\ &= \left(\frac{2\pi\hbar^4}{k_\mathrm{B}m_e^3}\right)^{1/2} T^{-1/2}\frac{\langle\Omega_{u\ell}\rangle}{\mathrm{g}_u} \\ &= \frac{8.62942\times10^{-6}}{T^{1/2}}\frac{\langle\Omega_{u\ell}\rangle}{\mathrm{g}_u} \quad [\mathrm{cm}^3\ \mathrm{s}^{-1}], \end{aligned}$$

effective collision strength:

$$\langle\Omega_{u\ell}\rangle \equiv \int_0^\infty \Omega_{u\ell}(E)e^{-E/k_\mathrm{B}T}d(E/k_\mathrm{B}T)$$

and the ***collisional excitation rate per unit volume per unit time*** is

$$\left(\frac{dn_u}{dt}\right)_{\ell\rightarrow u} = n_e n_\ell k_{\ell u}$$

$$k_{\ell u} \equiv \langle\sigma v\rangle_{\ell\rightarrow u}$$

$$k_{\ell u} = \int_{v_\mathrm{min}}^\infty v\sigma_{\ell u}(v)f(v)dv \quad \text{Here, } \frac{1}{2}m_e v_\mathrm{min}^2 = E_{u\ell}$$
$$= \left(\frac{2\pi\hbar^4}{k_\mathrm{B}m_e^3}\right)^{1/2} T^{-1/2}\frac{\langle\Omega_{u\ell}\rangle}{\mathrm{g}_\ell}\exp\left(-\frac{E_{u\ell}}{k_\mathrm{B}T}\right)$$

Here, $k_{\ell u}$ and $k_{u\ell}$ are the collisional rate coefficient for excitation and de-excitation coefficients in units of cm^3 s^{-1}, respectively. We also note that ***the rate coefficients for collisional excitation and de-excitation are related by***

$$k_{\ell u} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell}k_{u\ell}\exp\left(-\frac{E_{u\ell}}{k_\mathrm{B}T}\right) \qquad \langle\sigma v\rangle_{\ell\rightarrow u} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell}\langle\sigma v\rangle_{u\rightarrow\ell}\exp\left(-\frac{E_{u\ell}}{k_\mathrm{B}T}\right)$$

Sum rule for collision strengths

- Quantum mechanical sum rule for collision strengths for the case where one term consists of a singlet ($S = 0$ or $L = 0$) and the second consists of a multiplet: the collision strength of each fine structure level J is related to the total collision strength of the multiplet by

$$\Omega_{(SLJ,\ S'L'J')} = \frac{(2J'+1)}{(2S'+1)(2L'+1)}\Omega_{(SL,\ S'L')}$$

Here, $(2J'+1)$ is the statistical weight of an individual level in the multiplet, and $(2S'+1)(2L'+1)$ is the statistical weight of the multiplet term.

We can regard the collision strength as "shared" amongst these levels in proportion to the statistical weights of the individual levels $(g_J = 2J+1)$.

- ***The flux ratio between the lines in a multiplet is proportional to the ratio of their collision strengths, in a low density medium.*** Then, the flux ratio is determined by the ratio of their statistical weights.
  - C-like ions ($1s^2 2s^2 2p^2 \rightarrow 1s^2 2s^2 2p^2$) forbidden or inter combination transitions.

    ground states (triplet) - $^3P_0$: $^3P_1$: $^3P_2$ = 1 : 3 : 5

    excited states (singlets) - $^1D_2$, $^1S_1$
  - Li-like ions ($1s^2 2s^1 \rightarrow 1s^2 2p^1$) resonance transitions

    ground state (singlet) - $^2S_{1/2}$

    excited states (doublet) - $^2P_{3/2}$: $^2P_{1/2}$ = 2 : 1

Collisionally-Excited Emission Line

- Emission line flux
  - **In the low density limit**, the collisional rate between atoms and electrons is much slower than the (spontaneous) radiative de-excitation rate of of the excited level. Thus, we can balance the collisional feeding into level u by the rate of radiative transition back down to level ℓ. The level population is determined by

$$n_e n_\ell k_{\ell u} = A_{u\ell} n_u$$

$$\frac{n_u}{n_\ell} = \frac{n_e k_{\ell u}}{A_{u\ell}} = \frac{n_e}{A_{u\ell}} \beta \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} T^{-1/2} \exp\left(-\frac{E_{u\ell}}{kT}\right)$$

where $A_{u\ell}$ is the Einstein coefficient for spontaneous emission. The line emissivity is given by

$$\begin{aligned} 4\pi j_{u\ell} &= E_{u\ell} A_{u\ell} n_u = E_{u\ell} n_e n_\ell k_{\ell u} \\ &= n_e n_\ell E_{u\ell} \frac{8.62942 \times 10^{-6}}{T^{1/2}} \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right) \quad [\text{erg cm}^{-3}\ \text{s}^{-1}] \\ &\simeq \beta \chi n_e^2 E_{\ell u} T^{-1/2} \frac{\langle \Omega_{u\ell} \rangle}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right) \end{aligned}$$

Here, $\beta = \left(\frac{2\pi\hbar^4}{k m_e^2}\right)^{1/2} = 8.62942 \times 10^{-6}$

$\chi = n_\ell / n_e$

For low temperature, the exponential term dominates because few electrons have energy above the threshold for collisional excitation, so that the line rapidly fades with decreasing temperature.

At high temperature, the $T^{-1/2}$ term controls the cooling rate, so the line fades slowly with increasing temperature.

- **In high-density limit**, the level population are set by the Boltzmann equilibrium, and the line emissivity is

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{E_{u\ell}}{kT}\right)$$

$$\begin{aligned} 4\pi j_{u\ell} &= E_{\ell u} A_{u\ell} n_u \\ &= n_\ell E_{\ell u} A_{u\ell} \frac{g_u}{g_\ell} \exp\left(-\frac{E_{\ell u}}{kT}\right) \\ &\simeq \chi n_e E_{\ell u} A_{u\ell} \frac{g_u}{g_\ell} \exp\left(-\frac{E_{\ell u}}{kT}\right) \end{aligned}$$

Here, **the line flux scales as n_e rather than n_e^2, but the line flux tends to a constant value at high temperature.**

- **Critical density** is defined as **the density where the radiative depopulation rate matches the collisional de-excitation for the excited state.**

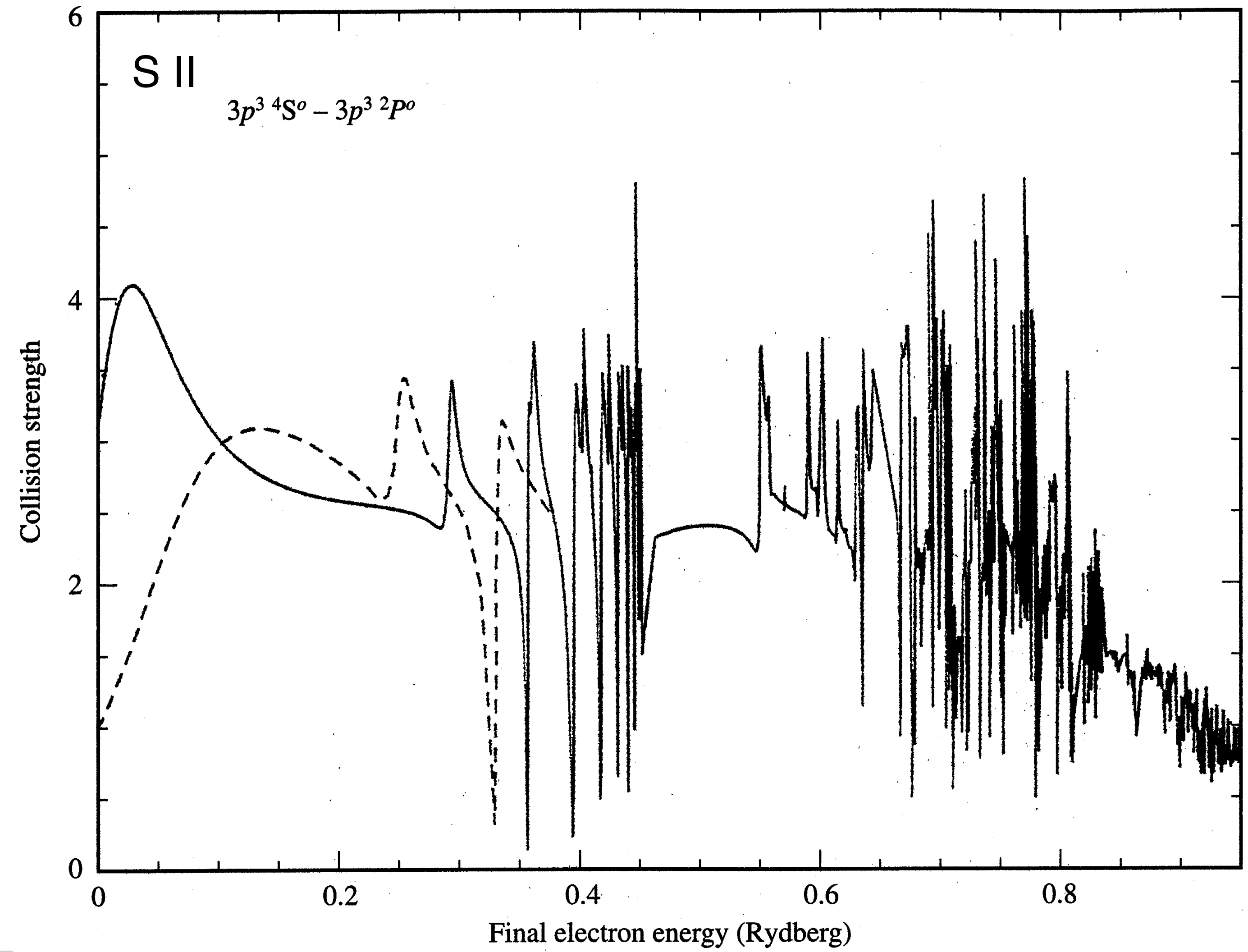
$$\begin{aligned} A_{u\ell} n_u &= n_e n_u k_{u\ell} \\ n_{\rm crit} &= \frac{A_{u\ell}}{k_{u\ell}} \end{aligned}$$

$$\rightarrow \quad \begin{aligned} n_{\rm crit} &= A_{u\ell} \frac{g_u}{\beta \langle \Omega_{u\ell} \rangle} T^{1/2} \\ &= 1.2 \times 10^3 \frac{A_{u\ell}}{10^{-4}\,{\rm s}^{-1}} \frac{g_u}{\langle \Omega_{u\ell} \rangle} \left(\frac{T}{10^4\,{\rm K}}\right)^{1/2} \quad [{\rm cm}^{-3}] \end{aligned}$$

- At densities higher than the critical density, collisional de-excitation becomes significant, and the forbidden lines will be weaker as the density increases.

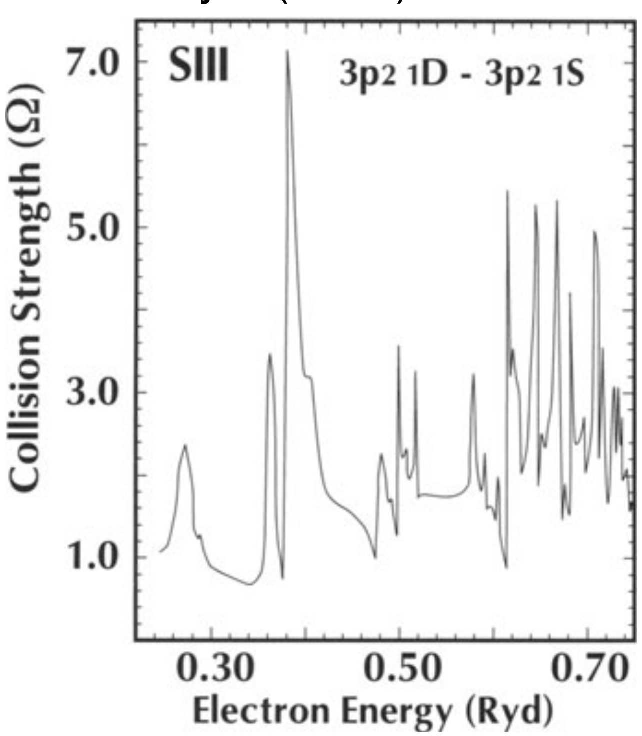
At around the critical density, the "line emissivity vs density" plotted in log-log scale changes slope from +2 to +1.

- Collision Strength
  - Quantum mechanical calculations show that (1) the resonance structure in the collision strengths is important and (2) the collision strength increases with energy for neutral species.

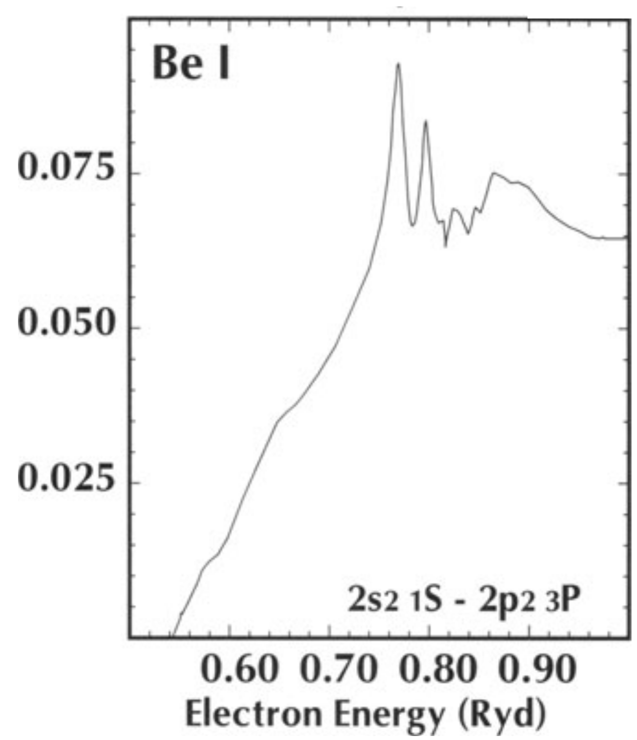


solid - Ramsbottom et al. (1996)
dashed - Cai & Pradhan (1993)

Tayal (1996)



Badnell (1999)



The ***effective collision strength,*** which is thermally averaged, has a value in a range of

$$\langle \Omega_{u\ell} \rangle = \int_0^\infty \Omega_{u\ell}(E) e^{-E/k_\mathrm{B}T} d(E/k_\mathrm{B}T)$$

$$10^{-2} < \langle \Omega_{u\ell} \rangle < 10$$

See Table F.1 to F.5 in [Draine]

- As can be seen in Tables and the formula, collisional de-excitation is negligible for resonance and most forbidden lines in the ISM.

Ion	ℓ	u	E_ℓ/k (K)	E_u/k (K)	$\lambda_{u\ell}$ (μm)	$n_{\rm H,crit}(u)$ $T = 100\,$K (cm^{-3})	$T = 5000\,$K (cm^{-3})
C II	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	91.21	157.74	2.0×10^3	1.5×10^3
C I	$^3\mathrm{P}_0$	$^3\mathrm{P}_1$	0	23.60	609.7	620	160
	$^3\mathrm{P}_1$	$^3\mathrm{P}_2$	23.60	62.44	370.37	720	150
O I	$^3\mathrm{P}_2$	$^3\mathrm{P}_1$	0	227.71	63.185	2.5×10^5	4.9×10^4
	$^3\mathrm{P}_1$	$^3\mathrm{P}_0$	227.71	326.57	145.53	2.3×10^4	8.4×10^3
Si II	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	413.28	34.814	1.0×10^5	1.1×10^4
Si I	$^3\mathrm{P}_0$	$^3\mathrm{P}_1$	0	110.95	129.68	4.8×10^4	2.7×10^4
	$^3\mathrm{P}_1$	$^3\mathrm{P}_2$	110.95	321.07	68.473	9.9×10^4	3.5×10^4

Table 17.1 in [Draine]

- However, it is not true for the 21 cm hyperfine structure line of hydrogen.
  - The critical density for 21cm line is

$$n_{\rm crit} \sim 10^{-3}\,(T/100\,{\rm K})^{-1/2}\ \ [{\rm cm}^{-3}]$$

$$A_{u\ell} = 2.88 \times 10^{-15}\ [{\rm s}^{-1}]$$

  - The hyperfine levels are thus essentially in collisional equilibrium in the CNM.

The collisional strengths and other atomic data are available in the CHIANTI atomic database (https://www.chiantidatabase.org/).

