Astrophysics

Lecture 07
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Four-velocity

The (infinitesimally small) difference between the coordinates of two events is also a four-vector. Dividing by the proper time yields a four-vector, the four-velocity:

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \longrightarrow \begin{aligned} U^0 &= \frac{c\,dt}{d\tau} = c\gamma_u \\ U^i &= \frac{dx^i}{d\tau} = \gamma_u u^i \end{aligned} \quad \text{or} \quad \boxed{\vec{U} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}}$$

where $\gamma_u \equiv \left(1 - u^2/c^2\right)^{-1/2}$, $u \equiv \left|\dfrac{d\mathbf{x}}{dt}\right|$

length of the four-velocity : $\boxed{\vec{U} \cdot \vec{U} = U^\mu U_\mu = -\left(\gamma_u c\right)^2 + \left(\gamma_u \mathbf{u}\right)^2 = -c^2}$

Transformation of the four-velocity:

$$\begin{aligned} U'^0 &= \gamma\left(U^0 - \beta U^1\right) \\ U'^1 &= \gamma\left(-\beta U^0 + U^1\right) \\ U'^2 &= U^2 \\ U'^3 &= U^3 \end{aligned} \longrightarrow \begin{aligned} \gamma_{u'} c &= \gamma\left(c\gamma_u - \beta\gamma_u u^1\right) \\ \gamma_{u'} u'^1 &= \gamma\left(-\beta c\gamma_u + \gamma_u u^1\right) \\ \gamma_{u'} u'^2 &= \gamma_u u^2 \\ \gamma_{u'} u'^3 &= \gamma_u u^3 \end{aligned}$$

The first two equations become:

$$\longrightarrow \begin{aligned} \gamma_{u'} &= \gamma\gamma_u\left(1 - vu'/c^2\right) \\ \gamma_{u'} u'^1 &= \gamma\gamma_u\left(u^1 - v\right) \end{aligned}$$

Note: γ denotes the factor for the relative velocity between two frames.
γ_u and $\gamma_{u'}$ are the factors for a velocity vector measured in K and K', respectively.

$\longrightarrow$

velocity component: $$u'^1 = \frac{u^1 - v}{1 - vu^1/c^2}$$ This is the previously derived formula.

speed: $$\gamma_{u'} = \gamma\gamma_u\left(1 - \frac{vu^1}{c^2}\right)$$ This is the transform for speed.

Here, $u^1 = u\cos\theta$ and $u'^1 = u'\cos\theta'$

Momentum and Energy

- Four-momentum of a particle with a mass m_0 is defined by

$$P^\mu \equiv m_0 U^\mu \qquad \begin{aligned} P^0 &= m_0 c \gamma_v \\ P^i &= \gamma_v m_0 \mathbf{v} \end{aligned}$$

- In the nonrelativistic limit,

$$P^0 c = m_0 c^2 \gamma = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = m_0 c^2 + \frac{1}{2} m_0 v^2 + \cdots$$

Therefore, we interpret $E \equiv P^0 c = \gamma_v m_0 c^2$ as the total energy of the particle.

The quantity $m_0 c^2$ is interpreted as the rest energy of the particle.

Then,

$$\mathbf{p} \equiv \gamma_v m_0 \mathbf{v}, \quad P^\mu = (E/c,\ \mathbf{p})$$ Here, **p** is the spatial component of the four-momentum.

Since $\vec{U}^2 = -c^2$, we obtain $\vec{P}^2 = -m_0^2 c^2$. Comparing with $\vec{P}^2 = -\frac{E^2}{c^2} + |\mathbf{p}|^2$, we obtain

$$\boxed{E^2 = m_0^2 c^4 + c^2 |\mathbf{p}|^2}$$

- Photons are massless, but we can still define

$$P^\mu = (E/c,\ \mathbf{p}), \quad E = |\mathbf{p}|\, c \quad \rightarrow \quad \vec{P}^2 = 0$$

Wavenumber vector and frequency

- Quantum relations:

$$E = h\nu = \hbar\omega \qquad \begin{pmatrix} \omega & = 2\pi\nu \\ k & = 2\pi/\lambda \end{pmatrix}$$

$$p = E/c = \hbar k$$

We can define four wavenumber vector:

$$\vec{k} = \frac{1}{\hbar}\vec{P} = \left(\frac{\omega}{c},\ \mathbf{k}\right)$$

Note that it's a null vector:

$$\vec{k}\cdot\vec{k} = |\mathbf{k}|^2 - \omega^2/c^2 = 0$$

Then, we obtain an invariant:

$$\vec{k}\cdot\vec{x} = k_\mu x^\mu = \mathbf{k}\cdot\mathbf{x} - \omega t$$

Therefore, **the phase of the plane wave is an invariant.**

- Transform of $\overrightarrow{k}$ gives the Doppler shift formula.

$$k'^0 = \gamma\left(k^0 - \beta k^1\right) \longrightarrow \omega' = \gamma\left(\omega - \beta c k^1\right) = \omega\gamma\left(1 - \frac{v}{c}\cos\theta\right)$$

$$k'^1 = \gamma\left(-\beta k^0 + k^1\right)$$

$$k'^2 = k^2$$

$$k'^3 = k^3$$

$$k^1 = (\omega/c)\cos\theta$$

* Tensor Analysis *

- Definition:

  zeroth-rank tensor : Lorentz invariant (scalar) $s' = s$

  first-rank tensor : four-vector $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

  second-rank tensor: $T'^{\mu\nu} = \Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau} T^{\sigma\tau}$

- Covariant components and mixed components:

$$T_{\mu\nu} = \eta_{\mu\sigma}\eta_{\nu\tau}T^{\sigma\tau} \qquad T^{\mu}{}_{\nu} = \eta_{\nu\tau}T^{\mu\tau} \qquad T_{\mu}{}^{\nu} = \eta_{\mu\sigma}T^{\sigma\nu}$$

- Transformation rules:

$$\begin{aligned} T'_{\mu\nu} &= \eta_{\mu\alpha}\eta_{\nu\beta}T'^{\alpha\beta} \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}T^{\gamma\delta} \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}\eta^{\gamma\sigma}\eta^{\delta\tau}T_{\sigma\tau} \\ &= \widetilde{\Lambda}_{\mu}{}^{\sigma}\widetilde{\Lambda}_{\nu}{}^{\tau}T_{\sigma\tau} \end{aligned}$$

$$\begin{aligned} T'^{\mu}{}_{\nu} &= \eta_{\nu\alpha}T'^{\mu\alpha} \\ &= \eta_{\nu\alpha}\Lambda^{\mu}{}_{\sigma}\Lambda^{\alpha}{}_{\delta}T^{\sigma\delta} \\ &= \eta_{\nu\alpha}\Lambda^{\mu}{}_{\sigma}\Lambda^{\alpha}{}_{\delta}\eta^{\delta\tau}T^{\sigma}_{\tau} \\ &= \Lambda^{\mu}{}_{\sigma}\widetilde{\Lambda}_{\nu}{}^{\tau}T^{\sigma}{}_{\tau} \end{aligned}$$

$$\begin{aligned} T'_{\mu}{}^{\nu} &= \eta_{\mu\alpha}T'^{\alpha\nu} \\ &= \eta_{\mu\alpha}\Lambda^{\alpha}{}_{\beta}\Lambda^{\nu}{}_{\tau}T^{\beta\tau} \\ &= \eta_{\mu\alpha}\Lambda^{\alpha}{}_{\beta}\Lambda^{\nu}{}_{\tau}\eta^{\beta\sigma}T_{\sigma}{}^{\tau} \\ &= \widetilde{\Lambda}_{\mu}{}^{\beta}\Lambda^{\nu}{}_{\tau}T_{\sigma}{}^{\tau} \end{aligned}$$

- Symmetric tensor = a tensor that is invariant under a permutation of its indices.

$$T^{\mu\nu} = T^{\nu\mu}$$

- Antisymmetric tensor : if it alternates sign when any two indices of the subset are interchanged.

$$T^{\mu\nu} = -T^{\nu\mu}$$

- Examples of the second-rank tensors

  A product of two vectors: $A^\mu B^\nu$

$$A'^\mu B'^\nu = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau A^\sigma B^\tau$$

  The Minkowski metric:

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  The Kronecker-delta:

$$\delta^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Higher-rank tensors
  - Addition: $A^\mu + B^\mu,\ F^{\mu\nu} + G^{\mu\nu}$
  - Multiplication: $A^\mu B^\nu,\ F^{\mu\nu} G_{\sigma\tau}$
  - Raising and Lowering Indices: The metric can be used to change contravariant indices into covariant ones, and vice versa, by the processes of raising and lowering.
  - Contraction: $A^\mu B_\nu \quad \rightarrow \quad A^\mu B_\mu$ scalar
    $T^{\mu\nu}{}_\sigma \quad \rightarrow \quad T^{\mu\nu}{}_\nu$ vector

$$T'^{\mu\nu}{}_\nu = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \widetilde{\Lambda}_\nu{}^\tau T^{\alpha\beta}{}_\tau = \Lambda^\mu{}_\alpha \delta^\tau{}_\beta T^{\alpha\beta}{}_\tau = \Lambda^\mu{}_\alpha T^{\alpha\beta}{}_\beta$$

  - Gradients of Tensor Fields: A tensor field is a tensor that is a function of the spacetime coordinates in Cartesian coordinate systems. The gradient operation $\partial/\partial x^\mu \equiv \partial_\mu$ acting on such a field produces a tensor field of on higher rank with μ as a new covariant index.

$$\lambda \quad \rightarrow \quad \frac{\partial \lambda}{\partial x^\mu} \equiv \partial_\mu \lambda \equiv \lambda_{,\,\mu} \quad \text{vector (gradient)} \qquad A^\mu \quad \rightarrow \quad \frac{\partial A^\mu}{\partial x^\mu} \equiv \partial_\mu A^\mu \equiv A^\mu_{,\,\mu} \quad \text{scalar (divergence)}$$

  - **Invariance of form or Lorentz covariance or covariance**: A fundamental property of a tensor equation is that if it is true in one Lorentz frame, then it is true in all Lorentz frames. Covariance plays a powerful role in helping decide what the proper equations of physics are.

