

# Astrophysics

Lecture 07

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# Four-velocity

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The (infinitesimally small) difference between the coordinates of two events is also a four-vector. Dividing by the proper time yields a four-vector, the four-velocity:

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \rightarrow U^0 = \frac{cdt}{d\tau} = c\gamma_u \quad \text{or} \quad U^i = \frac{dx^i}{d\tau} = \gamma_u u^i \quad \boxed{\vec{U} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}}$$

$d\tau = dt/\gamma$

where  $\gamma_u \equiv (1 - u^2/c^2)^{-1/2}$   
 $u \equiv \left| \frac{d\mathbf{x}}{dt} \right|$

length of the four-velocity :

$$\vec{U} \cdot \vec{U} = U^\mu U_\mu = -(\gamma_u c)^2 + (\gamma_u \mathbf{u})^2 = -c^2$$

Transformation of the four-velocity:

$$\begin{aligned} U'^0 &= \gamma (U^0 - \beta U^1) & \gamma_{u'} c &= \gamma (c\gamma_u - \beta\gamma_u u^1) & \rightarrow & \gamma_{u'} = \gamma\gamma_u (1 - vu'/c^2) \\ U'^1 &= \gamma (-\beta U^0 + U^1) & \gamma_{u'} u'^1 &= \gamma (-\beta c\gamma_u + \gamma_u u^1) & \gamma_{u'} u'^1 &= \gamma\gamma_u (u^1 - v) \\ U'^2 &= U^2 & \gamma_{u'} u'^2 &= \gamma_u u^2 \\ U'^3 &= U^3 & \gamma_{u'} u'^3 &= \gamma_u u^3 \end{aligned}$$

The first two equations become:

$$\begin{aligned} \gamma_{u'} &= \gamma\gamma_u (1 - vu'/c^2) \\ \gamma_{u'} u'^1 &= \gamma\gamma_u (u^1 - v) \end{aligned}$$

Note:  $\gamma$  denotes the factor for the relative velocity between two frames.  
 $\gamma_u$  and  $\gamma_{u'}$  are the factors for a velocity vector measured in  $K$  and  $K'$ , respectively.

velocity component:

$$u'^1 = \frac{u^1 - v}{1 - vu^1/c^2}$$

This is the previously derived formula.



speed:

$$\gamma_{u'} = \gamma\gamma_u \left( 1 - \frac{vu^1}{c^2} \right)$$

This is the transform for speed.

Here,  $u^1 = u \cos \theta$  and  $u'^1 = u' \cos \theta'$

# Momentum and Energy

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- Four-momentum of a particle with a mass  $m_0$  is defined by

$$P^\mu \equiv m_0 U^\mu \quad P^0 = m_0 c \gamma_v \quad P^i = \gamma_v m_0 \mathbf{v}$$

- In the nonrelativistic limit,

$$P^0 c = m_0 c^2 \gamma = m_0 c^2 \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} = m_0 c^2 + \frac{1}{2} m_0 v^2 + \dots$$

Therefore, we interpret  $E \equiv P^0 c = \gamma_v m_0 c^2$  as the total energy of the particle.

The quantity  $m_0 c^2$  is interpreted as the rest energy of the particle.

Then,

$\mathbf{p} \equiv \gamma_v m_0 \mathbf{v}$ ,  $P^\mu = (E/c, \mathbf{p})$  Here,  $\mathbf{p}$  is the spatial component of the four-momentum.

Since  $\vec{U}^2 = -c^2$ , we obtain  $\vec{P}^2 = -m_0^2 c^2$ . Comparing with  $\vec{P}^2 = -\frac{E^2}{c^2} + |\mathbf{p}|^2$ , we obtain

$$E^2 = m_0^2 c^4 + c^2 |\mathbf{p}|^2$$

- Photons are massless, but we can still define

$$P^\mu = (E/c, \mathbf{p}), \quad E = |\mathbf{p}| c \quad \rightarrow \quad \vec{P}^2 = 0$$

# Wavenumber vector and frequency

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- Quantum relations:

$$\begin{aligned} E &= h\nu = \hbar\omega \\ p &= E/c = \hbar k \end{aligned} \quad \left( \begin{array}{l} \omega = 2\pi\nu \\ k = 2\pi/\lambda \end{array} \right)$$

We can define four wavenumber vector:

$$\vec{k} = \frac{1}{\hbar} \vec{P} = \left( \frac{\omega}{c}, \mathbf{k} \right)$$

Then, we obtain an invariant:

$$\vec{k} \cdot \vec{x} = k_\mu x^\mu = \mathbf{k} \cdot \mathbf{x} - \omega t$$

Note that it's a null vector:

$$\vec{k} \cdot \vec{k} = |\mathbf{k}|^2 - \omega^2/c^2 = 0$$

Therefore, **the phase of the plane wave is an invariant.**

- Transform of  $\vec{k}$  gives the Doppler shift formula.

$$\begin{aligned} k'^0 &= \gamma (k^0 - \beta k^1) & \longrightarrow & \omega' = \gamma (\omega - \beta c k^1) = \omega \gamma \left( 1 - \frac{v}{c} \cos \theta \right) \\ k'^1 &= \gamma (-\beta k^0 + k^1) & & \uparrow \\ k'^2 &= k^2 \\ k'^3 &= k^3 \end{aligned}$$

# \* Tensor Analysis \*

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- Definition:

zeroth-rank tensor : Lorentz invariant (scalar)       $s' = s$

first-rank tensor : four-vector                           $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

second-rank tensor:     $T'^{\mu\nu} = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau T^{\sigma\tau}$

- Covariant components and mixed components:

$$T_{\mu\nu} = \eta_{\mu\sigma}\eta_{\nu\tau}T^{\sigma\tau} \quad T^\mu{}_\nu = \eta_{\nu\tau}T^{\mu\tau} \quad T_\mu{}^\nu = \eta_{\mu\sigma}T^{\sigma\nu}$$

- Transformation rules:

$$\begin{aligned} T'_{\mu\nu} &= \eta_{\mu\alpha}\eta_{\nu\beta}T'^{\alpha\beta} \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}\Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta T^{\gamma\delta} \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}\Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta \eta^{\gamma\sigma}\eta^{\delta\tau} T_{\sigma\tau} \\ &= \tilde{\Lambda}_\mu{}^\sigma \tilde{\Lambda}_\nu{}^\tau T_{\sigma\tau} \end{aligned}$$

$$\begin{aligned} T'^\mu{}_\nu &= \eta_{\nu\alpha}T'^{\mu\alpha} \\ &= \eta_{\nu\alpha}\Lambda^\mu{}_\sigma\Lambda^\alpha{}_\delta T^{\sigma\delta} \\ &= \eta_{\nu\alpha}\Lambda^\mu{}_\sigma\Lambda^\alpha{}_\delta \eta^{\delta\tau} T_\tau^\sigma \\ &= \Lambda^\mu{}_\sigma \tilde{\Lambda}_\nu{}^\tau T_\sigma^\tau \end{aligned}$$

$$\begin{aligned} T'_\mu{}^\nu &= \eta_{\mu\alpha}T'^{\alpha\nu} \\ &= \eta_{\mu\alpha}\Lambda^\alpha{}_\beta\Lambda^\nu{}_\tau T^{\beta\tau} \\ &= \eta_{\mu\alpha}\Lambda^\alpha{}_\beta\Lambda^\nu{}_\tau \eta^{\beta\sigma} T_\sigma{}^\tau \\ &= \tilde{\Lambda}_\mu{}^\beta \Lambda^\nu{}_\tau T_\sigma{}^\tau \end{aligned}$$

- Symmetric tensor = a tensor that is invariant under a permutation of its indices.

$$T^{\mu\nu} = T^{\nu\mu}$$

- Antisymmetric tensor : if it alternates sign when any two indices of the subset are interchanged.

$$T^{\mu\nu} = -T^{\nu\mu}$$

- Examples of the second-rank tensors

A product of two vectors:  $A^\mu B^\nu$

$$A'^\mu B'^\nu = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau A^\sigma B^\tau$$

The Minkowski metric:

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Kronecker-delta:

$$\delta^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Higher-rank tensors
  - Addition:  $A^\mu + B^\mu$ ,  $F^{\mu\nu} + G^{\mu\nu}$
  - Multiplication:  $A^\mu B^\nu$ ,  $F^{\mu\nu} G_{\sigma\tau}$
  - Raising and Lowering Indices: The metric can be used to change contravariant indices into covariant ones, and vice versa, by the processes of raising and lowering.
  - Contraction:  $A^\mu B_\nu \rightarrow A^\mu B_\mu$  scalar  
 $T^{\mu\nu}_\sigma \rightarrow T^{\mu\nu}_\nu$  vector

$$T'^{\mu\nu}_\nu = \Lambda^\mu_\alpha \Lambda^\nu_\beta \tilde{\Lambda}_\nu^\tau T^{\alpha\beta}_\tau = \Lambda^\mu_\alpha \delta^\tau_\beta T^{\alpha\beta}_\tau = \Lambda^\mu_\alpha T^{\alpha\beta}_\beta$$

- Gradients of Tensor Fields: A tensor field is a tensor that is a function of the spacetime coordinates in Cartesian coordinate systems. The gradient operation  $\partial/\partial x^\mu \equiv \partial_\mu$  acting on such a field produces a tensor field of on higher rank with  $\mu$  as a new covariant index.

$$\lambda \rightarrow \frac{\partial \lambda}{\partial x^\mu} \equiv \partial_\mu \lambda \equiv \lambda_{,\mu} \quad \text{vector (gradient)} \quad A^\mu \rightarrow \frac{\partial A^\mu}{\partial x^\mu} \equiv \partial_\mu A^\mu \equiv A^\mu_{,\mu} \quad \text{scalar (divergence)}$$

- **Invariance of form or Lorentz covariance or covariance:** A fundamental property of a tensor equation is that if it is true in one Lorentz frame, then it is true in all Lorentz frames. Covariance plays a powerful role in helping decide what the proper equations of physics are.

# [Covariance of Electromagnetic Phenomena]

- Equation of charge conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

The above equation can be written as a tensor equation,

$$\boxed{\frac{\partial j^\mu}{\partial x^\mu} = 0, \quad j^{\mu,\mu} = 0 \quad \text{or} \quad \partial_\mu j^\mu = 0}$$

if the **four-current** is defined by

$$j^\mu = \begin{pmatrix} \rho c \\ \mathbf{j} \end{pmatrix} \quad j_\mu = \begin{pmatrix} -\rho c \\ \mathbf{j} \end{pmatrix}$$

- Note that the Jacobian (determinant) of the transformation from  $x_\mu$  to  $x'_\mu$  is simply the determinant of  $\Lambda$ , which is unity. Therefore, the **four-volume element is an invariant.**

$$dx'_0 dx'_1 dx'_2 dx'_3 = \det \Lambda dx_0 dx_1 dx_2 dx_3 = dx_0 dx_1 dx_2 dx_3$$

Since  $\rho$  is the zeroth component of the four-current, **the charge element within a three-volume element is an invariant.**

$$de = \rho dx_1 dx_2 dx_3$$

It is also an empirical fact that  $e$  is invariant.