Collision strengths at T = 10⁴ K

Table 4.1 in The Interstellar Medium [Lequeux]

Ion	Transition l–u	λ µm	A_{ul} s^{-1}	Ω_{ul}	n_{crit} cm^{-3}
C I	$^3\mathrm{P}_0$–$^3\mathrm{P}_1$	609.1354	7.93×10^{-8}	–	(500)
	$^3\mathrm{P}_1$–$^3\mathrm{P}_2$	370.4151	2.65×10^{-7}	–	(3000)
C II	$^2\mathrm{P}_{1/2}$–$^2\mathrm{P}_{3/2}$	157.741	2.4×10^{-6}	1.80	47 (3000)
N II	$^3\mathrm{P}_0$–$^3\mathrm{P}_1$	205.3	2.07×10^{-6}	0.41	41
	$^3\mathrm{P}_1$–$^3\mathrm{P}_2$	121.889	7.46×10^{-6}	1.38	256
	$^3\mathrm{P}_2$–$^1\mathrm{D}_2$	0.65834	2.73×10^{-3}	2.99	7700
	$^3\mathrm{P}_1$–$^1\mathrm{D}_2$	0.65481	9.20×10^{-4}	2.99	7700
N III	$^2\mathrm{P}_{1/2}$–$^2\mathrm{P}_{3/2}$	57.317	4.8×10^{-5}	1.2	1880
O I	$^3\mathrm{P}_2$–$^3\mathrm{P}_1$	63.184	8.95×10^{-5}	–	2.3×10^4 (5×10^5)
	$^3\mathrm{P}_1$–$^3\mathrm{P}_0$	145.525	1.7×10^{-5}	–	3400 (1×10^5)
	$^3\mathrm{P}_2$–$^1\mathrm{D}_2$	0.63003	6.3×10^{-3}	–	1.8×10^6
O II	$^4\mathrm{S}_{3/2}$–$^2\mathrm{D}_{5/2}$	0.37288	3.6×10^{-5}	0.88	1160
	$^4\mathrm{S}_{3/2}$–$^2\mathrm{D}_{3/2}$	0.37260	1.8×10^{-4}	0.59	3890
O III	$^3\mathrm{P}_0$–$^3\mathrm{P}_1$	88.356	2.62×10^{-5}	0.39	461
	$^3\mathrm{P}_1$–$^3\mathrm{P}_2$	51.815	9.76×10^{-5}	0.95	3250
	$^3\mathrm{P}_2$–$^1\mathrm{D}_2$	0.50069	1.81×10^{-2}	2.50	6.4×10^5
	$^3\mathrm{P}_1$–$^1\mathrm{D}_2$	0.49589	6.21×10^{-3}	2.50	6.4×10^5
	$^1\mathrm{D}_2$–$^1\mathrm{S}_0$	0.43632	1.70	0.40	2.4×10^7
Ne II	$^2\mathrm{P}_{1/2}$–$^2\mathrm{P}_{3/2}$	12.8136	8.6×10^{-3}	0.37	5.9×10^5
Ne III	$^3\mathrm{P}_2$–$^3\mathrm{P}_1$	15.5551	3.1×10^{-2}	0.60	1.27×10^5
	$^3\mathrm{P}_1$–$^3\mathrm{P}_0$	36.0135	5.2×10^{-3}	0.21	1.82×10^4
Si II	$^2\mathrm{P}_{1/2}$–$^2\mathrm{P}_{3/2}$	34.8152	2.17×10^{-4}	7.7	(3.4×10^5)
S II	$^4\mathrm{S}_{3/2}$–$^2\mathrm{D}_{5/2}$	0.67164	2.60×10^{-4}	4.7	1240
	$^4\mathrm{S}_{3/2}$–$^2\mathrm{D}_{3/2}$	0.67308	8.82×10^{-4}	3.1	3270
S III	$^3\mathrm{P}_0$–$^3\mathrm{P}_1$	33.4810	4.72×10^{-4}	4.0	1780
	$^3\mathrm{P}_1$–$^3\mathrm{P}_2$	18.7130	2.07×10^{-3}	7.9	1.4×10^4
S IV	$^2\mathrm{P}_{1/2}$–$^2\mathrm{P}_{3/2}$	10.5105	7.1×10^{-3}	8.5	5.0×10^4
Ar II	$^2\mathrm{P}_{1/2}$–$^2\mathrm{P}_{3/2}$	6.9853	5.3×10^{-2}	2.9	1.72×10^6
Ar III	$^3\mathrm{P}_2$–$^3\mathrm{P}_1$	8.9914	3.08×10^{-2}	3.1	2.75×10^5
	$^3\mathrm{P}_1$–$^3\mathrm{P}_0$	21.8293	5.17×10^{-3}	1.3	3.0×10^4
Fe II	$^6\mathrm{D}_{7/2}$–$^6\mathrm{D}_{5/2}$	35.3491	1.57×10^{-3}	–	(3.3×10^6)
	$^6\mathrm{D}_{9/2}$–$^6\mathrm{D}_{7/2}$	25.9882	2.13×10^{-3}	–	(2.2×10^6)

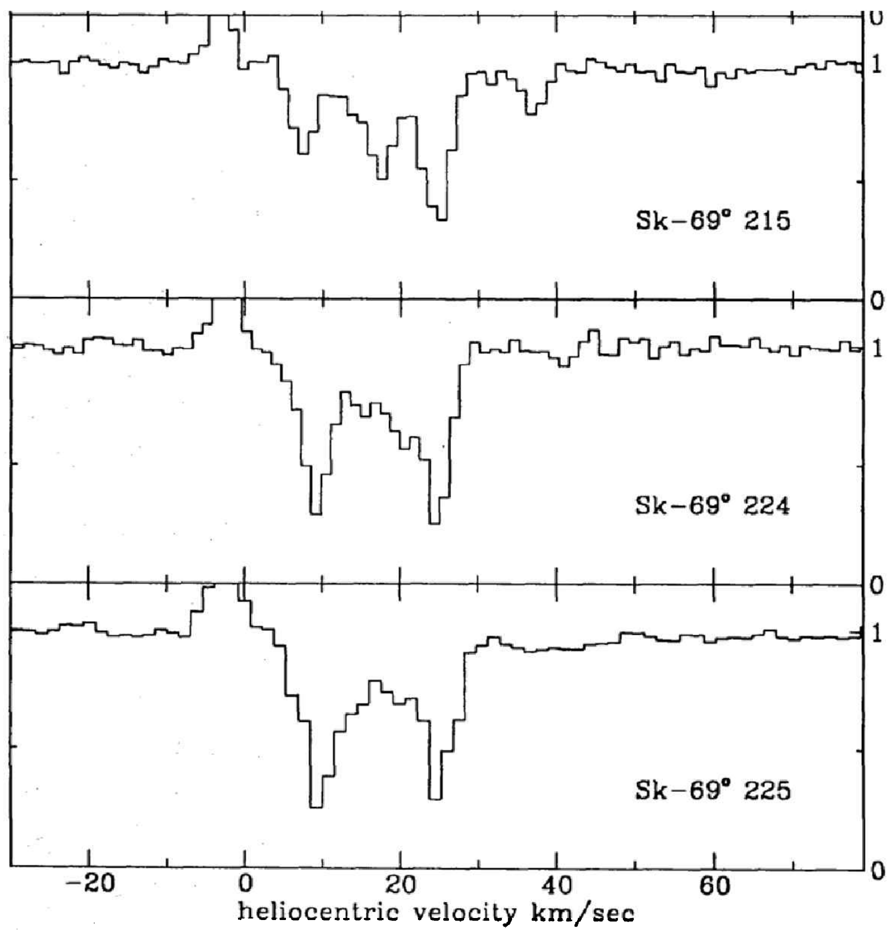
Overall Properties of the Cold Neutral Medium (CNM)

- Overall properties of the CNM
  - Temperature $T \sim 100\,\mathrm{K}$
  - Mean kinetic energy per particle $\langle E \rangle = (3/2)kT \sim 0.013\,\mathrm{eV}$
  - Number density
    - $n_{\mathrm{atom}} \sim 30\,\mathrm{cm}^{-3}$ for atoms
    - $n_e \sim 0.04\,\mathrm{cm}^{-3}$ for free electrons
  - Thermal velocity
    - $v_{\mathrm{th}}(\mathrm{H}) \sim 1.6\,\mathrm{km\,s^{-1}}$ for hydrogen atoms
    - $v_{\mathrm{th}}(e) \sim 67\,\mathrm{km\,s^{-1}}$ for free electrons
  - Mean free path
    - $\lambda_{\mathrm{mfp}}(\mathrm{HH}) \sim 0.74\,\mathrm{AU}$ for atom-atom collisions
    - $\lambda_{\mathrm{mfp}}(e\mathrm{H}) \sim 1700\,\mathrm{AU}$ for atom-electron collisions
    - $\lambda_{\mathrm{mfp}}(ee) \sim 1.9 \times 10^{-3}\,\mathrm{AU}$ for electron-electron collisions
  - Collisional time scale
    - $t_{\mathrm{coll}}(\mathrm{HH}) \sim 2.2\,\mathrm{yr}$ for atom-atom collisions
    - $t_{\mathrm{coll}}(e\mathrm{H}) \sim 120\,\mathrm{yr}$ for atom-electron collisions
    - $t_{\mathrm{coll}}(ee) \sim 1.2\,\mathrm{hr}$ for electron-electron collision

See lecture05 - collisional time scale.pdf
for the detailed calculations of the numerical values

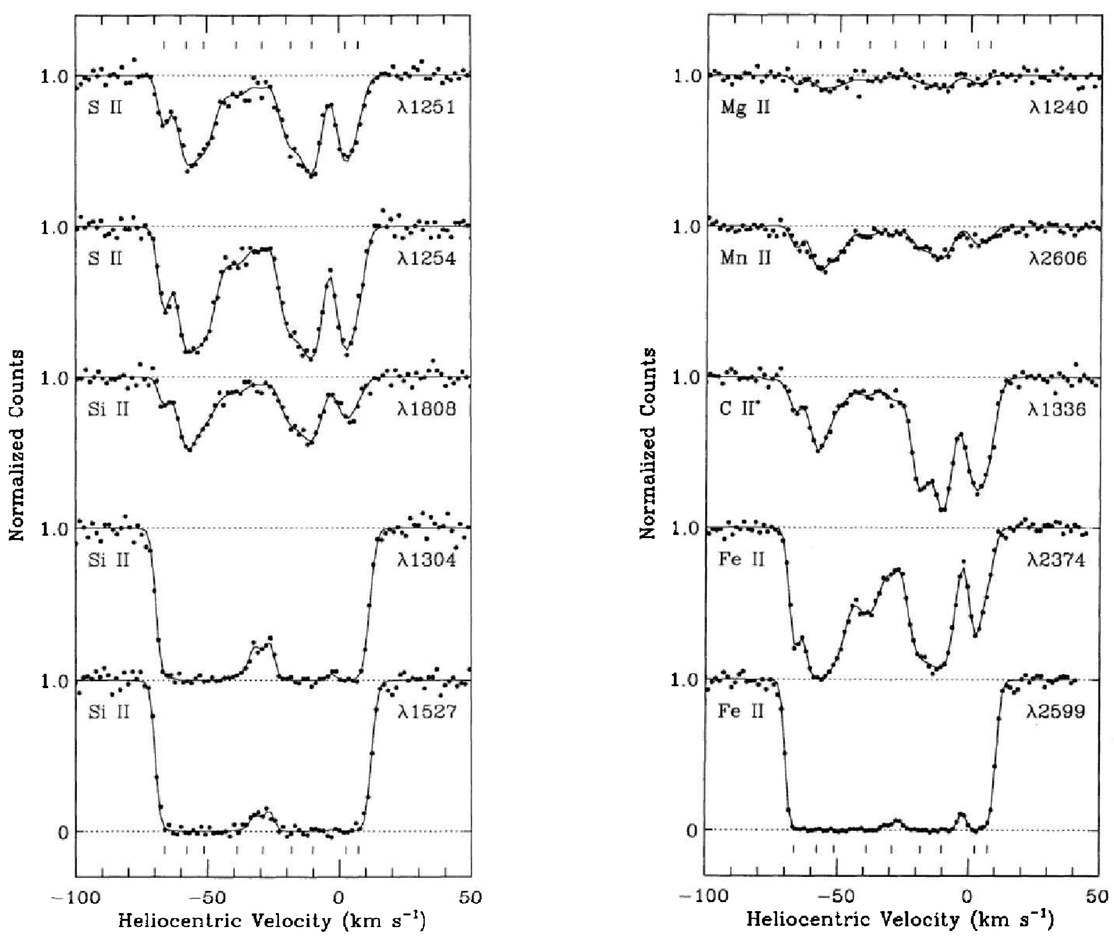
Observations of Absorption Lines Toward the CNM

- The CNM gives rise to a number of absorption features in the spectra of hot background stars (and quasars).
  - The most prominent absorption lines at visible wavelengths are Ca II K and H lines at $\lambda = 3933, 3968$ Å, and Na I D_1 and D_2 doublet lines at $\lambda = 5890, 5896$ Å.



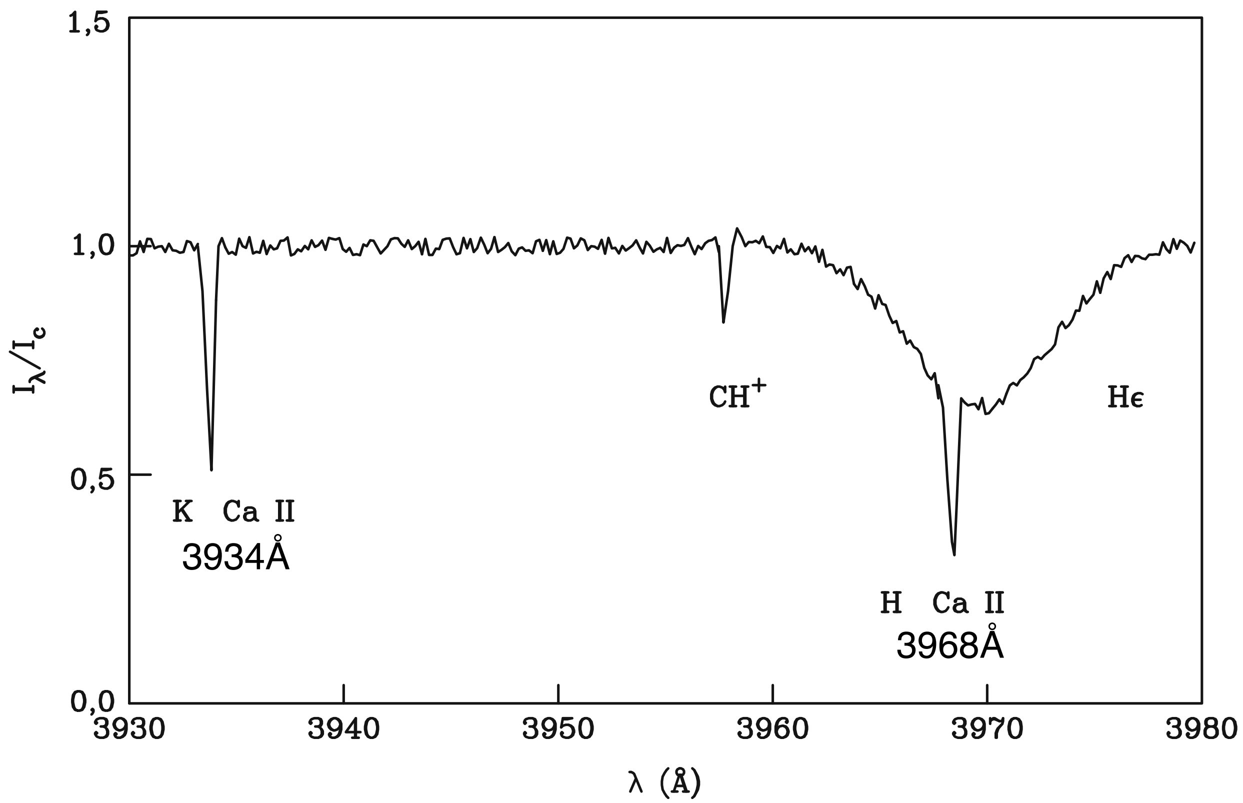
Na I D_2 interstellar absorption line seen along 3 lines of sight to stars in LMC (Molaro et al. 1993)

[Note] The cold gas is ~100 pc away from Earth, meaning that 5 arcmin corresponds to ~ 0.15 pc.



UV interstellar absorption lines toward an O-type star HD93521. (Spitzer & Fitzpatrick 1993)

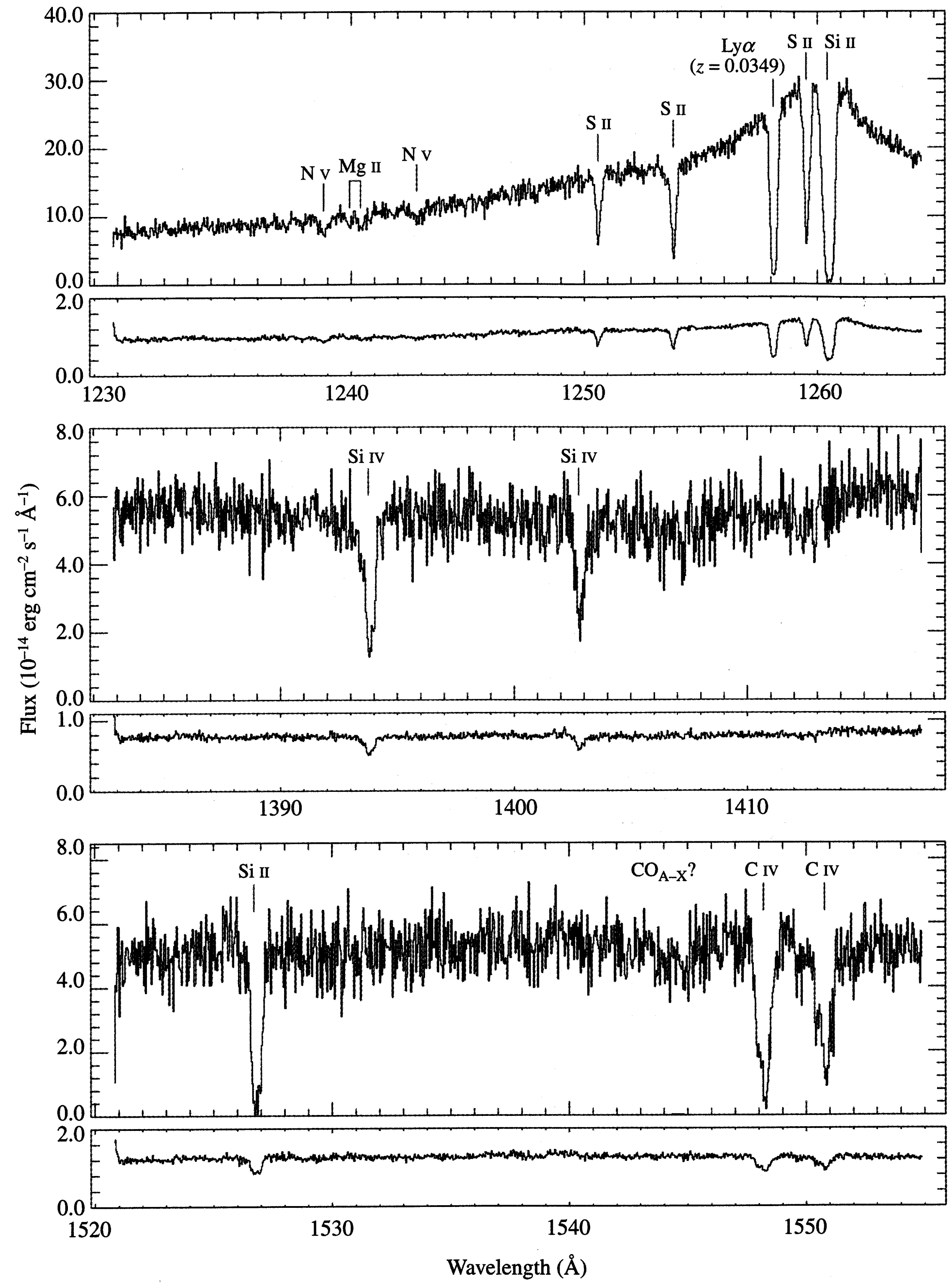
[Note] (1) multiple velocity components and (2) line saturation on Si II and Fe II.
The multiple velocities are due primarily to the differential rotation of our galaxy. (clouds at different distances)



Interstellar absorption lines in the spectrum of ζ Oph (O9.5V).