[Covariance of Electromagnetic Phenomena]

- Equation of charge conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(-\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

The above equation can be written as a tensor equation,

$$\frac{\partial j^\mu}{\partial x^\mu} = 0, \quad j^\mu{}_{,\,\mu} = 0 \quad \text{or} \quad \partial_\mu j^\mu = 0$$

if the **four-current** is defined by

$$j^\mu = \begin{pmatrix} \rho c \\ \\ \mathbf{j} \end{pmatrix} \qquad\qquad j_\mu = \begin{pmatrix} -\rho c \\ \\ \mathbf{j} \end{pmatrix}$$

- Note that the Jacobian (determinant) of the transformation from x_μ to x'_μ is simply the determinant of Λ, which is unity. Therefore, the **four-volume element** is an invariant.

$$dx'_0 dx'_1 dx'_2 dx'_3 = \det \Lambda dx_0 dx_1 dx_2 dx_3 = dx_0 dx_1 dx_2 dx_3$$

Since ρ is the zeroth component of the four-current, the charge element within a three-volume element is an invariant.

$$de = \rho dx_1 dx_2 dx_3$$

It is also an empirical fact that e is invariant.

- The set of vector and scalar wave equations in the Lorentz gauge is

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}$$
$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{4\pi}{c}(\rho c)$$

If we define the **four-potential**

$$A^\mu = \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix}, \quad A_\mu = \begin{pmatrix} -\phi \\ \mathbf{A} \end{pmatrix},$$

then the wave equations can be written as the tensor equations

$$\frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} = -\frac{4\pi}{c} j^\mu, \quad \partial_\nu \partial^\nu A^\mu = -\frac{4\pi}{c} j^\mu, \quad A^{\mu,\nu}_{,\nu} = -\frac{4\pi}{c} j^\mu$$

d'Alembertian operator: $\Box \equiv \frac{\partial^2}{\partial x^\nu x_\nu} \quad \rightarrow \quad \Box A^\mu = -\frac{4\pi}{c} j^\mu$

- The Lorentz gauge should be preserved under Lorentz transformations.

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \quad \rightarrow \quad \frac{\partial A^\mu}{\partial x^\mu} = 0 \ \text{ or } \ A^\mu{}_{\mu} = 0$$

- **Electromagnetic field tensor**:

The fields are expressed in terms of the potentials as
$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

The *x* components of the electric and magnetic fields are explicitly

$$\begin{aligned} E_x &= -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} = \partial^0 A^1 - \partial^1 A^0 \\ B_x &= \quad \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \partial^2 A^3 - \partial^3 A^2 \end{aligned}$$

These equations imply that the electric and magnetic fields, six components in all, are the elements of a **second-rank, antisymmetric field-strength tensor**, because a rank two antisymmetric tensor has exactly six independent components.

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \longrightarrow \begin{aligned} F^{0i} &= E_i \\ F^{i0} &= -E_i \\ F^{12} &= -F^{21} = B_3, \ \cdots \end{aligned}$$

covariant field-strength tensor

$$F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta} \longrightarrow \begin{aligned} F_{0i} &= \eta_{0\alpha}\eta_{i\beta}F^{\alpha\beta} = -F^{0i} \\ F_{i0} &= \eta_{i\alpha}\eta_{0\beta}F^{\alpha\beta} = -F^{i0} \\ F_{ij} &= \eta_{i\alpha}\eta_{j\beta}F^{\alpha\beta} = F^{ij} \end{aligned}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \longrightarrow \begin{aligned} F_{0i} &= -E_i \\ F_{i0} &= E_i \\ F_{12} &= -F_{21} = B_3, \quad \cdots \end{aligned}$$

- The two Maxwell equations containing sources (inhomogeneous equations):

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{j} \end{aligned} \longrightarrow \begin{aligned} \sum_{i=1}^{3} \partial_i E_i &= \frac{4\pi}{c} j^0 \\ \partial_2 B_3 - \partial_3 B_2 - \partial_0 E_1 &= \frac{4\pi}{c} j^1 \end{aligned} \longrightarrow \begin{aligned} -\sum_{i=1}^{3} \partial_i F^{i0} &= \frac{4\pi}{c} j^0 \\ -\partial_0 F^{01} - \partial_2 F^{21} - \partial_3 F^{31} &= \frac{4\pi}{c} j^1 \end{aligned}$$

$$\boxed{\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} j^\nu \quad \text{or} \quad \partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu} \qquad \partial^\mu F_{\mu\nu} = -\frac{4\pi}{c} j_\nu \quad \text{or} \quad \partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu$$

- The conservation of charge easily follows from the above equation and the asymmetric property.

$$\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu \partial_\mu \left(-F^{\nu\mu}\right) \underset{\substack{\uparrow \\ \text{index exchange}}}{=} -\partial_\mu \partial_\nu F^{\mu\nu} \qquad \therefore \boxed{\begin{aligned} &\partial_\mu \partial_\nu F^{\mu\nu} = 0 \\ &\partial_\nu j^\nu = -\frac{c}{4\pi} \partial_\nu \partial_\mu F^{\mu\nu} = 0 \end{aligned}}$$

- The "internal" Maxwell equations (homogeneous equations):

$$\begin{aligned} &\nabla \cdot \mathbf{B} = 0 \\ &\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \end{aligned} \longrightarrow \begin{aligned} &\sum_{i=1}^{3} \partial_i B_i = 0 \\ &\partial_2 E_3 - \partial_3 E_2 + \partial_0 B_1 = 0 \end{aligned} \longrightarrow \begin{aligned} &\partial_1 F^{23} + \partial_2 F^{31} + \partial_3 F^{12} = 0 \\ &\partial_2 F^{30} + \partial_3 F^{20} + \partial_0 F^{23} = 0 \end{aligned}$$

$$\boxed{\partial_\mu F^{\nu\sigma} + \partial_\nu F^{\sigma\mu} + \partial_\sigma F^{\mu\nu} = 0} \quad \text{or} \quad \partial^\mu F_{\nu\sigma} + \partial^\nu F_{\sigma\mu} + \partial^\sigma F_{\mu\nu} = 0$$

The equation can be written concisely as $F^{[\mu\nu,\sigma]} = 0$ or $F_{[\mu\nu,\sigma]} = 0$, where [ ] around indices denote all permutations of indices, with even permutation contributing with a positive sign and odd permutation with a negative sign, for example,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{[\nu,\mu]}$$

- Transformation of Electromagnetic Fields

- Since $F^{\mu\nu}$ is a second-rank tensor, its components transform in the usual way:

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} F^{\alpha\beta} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

For a pure boost along the *x*-axis:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow$$

$$\begin{aligned}
E'_x &= F'^{01} = \Lambda^0{}_0 \Lambda^1{}_1 F^{01} + \Lambda^0{}_1 \Lambda^1{}_0 F^{10} = \gamma^2 E_x - \beta^2\gamma^2 E_x = E_x \\
E'_y &= F'^{02} = \Lambda^0{}_0 \Lambda^2{}_2 F^{02} + \Lambda^0{}_1 \Lambda^2{}_2 F^{12} = \gamma E_y - \beta\gamma B_z \\
E'_z &= F'^{03} = \Lambda^0{}_0 \Lambda^3{}_3 F^{03} + \Lambda^0{}_1 \Lambda^3{}_3 F^{13} = \gamma E_z + \beta\gamma B_y \\
B'_x &= F'^{23} = \Lambda^2{}_2 \Lambda^3{}_3 F^{23} = B_x \\
B'_y &= F'^{31} = \Lambda^3{}_3 \left(\Lambda^1{}_0 F^{30} - \Lambda^1{}_1 F^{31} \right) = \beta\gamma E_z + \gamma B_y \\
B'_z &= F'^{12} = \Lambda^1{}_0 \Lambda^2{}_2 F^{03} + \Lambda^1{}_1 \Lambda^2{}_2 F^{12} = -\beta\gamma E_y + \gamma B_z
\end{aligned}$$

- In general,

$$\boxed{\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} & \qquad \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma \left(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B} \right) & \qquad \mathbf{B}'_{\perp} &= \gamma \left(\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E} \right) \end{aligned}}$$

The concept of a pure electric or pure magnetic is not Lorentz invariant.

- Lorenz invariants:

  dot product of F with itself or “square” of F:

$$F^{\mu\nu}F_{\mu\nu} = \sum_{i=1}^{3} F^{0i}F_{0i} + \sum_{i=1}^{3} F^{i0}F_{i0} + \sum_{i\neq j} F^{ij}F_{ij} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right)$$

  determinant of F:

$$\det F = (\mathbf{E}\cdot\mathbf{B})^2$$

[Relativistic Mechanics and the Lorentz Four-Force]

- We can define a **four-acceleration** a^μ in exactly the same way as we obtained the four-velocity.

$$a^\mu \equiv \frac{dU^\mu}{d\tau}$$

Note that the four-acceleration and four-velocity are orthogonal:

$$\vec{a} \cdot \vec{U} \equiv \frac{dU^\mu}{d\tau} U_\mu = \frac{1}{2}\frac{d}{d\tau}\left(U^\mu U_\mu\right) = \frac{1}{2}\frac{d}{d\tau}\left(-c^2\right) = 0$$

- We can also define the four-force F^μ from the Lorentz force, so as to obtain a relativistic form of Newton's equation.

$$\boxed{F^\mu \equiv m_0 a^\mu = \frac{dP^\mu}{d\tau}} \qquad \vec{F} = \frac{d\vec{P}}{d\tau} = \gamma \frac{d\vec{P}}{d\tau} = \gamma\left(\frac{1}{c}\frac{dE}{dt}, \ \frac{d\mathbf{p}}{dt}\right)$$

Since $\mathbf{F}_{\text{Lorentz}} = q\left[\mathbf{E} + \frac{1}{c}\left(\mathbf{v} \times \mathbf{B}\right)\right]$, the **Lorentz four-force** should involve (1) the electromagnetic field tensor and (2) the four-velocity and should also be (3) a four-vector and (4) proportional to the charge of the particle. Therefore, the simplest possibility is

$$\boxed{F^\mu_{\text{Lorentz}} = \frac{q}{c} F^{\mu\nu} U_\nu}$$

- Let's check to see if it is indeed what we want.

$$F^0_{\text{Lorentz}} = \frac{q}{c} F^{0\nu} U_\nu = \frac{q}{c} \sum_{i=1}^{3} E_i \gamma v_i = \frac{q}{c} \gamma \left(\mathbf{E} \cdot \mathbf{v} \right) \longrightarrow \frac{dE}{dt} = q \mathbf{E} \cdot \mathbf{v} \quad \text{: conservation of energy}$$

The rate of change of particle energy is the mechanical work done on the particle by the field.

$$\begin{aligned} F^1_{\text{Lorentz}} &= \frac{q}{c} F^{1\nu} U_\nu = \frac{q}{c} \left(F^{10} \left(-\gamma c \right) + F^{12} \gamma v_2 + F^{13} \gamma v_3 \right) \\ &= \frac{q}{c} \gamma \left(E_1 c + B_3 v_2 - B_2 v_3 \right) \end{aligned} \longrightarrow \frac{d\mathbf{p}}{dt} = q \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right)$$

Therefore, we obtained the desired expression for the four-Lorentz force.