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left( -\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

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- The set of vector and scalar wave equations in the Lorentz gauge is

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{j}$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{4\pi}{c} (\rho c)$$

If we define the **four-potential**

$$A^\mu = \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix}, \quad A_\mu = \begin{pmatrix} -\phi \\ \mathbf{A} \end{pmatrix},$$

then the wave equations can be written as the tensor equations

$$\frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} = -\frac{4\pi}{c} j^\mu, \quad \partial_\nu \partial^\nu A^\mu = -\frac{4\pi}{c} j^\mu, \quad A^{\mu,\nu} = -\frac{4\pi}{c} j^\mu$$

d'Alembertian operator:  $\square \equiv \frac{\partial^2}{\partial x^\nu \partial x_\nu} \rightarrow \boxed{\square A^\mu = -\frac{4\pi}{c} j^\mu}$

- The Lorentz gauge should be preserved under Lorentz transformations.

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \rightarrow \boxed{\frac{\partial A^\mu}{\partial x^\mu} = 0} \text{ or } A^\mu{}_\mu = 0$$

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- **Electromagnetic field tensor:**

The fields are expressed in terms of the potentials as  $\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$   
 $\mathbf{B} = \nabla \times \mathbf{A}$

The  $x$  components of the electric and magnetic fields are explicitly

$$\begin{aligned} E_x &= -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial\phi}{\partial x} = \partial^0 A^1 - \partial^1 A^0 \\ B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \partial^2 A^3 - \partial^3 A^2 \end{aligned}$$

These equations imply that the electric and magnetic fields, six components in all, are the elements of a **second-rank, antisymmetric field-strength tensor**, because a rank two antisymmetric tensor has exactly six independent components.

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$



$F^{0i} = E_i$   
 $F^{i0} = -E_i$   
 $F^{12} = -F^{21} = B_3, \dots$

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covariant field-strength tensor

$$F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}$$



$$F_{0i} = \eta_{0\alpha}\eta_{i\beta}F^{\alpha\beta} = -F^{0i}$$

$$F_{i0} = \eta_{i\alpha}\eta_{0\beta}F^{\alpha\beta} = -F^{i0}$$

$$F_{ij} = \eta_{i\alpha}\eta_{j\beta}F^{\alpha\beta} = F^{ij}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$



$$F_{0i} = -E_i$$

$$F_{i0} = E_i$$

$$F_{12} = -F_{21} = B_3, \dots$$

- The two Maxwell equations containing sources (inhomogeneous equations):

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{j}$$



$$\sum_{i=1}^3 \partial_i E_i = \frac{4\pi}{c} j^0$$

$$\partial_2 B_3 - \partial_3 B_2 - \partial_0 E_1 = \frac{4\pi}{c} j^1$$



$$-\sum_{i=1}^3 \partial_i F^{i0} = \frac{4\pi}{c} j^0$$

$$-\partial_0 F^{01} - \partial_2 F^{21} - \partial_3 F^{31} = \frac{4\pi}{c} j^1$$

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} j^\nu \quad \text{or} \quad \partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu$$

$$\partial^\mu F_{\mu\nu} = -\frac{4\pi}{c} j_\nu \quad \text{or} \quad \partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu$$

- The conservation of charge easily follows from the above equation and the asymmetric property.

$$\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu \partial_\mu (-F^{\nu\mu}) = -\partial_\mu \partial_\nu F^{\mu\nu}$$

↑  
index exchange

$$\therefore \boxed{\partial_\mu \partial_\nu F^{\mu\nu} = 0}$$

$$\partial_\nu j^\nu = -\frac{c}{4\pi} \partial_\nu \partial_\mu F^{\mu\nu} = 0$$

- The “internal” Maxwell equations (homogeneous equations):

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \longrightarrow \quad \sum_{i=1}^3 \partial_i B_i = 0$$

$$\partial_2 E_3 - \partial_3 E_2 + \partial_0 B_1 = 0 \quad \longrightarrow \quad \begin{aligned} \partial_1 F^{23} + \partial_2 F^{31} + \partial_3 F^{12} &= 0 \\ \partial_2 F^{30} + \partial_3 F^{20} + \partial_0 F^{23} &= 0 \end{aligned}$$

$$\boxed{\partial_\mu F^{\nu\sigma} + \partial_\nu F^{\sigma\mu} + \partial_\sigma F^{\mu\nu} = 0} \quad \text{or} \quad \partial^\mu F_{\nu\sigma} + \partial^\nu F_{\sigma\mu} + \partial^\sigma F_{\mu\nu} = 0$$

The equation can be written concisely as  $F^{[\mu\nu,\sigma]} = 0$  or  $F_{[\mu\nu,\sigma]} = 0$ , where [ ] around indices denote all permutations of indices, with even permutation contributing with a positive sign and odd permutation with a negative sign, for example,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{[\nu,\mu]}$$

# - Transformation of Electromagnetic Fields

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- Since  $F^{\mu\nu}$  is a second-rank tensor, its components transform in the usual way:

$$F'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} F^{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$$

For a pure boost along the  $x$ -axis:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{aligned} E'_x &= F'^{01} = \Lambda^0{}_0 \Lambda^1{}_1 F^{01} + \Lambda^0{}_1 \Lambda^1{}_0 F^{10} = \gamma^2 E_x - \beta^2 \gamma^2 E_x = E_x \\ E'_y &= F'^{02} = \Lambda^0{}_0 \Lambda^2{}_2 F^{02} + \Lambda^0{}_1 \Lambda^2{}_2 F^{12} = \gamma E_y - \beta \gamma B_z \\ E'_z &= F'^{03} = \Lambda^0{}_0 \Lambda^3{}_3 F^{03} + \Lambda^0{}_1 \Lambda^3{}_3 F^{13} = \gamma E_z + \beta \gamma B_y \\ B'_x &= F'^{23} = \Lambda^2{}_2 \Lambda^3{}_3 F^{23} = B_x \\ B'_y &= F'^{31} = \Lambda^3{}_3 (\Lambda^1{}_0 F^{30} - \Lambda^1{}_1 F^{31}) = \beta \gamma E_z + \gamma B_y \\ B'_z &= F'^{12} = \Lambda^1{}_0 \Lambda^2{}_2 F^{03} + \Lambda^1{}_1 \Lambda^2{}_2 F^{12} = -\beta \gamma E_y + \gamma B_z \end{aligned}$$

- In general,

$$\mathbf{E}'_{||} = \mathbf{E}_{||}$$

$$\mathbf{E}'_{\perp} = \gamma (\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B})$$

$$\mathbf{B}'_{||} = \mathbf{B}_{||}$$

$$\mathbf{B}'_{\perp} = \gamma (\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E})$$

The concept of a pure electric or pure magnetic is not Lorentz invariant.

- Lorenz invariants:
  - The dot product of  $F$  with itself or “square” of  $F$  is

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= \sum_{i=1}^3 F^{0i}F_{0i} + \sum_{i=1}^3 F^{i0}F_{i0} + \sum_{i \neq j} F^{ij}F_{ij} \\ &= 2(\mathbf{B}^2 - \mathbf{E}^2) \end{aligned}$$

Therefore,  $\mathbf{B}^2 - \mathbf{E}^2$  is invariant under Lorentz transformations.

- The determinant of  $F$  is invariant.

$$\det F = (\mathbf{E} \cdot \mathbf{B})^2$$

Thus  $\mathbf{E} \cdot \mathbf{B}$  is also an invariant.

- The determinant of any second-rank tensor is scalar, since

$$\begin{aligned} \det A_{\alpha\beta} &= \det \tilde{\Lambda}_\alpha^\mu \tilde{\Lambda}_\beta^\nu A'_{\mu\nu} \\ &= (\det \tilde{\Lambda})^2 \det A'_{\mu\nu} \\ &= \det A'_{\mu\nu} \end{aligned}$$

## [Relativistic Mechanics and the Lorentz Four-Force]

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- We can define a **four-acceleration**  $a^\mu$  in exactly the same way as we obtained the four-velocity.

$$a^\mu \equiv \frac{dU^\mu}{d\tau}$$

Note that the four-acceleration and four-velocity are orthogonal:

$$\vec{a} \cdot \vec{U} \equiv \frac{dU^\mu}{d\tau} U_\mu = \frac{1}{2} \frac{d}{d\tau} (U^\mu U_\mu) = \frac{1}{2} \frac{d}{d\tau} (-c^2) = 0$$

- We can also define the **four-force**  $F^\mu$  from the Lorentz force, so as to obtain a relativistic form of Newton's equation.

$$F^\mu \equiv m_0 a^\mu = \frac{dP^\mu}{d\tau}$$

$$\vec{F} = \frac{d\vec{P}}{d\tau} = \gamma \frac{d\vec{P}}{d\tau} = \gamma \left( \frac{1}{c} \frac{dE}{dt}, \frac{d\mathbf{p}}{dt} \right)$$

- Since  $\mathbf{F}_{\text{Lorentz}} = q \left[ \mathbf{E} + \frac{1}{c} (\mathbf{v} \times \mathbf{B}) \right]$ , the **Lorentz four-force** should involve (1) the electromagnetic field tensor and (2) the four-velocity and should also be (3) a four-vector and (4) proportional to the charge of the particle. Therefore, the simplest possibility is

$$F_{\text{Lorentz}}^\mu = \frac{q}{c} F^{\mu\nu} U_\nu$$

- Let's check to see if it is indeed what we want.

$$F_{\text{Lorentz}}^0 = \frac{q}{c} F^{0\nu} U_\nu = \frac{q}{c} \sum_{i=1}^3 E_i \gamma v_i = \frac{q}{c} \gamma (\mathbf{E} \cdot \mathbf{v}) \longrightarrow \frac{dE}{dt} = q \mathbf{E} \cdot \mathbf{v} : \text{conservation of energy}$$

The rate of change of particle energy is the mechanical work done on the particle by the field.

$$\begin{aligned} F_{\text{Lorentz}}^1 &= \frac{q}{c} F^{1\nu} U_\nu = \frac{q}{c} (F^{10}(-\gamma c) + F^{12}\gamma v_2 + F^{13}\gamma v_3) \\ &= \frac{q}{c} \gamma (E_1 c + B_3 v_2 - B_2 v_3) \end{aligned} \longrightarrow \frac{d\mathbf{p}}{dt} = q \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right)$$

Therefore, we obtained the desired expression for the four-Lorentz force.