Note that the Ca II H line occurs inside the Hε hydrogen line, which is much broader and of stellar origin.

Figure 4.6 in Astrophysics of the Interstellar Medium [Maciel]



Interstellar absorption lines toward the Seyfert 1 galaxy ESO 141-055.

Figure 5.5 in Physics and Chemistry of the Interstellar Medium [Kwok]

- The alkali metals (Li, K, and Na) and alkaline earth metals (Ca) produces absorption lines at visible wavelengths ($4000\,\text{Å} < \lambda < 7300\,\text{Å}$, $1.7\,\text{eV} < E < 3.1\,\text{eV}$); these elements have loosely bound outer electrons.

- Most other elements produce UV absorption lines ($\lambda < 4000\,\text{Å}$, $E > 3.1\,\text{eV}$).
  - Therefore, the study of the CNM was extensively made by the launch of orbiting UV telescopes (Corpernicus, IUE, etc).
  - In particular, Lyα ($\lambda = 1215.67\,\text{Å}$; $E = 10.2\,\text{eV}$) from hydrogen.

- Interstellar absorption lines at visible wavelengths were also found from neutral atoms such as Ca I, K I, Li I, ions such as Ti II, and diatomic molecules such as CH, NH, CN, CH^+ and C_2.
  - [Note] The first discovery of interstellar molecules was made by the detection of CH absorption at $\lambda \sim 4300\,(4315)\text{Å}$ (Swings & Rosenfeld 1937), not at radio wavelengths.
  - CH, NH, and CN are referred to as "**radicals**", in chemistry, meaning molecules that contain at least one unpaired electron. They quickly combine with one another, or with single atoms in laboratory. But, in the low density of the ISM, they have long lifetimes.

- The composition and excitation of interstellar gas can be studied using absorption lines that appear in the spectra of background stars (or other sources).
- The interstellar lines are typically narrow compared to spectral features produced by absorption in stellar photospheres, and in practice can be readily distinguished.
  - For instance, consider the Fraunhofer lines in the Sun's spectrum. The Ca H and K lines, with equivalent widths of 14Å and 19Å respectively, are the strongest absorption lines. The Na I D_1 and D_2 lines have equivalent widths of 0.6Å and 0.8Å.
  - For many interstellar lines, the equivalent width is sufficiently small that the mÅ is a convenient unit.
- It is normally possible to detect absorption only by the ground state (and perhaps the excited fine-structure levels of the ground electronic state) - the populations in the excited electronic states are too small to be detected in absorption.
- The widths of absorption lines are usually determined by Doppler broadening, with line widths of a few km s^{-1} (or $\Delta\lambda/\lambda \approx 10^{-5}$) - often observed in cool clouds.

- Absorption lines (and emission lines) contains a lots of information about number density, temperature, chemical abundances, ionization states, and excitation states.
- However, interpreting the information requires understanding the ways in which light interacts with baryonic matter, radiative transfer.
- **We need to know the line profile to analyze absorption lines.**

Optical Depth

- The optical depth in an absorption line can be written

$$\boxed{\tau_\nu = \frac{\pi e^2}{m_e c} f_{\ell u} \left(1 - \frac{n_u/g_u}{n_\ell/g_\ell}\right) N_\ell \phi_\nu \simeq \frac{\pi e^2}{m_e c} f_{\ell u} N_\ell \phi_\nu}$$

Here, $N_\ell \equiv \int n_\ell ds$ is the column density of the absorbers.

The line profile is given by $\phi_\nu = \frac{1}{\Delta\nu_{\rm D}\sqrt{\pi}} H(u,a)$, and its value at the line center is

$$\phi_\nu(\nu = \nu_{\ell u}) = \frac{1}{\nu_{\ell u}(b/c)\sqrt{\pi}} H(0,a)$$
$$\approx \frac{1}{\nu_{\ell u}(b/c)\sqrt{\pi}}$$

$$u = \frac{\nu - \nu_{\ell u}}{\Delta\nu_{\rm D}} = \frac{\nu - \nu_{u\ell}}{\nu_{\ell u}(b/c)}$$
$$= \frac{v}{b} \quad \left(v = \frac{\nu - \nu_{\ell u}}{\nu_{\ell u}} c, \;\; b = \sqrt{2} v_{\rm rms} = \sqrt{\frac{2k_{\rm B}T}{m}}\right)$$

The correction factor for stimulated emission is negligible for the optical lines. Then, dropping the correction factor, the optical depth can be written

$$\tau_\nu = \tau_0 H(u,a)$$

Here, τ_0 is the optical depth at the line center.

$$\tau_0 = \frac{\sqrt{\pi} e^2}{m_e c} f_{\ell u} \frac{\lambda_{\ell u}}{b} N_\ell$$

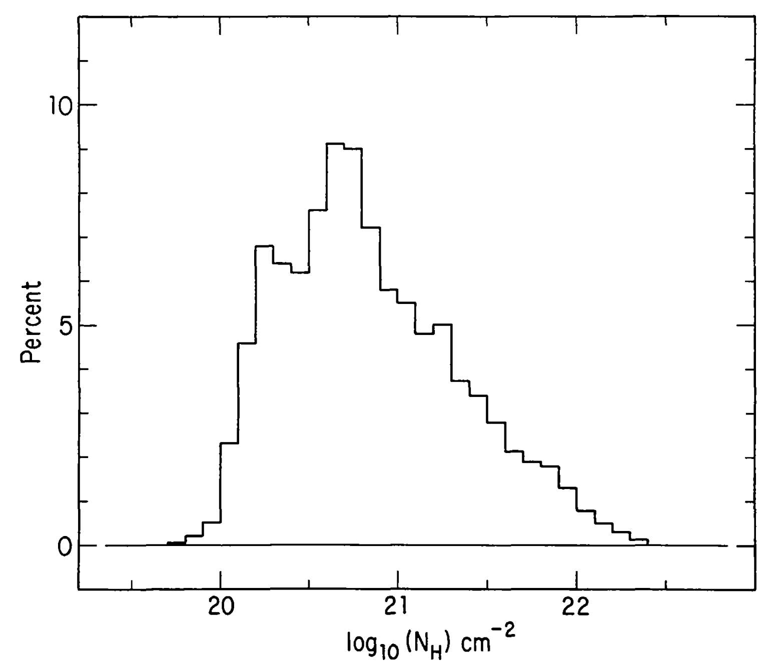
- The central optical depth for Lyα is

$$\boxed{\tau_0 = 0.7580 \left(\frac{N_\ell}{10^{13}\,\mathrm{cm}^{-2}}\right)\left(\frac{f_{\ell u}}{0.4164}\right)\left(\frac{\lambda_{\ell u}}{1215.67\,\text{Å}}\right)\left(\frac{10\,\mathrm{km\,s}^{-1}}{b}\right)}$$

  - In the WNM, Lyα will be optically thin $(\tau_0 < 1)$ when $N_\ell < 10^{13}\,\mathrm{cm}^{-2}$ and optically thick $(\tau_0 > 1)$ when $N_\ell > 10^{13}\,\mathrm{cm}^{-2}$.
  - In the CNM, Lyα will be optically thin when $N_\ell < 10^{12}\,\mathrm{cm}^{-2}$ and optically thick when $N_\ell > 10^{12}\,\mathrm{cm}^{-2}$.
  - In Milky Way, the total column density of hydrogen atom is $N_\ell \sim 10^{20} - 10^{21}\,\mathrm{cm}^{-2}$.

The percentage of the sky covered by H I at a given N_H.

Figure 4 in Dickey & Lockman (1990, ARA&A)



  - As a reference, the column density of the Earth's atmosphere, looking upward from sea level, is $N \sim 2 \times 10^{25}\,\mathrm{cm}^{-2}$.

Absorption Line Shapes

- Lyman α absorption line profiles for $b = 10\,\mathrm{km\,s^{-1}}$

  $$F_\nu/F_\nu(0) = e^{-\tau_0 H(u,a)}$$

  - When $\tau_0 < 1$, $F_\nu/F_\nu(0) \approx 1 - \tau_\nu$ and thus the shape of an absorption line resembles the upside-down Voight function.
  - When $\tau_0 \gg 1$, the absorption line saturates at its center and becomes increasingly "box-shaped."

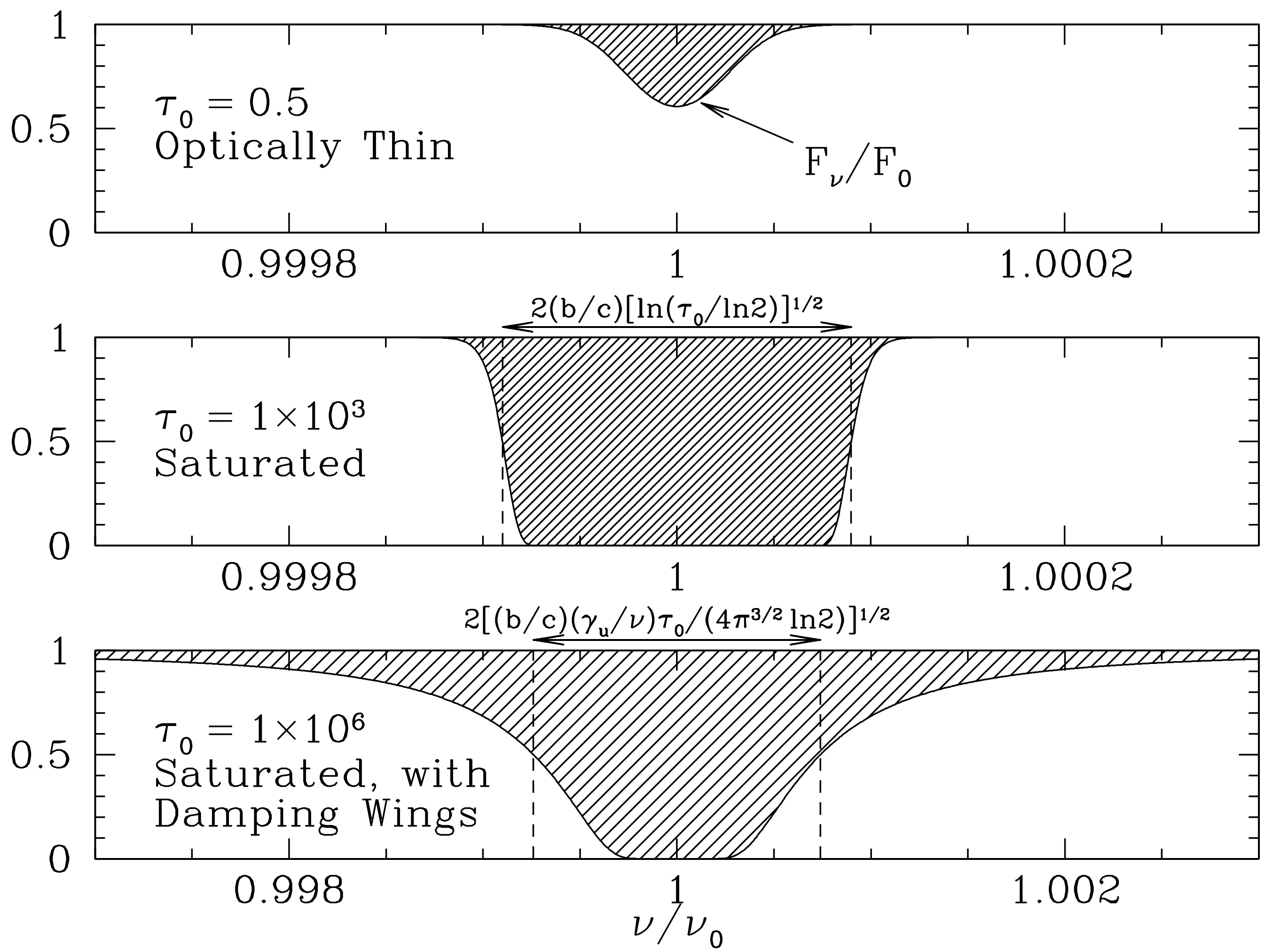
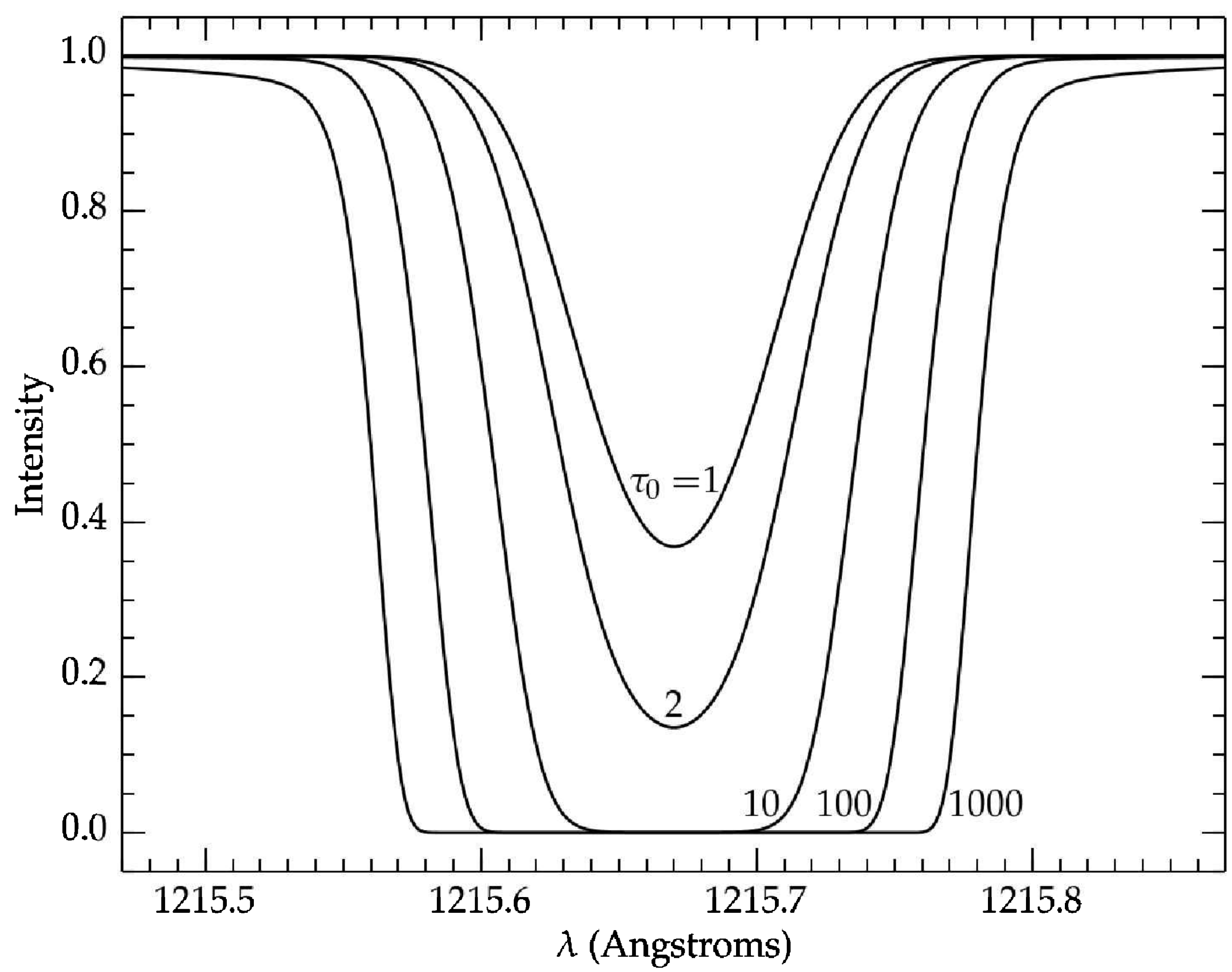


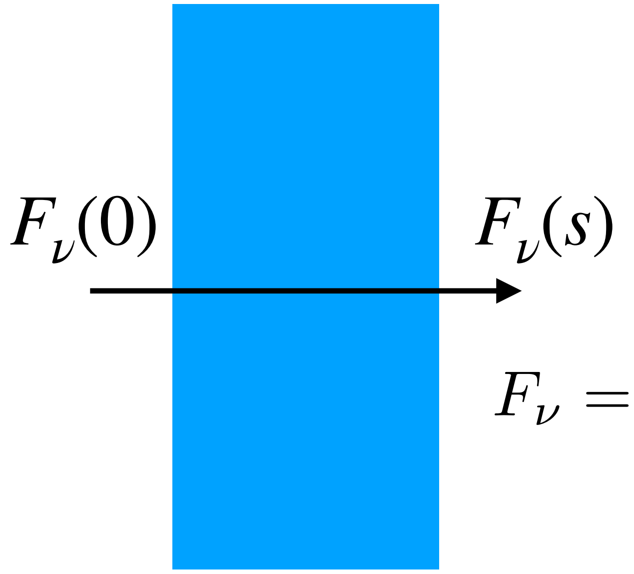
Figure 9.1 in [Draine]
Note the different abscissa in the lowest panel.