- Note that **the four-force is always orthogonal to the four-velocity**:

$$\vec{F} \cdot \vec{U} = m_0 \left(\vec{a} \cdot \vec{U} \right) = 0$$

It implies that **every four-force must have some velocity dependence**.

For the Lorentz four-force, in particular, we find

$$\vec{F}_{\text{Lorentz}} \cdot \vec{U} = \frac{q}{c} F^{\mu\nu} U_\mu U_\nu = 0$$

because $F^{\mu\nu}$ is antisymmetric and $U_\mu U_\nu$ is symmetric.

Mathematical Formulae

- Gamma function

$$\Gamma(x) \equiv \int_0^\infty t^{x-1} e^{-t} dt, \quad \Gamma(x) = (x-1)! = (x-1)\Gamma(x-2), \quad \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

- Euler-Mascheroni constant

$$\gamma \equiv \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right) = -\int_0^\infty e^{-x} \ln x dx = 0.577215664901532$$

- Modified Bessel function of the second kind

$$K_n(x) \equiv \frac{\Gamma(n+1/2)(2x)^n}{\sqrt{\pi}} \int_0^\infty \frac{\cos t}{(t^2+x^2)^{n+1/2}} dt$$

(1) $0 < x < \sqrt{n+1}$

$$K_n(x) \approx \begin{cases} -\ln\left(\frac{x}{2}\right) - \gamma & \text{if } n = 0 \\ \frac{\Gamma(n)}{2}\left(\frac{2}{x}\right)^n & \text{if } n > 0 \end{cases}$$

(2) $x \gg \left| n^2 - 1/4 \right|$

$$K_n(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + \frac{(4n^2-1)}{8x} \right]$$

Recurrence formulae

$$K_{n-1}(x) - K_{n+1}(x) = -\frac{2n}{x} K_n(x)$$

$$K_{n-1}(x) + K_{n+1}(x) = -2K_n'(x)$$

Integral formula

$$\begin{aligned} \int x K_n^2(x) dx &= \frac{1}{2}x^2 \left[K_n^2(x) - K_{n-1}(x) K_{n+1}(x) \right] \\ &= -x K_{n-1}(x) K_n(x) + \\ &\quad \frac{1}{2}x^2 \left[K_n^2(x) - K_{n-1}^2(x) \right] \end{aligned}$$

[Fields of a Uniformly Moving Charge]

- Let's find the fields of a charge moving with constant velocity υ along the x axis. In the rest frame of the particle the fields are

$$\mathbf{E}' = \left(E'_x,\, E'_y,\, E'_z\right) = \frac{q}{r'^3}\left(x',\, y',\, z'\right) \qquad \text{where} \qquad r' = \left(x'^2 + y'^2 + z'^2\right)^{1/2}$$

$$\mathbf{B}' = (0,\, 0,\, 0)$$

inverse transformation of the previous one:

$$\begin{aligned} \mathbf{E}_\parallel &= \mathbf{E}'_\parallel & \mathbf{B}_\parallel &= \mathbf{B}'_\parallel \\ \mathbf{E}_\perp &= \gamma\left(\mathbf{E}'_\perp - \boldsymbol{\beta}\times\mathbf{B}'\right) & \mathbf{B}_\perp &= \gamma\left(\mathbf{B}'_\perp + \boldsymbol{\beta}\times\mathbf{E}'\right) \end{aligned} \longrightarrow \begin{aligned} E_x &= \frac{qx'}{r'^3} & B_x &= 0 \\ E_y &= \gamma\frac{qy'}{r'^3} & B_y &= -\gamma\beta\frac{qz'}{r'^3} \\ E_z &= \gamma\frac{qz'}{r'^3} & B_z &= \gamma\beta\frac{qy'}{r'^3} \end{aligned}$$

Since $x' = \gamma(x - \upsilon t),\ y' = y,\ z' = z$, we obtain

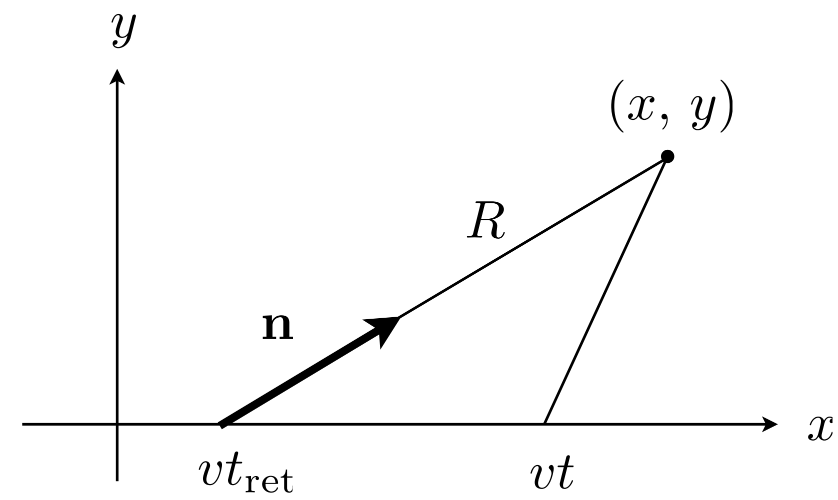
$$\boxed{\begin{aligned} E_x &= \gamma\frac{q\,(x - vt)}{r^3} & B_x &= 0 \\ E_y &= \gamma\frac{qy}{r^3} & B_y &= -\gamma\beta\frac{qz}{r^3} \\ E_z &= \gamma\frac{qz}{r^3} & B_z &= \gamma\beta\frac{qy}{r^3} \end{aligned}}$$

where $r = \left[(x - vt)^2 + y^2 + z^2\right]^2$

Is this equivalent to the fields given by the Lienard-Wiechert potentials?

- Velocity field from the retarded potential

- For simplicity, assume $z = 0$.



$$\mathbf{E} = (E_x,\, E_y,\, E_z) = \gamma \frac{q}{r^3}\,(x - vt,\, y,\, z)$$
$$= \gamma \frac{q}{\left(\gamma^2 \bar{x}^2 + y^2\right)^{3/2}}\,(\bar{x},\, y,\, 0) \qquad \text{where } \bar{x} \equiv x - vt$$

Let us first find where the retarded position of the particle is.

$$t_{\rm ret} \equiv t - R/c$$
$$R^2 = (x - vt_{\rm ret})^2 + y^2 = (\bar{x} + \beta R)^2 + y^2$$
$$\mathbf{n} = \frac{(\bar{x} + \beta R)}{R}\hat{\mathbf{x}} + \frac{y}{R}\hat{\mathbf{y}} = \left(\frac{\bar{x}}{R} + \beta\right)\hat{\mathbf{x}} + \frac{y}{R}\hat{\mathbf{y}}$$

(1) $$\mathbf{n} - \boldsymbol{\beta} = \frac{\bar{x}}{R}\hat{\mathbf{x}} + \frac{y}{R}\hat{\mathbf{y}}$$

$$\mathbf{E} = \gamma \frac{q}{\left(\gamma^2 \bar{x}^2 + y^2\right)^{3/2}}\,(\mathbf{n} - \boldsymbol{\beta})$$

$$R \longrightarrow \quad \left(1 - \beta^2\right)^2 R^2 - 2\bar{x}\beta R - \bar{x}^2 - y^2 = 0$$
$$R^2 - 2\bar{x}\gamma^2\beta R - \gamma^2\left(\bar{x}^2 + y^2\right) = 0$$

$$R = \gamma^2\beta\bar{x} \pm \left[\gamma^4\beta^2\bar{x}^2 + \gamma^2\left(\bar{x}^2 + y^2\right)\right]^{1/2}$$
$$= \gamma^2\beta\bar{x} \pm \gamma\left[\gamma^2\beta^2\bar{x}^2 + \left(\bar{x}^2 + y^2\right)\right]^{1/2}$$
$$= \gamma^2\beta\bar{x} \pm \gamma\left(\gamma^2\bar{x}^2 + y^2\right)^{1/2}$$

