Interstellar Medium (ISM)

Lecture 12

June 18 (Thursday), 2020

updated 06/13, 23:17

선광일 (Kwangil Seon)
KASI / UST

Molecular Clouds 2

- Interstellar CO
- CO to H_2 ratio
- Molecular Clouds: Observations

Excitation Temperature

- The excitation temperature for a given transition is defined as:

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} e^{-h\nu_{ul}/kT_{\rm exc}}$$

  - For pure rotational transitions, the excitation temperature is often called the rotation temperature.
  - For vibrational transitions, it is called the vibrational temperature.
  - This nomenclature is analogous to the "spin temperature" defined for the H I 21-cm hyperfine transition.

Interstellar CO

- Much of what we know about molecular gas comes from observations of “tracer” molecules such as carbon monoxide (CO).
- Fundamental frequencies
  - Vibrational frequency
    - The fundamental vibrational frequency corresponds to to a wavelength:

$\lambda_0 = c/\nu_0 \approx 4.6\mu\mathrm{m}$ This energy is ~50% of the energy in the H_2 fundamental frequency.

  - Rotational frequency:
    - CO J=1-0, 2.60mm emission is the most important line used in studies of molecular gas.

$$\lambda = 2.60\,\mathrm{mm} \rightarrow \nu = 115.3\,\mathrm{GHz} \rightarrow h\nu = 4.767 \times 10^{-4}\,\mathrm{eV} \rightarrow h\nu/k = 5.532\,\mathrm{K}$$

- The line width b that we measure for molecular line emission is often much broader than the thermal value $b \sim 0.1(T_{\rm gas}/20\,\mathrm{K})^{1/2}\,\mathrm{km\,s^{-1}}$ that we expect. The large broadening is similar for molecules of different molecular mass, suggesting that the broadening is due to turbulent motions within the cloud.

Virial Mass Estimate

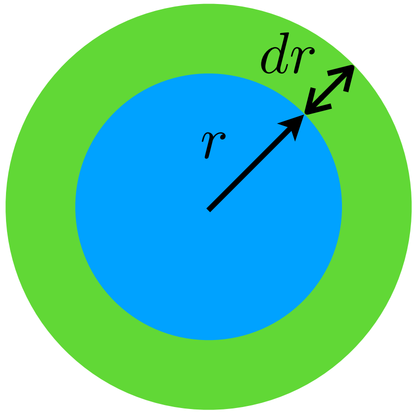
- A fundamental property is that GMCs are gravitationally bound and in viral equilibrium.
  - Their masses can then be estimated using line widths as a measure of the could velocities, following the arguments of Solomon et al.
  - The virial theorem provides a general equation that relates the average over time of the total kinetic energy of a stable, self-gravitating system of discrete particles, with the total potential energy of the system.

$$2\langle K\rangle + \langle U\rangle = 0$$

  - For a uniform density sphere with a mass M and radius R, the gravitational potential energy is

$$U = -\frac{3}{5}\frac{GM^2}{R}$$

$$\begin{aligned} U &= -\int_0^R \frac{GM_r dM_r}{r} \\ &= -\int_0^R \frac{G}{r}\left(\frac{4\pi}{3}r^3\rho\right)4\pi r^2\rho dr \\ &= -\frac{(4\pi)^2}{3\times 5}G\rho^2 R^5 = -\frac{3}{5}\frac{GM^2}{R} \end{aligned}$$



Density $\rho = \frac{M}{(4\pi/3)R^3}$

Mass within a radius r

$$M_r = \int_0^r \rho(4\pi r'^2 dr') = \frac{4\pi}{3}r^3\rho$$

Mass between a shell (r, r+dr)

$$dM_r = \rho(4\pi r^2 dr)$$

- If the cloud is in equilibrium, with self-gravity being balanced by turbulent pressure, the virial theorem states that the turbulent velocity, in one dimension, to be

$$\langle K \rangle = \sum_i \frac{3}{2} m_i \langle v_i^2 \rangle = \frac{3}{2} \sum_i m_i \sigma_v^2 = \frac{3}{2} M \sigma_v^2$$

Here, σ_v is the rms velocity dispersion.

$$2\langle K \rangle + \langle U \rangle = 0 \quad \rightarrow \quad \sigma_v^2 = \frac{1}{5} \frac{GM}{R}$$

- Therefore, we obtain the total mass of the self-gravitation cloud in terms of the line width (broadening parameter):

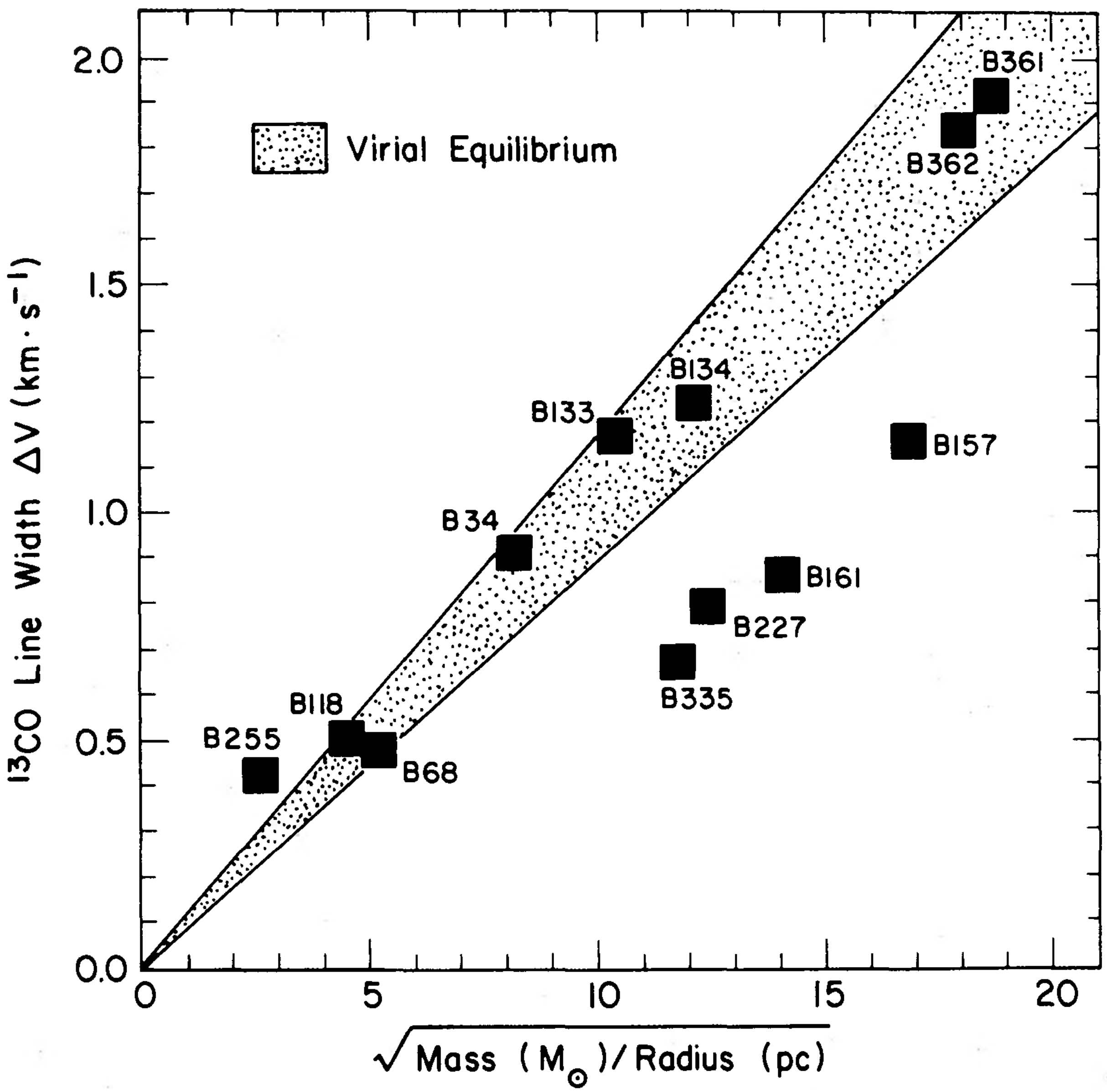
virial mass: $$M = \frac{5b^2 R}{2G} \approx 600\, M_\odot \left(\frac{b}{1\,\mathrm{km\,s^{-1}}} \right)^2 \left(\frac{R}{1\,\mathrm{pc}} \right)$$ Here, $b = \sqrt{2}\sigma_v$

- Note that the virial theorem for a uniform density clouds indicates that the following size-linewidth correlation if the density is constant. But, it appears that the density is also controlled by turbulence.

$$b^2 = \frac{2}{5} \frac{GM}{R} = \frac{8\pi}{15} G\rho R^2 \quad \rightarrow \quad b = \left(\frac{8\pi}{15} G\rho \right)^{1/2} R$$

- The ***line width - (mass/radius) correlation***:

$$b^2 = \left(\frac{2G}{5}\right)^{1/2} \left(\frac{M}{R}\right)^{1/2}$$



The ^{13}CO line width versus (mass/radius)$^{1/2}$ for globules with $T < 10$K.