- Note that **the four-force is always orthogonal to the four-velocity**:

$$\vec{F} \cdot \vec{U} = m_0 (\vec{a} \cdot \vec{U}) = 0$$

It implies that **every four-force must have some velocity dependence**, although this dependence might become negligible in the nonrelativistic limit.

For the Lorentz four-force, in particular, we find

$$\vec{F}_{\text{Lorentz}} \cdot \vec{U} = \frac{q}{c} F^{\mu\nu} U_\mu U_\nu = 0$$

because  $F^{\mu\nu}$  is antisymmetric and  $U_\mu U_\nu$  is symmetric.

$$\boxed{\frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v} = q \mathbf{E} \cdot \mathbf{v}}$$

# Mathematical Formulae

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- Gamma function

$$\Gamma(x) \equiv \int_0^\infty t^{x-1} e^{-t} dt, \quad \Gamma(x) = (x-1)! = (x-1)\Gamma(x-2), \quad \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

- Euler-Mascheroni constant

$$\gamma \equiv \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \ln(n) \right) = - \int_0^\infty e^{-x} \ln x dx = 0.577215664901532$$

- Modified Bessel function of the second kind

$$K_n(x) \equiv \frac{\Gamma(n+1/2)(2x)^n}{\sqrt{\pi}} \int_0^\infty \frac{\cos t}{(t^2 + x^2)^{n+1/2}} dt$$

$$(1) \quad 0 < x < \sqrt{n+1}$$

$$K_n(x) \approx \begin{cases} -\ln\left(\frac{x}{2}\right) - \gamma & \text{if } n = 0 \\ \frac{\Gamma(n)}{2} \left(\frac{2}{x}\right)^n & \text{if } n > 0 \end{cases}$$

$$(2) \quad x \gg |n^2 - 1/4|$$

$$K_n(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \left[ 1 + \frac{(4n^2 - 1)}{8x} \right]$$

## Recurrence formulae

$$K_{n-1}(x) - K_{n+1}(x) = -\frac{2n}{x} K_n(x)$$

$$K_{n-1}(x) + K_{n+1}(x) = -2K'_n(x)$$

## Integral formula

$$\begin{aligned} \int x K_n^2(x) dx &= \frac{1}{2} x^2 [K_n^2(x) - K_{n-1}(x)K_{n+1}(x)] \\ &= -x K_{n-1}(x) K_n(x) + \\ &\quad \frac{1}{2} x^2 [K_n^2(x) - K_{n-1}^2(x)] \end{aligned}$$

## [Fields of a Uniformly Moving Charge]

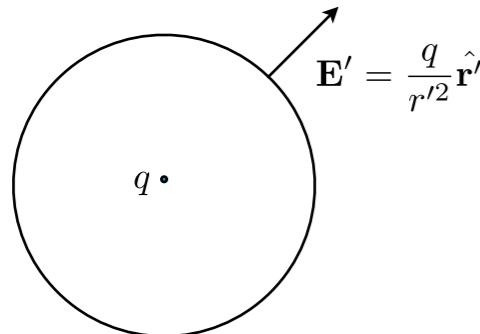
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- Recall that the electromagnetic field from a single moving charge is given by

|   |   |
|---|---|
| <span style="color: red;">velocity field</span>   | <span style="color: red;">acceleration field</span> |
| $\mathbf{E}(\mathbf{r}, t) = q \left[ \frac{(\mathbf{n} - \boldsymbol{\beta})(1 - \beta^2)}{\kappa^3 R^2} \right]_{\text{ret}} + \frac{q}{c} \left[ \frac{\mathbf{n}}{\kappa^3 R} \times \left\{ (\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right\} \right]_{\text{ret}}$ |   |
| $\mathbf{B}(\mathbf{r}, t) = [\mathbf{n} \times \mathbf{E}(\mathbf{r}, t)]_{\text{ret}}$  |   |

- The velocity field is caused by a motion of the charge relative to the observer frame. Hence, the velocity field would be obtained by Lorentz transforming the static electromagnetic field (Coulomb field).

- Let's find the fields of a charge moving with constant velocity  $v$  along the  $x$  axis. In the rest frame  $K'$  of the particle the fields are given by



### Coulomb field in $K'$

$$\mathbf{E}' = (E'_x, E'_y, E'_z) = \frac{q}{r'^3} (x', y', z') \quad \text{where} \quad r' = (x'^2 + y'^2 + z'^2)^{1/2}$$

$$\mathbf{B}' = (0, 0, 0)$$

Inverse transformation of the previous one:

$$\begin{aligned} \mathbf{E}_{\parallel} &= \mathbf{E}'_{\parallel} & \mathbf{B}_{\parallel} &= \mathbf{B}'_{\parallel} \\ \mathbf{E}_{\perp} &= \gamma (\mathbf{E}'_{\perp} - \boldsymbol{\beta} \times \mathbf{B}') & \mathbf{B}_{\perp} &= \gamma (\mathbf{B}'_{\perp} + \boldsymbol{\beta} \times \mathbf{E}') \end{aligned}$$



$$\begin{aligned} E_x &= \frac{qx'}{r'^3} & B_x &= 0 \\ E_y &= \gamma \frac{qy'}{r'^3} & B_y &= -\gamma \beta \frac{qz'}{r'^3} \\ E_z &= \gamma \frac{qz'}{r'^3} & B_z &= \gamma \beta \frac{qy'}{r'^3} \end{aligned}$$

Now, we need to express these equations in terms of the unprimed coordinates. Since  $x' = \gamma(x - vt)$ ,  $y' = y$ ,  $z' = z$ , we obtain

|                                      |                                      |
|--------------------------------------|--------------------------------------|
| $E_x = \gamma \frac{q(x - vt)}{r^3}$ | $B_x = 0$                            |
| $E_y = \gamma \frac{qy}{r^3}$        | $B_y = -\gamma \beta \frac{qz}{r^3}$ |
| $E_z = \gamma \frac{qz}{r^3}$        | $B_z = \gamma \beta \frac{qy}{r^3}$  |

where  $r = [(x - vt)^2 + y^2 + z^2]^{1/2}$

### Question:

Is this equivalent to the velocity field given by the Lienard-Wiechert potentials?

## - Velocity field from the retarded potential

---

- For simplicity, assume  $z = 0$ .

$$\begin{aligned}\mathbf{E} &= (E_x, E_y, E_z) = \gamma \frac{q}{r^3} (x - vt, y, z) \\ &= \gamma \frac{q}{(\gamma^2 \bar{x}^2 + y^2)^{3/2}} (\bar{x}, y, 0) \quad \text{where } \bar{x} \equiv x - vt\end{aligned}$$

Let us first find where the **retarded position** of the particle is.

$$t_{\text{ret}} \equiv t - R/c$$

$$R^2 = (x - vt_{\text{ret}})^2 + y^2 = (\bar{x} + \beta R)^2 + y^2$$

$$R \rightarrow (1 - \beta^2)^2 R^2 - 2\bar{x}\beta R - \bar{x}^2 - y^2 = 0$$

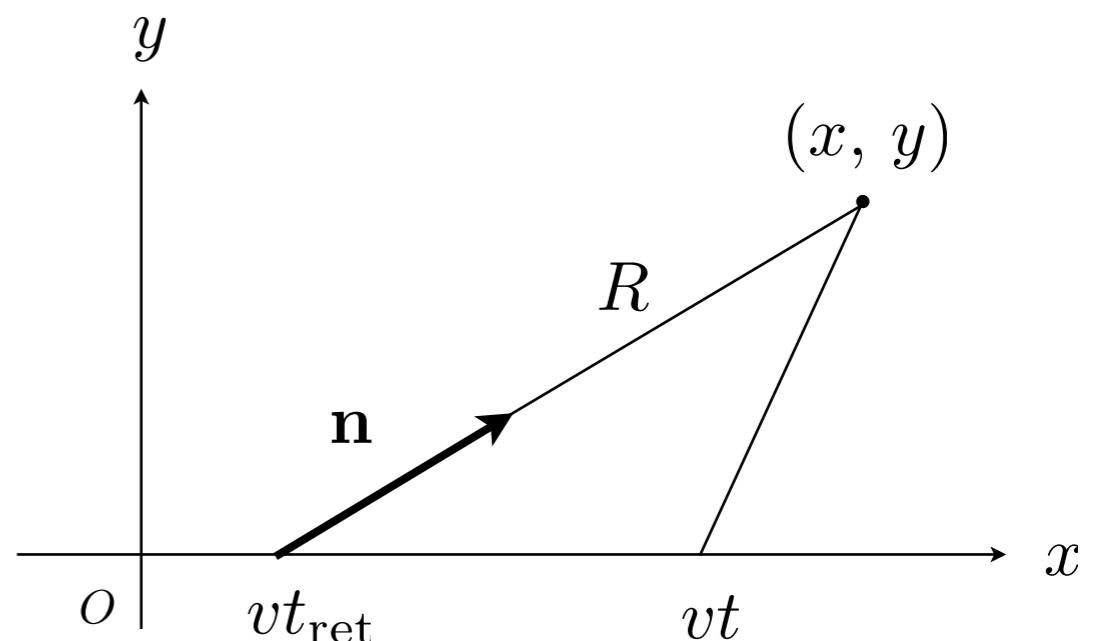
$$R^2 - 2\bar{x}\gamma^2\beta R - \gamma^2(\bar{x}^2 + y^2) = 0$$

$$R = \gamma^2\beta\bar{x} \pm [\gamma^4\beta^2\bar{x}^2 + \gamma^2(\bar{x}^2 + y^2)]^{1/2}$$

$$= \gamma^2\beta\bar{x} \pm \gamma[\gamma^2\beta^2\bar{x}^2 + (\bar{x}^2 + y^2)]^{1/2}$$

$$= \gamma^2\beta\bar{x} \pm \gamma(\gamma^2\bar{x}^2 + y^2)^{1/2}$$

positive solution  $\rightarrow R = \gamma^2\beta\bar{x} + \gamma(\gamma^2\bar{x}^2 + y^2)^{1/2}$



- Let us first find where the **retarded position** of the particle is.

$$(1) \quad \mathbf{n} = \frac{(x - vt + vR/c) \hat{\mathbf{x}} + y \hat{\mathbf{y}}}{R}$$

$$= \frac{(\bar{x} + \beta R)}{R} \hat{\mathbf{x}} + \frac{y}{R} \hat{\mathbf{y}}$$

$$= \left( \frac{\bar{x}}{R} + \beta \right) \hat{\mathbf{x}} + \frac{y}{R} \hat{\mathbf{y}}$$

$$\mathbf{n} - \boldsymbol{\beta} = \frac{\bar{x}}{R} \hat{\mathbf{x}} + \frac{y}{R} \hat{\mathbf{y}}$$

$$\mathbf{E} = \gamma \frac{q}{(\gamma^2 \bar{x}^2 + y^2)^{3/2}} (\mathbf{n} - \boldsymbol{\beta})$$

