

Interstellar Medium (ISM)

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Intermediate States - expansion phase

- Assumptions:
 - The shocked gas layer is thin.
 - The ionization front follows the shock front and the expansion velocity of ionized sphere is approximately the same as the shock velocity.

$$V_{\text{I-front}} \approx V_{\text{shock}} \quad \frac{dR}{dt} = V_s$$

- Expansion:
 - The pressure behind a strong “isothermal” shock (high Mach number) is related to the shock velocity:

$$P_s = \rho_0 V_s^2 = n_0 m_H V_s^2$$

- Now assume that the pressure behind the shock wave is equal to the pressure of the ionized gas (pressure equilibrium).

$$P_i = 2n_i kT = n_i m_H c_i^2 \quad \left(c_i^2 \equiv \frac{2kT}{m_H} \right) \quad \text{for fully-ionized hydrogen gas}$$

- Then, the shock velocity is given by

$$P_s = P_i \rightarrow V_s^2 = \frac{n_i}{n_0} c_i^2 \rightarrow \frac{V_s^2}{c_i^2} = \frac{n_i}{n_0}$$

- We assume that the amount of fresh neutral gas to be ionized is very small. Then, the ionization balance for the region within R gives

$$Q_0 = \frac{4\pi}{3} R^3 n_i^2 \alpha_B \quad \rightarrow \quad R^3 = \frac{3Q_0}{4\pi n_i^2 \alpha_B} = R_s^3 \left(\frac{n_0}{n_i} \right)^2 \quad R_s = \text{Strömgren radius for the initial stage.}$$

- Combining with $\frac{V_s^2}{c_i^2} = \frac{n_i}{n_0}$, the equation for the expansion of the ionization front is

$$\begin{aligned} R^3 &= R_s^3 \left(\frac{c_i}{V_s} \right)^4 & \leftarrow \frac{dR}{dt} = V_s \\ \rho \equiv R/R_s, \quad \tau \equiv c_i t / R_s &\longrightarrow \quad \rho^3 \left(\frac{d\rho}{d\tau} \right)^4 = 1 & \rightarrow \boxed{\rho^{3/4} \frac{d\rho}{d\tau} = 1} \end{aligned}$$

- For a suitable boundary condition, we assume that the initial Strömgren sphere is set up at τ_0 (a very small fraction of the lifetime of the H II region):

$$R = R_s \text{ at } \tau = \tau_0$$

Then, the solution of the differential equation is

$$\rho = \left[1 + \frac{7}{4}(\tau - \tau_0) \right]^{4/7}$$

$$R = R_s \left(1 + \frac{7}{4} \frac{t - t_0}{R_s/c_i} \right)^{4/7}$$

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- Expanding velocity is

$$\frac{dR}{dt} = c_i \left(1 + \frac{7}{4} \frac{t - t_0}{R_s/c_i} \right)^{-3/7}$$

- What is the time scale to reach the pressure equilibrium?

$$R(t_{\text{eq}}) = R_f \approx 34R_s$$

$$R_s \left(1 + \frac{7}{4} \frac{t_{\text{eq}}}{R_s/c_i} \right)^{4/7} \approx 34R_s$$

$$t_{\text{eq}} \approx 273 (R_s/c_i)$$

- The expanding velocity at this point is:

$$V_s = \frac{dR}{dt} = 0.71 c_i \quad \text{at} \quad t_{\text{eq}} = 273R_s/c_i$$

Timescales for typical HII region

- Let's examine the case of an O7V star with

$$Q_0 = 10^{49} \text{ s}^{-1}, \quad n_0 = 10^2 \text{ cm}^{-3}, \quad T = 10^4 \text{ K}$$

- Initial state: recombination time scale $t_{\text{rec}} = (n_0 \alpha_B)^{-1}$

$$R \approx R_s \text{ at } t = t_{\text{rec}}$$

$$R_s \approx 3 \text{ pc} (\approx 10^{19} \text{ cm})$$

$$t_{\text{rec}} \approx 1000 \text{ yr}$$

$$\begin{aligned} t &\lesssim t_{\text{rec}} \\ R(t) &= R_s \left(1 - e^{-t/t_{\text{rec}}}\right)^{1/3} \\ \frac{dR}{dt} &= \frac{R_s}{3t_{\text{rec}}} \frac{e^{-t/t_{\text{rec}}}}{(1 - e^{-t/t_{\text{rec}}})^{2/3}} \end{aligned}$$

- Expansion phase: expansion timescale $t_{\text{exp}} = R_s/c_i$

expansion velocity : $V_s \leq 0.65 c_i$ at $t \geq t_{\text{exp}}$

$$c_i \approx 10 \text{ km s}^{-1}$$

$$t_{\text{exp}} \approx 3 \times 10^5 \text{ yr} \rightarrow t_{\text{exp}} \approx 200 t_{\text{rec}}$$

$$\begin{aligned} t_{\text{rec}} &\lesssim t \lesssim t_{\text{eq}} \\ R &= R_s \left(1 + \frac{7}{4} \frac{t - t_0}{R_s/c_i}\right)^{4/7} \\ \frac{dR}{dt} &= c_i \left(1 + \frac{7}{4} \frac{t - t_0}{R_s/c_i}\right)^{-3/7} \end{aligned}$$

- Final state: equilibrium timescale $t_{\text{eq}} \approx 273 R_s/c_i$ (from expansion phase model)

$$R = R_f \text{ at } t = t_{\text{eq}}$$

$$R_f/R_s \approx 34$$

$$t_{\text{eq}} \approx 10^8 \text{ yr} \rightarrow t_{\text{eq}} \approx 300 t_{\text{exp}}$$

Does the Stromgren sphere reach pressure equilibrium?

- Main-sequence lifetime of an ionizing star

$$t_{\text{MS}} \approx 10^{10} \left(\frac{M}{M_{\odot}} \right)^{-2} \text{ yr} \quad t_{\text{MS}} \approx 10^7 \text{ yr} \text{ for } M \approx 15M_{\odot}$$

- Size

- During the lifetime of an O star, which is less than 10 Myr, interstellar gas moving at 10 km/s will travel less than 100 pc, which is comparable with the diameter of the larger H II regions.
- Thus, *before an H II region has expanded very far, its central energy source will be extinguished.*

- Time Scale:

- Main-sequence lifetime of an ionizing star is 10 times smaller than the time scale for the pressure equilibrium:

$$t_{\text{MS}} \approx 10^7 \text{ yr} \ll t_{\text{eq}} \approx 10^8 \text{ yr}$$

- *It is unlikely that the final state (pressure equilibrium) of H II region can be reached during lifetime of star.*

Gas Dynamics

- Gas Dynamics / Shock

Introduction to Gas Dynamics

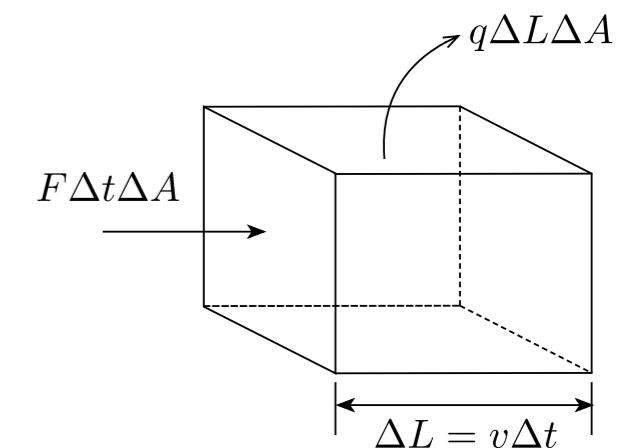
- Assumption for hydrodynamics:
 - particle mean free path << size of the region
 - We will describe the equations for conservation of mass, momentum and energy, in 1D space.

- ***Definition***

- **Flux of a hydrodynamic quantity q** (for instance, density):

Fluid moves a distance ΔL during a time interval Δt with a velocity v .

$$F\Delta t\Delta A = q\Delta L\Delta A \rightarrow F = qv$$



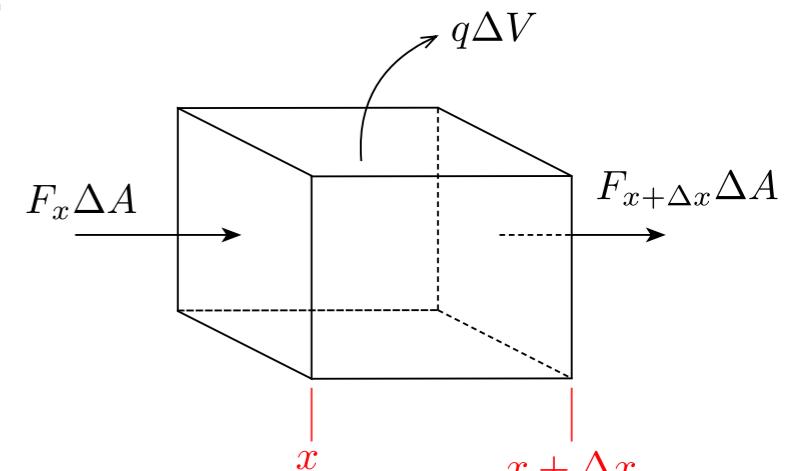
- ***Conservation equation for a quantity q***

- change of the quantity within a volume ΔV for a time interval Δt :

Here, Δt and Δx are independent.

$$\frac{q\Delta V|_{t+\Delta t} - q\Delta V|_t}{\Delta t} = F\Delta A|_x - F\Delta A|_{x+\Delta x}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial F}{\partial x} \rightarrow \frac{\partial q}{\partial t} = -\frac{\partial(qv)}{\partial x}$$



- Here, no sources or sinks of the quantity within ΔV were assumed. If any, the loss and gain terms should be added in the right-hand side.

Mass Conservation

- Conservation equations

- ***Mass conservation (continuity equation)***

- ▶ mass within a volume $dV = \rho dV$
- ▶ no sources or sinks of material within dV
- ▶ Consider the mass per unit area (dA), contained in the volume

$$\rho dV/dA = \rho dx \quad \longrightarrow \quad \frac{\partial}{\partial t}(\rho dx) = \overbrace{\rho u}^{\text{incoming}} - \overbrace{(\rho + d\rho)(u + du)}^{\text{outgoing}}$$

$$= -(\rho du + ud\rho + \cancel{d\rho du})$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$

- ▶ Mass loss and gain terms should be added in the right-hand side, if necessary.

Momentum Conservation

- **Momentum conservation (Euler's equation)**

- ▶ momentum within dV (per unit area) = $(\rho dV)u/dA = \rho dx u$
= change of momentum due to fluid flow and gas pressure acting on the surface of dV

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u dx) &= \underbrace{\rho u^2}_{\text{incoming}} - \underbrace{(\rho + d\rho)(u + du)^2}_{\text{outgoing}} + \underbrace{P}_{\text{incoming}} - \underbrace{P + dP}_{\text{outgoing}} \\ &= \rho u^2 - \left(\rho u^2 + 2\rho u du + \cancel{\rho du^2} + u^2 d\rho + \cancel{2ud\rho du} + \cancel{d\rho du^2} \right) - dP \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) &= -2\rho u \frac{\partial u}{\partial x} - u^2 \frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x} \\ \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} &= -\rho u \frac{\partial u}{\partial x} - u \left(\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} \right) - \frac{\partial P}{\partial x} \end{aligned}$$

Using mass conservation, $\frac{\partial u}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$

$$\rho \frac{\partial u}{\partial t} = -\rho u \frac{\partial u}{\partial x} - \frac{\partial P}{\partial x}$$

or

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) &= -2\rho u \frac{\partial u}{\partial x} - u^2 \frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x} \\ &= -\frac{\partial}{\partial x}(\rho u^2) - \frac{\partial P}{\partial x} \end{aligned}$$

$$\frac{\partial}{\partial t}(\rho u) = -\frac{\partial}{\partial x} (\rho u^2 + P)$$

- ▶ Further terms could be added in the right-hand side, accounting for forces due to gravity, magnetic fields, radiation field, and viscosity.

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- ▶ The following quantity is sometimes known as **Bernoulli's constant**.

$$\rho u^2 + P$$

One may use it to understand why, for example, fast winds engulfing a house causes it to **explode**, rather than **implode**, because the pressure external to the house becomes lower than its value inside it.

- ▶ Viscous force is due to “internal friction” in the fluid (resistivity of the fluid to the flow), as two adjacent fluid parcels move relative to each other.)

$$\text{viscous force} \propto \frac{\partial^2 u}{\partial x^2}$$

The viscous force is usually much smaller than force due to gas pressure, but important in high-speed flows with large velocity gradients, as in accretion disks.

Gas Dynamics - Energy Conservation

- ***Energy conservation***

- ▶ The first law of thermodynamics states that

heat added in a system = change in internal energy + work done on surroundings

$$dQ = dU + PdV$$

- ▶ Internal energy (per particle) for ideal gas is

$U/N = \frac{3}{2}kT$ for monatomic gas (translation about 3 axes)

$U/N = \frac{5}{2}kT$ for diatomic gas (+rotation about 2 axes)

$U/N = 3kT$ for polyatomic gas (+rotation about 3 axes)

Here, N is the number of particles.

An ideal gas is a theoretical gas composed of many randomly moving point particles whose only interactions are perfectly elastic collisions (no viscosity or heat conduction).

- ▶ In general, the internal energy per particle is

$$U/N = \frac{f}{2}kT \quad (f = \text{degree of freedom})$$

At high temperature, molecules have access to an increasing number of vibrational degrees of freedom, as they start to bend and stretch.

- The ideal gas law (the equation of state) for a perfect Maxwellian distribution.

$$PV = NkT$$

$$P = \frac{N}{V} kT$$

- **Specific heat capacity** is the amount of **heat energy required to raise the temperature of a material per unit of mass**.

- ▶ specific heat capacity **at constant volume**:

$$c_V \equiv \frac{1}{M} \left(\frac{\partial Q}{\partial T} \right)_V = \frac{1}{M} \left(\frac{\partial U}{\partial T} \right)_V$$

$$c_V = \frac{f}{2} \frac{k}{m}$$

M = total mass

$m = M/N$ = mass per particle

$m = \mu m_H$

(μ = mean atomic weight per particle)

- ▶ specific heat capacity **at constant pressure**:

$$c_P \equiv \frac{1}{M} \left(\frac{\partial Q}{\partial T} \right)_P = \frac{1}{M} \left(\frac{\partial U}{\partial T} \right)_P + \frac{P}{M} \left(\frac{\partial V}{\partial T} \right)_P = \frac{1}{M} \frac{f}{2} Nk + \frac{P}{M} \frac{Nk}{P}$$

$$\therefore c_P = \frac{f+2}{2} \frac{k}{m} = c_V + \frac{k}{m}$$

- ▶ Ratio of specific heat capacities:

$$\gamma \equiv \frac{c_P}{c_V} = \frac{f+2}{f} = \frac{5}{3} \text{ for monatomic gas}$$

$$= \frac{7}{5} \text{ for diatomic gas}$$

$$= \frac{4}{3} \text{ for polyatomic gas}$$

γ is called the **adiabatic index**.

$$c_P > c_V$$

This inequality implies that when pressure is held constant, some of the added heat goes into PdV work instead of into internal energy.