Fig 3, Leung et al. (1982, ApJ, 262, 583)

- The ***Mass - CO luminosity correlation***
  - Let $I_{\rm CO}$ the CO brightness temperature integrated over the line profile:

$$I_{\rm CO} \equiv \int T_{\rm CO} dv \approx T_{\rm CO} b \quad (b \approx \Delta v)$$

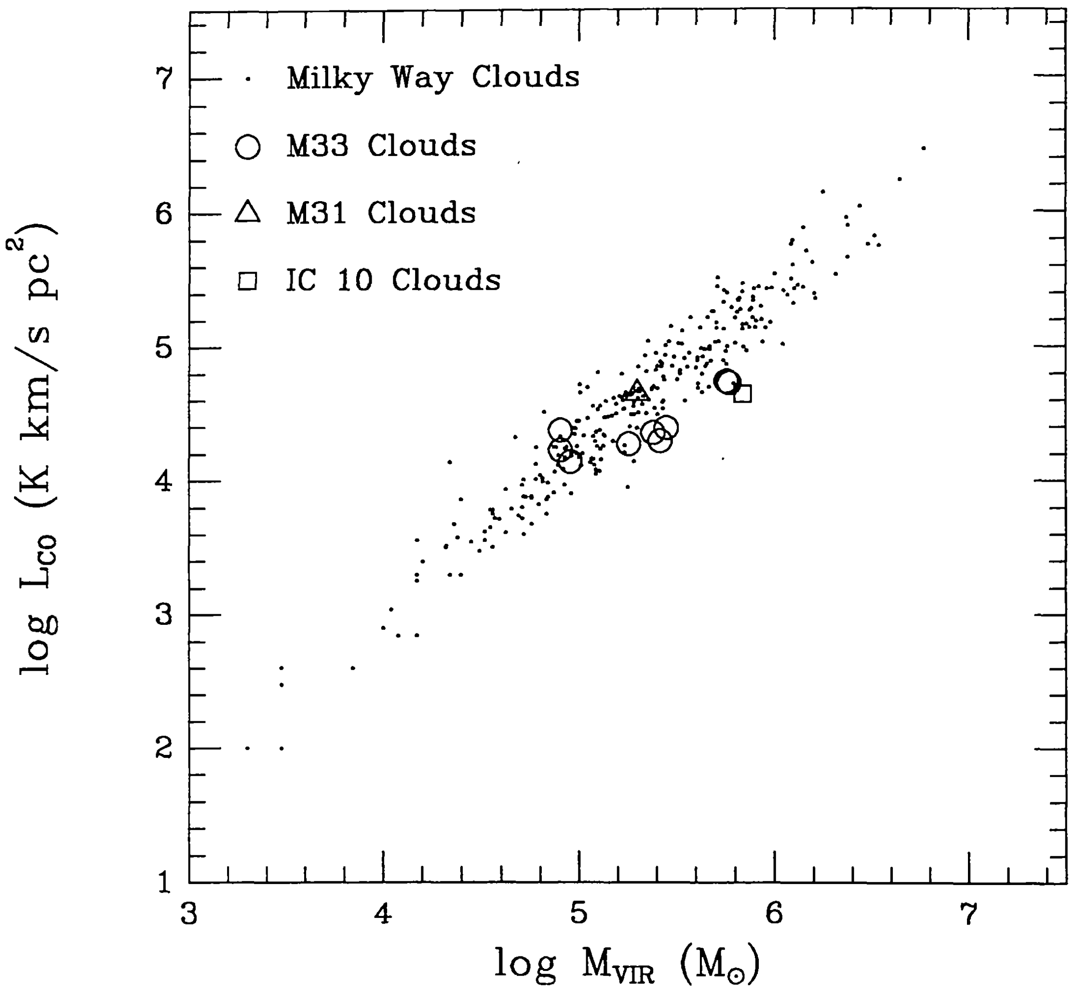
  - The CO luminosity of a cloud is:

$$L_{\rm CO} \equiv D^2 \int I_{\rm CO} d\Omega \approx \pi R^2 T_{\rm CO} b$$
$$= \sqrt{\frac{2\pi^2}{5} G M R^3} = \sqrt{\frac{6\pi}{20} G} \frac{T_{\rm CO}}{\sqrt{\rho}} M$$

Here, D = distance, and virial condition: $b \approx \sqrt{\frac{2}{5}\frac{GM}{R}}$

  - To the extend that the ratio $\sqrt{\rho}/T_{\rm CO}$ does not vary in the mean from cloud to cloud, this indicates that the total CO luminosity is directly proportional to the total mass of molecular clouds.
  - ***This correlation supports the fundamental assumption that GMCs are in virial equilibrium.***

$$L_{\rm CO} \approx M \left(\frac{6\pi}{20} G\right)^{1/2} \frac{T_{\rm CO}}{\sqrt{\rho}}$$



Comparison of CO luminosities and virial masses of molecular clouds

Fig 1, Young & Scoville (1991 ARA&A)

Critical Density of CO (J = 1-0)

- Conditions that are favorable for formation of CO are usually favorable for the formation of H_2.
  - Therefore, the main colliders that excite and de-excite the rotational states of CO are hydrogen molecules rather than hydrogen atoms of free electrons.
  - The collisional de-excitation rate coefficient of CO (J = 1) for collisions with H_2 collisions is

$$k_{10} \approx 6 \times 10^{-11} \, (T/100\,\mathrm{K})^{0.2} \, \mathrm{cm}^3 \, \mathrm{s}^{-1}$$

- The Einstein A coefficient for a rotational transition J → J-1 is given by

$$A_{J\to J-1} = \frac{128\pi^3}{3\hbar} \left(\frac{B_0}{hc}\right)^3 \mu_0^2 \frac{J^4}{J+1/2} = 7.16 \times 10^{-8}\,\mathrm{s}^{-1} \ \text{ for } J = 1 \to 0$$

- At the fundamental frequency, the dominant background radiation is the CMB.
  - The good news is that the 2.6 mm line of CO is radiatively excited by blackbody radiation, which is analytically tractable.
  - The bad news is that we can't use the Rayleigh-Jeans limit, as we did studying the 21 cm line.
  - The photon occupation number is: $$\bar{n}_\gamma = \frac{1}{e^{5.532\,\mathrm{K}/2.725\,\mathrm{K}} - 1} = 0.151 \qquad (T_{\mathrm{rad}} = T_{\mathrm{CMB}} = 2.725\,\mathrm{K})$$

- The critical density at which collisional de-excitation equals to radiative de-excitation is given by

$$n_{\mathrm{crit,CO}} = \frac{(1+\bar{n}_\gamma)A_{10}}{k_{10}} \approx 1400 (T/100\,\mathrm{K})^{-0.2}\,\mathrm{cm}^{-3}$$

In the central dense cores of molecular clouds with $n > n_{\mathrm{crit}}$, collisional excitation will dominate, and J = 1 level is expected to be thermalized.

Optical Thickness of CO (J = 1-0)

- The net absorption coefficient, taking into account both absorption and stimulated emission is:

$$\kappa_\nu = n_0\sigma_{01}\left(1-\frac{g_0}{g_1}\frac{n_1}{n_0}\right), \quad \sigma_{01} = \frac{g_1}{g_0}\frac{c^2}{8\pi\nu_{10}^2}A_{10}\phi_\nu, \quad \frac{n_1}{n_0} = \frac{g_1}{g_0}\exp\left(-\frac{k\nu_{10}}{kT_{\rm exc}}\right) \qquad g_J = 2J+1$$

- The optical depth of the thermally broadened CO line can be written as

$$\phi_\nu = \frac{1}{\sqrt{\pi}}\frac{\lambda}{b}e^{-\frac{v^2}{b^2}}$$

$$\tau_\nu = \int \kappa_\nu ds = \tau_0 e^{-v^2/b^2}$$

optical depth at line center =

$$\tau_0 = \frac{g_1}{g_0}\frac{c^2}{8\pi^{3/2}\nu_{10}^3}A_{10}\frac{c}{b}\left(1-e^{-h\nu_{10}/kT_{\rm exc}}\right)N_0$$

$$\tau_0 = 297\left(\frac{N_{\rm H}}{10^{21}\,{\rm cm}^{-2}}\right)\left[\frac{n({\rm CO})/n_{\rm H}}{7\times10^{-5}}\right]\left[\frac{n(J=0)}{n({\rm CO})}\right]\left(\frac{2\,{\rm km\,s^{-1}}}{b}\right)\left(1-e^{-5.532\,{\rm K}/{\rm T}_{\rm exc}}\right)$$

Here, $N_{\rm H}$ is the column density of H nucleon in the J = 0 rotational level. Recall that the cosmic abundance of carbon is $n_{\rm C}/n_{\rm H} = 2.95\times10^{-4}$ and a fraction $f_{\rm CO}\approx 0.25$ of the carbon is in CO molecules.

- The above equation seems to indicate that the CO J=1-0 line is optically thick. However, we need to estimate the fraction of the CO that is in the J = 0 level.

- The fraction of CO in a given rotational level J will be:

$$\frac{n(\mathrm{CO},J)}{n(\mathrm{CO})} = \frac{(2J+1)e^{-B_0 J(J+1)/kT_{\rm exc}}}{\sum_J (2J+1)e^{-B_0 J(J+1)/kT_{\rm exc}}}$$

Here, B_0 is the “rotation constant.”

- We can approximate the partition function in the denominator by

$$Z = \sum_{J=0}^{\infty}(2J+1)e^{-B_0 J(J+1)/kT_{\rm exc}} \approx \left[1+(kT_{\rm exc}/B_0)^2\right]^{1/2}$$

This approximation is exact in the limits $kT_{\rm exc}/B_0 \to 0$ and $kT_{\rm exc}/B_0 \gg 1$, and accurate to within 6% for all $T_{\rm exc}$.

$$\tau_0 = 297\left(\frac{N_{\rm H}}{10^{21}\,{\rm cm}^{-2}}\right)\left[\frac{n(\mathrm{CO})/n_{\rm H}}{7\times10^{-5}}\right]\frac{\left(1-e^{-5.532\,{\rm K}/{\rm T}_{\rm exc}}\right)}{\left[1+(T_{\rm exc}/2.77\,{\rm K})^2\right]^{1/2}}\left(\frac{2\,{\rm km\,s^{-1}}}{b}\right)$$

Here, we used $B_0/k = 2.77\,{\rm K}$ for $^{12}C^{16}O$.

$$\tau_0 \approx 50\left(\frac{N_{\rm H}}{10^{21}\,{\rm cm}^{-2}}\right)\left[\frac{n(\mathrm{CO})/n_{\rm H}}{7\times10^{-5}}\right]\left(\frac{2\,{\rm km\,s^{-1}}}{b}\right)$$

for a typical CO rotation temperature $T_{\rm exc} \approx 8\,{\rm K}$.