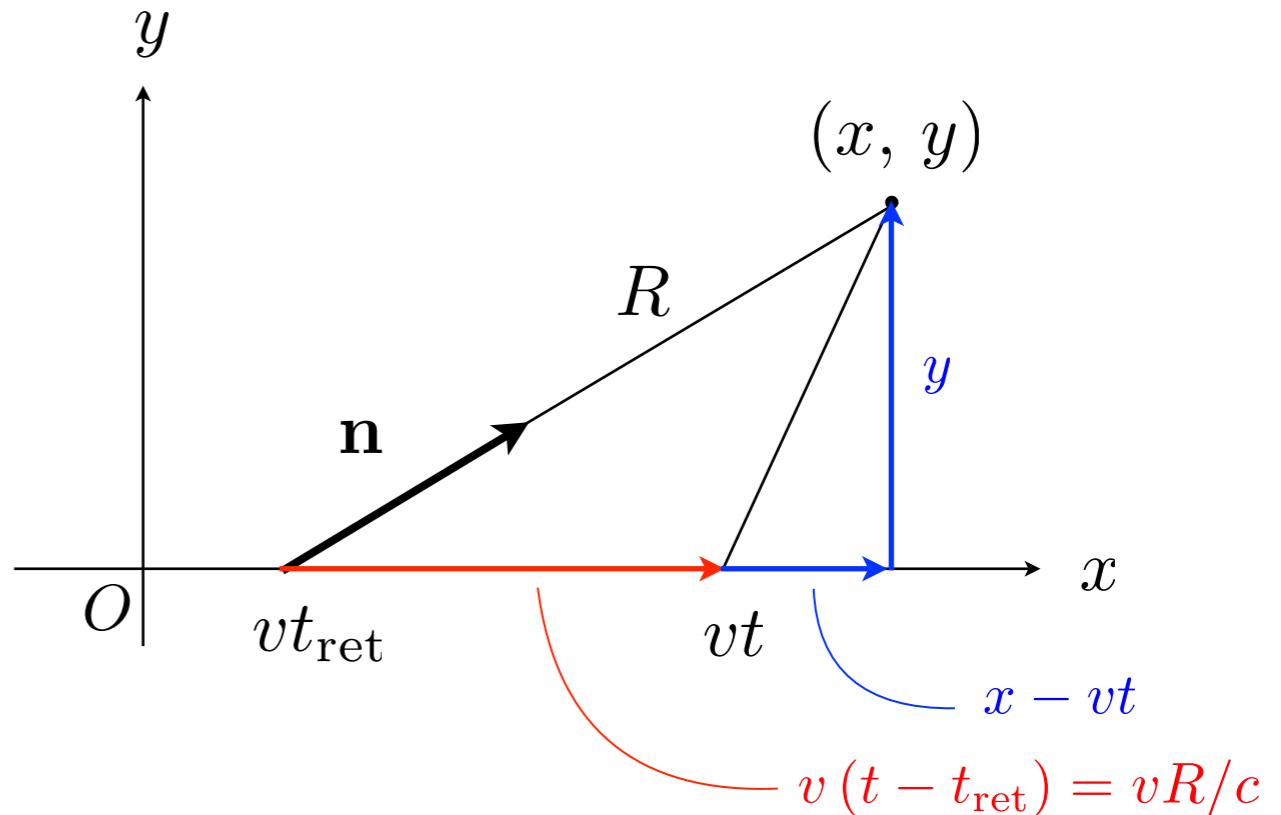
$$(2) \quad (\gamma^2 \bar{x}^2 + y^2)^{1/2} = \frac{R - \gamma^2 \beta \bar{x}}{\gamma} = R \gamma \left( \frac{1}{\gamma^2} - \frac{\beta \bar{x}}{R} \right)$$

$$= R \gamma \left( 1 - \beta^2 - \frac{\beta \bar{x}}{R} \right)$$

$$= R \gamma \left[ 1 - \beta \left( \frac{\bar{x}}{R} + \beta \right) \right]$$

$$= R \gamma (1 - \mathbf{n} \cdot \boldsymbol{\beta}) = R \gamma \kappa$$

$$R = \gamma^2 \beta \bar{x} + \gamma (\gamma^2 \bar{x}^2 + y^2)^{1/2}$$



$$\therefore \boxed{\begin{aligned} \mathbf{E} &= q \frac{(\mathbf{n} - \boldsymbol{\beta})}{\gamma^2 \kappa^3 R^2} \\ &= q \frac{(\mathbf{n} - \boldsymbol{\beta})(1 - \beta^2)}{\kappa^3 R^2} \end{aligned}}$$

This is identical to the velocity field component!

# - Time-dependence of the electric field at a point

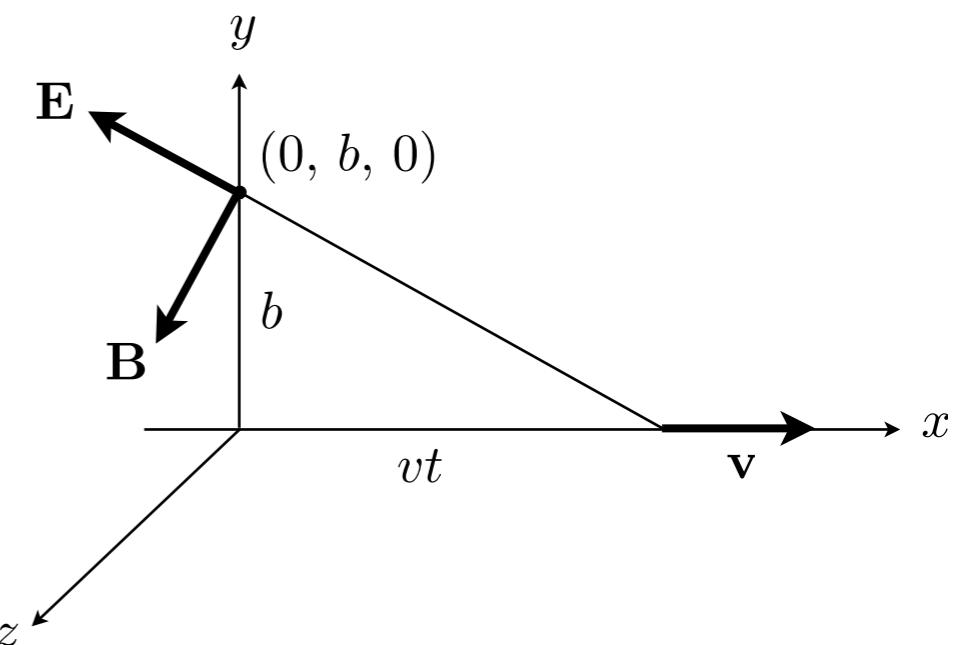
- For simplicity, let us choose the field point to be at  $(0, b, 0)$ . This involves no loss in generality. Then,

$$E_x = -\frac{q\gamma vt}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = -\frac{q}{b^2} \frac{\gamma vt/b}{(\gamma^2 v^2 t^2/b^2 + 1)^{3/2}}$$

$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = \frac{q\gamma}{b^2} \frac{1}{(\gamma^2 v^2 t^2/b^2 + 1)^{3/2}}$$

$$E_z = 0$$

$$B_x = 0, \quad B_y = 0, \quad B_z = \beta E_y$$



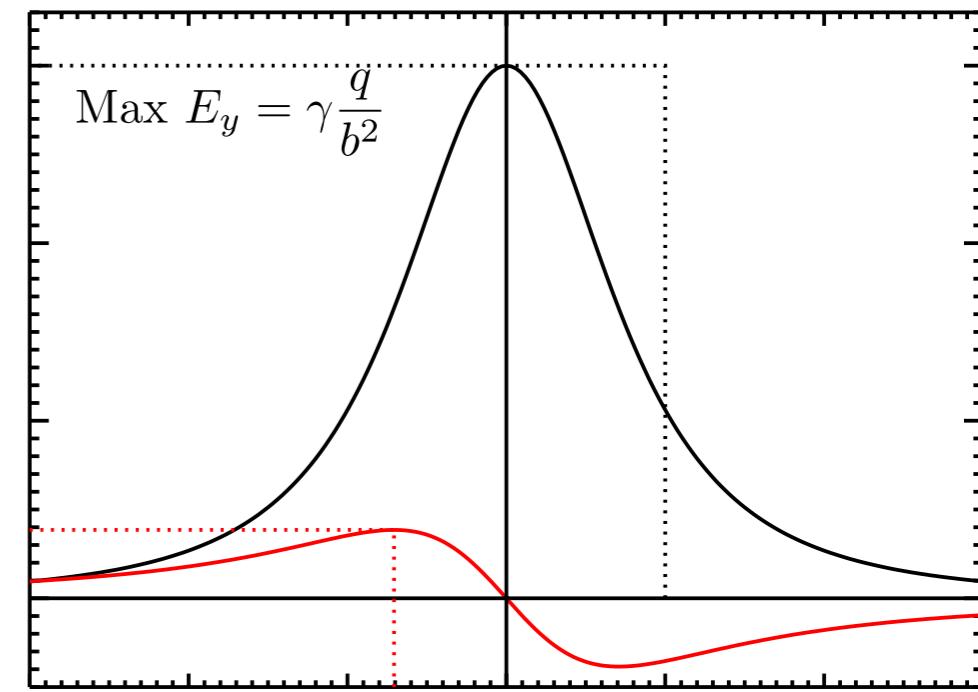
(1) The fields are strong only during a time interval of  $|vt| \lesssim b/\gamma$ , implying that the field of the moving charge are concentrated in the plane transverse to its motion into an angle of order  $1/\gamma$ .

$$\Delta\theta \approx v\Delta t/b \sim 1/\gamma$$

(2) As  $\gamma \gg 1 \rightarrow |E_x|_{\max} \ll |E_y|_{\max}$

The field of a highly relativistic charge appears to be a pulse of radiation traveling in the same direction as the charge and confined to the transverse plane.

$$\begin{aligned} \text{Max } E_x \\ = \frac{2}{3^{3/2}} \frac{q}{b^2} \end{aligned}$$



$$|t| = \frac{1}{\sqrt{2}\gamma v} \quad |t| \approx \frac{b}{\gamma v}$$

see the animation:

[https://seoncafe.github.io/Teaching\\_files/2022b\\_astrophysics/ani\\_pulse.mp4](https://seoncafe.github.io/Teaching_files/2022b_astrophysics/ani_pulse.mp4)