(2) $$\left(\gamma^2\bar{x}^2 + y^2\right)^{1/2} = \frac{R - \gamma^2\beta\bar{x}}{\gamma} = R\gamma\left(\frac{1}{\gamma^2} - \frac{\beta\bar{x}}{R}\right)$$
$$= R\gamma\left(1 - \beta^2 - \frac{\beta\bar{x}}{R}\right)$$
$$= R\gamma\left[1 - \beta\left(\frac{\bar{x}}{R} + \beta\right)\right]$$
$$= R\gamma\,(1 - \mathbf{n}\cdot\boldsymbol{\beta}) = R\gamma\kappa$$

positive solution $\rightarrow$ $R = \gamma^2\beta\bar{x} + \gamma\left(\gamma^2\bar{x}^2 + y^2\right)^{1/2}$

$$\therefore\ \boxed{\mathbf{E} = q\frac{(\mathbf{n} - \boldsymbol{\beta})}{\gamma^2\kappa^3 R^2} = q\frac{(\mathbf{n} - \boldsymbol{\beta})\left(1 - \beta^2\right)}{\kappa^3 R^2} \quad \text{: velocity field}}$$

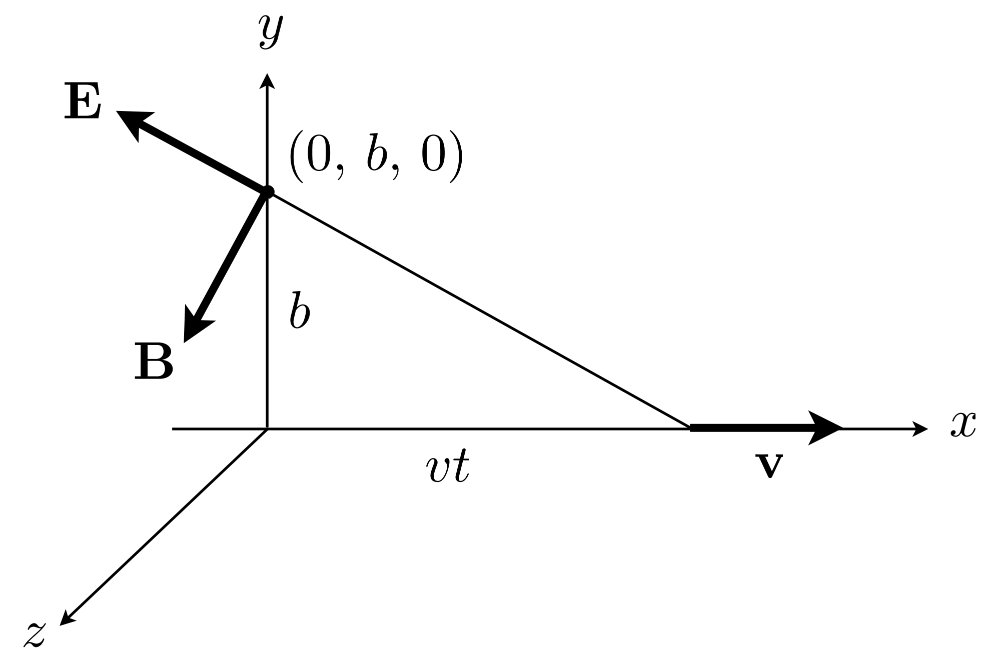
- Time-dependence of the electric field at a point

- Let us choose the field point to be at $(0, b, 0)$.

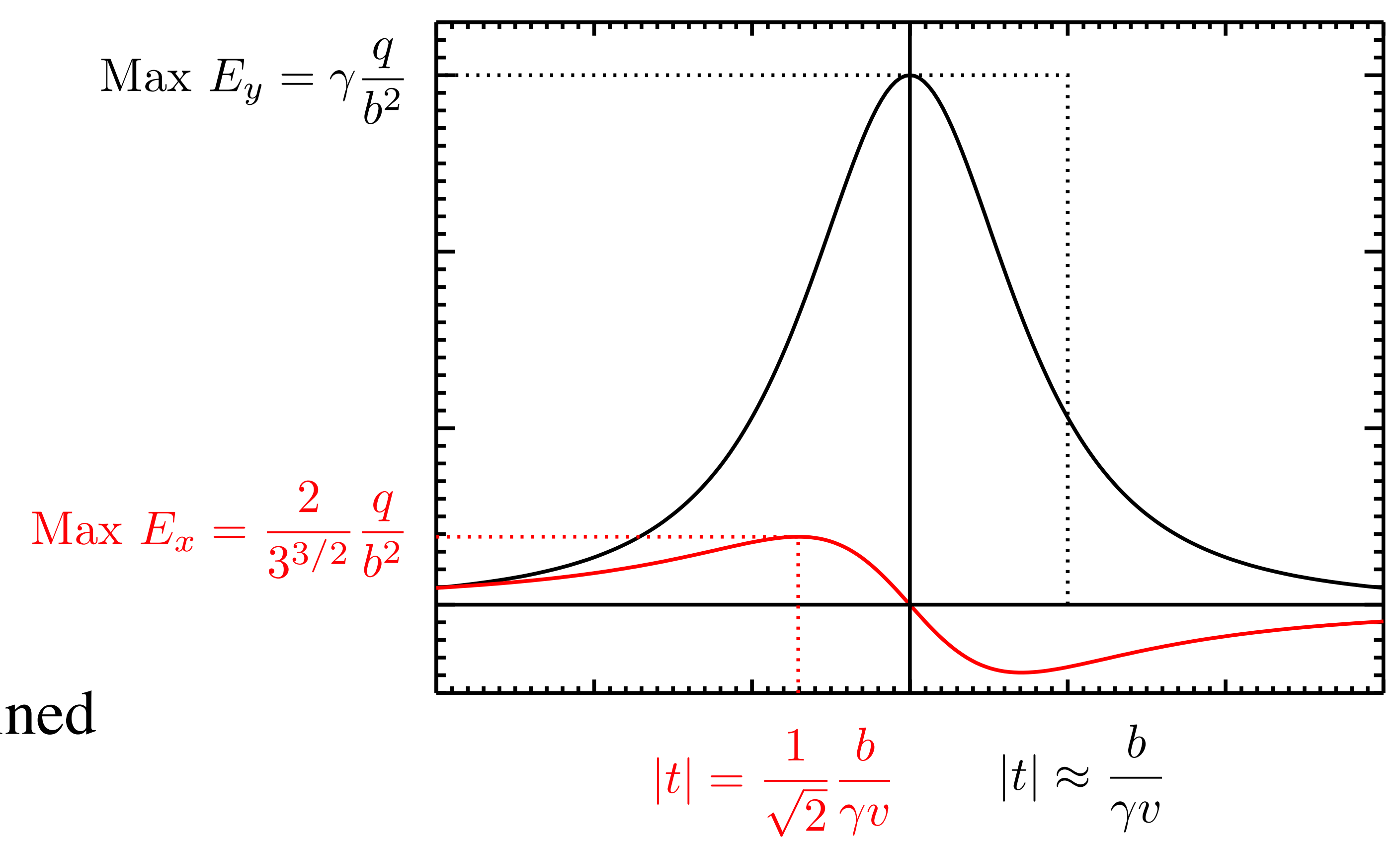
  This involves no loss in generality. Then,

$$E_x = -\frac{q\gamma vt}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = -\frac{q}{b^2}\frac{\gamma vt/b}{(\gamma^2 v^2 t^2/b^2 + 1)^{3/2}}$$
$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = \frac{q\gamma}{b^2}\frac{1}{(\gamma^2 v^2 t^2/b^2 + 1)^{3/2}}$$
$$E_z = 0$$

$$B_x = 0$$
$$B_y = 0$$
$$B_z = \beta E_y$$



As $\gamma \gg 1 \;\rightarrow\; |E_x| \ll E_y$



The field of a highly relativistic charge appears to be a pulse of radiation traveling in the same direction as the charge and confined to the transverse plane.

- Spectrum of the pulse

- Spectrum of this pulse of virtual radiation.

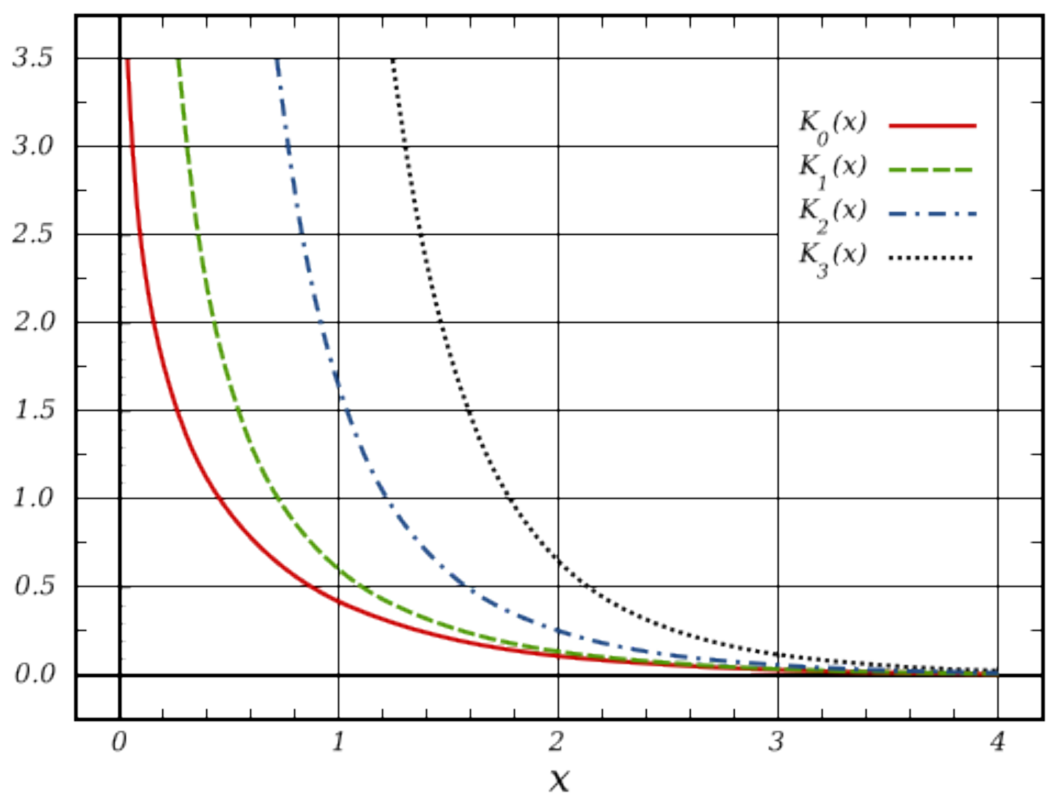
$$
\begin{aligned}
\hat{E}(\omega) &= \frac{1}{2\pi}\int E_y(t)e^{i\omega t}dt \\
&= \frac{q\gamma b}{2\pi}\int_{-\infty}^{\infty}\left(\gamma^2v^2t^2+b^2\right)^{-3/2}e^{i\omega t}dt = \frac{q\gamma b}{2\pi}\int_0^{\infty}\left(\gamma^2v^2t^2+b^2\right)^{-3/2}\left(e^{i\omega t}+e^{-i\omega t}\right)dt \\
&= \frac{q\gamma b}{\pi}\int_0^{\infty}\left(\gamma^2v^2t^2+b^2\right)^{-3/2}\cos\omega t dt
\end{aligned}
$$