- Thus, **the CO 1-0 transition is expected to be often quite optically thick.**

Radiative Trapping

- Radiative Trapping:
  - In many situation of astrophysical interest, there is sufficient gas present so that, for some species X, a photon emitted in a transition $X_u \rightarrow X_\ell$ will have a high probability of being absorbed by another X_ℓ somewhere nearby, and, therefore, a low probability of escaping from the emitting region.
  - This phenomenon — referred to as radiative trapping — have two effects.
    - (1) it reduces the emission in the $X_u \rightarrow X_\ell$ photons emerging from the region, and
    - (2) it acts to increase the level of excitation of species X (relative to what it would be were the emitted photons to escape freely).
  - An exact treatment of the effects of radiative trapping is a complex problem of coupled radiative transfer and excitation — it is nonlocal, because photons emitted from one point in the cloud affect the level populations at other points.
  - The ***escape probability approximation*** is a simple way to take into account the effects of radiative trapping.

• *Escape Probability Approximation*

- Suppose that a some point $\mathbf{r}$ in the cloud, the optical depth $\tau_\nu(\hat{\mathbf{n}}, \mathbf{r})$ in direction $\hat{\mathbf{n}}$ and at frequency ν is know.
- We define the direction-averaged “escape probability” for photons with frequency ν emitted from location $\mathbf{r}$:

$$\bar{\beta}_\nu(\mathbf{r}) \equiv \int \frac{d\Omega}{4\pi} e^{-\tau_\nu(\hat{\mathbf{n}}, \mathbf{r})}$$

  Now define the direction-averaged and frequency-averaged escape probability:

$$\langle \beta(\mathbf{r}) \rangle = \int \phi_\nu \bar{\beta}_\nu(\mathbf{r}) d\nu \qquad \text{Normalized line profile} = \int \phi_\nu d\nu = 1$$

- Now we make two approximations:
  - (1) we will approximate the excitation in the cloud as uniform.
  - (2) we make the “on-the-spot” approximation: we assume that if a radiated photon is going to be absorbed, it will be absorbed so close to the point of emission that we can approximate it as being absorbed at the point of emission.
  - These approximations replace a difficult nonlocal excitation problem with a much simpler local one! This is called the escape probability approximation.

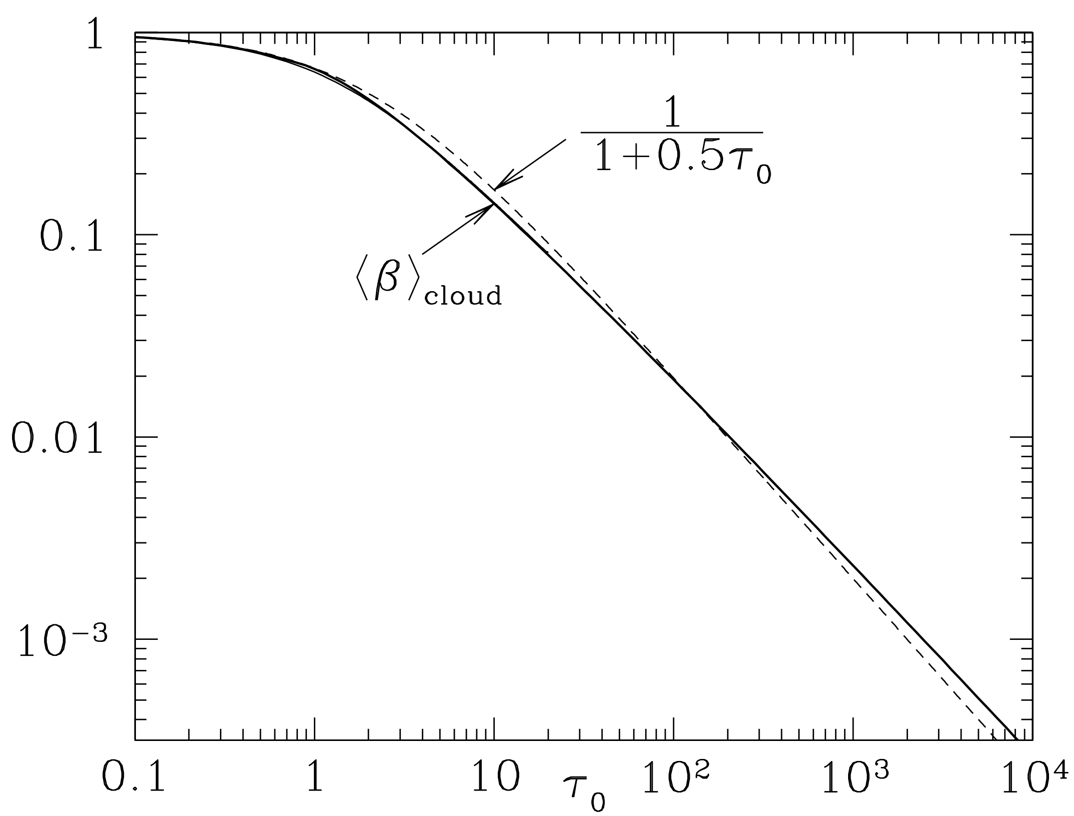
- Homogeneous Static Spherical Cloud.

  - The angle-averaged escape probability $\bar{\beta}$ depends on the geometry and velocity structure of the region.

    For the case of a finite cloud, $\bar{\beta}$ will depend on position — it will be highest at the cloud boundary, and smallest at the cloud center.

  - We now define the escape probability averaged over the line profile and over the cloud volume. for a uniform density spherical cloud,

$$\langle \beta \rangle_{\text{cloud}} = \frac{1}{4\pi R^3/3} \int_0^R \langle \beta(\mathbf{r}) \rangle \, 4\pi r^2 dr$$



The left figure shows the mass-averaged escape probability calculated numerically for the case of a homogeneous spherical cloud, as a function of the optical depth τ_0 at line-center from the center of the cloud to the surface. The gas is assumed to have a Gaussian velocity distribution. [Fig 19.1, Draine]

A satisfactory approximation is provided by the simple fitting function.

$$\boxed{\langle \beta \rangle_{\text{cloud}} = \frac{1}{1 + 0.5\tau_0}}$$

Excitation Temperature of CO (J = 1-0)

- No approximation is valid for CO.
  - We can't use the handy $kT_{\rm exc} \ll h\nu_{u\ell}$ approximation that was useful for Lyα line.
  - Neither can we use the $kT_{\rm exc} \gg h\nu_{u\ell}$ approximation that was used for 21-cm line.
- The radiative transfer equation is the usual

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

  - Emissivity and absorption coefficient are

$$j_\nu = n_1 \frac{A_{10}}{4\pi} h\nu_{10}\phi_\nu \qquad \kappa_\nu = n_0 \frac{g_1}{g_0} \frac{c^2}{8\pi\nu_{10}^2} A_{10}\phi_\nu \left(1 - \frac{g_0}{g_1}\frac{n_1}{n_0}\right), \quad \frac{n_1}{n_0} = \frac{g_1}{g_0} \exp\left(-\frac{k\nu_{10}}{kT_{\rm exc}}\right)$$

  - The source function can be written as

$$S_\nu \equiv \frac{j_\nu}{\kappa_\nu} = \frac{2h\nu_{10}^3}{c^2} \frac{1}{\exp\left(h\nu_{10}/kT_{\rm exc}\right) - 1} = B_\nu(T_{\rm exc})$$

  - If the excitation (rotation) temperature is constant over the entire region of emission, we have the solution:

$$I_\nu = I_\nu(0) e^{-\tau_\nu} + S_\nu \left(1 - e^{-\tau_\nu}\right)$$

  - The background radiation is the Cosmic Microwave Background: $I_\nu(0) = B_\nu(T_{\rm rad})$

- Now, we do perform the "on" observation toward a molecular cloud, and "off" observation over to a black sky.

$$I_\nu(\mathrm{on}) = \frac{2h\nu^3}{c^2}\left[\frac{e^{-\tau_\nu}}{\exp\left(h\nu/kT_{\mathrm{rad}}\right)-1} + \frac{1-e^{-\tau_\nu}}{\exp\left(h\nu/kT_{\mathrm{exc}}\right)-1}\right]$$
$$I_\nu(\mathrm{off}) = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(h\nu/kT_{\mathrm{rad}}\right)-1}$$

- The difference between the two observations is

$$\begin{aligned}\Delta I_\nu &= I_\nu(\mathrm{on}) - I_\nu(\mathrm{off}) \\ &= \frac{2h\nu^3}{c^2}\left[\frac{1}{\exp\left(h\nu/kT_{\mathrm{exc}}\right)-1} - \frac{1}{\exp\left(h\nu/kT_{\mathrm{rad}}\right)-1}\right]\left(1-e^{-\tau_\nu}\right)\end{aligned}$$

- It is customary to express the intensity in terms of an antenna temperature:

$$T_{\mathrm{A}} \equiv \frac{c^2}{2k}\frac{\Delta I_\nu}{\nu^2} \approx \frac{h\nu}{k}\left[\frac{1}{\exp\left(h\nu/kT_{\mathrm{exc}}\right)-1} - \frac{1}{\exp\left(h\nu/kT_{\mathrm{rad}}\right)-1}\right]$$ for an optical thick cloud $\tau_\nu \gg 1$.