# - Spectrum of the pulse

---

- Spectrum of this pulse is given by (if the x-axis component is ignored)

$$\begin{aligned}
 \hat{E}(\omega) &= \frac{1}{2\pi} \int E_y(t) e^{i\omega t} dt \\
 &= \frac{q\gamma b}{2\pi} \int_{-\infty}^{\infty} (\gamma^2 v^2 t^2 + b^2)^{-3/2} e^{i\omega t} dt = \frac{q\gamma b}{2\pi} \int_0^{\infty} (\gamma^2 v^2 t^2 + b^2)^{-3/2} (e^{i\omega t} + e^{-i\omega t}) dt \\
 &= \frac{q\gamma b}{\pi} \int_0^{\infty} (\gamma^2 v^2 t^2 + b^2)^{-3/2} \cos \omega t dt
 \end{aligned}$$

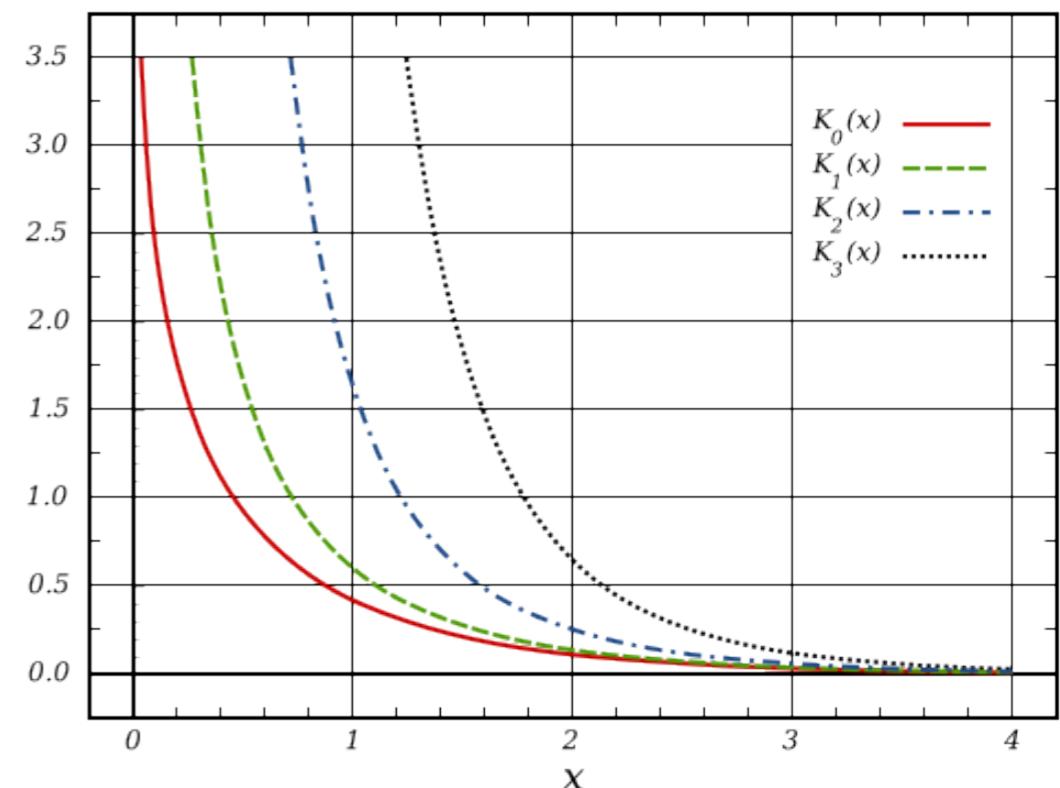
This integral can be done in terms of the modified Bessel function:

$$K_n(x) \equiv \frac{\Gamma(n + 1/2)(2x)^n}{\sqrt{\pi}} \int_0^{\infty} \frac{\cos t}{(t^2 + x^2)^{n+1/2}} dt \quad \text{Gamma function: } \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

$$\begin{aligned}
 \hat{E}(\omega) &= \frac{q\gamma b}{\pi} \left( \frac{\gamma^2 v^2}{\omega^2} \right)^{-3/2} \frac{1}{\omega} \int_0^{\infty} \left( \omega^2 t^2 + \frac{b^2 \omega^2}{\gamma^2 v^2} \right)^{-3/2} (\cos \omega t) d(\omega t) \\
 &= \frac{q}{\pi b v} \frac{b \omega}{\gamma v} K_1 \left( \frac{b \omega}{\gamma v} \right)
 \end{aligned}$$

Thus the spectrum is

$$\frac{dW}{dA d\omega} = c |\hat{E}(\omega)|^2 = \frac{q^2}{\pi^2 b^2 v^2} \left( \frac{b \omega}{\gamma v} \right)^2 K_1^2 \left( \frac{b \omega}{\gamma v} \right)$$



- The spectrum starts cut off for  $\omega > \gamma v/b$ .

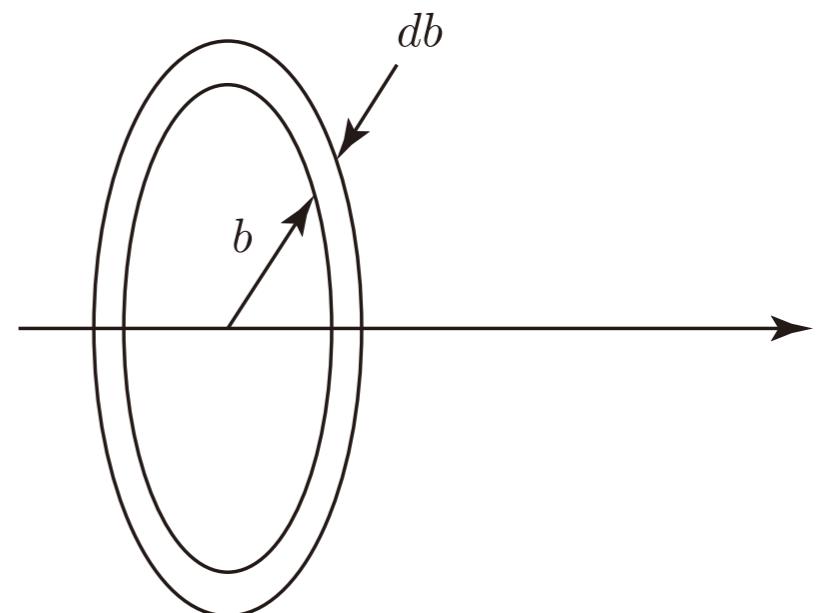
This is predicted from the uncertainty principle, since the pulse is confined roughly to a time interval of order  $\Delta t \approx b/\gamma v$ .

$$\Delta\omega \sim \frac{1}{\Delta t} \sim \gamma v/b$$

- Total energy per unit frequency range is obtained by

$$\frac{dW}{d\omega} = 2\pi \int_{b_{\min}}^{b_{\max}} \frac{dW}{dAd\omega} b db$$

$$b_{\max} \rightarrow \infty, \quad b_{\min} \neq 0 \text{ (a finite value)}$$



The lower limit has been chosen not as zero but as some minimum distance  $b_{\min}$ , such that the approximation of the field by means of classical electrodynamics and a point charge is valid.

$$\begin{aligned} \frac{dW}{d\omega} &= \frac{2q^2c}{\pi v^2} \int_x^\infty y K_1^2(y) dy \\ &= \frac{2q^2c}{\pi v^2} \left[ x K_0(x) K_1(x) - \frac{1}{2} x^2 (K_1^2(x) - K_0^2(x)) \right] \end{aligned} \quad \text{where} \quad y \equiv \frac{\omega b}{\gamma v} \quad \text{and} \quad x \equiv \frac{\omega b_{\min}}{\gamma v}$$

- Two limiting cases:

$$(1) \quad \omega \ll \frac{\gamma v}{b_{\min}} \quad (x \ll 1),$$

$$\begin{aligned} & x K_0(x) K_1(x) - \frac{1}{2} x^2 (K_1^2(x) - K_0^2(x)) \\ & \approx x (-\ln(x/2) - \gamma) \frac{1}{x} - \frac{x^2}{2} \left[ \frac{1}{x^2} - (\ln(x/2) + \gamma)^2 \right] \\ & \approx \ln \left[ \frac{2}{x} e^{-(\gamma+1/2)} \right] \\ & = \ln \left( \frac{0.68}{x} \right) \end{aligned}$$



$$\begin{aligned} & \omega \ll \frac{\gamma v}{b_{\min}} \\ & \frac{dW}{d\omega} = \frac{2q^2 c}{\pi v^2} \ln \left( 0.68 \frac{\gamma v}{\omega b_{\min}} \right) \end{aligned}$$

$$(2) \quad \omega \gg \frac{\gamma v}{b_{\min}} \quad (x \gg 1),$$

$$\begin{aligned} & x K_0(x) K_1(x) - \frac{1}{2} x^2 (K_1^2(x) - K_0^2(x)) \\ & \approx x \frac{\pi}{2x} e^{-2x} - \frac{1}{2} x^2 \frac{\pi}{2x} e^{-2x} \left[ \left( \frac{3}{8x} \right)^2 - \left( \frac{1}{8x} \right)^2 \right] \\ & = \frac{\pi}{4} e^{-2x} \end{aligned}$$



$$\begin{aligned} & \omega \gg \frac{\gamma v}{b_{\min}} \\ & \frac{dW}{d\omega} = \frac{q^2 c}{2v^2} \exp \left( -\frac{2\omega b_{\min}}{\gamma v} \right) \end{aligned}$$

## [Emission from Relativistic Particles]

---

- Total emitted power:

Imagine **an instantaneous rest frame  $K'$** , such that the particle has zero velocity at a certain time. We can then calculate the radiation emitted by use of the dipole (Larmor) formula.

Suppose that the particle emits a total amount of energy  $dW'$  in this frame in time  $dt'$ . The momentum of this radiation is zero,  $d\mathbf{p}' = 0$ , because the emission is symmetrical in the frame.