This integral can be done in terms of the modified Bessel function:

$$K_n(x) \equiv \frac{\Gamma(n+1/2)(2x)^n}{\sqrt{\pi}}\int_0^{\infty}\frac{\cos t}{(t^2+x^2)^{n+1/2}}dt$$

Gamma function: $\Gamma(3/2)=\frac{1}{2}\Gamma(1/2)=\frac{\sqrt{\pi}}{2}$

$$
\begin{aligned}
\hat{E}(\omega) &= \frac{q\gamma b}{\pi}\left(\frac{\gamma^2v^2}{\omega^2}\right)^{-3/2}\frac{1}{\omega}\int_0^{\infty}\left(\omega^2t^2+\frac{b^2\omega^2}{\gamma^2v^2}\right)^{-3/2}(\cos\omega t)\,d(\omega t) \\
&= \frac{q}{\pi bv}\frac{b\omega}{\gamma v}K_1\left(\frac{b\omega}{\gamma v}\right)
\end{aligned}
$$

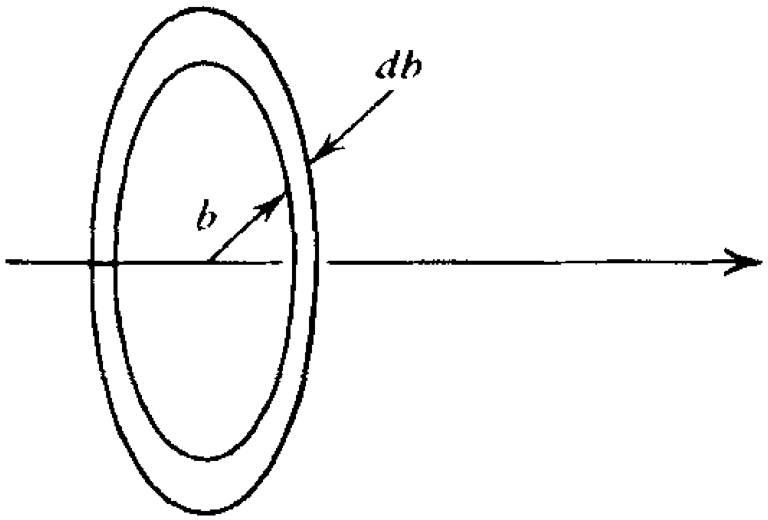


Thus the spectrum is

$$\frac{dW}{dA\,d\omega} = c\left|\hat{E}(\omega)\right|^2 = \frac{q^2}{\pi^2b^2v^2}\left(\frac{b\omega}{\gamma v}\right)^2K_1^2\left(\frac{b\omega}{\gamma v}\right)$$

The spectrum starts cut off for $\omega > \gamma v/b$.

$$\Delta\omega \sim \frac{1}{\Delta t} \sim \gamma v/b$$



- Total energy per unit frequency range is obtained by

$$\frac{dW}{d\omega} = 2\pi \int_{b_{\min}}^{b_{\max}} \frac{dW}{dA d\omega} b db$$

The lower limit has been chosen as some minimum distance, such that the approximation of the field by means of classical electrodynamics and a point charge is valid.

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2} \int_x^\infty y K_1^2(y) dy$$
$$= \frac{2q^2c}{\pi v^2} \left[x K_0(x) K_1(x) - \frac{1}{2} x^2 \left(K_1^2(x) - K_0^2(x) \right) \right]$$

where $y \equiv \dfrac{\omega b}{\gamma v}$ and $x \equiv \dfrac{\omega b_{\min}}{\gamma v}$

- Two limiting cases:

$$(1)\quad \omega \ll \frac{\gamma v}{b_{\min}}\ (x \ll 1),$$

$$\begin{aligned}
&xK_0(x)K_1(x) - \frac{1}{2}x^2\left(K_1^2(x) - K_0^2(x)\right)\\
&\quad\approx x\left(-\ln(x/2)-\gamma\right)\frac{1}{x} - \frac{x^2}{2}\left[\frac{1}{x^2} - \left(\ln(x/2)+\gamma\right)^2\right]\\
&\quad\approx \ln\left[\frac{2}{x}e^{-(\gamma+1/2)}\right]\\
&\quad= \ln\left(\frac{0.68}{x}\right)
\end{aligned}
\quad\longrightarrow\quad \frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2}\ln\left(0.68\frac{\gamma v}{\omega b_{\min}}\right)$$

$$(2)\quad \omega \gg \frac{\gamma v}{b_{\min}}\ (x \gg 1),$$

$$\begin{aligned}
&xK_0(x)K_1(x) - \frac{1}{2}x^2\left(K_1^2(x) - K_0^2(x)\right)\\
&\quad\approx x\frac{\pi}{2x}e^{-2x} - \frac{1}{2}x^2\frac{\pi}{2x}e^{-2x}\left[\left(\frac{3}{8x}\right)^2 - \left(\frac{1}{8x}\right)^2\right]\\
&\quad= \frac{\pi}{4}e^{-2x}
\end{aligned}
\quad\longrightarrow\quad \frac{dW}{d\omega} = \frac{q^2c}{2v^2}\exp\left(-\frac{2\omega b_{\min}}{\gamma v}\right)$$

[Emission from Relativistic Particles]

- Total emitted power:

Imagine **an instantaneous rest frame** K', such that the particle has zero velocity at a certain time. We can then calculate the radiation emitted by use of the dipole (Larmor) formula.

Suppose that the particle emits a total amount of energy dW' in this frame in time dt'. The momentum of this radiation is zero, $d\mathbf{p}' = 0$, because the emission is symmetrical in the frame.

The energy in a frame K moving with velocity $-\mathbf{v}$ w.r.t. the particle is:

$$dW = \gamma dW' \longleftarrow dE = c dP^0 = c\tilde{\Lambda}^0{}_\mu dP'^\mu = c\tilde{\Lambda}^0{}_0 dP^0 = \gamma dE'$$

The time interval dt is simply

$$dt = \gamma dt'$$

The total power emitted in frames K and K' are given by

$$P = \frac{dW}{dt}, \quad P' = \frac{dW'}{dt'}$$

Thus **the total emitted power is a Lorentz invariant** for any emitter that emits with front-back symmetry in its instantaneous rest frame.

$$P = P'$$

- **the Larmor formula in covariant form:**

  Recall that $\overrightarrow{a} \cdot \overrightarrow{U} = 0,$ and because $\overrightarrow{U} = \left(c,\, \mathbf{0}\right)$ in the instantaneous rest frame of the particle, we have

$$a'_0 = 0 \quad \rightarrow \quad \left|\mathbf{a}'\right|^2 = a'_k a'^k = a'_\mu a'^\mu = \vec{a} \cdot \vec{a}$$

  Therefore,

$$P' = \frac{2q^2}{3c^3} \left|\mathbf{a}'\right|^2 \quad \longrightarrow \quad P = \frac{2q^2}{3c^3} \vec{a} \cdot \vec{a}$$

- Expression of *P* in terms of the three-vector acceleration

  Recall

$$dt = \gamma \left(dt' + \frac{v}{c^2} dx'_\parallel \right)$$

$$u_\parallel = \frac{u'_\parallel + v}{1 + v u'_\parallel / c^2}$$

$$u_\perp = \frac{u'_\perp}{\gamma \left(1 + v u'_\parallel / c^2 \right)}$$

$$\longrightarrow$$

$$\sigma \equiv \left(1 + u u'_\parallel / c^2 \right)$$

$$dt = \gamma dt' \sigma$$

$$u_\parallel = \frac{u'_\parallel + v}{\sigma}$$

$$u_\perp = \frac{u'_\perp}{\gamma \sigma}$$

$$\longrightarrow$$

$$dt = \gamma dt' \sigma$$

$$\begin{aligned} du_\parallel &= \frac{du'_\parallel}{\sigma} - \frac{u'_\parallel + v}{\sigma^2} \frac{v}{c^2} du'_\parallel \\ &= \frac{du'_\parallel}{\sigma^2} \left(1 - \frac{v^2}{c^2} \right) = \frac{du'_\parallel}{\gamma^2 \sigma^2} \end{aligned}$$

$$\begin{aligned} du_\perp &= \frac{du'_\perp}{\gamma \sigma} - \frac{u'_\perp}{\gamma \sigma^2} \frac{v}{c^2} du'_\parallel \\ &= \frac{1}{\gamma \sigma^2} \left(\sigma du'_\perp - \frac{v u'_\perp}{c^2} du'_\parallel \right) \end{aligned}$$

Hence,

$$a_\parallel = \frac{du_\parallel}{dt} = \frac{1}{\gamma^3\sigma^3}\frac{du'_\parallel}{dt'}$$

$$a_\perp = \frac{du_\perp}{dt} = \frac{1}{\gamma^2\sigma^3}\left(\sigma\frac{du'_\perp}{dt'} - \frac{vu'_\perp}{c^2}\frac{du'_\parallel}{dt'}\right)$$