- We can solve this for excitation temperature:

$$\frac{5.532\,\mathrm{K}}{T_{\mathrm{exc}}} \approx \ln\left(1 + \frac{1}{T_{\mathrm{A}}/5.532\,\mathrm{K} + 0.151}\right)$$

- In our Galaxy, it is often found that $T_{\mathrm{A}} \sim 5\,\mathrm{K}$ at the line center, implying $T_{\mathrm{exc}} \sim 8\,\mathrm{K}$.

^{13}CO Column Density & ^{13}CO to H_2

- The ^{12}CO 2.6 mm line is optically thick. All the 2.6 mm emission that we see comes from a thin skin at the surface of the molecular cloud.
  - The column density of CO can be more easily derived from the optically thin ^{13}CO line. The rarer isotopes will refer to gas deeper in the molecular cloud.
  - The cosmic ratio of ^{13}CO to ^{12}CO is ~ 0.011. Hence, ^{13}CO is optically thin.
  - The frequency of ^{13}CO J = 1-0 transition is 110.20 GHz, 4.6% lower than that for ^{12}CO.

$$\frac{\mu(^{13}\mathrm{C}^{16}\mathrm{O})}{\mu(^{12}\mathrm{C}^{16}\mathrm{O})} = \frac{13 \times 16/(13+16)}{12 \times 16/(12+16)} \approx 1.0460 \qquad \frac{B_0(^{13}\mathrm{C}^{16}\mathrm{O})}{B_0(^{12}\mathrm{C}^{16}\mathrm{O})} = \frac{\mu(^{12}\mathrm{C}^{16}\mathrm{O})}{\mu(^{13}\mathrm{C}^{16}\mathrm{O})} \approx 1/1.0460 = 0.9560$$

  - We assume that T_{ext} of ^{13}CO is the same as that of ^{12}CO. This is generally a safe assumption, since the two isotopes of carbon are intermingled in the ISM.

$$T_{\mathrm{A}}(^{13}\mathrm{CO}) \approx \frac{h\nu}{k}\left[\frac{1}{\exp\left(h\nu/kT_{\mathrm{exc}}\right)-1} - \frac{1}{\exp\left(h\nu/kT_{\mathrm{rad}}\right)-1}\right]\tau_\nu$$

$h\nu/k = 5.140\,\mathrm{K}$ for the ^{13}CO J = 1-0 transition.

  - The integrated line strength is then

$$\int T_{\mathrm{A}}(^{13}\mathrm{CO})dv \approx 5.140\,\mathrm{K}\left(\frac{1}{e^{5.140\,\mathrm{K}/T_{\mathrm{exc}}}-1} - 0.1787\right)\int \tau_\nu dv$$

- The optical depth, integrated over the thermally broadened line, is

$$\int \tau_\nu dv = \sqrt{\pi} b \tau_0 = \frac{1}{8\pi} \frac{g_1}{g_0} \frac{c^3}{\nu^3} A_{10} \left(1 - e^{-h\nu/kT_{\rm exc}}\right) N_0(^{13}{\rm CO})$$

$$\tau_0 = \frac{g_1}{g_0} \frac{c^2}{8\pi^{3/2}\nu_{10}^3} A_{10} \frac{c}{b} \left(1 - e^{-h\nu_{10}/kT_{\rm exc}}\right) N_0$$

Here, $N_0(^{13}{\rm CO})$ is the column density of ^{13}CO in its rotational ground state with J = 0.

Note: $$\begin{aligned} A_{10} &= 7.20 \times 10^{-8}\,{\rm s}^{-1} \quad \text{for } ^{12}{\rm CO} \\ &= 6.29 \times 10^{-8}\,{\rm s}^{-1} \quad \text{for } ^{13}{\rm CO} \end{aligned}$$

$$\int \tau_\nu dv \approx 1.52 \times 10^{-15} \left(1 - e^{-5.140\,{\rm K}/T_{\rm exc}}\right) N_0(^{13}{\rm CO}) \left[\frac{{\rm km\,s^{-1}}}{{\rm cm^{-2}}}\right]$$

- Therefore, we find:

$$N_0(^{13}{\rm CO}) \approx 2.9 \times 10^{14} \frac{\int T_{\rm A}(^{13}{\rm CO}) dv}{1\,{\rm K\,km\,s^{-1}}} \left[{\rm cm^{-2}}\right] \qquad \text{when} \qquad T_{\rm exc} \approx 8\,{\rm K}$$

X-factor (^{13}CO to H_2)

- X-factor (^{13}CO to H_2)
  - The fraction of ^{13}CO in its rotational ground state:

$$f_{J=0} = \frac{1}{\sum_J (2J+1) e^{-B_0 J(J+1)/kT_{\rm exc}}} \approx \left[\frac{1}{1+\left(\frac{T_{\rm exc}}{2.766/1.0460\,{\rm K}}\right)^2}\right]^{1/2} \approx 0.314$$

  - Then, we obtain the CO to H_2 ratio:

$$\begin{aligned} N({\rm H_2}) =& 1.8\times10^6 \\ & \times \frac{1}{2}\frac{n_{\rm H}/n_{\rm C}}{(1/3.0\times10^{-4})}\frac{n({\rm C})/n(^{12}{\rm CO})}{4}\frac{n(^{12}{\rm CO})/n(^{13}{\rm CO})}{90}\frac{N(^{13}{\rm CO})/N_0(^{13}{\rm CO})}{3}N_0(^{13}{\rm CO}) \end{aligned}$$

$$N({\rm H_2}) \approx 5\times10^{20}\left[\frac{\int T_{\rm A}(^{13}{\rm CO})dv}{1\,{\rm K\,km\,s^{-1}}}\right] \quad [{\rm cm^{-2}}]$$

$$X_{^{13}{\rm CO}} \equiv \frac{N({\rm H_2})}{W_{^{13}{\rm CO}}} \approx 5\times10^{20}\left[\frac{\rm cm^{-2}}{\rm K\,km\,s^{-1}}\right] \longleftarrow \qquad W_{^{13}{\rm CO}} \equiv \int T_{\rm A}(^{1213}{\rm CO})dv$$

X-factor (^{12}CO to H_2) - Correct Approach

(Chapter 19 of Draine's book)

- X-factor
  - It would be useful if we had a reliable way to convert from something easy to measure (the antenna temperature integrated over the line of ^{12}CO) to something difficult to measure (the column density $N(H_2)$ of molecular hydrogen).

$$W_{\mathrm{CO}} \equiv \int T_{\mathrm{A}}(^{12}\mathrm{CO})dv$$

$$\boxed{X_{\mathrm{CO}} \equiv \frac{N(\mathrm{H}_2)}{W_{\mathrm{CO}}}}$$

  - If the cloud is larger than our antenna beam, we can relate W_{CO} to $N(\mathrm{H}_2)$.
    - Recall that the optical depth at line-center from center to edge of the spherical cloud:

$$\tau_0 = \frac{g_1}{g_0}\frac{A_{10}c^2}{8\pi^{3/2}\nu_{10}^3}\frac{c}{b}\left(1 - e^{-h\nu_{10}/kT_{\mathrm{exc}}}\right)n_0 R$$

Use virial theorem:

$$b = \left(\frac{2G}{5}\frac{M}{R}\right)^{1/2}$$

$$\tau_0 = \frac{g_1}{g_0}\frac{A_{10}\lambda_{10}^3}{8\pi}\left(\frac{5}{2\pi G}\right)^{1/2}\frac{n_0 R^{3/2}}{M^{1/2}}\left(1 - e^{-h\nu_{10}/kT_{\mathrm{exc}}}\right)$$

    - The luminosity of the cloud in a spectral line (J = 1-0) is:

$$L_{10} = \int dr 4\pi r^2 n_1 A_{10} h\nu_{10}\langle\beta\rangle_{\mathrm{cloud}}$$
$$= \int dV 4\pi j_{10}\langle\beta\rangle_{\mathrm{cloud}}$$

Emissivity: $4\pi j_{10} = n_1 A_{10} h\nu_{10}$

Escape probability averaged over the line profile and over the cloud volume:

$$\langle\beta\rangle_{\mathrm{cloud}} = \frac{1}{4\pi R^3/3}\int_0^R \langle\beta(\mathbf{r})\rangle 4\pi r^2 dr$$

- Using the escape probability $\langle\beta\rangle_{\rm cloud} \approx \dfrac{1}{1+0.5\tau_0}$

$$L_{10} \approx \frac{4\pi}{3} R^3 n_1 A_{10} h\nu_{10} \frac{2}{\tau_0}$$

Here, we assume $\tau_0 \gg 1$, thus $\langle\beta\rangle_{\rm cloud} \approx 2/\tau_0$.

$$\begin{aligned} L_{10} &\approx \frac{64\pi^2}{3} \left(\frac{2\pi G}{5}\right)^{1/2} \frac{hc}{\lambda_{10}^4} M^{1/2} R^{3/2} \frac{n_1}{n_0} \frac{g_0}{g_1} \frac{1}{1-e^{-h\nu_{10}/kT_{\rm exc}}} \\ &\approx \frac{64\pi^2}{3} \left(\frac{2\pi G}{5}\right)^{1/2} \frac{hc}{\lambda_{10}^4} M^{1/2} R^{3/2} \frac{1}{e^{h\nu_{10}/kT_{\rm exc}}-1} \end{aligned}$$

- The line luminosity per unit mass is:

$$\boxed{\frac{L_{10}}{M} \approx 32\pi^2 \left(\frac{2G}{15}\right)^{1/2} \frac{hc}{\lambda_{10}^4} \frac{1}{\rho^{1/2}} \frac{1}{e^{h\nu_{10}/kT_{\rm exc}}-1}} \longleftarrow \quad M = \frac{4\pi R^3}{3}\rho$$

Note that for a give spectral line, the luminosity per unit mass depends on the cloud density and on the excitation temperature.