- Energy Conservation - limiting cases

- **Adiabatic flow** - negligible heat transport (Internal energy is changed only by work).

$$dQ = dU + PdV = Mc_VdT + PdV$$

$$dQ = 0$$

$$\rightarrow PdV = -Mc_VdT$$

$$PV = NkT$$

$$\rightarrow VdP + PdV = NkdT$$

We combine two equations and eliminate dT term:

$$\begin{aligned} VdP + PdV &= -\frac{Nk}{Mc_V} PdV \\ &= -\frac{k}{m c_V} PdV \end{aligned}$$



$$\begin{aligned} VdP &= -\left(1 + \frac{k}{m c_V}\right) PdV \\ &= -\frac{1}{c_V} \left(c_V + \frac{k}{m}\right) PdV \\ &= -\gamma PdV \end{aligned}$$



$$\frac{dP}{P} = -\gamma \frac{dV}{V}$$

We can rewrite this in terms of density:

$$\rho V = M$$

$$\rightarrow \rho dV + Vd\rho = 0$$

$$\rightarrow \frac{d\rho}{\rho} = -\frac{dV}{V}$$

$$\longrightarrow \frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$

In summary,

$$\frac{dP}{P} = -\gamma \frac{dV}{V}$$

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$

$$P \propto \rho^\gamma$$

$$P \propto V^{-\gamma}$$

$$\rightarrow T \propto V^{-(\gamma-1)}$$

adiabatic heating/cooling

- ▶ **Isothermal flow** - extremely efficient cooling (heat transport).

heat transport timescale << dynamic timescale

This implies the balance between heating and cooling, hence a constant temperature.

From the ideal gas law,

$$P = \frac{N}{V} kT = \rho \frac{kT}{m}$$

$$\begin{aligned} P &\propto \rho \\ P &\propto V^{-1} \end{aligned}$$

- ▶ **In general**, we have

$$\begin{aligned} P &\propto \rho^\gamma \\ P &\propto V^{-\gamma} \end{aligned}$$

$(\gamma = 1$ for isothermal gas)

A gas that has an equation of state with this power-law form is called a **polytope**, from the Greek polytropos, meaning “turning many ways” or “versatile.”

(A polystrope should not be confused with a polytrope, which is the n-dimensional generalization of a 2D polygon and 3D polyhedron.)

- **Specific internal energy** of the gas (**per unit mass**):

$$\begin{aligned}\epsilon &\equiv U/M \\ U/N &= \frac{f}{2}kT\end{aligned}\longrightarrow \epsilon = \frac{f}{2} \frac{kT}{m} \quad \text{or} \quad \epsilon = \frac{1}{\gamma-1} \frac{kT}{m} = \frac{1}{\gamma-1} \frac{P}{\rho}$$

- **Total Energy (per unit volume)**:

► **Internal energy per unit volume:**

$$\mathcal{E}_{\text{int}} = \rho\epsilon = \frac{1}{\gamma-1}P$$

► **Kinetic energy due to bulk motion, per unit volume:**

$$\mathcal{E}_{\text{kin}} = \rho \frac{u^2}{2}$$

► **Work on unit volume:**

$$\mathcal{E}_{\text{mech}} = \frac{PdV}{dV} = P$$

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_{\text{int}} + \mathcal{E}_{\text{kin}} + \mathcal{E}_{\text{mech}} \\ &= \rho \left(\frac{u^2}{2} + \epsilon \right) + P\end{aligned}\longrightarrow \mathcal{E} = \rho \frac{u^2}{2} + \frac{\gamma}{\gamma-1}P$$

- **Energy conservation:**

$$\frac{\partial \mathcal{E}}{\partial t} = -\frac{\partial(u\mathcal{E})}{\partial x}$$

$$\frac{\partial}{\partial t} \left(\rho \frac{u^2}{2} + \frac{\gamma}{\gamma-1}P \right) = -\frac{\partial}{\partial x} \left[u \left(\rho \frac{u^2}{2} + \frac{\gamma}{\gamma-1}P \right) \right]$$

Sound Wave

- Suppose that we are surrounded by an ideal gas with a plane parallel symmetry:
 - We consider a region where the gas has initially a uniform density, pressure, and no bulk velocity: $\rho_0, P_0, u_0 = 0$

In the uniform gas, we introduce small perturbations of the form:

$$\begin{array}{ll} \rho(x, t) = \rho_0 + \rho_1(x, t) & P_1 = P - P_0 \\ u(x, t) = u_1(x, t) & \propto (\rho_0 + \rho_1)^\gamma - \rho_0^\gamma \\ P(x, t) = P_0 + P_1(x, t) & \propto \gamma \rho_0^{\gamma-1} \rho_1 \end{array} \longrightarrow \quad \longrightarrow \quad P_1 = \frac{\gamma P_0}{\rho_0} \rho_1$$

We obtain:

$$\begin{array}{ccc} \frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} & \xrightarrow{\hspace{1cm}} & \frac{\partial \rho_1}{\partial t} = -\rho_0 \frac{\partial u_1}{\partial x} \\ \rho \frac{\partial u}{\partial t} = -\rho u \frac{\partial u}{\partial x} - \frac{\partial P}{\partial x} & \xrightarrow{\hspace{1cm}} & \rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial P_1}{\partial x} = -\frac{\gamma P_0}{\rho_0} \frac{\partial \rho_1}{\partial x} \end{array} \quad \boxed{\frac{\partial^2 \rho_1}{\partial t^2} = -\frac{\gamma P_0}{\rho_0} \frac{\partial^2 \rho_1}{\partial x^2}}$$

- The resulting equation represents a **sound wave (acoustic wave)** with a constant sound speed:

$$c_s = \left(\frac{\gamma P}{\rho} \right)^{1/2} = \left(\frac{\gamma k T}{m} \right)^{1/2}$$

$$c_s \propto \rho^{(\gamma-1)/2}$$

For $\gamma > 1$ sound travels more rapidly in a denser gas.

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- The sound speed is of the same order as the mean thermal velocity:

$$c_s = 1.2 \text{ km s}^{-1} \left(\frac{\gamma}{5/3} \right)^{1/2} \left(\frac{m}{m_p} \right)^{-1/2} \left(\frac{T}{100 \text{ K}} \right)^{1/2}$$

$(m_p = \text{proton mass})$

- **Sound crossing time:**

- ▶ sound crossing time = time it takes for a signal to cross a region of size L :

$$t_{\text{cross}} = L/c_s$$

- ▶ A small pressure gradient tends to be smoothed out within the sound crossing time. Generally, when a stationary gas is disturbed, the resultant changes in velocity, density, pressure, and temperature are communicated downstream at the sound speed.

Fast changes occurring on timescales $\ll t_{\text{cross}}$ will survive, and a shock front forms.

Slow changes occurring on timescales $\gg t_{\text{cross}}$ will be damped.

- **Mach number** = gas velocity / sound speed

$$\mathcal{M} \equiv u/c_s$$

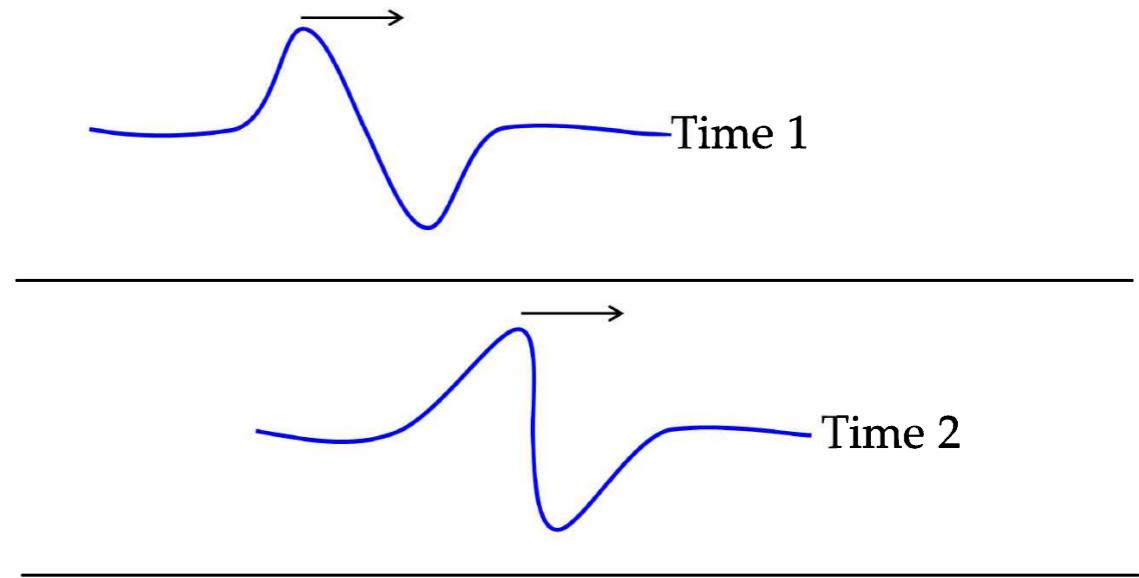
$\mathcal{M} > 1$	supersonic
$\mathcal{M} < 1$	subsonic

Shock

- Shock

- A low-amplitude sound wave traveling through a medium will be adiabatic; that is it will not increase the entropy of the gas through which it passes.
- For an adiabatic process, the equation of state for the gas is

$$c_s \propto \rho^{(\gamma-1)/2}$$



- Thus, for $\gamma > 1$, sound travels more rapidly in a denser gas.
- ***For a supersonic gas, the motion itself is faster than the speed of communication, and instead of a smooth transition, the physical quantities (density, pressure, and temperature) undergo a sudden change in values over a small distance.*** This phenomenon is referred to as a **shock**.
- We define the shock front as the region over which the velocity, density, and pressure of the gas undergo sudden changes. The shock front is a layer whose thickness is comparable to the mean free path between particle collisions.
- The ordinary sound that we hear every day will not, in practice, steepen into shocks.
- However, high amplitude pressure fluctuations will rapidly steepen into shocks.

Shock Front

- Jump condition (***Rankine-Hugoniot conditions***)

- Let

ρ = mass density, T = temperature,

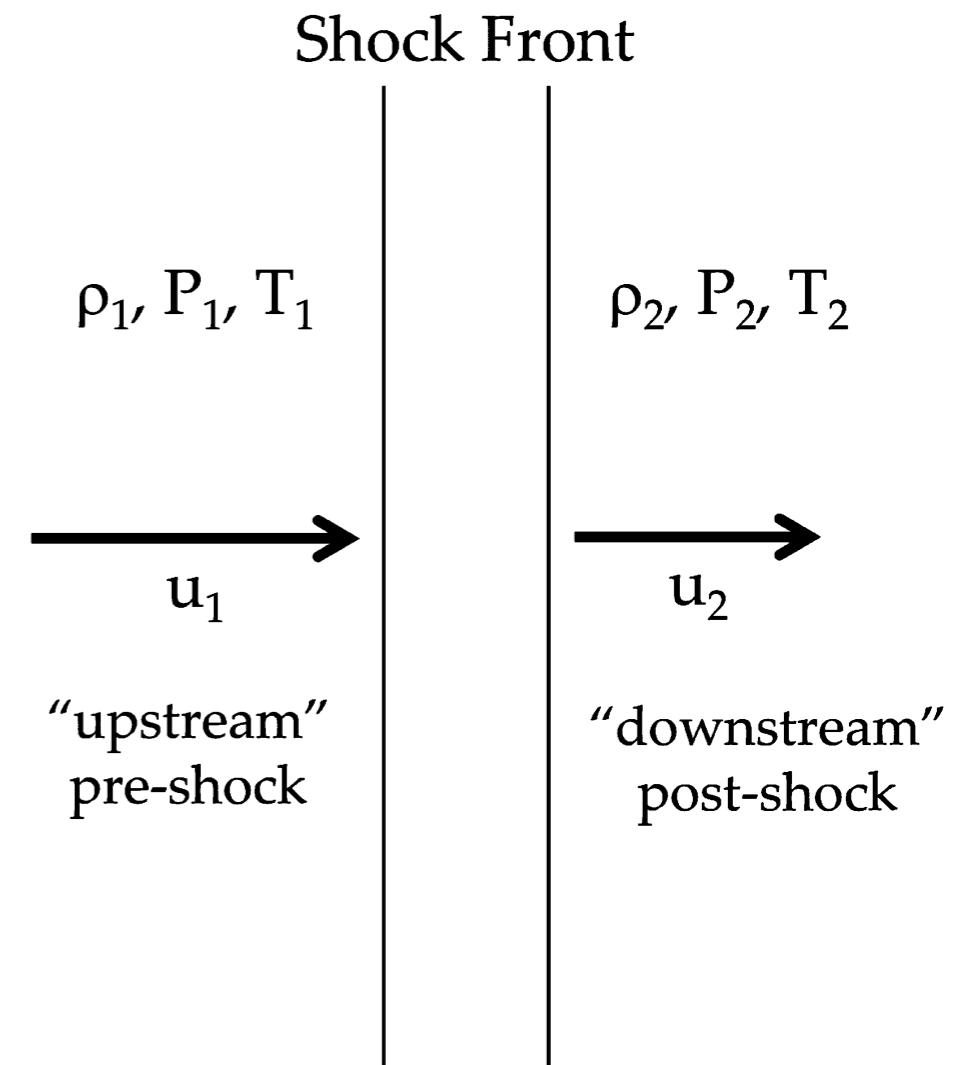
m = mean molecular mass

- If a patch is small compared to the shock front's radius of curvature, then we can treat the shock front as if it has ***plane parallel*** symmetry.

- ***It is convenient to use a frame of reference in which the shock front is stationary.***

- Let us consider a shock propagating with velocity V_s into a gas that is previously at rest. In the frame of reference of the shock, the gas in the pre-shock region is approaching at a velocity of $-V_s$.

- In this frame, the bulk velocity $u_1 = -V_s$ of the pre-shock (upstream) gas toward the shock front. The bulk velocity u_2 of the post-shock (downstream) gas points away from the shock front.



Plane parallel steady-state shock,
in the reference frame of the shock
front.

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- Let's consider a steady state solution.
 - The gas properties immediately before being shocked (“1”) and immediately after being shocked (“2”) are obtained from the conservation laws:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

$$u_1 \left(\rho_1 \frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} P_1 \right) = u_2 \left(\rho_2 \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} P_2 \right)$$

Dividing the third equation with the first equation:

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}$$

In summary,

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}$$

Here, we assume that an adiabatic index is the same on both sides of the shock front.

- From the three equations, we should be able to derive the changes, ρ_2/ρ_1 , u_2/u_1 , and P_2/P_1 across the shock.