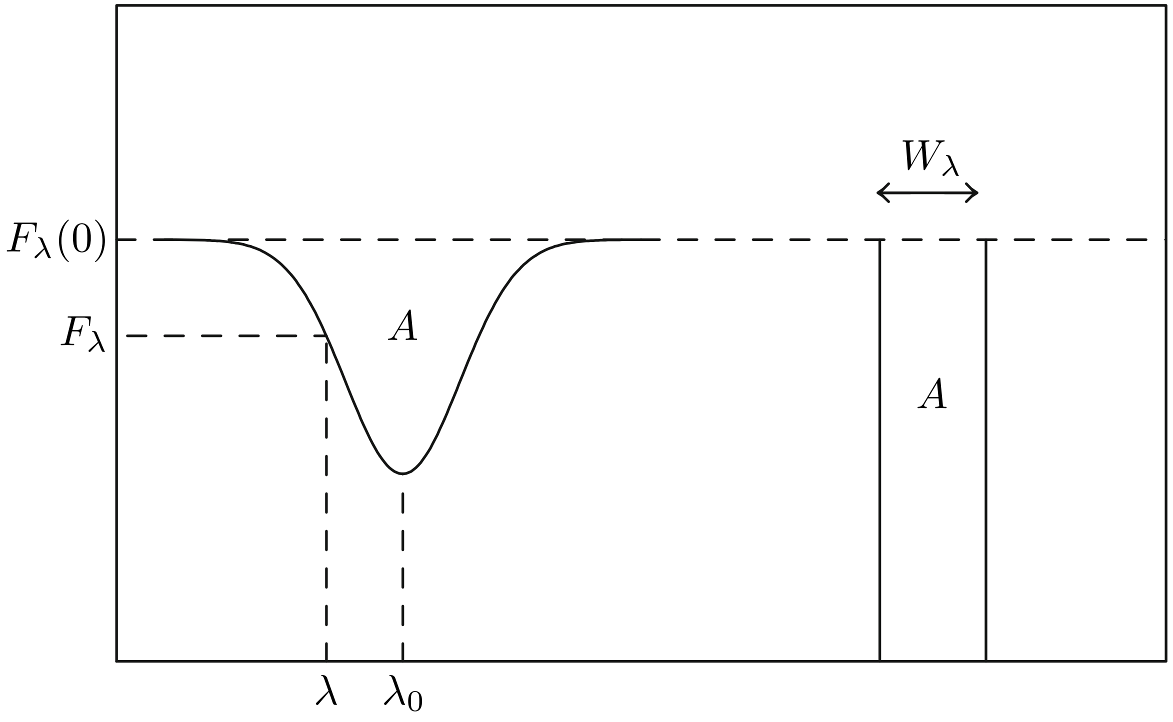
Lyman α absorption lines for b = 10 km s^{-1}.
Figure 2.6 in [Ryden]

Equivalent Width & Curve of growth

- ***Equivalent width***
  - The spectrograph often lack the spectral resolution to resolve the profiles of narrow lines, but can measure the total amount of “missing power” resulting from a narrow absorption line.
  - *The equivalent width is a measure of the strength of an absorption line, in terms of “missing power” in the unresolved absorption line.*



$$F_\nu = F_\nu(0)e^{-\tau_\nu}$$



- ***Curve of growth***
  - The curve of growth refers to the numerical relation between the observed equivalent width and the underlying optical depth (or the column density) of the absorber.

Equivalent Width

- Suppose that we measure the energy flux density F_ν using an aperture of solid angle $\Delta\Omega$. Then, we obtain the flux density at the observer:

$$F_\nu = F_\nu(0)e^{-\tau_\nu} + B_\nu(T_{\rm exc})\Delta\Omega(1 - e^{-\tau_\nu})$$

Here, $F_\nu(0)$ is the flux density of the background light source.

- At optical frequencies and in the neutral medium, nearly all atoms are in their ground state. Thus, we normally have $n_u/n_\ell \ll 1,\ B_\nu(T_{\rm exc})\Delta\Omega \ll F_\nu(0)$. Then, we can neglect the emission from the ISM.

$$\frac{h\nu}{k_{\rm B}T_{\rm exc}} = \frac{6000\,\text{Å}}{\lambda}\frac{2.4\times10^4\,{\rm K}}{T_{\rm exc}}$$

$$F_\nu = F_\nu(0)e^{-\tau_\nu}$$

- If the background spectrum is smooth, we can define the ***dimensionless equivalent width*** and the ***wavelength equivalent width*** as follows:

$$W \equiv \int \frac{d\nu}{\nu_0}\left[1 - \frac{F_\nu}{F_\nu(0)}\right] = \int \frac{d\nu}{\nu_0}\left(1 - e^{-\tau_\nu}\right)$$

$$W_\lambda \equiv \int d\lambda\left(1 - e^{-\tau_\lambda}\right) \approx \lambda_0 W$$

  - The equivalent width is the width of a straight-sided, perfectly black absorption line that has the same integrated flux deficit as the actual absorption line.

Overall Shape of the Curve of Growth

The Curve of Growth

= the relation between optical depth at line center and equivalent width

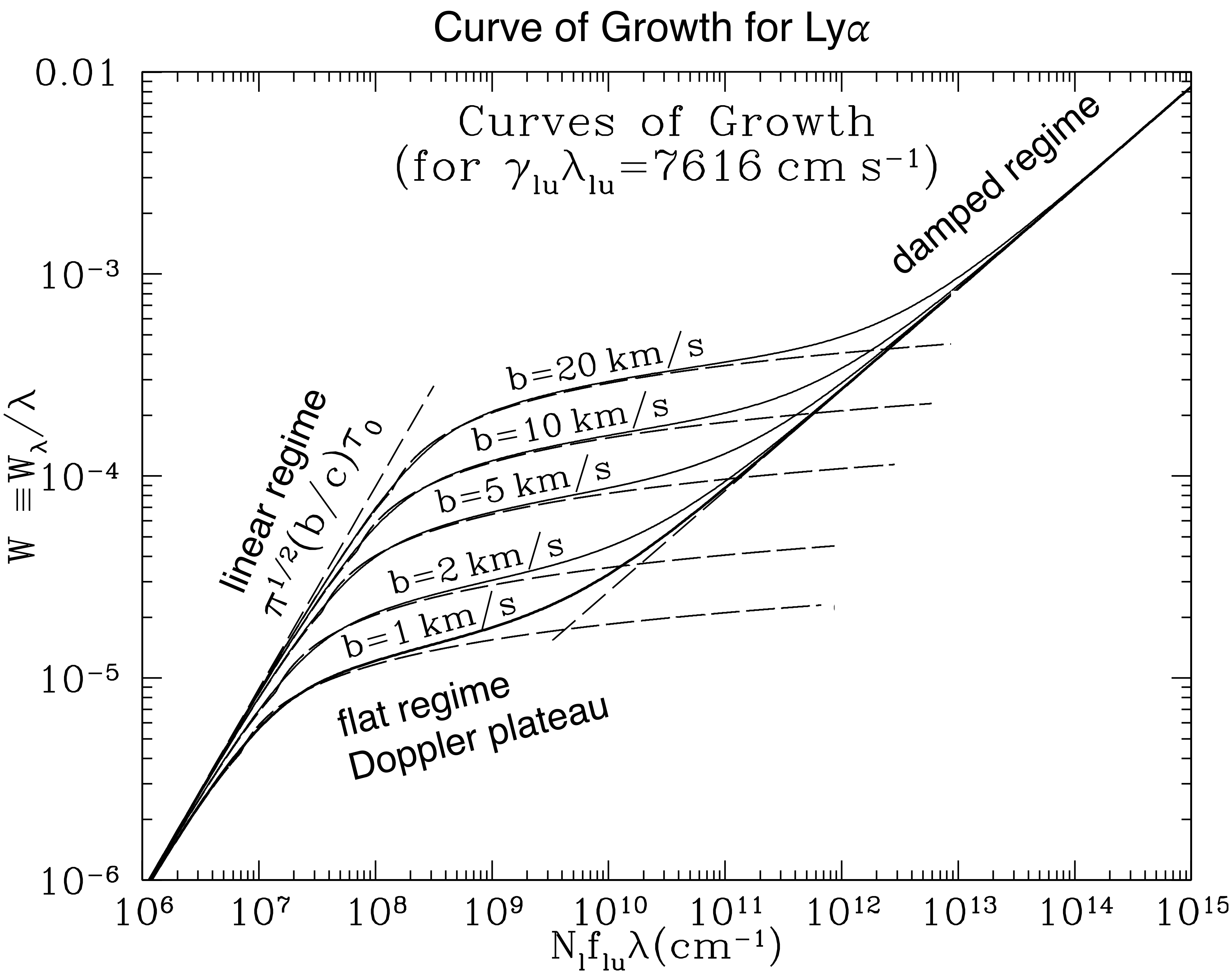


Figure 9.2 in [Draine]

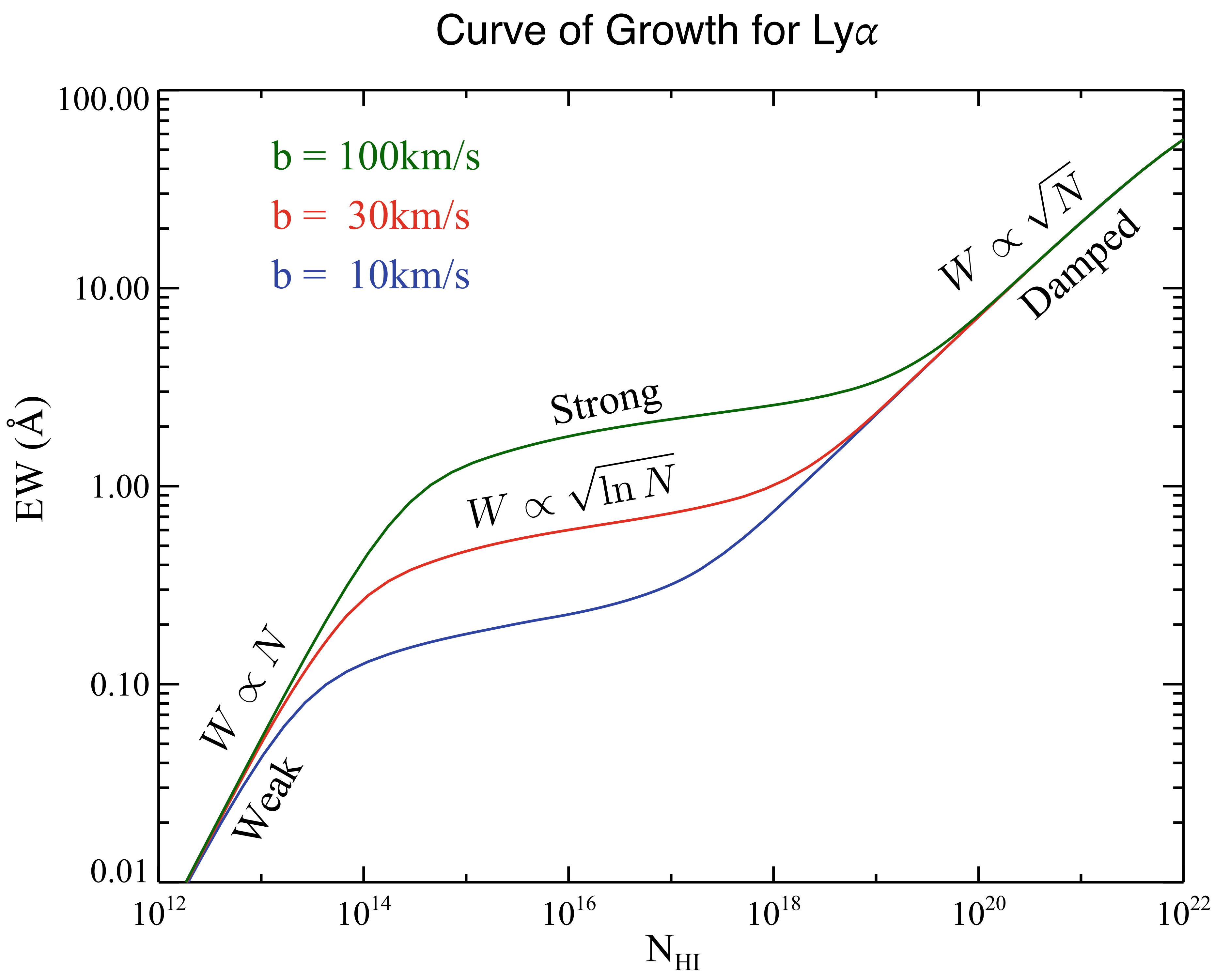


Figure 2.10 in Chap. 2 [Prochaska]
Lyman-alpha as An Astrophysical and Cosmological Tool

Detailed Analysis of The Curve of Growth

- Equivalent width: $$W = \int_{-\infty}^{\infty} \frac{d\nu}{\nu_0}\left(1 - e^{-\tau_\nu}\right) = \frac{b}{c}\int_{-\infty}^{\infty} du\left(1 - e^{-\tau_\nu}\right)$$

- ***Optically Thin Absorption,*** $\tau_0 \lesssim 1$ ***(linear regime)***

$$W = \frac{b}{c}\int_{-\infty}^{\infty} du\left(\tau_\nu - \frac{\tau_\nu^2}{2} + \cdots\right) \approx \frac{b}{c}\int_{-\infty}^{\infty} du\left(\tau_0 e^{-u^2} - \tau_0^2\frac{e^{-2u^2}}{2} + \cdots\right)$$

$$\tau_\nu = \tau_0 H(u,a) \approx \tau_0 e^{-u^2} \quad \text{if} \quad a \ll 1$$

$$= \sqrt{\pi}\frac{b}{c}\tau_0\left(1 - \frac{\tau_0}{2\sqrt{2}} + \cdots\right)$$

$$W \approx \sqrt{\pi}\frac{b}{c}\frac{\tau_0}{1 + \tau_0/(2\sqrt{2})} \longleftarrow 1 - x \approx \frac{1}{1+x}$$

$$= \frac{\pi e^2}{m_e c^2} N_\ell f_{\ell u}\lambda_{\ell u}\frac{1}{1 + \tau_0/(2\sqrt{2})} \longleftarrow \tau_0 = \frac{\sqrt{\pi}e^2}{m_e c} f_{\ell u}\frac{\lambda_{\ell u}}{b} N_\ell$$

$$W = 4.48 \times 10^{-6}\left(\frac{N_\ell}{10^{12}\,\mathrm{cm}^{-2}}\right)\left(\frac{f_{\ell u}}{0.4164}\right)\left(\frac{\lambda_{\ell u}}{1215.67\,\text{Å}}\right)$$

$$N_\ell = 1.84 \times 10^{12}\,\mathrm{cm}^{-2}\left(\frac{0.4164}{f_{\ell u}}\right)\left(\frac{1215.67\,\text{Å}}{\lambda_{\ell u}}\right)^2\left(\frac{W_\lambda}{0.01\,\text{Å}}\right) \quad \text{if} \quad \tau_0 \lesssim 1$$

$$\leftarrow \quad W \approx \frac{W_\lambda}{\lambda_{\ell u}}$$

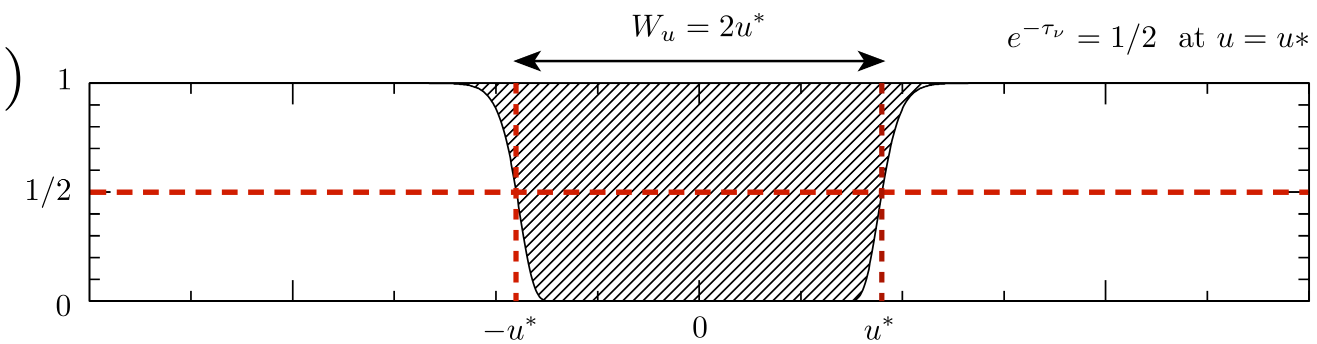
The measurement of *W* allows us to determine *N*, provided that the line is optical thin.