- Now, we relate (1) the mass to the average column density of hydrogen necleon and (2) the luminosity to the average antenna (brightness) temperature.

hydrogen column density: $N_{\rm H} = \dfrac{\Sigma}{1.4 m_{\rm H}} = \dfrac{\frac{M}{\pi R^2}}{1.4 m_{\rm H}}$

$\Sigma =$ surface density

Antenna temperature averaged over the projected surface area:

$$\int T_{\rm A} dv \equiv \frac{\int\int \frac{c^2}{2k\nu^2} I_\nu dv dA \left<\beta\right>_{\rm cloud}}{\int dA}$$

Here,

dA = projected area

ds = pathlength along a sightline

From the radiative transfer equation: $I_\nu = \int j_\nu ds$

$$\int T_{\rm A} dv = \frac{1}{\pi R^2} \int \left[\int \frac{c^2}{2k\nu^2} j_\nu \left<\beta\right>_{\rm cloud} dv \right] dV$$
$$= \frac{1}{\pi R^2} \frac{1}{4\pi} \frac{c^3}{2k\nu_{10}^3} L_{10}$$

$$\frac{v}{c} \equiv \frac{\nu - \nu_{10}}{\nu_{10}}$$

$$j_\nu dv = j_\nu \left| \frac{dv}{d\nu} \right| d\nu = j_\nu \frac{c}{\nu_{10}} d\nu$$

- Then, we can relate the antenna temperature to the total H column density (averaged over the beam solid angle):

$$\frac{N_{\rm H}}{\int T_{\rm A} dv} = \frac{8\pi k}{1.4 m_{\rm H}} \frac{M}{L_{10}}$$
$$= \frac{1}{4\pi} \frac{k\lambda_{10}}{hc} \left(\frac{15}{2.8 G m_{\rm H}} \right)^{1/2} (n_{\rm H})^{1/2} \left(e^{h\nu_{10}/kT_{\rm exc}} - 1 \right)$$

Here, we used:
$\rho = 1.4 \times m_{\rm H} n_{\rm H}$

- If we assume $N(\mathrm{H_2}) = N_\mathrm{H}/2$, the above equation becomes:

$$X_\mathrm{CO} = \frac{N(\mathrm{H_2})}{\int T_\mathrm{A} dv} = \frac{1}{8\pi}\frac{k\lambda_{10}}{hc}\left(\frac{15}{2.8Gm_\mathrm{H}}\right)^{1/2}(n_\mathrm{H})^{1/2}\left(e^{h\nu_{10}/kT_\mathrm{exc}} - 1\right)$$
$$= 1.58 \times 10^{20}\left(\frac{n_\mathrm{H}}{10^3\,\mathrm{cm^{-3}}}\right)^{1/2}\left(e^{5.532\,\mathrm{K}/T_\mathrm{exc}} - 1\right)\left[\frac{\mathrm{cm^{-2}}}{\mathrm{K\,km\,s^{-1}}}\right]$$

- For $n_\mathrm{H} = 10^3\,\mathrm{cm^{-3}}$ and $T_\mathrm{exc} = 8\,\mathrm{K}$, this yields:

$$X_\mathrm{CO} = \frac{N(\mathrm{H_2})}{\int T_\mathrm{A} dv} = 1.56 \times 10^{20}\left(\frac{n_\mathrm{H}}{10^3\,\mathrm{cm^{-3}}}\right)^{1/2}\left[\frac{\mathrm{cm^{-2}}}{\mathrm{K\,km\,s^{-1}}}\right]$$

Dame et al. (2001) observationally find $X_\mathrm{CO} = (1.8 \pm 0.3) \times 10^{20}\,\mathrm{cm^{-2}/K\,km\,s^{-1}}$, where infrared emission from dust (Schlegel et al. 1998) was used as a mass tracer.

The most recent determination using gamma rays finds

$X_\mathrm{CO} = (1.76 \pm 0.04) \times 10^{20}\,\mathrm{cm^{-2}/K\,km\,s^{-1}}$ for the Orion A GMC (Okumura et al. 2009)

<u>These results suggests that $n_\mathrm{H} = 10^3\,\mathrm{cm^{-3}}$ and $T_\mathrm{exc} = 8\,\mathrm{K}$ may be representative of self-gravitating molecular clouds in the local ISM.</u>

X-factor (^{12}CO to H_2) - Approach of Ryden

- X-factor
  - Since a typical molecular cloud is optically thick at the J = 1-0 rotational transition of ^{12}CO, we use the radiative transfer solution for an opaque object:

$$I_\nu = \frac{2h\nu^3}{c^2}\left[\frac{e^{-\tau_\nu}}{\exp\left(h\nu/kT_{\rm rad}\right)-1} + \frac{1-e^{-\tau_\nu}}{\exp\left(h\nu/kT_{\rm exc}\right)-1}\right] \quad \rightarrow \quad \boxed{I_\nu \approx \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(h\nu/kT_{\rm exc}\right)-1}}$$

  - Expressed in terms of the antenna temperature, this is

$$T_{\rm A} \equiv \frac{c^2}{2k\nu^2} I_\nu \approx \frac{h\nu}{k}\frac{1}{\exp\left(h\nu/kT_{\rm exc}\right)-1}$$

However, this is wrong by a geometrical factor because we are dealing with a sphere. Toward outer boundary of the sphere, the optical depth will be thin.

  - The observed emission lines is narrow, we can assume that $\nu = \nu_{10}$ over the entire line width.

$$\int T_{\rm A} dv \approx \sqrt{\pi} b T_{\rm A}(\nu_{10}) \approx \sqrt{\pi}\frac{h\nu_{10}}{k}\frac{b}{\exp\left(h\nu_{10}/kT_{\rm exc}\right)-1}$$

assuming Gaussian line profile

$$T_{\rm A} = T_{\rm A}(0)e^{-v^2/b^2}$$

  - Then, the X-factor is

$$X_{\rm CO} = \frac{N({\rm H}_2)}{b}\frac{1}{\sqrt{\pi}}\frac{k}{h\nu_{10}}\left(e^{h\nu_{10}/kT_{\rm exc}} - 1\right)$$

- Now let's use virial theorem for a uniform, spherical cloud:

$$\sigma_v^2 = \frac{GM_c}{5R_c}, \quad b = \sqrt{2}\sigma_v \quad \rightarrow \quad \boxed{M_c = \frac{5}{2}\frac{R_c b^2}{G}} \qquad M_c \equiv \frac{4\pi}{3}R_c^3\rho_c \quad \rightarrow \quad \boxed{\frac{b}{R_c} = \left(\frac{8\pi}{15}G\rho_c\right)^{1/2}}$$

- Surface density:

$$\Sigma = \frac{M_c}{\pi R_c^2} \xrightarrow{M_c = \frac{5}{2}\frac{R_c b^2}{G}} \Sigma = \frac{5}{2\pi G}\frac{b^2}{R_c}$$

- Column density of hydrogen nucleon in terms of velocity dispersion:

$$N(\mathrm{H}) \approx \frac{\Sigma}{1.4 \times m_{\mathrm{H}}} = \frac{5}{2.8\pi}\frac{1}{Gm_{\mathrm{H}}}\frac{b^2}{R_c} \qquad N(\mathrm{H_2}) = \frac{1}{2}N(\mathrm{H})$$

- The X-factor is:

$$\frac{N(\mathrm{H_2})}{W_{\mathrm{CO}}} = \frac{5}{2 \times 2.8\pi\sqrt{\pi}}\frac{1}{Gm_{\mathrm{H}}}\frac{k}{h\nu_{10}}\frac{b}{R_c}\left(e^{h\nu_{10}/kT_{\mathrm{exc}}} - 1\right)$$

$$\frac{b}{R_c} = \left(\frac{8\pi}{15}G\rho_c\right)^{1/2} \approx \left(\frac{8\pi}{15}Gm_{\mathrm{H}}n_{\mathrm{H}} \times 1.4\right)^{1/2}$$

$$\frac{N(\mathrm{H})}{W_{\mathrm{CO}}} = \frac{1}{2\pi}\frac{k}{h\nu_{10}}\left(\frac{20}{3}\frac{1}{2.8Gm_{\mathrm{H}}}\right)^{1/2} n_{\mathrm{H}}^{1/2}\left(e^{h\nu_{10}/kT_{\mathrm{exc}}} - 1\right)$$

This result is too large by a factor or 8/3 = 2.67.

Molecular Clouds: Observations

- Cloud Structure
  - Local density estimates using line ratios often give larger densities than global mean densities found by averaging the observed molecular column densities along the line of sight.
  - The interpretation of this is that the clouds are very clumpy, with the dense cores having typical sizes of < 1 pc or smaller, and densities > 10^6 cm^{-3}.
  - The overall cloud extends for 3 — 20 pc on average, with a mean density of 10^{3-4} cm^{-3}.
  - Most molecular clouds show a number of discernible cores. These are often detected as sources of molecular lines with high critical densities (e.g., CS), while the general cloud is mapped using lines of lower critical density (mainly CO).
  - Within the galaxies, molecular clouds are most often seen organized into complexes with sizes from 20 pc to 100 pc, and overall H^2 masses of 10^{4-6} M_{sun}. The distinction between “clouds” and “complexes” in terms of sizes and masses is somewhat artificial. A more precise statement would be that we see a wide range of structures, from single small clouds to large complexes of clouds, with many complexes arrayed along the spiral arms of the Galaxy.

Molecular Clouds: Cloud Categories

- Cloud Categories (based on the total surface density)
  - Individual clouds are separated into categories based on their optical appearance: diffuse, translucent, or dark, depending on the visual extinction Av through the cloud.