It is convenient to use a dimensionless number, the Mach number of the upstream:

$$\mathcal{M}_1 = u_1/c_1, \quad c_1^2 = \frac{\gamma P_1}{\rho_1} \quad \rightarrow \quad P_1 = \frac{u_1^2 \rho_1}{\gamma \mathcal{M}_1^2}$$

(1) To find the equation for densities:

$$\begin{aligned} \rho_1 u_1^2 + P_1 &= \rho_2 u_2^2 + P_2 \\ \rho_1 u_1 = \rho_2 u_2 \text{ and } P_1 = \frac{u_1^2 \rho_1}{\gamma \mathcal{M}_1^2} &\rightarrow \rho_1 u_1^2 + u_1^2 \frac{\rho_1}{\gamma \mathcal{M}_1^2} = \frac{(\rho_1 u_1)^2}{\rho_2} + P_2 \\ &\rightarrow P_2 = \rho_1 u_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right) \end{aligned}$$

Inserting these relations into the energy conservation equation:

$$\begin{aligned} \frac{u_1^2}{2} + \frac{\gamma}{\gamma-1} \frac{P_1}{\rho_1} &= \frac{u_2^2}{2} + \frac{\gamma}{\gamma-1} \frac{P_2}{\rho_2} \\ \rightarrow \frac{u_1^2}{2} + \frac{1}{\gamma-1} \frac{u_1^2}{\mathcal{M}_1^2} &= \frac{1}{2} \left(\frac{\rho_1 u_1}{\rho_2} \right)^2 + \frac{\gamma}{\gamma-1} \frac{\rho_1 u_1^2}{\rho_2} \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right) \\ \rightarrow \frac{1}{2} + \frac{1}{\gamma-1} \frac{1}{\mathcal{M}_1^2} &= \frac{1}{2} \left(\frac{\rho_1}{\rho_2} \right)^2 + \frac{\gamma}{\gamma-1} \frac{\rho_1}{\rho_2} \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right) \end{aligned}$$



$$ax^2 + bx - c = 0$$

where $x = \frac{\rho_1}{\rho_2}$

$$\begin{aligned} a &= \frac{1}{2} - \frac{\gamma}{\gamma-1} \\ b &= \frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_1^2} \\ c &= \frac{1}{2} + \frac{1}{(\gamma-1)\mathcal{M}_1^2} \end{aligned}$$

$$x = \frac{b^2 \pm \sqrt{b^2 + 4ac}}{2a}$$

$$\frac{\rho_1}{\rho_2} = \frac{-\left[\frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_0^2}\right] \pm \frac{\mathcal{M}_1^2 - 1}{\mathcal{M}_1^2(\gamma-1)}}{1 - \frac{2\gamma}{\gamma-1}}$$

→

$$\frac{\rho_1}{\rho_2} = 1 \quad \text{or} \quad \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2}$$

$$\therefore \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}$$

(2) Now, we obtain the equation for pressures:

Divide the following equation

$$P_2 = \rho_1 u_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right)$$

with this

$$P_1 = \frac{u_1^2 \rho_1}{\gamma \mathcal{M}_1^2}$$



$$\frac{P_2}{P_1} = \gamma \mathcal{M}_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right)$$

$$= \gamma \mathcal{M}_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2} \right)$$

$$\therefore \frac{P_2}{P_1} = \frac{2\gamma \mathcal{M}_1^2 - (\gamma-1)}{\gamma+1}$$

(3) Using the ideal gas law:

$$P = \frac{\rho k T}{m} \quad \rightarrow \quad \frac{T_2}{T_1} = \frac{\rho_1}{\rho_2} \frac{P_2}{P_1}$$

Using the equations for densities and pressures:

$$\therefore \frac{T_2}{T_1} = \frac{[(\gamma-1)\mathcal{M}_1^2 + 2][2\gamma \mathcal{M}_1^2 - (\gamma-1)]}{(\gamma+1)^2 \mathcal{M}_1^2}$$

In summary, we obtain the jump conditions:

$$\begin{aligned}\frac{\rho_2}{\rho_1} &= \frac{(\gamma + 1)\mathcal{M}_1^2}{(\gamma - 1)\mathcal{M}_1^2 + 2} = \frac{u_1}{u_2} \\ \frac{P_2}{P_1} &= \frac{2\gamma\mathcal{M}_1^2 - (\gamma - 1)}{\gamma + 1} \\ \frac{T_2}{T_1} &= \frac{[(\gamma - 1)\mathcal{M}_1^2 + 2][2\gamma\mathcal{M}_1^2 - (\gamma - 1)]}{(\gamma + 1)^2\mathcal{M}_1^2}\end{aligned}$$

In the **lab frame**, let V_s = shock velocity, v_1, v_2 = gas velocities in upstream (pre-shock) and downstream (post-shock), respectively ($v_1 = 0$) .

Using $u_1 = -V_s$ and $u_2 = v_2 - V_s$, we have

$$\frac{-V_s}{v_2 - V_s} = \frac{(\gamma + 1)\mathcal{M}_1^2}{(\gamma - 1)\mathcal{M}_1^2 + 2}$$

Downstream velocity in the lab frame:

$$v_2 = \frac{2(\mathcal{M}_1^2 - 1)}{(\gamma + 1)\mathcal{M}_1^2} V_s$$

Note a typo in Equation (16.12) of Kwok's book.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)\mathcal{M}_1^2}{(\gamma - 1)\mathcal{M}_1^2 + 2} = \frac{u_1}{u_2}$$

$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma - 1)}{\gamma + 1}$$

$$\frac{T_2}{T_1} = \frac{[(\gamma - 1)\mathcal{M}_1^2 + 2][2\gamma\mathcal{M}_1^2 - (\gamma - 1)]}{(\gamma + 1)^2\mathcal{M}_1^2}$$

For a strong shock: $\mathcal{M}_1 \gg 1$

$$P_2 \approx \frac{2\gamma\mathcal{M}_1^2}{\gamma + 1} P_1 \xrightarrow{P_1 = c_1^2 \frac{\rho_1}{\gamma}} \frac{2\gamma(u_1/c_1)^2}{\gamma + 1} c_1^2 \frac{\rho_1}{\gamma}$$

$$T_2 \approx \frac{2(\gamma - 1)\gamma}{(\gamma + 1)^2} \mathcal{M}_1^2 T_1 = \frac{2(\gamma - 1)\gamma}{(\gamma + 1)^2} \left(\frac{u_1}{c_1}\right)^2 T_1$$

speed of the downstream in the laboratory frame:

$$\frac{\rho_2}{\rho_1} \simeq \frac{\gamma + 1}{\gamma - 1}$$

$$\frac{u_2}{u_1} \simeq \frac{\gamma - 1}{\gamma + 1}$$

$$P_2 \simeq \frac{2}{\gamma + 1} \rho_1 u_1^2$$

$$T_2 \simeq \frac{2(\gamma - 1)}{(\gamma + 1)^2} \frac{m}{k} u_1^2$$

$$v_2 \simeq \frac{2}{(\gamma + 1)} V_s$$

monatomic gas: $\gamma = 5/3$

$$\frac{\rho_2}{\rho_1} \simeq 4$$

$$\frac{u_2}{u_1} \simeq \frac{1}{4}$$

$$P_2 \simeq \frac{3}{4} \rho_1 u_1^2$$

$$T_2 \simeq \frac{3}{16} \frac{m}{k} u_1^2$$

$$v_2 \simeq \frac{3}{4} V_s$$

For an isothermal shock: $\gamma = 1$

speed of the downstream in the laboratory frame:

$$\frac{\rho_2}{\rho_1} = \mathcal{M}_1^2 = \frac{u_1}{u_2}$$

$$P_2 = \mathcal{M}_1^2 P_1 = \rho_1 u_1^2$$

$$T_2 = T_1$$

$$v_2 = \left(1 - \frac{1}{\mathcal{M}_1^2}\right) V_s$$

$$\begin{aligned} u_1 u_2 &= c_1^2 \\ c_2 &= c_1 \end{aligned}$$

- Consider a strong shock
 - **No matter how strong the shock is, the gas can only be compressed by a factor of at most 4:**

$$\frac{\rho_2}{\rho_1} \approx 4 \quad \text{for } \gamma = 5/3$$

(monatomic gas)

$$P_2 \approx \frac{3}{4} \rho_1 u_1^2$$

$$T_2 \approx \frac{3}{16} \frac{m}{k} u_1^2$$

Note that the mean molecular mass (mass per particle) is

$$m = \frac{1.4m_{\text{H}}}{1.1} = 1.273m_{\text{H}} \quad \text{for neutral gas}$$

$$m = \frac{1.4m_{\text{H}}}{2.3} = 0.609m_{\text{H}} \quad \text{for ionized gas}$$

$n \simeq 2.3n_{\text{H}}$

for ionized gas,
one electron from an ionized hydrogen
two electrons from a doubly-ionized helium.

- In the lab frame, V_s = shock velocity, v_1 , v_2 = gas velocities in upstream and downstream, respectively.

$$u_1 = v_1 - V_s = -V_s \quad (v_1 = 0)$$

$$u_2 = v_2 - V_s$$

- Then, the post-shock velocity is

$$\frac{u_2}{u_1} = \frac{v_s - V_s}{-V_s} = \frac{1}{4} \quad \Rightarrow \quad v_2 = \frac{3}{4} V_s$$

- Hence, **the post-shock moves in the same direction as the shock front with a velocity of 3/4 of the shock velocity.**

- Then, the post-shock pressure, temperature, specific internal energy, and specific kinetic energy are, respectively,

$$P_2 = \frac{3}{4} \rho_1 V_s^2$$

$$T_2 = \frac{3m}{16k} V_s^2$$

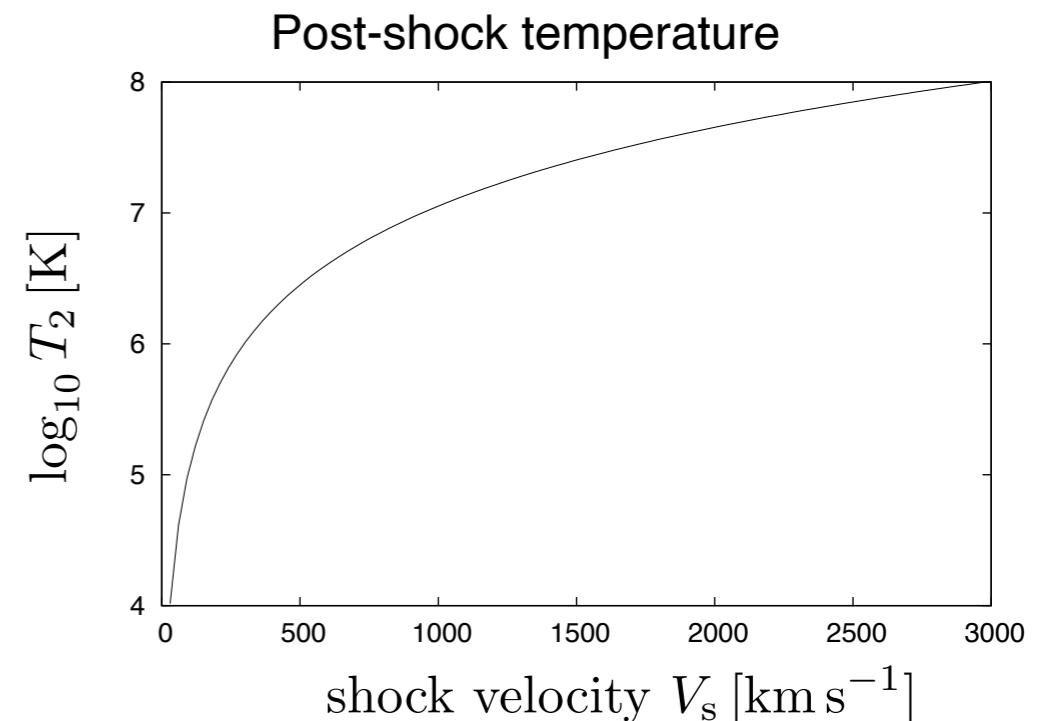
for $\gamma = 5/3$

$$\frac{\mathcal{E}_{\text{int},2}}{\rho_2} = \epsilon_2 = \frac{3 P_2}{2 \rho_2} = \frac{3}{2} \frac{(3/4) \rho_1 V_s^2}{4 \rho_1}$$

$$\frac{\mathcal{E}_{\text{kin},2}}{\rho_2} = \frac{1}{2} v_2^2$$

$$\frac{\mathcal{E}_{\text{int},2}}{\rho_2} = \frac{9}{32} V_s^2$$

$$\frac{\mathcal{E}_{\text{kin},2}}{\rho_2} = \frac{9}{32} V_s^2$$



- A strong shock can produce very high pressures and temperatures. An interstellar shock front with propagation speed $V_s \sim 1000 \text{ km s}^{-1}$ (typical for a supernova shock wave) produces shock heated gas with

$$T_2 \approx 1.38 \times 10^7 \text{ K} \left(\frac{m}{0.609 m_H} \right) \left(\frac{V_s}{1000 \text{ km s}^{-1}} \right)^2$$

or $T_2 \approx 1.38 \times 10^5 \text{ K} \left(\frac{m}{0.609 m_H} \right) \left(\frac{V_s}{100 \text{ km s}^{-1}} \right)^2$

assuming the shocks gas is fully ionized hydrogen.

- In general, shock fronts convert supersonic gas into subsonic gas in the shock's frame of reference. Shocks increase density, pressure, and temperature, and decrease bulk velocity relative to the shock front. *Shocks act as entropy generators.*

Hot Ionized Medium

- Hot Gas Cooling
- Supernova Remnant
- Local Hot Bubble

General Properties of the HIM

- Hot Ionized Medium, coronal gas
 - About half the volume of the ISM in our Galaxy is occupied by the HIM.
 - Temperature $\sim 10^6$ K.
 - Typical ion number density $n \sim 0.004 \text{ cm}^{-3}$
 - It provides only $\sim 0.2\%$ of the mass of the ISM, despite being the largest contributor to its volume.
 - The HIM is hot because it has been heated by shock fronts that result from supernova explosions.
 - ***We live in the “Local Bubble”, which is $\sim 100 \text{ pc}$ in size. The Local Bubble is thought to have been blown by a supernova that went off $\sim 10 \text{ Myr}$ ago.***

Collisional Ionization Equilibrium

- CIE
 - CIE assumes that the plasma is in a steady state, and that collisional ionization, charge exchange, radiative recombination, and dielectronic recombination are the only processes altering the ionization balance.
 - ▶ Note that the reverse process to collisional ionization is a three-body recombination, which is unlikely to occur.
 - The ionization fractions for each element depend only on the gas temperature, with no dependence on the gas density.
- Ionization fraction
 - For hydrogen, the balance equation is : ionization rate = recombination rate

$$n_e n(\text{H}^0) k_{\text{ci}, \text{H}} = n_e n(\text{H}^+) \alpha_{\text{A}, \text{H}} \quad n(\text{H}^0) + n(\text{H}^+) = n(\text{H})$$

- The rate coefficients for collisional ionization and radiative recombination are:

$$k_{\text{ci}, \text{H}} = 5.849 \times 10^{-9} T_4^{1/2} e^{-15.782/T_4} [\text{cm}^3 \text{s}^{-1}]$$

$$\begin{aligned} \alpha_{\text{A}, \text{H}} &= 4.13 \times 10^{-13} T_4^{-0.7131 - 0.0115 \ln T_4} [\text{cm}^3 \text{s}^{-1}] \quad \text{for } 30 \text{ K} < T < 3 \times 10^4 \text{ K} \\ &= 5 \times 10^{-16} T_7^{-1.5} \quad \text{for } T > 10^6 \text{ K} \end{aligned} \quad [\text{from Draine}]$$

$$\alpha_{\text{A}, \text{H}} = 1.269 \times 10^{-13} [\text{cm}^3 \text{s}^{-1}] \frac{x^{1.503}}{(1 + (x/0.522)^{0.47})^{1.923}} \quad \text{where } x = 2 \times 157807 \text{ K}/T \quad [\text{Hui \& Gendin 1997, MNRAS}]$$