- ***Flat Portion of the Curve of Growth,*** $1 < \tau_0 \lesssim \tau_{\rm damp}$
  - Now consider what happens when an absorption line is optically thick, but not so optically thick that the the broad Lorentz wing ν^{-2} provide a significant contribution to the absorption.
  - The optical depth at which the wing become important is called the ***damping optical depth*** $\tau_{\rm damp}$.

$$W = \frac{b}{c}\int_{-\infty}^{\infty} du \left[1 - \exp\left(-\tau_0 e^{-u^2}\right)\right]$$

  - The absorption line shape is almost "box-shaped." We assume that the term in square brackets equals "1" until a certain value u_* and then, suddenly drops to "0".
  - We define u_* to be the location at half maximum of the square brackets:

$$\exp\left(-\tau_0 e^{-u_*^2}\right) = \frac{1}{2} \rightarrow u_*^2 = \ln\left(\tau_0 / \ln 2\right)$$



  - Then, we have

$$W \approx \frac{b}{c}\int_{-u_*}^{u_*} du = \frac{b}{c}(2u_*) \longrightarrow W \approx \frac{2b}{c}\sqrt{\ln\left(\tau_0/\ln 2\right)}$$

  - Note that W is very insensitive to τ_0 (and thus N_ℓ) in this regime. Because W increases so slowly with increasing N_ℓ, this is referred to as the flat portion of the curve of growth.

- Inverting the above equation, we obtain

$$\tau_0 \approx (\ln 2) \exp\left[\left(\frac{cW}{2b}\right)^2\right]$$

$$N_\ell \approx \frac{\ln 2}{\sqrt{\pi}} \frac{m_e c}{e^2} \frac{b}{f_{\ell u}\lambda_{\ell u}} \exp\left[\left(\frac{cW}{2b}\right)^2\right]$$

$$N_\ell \approx 9.15 \times 10^{12}\,\mathrm{cm}^{-2} \left(\frac{0.4164}{f_{\ell u}}\right)\left(\frac{1215.67\,\text{Å}}{\lambda_{\ell u}}\right)\left(\frac{b}{10\,\mathrm{km\,s^{-1}}}\right)$$
$$\times \exp\left[0.0152\left(\frac{1215.67\,\text{Å}}{\lambda_{\ell u}}\right)^2\left(\frac{10\,\mathrm{km\,s^{-1}}}{b}\right)^2\left(\frac{W_\lambda}{0.01\,\text{Å}}\right)^2\right]$$

- The column density at a given equivalent width depends on the temperature, and thus on the thermal broadening.
- Any error in evaluating W_λ (from misestimating the continuum flux, for instance) will propagate exponentially into an error in N_ℓ.
- Therefore, it is advised not to use the above equation unless you have a very good idea of what the equivalent width W_λ and the thermal broadening b are for the line in question.

- ***Damped Portion of the Curve of Growth,*** $\tau_0 \gtrsim \tau_{\rm damp}$ ***(square-root regime)***
  - In this regime, the Doppler core of the line is totally saturated, but the "damping wing" of the Voigt profile start to contribute significantly to the equivalent width.

$$W = \frac{b}{c}\int_{-\infty}^{\infty} du \left[1 - \exp\left(-\tau_0 \frac{a}{\sqrt{\pi}u^2}\right)\right]$$

change of variables: Let $\tau_0 \dfrac{a}{\sqrt{\pi}u^2} = \dfrac{1}{x^2} \quad \rightarrow \quad u = \left(\dfrac{\tau_0 a}{\sqrt{\pi}}\right)^{1/2} x$

$$W = \frac{b}{c}\left(\frac{\tau_0 a}{\sqrt{\pi}}\right)^{1/2} \int_{-\infty}^{\infty} dx \left[1 - \exp\left(-1/x^2\right)\right] = \frac{b}{c}\left(\frac{\tau_0 a}{\sqrt{\pi}}\right)^{1/2} 2\sqrt{\pi}$$

$$\begin{aligned} I &\equiv \int_{-\infty}^{\infty} dx \left(1 - e^{-1/x^2}\right) = 2\int_0^{\infty} dx \left(1 - e^{-1/x^2}\right) \\ &= 2\int_0^{\infty} \frac{dy^2}{y^2}\left(1 - e^{-y^2}\right) \\ &= 2\left[-\frac{1}{y}\left(1 - e^{-y^2}\right)\right]_0^{\infty} + 2\int_0^{\infty} \frac{1}{y}\left(2ye^{-y^2}\right) dy \\ &= \lim_{y\to 0} \frac{2}{y}\left(1 - e^{-y^2}\right) + 4\int_0^{\infty} e^{-y^2} dy \\ &= \lim_{y\to 0} \frac{2}{y}\left(1 - 1 + y^2\right) + 2\int_{-\infty}^{\infty} e^{-y^2} dy = 2\sqrt{\pi} \end{aligned}$$

Therefore, we have

$$W = \frac{b}{c}\left(4\sqrt{\pi}\tau_0 a\right)^{1/2}$$

$$a = \frac{\gamma_{\ell u}}{4\pi}\frac{1}{\nu_{\ell u}(b/c)} = \frac{\gamma_{\ell u}}{4\pi}\frac{\lambda_{\ell u}}{b}$$

$$W = \sqrt{\frac{b}{c}\frac{\tau_0}{\sqrt{\pi}}\frac{\gamma_{\ell u}\lambda_{\ell u}}{c}} \longrightarrow \tau_0 = \sqrt{\pi}\frac{c}{b}\frac{c}{\gamma_{\ell u}\lambda_{\ell u}} W^2$$

- In the book of Draine, he chooses the FWHM interval for the integration.

$$\exp\left(-\tau_0 \frac{a}{\sqrt{\pi}u_*^2}\right) = \frac{1}{2} \longrightarrow u_*^2 = \frac{\tau_0}{\sqrt{\pi}} \frac{a}{\ln 2}$$

$$a = \frac{\gamma_{\ell u}}{4\pi} \frac{1}{\nu_{\ell u}(b/c)} = \frac{\gamma_{\ell u}}{4\pi} \frac{\lambda_{\ell u}}{b}$$

$$W \approx \frac{b}{c} \int_{-u_*}^{u_*} du = \frac{b}{c}(2u_*) \longrightarrow W \approx \frac{2b}{c} \sqrt{\frac{\tau_0}{\sqrt{\pi}} \frac{a}{\ln 2}}$$

$$= \frac{1}{\sqrt{\pi \ln 2}} \sqrt{\frac{b}{c} \frac{\tau_0}{\sqrt{\pi}} \frac{\gamma_{\ell u}\lambda_{\ell u}}{c}}$$

- He note that this value is smaller than by a factor of $\sqrt{\pi \ln 2} = 1.476$. Multiplying by this factor, he obtain the same result as ours.
- This is equivalent to choosing the followings:

$$\exp\left(-\tau_0 \frac{a}{\sqrt{\pi}u_*^2}\right) = \exp\left(-\frac{1}{\pi}\right) = 0.7274$$

$$u_*^2 = \frac{\tau_0}{\sqrt{\pi}}(\pi a) = \pi \ln 2 \left(\frac{\tau_0}{\sqrt{\pi}} \frac{a}{\ln 2}\right)$$

$$N_\ell = \frac{m_e c^3}{e^2} \frac{1}{f_{\ell u} \gamma_{\ell u} \lambda_{\ell u}^2} W^2 = \frac{m_e c^3}{e^2} \frac{1}{f_{\ell u} \gamma_{\ell u} \lambda_{\ell u}^4} W_\lambda^2$$

$$N_\ell = 1.867 \times 10^{18}\,\mathrm{cm}^{-2} \left(\frac{0.4164}{f_{\ell u}}\right) \left(\frac{6.265 \times 10^8\,\mathrm{s}^{-1}}{\gamma_{\ell u}}\right) \left(\frac{1215.67\,\text{Å}}{\lambda_{\ell u}}\right)^4 \left(\frac{W_\lambda}{1\,\text{Å}}\right)^2$$

- The equivalent width is proportional to the square-root of the optical depth. The column density (optical depth) is proportional to the square of the measured equivalent width.
- Furthermore, the column density is independent of the thermal broadening.

• The ***damping optical depth*** at which the transition from the flat to the damped portion of the curve of growth occurs are obtained by setting $W^{\rm flat} = W^{\rm sq.-root}$.

$$\frac{2b}{c}\sqrt{\ln\left(\tau_{\rm damp}/\ln 2\right)} = \sqrt{\frac{b}{c}\frac{\tau_{\rm damp}}{\sqrt{\pi}}\frac{\gamma_{\ell u}\lambda_{\ell u}}{c}}$$

$$\begin{aligned}\tau_{\rm damp} &= 4\sqrt{\pi}\frac{b}{\gamma_{\ell u}\lambda_{\ell u}}\ln\left(\tau_{\rm damp}/\ln 2\right) \\ &\approx 4\sqrt{\pi}\frac{b}{\gamma_{\ell u}\lambda_{\ell u}}\ln\left(\frac{4\sqrt{\pi}}{\ln 2}\frac{b}{\gamma_{\ell u}\lambda_{\ell u}}\right)\end{aligned}$$

An approximate solution for

$$x = C\ln(x/\ln 2) \text{ with } C \gg 1$$

is $x \approx C\ln(C/\ln 2)$.

This solution is found to underestimate $\tau_{\rm damp}$ by a factor of ~1.4 (~1.3) for b = 1 (10) km s^{-1}.

$$C \equiv 4\sqrt{\pi}\frac{b}{\gamma_{\ell u}\lambda_{\ell u}} = 93.1\left(\frac{b}{1\,\mathrm{km\,s^{-1}}}\right)\left(\frac{6.265\times10^{8}\,\mathrm{s^{-1}}}{\gamma_{\ell u}}\right)\left(\frac{1215.67\,\text{Å}}{\lambda_{\ell u}}\right)$$

$$\begin{aligned}
\tau_{\mathrm{damp}} &\approx 93.1\left(\frac{b}{1\,\mathrm{km\,s^{-1}}}\right)\left(\frac{7616\,\mathrm{cm\,s^{-1}}}{\gamma_{\ell u}\lambda_{\ell u}}\right)\ln\left[134\left(\frac{b}{1\,\mathrm{km\,s^{-1}}}\right)\left(\frac{7616\,\mathrm{cm\,s^{-1}}}{\gamma_{\ell u}\lambda_{\ell u}}\right)\right] \\
&= 456\left(\frac{b}{1\,\mathrm{km\,s^{-1}}}\right)\ln\left[1+0.204\ln\left(\frac{b}{1\,\mathrm{km\,s^{-1}}}\right)\right] \\
&= 635\left(\frac{b}{1.3\,\mathrm{km\,s^{-1}}}\right)\ln\left[1+0.194\ln\left(\frac{b}{1.3\,\mathrm{km\,s^{-1}}}\right)\right] \\
&= 931\left(\frac{b}{10\,\mathrm{km\,s^{-1}}}\right)\ln\left[1+0.139\ln\left(\frac{b}{10\,\mathrm{km\,s^{-1}}}\right)\right]
\end{aligned}$$

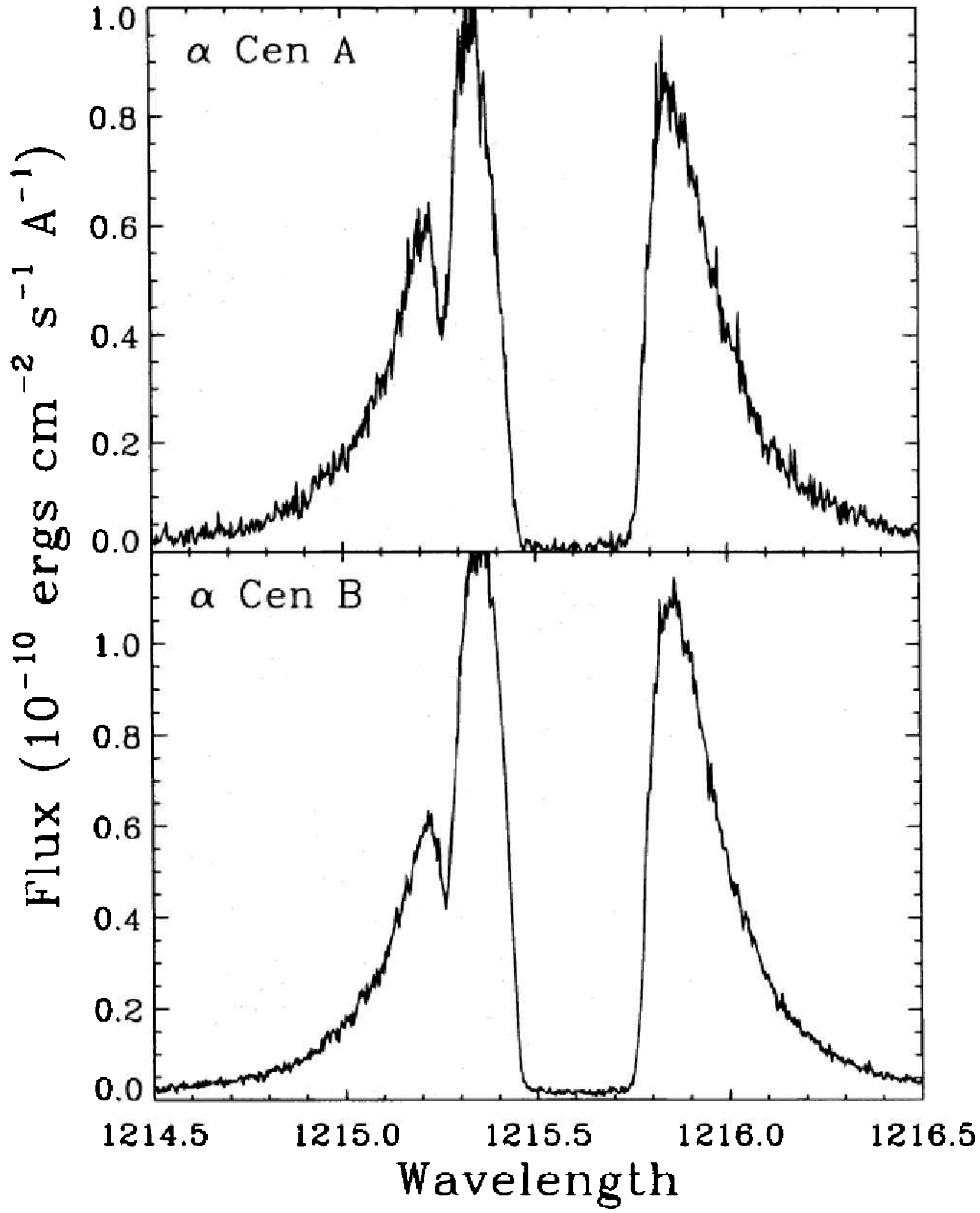
$$[N_\ell]_{\mathrm{damp}} \approx \frac{4m_e c}{e^2}\frac{b^2}{f_{\ell u}\gamma_{\ell u}\lambda_{\ell u}^2}\ln\left[\frac{4\sqrt{\pi}}{\ln 2}\frac{b}{\gamma_{\ell u}\lambda_{\ell u}}\right]$$

$$[N_\ell]_{\mathrm{damp}} = 1.23\times10^{16}\,[\mathrm{cm^{-2}}]\left(\frac{0.4164}{f_{\ell u}}\right)\left(\frac{1215.67\,\text{Å}}{\lambda_{\ell u}}\right)\left(\frac{b}{10\,\mathrm{km\,s^{-1}}}\right)\left(\frac{\tau_{\mathrm{damp}}}{931}\right)$$

Observations: H and D Lyα

- The study of Lyα absorption line is useful to studying the cold clouds in our galaxy. Lyα tends to be optically thick.
- α Cen A and B (d = 1.34 pc) have broad Lyα "emission" lines from their hot chromospheres.
  - Superposed on the emission lines are optically thick absorption lines by the ISM.
  - For both α Cen A and B, $W_\lambda \approx 0.3\,\text{Å}$.

    $$\tau_0 = 68,000, \;\; b = 11.8\,\mathrm{km\,s^{-1}} \;\; (T = 8300\,\mathrm{K})$$
    $$N_\ell = 1.1 \times 10^{18}\,\mathrm{cm^{-2}}$$
  - This represents the regime where the flat part of the curve of growth gives way to the square-root part.
  - The stars are within our Local "Hot" Bubble so that the temperature is high. The column density imply that a density of 0.25 cm^{-3}.



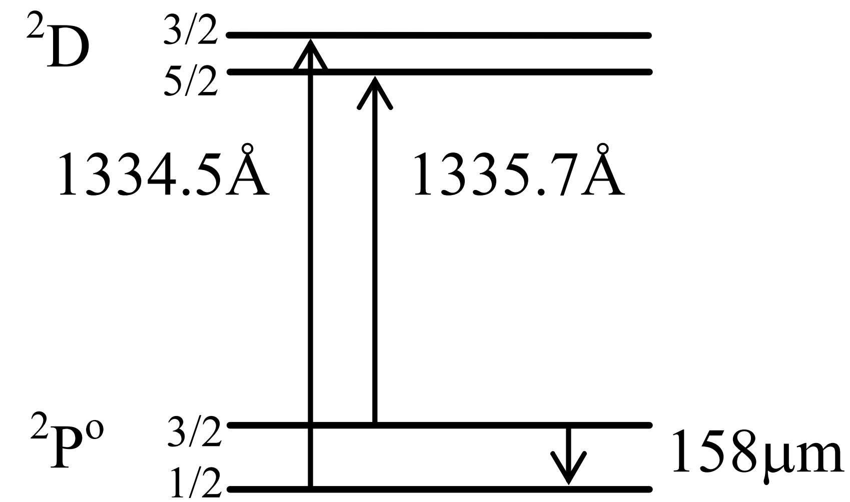
Lyα lines toward α Cen A (above) and α Cen B (below).