Category	A_V (mag)	Examples
Diffuse Molecular Cloud	$\lesssim 1$	ζ Oph cloud, $A_V = 0.84$ [a]
Translucent Cloud	1 to 5	HD 24534 cloud, $A_V = 1.56$ [b]
Dark Cloud	5 to 20	B68 [c], B335 [d]
Infrared Dark Cloud (IRDC)	20 to $\gtrsim 100$	IRDC G028.53-00.25 [e]

[a] van Dishoeck & Black (1986).
[b] Rachford et al. (2002).
[c] Lai et al. (2003).
[d] Doty et al. (2010).
[e] Rathborne et al. (2010).

[Table 32.1, Draine]

  - Diffuse and translucent clouds have sufficient UV radiation to keep gas-phase carbon mainly photo ionized throughout the cloud.
    - Such clouds are usually pressure-confined, although self-gravity may be significant in some cases.
  - The typical dark clouds have Av ~ 10 mag, and is self-gravitating. Some dark clouds contain dense regions that are extremely opaque, with Av > 20 mag.
  - Infrared Dark Clouds are opaque even at 8 μm, and can be seen in silhouette against a background of diffuse 8 μm emission from PAHs in the ISM.

- Terminology for Cloud Complexes and Their Components

Categories	Size (pc)	n_{H} (cm^{-3})	Mass $(M_\odot)$	Linewidth $(\mathrm{km\,s}^{-1})$	A_V (mag)	Examples
GMC Complex	$25-200$	$50-300$	$10^5-10^{6.8}$	$4-17$	$3-10$	M17, W3, W51
Dark Cloud Complex	$4-25$	10^2-10^3	$10^3-10^{4.5}$	$1.5-5$	$4-12$	Taurus, Sco-Oph
GMC	$2-20$	10^3-10^4	$10^3-10^{5.3}$	$2-9$	$9-25$	Orion A, Orion B
Dark Cloud	$0.3-6$	10^2-10^4	$5-500$	$0.4-2$	$3-15$	B5, B227
Star-forming Clump	$0.2-2$	10^4-10^5	$10-10^3$	$0.5-3$	$4-90$	OMC-1, 2, 3, 4
Core	$0.02-0.4$	10^4-10^6	$0.3-10^2$	$0.3-2$	$30-200$	B335, L1535

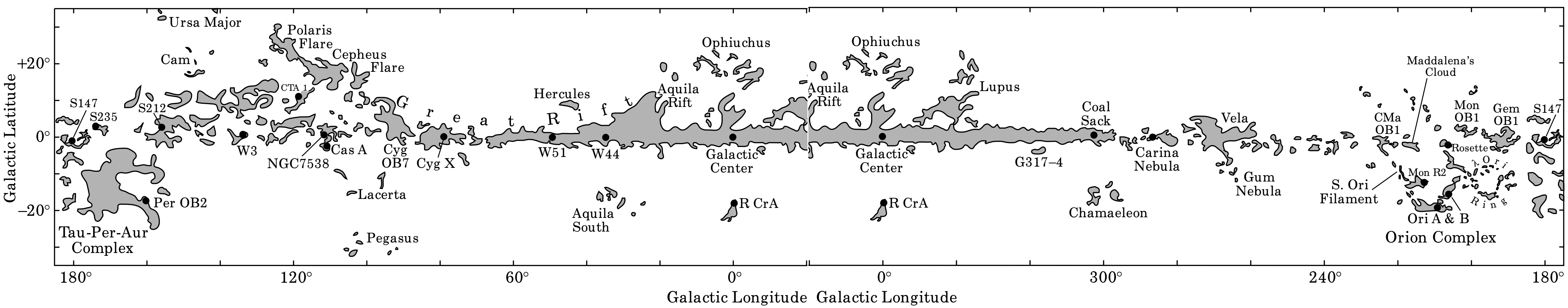
[Table 32.2, Draine]

- The **giant molecular cloud (GMC)** and **dark cloud** categories are distinguished mainly by total mass.
- Groups of distinct clouds are referred to as **cloud complexes**.
  - Molecular clouds are sometimes found in isolation, but in many cases molecular clouds are grouped together into complexes.
  - Since large clouds generally have substructure, the distinction between "cloud" and "cloud complex" is somewhat arbitrary.
  - Delineation of structure in cloud complexes is guided by the intensities and radial velocities of molecular lines (e.g., CO J = 1-0) as well as maps of thermal emission from dust at subs wavelengths.

Categories	Size (pc)	n_H (cm^{-3})	Mass ($M_\odot$)	Linewidth ($km\,s^{-1}$)	A_V (mag)	Examples
GMC Complex	$25-200$	$50-300$	$10^5-10^{6.8}$	$4-17$	$3-10$	M17, W3, W51
Dark Cloud Complex	$4-25$	10^2-10^3	$10^3-10^{4.5}$	$1.5-5$	$4-12$	Taurus, Sco-Oph
GMC	$2-20$	10^3-10^4	$10^3-10^{5.3}$	$2-9$	$9-25$	Orion A, Orion B
Dark Cloud	$0.3-6$	10^2-10^4	$5-500$	$0.4-2$	$3-15$	B5, B227
Star-forming Clump	$0.2-2$	10^4-10^5	$10-10^3$	$0.5-3$	$4-90$	OMC-1, 2, 3, 4
Core	$0.02-0.4$	10^4-10^6	$0.3-10^2$	$0.3-2$	$30-200$	B335, L1535

[Table 32.2, Draine]

- Structures within a cloud (self-gravitating entities) are described as **clumps**.
    - Clumps may or may not be forming stars; in the former case they are termed **star-forming clumps**. **Cores** are density peaks within star-forming clumps that will form a single star or a binary star.
- GMC and GMC complex
    - Much of the molecular mass is found in large clouds known as “giant molecular clouds”, with masses ranging from ~ 10^3 $M_\odot$ to ~ $2x10^5$ $M_\odot$. Theses have reasonably well-defined boundaries.
    - A GMC complex is a gravitationally bound group of GMCs (and smaller clouds) with a total mass $\gtrsim 10^{5.3} M_\odot$.



Locations of prominent molecular clouds along the Milky Way

[Fig 32.2, Draine, Dame et al. (2001)]

- Overall mass distribution of GMCs in the Milky Way
  - CO line surveys can detect GMCs at large distances, allowing the total number in the Galaxy to be estimated.
  - The overall mass distribution of GMCs in the Milky Way (excluding the molecular material within a few hundred pc of the Galactic center) can be approximated by a power-law:

$$\frac{dN_{\rm GMC}}{d\ln M_{\rm GMC}} \approx N_u \left(\frac{M_{\rm GMC}}{M_u}\right) \quad \text{for} \quad 10^3\, M_\odot \lesssim M_{\rm GMC} < M_u$$

$$M_u \approx 6 \times 10^6 M_\odot$$

$$N_u \approx 63$$

$$\alpha \approx 0.6$$

  - Most of the mass is in the most massive GMCs:
    - ~ 80% of the molecular mass is in GMCs with $M > 10^5\, M_\odot$.

- Star Counts
  - Molecular clouds were originally discovered by star counts.
    - Herschel (1785) noticed that there were patches along the Milky Way where very few stars were seen. He incorrectly attributed this to a real absence of stars.
    - We now understand that the apparent deficiency of stars is the result of obscuration by dusty clouds.
    - Star counts using background stars is a good way to study the cloud structure.
    - Because the visual obscuration can be very large, studies of dark clouds using star counts are now usually done in the J, H, or K bands.

Gas Surface Density in the Milky Way

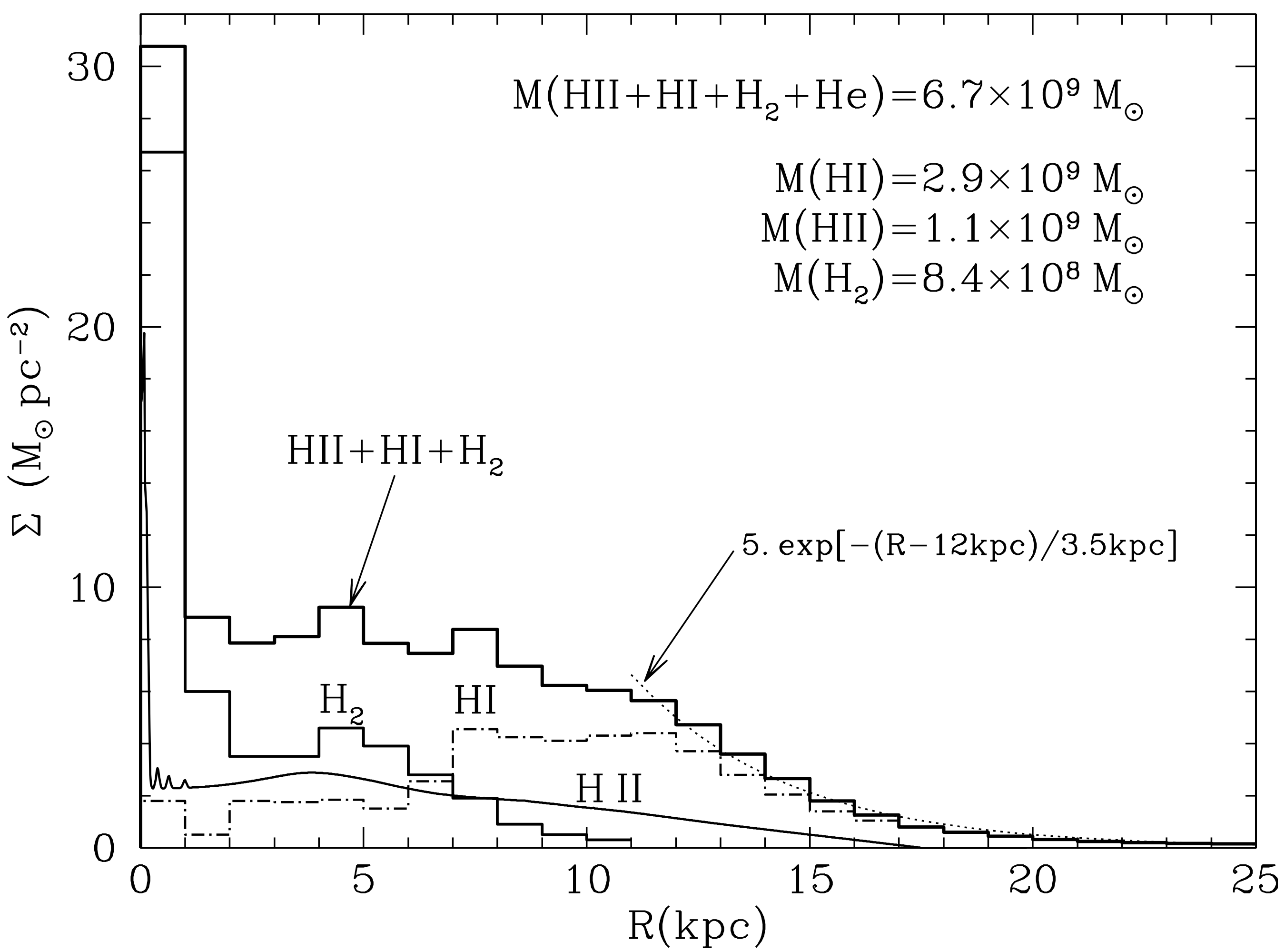
- The most common way to study molecular gas is through molecular line emission, and the primary line used is the J = 1-0 transition of CO.
  - This transition is often optically thick, but the CO 1-0 luminosity of a cloud is approximately proportional to the total mass.
  - Velocity-resolved mapping of CO 1-0 together with an assumed rotation curve and an adopted value of the "CO to H_2 conversion factor" X_{CO} have been used to infer the surface density of H_2 over the Milky Way disk.