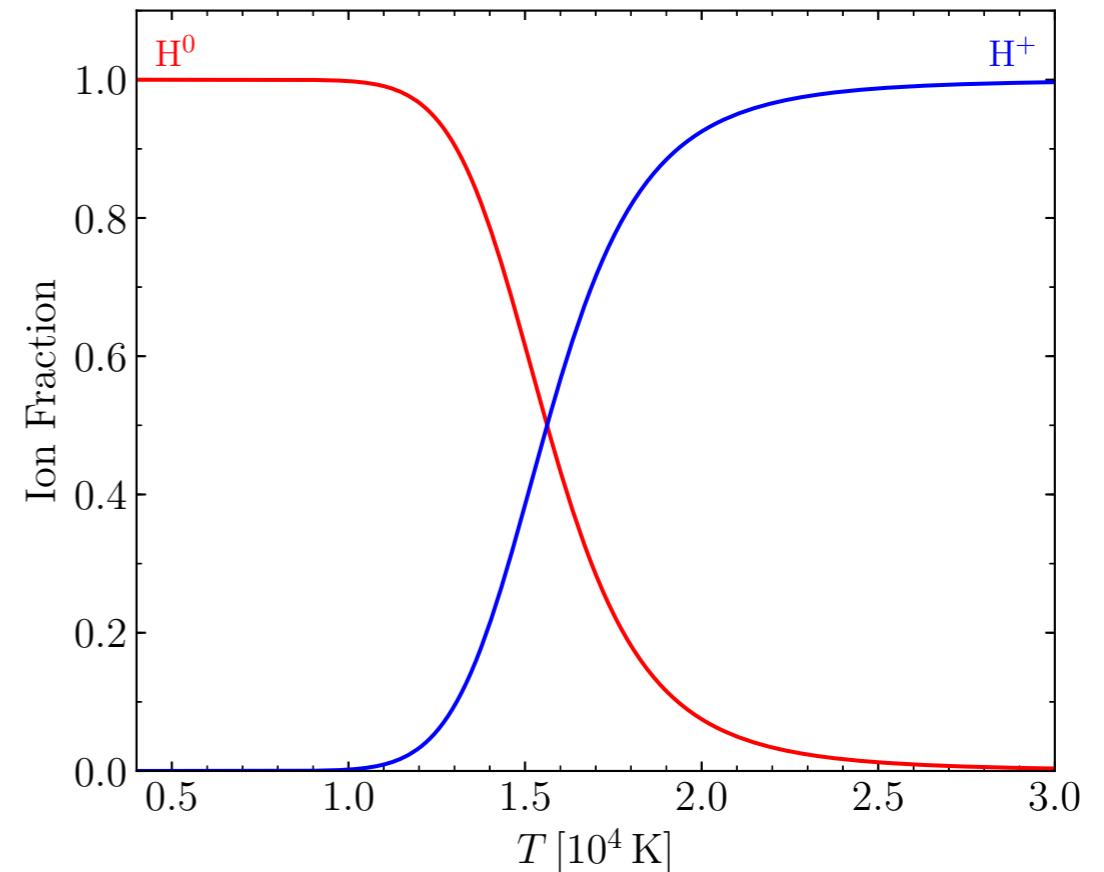
- The ionization fraction is

$$\begin{aligned} x &\equiv \frac{n(\text{H}^+)}{n(\text{H}^0) + n(\text{H}^+)} \\ &= \frac{k_{\text{ci},\text{H}}}{k_{\text{ci},\text{H}} + \alpha_{\text{A},\text{H}}} \end{aligned}$$

- The ion fractions are

$$x \approx 0.002 \text{ at } T = 10^4 \text{ K}$$

$$1 - x \approx 3 \times 10^{-7} \text{ at } T = 10^6 \text{ K}$$



H II regions with $T = 10^4$ K are photoionized by UV photons from hot stars.

Hydrogen gas with $T = 10^6$ K is almost entirely collisionally ionized.

- For Helium, the balance equations are:

$$n(\text{He}^+) \alpha_{10} = n(\text{He}^0) k_{01}$$

$$n(\text{He}^+) k_{12} = n(\text{He}^{2+}) \alpha_{21}$$

$$n(\text{He}) = n(\text{He}^0) + n(\text{He}^+) + n(\text{He}^{2+})$$

Here, ij indicates $X^{i+} \rightarrow X^{j+}$.

- The rate coefficients are

$$k_{01} = 2.39 \times 10^{-11} T^{1/2} e^{-285,335/T}$$

from Cen (1992, ApJS)

$$k_{12} = 5.68 \times 10^{-12} T^{1/2} e^{-631,515/T}$$

$$\alpha_{10} = 1.50 \times 10^{-10} T^{-0.6353} \text{ radiative recombination}$$

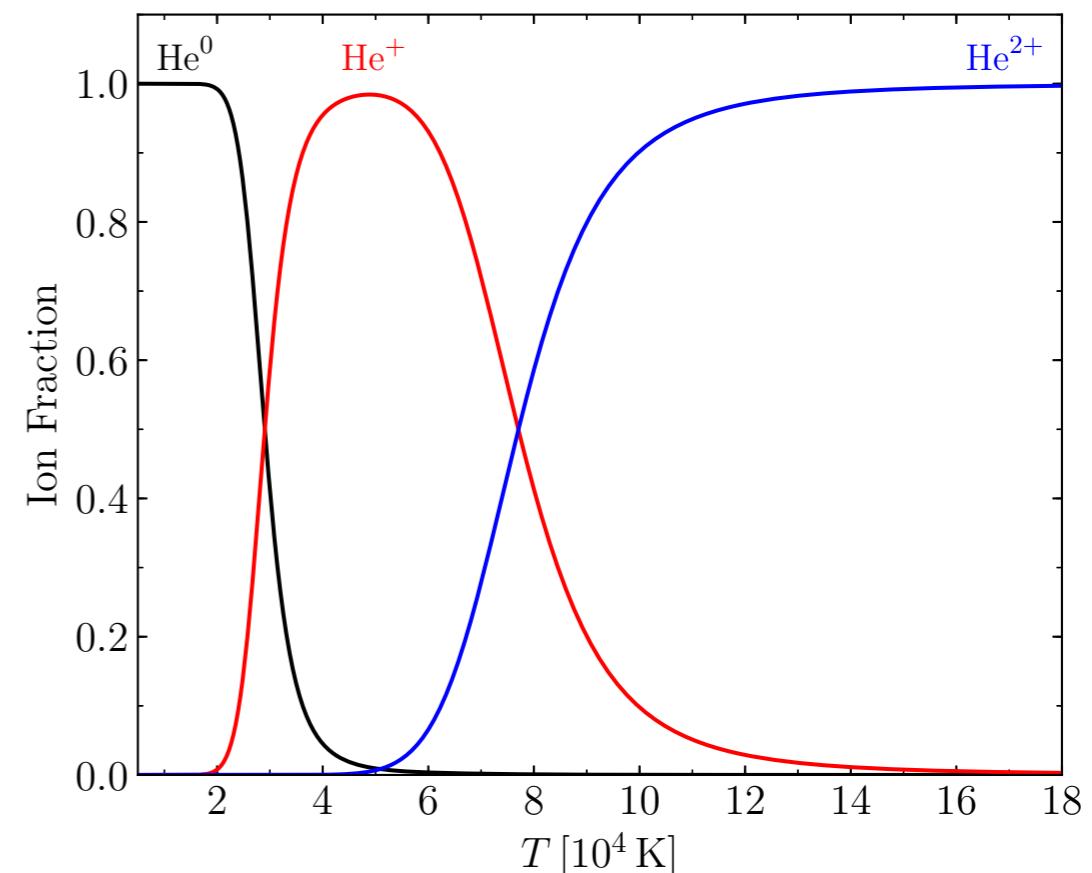
$$+ 1.9 \times 10^{-3} T^{-1.5} e^{-470,000/T} (1 + 0.3e^{-94,000/T}) \text{ dielectronic recombination (but not significant)}$$

$$\alpha_{21} = 3.36 \times 10^{-10} T^{-1/2} T_3^{-0.2} / (1 + T_6^{0.7})$$

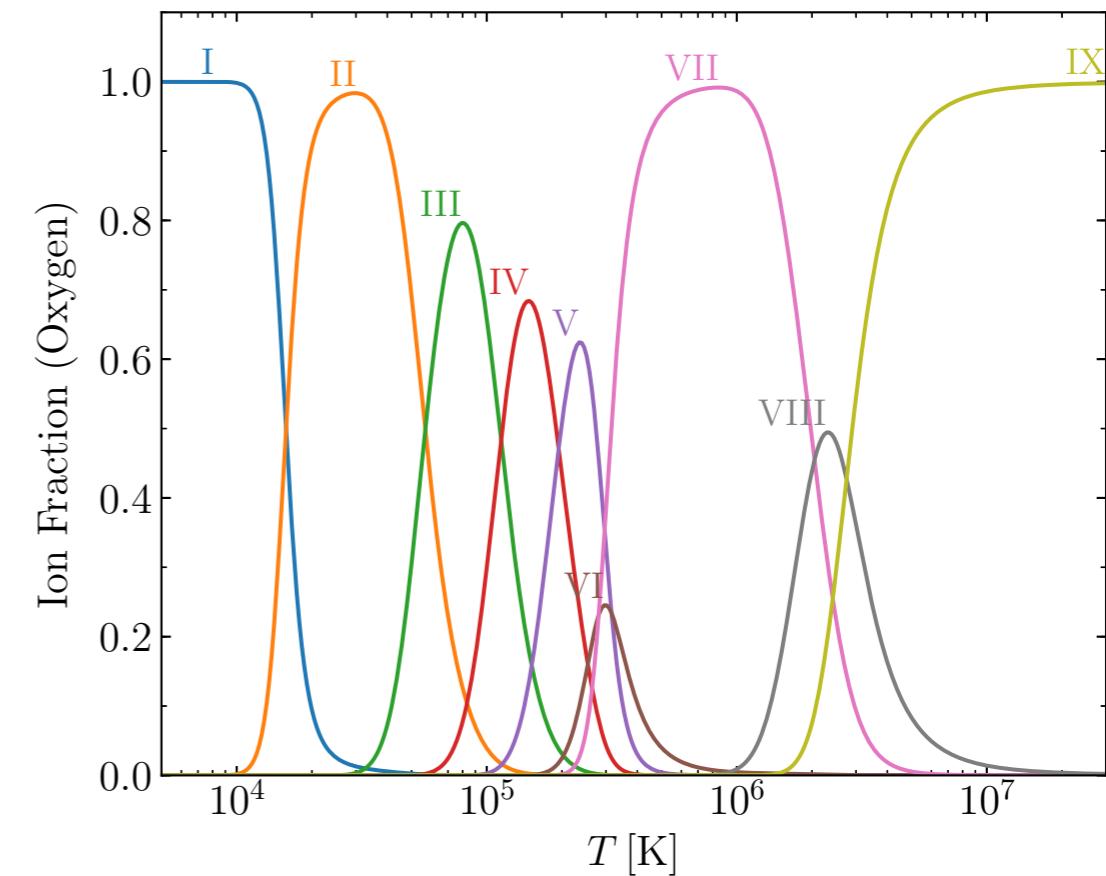
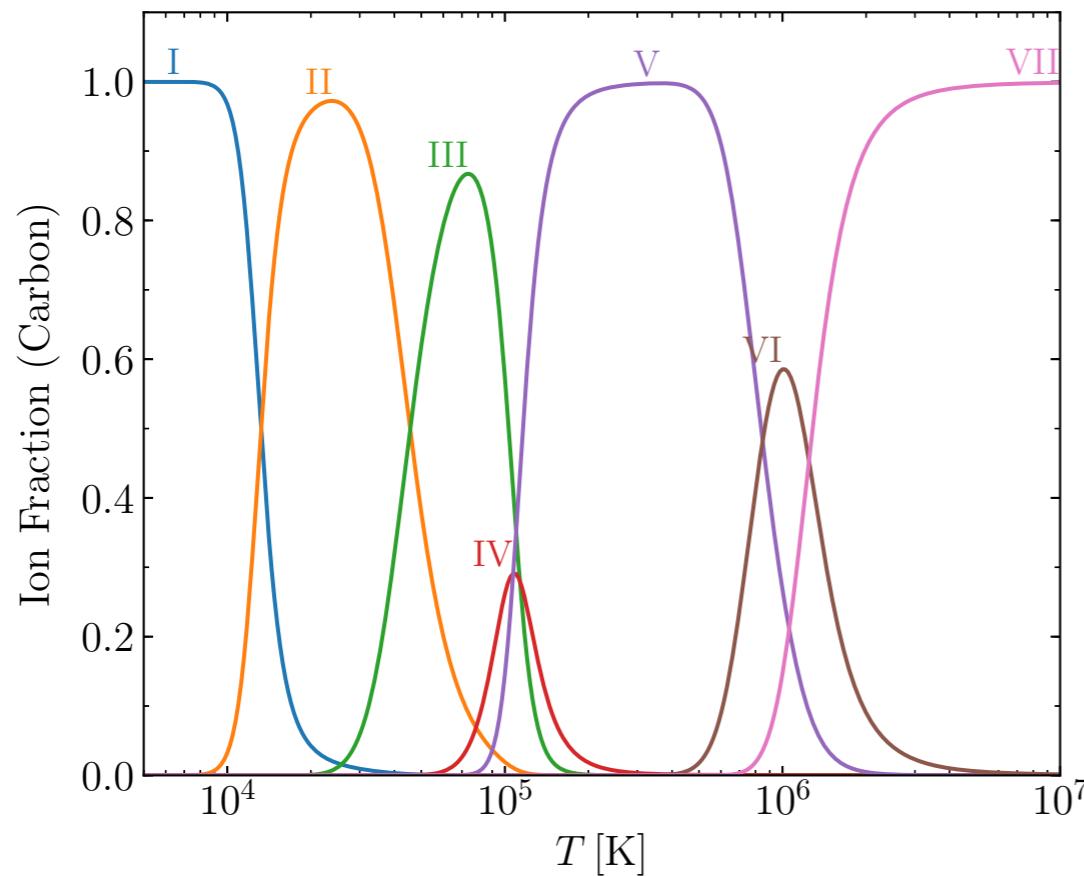
- Using the above rate coefficients, the ionization fractions can be estimated as follows:

$$x \equiv \frac{n(\text{He}^+)}{n(\text{He})} = \frac{1}{1 + \alpha_{10}/k_{01} + k_{12}/\alpha_{21}}$$

$$y \equiv \frac{n(\text{He}^{2+})}{n(\text{He})} = \frac{k_{12}}{\alpha_{21}} x$$



- Heavy Elements
 - ▶ The calculation is usually done numerically, for instance, using CHIANTI
[CHIANTI: https://www.chiantidatabase.org/](https://www.chiantidatabase.org/)
 - ▶ For instance, the ion fractions of Carbon and Oxygen as a function of temperature are:



- At $T \sim 10^6$ K, we expect a mix of C V, C VI, and C VII.
- At $T \sim 4 \times 10^6$ K and higher, almost all the carbon will be in the form of fully ionized C VII.

The figures were calculated using CHIANTI.