$\longrightarrow$

Transformation of three-vector acceleration:

$$\boxed{\begin{aligned} a_\parallel &= \frac{1}{\gamma^3\sigma^3}a'_\parallel \\ a_\perp &= \frac{1}{\gamma^2\sigma^3}\left(\sigma a'_\perp - \frac{vu'_\perp}{c^2}a'_\parallel\right) \\ \text{where}\quad & \sigma \equiv \left(1 + \frac{vu'_\parallel}{c^2}\right) \end{aligned}}$$

In an instantaneous rest frame of a particle,

$$u'_\parallel = u'_\perp = 0, \quad \sigma = 1$$

$$\boxed{\begin{aligned} a'_\parallel &= \gamma^3 a_\parallel \\ a'_\perp &= \gamma^2 a_\perp \end{aligned}}$$

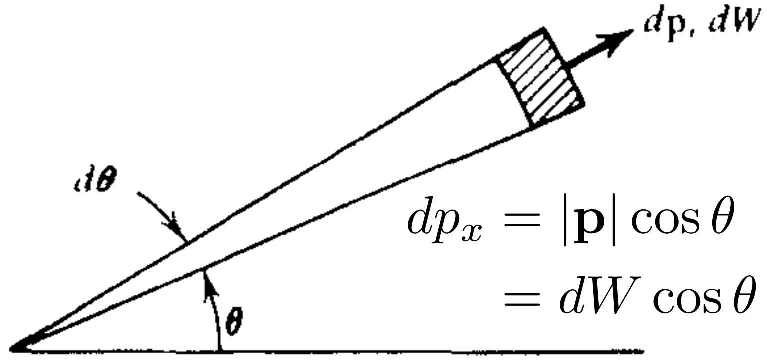
Note $\tan\theta'_a \equiv \dfrac{a'_\perp}{a'_\parallel} = \dfrac{1}{\gamma}\dfrac{a_\perp}{a_\parallel} = \dfrac{1}{\gamma}\tan\theta_a$

Thus we can write

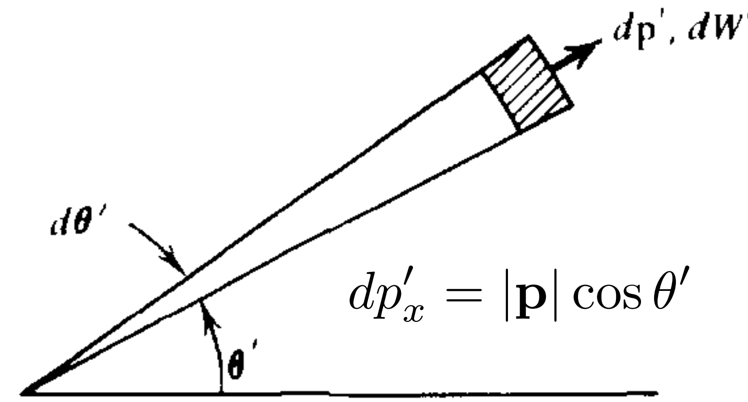
$$P = \frac{2q^2}{3c^3}\left|\mathbf{a}'\right|^2 = \frac{2q^2}{3c^3}\left(a'^2_\parallel + a'^2_\perp\right) \longrightarrow \boxed{P = \frac{2q^2}{3c^3}\gamma^4\left(\gamma^2 a^2_\parallel + a^2_\perp\right)}$$

Differential Power

- <u>Angular Distribution of Emitted and Received Power</u>



$$dp_x = |\mathbf{p}| \cos\theta$$
$$= dW \cos\theta$$



$$dp'_x = |\mathbf{p}| \cos\theta'$$

Note:

$d\phi' = d\phi$

In the instantaneous rest frame of the particle, let us consider an amount of energy dW' that is emitted into the solid angle $d\Omega' = \sin\theta' d\theta' d\phi'$ (see the above figure).

$$\mu \equiv \cos\theta \quad \rightarrow \quad d\Omega = d\mu d\phi$$
$$\mu' \equiv \cos\theta' \quad \rightarrow \quad d\Omega' = d\mu' d\phi'$$

Recall the aberration formula:

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \quad \rightarrow \quad \mu = \frac{\mu' + \beta}{1 + \beta\mu'} \quad \rightarrow \quad \mu' = \frac{\mu - \beta}{1 - \beta\mu}$$

$$\boxed{\begin{array}{lcl} d\mu = \dfrac{d\mu'}{\gamma^2 (1 + \beta\mu')^2} & & d\Omega = \dfrac{d\Omega'}{\gamma^2 (1 + \beta\mu')^2} \\ & \longrightarrow & \\ d\mu' = \dfrac{d\mu}{\gamma^2 (1 - \beta\mu)^2} & & d\Omega' = \dfrac{d\Omega}{\gamma^2 (1 - \beta\mu)^2} \end{array}}$$

- <u>Power</u>

Recall that energy and momentum form a four-vector

$$\vec{P} = (E/c,\ \mathbf{p}) \quad \text{and} \quad |\mathbf{p}| = E/c \quad \longrightarrow \quad dW = \gamma(dW' + v dp'_x) = \gamma(1+\beta\mu')dW'$$

$$\therefore\ dW = \gamma(1+\beta\mu')dW'$$
$$dW' = \gamma(1-\beta\mu)dW$$

$$\boxed{\frac{dW}{d\Omega} = \gamma^3 (1+\beta\mu')^3 \frac{dW'}{d\Omega'}, \quad \frac{dW'}{d\Omega'} = \gamma^3 (1-\beta\mu)^3 \frac{dW}{d\Omega}}$$

In the rest frame, **the power emitted in a unit time interval** is $\dfrac{dP'}{d\Omega'} \equiv \dfrac{dW'}{dt' d\Omega'}$

However, in the observer's frame, there are two possible choices for the time interval to calculate the power.

(1) <u>$dt = \gamma dt'$:</u> $\quad dt_A = \gamma(1-\beta\mu)dt'$ or $dt_A = \gamma^{-1}(1+\beta\mu')^{-1}dt'$

This is the time interval during which the emission occurs. With this choice we obtain **the emitted power**.

(2) <u>$dt_A = \gamma(1-\beta\mu)dt'$ or $dt_A = \gamma^{-1}(1+\beta\mu')^{-1}dt'$:</u>

This is the time interval of the radiation as received by a stationary observer in K. With this choice we obtain **the received power**.

- Thus we obtain the two results:

$$\boxed{\begin{aligned} \frac{dP_e}{d\Omega} &= \gamma^2 (1+\beta\mu')^3 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4 (1-\beta\mu)^3} \frac{dP'}{d\Omega'} \\ \frac{dP_r}{d\Omega} &= \gamma^4 (1+\beta\mu')^4 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4 (1-\beta\mu)^4} \frac{dP'}{d\Omega'} \end{aligned}}$$

P_r is the power actually measured by an observer. It has the expected symmetry property of yielding the inverse transformation by interchanging primed and unprimed variables, along with a change of sign of β.

P_e is used in the discussion of emission coefficient.

In practice, **the distinction between emitted and received power is often not important, since they are equal in an average sense for stationary distributions of particles.**

- Beaming effect:

If the radiation if isotropic in the particle's frame, then the angular distribution in the observer's frame will be highly peaked in the forward direction for highly relativistic velocities.

The factor $\gamma^{-4}(1-\beta\mu)^{-4}$ is sharply peaked near $\theta \approx 0$ with an angular scale of order $1/\gamma$.

$$\gamma^{-4}(1-\beta\mu)^{-4} \approx \gamma^{-4}\left[1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right)\right]^{-4} = \gamma^{-4}\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^{-4} = \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^{-4}$$

- Dipole emission from a slowly moving particle

$$\frac{dP'}{d\Omega'} = \frac{q^2 a'^2}{4\pi c^3} \sin^2 \Theta'$$

$\Theta' =$ the angle between the acceleration and the direction of emission.

Using $a'_{\parallel} = \gamma^3 a_{\parallel},\ a'_{\perp} = \gamma^2 a_{\perp}$ and $\dfrac{dP_r}{d\Omega} = \gamma^{-4} (1-\beta\mu)^{-4} \dfrac{dP'}{d\Omega'}$, we obtain