$$X_{\rm CO} = 1.8 \times 10^{20}\,{\rm H_2\,cm^{-2}/K\,km\,s^{-1}}$$

Gas surface densities as a function of galactocentric radius. The Sun is assumed to be at R = 8.5 kpc.

- Surface density of H2 estimated from CO 1-0 observations (Nakanishi & Sofue 2006).
- Surface density of H II derived from pulsar dispersion measures (Cordes & Lazio 2003).
- Surface density of H I from 21-cm studies (Nakanishi & Sofue 2003)

[Fig 32.4, Draine]



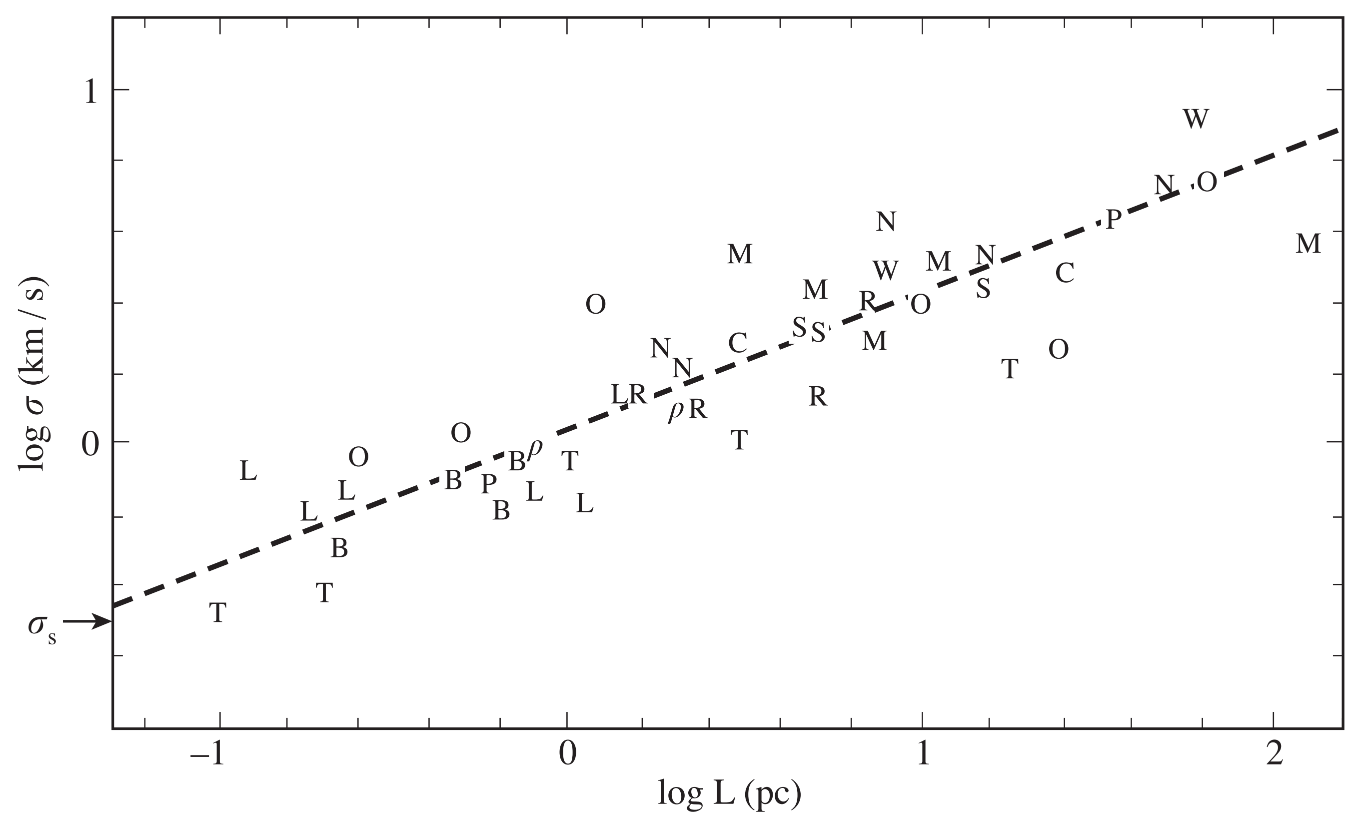
Size-Linewidth Relation in Molecular Clouds

- Larson (1981) noted that observations of molecular clouds in spectral lines of CO, H_2CO, NH_3, OH, and other species, were broadly consistent with a size-linewidth relation, where a density peak of characteristic size L tends to have a 3D velocity dispersion given by

$$\sigma_v^* \approx 1.10\, L_{\rm pc}^{\gamma}\, {\rm km\, s^{-1}}, \quad \gamma \approx 0.38 \quad \text{for } 0.01 \lesssim L_{\rm pc} \lesssim 10^2 \quad (L_{\rm pc} = L/{\rm pc})$$

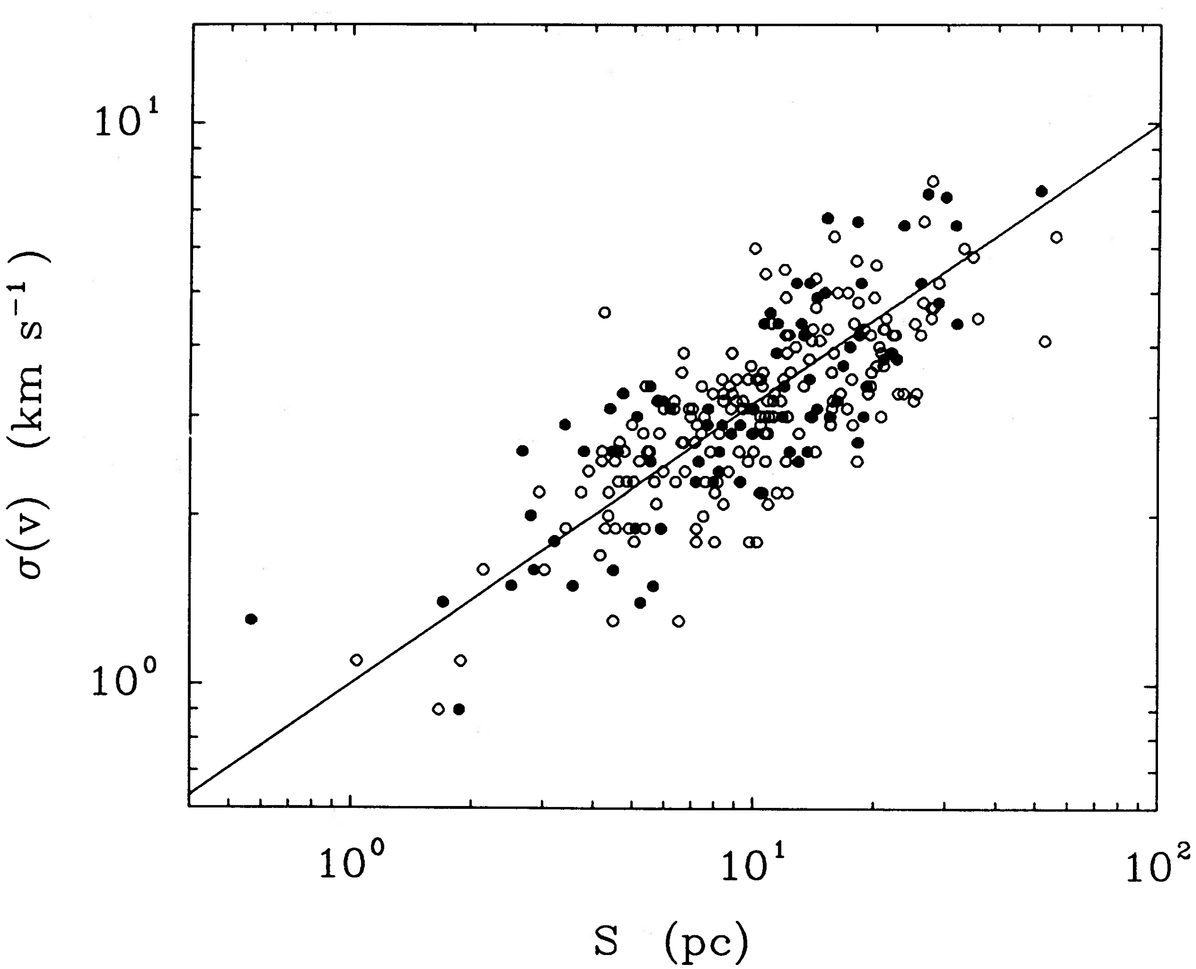
where *L* is the maximum projected dimension of the density peak.