Cooling in CIE

- ***Cooling function***

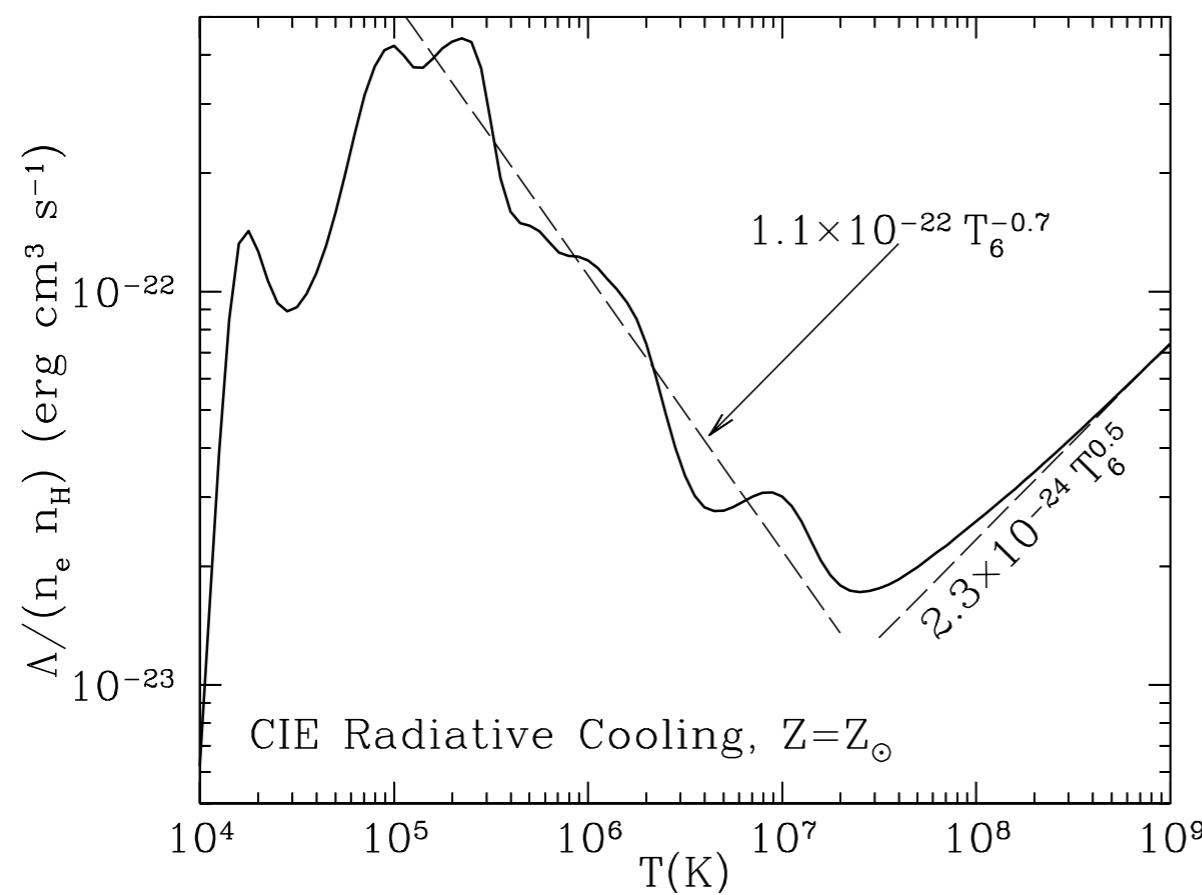
- At temperature $T > 10^4$ K, ionization of hydrogen provides enough free electrons so that collisional excitation of atoms or ions is dominated by electron collisions.
- At low densities, every collisional excitation is followed by a radiative decay, and ***the rate of removal of thermal energy per unit volume*** can be written:

$$\Lambda = n_e n_H f_{\text{cool}}(T)$$

The ***radiative cooling function*** $f_{\text{cool}}(T) \equiv \Lambda / (n_H n_e)$ is a function of ***temperature*** and of the ***elemental abundances*** relative to hydrogen.

- ***At high densities***, radiative cooling can be suppressed by collisional deexcitation, and the cooling function will then depend on density, in addition to T and elemental abundances.
- ***If ionizing radiation is present***, the ionization balance may depart from CIE, and the radiative cooling function will also depend on the spectrum and intensity of the ionizing radiation.

Fig 34.1
Draine



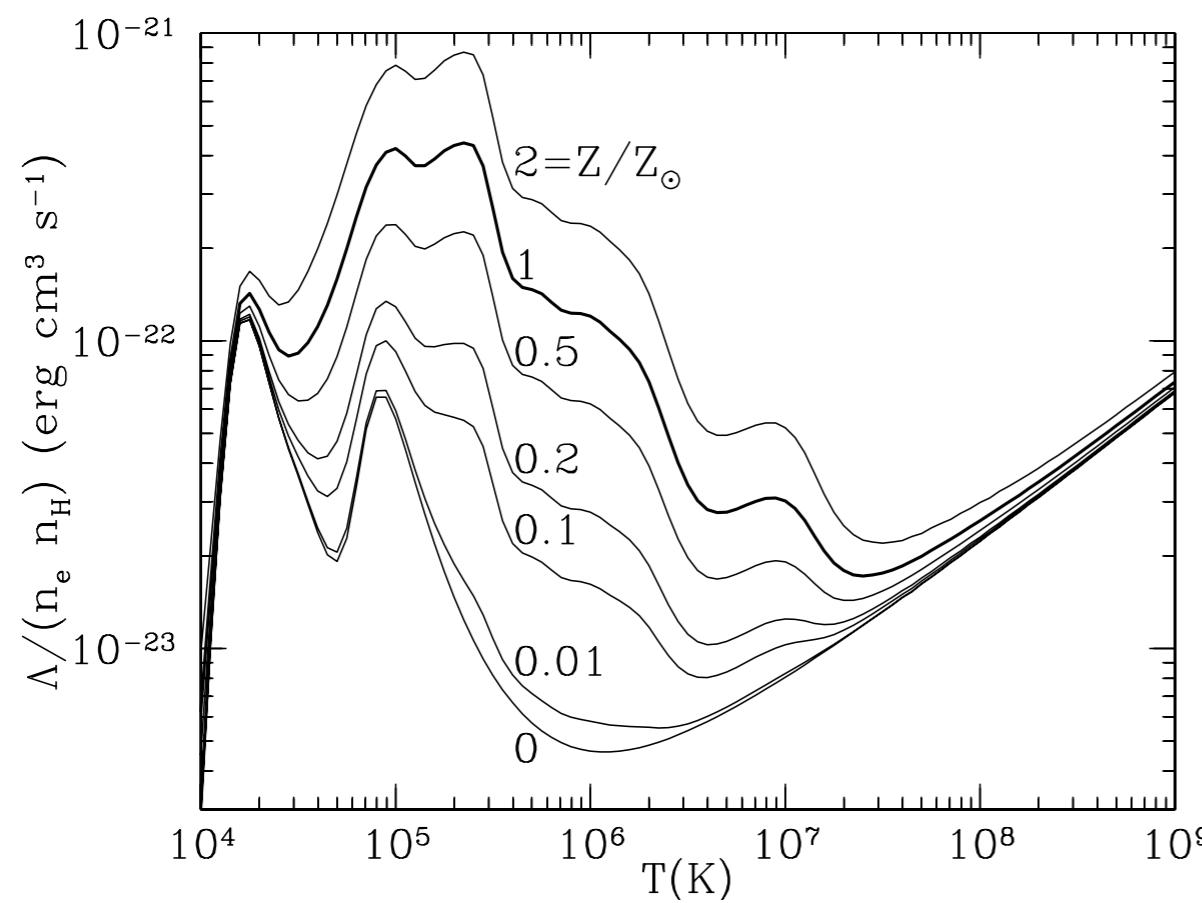
Radiative Cooling Function for solar-abundance

- At $T < 10^7$ K, the cooling is dominated by collisional excitation of bound electrons.
- At high temperatures, the ions are fully stripped of electrons, and bremsstrahlung (free-free) cooling dominates.

$$\Lambda/n_e n_H \approx 1.1 \times 10^{-22} T_6^{-0.7} \text{ [erg cm}^3 \text{ s}^{-1}\text{]}$$

$$(10^5 < T < 10^{7.3} \text{ K})$$

Fig 34.2
Draine



Cooling Function for different abundances

- In most applications, the abundances of elements beyond He can be assumed to be scaled up and down together.

The cooling functions were calculated using CHIANTI.

Fig 34.3
Draine

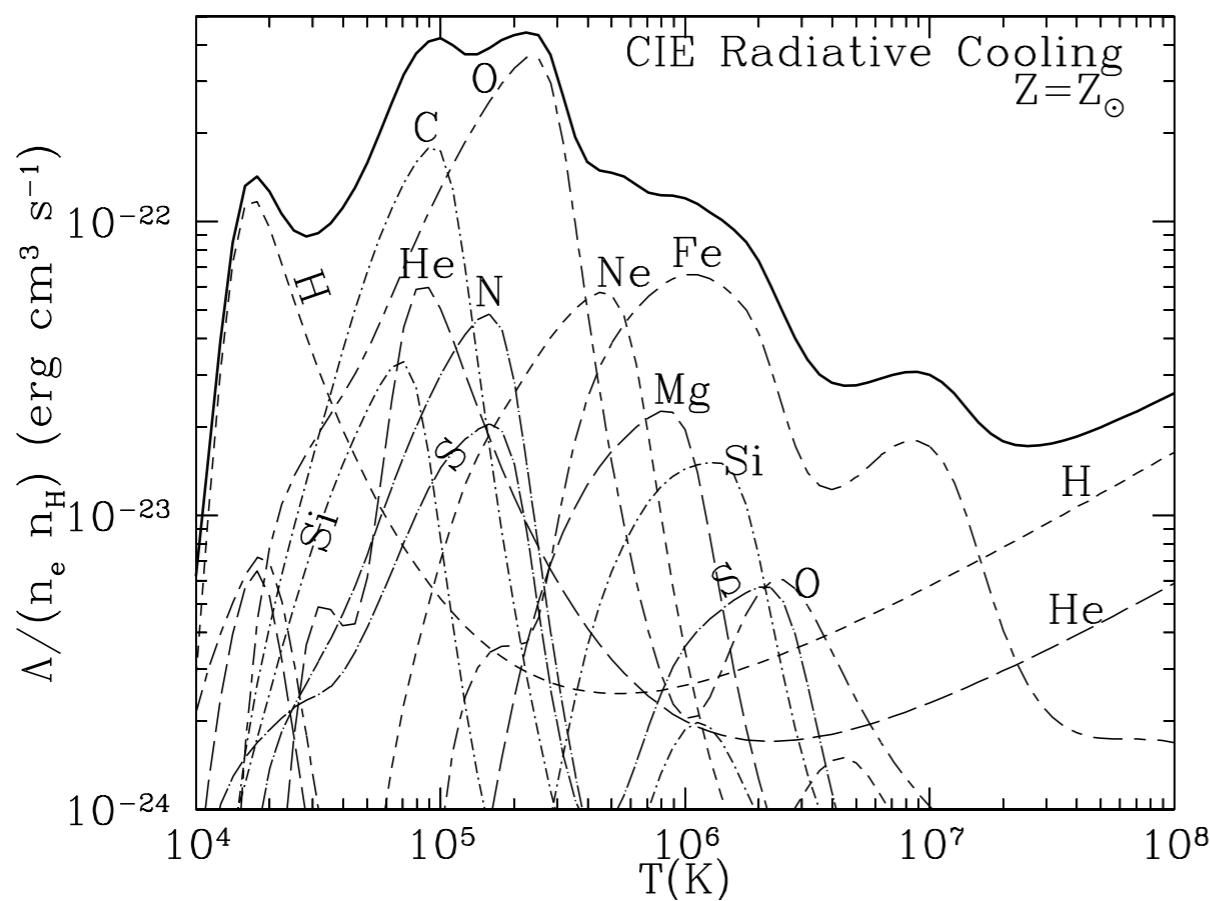
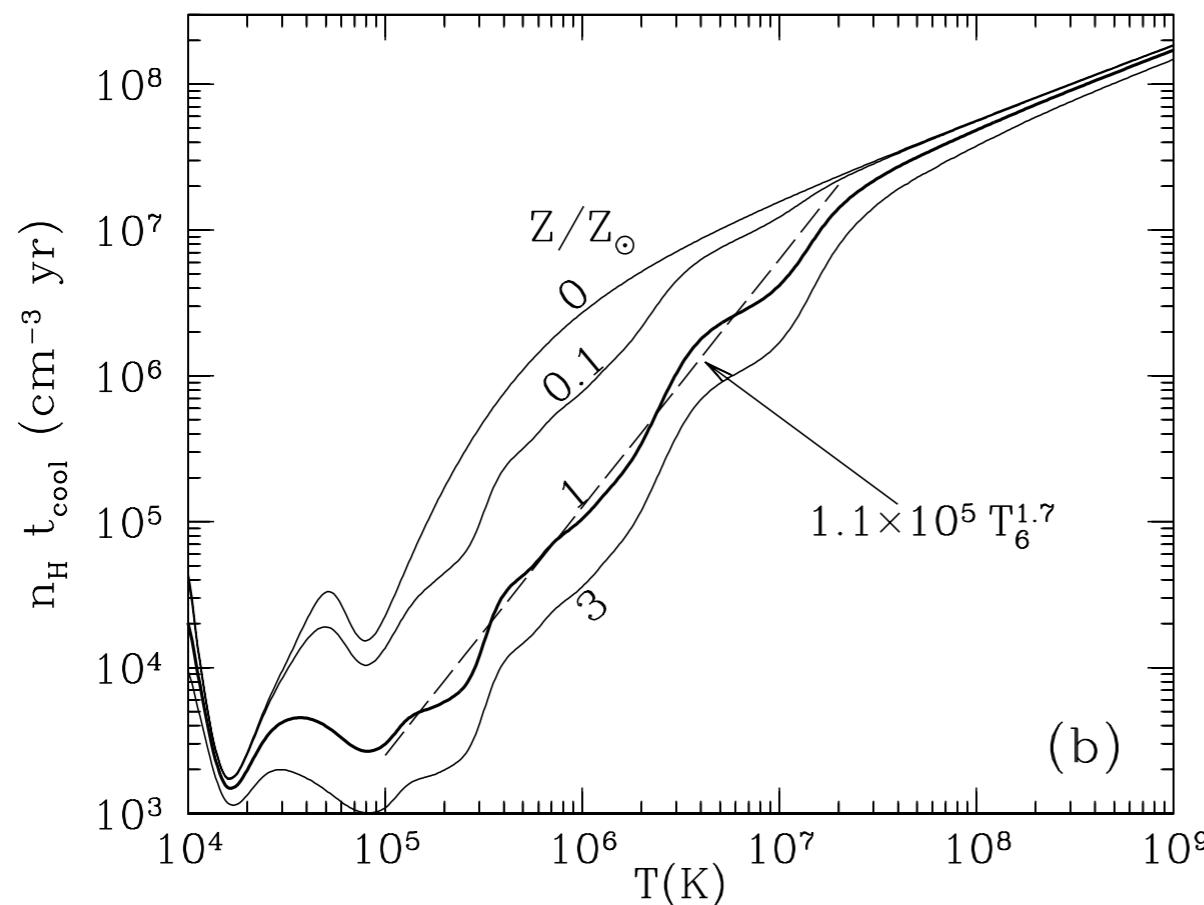


Fig 34.4
Draine



Radiative Cooling Function, with contributions from selected elements. In this calculation, the solar abundance is assumed.

- At $10^4 \text{ K} < T < 3 \times 10^4 \text{ K}$, cooling occurs mainly by Ly α emission from collisionally excited H atoms.
- At $3 \times 10^4 \text{ K} < T < 2 \times 10^7 \text{ K}$, cooling occurs mainly by permitted UV lines from collisionally excited heavy ions.
- For $10^{5.8} \text{ K} < T < 10^{7.2} \text{ K}$, the cooling is dominated by Mg, Si, and Fe - elements that in cold gas are normally depleted by factors of 5 or more.