Figure 2.8 in [Ryden], (Linsky & Wood 1996)

- On the left-hand slope of the Lyα emission lines, there is an optically thin absorption line. This is the absorption line of deuterium Lyα.
  - The deuterium Lyα lines are slightly blueshifted. wavelength for H Lyα = 1215.67Å, wavelength for deuterium Lyα is 1215.24Å.
  - The deuterium Lyα lines are optical thin, with $\tau_0 = 0.68$, and thus easier to interpret than the H Lyα from ordinary hydrogen. The column density of deuterium toward α Cen A and B, is $N_\ell = 6.1 \times 10^{12}\,\mathrm{cm}^{-2}$, giving a deuterium to hydrogen ratio $D/H \approx 6 \times 10^{-6}$. This is lower than the usual $D/H \approx 1.6 \times 10^{-5}$ in the Local Bubble, and is much less than the primordial value of $D/H \approx 2.5 \times 10^{-5}$.

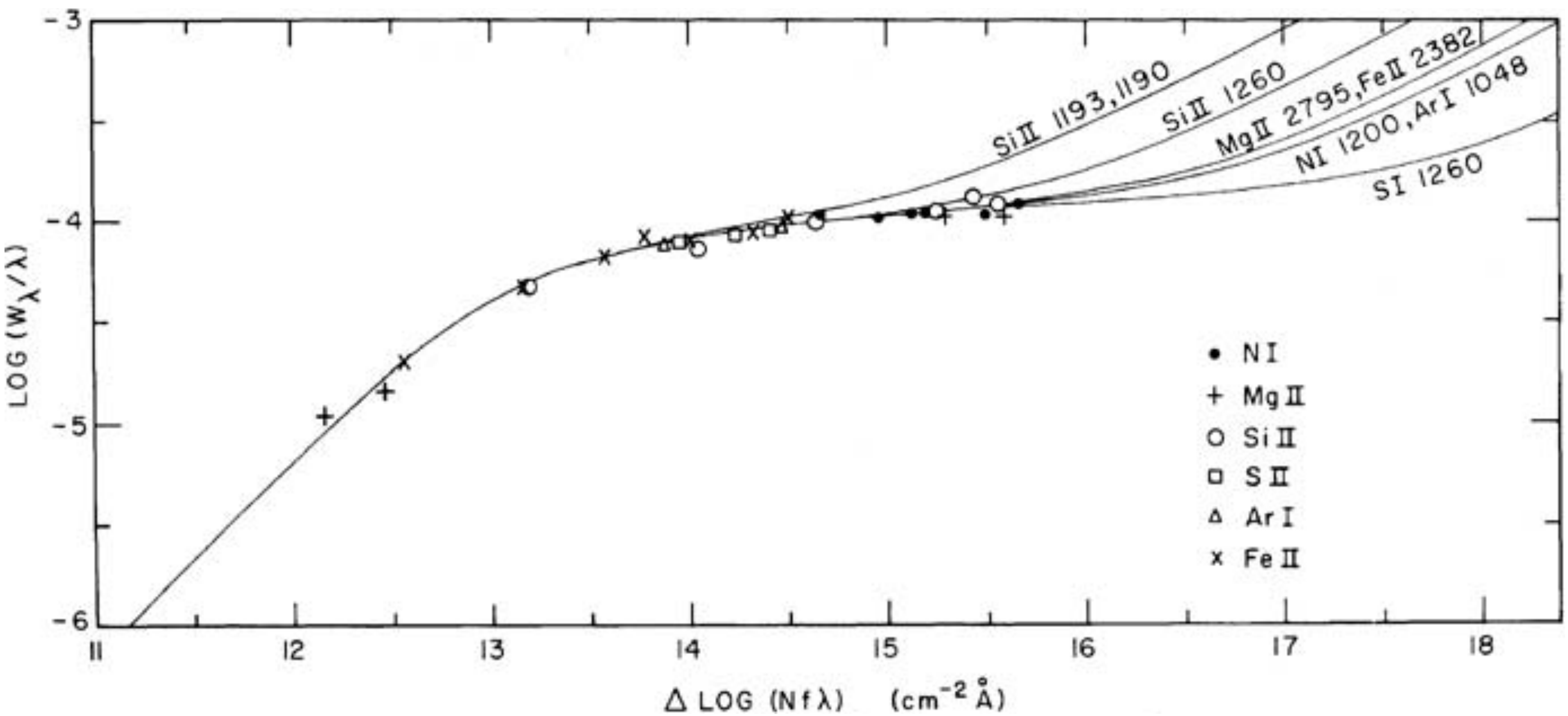
- For stars outside the Local Bubble, at d ~ 100 pc, Lyα lines are in the square-root part of the curve of growth, with $W_\lambda \sim 10\,\text{Å},\ N_\ell \sim 2 \times 10^{20}\,\mathrm{cm}^{-2},\ n_\ell \sim 0.6\,\mathrm{cm}^{-3}$.
- We observe in the visible and UV spectrum of many stars a substantial number of atomic interstellar absorption lines.
- The coexistence of Ca^0 and Ca^+ lines also allows us to obtain the degree of ionization of the corresponding cloud. We observe that Ca+ is much more abundant than Ca^0 (ionization potential of Ca is 6.11 eV).

Observations: Absorption lines from fine-structure levels



- We also observe several absorption lines with wavelength close to each other, which comes from fine-structure levels of the fundamental ground state of the same atom or ion.
  - We can then directly obtain the relative populations of these levels. This gives valuable information on those physical parameters that determine their excitation, essentially the electron density.
  - For examine, C II 1334.57Å and C II* 1335.70Å lines, which unfortunately are often saturated.
  - Morton (1975)'s observations of the C II and C II* ratio showed a significant population of atoms in $^2P^o_{3/2}$. This suggested that [C II] 157.7 µm line could be a strong cooling line in H I regions. It was not until the 1980s that the Kuiper Airborne Observatory detected Far-IR [C II] emission from the ISM, as predicted by Morton.
  - We may determine the cooling rate of the diffuse CNM due to [C II] 157.7 µm by observing the C II* 1335.70Å absorption line.
  - This observation gaves a cooling rate of $\sim 3.5 \times 10^{-26}\,\mathrm{erg\,s^{-1}}$ per hydrogen nucleus (Pottasch et al. 1979; Gry et al. 1992), which is in agreement with the more direct determination of Bennett et al. (1994)

- Observed curve of growth for the ISM in front of the star ζ Oph (Morton1975, ApJ)



  - As can be seen in the above figure, most observed absorption lines lie on the Doppler plateau.
  - Therefore, a better reduction technique than the use of curves of growth would be to adopt a fitting technique for the line profiles. This technique is the only one that can be used for complex line profiles.
- So, optical and UV absorption lines provide us useful information about the cold regions in the ISM. How much of each element and isotope is present? How hot is the gas? What are the integrated densities along the line of sight?

Observations: The Gas Phase Abundances

- The gas phase abundances of many elements relative to H have been determined on many different nightlines using interstellar absorption lines.
  - The observed gas-phase abundances vary from one sightline to another, which is presumed to reflect primarily variations in the amounts of various elements trapped in dust grains. Such removal of elements from the gas is known as ***interstellar depletion***.
  - Some elements, line Fe, are extremely under abundant in the gas phase, with gas-phase abundances that are typically only a few percent of the solar abundance.

Element	Solar system 12 + log(X/H)	Stars 12 + log(X/H)	H II 12 + log(X/H)	T_c^1 K	ζ Oph cold [X/H]	ζ Oph warm [X/H]
H	12.00	12.00	12.00	–	–	–
D	7.53	–	–	–	–0.33:[2]	
He	10.99	–	10.95	–	–	–
Li	3.31	–	–	1 225	–1.58	–
B	2.88	–	–	650	–0.93	–
C	8.55	8.33	8.60	75	–0.41	–
N	7.97	7.82	7.89	120	–0.07	–
O	8.87	8.66	8.77	180	–0.39	0.00
Ne	–	–	8.03	–	–	–
Na	6.31	–	–	970	–0.95	–
Mg	7.58	7.40	–	1 340	–1.55	–0.89
Si	7.55	7.27	–	1 311	–1.31	–0.53
P	5.57	–	–	1 151	–0.50	–0.23
S	7.27	7.09	7.31	648	+0.18	–
Ar	6.56	–	–	25	–0.48	–
K	5.13	–	–	1 000	–1.09	–
Ca	6.34	6.20	–	1 518	–3.73	–
Ti	4.93	4.81	–	1 549	–3.02	–1.31
Fe	7.50	7.43	6.59	1 336	–2.27	–1.25

[1] Condensation temperature at thermal and chemical equilibrium, appropriate for the Solar nebula with an initial gas pressure of 10^{-4} bar. ($1\,\mathrm{bar} = 10^6\,\mathrm{dyn\,cm^{-2}}$)

[2] For lines of sight other than that of ζ Oph: Linsky et al. (1995) .

The gas phase abundances along two lines of sight, compared to abundances in the Solar system.

Abundances are given as 12 + log(X/H), X being the chemical symbol for the element and H that of hydrogen.

The deficiencies in columns 6 and 7 are expressed as $[X/H] = \log(X/H) - \log(X/H)_\odot$.

The data come mainly from Savage & Sembach (1996) and from Snow & Witt (1996).

Table 4.2 in The Interstellar Medium [Lequeux]

Gas-phase abundance vs. Condensation Temperature

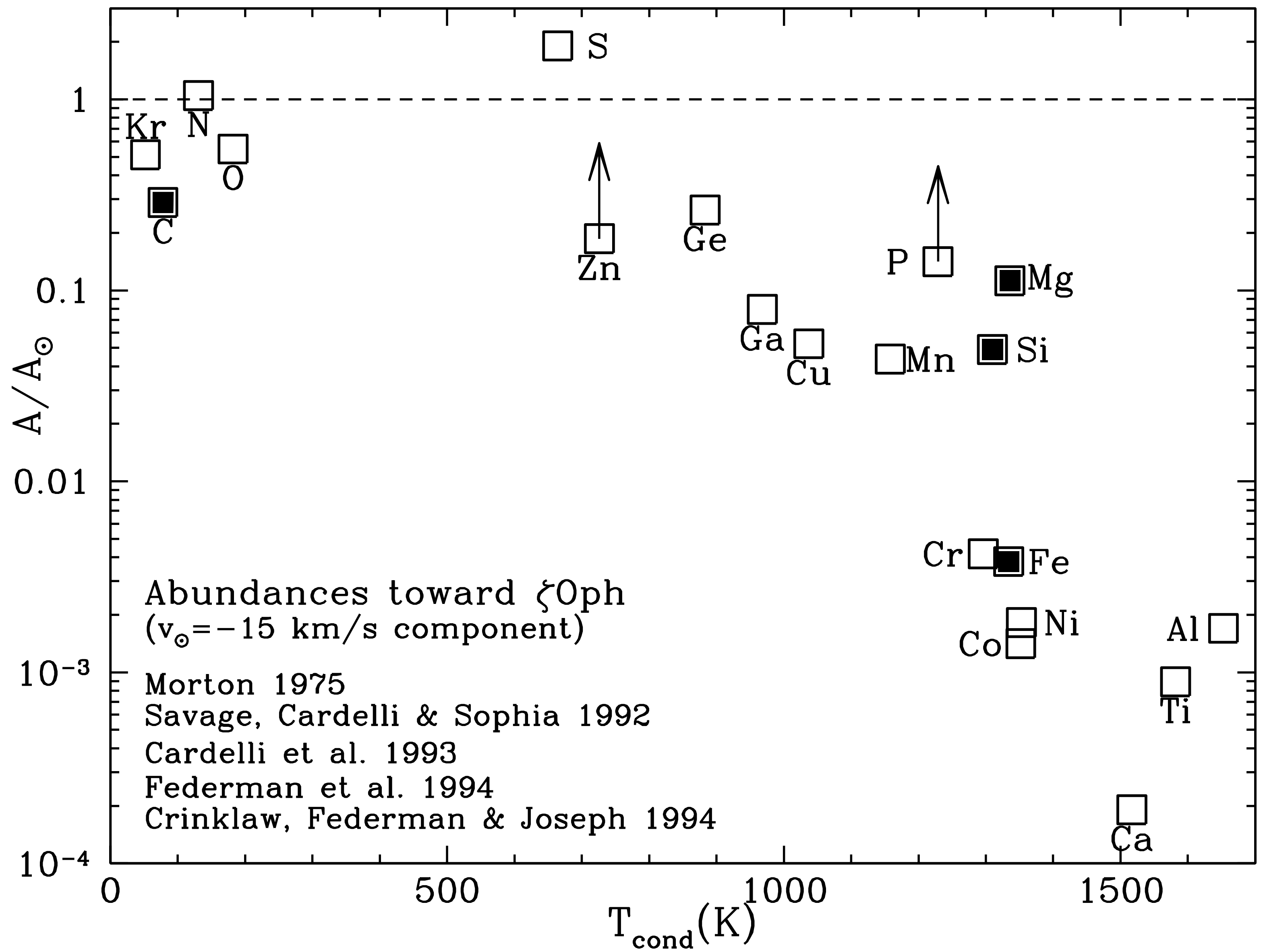


Figure 23.1 in [Draine]

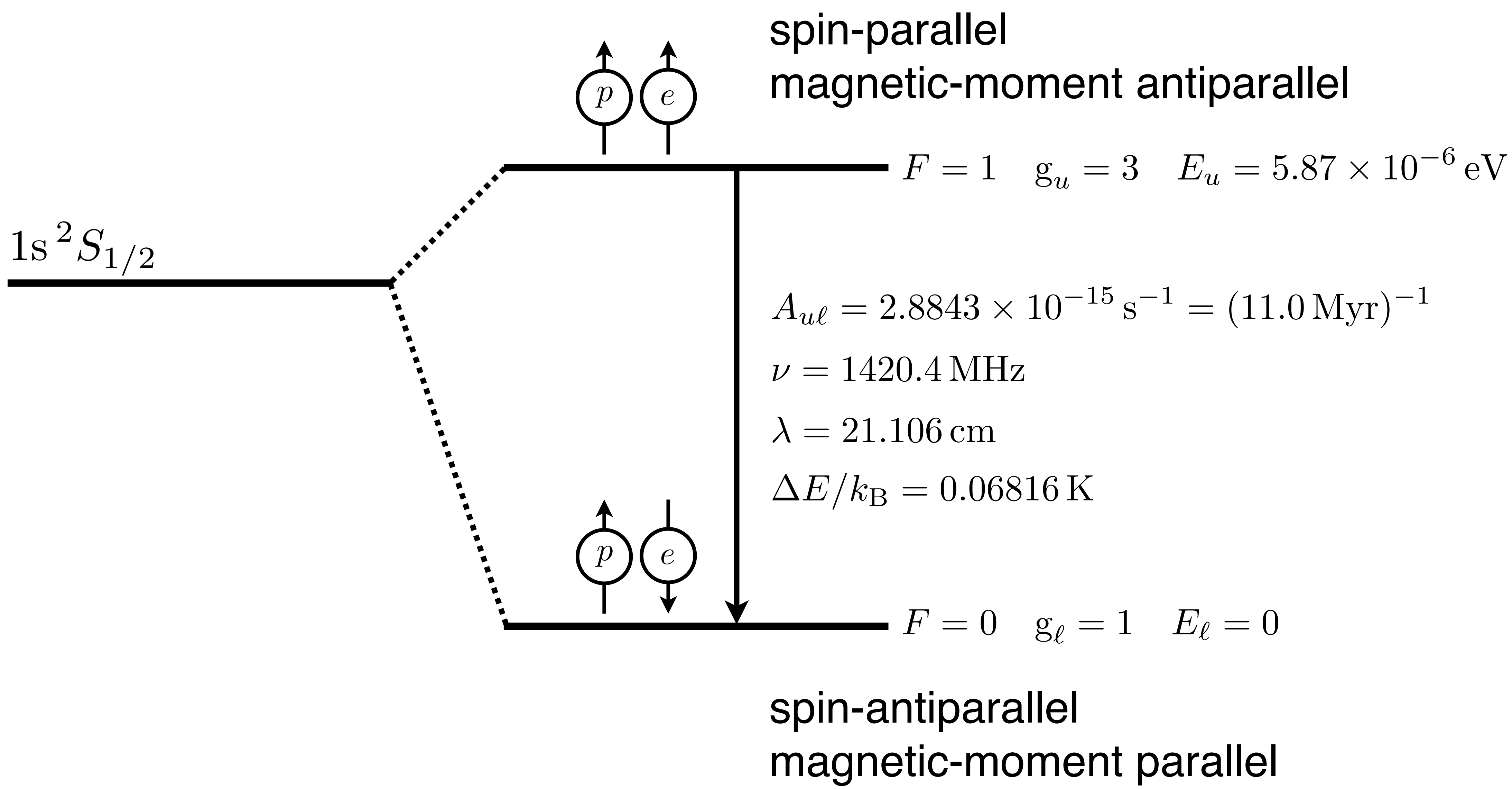
Gas-phase abundances (relative to solar) in the diffuse cloud toward ζ Ophiuchi (O9.5V star, 138 pc), plotted versus "condensation temperature". Solid symbols: major grain constituents C, Mg, Si, Fe. The apparent overabundance of S may be due to observational error, but may be arise because of S II absorption in the H II region around ζ Oph. There's a strong tendency for elements with high $T_{\rm cond}$ to be under abundant in the gas phase, presumably because most of the atoms are in solid grains.