- Larson noted that the power-law index $\gamma \approx 0.38$ is curiously close to the index 1/3 found by Kolmogorov for a turbulent cascade in an incompressible fluid.
- It therefore is tempting to refer to the observed fluid motion as “turbulence,” although in realty the motions are some combination of thermal motions, rotation, MHD waves, and turbulence.

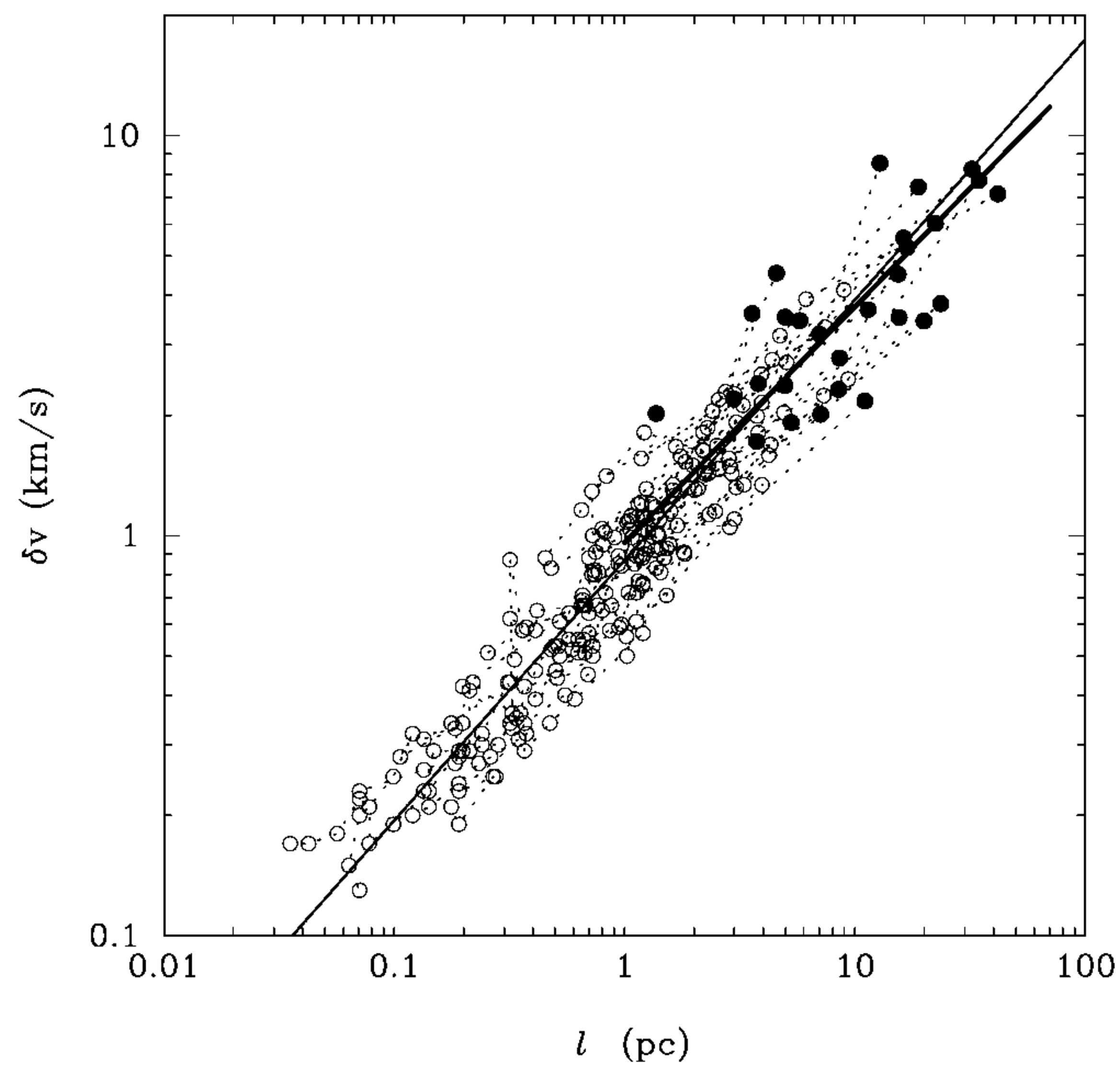


The 3D internal velocity dispersion versus maximum linear dimension L of the density peak.

Fig 32.5, Draine; Larson (1981)



Molecular cloud velocity dispersion as a function of size S for 273 clouds in the Galaxy
Fig 1, Solomon et al. (1987)



Velocity dispersion versus size S from PCA decomposition of ^{12}CO J = 1-0 imaging observations of 27 individual molecular clouds

Fig 1, Heyer & Brunt (2004)

- However, the power-law index $\gamma \approx 0.38$ has been questioned.
    - Solomon et al. (1987) found $\sigma_v^* \approx 1.0\, L_{\rm pc}^{0.5}\,{\rm km\,s^{-1}}$ from a study of 273 molecular clouds.
    - Heyer & Brunt (2004) found $\sigma_v^* \approx 0.96\, L_{\rm pc}^{0.59}\,{\rm km\,s^{-1}}$.
- Note that
    - For $L \gtrsim 0.02\,{\rm pc}$, $\sigma_v^* \gtrsim (kT/\mu)^{1/2} \approx 0.23 (T/15\,{\rm K})^{1/2}\,{\rm km\,s^{-1}}$ $\Rightarrow$ The linewidth is supersonic.
    - For $L \lesssim 0.02\,{\rm pc}$, the line width is nearly thermal, with only a small contribution from rotation, waves or turbulence.
- The density peaks are generally self-gravitating. If we assume them to be in approximate virial equilibrium, and consider only the kinetic energy associated with fluid motions, we can estimate the clump mass using the virial theorem.
    - For a uniform density sphere with a diameter *L*, we can obtain the follows:

$$\sigma_v^{*2} = 3\sigma_v^2 = 6GM/5L \quad (L = 2R) \longrightarrow$$

for $\gamma \approx 0.38$

clump mass:	$M \approx \dfrac{5\sigma_v^{*2} L}{6G} \approx 230\, L_{\rm pc}^{2\gamma+1}\, M_\odot$	$\rightarrow$	$230\, L_{\rm pc}^{1.76}\, M_\odot$
density:	$n_{\rm H} \approx M/[(4\pi/3)(L/2)^3 1.4 m_{\rm H}] \approx 1.3\times10^4\, L_{\rm pc}^{2\gamma-2}\,{\rm cm^{-3}}$	$\rightarrow$	$1.3\times10^4\, L_{\rm pc}^{-1.24}\,{\rm cm^{-3}}$
column density:	$N_{\rm H} = n_{\rm H} L \approx 4.0\times10^{22}\, L_{\rm pc}^{2\gamma-1}\,{\rm cm^{-2}}$	$\rightarrow$	$4.0\times10^{22}\, L_{\rm pc}^{-0.24}\,{\rm cm^{-2}}$

- We recall that for the dust in diffuse clouds, $A_V/N_{\rm H} = 1.87 \times 10^{21}\,{\rm cm}^2$, and we would have

$$A_V \approx 21\, L_{\rm pc}^{2\gamma-1}\,{\rm mag} \quad \to \quad 21\, L_{\rm pc}^{-0.24}\,{\rm mag}$$

The dust in dense clouds differs from that in the diffuse ISM, but this would give a reasonable estimate of the visual extinction through the cloud.

- If $\gamma \approx 0.38$, then a GMC complex with diameter $L \approx 50\,{\rm pc}$ would have

$$\begin{aligned} M &\approx 2 \times 10^5\, M_\odot \\ n_{\rm H} &\approx 100\,{\rm cm}^{-3} \\ A_V &\approx 8\,{\rm mag} \end{aligned}$$

whereas a core with diameter $L \approx 0.1\,{\rm pc}$ would have

$$\begin{aligned} M &\approx 4\, M_\odot \\ n_{\rm H} &\approx 2 \times 10^5\,{\rm cm}^{-3} \\ A_V &\approx 40\,{\rm mag} \end{aligned}$$

- Comments on the exponent:
  - If $\gamma < 0.5$, smaller clouds tend to have a higher column density, meaning to be darker.
  - If $\gamma = 0.5$, small clouds and large clouds would all have the same column density and dust extinction (Av).
  - However, observationally, smaller structures tend to have higher Av. This is consistent with exponent $\gamma < 0.5$.
  - Therefore, the Larson's exponent would be better constant with the observations.

- Expressing the above relations with density as the independent variables, we obtain

$$L_{\rm pc} = (n_3/13)^{1/(2\gamma-2)} \quad \rightarrow 7.94\, n_3^{-0.81} \quad ,$$

$$\sigma_v = 1.1(n_3/13)^{\gamma/(2\gamma-2)}\,\mathrm{km\,s^{-1}} \quad \rightarrow 2.43\, n_3^{-0.31}\,\mathrm{km\,s^{-1}} \quad ,$$

$$M = 230(n_3/13)^{(2\gamma+1)/(2\gamma-2)}\, M_\odot \quad \rightarrow 8940\, n_3^{-1.42}\, M_\odot \quad ,$$

$$A_V = 21(n_3/13)^{(2\gamma-1)/(2\gamma-2)}\,\mathrm{mag} \quad \rightarrow 13\, n_3^{0.19}\,\mathrm{mag} \quad ,$$