Cooling time
(using the formula given in next slides)

$$t_{\text{cool}} \approx 1.1 \times 10^5 T_6^{1.7} (n_{\text{H}}/\text{cm}^{-3})^{-1} [\text{yr}]$$

$$(10^5 \lesssim T \lesssim 10^{7.3} \text{ K})$$

for isochoric cooling (constant density)

Cooling Time Scale [isobaric / isochoric]

- Cooling Time scale for two important cases:

- The **first law of thermodynamics** states that:

Heat added in a system:

$$dQ = dU + PdV$$

- Using a heating and cooling rate per volume Γ and Λ , the change in heat is

$$dQ = (\Gamma - \Lambda)Vdt$$

- The change in the internal energy is

$$dU = (\Gamma - \Lambda)Vdt - PdV$$

- When there is no external heating ($\Gamma = 0$), the equation for an ideal gas with a degree of freedom f becomes:

$$\begin{aligned} U &= \frac{f}{2}NkT \\ PV &= NkT \end{aligned} \quad \longrightarrow \quad \boxed{d\left(\frac{f}{2}NkT\right) = -\Lambda Vdt - PdV}$$

- Consider the case of constant pressure or constant volume:

$$PdV = d(PV) - VdP = d(NkT) \quad \text{for } \textbf{isobaric cooling (constant pressure)}$$

$$PdV = 0 \quad \text{for } \textbf{isochoric cooling (constant density or volume)}$$

Therefore,

$$\frac{d}{dt} \left(\frac{f+2}{2} NkT \right) = -\Lambda V \quad \text{for } \textbf{isobaric cooling (constant pressure)}$$

$$\frac{d}{dt} \left(\frac{f}{2} NkT \right) = -\Lambda V \quad \text{for } \textbf{isochoric cooling (constant density or volume)}$$

The cooling time scale are then:

$$t_{\text{cool}} \equiv \frac{T}{|dT/dt|} \Rightarrow t_{\text{cool}} = \frac{f+2}{2} \frac{nkT}{\Lambda} \quad \text{for isobaric cooling}$$

$$n \equiv N/V \quad = \frac{f}{2} \frac{nkT}{\Lambda} \quad \text{for isochoric cooling}$$

Here, the number density includes all particles (molecules, atoms, ions, electrons)

Time Scales in the HIM

- **Cooling time scale:**

- In the HIM with temperatures $T \sim 10^6 - 10^7$ K, the cooling time scale is:

$$t_{\text{cool}} = \frac{5}{2} \frac{n k T}{\Lambda}$$

$$\begin{aligned} n_e &\approx 1.2 n_H \\ n &\approx 2.3 n_H \end{aligned}$$

For fully ionized gas,
one electron from an ionized hydrogen
two electrons from a doubly-ionized helium.

- The cooling time at $T \sim 10^6$ K is

$$\begin{aligned} t_{\text{cool}} &\approx 48 \text{ [Myr]} T_6^{1.7} \left(\frac{n_H}{0.004 \text{ cm}^{-3}} \right)^{-1} \\ &\approx 0.19 \text{ [Myr]} T_6^{1.7} \left(\frac{n_H}{1 \text{ cm}^{-3}} \right)^{-1} \end{aligned}$$

$$t_{\text{cool}} = \frac{5}{2} \frac{2.3}{1.2} \frac{k T}{\Lambda / (n_e n_H)} \frac{1}{n_H}$$

← $\Lambda / n_e n_H \approx 1.1 \times 10^{-22} T_6^{-0.7} \text{ [erg cm}^3 \text{ s}^{-1}\text{]}$

The HIM frequently doesn't have time to cool thoroughly before another supernova shock wave comes through to heat it again.

- At $T \sim 10^7$ K, the cooling time is

$$\begin{aligned} t_{\text{cool}} &\approx 7.2 \text{ [Gyr]} T_7^{1/2} \left(\frac{n_H}{0.004 \text{ cm}^{-3}} \right)^{-1} \\ &\approx 29 \text{ [Myr]} T_7^{1/2} \left(\frac{n_H}{1 \text{ cm}^{-3}} \right)^{-1} \end{aligned}$$

← $\Lambda / n_e n_H \approx 2.3 \times 10^{-24} T_6^{0.5} \text{ [erg cm}^3 \text{ s}^{-1}\text{]}$

Given the low density of the HIM, the cooling time of gas is comparable to the age of our galaxy (~13 Gyr, only after 0.8 Gyr after the Big Bang; Xiang & Rix, 2020, Nature, 603, 599).

-
- ***Recombination and Ionization Time scale:***
 - If collisional ionization could somehow be turned off, the recombination time scale is
$$t_{\text{rec}} = \frac{1}{n_e \alpha_{A,H}} \approx 0.6 \text{ [Gyr]} \left(\frac{n_H}{0.004 \text{ cm}^{-3}} \right)^{-1} \quad [T \approx 10^6 \text{ K}] \quad \alpha_{A,H} \approx 1.5 \times 10^{-14} \text{ [cm}^3 \text{ s}^{-1}\text{]}$$
 - If collisional ionization could suddenly switch on, the collisional ionization time scale is
$$t_{\text{ci}} = \frac{1}{n_e k_{\text{ci}}} \approx 160 \text{ [yr]} \left(\frac{n_H}{0.004 \text{ cm}^{-3}} \right)^{-1} \quad [T \approx 10^6 \text{ K}] \quad k_{\text{ci},H} \approx 5.0 \times 10^{-8} \text{ [cm}^3 \text{ s}^{-1}\text{]}$$
 - These times scales indicates that
 - ▶ If cold neutral hydrogen gas is shock-heated to $\sim 10^6$ K in a time $t_{\text{heat}} \ll t_{\text{ci}}$, it will take a time $t \sim t_{\text{ci}}$ for hydrogen to become ionized. ***During this time interval, the hydrogen will be out of collisional ionization equilibrium (under-ionized than in CIE).***
 - ▶ If highly ionized gas at $\sim 10^6$ K is cooled on a timescale $t_{\text{cool}} \ll t_{\text{rec}}$, and the heating source is turned off, it will take a time $t \sim t_{\text{rec}}$ for the hydrogen to recombine. ***During the intervening time, the gas will be out of CIE (over-ionized than in CIE).*** This is sometimes called “***delayed recombination***”.

- If a gradually cooling gas of the HIM to be remained in CIE, we require $t_{\text{rec}} < t_{\text{cool}}$

Assuming the recombination rate coefficient at high temperatures

$$\alpha_{A,H} \approx 5 \times 10^{-16} \text{ cm}^3 \text{ s}^{-1} (T/10^7 \text{ K})^{-1.5}$$

- At $T \sim 10^6 \text{ K}$,

$$t_{\text{rec}} = \frac{1}{n_e \alpha_{A,H}} \approx 0.5 \text{ [Gyr]} (T/10^6 \text{ K})^{1.5} (n_{\text{H}}/0.004 \text{ cm}^{-3})^{-1}$$

$$t_{\text{cool}} \approx 48 \text{ [Myr]} T_6^{1.7} (n_{\text{H}}/0.004 \text{ cm}^{-3})^{-1}$$

$$\Rightarrow \frac{t_{\text{rec}}}{t_{\text{cool}}} \approx 10 (T/10^6 \text{ K})^{-0.2}$$

- At $T \sim 10^7 \text{ K}$,

$$t_{\text{rec}} = \frac{1}{n_e \alpha_{A,H}} \approx 16 \text{ [Gyr]} (T/10^7 \text{ K})^{1.5} (n_{\text{H}}/0.004 \text{ cm}^{-3})^{-1}$$

$$t_{\text{cool}} \approx 7.2 \text{ [Gyr]} T_7^{0.5} (n_{\text{H}}/0.004 \text{ cm}^{-3})^{-1}$$

$$\Rightarrow \frac{t_{\text{rec}}}{t_{\text{cool}}} \approx 2.2 (T/10^7 \text{ K})$$

$t_{\text{rec}}(T \approx 10^6 \text{ K}) \ll t_{\text{rec}}(T \approx 10^7 \text{ K}), \quad t_{\text{cool}}(T \approx 10^6 \text{ K}) \ll t_{\text{cool}}(T \approx 10^7 \text{ K})$

Therefore, ***in the extremely hot regions, the hotter the gas is, the further away it is from CIE.***

Cooling in Shocked Gas

- The hot shocked gas is out of equilibrium, and will start to cool. Thus, the shock will be followed by a radiative zone in which the shock heated gas cools down by radiating away photons.

- At high temperatures $T > 2 \times 10^7 \text{ K}$

- The cooling is dominated by bremsstrahlung (free-free radiation), for which the *specific* cooling rate (per mass) is

$$\mathcal{L} = 2.7 [\text{erg g}^{-1} \text{s}^{-1}] \left(\frac{T}{10^7 \text{ K}} \right)^{1/2} \left(\frac{n_{\text{H}}}{1 \text{ cm}^{-3}} \right)$$

assuming a gas of fully ionized hydrogen. The *specific* internal energy of ionized hydrogen (per mass) is

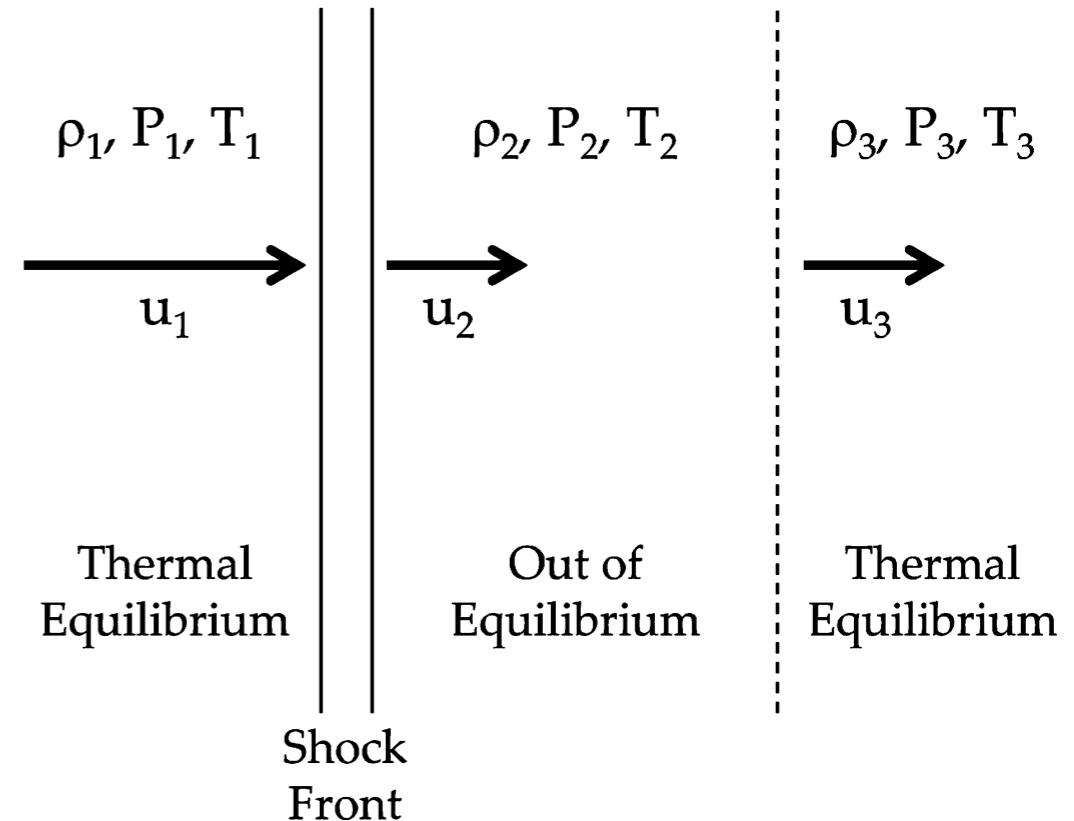
$$\epsilon = \frac{3}{2} \frac{(2n_{\text{H}})kT}{n_{\text{H}}m_{\text{H}}} \approx 2.5 \times 10^{15} [\text{erg g}^{-1}] \left(\frac{T}{10^7 \text{ K}} \right)$$

- Then, the bremsstrahlung cooling time is

$$t_{\text{cool}} = \frac{\epsilon}{\mathcal{L}} \approx 29 [\text{Myr}] \left(\frac{T}{10^7 \text{ K}} \right)^{1/2} \left(\frac{n_{\text{H}}}{1 \text{ cm}^{-3}} \right)^{-1}$$

$$\approx 34 [\text{Myr}] \left(\frac{V_s}{1000 \text{ km s}^{-1}} \right) \left(\frac{n_{\text{H}}}{1 \text{ cm}^{-3}} \right)^{-1}$$

$\curvearrowright T \simeq \frac{3}{16} \frac{m}{k} V_s^2$



The structure of a plane parallel radiative shock
[Figure 5.3 Ryden]

-
- ▶ During this time, the gas will move a distance, relative to the shock front:

$$R_{\text{cool}} \approx u_2 t_{\text{cool}} \approx \frac{u_1}{4} t_{\text{cool}}$$

$$\approx 8.7 \text{ [kpc]} \left(\frac{V_s}{1000 \text{ km s}^{-1}} \right)^2 \left(\frac{n_{\text{H}}}{1 \text{ cm}^{-3}} \right)^{-1}$$

This implies that ***the approximate thickness of the radiative zone for a strong shock is a long distance compared to the scale height of the ISM in our galaxy.*** Thus, the hot gas produced by high-speed shocks doesn't have time to cool before the shock runs out of gas to shock. << **No cooling of the shock gas in our galaxy>>**

- At lower temperature ($10^5 \text{ K} < T < 2 \times 10^7 \text{ K}$), corresponding to slower shock speeds ($80 \text{ km s}^{-1} < u_1 = V_s < 1200 \text{ km s}^{-1}$)
- ▶ The collisionally excited lines do most of the cooling. A useful approximation for the cooling rate gives

$$t_{\text{cool}} \approx 6600 \text{ [yr]} \left(\frac{V_s}{100 \text{ km s}^{-1}} \right)^{3.4} \left(\frac{n_{\text{H}}}{1 \text{ cm}^{-3}} \right)^{-1}$$

- ▶ This yields a thickness for the radiative zone.

$$R_{\text{cool}} \approx \frac{V_s}{4} t_{\text{cool}} = 0.17 \text{ [pc]} \left(\frac{V_s}{100 \text{ km s}^{-1}} \right)^{4.4} \left(\frac{n_{\text{H}}}{1 \text{ cm}^{-3}} \right)^{-1}$$

- ▶ These shorter time scales and length scales mean that ***radiative cooling is more effective at changing the structure of slower shocks.***

Effects of Supernovae on the ISM

- Consider the simplest case of a spherically symmetric explosion of a star in a uniform density and temperature.

- Free-Expansion Phase**

- Typical supernovae (SNe) explosion ejects a kinetic energy of

$$E_{51} \equiv E_0 / (10^{51} \text{ erg}) \approx 1$$

- The ejecta mass ranges from $M_{\text{ej}} \sim 1.4 M_{\odot}$ (Type Ia, white dwarf) to $M_{\text{ej}} \sim 10 - 20 M_{\odot}$ (Type II, core collapse of massive stars).
- The ejecta will have a rms velocity of

$$v_{\text{ej}} = \left(\frac{2E_0}{M_{\text{ej}}} \right)^{1/2} = 1.00 \times 10^4 \text{ km s}^{-1} E_{51}^{1/2} \left(\frac{M_{\text{ej}}}{M_{\odot}} \right)^{-1/2}$$

$M_{\text{ej}} \sim 10 M_{\odot}$ for core collapse supernovae (Type II, Type Ib, and Type Ic), powered by the gravitational collapse of the dense core of an evolved massive star.