The energy in a frame  $K$  moving with velocity  $-\mathbf{v}$  w.r.t. the particle is:

$$dW = \gamma dW' \quad \longleftarrow \quad dE = cdP^0 = c\tilde{\Lambda}^0_{\mu} dP'^{\mu} = c\tilde{\Lambda}^0_0 dP^0 = \gamma dE'$$

The time interval  $dt$  is simply

$$dt = \gamma dt'$$

The total power emitted in frames  $K$  and  $K'$  are given by

$$P = \frac{dW}{dt}, \quad P' = \frac{dW'}{dt'}$$

Thus **the total emitted power is a Lorentz invariant** for any emitter that emits with front-back symmetry in its instantaneous rest frame.

$$P = P'$$

- **The Larmor formula in covariant form:**

Recall that  $\vec{a} \cdot \vec{U} = 0$ , and because  $\vec{U} = (c, \mathbf{0})$  in the instantaneous rest frame of the particle, we have

$$a'_0 = 0 \quad \rightarrow \quad |\mathbf{a}'|^2 = a'_k a'^k = a'_\mu a'^\mu = \vec{a} \cdot \vec{a}$$

Therefore,

$$\begin{array}{ccc} P' = \frac{2q^2}{3c^3} |\mathbf{a}'|^2 & \longrightarrow & P = \frac{2q^2}{3c^3} \vec{a} \cdot \vec{a} \\ \text{nonrelativistic} & & \text{relativistic} \end{array}$$

- Expression of  $P$  in terms of the three-vector acceleration

Recall

$$\boxed{\begin{aligned} dt &= \gamma \left( dt' + \frac{v}{c^2} dx'_{||} \right) \\ u_{||} &= \frac{u'_{||} + v}{1 + vu'_{||}/c^2} \\ u_\perp &= \frac{u'_\perp}{\gamma (1 + vu'_{||}/c^2)} \end{aligned}}$$

$$\begin{aligned} \sigma &\equiv \left( 1 + vu'_{||}/c^2 \right) \\ dt &= \gamma dt' \sigma \\ u_{||} &= \frac{u'_{||} + v}{\sigma} \\ u_\perp &= \frac{u'_\perp}{\gamma \sigma} \end{aligned}$$

$$\begin{aligned} dt &= \gamma dt' \sigma \\ du_{||} &= \frac{du'_{||}}{\sigma} - \frac{u'_{||} + v}{\sigma^2} \frac{v}{c^2} du'_{||} \\ &= \frac{du'_{||}}{\sigma^2} \left( 1 - \frac{v^2}{c^2} \right) = \frac{du'_{||}}{\gamma^2 \sigma^2} \\ du_\perp &= \frac{du'_\perp}{\gamma \sigma} - \frac{u'_\perp}{\gamma \sigma^2} \frac{v}{c^2} du'_{||} \\ &= \frac{1}{\gamma \sigma^2} \left( \sigma du'_\perp - \frac{vu'_\perp}{c^2} du'_{||} \right) \end{aligned}$$

Hence,

Transformation of three-vector acceleration:

$$a_{\parallel} = \frac{du_{\parallel}}{dt} = \frac{1}{\gamma^3 \sigma^3} \frac{du'_{\parallel}}{dt'}$$

$$a_{\perp} = \frac{du_{\perp}}{dt} = \frac{1}{\gamma^2 \sigma^3} \left( \sigma \frac{du'_{\perp}}{dt'} - \frac{vu'_{\perp}}{c^2} \frac{du'_{\parallel}}{dt'} \right) \quad \longrightarrow$$

$$a_{\parallel} = \frac{1}{\gamma^3 \sigma^3} a'_{\parallel}$$

$$a_{\perp} = \frac{1}{\gamma^2 \sigma^3} \left( \sigma a'_{\perp} - \frac{vu'_{\perp}}{c^2} a'_{\parallel} \right)$$

$$\text{where } \sigma \equiv \left( 1 + \frac{vu'_{\parallel}}{c^2} \right)$$

In an instantaneous rest frame of a particle,

$$u'_{\parallel} = u'_{\perp} = 0, \quad \sigma = 1$$

$$a'_{\parallel} = \gamma^3 a_{\parallel}$$

$$a'_{\perp} = \gamma^2 a_{\perp}$$

Note

$$\tan \theta'_a \equiv \frac{a'_{\perp}}{a'_{\parallel}} = \frac{1}{\gamma} \frac{a_{\perp}}{a_{\parallel}} = \frac{1}{\gamma} \tan \theta_a$$

Thus we can write

$$P = \frac{2q^2}{3c^3} |\mathbf{a}'|^2 = \frac{2q^2}{3c^3} \left( a'^2_{\parallel} + a'^2_{\perp} \right) \quad \longrightarrow$$

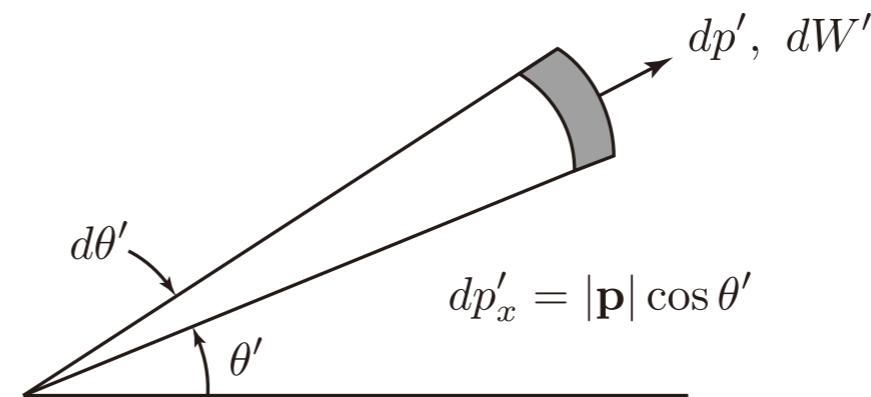
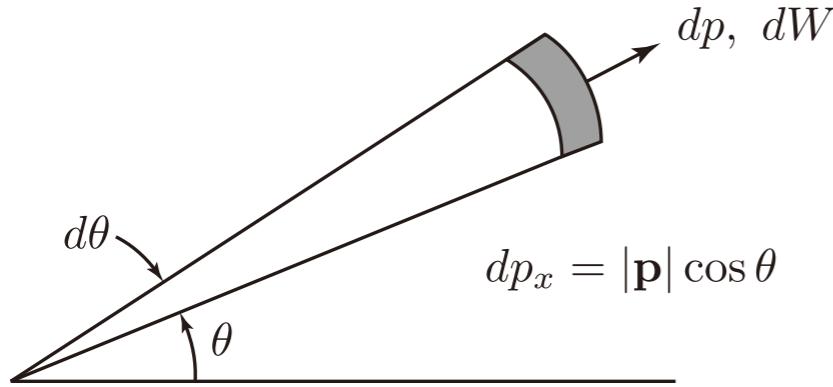
$$P = \frac{2q^2}{3c^3} \gamma^4 \left( \gamma^2 a_{\parallel}^2 + a_{\perp}^2 \right)$$

nonrelativistic

relativistic

# Differential Power

- Angular Distribution of Emitted and Received Power



Note:  $d\phi' = d\phi$

In the instantaneous rest frame of the particle, let us consider an amount of energy  $dW'$  that is emitted into the solid angle  $d\Omega' = \sin \theta' d\theta' d\phi'$  (see the above figure).

$$\mu \equiv \cos \theta \rightarrow d\Omega = d\mu d\phi$$

$$\mu' \equiv \cos \theta' \rightarrow d\Omega' = d\mu' d\phi'$$

Recall the aberration formula:

$$\cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'} \rightarrow \mu = \frac{\mu' + \beta}{1 + \beta \mu'} \rightarrow \mu' = \frac{\mu - \beta}{1 - \beta \mu}$$

$$d\mu = \frac{d\mu'}{\gamma^2 (1 + \beta \mu')^2}$$

$$d\mu' = \frac{d\mu}{\gamma^2 (1 - \beta \mu)^2}$$

$$d\Omega = \frac{d\Omega'}{\gamma^2 (1 + \beta \mu')^2}$$

$$d\Omega' = \frac{d\Omega}{\gamma^2 (1 - \beta \mu)^2}$$

Recall that energy and momentum form a four-vector

$$\vec{P} = (E/c, \mathbf{p}) \quad \text{and} \quad |\mathbf{p}| = E/c \quad \longrightarrow \quad dW = \gamma(dW' + vdp'_x) = \gamma(1 + \beta\mu')dW'$$

$$\therefore dW = \gamma(1 + \beta\mu')dW', \quad dW' = \gamma(1 - \beta\mu)dW$$

$$\frac{dW}{d\Omega} = \gamma^3 (1 + \beta\mu')^3 \frac{dW'}{d\Omega'}, \quad \frac{dW'}{d\Omega'} = \gamma^3 (1 - \beta\mu)^3 \frac{dW}{d\Omega}$$

In the rest frame, **the power emitted in a unit time interval** is

$$\frac{dP'}{d\Omega'} \equiv \frac{dW'}{dt'd\Omega'}$$

However, in the observer's frame, there are two possible choices for the time interval to calculate the power.

(1)  $dt = \gamma dt'$ :

This is the time interval during which the emission occurs. With this choice we obtain **the emitted power**.

(2)  $dt_A = \gamma(1 - \beta\mu)dt'$  or  $dt_A = \gamma^{-1}(1 + \beta\mu')^{-1}dt'$ :

This is the time interval of the radiation as received by a stationary observer in  $K$ . With this choice we obtain **the received power**.

- Thus we obtain the two results:

$$\frac{dP_e}{d\Omega} = \gamma^2 (1 + \beta\mu')^3 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4 (1 - \beta\mu)^3} \frac{dP'}{d\Omega'}$$

$$\frac{dP_r}{d\Omega} = \gamma^4 (1 + \beta\mu')^4 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4 (1 - \beta\mu)^4} \frac{dP'}{d\Omega'}$$

$P_r$  is the power actually measured by an observer. It has the expected symmetry property of yielding the inverse transformation by interchanging primed and unprimed variables, along with a change of sign of  $\beta$ .

$P_e$  is used in the discussion of emission coefficient.

In practice, **the distinction between emitted and received power is often not important, since they are equal in an average sense for stationary distributions of particles.**

- Beaming effect:

If the radiation is isotropic in the particle's frame, then the angular distribution in the observer's frame will be highly peaked in the forward direction for highly relativistic velocities.

The factor  $\gamma^{-4} (1 - \beta\mu)^{-4}$  is sharply peaked near  $\theta \approx 0$  with an angular scale of order  $1/\gamma$ .

$$\gamma^{-4} (1 - \beta\mu)^{-4} \approx \gamma^{-4} \left[ 1 - \left( 1 - \frac{1}{2\gamma^2} \right) \left( 1 - \frac{\theta^2}{2} \right) \right]^{-4} = \gamma^{-4} \left( \frac{1}{2\gamma^2} + \frac{\theta^2}{2} \right)^{-4} = \left( \frac{2\gamma}{1 + \gamma^2\theta^2} \right)^4$$

- Angular distribution of the dipole emission from a slowly moving particle in the instantaneous rest frame is

$$\frac{dP'}{d\Omega'} = \frac{q^2 a'^2}{4\pi c^3} \sin^2 \Theta'$$

$\Theta'$  = the angle between the acceleration and the direction of emission.