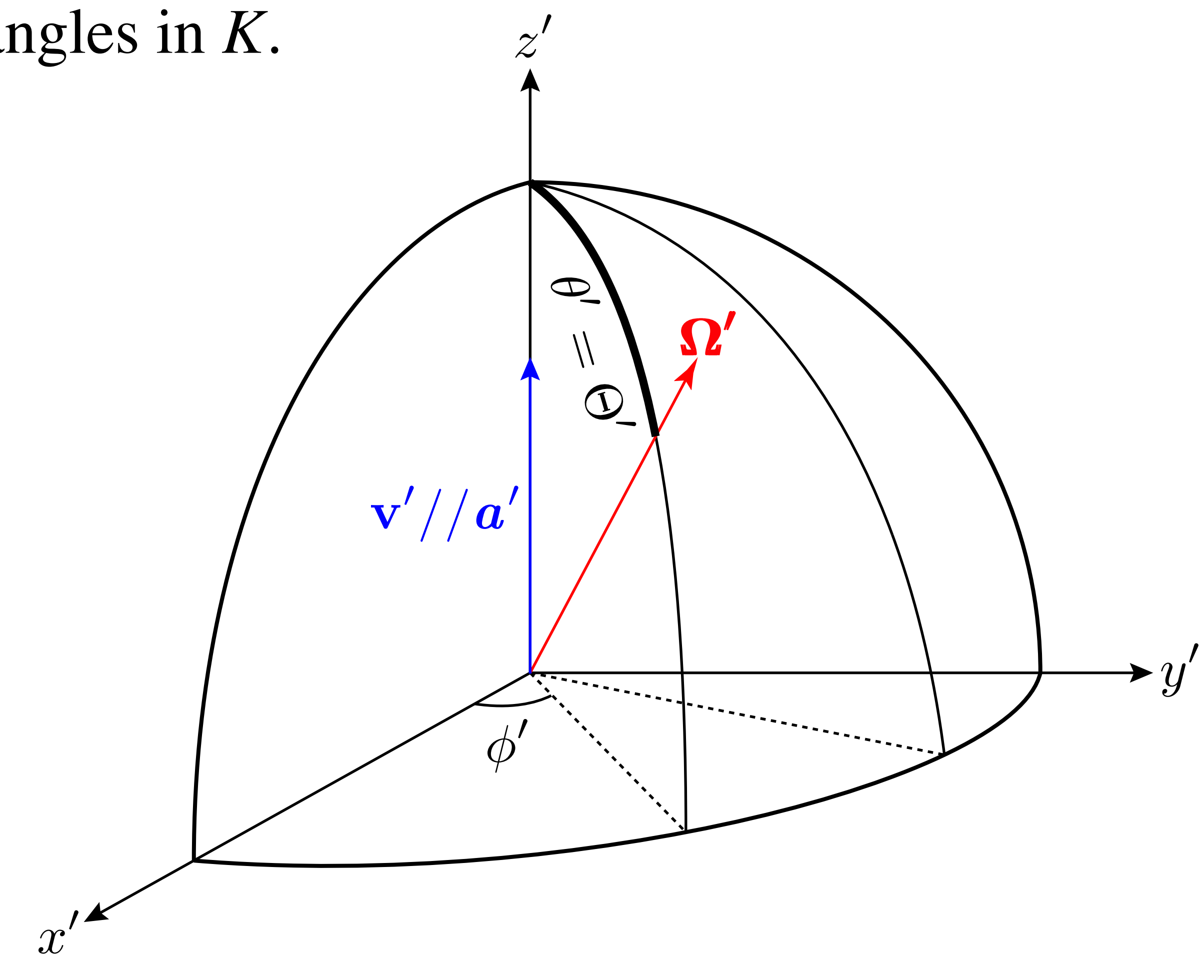
$$\boxed{\frac{dP_r}{d\Omega} = \frac{q^2}{4\pi c^3} \frac{\left(\gamma^2 a_{\parallel}^2 + a_{\perp}^2\right)}{(1-\beta\mu)^4} \sin^2 \Theta'}$$

To use this formula, we must relate Θ' to the angles in K.

(1) Acceleration parallel to velocity: $\Theta' = \theta',\ a_{\perp} = 0$

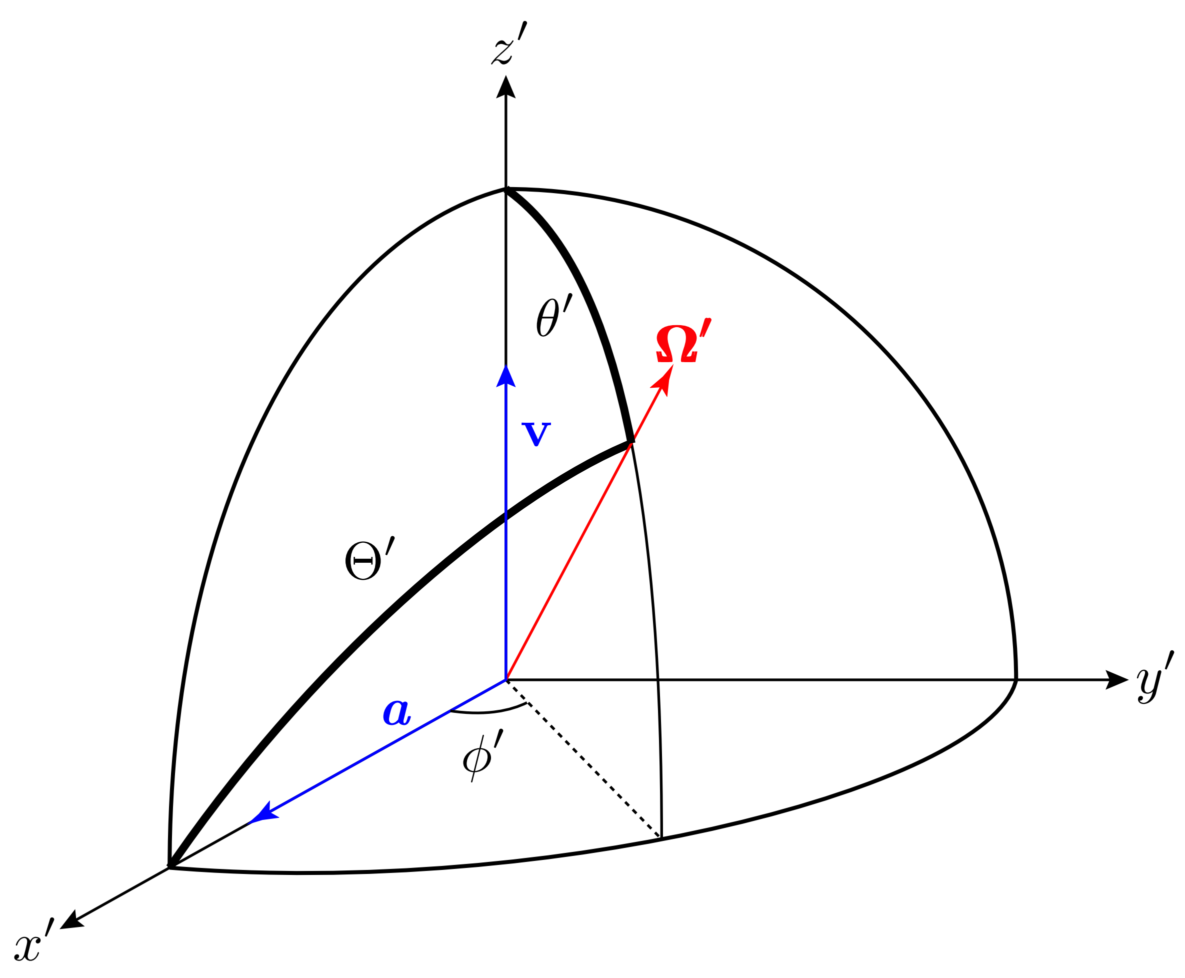
$$\sin^2 \Theta' = 1 - \mu'^2 = 1 - \left(\frac{\mu - \beta}{1 - \beta\mu}\right)^2 = \frac{1-\mu^2}{\gamma^2 (1-\beta\mu)^2}$$

$$\longrightarrow \quad \frac{dP_{r\parallel}}{d\Omega} = \frac{q^2 a_{\parallel}^2}{4\pi c^3} \frac{1-\mu^2}{(1-\beta\mu)^6}$$



(2) Acceleration perpendicular to velocity: $\cos\Theta' = \sin\theta' \cos\phi'$ (when a is in x-direction in the figure)

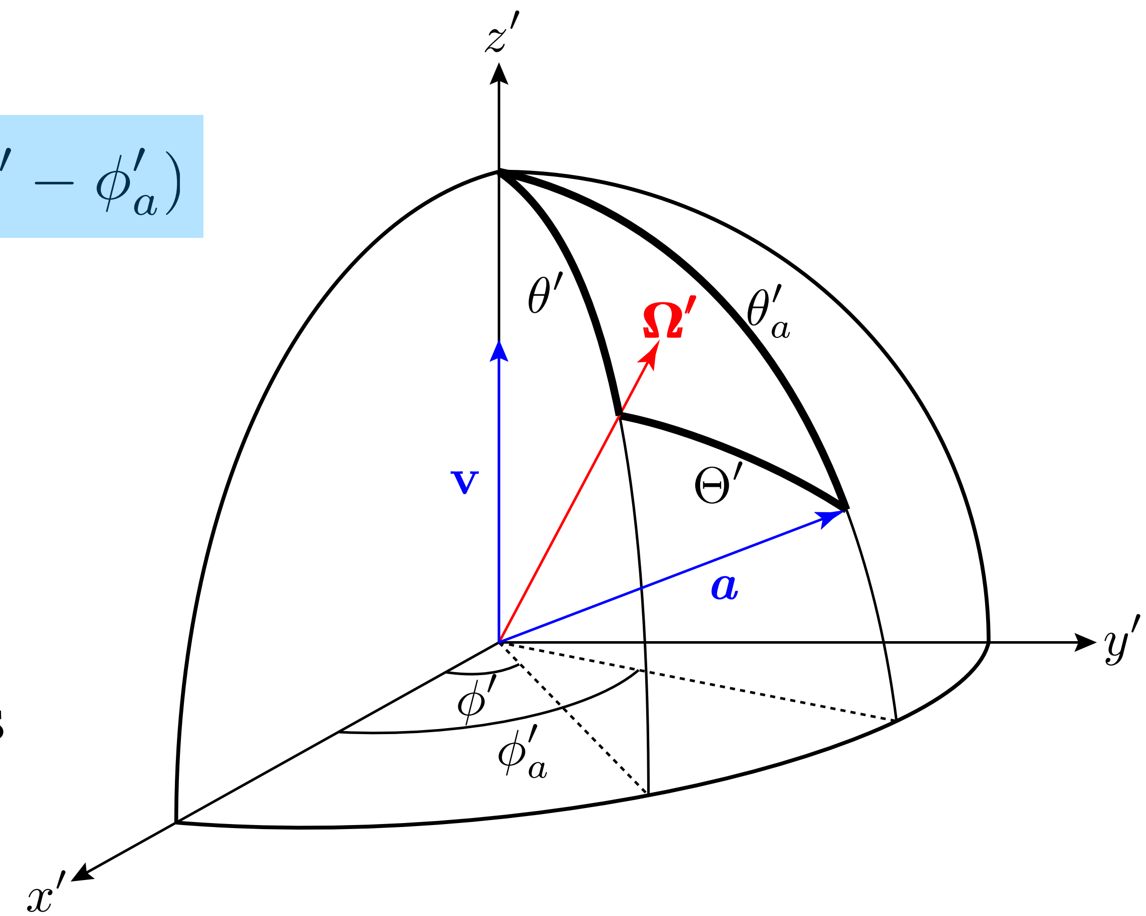
$$\sin^2\Theta' = 1 - \frac{\left(1-\mu^2\right)\cos^2\phi}{\gamma^2\left(1-\beta\mu\right)^2} \longrightarrow \frac{dP_{r\perp}}{d\Omega} = \frac{q^2 a_\perp^2}{4\pi c^3}\frac{1}{\left(1-\beta\mu\right)^4}\left[1 - \frac{\left(1-\mu^2\right)\cos^2\phi}{\gamma^2\left(1-\beta\mu\right)^2}\right]$$