$M_{\text{ej}} \sim 1 M_{\odot}$ for the thermonuclear explosion of a white dwarf (Type Ia)

An explosion produces an energy $\sim 10^{53}$ erg in neutrinos and $\sim 10^{49}$ erg in photons. These energies are not effectively deposited in the ISM.

$$\mathcal{M}_1 = \frac{v_{\text{ej}}}{c_1} \approx \frac{1 \times 10^4 \text{ km/s}}{1 \text{ km/s}} \gg 1$$

- ▶ This velocity is far greater than the sound speed in the surrounding material, and the expanding ejecta will drive a fast shock into the circumstellar medium.
- All of the matter interior to this shock surface is referred to as the **supernova remnant** (SNR).
- **The density of the ejecta far exceeds the density of the circumstellar medium**, and the ejecta continue to **expand ballistically at nearly constant velocity**. — this is referred to as the “free expansion phase.” The ejecta are not slowed down significantly until they have swept up a mass of interstellar gas comparable to the mass of the ejecta.

- At the early times of the free-expanding phase, there is only one shock, which propagating outward into the ambient medium.
- *As the ejecta expands, its density drops and less hot, thanks to adiabatic cooling. Then, the pressure ($P_2 \approx \mathcal{M}_1^2 P_1$) of the shocked circumstellar/interstellar medium soon exceeds the thermal pressure in the ejecta.*

$$\rho_{\text{ej}} = \frac{M_{\text{ej}}}{(4\pi/3)R^3} = \frac{M_{\text{ej}}}{(4\pi/3)(v_{\text{ej}}t)^3} \propto t^{-3}$$

- ▶ As the pressure in the ejecta drops, a reverse shock is driven into the ejecta. The SNR now contains two shocks: the ***original outward-propagating shock*** (the blastwave) and the ***reverse shock propagating inward***, slowing and shock-heating the ejecta (which had previously been cooled adiabatic expansion).
- ***End of the free-expanding phase:*** The reverse shock becomes important when the expanding ejecta material has swept up a mass of circumstellar or interstellar matter comparable to the ejecta mass.
 - ▶ The radius of the blastwave and the time when this occurs are:

$$R_1 = \left(\frac{M_{\text{ej}}}{(4\pi/3)\rho_0} \right)^{1/3} = 5.88 \times 10^{18} \text{ cm} \left(\frac{M_{\text{ej}}}{M_{\odot}} \right)^{1/3} n_0^{-1/3}$$

$$t_1 \approx \frac{R_1}{v_{\text{ej}}} = 186 \text{ yr} \left(\frac{M_{\text{ej}}}{M_{\odot}} \right)^{5/6} E_{51}^{-1/2} n_0^{-1/3}$$

$$R_1 = 0.525 \text{ pc} (M_{\text{ej}}/M_{\odot})^{1/3} n_0^{-1/3}$$

$\rho_0 \simeq 1.4m_p n_0$ → density of the ambient (neutral) medium

n_0 → number density of hydrogen in the ambient medium

- ▶ The free-expansion phase applies only for $t \lesssim t_1$

-
- ***Sedov-Taylor Phase (Energy-Conserving Phase, Adiabatic Phase)***
 - For $t \gtrsim t_1$, the reverse shock has reached the center of the remnant, all of the ejecta are now very hot, and the free-expansion phase is over.
 - ▶ The pressure in the SNR is far higher than the pressure in the surrounding medium.
 - ▶ The hot gas may have been emitting radiation, but if the density is low, the radiative losses at early times are negligible.
 - ▶ We can idealize the explosion as a large amount of energy released instantaneously at a point (blast wave), in the form of kinetic energy plus radiation.
 - The mass of shocked gas is much greater than the ejecta mass. The pressure P_2 behind the shock is much greater than the pressure P_1 of the surrounding gas. Hence, the properties of the shock front no longer depend on the ejecta mass and the surrounding gas pressure. The shock expands as if it were traveling into a pressureless medium.
 - The SNR now enters a phase where
 - ▶ We can neglect (1) the mass of the ejecta ($M_{ej} \ll (4\pi/3)R^3\rho_0$), (2) radiative losses, and (3) the pressure in the ambient medium.
 - ▶ ***The shock front properties depend only on the energy of explosion E and the density ρ_0 of the ambient medium.***
 - ▶ This phase can be approximated by idealizing the problem as a “point explosion” injecting only energy E_0 (zero ejecta mass) into a uniform-density zero-temperature medium of density ρ_0 , as in nuclear explosion.

- **Self-similar Solution:** In this phase, only the scale of the pressure and the length scale evolve with time, but the shape of the pressure and density as a function of position remains unaltered. These motions are called self-similar.
- The evolution during this phase is determined only by ***the energy of explosion E_0 , the density of interstellar gas ρ_0 , and the elapsed time from the explosion t .***

We do simple dimensional analysis to find out the form of the time evolution of the remnant. Let the explosion occur at $t = 0$ and the radius of the shock front be R_s . Suppose that

$$\begin{aligned} E &= [\text{mass}] \times [\text{length}]^2 / [\text{time}]^2 \\ \rho &= [\text{mass}] / [\text{length}]^3 \end{aligned}$$

$$R_s = AE^\alpha \rho_0^\beta t^\eta \quad (A = \text{a dimensionless constant})$$

- ▶ By equating the powers of mass, length, and time, we obtain

$$\text{mass : } 0 = \alpha + \beta$$

$$\text{length : } 1 = 2\alpha - 3\beta \quad \longrightarrow \quad \alpha = 1/5, \beta = -1/5, \eta = 2/5$$

$$\text{time : } 0 = -2\alpha + \eta$$

- ▶ Then, the solution for the shock-front radius is given by

$$R_s = A \left(\frac{Et^2}{\rho_0} \right)^{1/5} \quad A = 1.15167 \text{ for } \gamma = 5/3$$

for a monatomic gas.
This value is obtained from the exact solution.

-
- ▶ The solution gives ***the radius and velocity of the shock front***, and ***temperature of the post-shock gas***:

$$R_s = 1.54 \times 10^{19} [\text{cm}] E_{51}^{1/5} n_0^{-1/5} t_3^{2/5}$$

$$V_s = 1950 [\text{km s}^{-1}] E_{51}^{1/5} n_0^{-1/5} t_3^{-3/5}$$

$$T_s = 5.25 \times 10^7 [\text{K}] E_{51}^{2/5} n_0^{-2/5} t_3^{-6/5}$$

$$1.54 \times 10^{19} \text{ cm} = 5 \text{ pc}$$

$$t_3 \equiv t/10^3 \text{ yr}$$

$$V_s \equiv \frac{dR_s}{dt} = \frac{2}{5} A E^{1/5} \rho_0^{-1/5} t_3^{-3/5}$$

$$T_s = \frac{3}{16} m V_s^2 / k \quad \left(m \simeq \frac{1.4}{2.3} m_p \right) \quad \rho_0 = m n_0$$

m = mass per particle in the fully ionized, post-shock region.

- ▶ We have been assuming that the internal structure of the remnant is given by a similarity solution: by this we mean that the density, velocity, and pressure can be written

$$\rho(r) = \rho_0 f(x)$$

$$v(r) = \frac{R_s}{t} g(x)$$

$$P(r) = \frac{\rho_0 R_s^2}{t^2} h(x)$$

$$x \equiv \frac{r}{R_s}$$

$f(x), g(x), h(x)$ are dimensionless functions.

Inserting these into the fluid equations, with the Rankine-Hugoniot relations for boundary conditions, Taylor (1950) and Sedov (1959) found the solution for the dimensionless functions [$f(x)$, $g(x)$, $h(x)$, and A] independently, in connection with the development of nuclear weapons.

- ***Alternative approach:***

- For the strong shock, the specific internal (thermal) and the kinetic energies just behind the strong shock are given by, respectively:

$$\epsilon_{\text{int}} = \epsilon_{\text{kin}} = \frac{9}{32} V_s^2$$

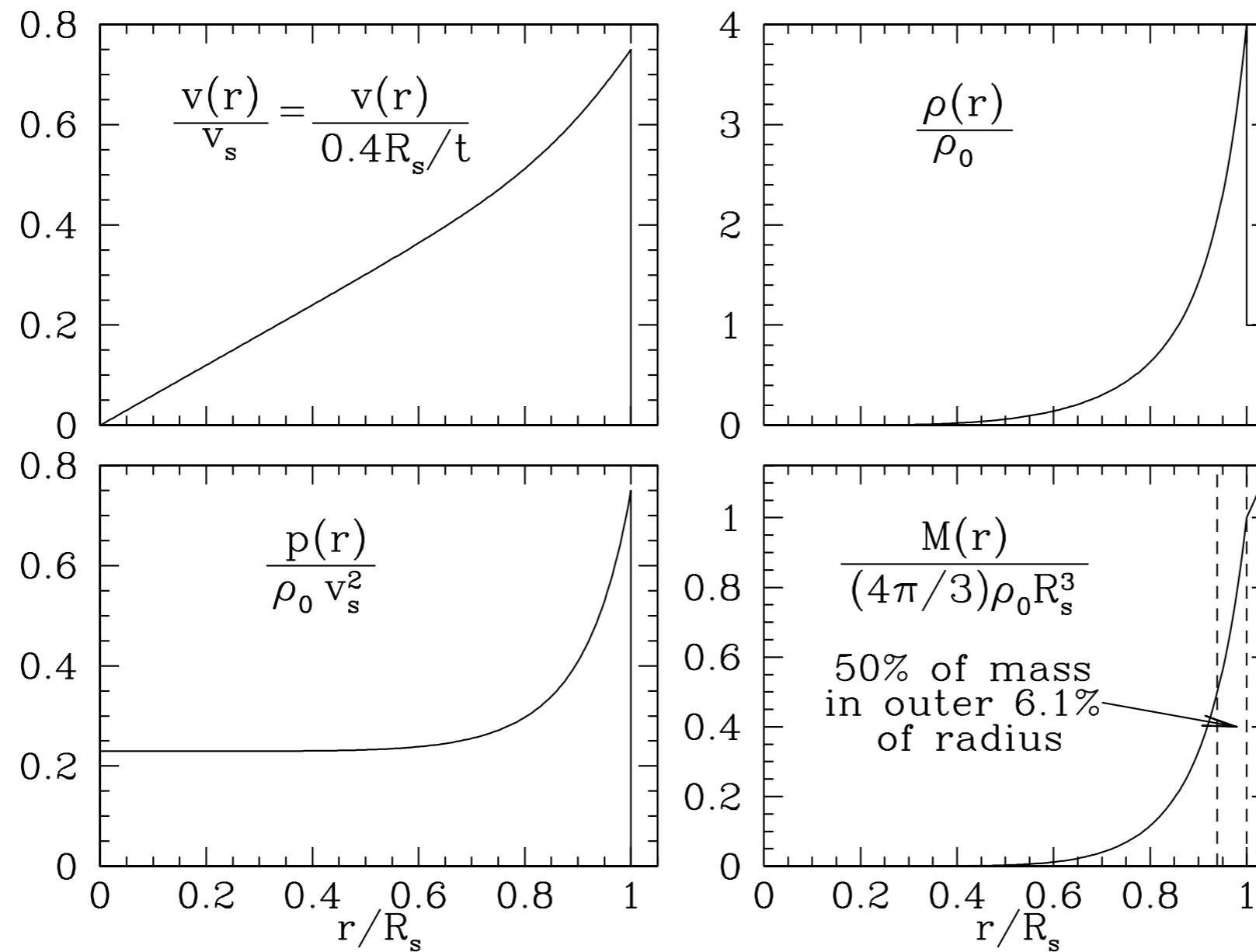
- Since the blast wave is slowing down with time, the specific energy of the internal gas varies with radius within the bubble of hot gas.
- Provided that most of the mass in the supernova remnant is made up of shocked interstellar gas and the mass of the stellar eject can be neglected, the total energy in the bubble of hot gas E_0 , which is equal to the energy injected by the supernova explosion since the radiative losses are negligible, is therefore

$$\begin{aligned} E_0 &= \phi \frac{4\pi}{3} R^3 \rho_0 (\epsilon_{\text{int}} + \epsilon_{\text{kin}}) \\ &= \phi \frac{3\pi}{4} R^3 \rho_0 \left(\frac{dR}{dt} \right)^2 \end{aligned}$$

ρ_0 = density of ambient medium
 R = the radius of the shock front at time t

where ϕ is a structure parameter, a numerical factor of order unity which accounts for the radial dependence of specific energy within the bubble. This equation represents the equation of motion of the shock front.

$$R = \left(\frac{25}{3\pi\phi} \right)^{1/5} \left(\frac{E_0 t^2}{\rho_0} \right)^{1/5} \quad A \equiv \left(\frac{25}{3\pi\phi} \right)^{1/5} = 1.15167 \text{ for } \gamma = 5/3$$



as $r \rightarrow 0$

$$\rho(r)/\rho_0 \rightarrow 0$$

$$T(r)/T_s \rightarrow \infty$$

$$P(r)/P_0 \rightarrow 0.306$$

Sedov-Taylor solution for $\gamma = 5/3$.

The temperature profile (not shown) can be obtained from the ratio of the pressure and density profiles. **The density falls inward, and the temperature rises inward.**

[Figure 39.1, Draine]

- ***End of the Sedov-Taylor phase***

- ▶ The hot gas interior to the shock front is, of course, radiating energy. When the radiative losses become important, a dense shell forms behind the shock and the SNR will leave the Sedov-Taylor phase and enter a “radiative” phase. In the radiative phase, the gas in the shell just interior to the shock front is now able to cool to temperatures much lower than the temperature $T_s = (3/16)mV_s^2/k$ at the shock front.
- ▶ In order to calculate the time scale, we need to estimate the cooling rate:

cooling function at a radius r :

$$\Lambda(r) \approx C [T(r)/10^6 \text{ K}]^{-0.7} n_{\text{H}}(r)n_e(r), \quad C = 1.1 \times 10^{-22} \text{ erg cm}^3 \text{ s}^{-1}, \quad n_e \simeq 1.2n_{\text{H}}$$

cooling rate at a time t , integrated over the volume of the SNR:

$$\begin{aligned} \frac{dE}{dt} &= - \int_0^{R_s} \Lambda(r) 4\pi r^2 dr && \xleftarrow{\hspace{1cm}} n_{\text{H}}n_e = 1.2n_{\text{H}}^2 = 1.2n_0^2 [\rho(r)/\rho_0(r)]^2 \\ &= -1.2Cn_0^2 (T_s/10^6 \text{ K})^{-0.7} \frac{4\pi}{3} R_s^3 \langle (\rho/\rho_0)^2 (T_s/T)^{0.7} \rangle \end{aligned}$$

Here, $n_0 \equiv n_{\text{H}}(r = R_s)$ is the hydrogen density of the ambient medium at $r = R_s$.

$T_s \equiv T(r = R_s)$ is the temperature of the post-shock region at $r = R_s$.

$\langle \dots \rangle$ denotes a volume-weighted average over the SNR.

From the Sedov-Taylor solution, we obtain $\langle (\rho/\rho_0)^2 (T_s/T)^{0.7} \rangle = 1.817$.