Using  $a'_{||} = \gamma^3 a_{||}$ ,  $a'_{\perp} = \gamma^2 a_{\perp}$  and  $\frac{dP_r}{d\Omega} = \gamma^{-4} (1 - \beta\mu)^{-4} \frac{dP'}{d\Omega'}$ , we obtain

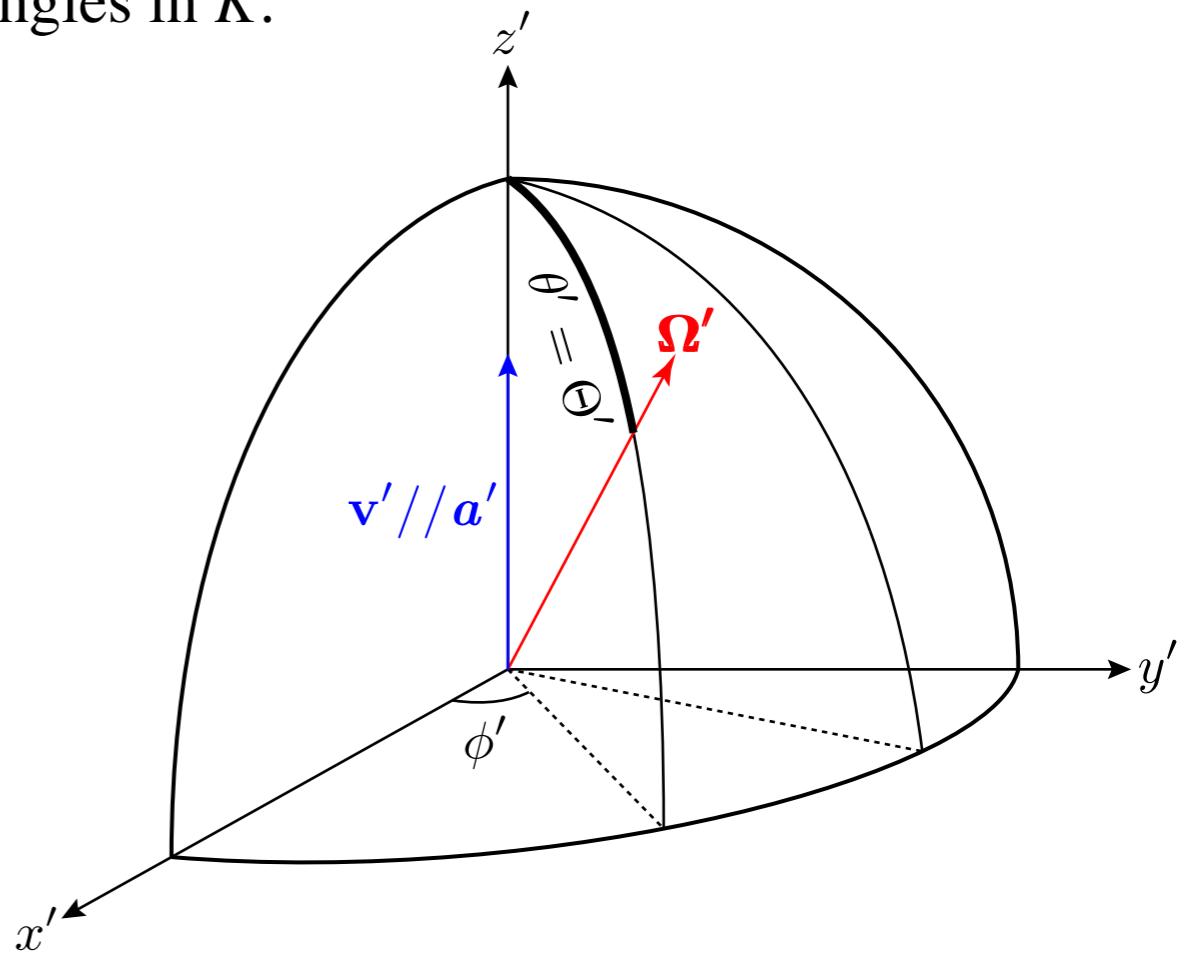
$$\boxed{\frac{dP_r}{d\Omega} = \frac{q^2}{4\pi c^3} \frac{(\gamma^2 a_{||}^2 + a_{\perp}^2)}{(1 - \beta\mu)^4} \sin^2 \Theta'}$$

To use this formula, we must relate  $\Theta'$  to the angles in  $K$ .

(1) Acceleration parallel to velocity:  $\Theta' = \theta'$ ,  $a_{\perp} = 0$

$$\sin^2 \Theta' = 1 - \mu'^2 = 1 - \left( \frac{\mu - \beta}{1 - \beta\mu} \right)^2 = \frac{1 - \mu^2}{\gamma^2 (1 - \beta\mu)^2}$$

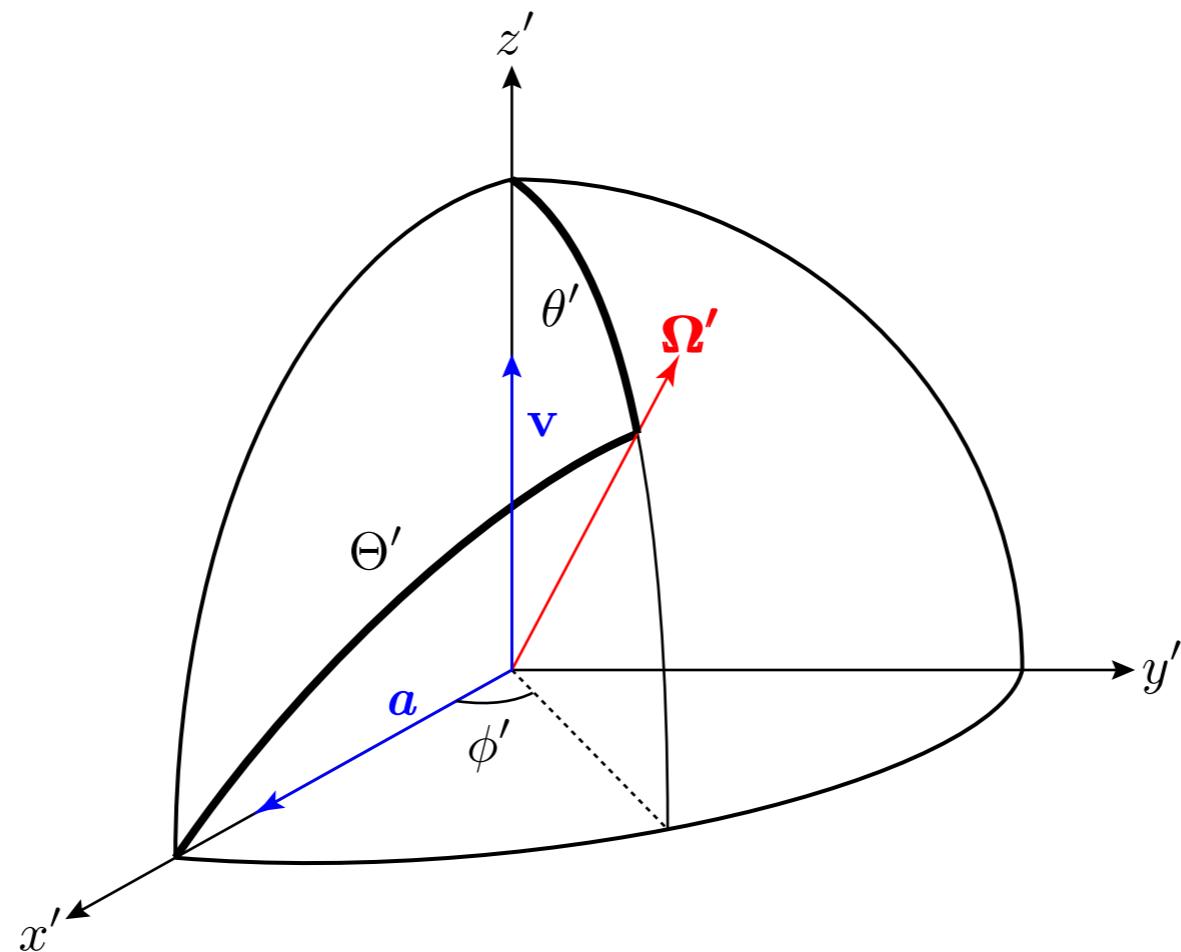
$$\rightarrow \frac{dP_{r||}}{d\Omega} = \frac{q^2 a_{||}^2}{4\pi c^3} \frac{1 - \mu^2}{(1 - \beta\mu)^6}$$



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(2) Acceleration perpendicular to velocity:  $\cos \Theta' = \sin \theta' \cos \phi'$  (when  $a$  is in  $x$ -direction in the figure)

$$\sin^2 \Theta' = 1 - \frac{(1 - \mu^2) \cos^2 \phi}{\gamma^2 (1 - \beta\mu)^2} \quad \longrightarrow \quad \frac{dP_{r\perp}}{d\Omega} = \frac{q^2 a_\perp^2}{4\pi c^3} \frac{1}{(1 - \beta\mu)^4} \left[ 1 - \frac{(1 - \mu^2) \cos^2 \phi}{\gamma^2 (1 - \beta\mu)^2} \right]$$



(3) In general

$$\cos \Theta' = \mu' \mu'_a + (1 - \mu'^2)^{1/2} (1 - \mu_a'^2)^{1/2} \cos (\phi' - \phi'_a)$$

$$\mathbf{r}_1 = (\sin \theta_1 \cos \phi_1, \sin \theta_1 \sin \phi_1, \cos \theta_1)$$