Condensation temperature : temperature at which 50% of the element in question would be incorporated into solid material in a gas of solar abundances, at LTE at a pressure $p = 10^2$ dyn cm^{-2} (Lodders 2003).

21 cm hyperfine line

- The CNM and WNM, taken together, provide over half the mass of the ISM.
  - H is the most abundant element in the universe. In the CNM and WNM, most of the hydrogen is in the form of neutral atoms.
  - The Lyα line of H provides a useful probe of the properties of the CNM and WNM. However, at its wavelength the Earth's atmosphere is highly opaque, and thus observing Lyα absorption requires orbiting UV satellites. In addition, Lyα can be seen in absorption only along those lines of sight toward sources with a high UV flux.
  - To do a global survey of atomic hydrogen in the galaxy, we need some way of easily detecting radiation from hydrogen, regardless of its kinetic temperature or number density.
  - Such a way was first found in 1944, by Henk van de Hulst.
    - He attempted to find emission lines at the wavelengths ~ 1 cm to 20 m, at which the Earth's atmosphere is transparent. He then realized that the hyperfine structure line resulting from a flip of the electron spin within a hydrogen atom should have a wavelength of 21 cm.
    - This was confirmed by Harold Ewen and Edward Purcell in 1951, when they first detected 21 cm emission from the Milky Way.

Hyperfine splitting of the 1s ground state of atomic H



Difference between Lyα and 21 cm transitions

- The excitation energy for Lyα (E = 10.2 eV, E/k = 118,000 K) is much higher than the kinetic temperature of the neutral ISM.

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{118,000\,\mathrm{K}}{T}\right) = 1.7 \times 10^{-51} \text{ at } T = 1000\,\mathrm{K}$$

  - Collisional excitation is unimportant, and most hydrogen atoms are in the lower level of the Lyα transition.
  - The Lyα has a higher energy by a factor of 1.7×10^6 than the 21 cm.

- The excitation energy for 21 cm is ~ 5.9 μeV, and its equivalent temperature E/k = 0.068 K is much lower than the temperature of the cosmic microwave background.
  - Even the CMB is able to populate the upper level.
  - If collisions are frequent, then the spin temperature will be solely determined by collisions, and thus will be a good tracer of the gas kinetic temperature.
  - Thus, there is ample opportunity to populate the upper energy level of the 21 cm hyperfine transition. The level populations for the 21 cm levels, since $T_{\mathrm{exc}} \gg 0.068\,\mathrm{K}$ in all circumstances of the ISM.

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} e^{-h\nu/kT_{\mathrm{exc}}} = 3\,e^{-0.068\,\mathrm{K}/T_{\mathrm{exc}}} \simeq 3 \longrightarrow \boxed{n_u \simeq \frac{3}{4} n_{\mathrm{H}}, \; n_\ell \simeq \frac{1}{4} n_{\mathrm{H}}}$$

  - However, in many cases (in particular in WNM), the hyperfine levels may not be in excitation equilibrium. Radio astronomers use the term ***spin temperature*** for 21 cm rather than the "excitation temperature."

Emissivity and Optical Depth

- ***Emissivity***:
  - The upper level contains ~ 75% of the H I under all conditions of interest, and thus *the 21-cm emissivity is effectively independent of the spin temperature*.

$$j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell}\phi_\nu \simeq \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} n_{\rm H} \phi_\nu \qquad \left(n_u \simeq \frac{3}{4} n_{\rm H}\right)$$

- ***Optical depth***

$$\kappa_\nu = n_\ell \sigma_{\ell u} - n_u \sigma_{u\ell} = n_\ell \sigma_{\ell u} \left(1 - e^{-h\nu_{u\ell}/kT_{\rm spin}}\right)$$

Because $h\nu_{u\ell}/kT_{\rm spin} \ll 1$ for all conditions of interest, the correction for stimulated emission is very important!

$$\kappa_\nu \simeq n_\ell \sigma_{\ell u} \frac{h\nu_{u\ell}}{kT_{\rm spin}} \ll n_\ell \sigma_{\ell u} \longleftarrow e^{-h\nu_{u\ell}/kT_{\rm spin}} \simeq 1 - k\nu_{u\ell}/kT_{\rm spin}$$

$$\kappa_\nu \simeq \left(\frac{1}{4} n_{\rm H}\right) \left(\frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_\ell \phi_\nu\right) \frac{h\nu_{u\ell}}{kT_{\rm spin}} \qquad \left(n_\ell \simeq \frac{1}{4} n_{\rm H}\right)$$

$$= \frac{3}{32\pi} A_{u\ell} \frac{hc\,\lambda_{u\ell}}{kT_{\rm spin}} n_{\rm H} \phi_\nu$$

The absorption coefficient is inversely proportional to the spin temperature.

- The damping constant of the 21 cm line profile is extremely small, and thus we can assume that *the line profile is a Gaussian*.

$$a = \frac{\gamma_{u\ell}}{4\pi}\frac{\lambda_{u\ell}}{b} = 4.844\times 10^{-20}\left(\frac{\gamma_{u\ell}}{2.8843\times 10^{-15}\,\mathrm{s}^{-1}}\right)\left(\frac{\lambda_{u\ell}}{21.106\,\mathrm{cm}}\right)\left(\frac{1\,\mathrm{km\,s}^{-1}}{b}\right)$$

- Hence,

$$\phi_\nu = \frac{1}{\sqrt{\pi}\Delta\nu_D}H(u,a) \simeq \frac{c}{\sqrt{\pi}\nu_{\ell u}b}e^{-u^2}$$

$$\left(u = v/b,\ \ b = \sqrt{2}v_{\rm rms} = \sqrt{2kT_{\rm gas}/m_{\rm H}}\right)$$

$$\tau_\nu = \kappa_\nu s = \frac{3}{32\pi}A_{u\ell}\frac{hc\,\lambda_{u\ell}}{kT_{\rm spin}}N_{\rm H\,I}\phi_\nu$$

$$= \frac{3}{32\pi}\frac{1}{\sqrt{\pi}}\frac{A_{u\ell}\lambda_{u\ell}^2}{b}\frac{hc}{kT_{\rm spin}}N_{\rm H\,I}e^{-u^2}$$

$$= 3.111\left(\frac{N_{\rm H\,I}}{10^{21}\,\mathrm{cm}^{-2}}\right)\left(\frac{100\,\mathrm{K}}{T_{\rm spin}}\right)\left(\frac{1\,\mathrm{km\,s}^{-1}}{b}\right)e^{-u^2}$$

or $$\tau_\nu = 2.201\left(\frac{N_{\rm H\,I}}{10^{21}\,\mathrm{cm}^{-2}}\right)\left(\frac{100\,\mathrm{K}}{T_{\rm spin}}\right)\left(\frac{1\,\mathrm{km\,s}^{-1}}{b/\sqrt{2}}\right)e^{-u^2}$$

$N_{\rm H\,I} \equiv \int n_{\rm H} ds$ is the column density of H I.

$\sim 10^{21}\,\mathrm{cm}^{-21}$ toward the Galactic disk.

Some lines of sight through our galaxy (at high galactic latitude) are optically thin and other lines of sight (at low galactic latitude) are optically thick at 21 cm.

- ***Self-absorption in the 21-cm line can be important*** in many sightlines in the ISM.
- ***The optical depth is inversely proportional to the spin temperature.***

- Typical optical depths of the 21-cm line:

$$\tau_0 = 0.311 \left(\frac{N_{\mathrm{H\,I}}}{10^{21}\,\mathrm{cm}^{-2}}\right)\left(\frac{100\,\mathrm{K}}{T_{\mathrm{spin}}}\right)\left(\frac{10\,\mathrm{km\,s}^{-1}}{b}\right)$$

- In the CNM, a typical spin temperature is $T_{\mathrm{spin}} \approx 50 - 100\,K$:

$$\tau_0^{\mathrm{CNM}} \approx 0.3 - 0.6$$
$$e^{-\tau_0} \approx 0.55 - 0.74$$

The CNM is in general optically thin, but show significant absorption.

- In the WNM, a typical spin temperature is $T_{\mathrm{spin}} \approx 5000 - 8000\,K$:

$$\tau_0^{\mathrm{WNM}} \approx 0.004 - 0.006$$
$$e^{-\tau_0} \approx 0.995$$

The 21-cm absorption is negligible in the WNM.

A typo in page 59 of Ryden's book: For thermal broadening b values typical of the cold ~~warm~~ neutral medium and excitation temperatures $T_{\mathrm{exc}} \sim 100\,\mathrm{K}$, lines of sight with $N_{\mathrm{HI}} > 10^{21}\,\mathrm{cm}^{-2}$ show significant absorption. (Remember from Section 2.3 that Lyman α becomes

[1] Column Density Determination

- ***Radio astronomers express the line profile as a function of radial velocity rather than of frequency.*** This is logical because line broadening is only caused by the Doppler effect, and its natural width being extremely narrow since the lifetime of the upper level is only limited by collisions which is rare in the diffuse medium.
- We first define the column density per velocity interval.

$$\frac{dN_{\rm HI}}{dv} = N_{\rm HI}\phi_v = N_{\rm HI}\frac{1}{\lambda_{u\ell}}\phi_\nu \qquad \phi_\nu = \phi_v\left|\frac{dv}{d\nu}\right| = \phi_v\frac{c}{\nu_{u\ell}} = \lambda_{u\ell}\phi_v$$

- The column density can be written:

$$\tau_\nu = \frac{3}{32\pi}A_{u\ell}\frac{hc\,\lambda_{u\ell}}{kT_{\rm spin}}N_{\rm H\,I}\phi_\nu \longrightarrow \tau(v) = \frac{3}{32\pi}A_{u\ell}\frac{hc\,\lambda_{u\ell}^2}{kT_{\rm spin}(v)}\frac{dN_{\rm HI}}{dv}$$

$$\frac{dN_{\rm HI}}{dv} = \frac{32\pi}{3}\frac{k}{A_{u\ell}hc\lambda_{u\ell}^2}T_{\rm spin}(v)\tau(v)$$

$$= 1.813\times10^{18}\,\frac{T_{\rm spin}(v)\tau(v)}{\rm K}\quad\left[\frac{\rm cm^{-2}}{\rm km\,s^{-1}}\right]$$

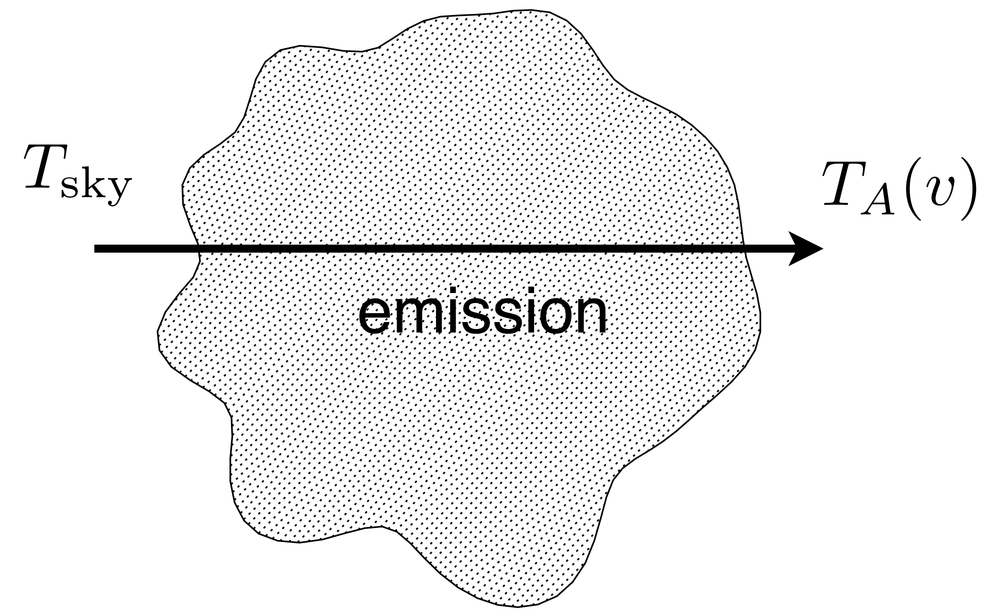
$$N_{\rm HI} = \int dv\frac{dN_{\rm HI}}{dv}$$

- This indicates that ***we need to know not only the optical depth but also the spin temperature to evaluate the column density***. However, in an optically thin limit, we will show that the dependency on the spin temperature is removed.

- ***Optically thin case:*** Suppose we are looking through an optically thin layer of neutral hydrogen toward a ***"dark sky"***, which is fainter than the hydrogen cloud, with an antenna temperature $T_{\rm sky}$.
  - In the optically thin limit, the RT equation becomes

$$\begin{aligned} T_A(v) &= T_{\rm sky} e^{-\tau_v} + T_{\rm spin}(v)\left(1 - e^{-\tau_v}\right) \\ &= T_{\rm sky} + \left(T_{\rm spin}(v) - T_{\rm sky}\right)\left(1 - e^{-\tau_v}\right) \\ &\approx T_{\rm sky} + T_{\rm spin}(v)\tau_v \quad \leftarrow \tau_\nu \ll 1,\ \ T_{\rm sky} \ll T_{\rm spin}(v) \end{aligned}$$

$$\tau(v) \approx \frac{T_A(v) - T_{\rm sky}}{T_{\rm spin}(v)}$$

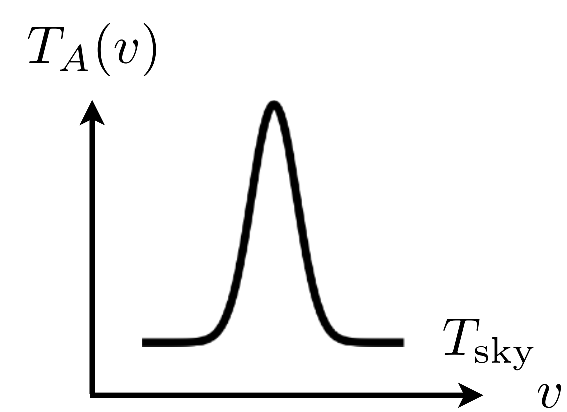


  - The column density per unit velocity interval is

$$\begin{aligned} \frac{dN_{\rm HI}}{dv} &\approx \frac{32\pi}{3}\frac{k}{A_{u\ell}hc\lambda_{u\ell}^2}\left[T_A(v) - T_{\rm sky}\right] \\ &= 1.813\times10^{18}\,\frac{T_A(v) - T_{\rm sky}}{\rm K}\ \left[\frac{\rm cm^{-2}}{\rm km\,s^{-1}}\right] \end{aligned}$$

$$N_{\rm HI} = \int dv \frac{dN_{\rm HI}}{dv}$$

We measure the antenna temperature of the dark sky ($T_{\rm sky}$) from the continuum at frequencies well above and below the 21-cm emission feature.



  - Therefore, *the intensity integrated over the line profile gives us the total H I column density without need to know $T_{\rm spin}$, provided that self-absorption is not important*.

- ***Alternative approach:***
  - If we now neglect absorption, then

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu \approx j_\nu \longrightarrow I_\nu = I_\nu(0) + \int j_\nu ds = I_\nu(0) + \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell}\phi_\nu N_{\rm HI}$$

  - Now suppose that $I_\nu(0)$ is known independently. We can then integrate the intensity over the line

$$\int \left[I_\nu - I_\nu(0)\right] d\nu = \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} N_{\rm HI}$$

  - This can be expressed in terms of antenna temperature T_A and relative velocity $v = \left[(\nu - \nu_{u\ell})/\nu_{u\ell}\right] c$

$$\begin{aligned}\int \left[T_A - T_A(0)\right] dv &= \int \frac{c^2}{2k\nu^2} \left[I_\nu - I_\nu(0)\right] \frac{c}{\nu_{u\ell}} d\nu \\ &\approx \frac{c^3}{2k\nu_{u\ell}^3} \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} N_{\rm HI} \\ &= C_0^{-1} N_{\rm HI}\end{aligned}$$

$$\begin{aligned}C_0 &\equiv \frac{32\pi}{3} \frac{k}{hc\lambda_{u\ell}^2 A_{u\ell}} \\ &= 1.813 \times 10^{18} \left[\frac{\rm cm^{-2}}{\rm K\,km\,s^{-1}}\right]\end{aligned}$$

$$C_0^{-1} = 5.516 \times 10^{-19} \left[\frac{\rm K\,km\,s^{-1}}{\rm cm^{-2}}\right]$$

- We, then, obtain the same equation as before:

$$N_{\rm HI} \approx C_0 \int [T_A - T_A(0)]\, dv$$
$$= 1.813 \times 10^{18} \int \frac{T_A - T_A(0)}{\rm K\,km\,s^{-1}} dv \;\; [{\rm cm^{-2}}]$$

- Here, we did not use the relation between the optical depth and column density.
  - In the first method, we assumed that $\tau_\nu \ll 1$ and $I_\nu(0) \ll S_\nu$:

$$\begin{aligned} I_\nu &= I_\nu(0)e^{-\tau_\nu} + S_\nu \left(1 - e^{-\tau_\nu}\right) \\ &= I_\nu(0) + [S_\nu - I_\nu(0)] \left(1 - e^{-\tau_\nu}\right) \\ &\approx I_\nu(0) + S_\nu \tau_\nu \end{aligned}$$

  - In the second method, we completely ignored the absorption.

$$\begin{aligned} \frac{dI_\nu}{ds} &= -\kappa_\nu I_\nu + j_\nu \\ &\approx j_\nu \end{aligned}$$

This may be a zeroth order approximation.