-
- Now, integrate the energy loss rate over a time interval t :

$$\Delta E(t) = -1.2 \times (1.817) C \frac{4\pi}{3} n_0^2 \int_0^t dt R_s^3 (T_s/10^6 \text{ K})^{-0.7}$$

Using the previous solutions, we obtain the fractional energy loss by time t :

$$\begin{aligned} R_s &= 1.54 \times 10^{19} [\text{cm}] E_{51}^{1/5} n_0^{-1/5} t_3^{2/5} \\ T_s &= 5.25 \times 10^7 [\text{K}] E_{51}^{2/5} n_0^{-2/5} t_3^{-6/5} \end{aligned}$$

$$\frac{\Delta E(t)}{E_0} \approx -2.38 \times 10^{-6} n_0^{1.68} E_{51}^{-0.68} t_3^{3.04}$$

- If we suppose that the SNR enter the “radiative phase” when

$$\Delta E(t_{\text{rad}})/E_0 \approx -1/3$$

we can solve for the cooling time t_{rad} :

$$t_{\text{rad}} = 49.3 \times 10^3 [\text{yr}] E_{51}^{0.22} n_0^{-0.55}$$

end of the Sedov-Taylor phase

- The radius and shock speed at the end of the Sedov-Taylor phase are:

$$7.32 \times 10^{19} \text{ cm} = 23.7 \text{ pc}$$

$$\begin{aligned} R_s(t_{\text{rad}}) &= 7.32 \times 10^{19} [\text{cm}] E_{51}^{0.29} n_0^{-0.42} \\ V_s(t_{\text{rad}}) &= 188 [\text{km s}^{-1}] (E_{51} n_0^2)^{0.07} \end{aligned}$$

Post-shock temperature is:

$$\begin{aligned} T_s(t_{\text{rad}}) &= 4.86 \times 10^5 [\text{K}] (E_{51} n_0^2)^{0.13} \\ kT_s(t_{\text{rad}}) &= 41 [\text{eV}] (E_{51} n_0^2)^{0.13} \end{aligned}$$

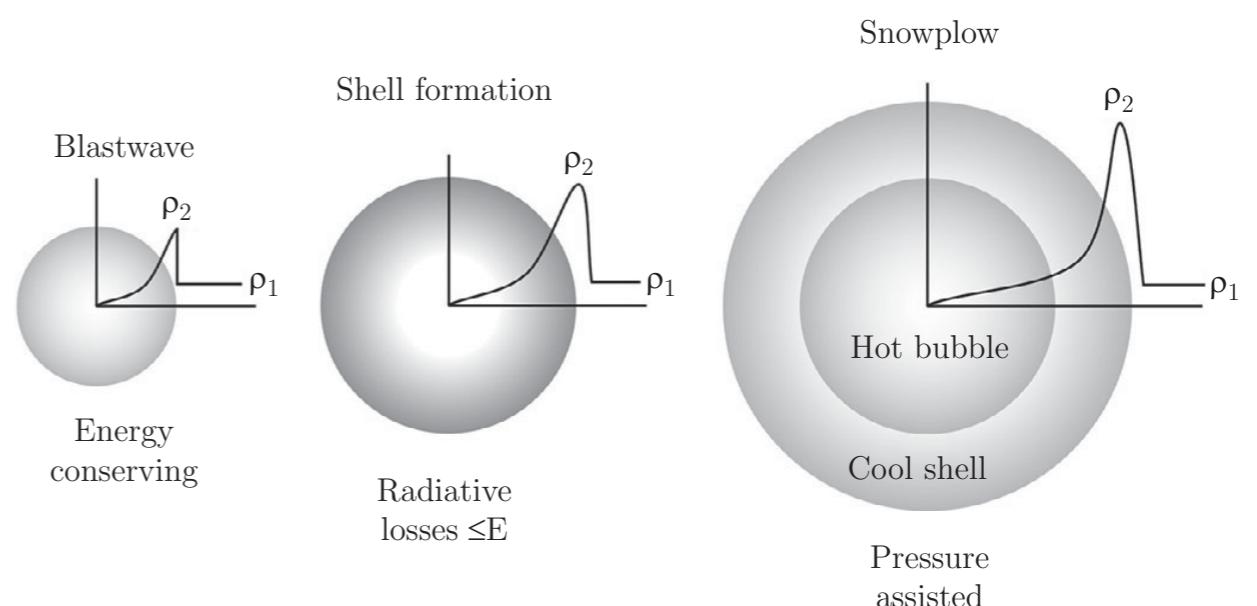
- **Snowplow Phase (Radiative Phase)**

- At $t \approx t_{\text{rad}}$
 - ▶ Cooling cause the thermal pressure just behind the shock to drop suddenly, and the shock wave briefly stalls. However, ***the very hot gas in the interior of the SNR has not cooled.***
 - ▶ The SNR now enters the snowplow phase, with a dense shell of cool gas enclosing a hot central volume where radiative cooling is unimportant.
 - ▶ This is called the snowplow phase because the expanding dense shell scoops up additional mass as it expands (the mass of the dense shell increases as it “sweeps up” the ambient gas). Since the gas cools well the shell will be thin and so the gas in the shell has a radial velocity that is almost the same as the shock front speed.
 - ▶ Most of the mass of a supernova remnant in the snowplow phase is in the dense shell. The inner region contains gas at very high temperatures. Thus, despite its low density, the gas in the inner region has a significant pressure, which push the dense shell outward.
 - ▶ The gas in the hot center cools by adiabatic expansion, and thus the pressure is given by

$$P \propto V^{-\gamma} \propto R_s^{-3\gamma} = R_s^{-5}$$

- ▶ So that the pressure in the interior evolves as

$$P_i = P_0(t_{\text{rad}}) \left(\frac{R_{\text{rad}}}{R_s} \right)^5$$



Transition from the Sedov-Taylor phase to the snowplow phase [Figure 5.5, Ryden]

- In the initial phase, the pressure exerted by the hot center causes the “radial momentum” of the shell to increases:

M_s = mass of the shell = mass of the ambient medium swept by the SNR

$$\frac{d}{dt} (M_s V_s) \approx P_i 4\pi R_s^2 = 4\pi P_0(t_{\text{rad}}) R_{\text{rad}}^5 R_s^{-3}$$

$$\begin{aligned} M_s &= \left(\frac{4\pi}{3} R_s^3 \right) \rho_0 \\ V_s &= \frac{dR_s}{dt} \end{aligned} \quad \longrightarrow \quad \frac{d}{dt} \left(R_s^3 \frac{dR_s}{dt} \right) \propto R_s^{-3}$$

Suppose that there is a power-law solution:

$$R_s \propto t^\eta$$

$$4\eta - 2 = -3\eta \Rightarrow \eta = 2/7$$

$$\begin{aligned} R_s &\approx R_s(t_{\text{rad}}) (t/t_{\text{rad}})^{2/7} \\ V_s &\approx \frac{2}{7} \frac{R_s}{t} = \frac{2}{7} \frac{R_s(t_{\text{rad}})}{t_{\text{rad}}} \left(\frac{t}{t_{\text{rad}}} \right)^{-5/7} \end{aligned}$$

- ▶ Because the effect of the internal pressure has been included, this solution is referred to as the ***pressure-modified (or pressure-driven) snowplow phase***.
- ▶ Note that with this construction, $R_s(t)$ is continuous from the Sedov-Taylor phase to the pressure-modified snowplow phase, but $V_s(t)$ undergoes a discontinuous drop by $2/7 \sim 29\%$ at $t = t_{\text{rad}}$.

- In the late phase of evolution, the stored thermal energy in the central region has been entirely radiated away, and only the momentum of the dense shell keeps the remnant expanding into the ISM.

$$\frac{d}{dt} (M_s V_s) = 0 \quad M_s = \text{mass of the shell}$$

$$M_s = \left(\frac{4\pi}{3} R_s^3 \right) \rho_0$$

$$V_s = \frac{dR_s}{dt} \quad \longrightarrow \quad R_s^3 \frac{dR_s}{dt} = \text{constant}$$

$$R_s \approx R_s(t_{\text{rad}}) (t/t_{\text{rad}})^{1/4} \quad \text{for } t \gg t_{\text{rad}}$$

$$V_s \approx \frac{1}{4} \frac{R_s}{t} = \frac{1}{4} R_s(t_{\text{rad}}) (t/t_{\text{rad}})^{-3/4}$$

- ▶ This phase is called the ***momentum-conserving snowplow phase***. Towards the end of this phase, the expansion velocity becomes sonic or subsonic with respect to the interstellar sound speed.
- ***In summary, the supernova expansion velocity is continuously slowed by its interaction with the surrounding medium.***

$$R \propto t \rightarrow R \propto t^{2/5} \rightarrow R \propto t^{2/7} \rightarrow R \propto t^{1/4}$$

- ***Fadeaway (Merging with the ISM)***

- For typical ISM parameters, the shock speed at the beginning of the snowplow phase is

$$V_s(t_{\text{rad}}) = 188 \text{ [km s}^{-1}\text{]} (E_{51} n_0^2)^{0.07}$$

- This results in a very strong shock when propagating through interstellar gas with $T < 10^4$ K. However, the shock front gradually slows, and the shock compression declines. This proceeds until the shock speed approaches the effective sound speed in the gas and at this point the compression ratio approaches 1, and the shock wave turns into a sound wave.
- Setting $V_s \approx c_s$ gives the a “fadeaway time”

$$V_s \approx \frac{2}{7} \frac{R_s(t_{\text{rad}})}{t_{\text{rad}}} \left(\frac{t}{t_{\text{rad}}} \right)^{-5/7} \longrightarrow t_{\text{fade}} \approx \left(\frac{(2/7)R_{\text{rad}}/t_{\text{rad}}}{c_s} \right)^{7/5} t_{\text{rad}}$$

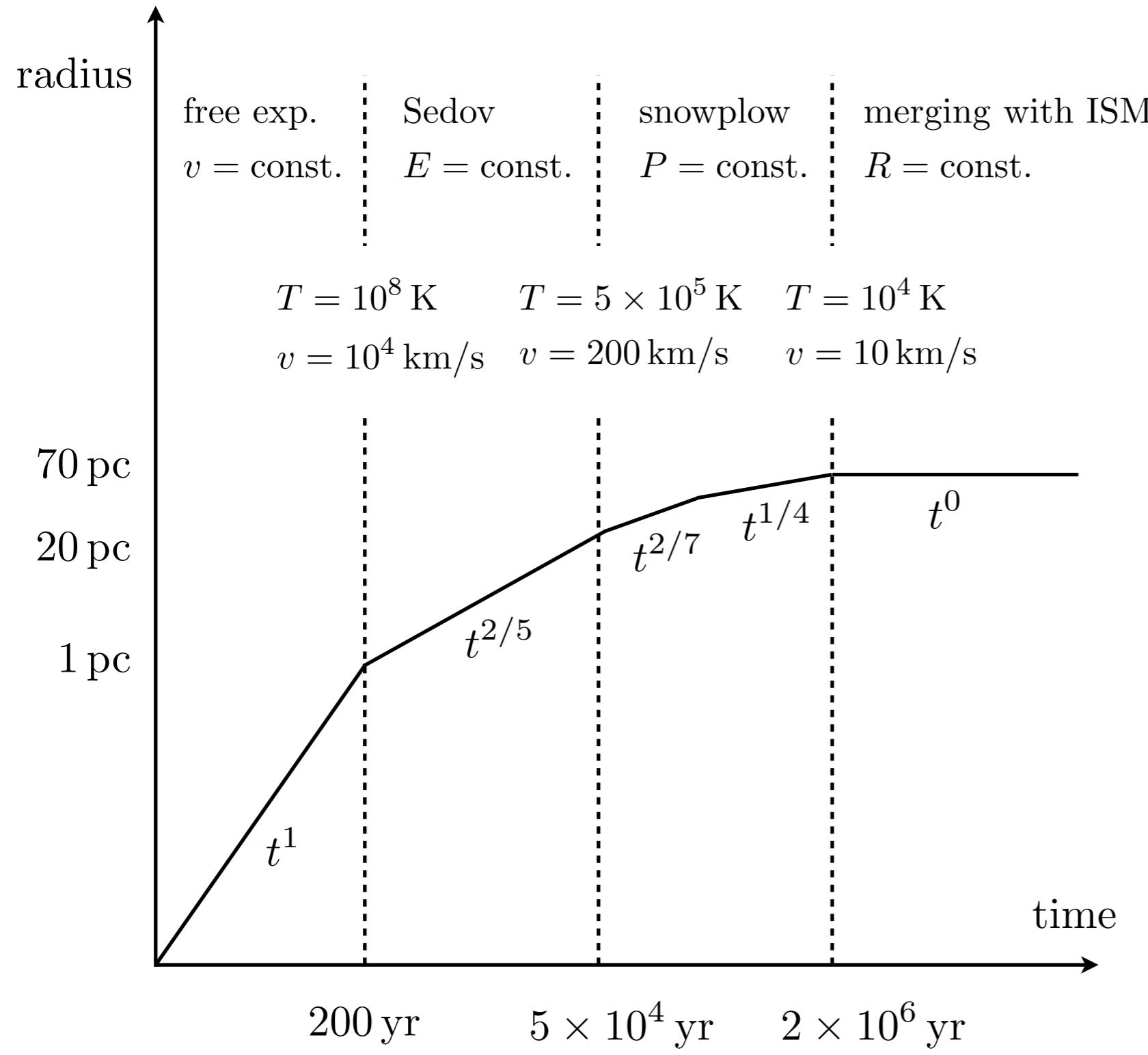
$$t_{\text{fade}} \approx 1.87 \times 10^6 \text{ [yr]} E_{51}^{0.32} n_0^{-32} \left(\frac{c_s}{10 \text{ km s}^{-1}} \right)^{-7/5} \quad \text{end of the snowplow phase}$$

$$R_{\text{fade}} \equiv R_s(t_{\text{fade}}) \approx \frac{7}{2} V_s t_{\text{fade}} \approx \frac{7}{2} c_s t_{\text{fade}}$$

$$\approx 2.07 \times 10^{20} \text{ [cm]} E_{51}^{0.32} n_0^{-0.37} \left(\frac{c_s}{10 \text{ km s}^{-1}} \right)^{-2/5}$$

$$2.07 \times 10^{20} \text{ cm} = 67.1 \text{ pc}$$

Illustration of the phases in the dynamics of a SNR



Four Phases of the SNR

1. Free Expansion Phase
2. Sedov-Taylor Phase
3. Snowplow Phase
(momentum-driven)
(momentum-conserving)
4. Merging Phase

Rough estimates for the temperatures and velocities at the end of each phase are given.

Fig 4.9, Rosswog & Bruggen,
Introduction to High-Energy Astrophysics
(slightly modified)

Homework (due date: 05/29)

[Q14]

The “cooling time” $\tau_{\text{cool}} \equiv |d \ln T / dt|^{-1}$. Suppose the power radiated per unit volume Λ can be approximated by

$$\Lambda \approx A n_{\text{H}} n_e \left[T_6^{-0.7} + 0.021 T_6^{1/2} \right]$$

for gas of cosmic abundances, where $A = 1.1 \times 10^{-22} \text{ erg cm}^3 \text{ s}^{-1}$, and $T_6 \equiv T / 10^6 \text{ K}$. Assume the gas to have $n_{\text{He}} = 0.1 n_{\text{H}}$, with both H and He fully ionized.

Compute the cooling time (at constant pressure) due to radiative cooling

- (a) in a supernova remnant at $T = 10^7 \text{ K}$, $n_{\text{H}} = 10^{-2} \text{ cm}^{-3}$.
- (b) for intergalactic gas within a dense galaxy cluster (the “intracluster medium”) with $T = 10^8 \text{ K}$, $n_{\text{H}} = 10^{-3} \text{ cm}^{-3}$.

[Q15]

Consider a strong shock wave propagating into a medium that was initially at rest. Assume the gas to be monatomic ($\gamma = 5/3$). Consider the material just behind the shock front. The gas has an energy density u_{thermal} from random thermal motions, and an energy density u_{flow} from the bulk motion of the shocked gas. If cooling is negligible, calculate the ratio $u_{\text{flow}}/u_{\text{thermal}}$ in the frame of reference where the shock front is stationary.