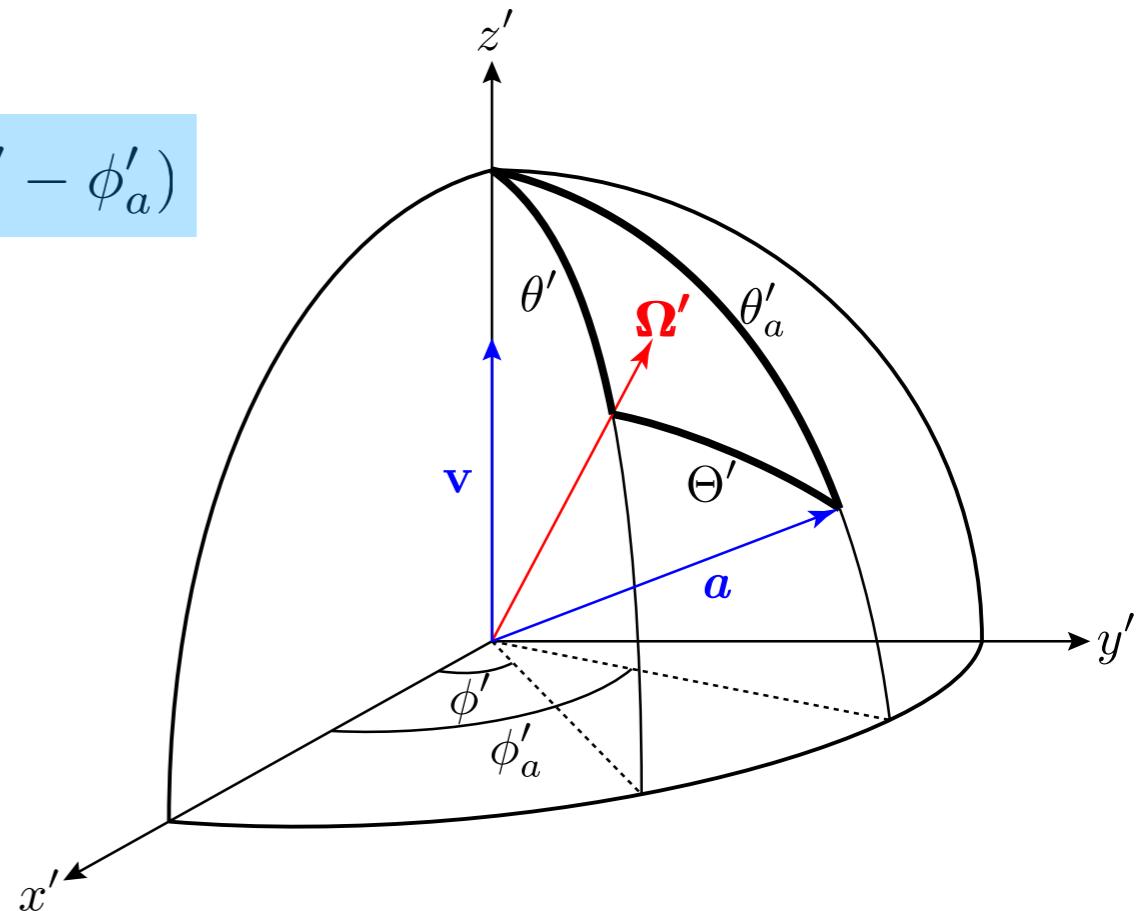
$$\mathbf{r}_2 = (\sin \theta_2 \cos \phi_2, \sin \theta_2 \sin \phi_2, \cos \theta_2)$$

$$\cos \Theta = \mathbf{r}_1 \cdot \mathbf{r}_2$$

$$= \mu_1 \mu_2 + \sqrt{1 - \mu_1^2} \sqrt{1 - \mu_2^2} (\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2)$$

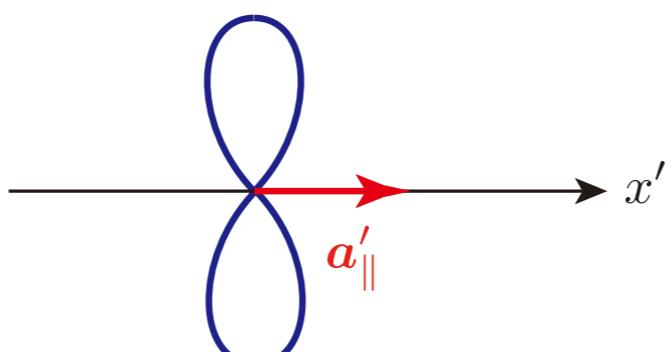
$$= \mu_1 \mu_2 + \sqrt{1 - \mu_1^2} \sqrt{1 - \mu_2^2} \cos (\phi_1 - \phi_2)$$

- In the extreme relativistic limit, the radiation becomes strongly peaked in the forward direction.

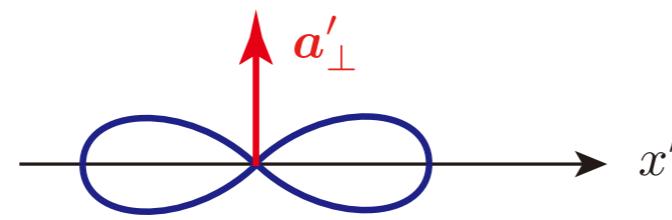


particle's rest frame:

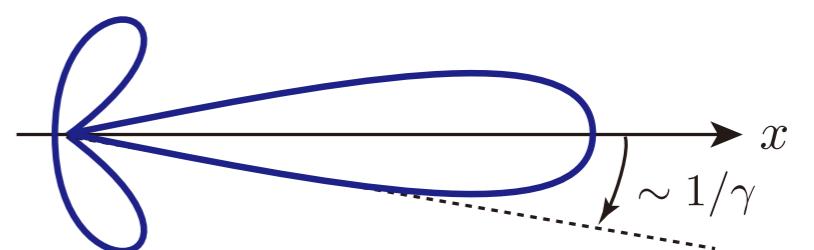
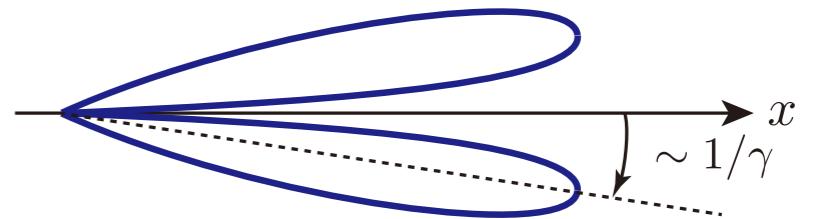
parallel acceleration:



perpendicular acceleration:



observer's frame:



## [Invariant Phase Volumes and Specific Intensity]

- **Phase volume**

Consider a group of particles that occupy a slight spread in position and in momentum at a particular time. In a rest frame comoving with the particles, they occupy a spatial volume element  $d^3\mathbf{x}' = dx'dy'dz'$  and a momentum volume element  $d^3\mathbf{p}' = dP'_x dP'_y dP'_z$ .

$$\begin{aligned} d^3\mathbf{x}' &= dx'dy'dz' && \text{phase volume in the comoving frame:} \\ d^3\mathbf{p}' &= dP'_x dP'_y dP'_z &\longrightarrow& d\mathcal{V}' \equiv d^3\mathbf{x}'d^3\mathbf{p}' = dx'dy'dz'dP'_x dP'_y dP'_z \end{aligned}$$

In the observer's frame,  $dx = \frac{1}{\gamma}dx'$  (length contraction),  $dy = dy'$ ,  $dz = dz'$

$$dp_x = \gamma(dp'_x + \beta dP'_0), \quad dp_y = dp'_y, \quad dp_z = dp'_z$$

We note that  $dP'_0 = 0 + \mathcal{O}(dp'^2)$  because the velocities are near zero in the comoving frame and the energy is quadratic in velocity. Therefore, we have  $dp_x = \gamma dp'_x$ . By combining all these, it is found that **the phase-space volume is invariant**.

$$d\mathcal{V}' \equiv d^3\mathbf{x}'d^3\mathbf{p}' = d^3\mathbf{x}d^3\mathbf{p} \equiv d\mathcal{V}$$

: Lorentz invariant

This contains no reference to particle mass, and therefore it has applicability to photons. **The phase space density is an invariant**, since the number of particles within the phase volume element is a countable quantity and itself invariant.

$$f \equiv \frac{dN}{d\mathcal{V}} = \frac{dN}{d^3\mathbf{x}d^3\mathbf{p}}$$

: Lorentz invariant

- **Specific Intensity and Source Function**

Definition of the energy density per unit solid angle per frequency range.

$$\begin{aligned} h\nu \frac{dN}{dV} d^3\mathbf{p} d^3\mathbf{x} &= h\nu \frac{dN}{dV} p^2 dp d\Omega d^3\mathbf{x} & u_\nu(\Omega) &= h\nu \frac{dN}{dV} p^2 \frac{dp}{d\nu} & \leftarrow p = \frac{h\nu}{c} \\ &= u_\nu(\Omega) d\Omega d\nu d^3\mathbf{x} & \longrightarrow &= h\nu \left( \frac{h\nu}{c} \right)^2 \left( \frac{h}{c} \right) \frac{dN}{dV} \end{aligned}$$

Since  $u_\nu(\Omega) = I_\nu/c$ , we find that

$$\frac{I_\nu}{\nu^3} = \text{ Lorentz invariant}$$

Because the source function occurs in the transfer equation as the difference  $I_\nu - S_\nu$ , the source function must have the same transformation properties as the intensity.

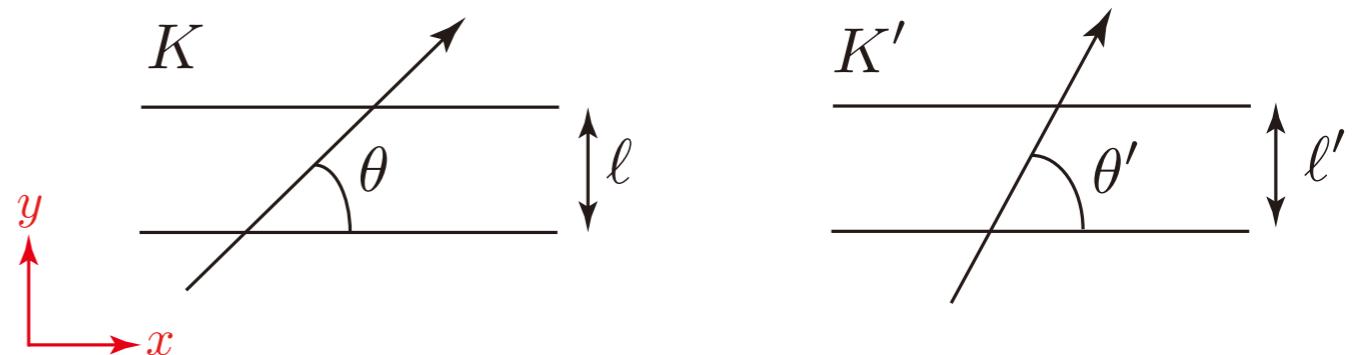
- **Optical Depth, Absorption Coefficient and Emission Coefficient**

The optical depth must be an invariant, since  $\exp(-\tau)$  gives the fraction of photons passing through the material, and this involves simple counting.

$$\tau = \text{ Lorentz invariant}$$

- **Absorption Coefficient and Emission Coefficient**

Consider the optical depth in two frames:



Then, the optical depth is

$$\tau_\nu = \frac{\ell \alpha_\nu}{\sin \theta} = \frac{\ell}{\nu \sin \theta} \nu \alpha_\nu = \text{Lorentz invariant}$$

Note that  $\nu \sin \theta$  is proportional to the  $y$  component of the photon four-momentum  $\vec{k} = (\omega/c, \mathbf{k})$ .

Both  $k_y$  and  $\ell$  are the same in both frames, being perpendicular to the motion. Therefore, we have

$$\nu \sin \theta \propto k_y, \quad k_y = k'_y, \quad \ell' = \ell$$

$$\nu \alpha_\nu = \text{Lorentz invariant}$$

Finally, we obtain the transformation of the emission coefficient from the definition of the source function:

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu}$$

$$\frac{j_\nu}{\nu^2} = \text{Lorentz invariant}$$

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- **Number density**

$$n = \frac{dN}{dV} = \frac{dN}{d^3\mathbf{x}} = \frac{dN'}{\gamma^{-1}d^3\mathbf{x}'} = \gamma n'$$

$$n = \gamma n'$$