(3) In general

$$\cos\Theta' = \mu'\mu'_a + \left(1-\mu'^2\right)^{1/2}\left(1-\mu_a'^2\right)^{1/2}\cos\left(\phi'-\phi'_a\right)$$

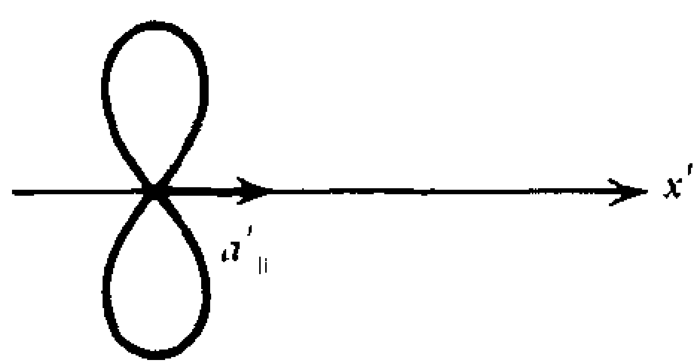
$$\mathbf{r}_1 = (\sin\theta_1\cos\phi_1,\ \ \sin\theta_1\sin\phi_1,\ \ \cos\theta_1)$$
$$\mathbf{r}_2 = (\sin\theta_2\cos\phi_2,\ \ \sin\theta_2\sin\phi_2,\ \ \cos\theta_2)$$
$$\begin{aligned}\cos\Omega &= \mathbf{r}_1\cdot\mathbf{r}_2\\ &= \mu_1\mu_2 + \sqrt{1-\mu_1}\sqrt{1-\mu_2}\,(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)\end{aligned}$$

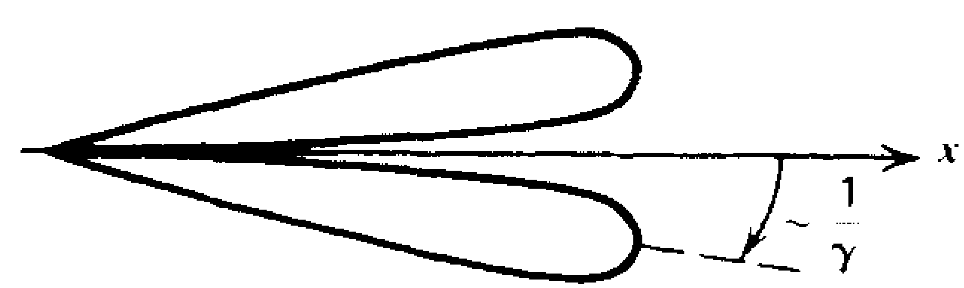


- In the extreme relativistic limit, the radiation becomes strongly peaked in the forward direction.

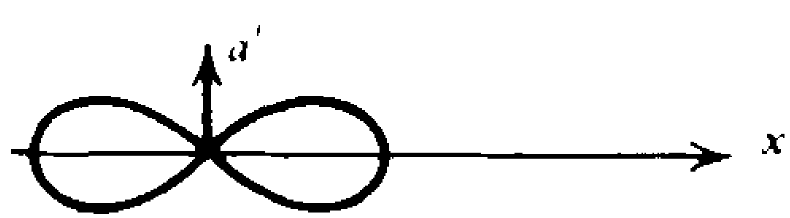
particle's rest frame: observer's frame:

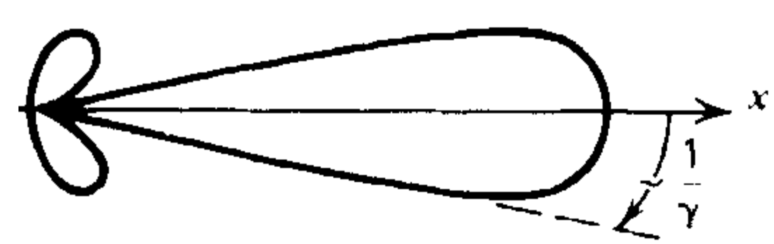
parallel acceleration:





perpendicular acceleration:





[Invariant Phase Volumes and Specific Intensity]

- **Phase volume**

Consider a group of particles that occupy a slight spread in position and in momentum at a particular time. In a rest frame comoving with the particles, they occupy a spatial volume element and a momentum volume element.

$$d^3\mathbf{x}' = dx'dy'dz'$$
$$d^3\mathbf{p}' = dp'_x dp'_y dp'_z$$

$\longrightarrow$ phase volume in the comoving frame:

$$d\mathcal{V}' \equiv d^3\mathbf{x}' d^3\mathbf{p}' = dx'dy'dz'dp'_x dp'_y dp'_z$$

In the observer's frame, $dx = \frac{1}{\gamma}dx'$ (length contraction), $dy = dy'$, $dz = dz'$

$$dp_x = \gamma\left(dp'_x + \beta dP'_0\right), \quad dp_y = dp'_y, \quad dp_z = dp'_z$$

We note that $dP'_0 = 0 + \mathcal{O}(dp'^2_x)$ because the velocities are near zero in the comoving frame and the energy is quadratic in velocity. Therefore, we have $dp_x = \gamma dp'_x$.

$$\boxed{d\mathcal{V}' \equiv d^3\mathbf{x}' d^3\mathbf{p}' = d^3\mathbf{x} d^3\mathbf{p} \equiv d\mathcal{V}}$$: Lorentz invariant

This contains no reference to particle mass, and therefore it has applicability to photons.

The phase space density

$$\boxed{f \equiv \frac{dN}{d\mathcal{V}} = \frac{dN}{d^3\mathbf{x} d^3\mathbf{p}}}$$

is an invariant, since the number of particles within the phase volume element is a countable quantity and itself invariant.

- **Specific Intensity and Source Function**

Definition of the energy density per unit solid angle per frequency range.

$$h\nu \frac{dN}{d\mathcal{V}} p^2 dp d\Omega d^3\mathbf{x} = u_\nu(\Omega) d\Omega d\nu d^3\mathbf{x} \longrightarrow \quad u_\nu(\Omega) = h\nu \frac{dN}{d\mathcal{V}} p^2 \frac{dp}{d\nu} \quad \leftarrow \quad p = \frac{h\nu}{c}$$

$$= h\nu \left(\frac{h\nu}{c}\right)^2 \left(\frac{h}{c}\right) \frac{dN}{d\mathcal{V}}$$

Since $u_\nu(\Omega) = I_\nu / c$, we find that $\boxed{\dfrac{I_\nu}{\nu^3} = \text{ Lorentz invariant}}$

Because the source function occurs in the transfer equation as the difference $I_\nu - S_\nu$, the source function must have the same transformation properties as the intensity.

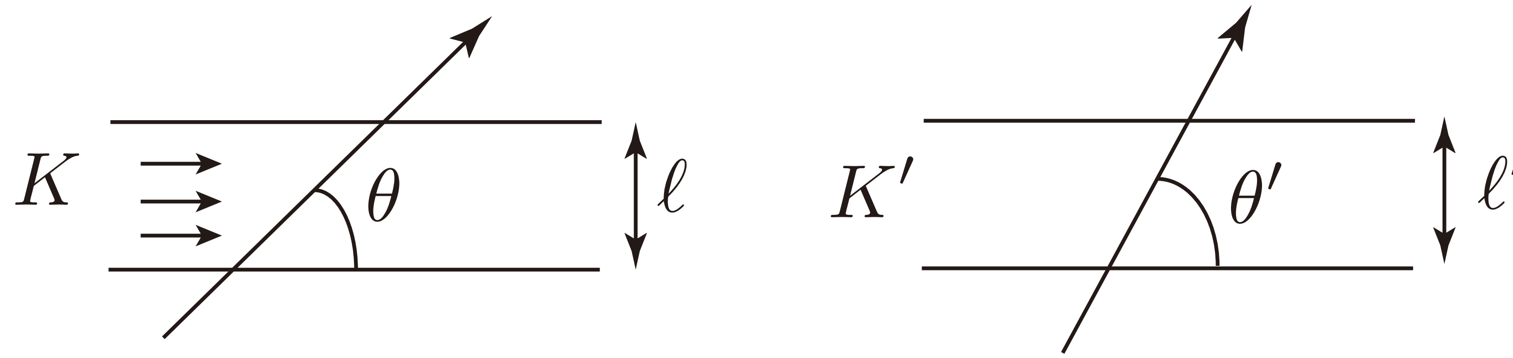
- **Optical Depth, Absorption Coefficient and Emission Coefficient**

The optical depth must be an invariant, since $\exp(-\tau)$ gives the fraction of photons passing through the material, and this involves simple counting.

$$\boxed{\tau = \text{ Lorentz invariant}}$$

- **Absorption Coefficient and Emission Coefficient**

Consider the optical depth in two frames:



Then, the optical depth is

$$\tau_\nu = \frac{\ell \alpha_\nu}{\sin\theta} = \frac{\ell}{\nu \sin\theta}\nu\alpha_\nu = \text{ Lorentz invariant}$$

Note that $\nu \sin\theta$ is proportional to the y component of the photon four-momentum $\vec{k} = (\omega/c,\ \mathbf{k})$.

Both k_y and ℓ are the same in both frames, being perpendicular to the motion. Therefore, we have

$$\nu \sin\theta \propto k_y, \quad k_y = k'_y, \quad \ell' = \ell$$

$$\boxed{\nu\alpha_\nu = \text{ Lorentz invariant}}$$

Finally, we obtain the transformation of the emission coefficient from the definition of the source function:

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu}$$

$$\boxed{\frac{j_\nu}{\nu^2} = \text{ Lorentz invariant}}$$