[2] Spin Temperature Determination

- To derive the spin temperature, we need to combine the emission observation with an absorption observation, which is so called “emission-absorption” method.

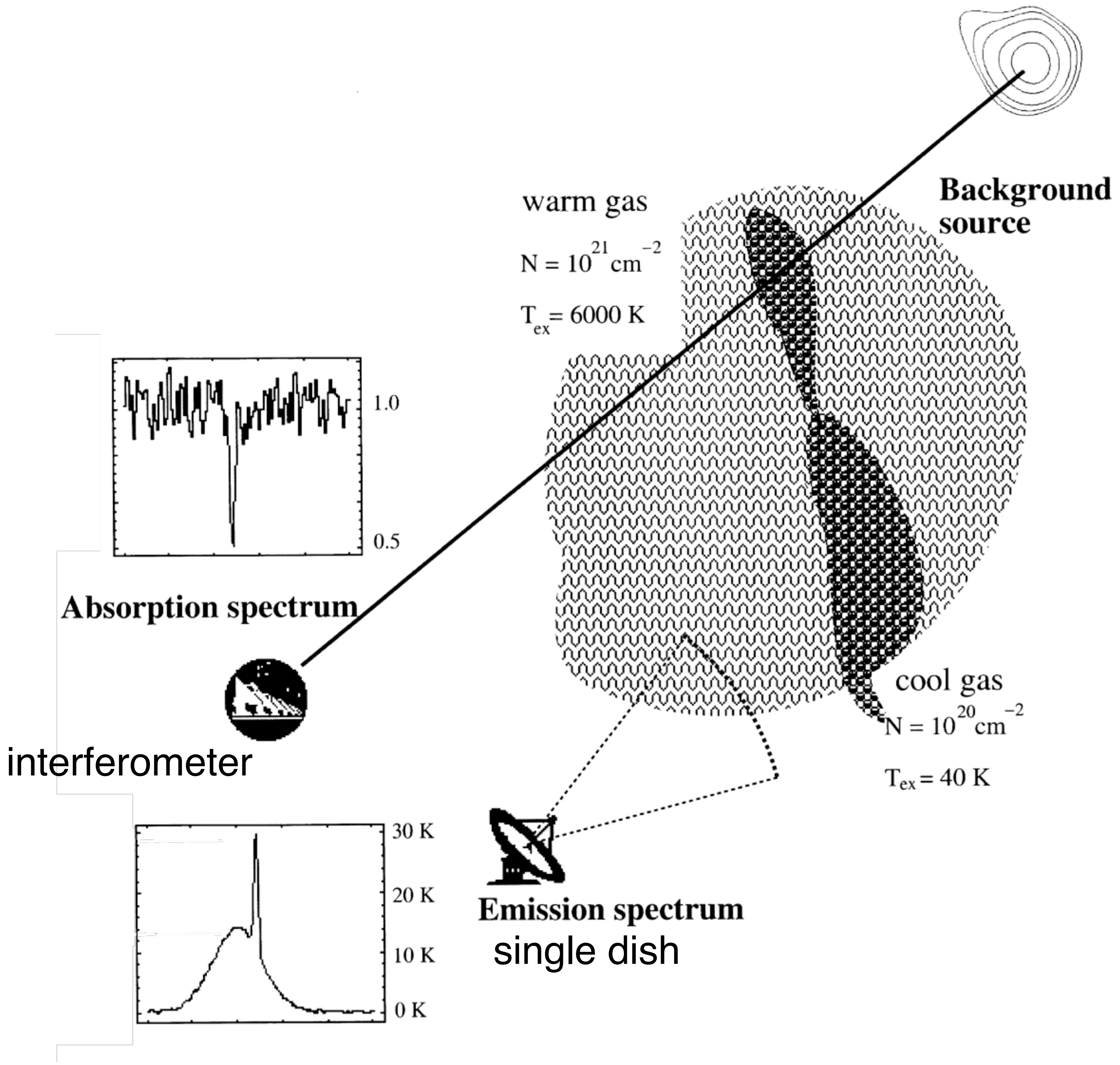
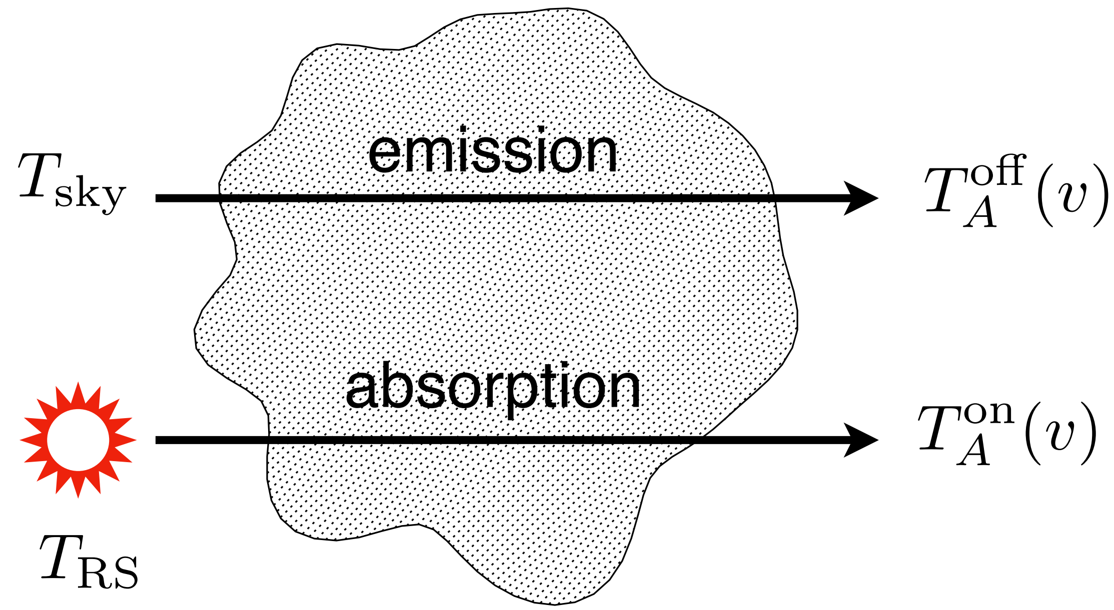


Figure 15 in Dickey et al. (2000, ApJ)

- In cases where we have a ***"bright" background radio source*** with a continuum spectrum (a typical radio-loud quasar or an active galactic nucleus, or a radio galaxy), we can study both emission and absorption by the foreground ISM in our galaxy by comparing ***"on-source" and "off-source" observations***.

- The spectra measured on the blank sky and on the radio source are, respectively,

$$T_A^{\rm on}(v) = T_{\rm RS} e^{-\tau_v} + T_{\rm spin}(v)\left(1 - e^{-\tau_v}\right)$$
$$T_A^{\rm off}(v) = T_{\rm sky} e^{-\tau_v} + T_{\rm spin}(v)\left(1 - e^{-\tau_v}\right)$$



- These two equation can be solved for the two unknowns, $\tau(v)$ and $T_{\rm spin}(v)$.

$$\tau(v) = \ln\left[\frac{T_{\rm RS} - T_{\rm sky}}{T_A^{\rm on}(v) - T_A^{\rm off}(v)}\right]$$

$$T_{\rm spin}(v) = \frac{T_A^{\rm off}(v) T_{\rm RS} - T_A^{\rm on}(v) T_{\rm sky}}{(T_{\rm RS} - T_{\rm sky}) - \left[T_A^{\rm on}(v) - T_A^{\rm off}(v)\right]}$$

We can also derive the column density from these two quantities for an optically thick cloud.

The solution gives, in general, the spin temperature as a function of velocity.

- We usually consider a case where ***the radio source is "much" brighter than the spin temperature of the intervening hydrogen cloud***.
  - The RT equations for the "on-source" and "off-source" measurements can be written:

$$\boxed{\text{assumptions}: T_{\rm RS} \gg T_{\rm spin}}$$

$$T_A^{\rm on} = T_{\rm RS} - T_{\rm RS}\left(1-e^{-\tau}\right) + T_{\rm spin}\left(1-e^{-\tau}\right) \approx T_{\rm RS}e^{-\tau}$$

$$(1)\quad T_A^{\rm on}(v) = T_{\rm RS}e^{-\tau_v} + T_{\rm spin}(v)\left(1-e^{-\tau_v}\right)$$

$$(2)\quad T_A^{\rm off}(v) = T_{\rm sky}e^{-\tau_v} + T_{\rm spin}(v)\left(1-e^{-\tau_v}\right)$$

$$\longrightarrow$$

$$T_A^{\rm on}(v) \approx T_{\rm RS}\, e^{-\tau_v}$$

$$T_A^{\rm off}(v) = T_{\rm sky} + \left(T_{\rm spin} - T_{\rm sky}\right)\left(1-e^{-\tau_v}\right)$$

$$(1)\quad \frac{T_{\rm RS} - T_A^{\rm on}(v)}{T_{\rm RS}} \approx 1 - e^{-\tau_v}$$

$$(2)\quad \Delta T_A^{\rm off}(v) = \Delta T_{\rm spin}(v)\left(1-e^{-\tau_v}\right)$$

Here, $\Delta T_A^{\rm off}(v) \equiv T_A^{\rm off}(v) - T_{\rm sky} \approx T_A^{\rm off}$

$$\Delta T_{\rm spin}(v) \equiv T_{\rm spin}(v) - T_{\rm sky} \approx T_{\rm spin}$$

$$\left(T_{\rm sky} \approx 3\,{\rm K}\right)$$

- ***Equivalent Width:***
  - Using the absorption spectrum from the "on-source" observation, we can "approximately" obtain the "velocity equivalent width."

$$W_v = \int dv \left(1 - e^{-\tau_v}\right)$$
$$\approx \int dv \left[\frac{T_{\rm RS} - T_A^{\rm on}(v)}{T_{\rm RS}}\right]$$

$$\text{Note}: W_v = c \int \frac{d\nu}{\nu_{u\ell}} \left(1 - e^{-\tau_\nu}\right) = cW$$

For a weak absorption line, W_v is an upper limit.

- ***Spin Temperature:***
  - Combining the two equations (1) and (2), we can obtain two spin temperatures. The first one is the line-of-sight average spin temperature, and the second the spin temperature in a velocity channel.

$$\langle \Delta T_{\rm spin} \rangle \approx \frac{\int \Delta T_A^{\rm off}(v) dv}{\int \left(1 - e^{-\tau_v}\right) dv}$$

$$\Delta T_{\rm spin}(v) = \frac{\Delta T_A^{\rm off}(v)}{\left(1 - e^{-\tau_v}\right)}$$

$\longleftarrow$ assuming $\Delta T_{\rm spin} =$ constant.

$$\int dv \left[\Delta T_A^{\rm off}(v)\right] = \Delta T_{\rm spin} \int dv \left(1 - e^{-\tau_v}\right)$$

For a weak absorption line, $\Delta T_{\rm spin}$ is an lower limit.

- ***In an optically thin limit,***
  - We know the relation between the antenna temperature and column density:

$$N_{\rm HI} \approx C_0 \int \Delta T_A^{\rm off}(v) dv \qquad \frac{dN_{\rm HI}}{dv} \approx C_0 \Delta T_A^{\rm off}(v)$$

  - Then, we can express the spin temperature in terms of column density and equivalent width (absorption profile):

$$\langle \Delta T_{\rm spin} \rangle = \frac{1}{W_v} \int \Delta T_A^{\rm off}(v) dv = \frac{C_0^{-1}}{W_v} N_{\rm HI}$$

$$\langle \Delta T_{\rm spin} \rangle \approx 0.5516 \, \frac{N_{\rm HI}/10^{18}\,{\rm cm}^{-2}}{W_v/{\rm km\,s}^{-1}} \; [{\rm K}]$$

$$\Delta T_{\rm spin}(v) = \frac{C_0^{-1}}{(1 - e^{-\tau_v})} \frac{dN_{\rm HI}}{dv}$$

$$C_0^{-1} = 5.516 \times 10^{-19} \; \left[\frac{{\rm K\,km\,s}^{-1}}{{\rm cm}^{-2}} \right]$$

Homework (due date: 04/17)

from Draine's problems

[Q5] **1** A local H I cloud is interposed between us and the cosmic microwave background with temperature $T_{\rm CMB} = 2.7255\,{\rm K}$. Suppose that the H I in the cloud has a spin temperature $T_{\rm spin} = 50\,{\rm K}$, and that the optical depth at line-center (of the 21 cm line) is $\tau = 0.1$. The cloud is extended. We observe the cloud with a radio telescope.

(a) What will be the (absolute) brightness temperature T_B at line-center of the 21 cm line? Express your answer in deg K. You may assume that $h\nu \ll kT_B$.

(b) What will be the (absolute) intensity at line-center of the 21 cm line? Express your answer in $\,{\rm Jy\,sr^{-1}}$.

2 Consider a photon of frequency $h\nu$ entering a slab of material containing two-level atoms with excitation temperature $T_{u\ell}$. At the frequency of the photon, let the optical depth of the slab be τ.

(a) Let $P_{\rm abs}$ be the probability that the original photon will undergo absorption before exiting from the slab. Give an expression for $P_{\rm abs}$ in terms of τ

Hint: here, τ is the optical depth for pure absorption.

(b) Consider a photon that crossed the slab without being absorbed. Let $P_{\rm stim.em.}$ be the probability that the incident photon will stimulate emission of one or more photons. Give an expression for $P_{\rm stim.em.}$ in terms of τ and $h\nu/kT_{u\ell}$.

3 Suppose that we have a molecule with three energy levels – denoted 0, 1, 2 – ordered according to increasing energy, $E_0 < E_1 < E_2$. Let g_0, g_1, g_2 be the degeneracies of the levels. Supppose that there is radiation present with $h\nu = E_2 - E_0$, due to an external source plus emission in the $2 \rightarrow 0$ transition.

Let ζ_{02} be the absorption probability per unit time for a molecule in level 0, with a transition to level 2. Let A_{20}, A_{21}, and A_{10} be the Einstein A coefficients for decays 2→0, 2→1, and 1→0 by spontaneous emission of a photon. Ignore collisional processes.

Hint: this mean no external radiation field for $2 \rightarrow 1$ and $1 \rightarrow 0$

(a) Ignoring possible absorption of photons in the $2 \rightarrow 1$ and $1 \rightarrow 0$ transitions, obtain an expression for the ratio n_1/n_0, where n_i is the number density of molecules in level i.

(b) How large must ζ_{02} be for this molecule to act as a maser in the 1→0 transition?

(c) Is it possible for this system to have maser emission in the 2→1 transition? If so, what conditions must be satisfied?

Hint: find a relation between A_{21} and A_{10} to make the solution self-consistent.

[Q6] Voigt profile

- We want to derive an approximate formula for the transition point from the Gaussian core to the Lorentz wing, which is defined by

$$u^2 = \ln\left(\sqrt{\pi}/a\right) + \ln u^2 \qquad \text{or} \qquad x = \ln\left(\sqrt{\pi}/a\right) + \ln x, \ \text{where } x \equiv u^2$$

The above equation can be expressed in the form:

$$x = g(x) \text{ where } g(x) = \ln x + \ln\left(\sqrt{\pi}/a\right)$$

This equation can be solved using ***"Fixed Point Iteration Method."*** Starting from any initial point x_0, the following recursive process gives an approximate solution of the equation.

$$\boxed{x_{n+1} = g(x_n)}$$

(1) Find a numerical solution x_* for Lyα line with $b = 10\,\mathrm{km\,s^{-1}}$, which is appropriate for Lyα in the WNM with $T \sim 10000$ K.

(2) Let's denote the width parameter as a_* for $b = 10\,\mathrm{km\,s^{-1}}$. This means that

$$x_* = \ln x_* + \ln\left(\sqrt{\pi}/a_*\right)$$

Now, for any parameter a which is different from a_*, you may express the constant term in $g(x)$ as follows:

$$\ln\left(\sqrt{\pi}/a\right) = \ln\left(a_*/a\right) + \ln\left(\sqrt{\pi}/a_*\right)$$

To find the solution for $a \neq a_*$ $(\mathrm{but},\ a \approx a_*)$, choose an initial guess to be $x_0 = x_*$. Show that the solution for any a can be expressed as (after only a single iteration):

$$x_1 = x_* + \ln\left(a_*/a\right)$$

Insert numerical values into the above equation and compare it with Eq. (2.39) in Ryden's book (our textbook).

(3) Insert numerical values into the above equation compare it with the results in this lecture note and Eq. (6.42) in Draine's book.

[Note]

- An approximate solution for the damping optical depth, defined by $x = C\ln\left(x/\ln 2\right)$, can be obtained using "Fixed Point Iteration Method."

Homework

[Q7]

- Measurements of the equivalent width of the absorption Na I D lines at λ = 5890Å in the direction of star HD 190066 (type B1I) give the result $W \sim$ 400 mÅ.

(1) Assume this is a weak line and calculate the column density of neutral Na atoms in the direction of the star. Show that in this case, the following relation is valid:

$$N \simeq \frac{11.3\left(W_\lambda/\mathrm{m\AA}\right)}{(\lambda/\mathrm{cm})^2 f_{u\ell}}\,[\mathrm{cm}^{-2}]$$

Here, use $f_{u\ell} = 0.65$.

(2) Analysis of the line saturation suggests a correction factor of the order of 6 for the column density. Apply this factor to the above result and estimate the Na total column density, assuming that 99% of the sodium atoms are ionized.