

# Astrophysics

Lecture 01

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선광일 (Kwang-Il Seon)  
UST / KASI

# Overview

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## Astrophysics will

- link astrophysical systems with astronomical observables
- cover many areas of physics and astrophysics (electrodynamics, quantum mechanics, statistical mechanics, relativity...)

## Textbooks

- Radiative Processes in Astrophysics (Rybicki & Lightman)
- 천체물리학 (구본철, 김웅태) : 품절 (out-of-stock)
- Astrophysics Processes (Hale Bradt)

# Fundamentals of Radiative Transfer

# Electromagnetic Radiation

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## Particle/wave duality

classically: electromagnetic waves

- speed of light:  $c = 3 \times 10^{10} \text{ cm s}^{-1}$
- wavelength and frequency:  $\lambda = c/\nu$

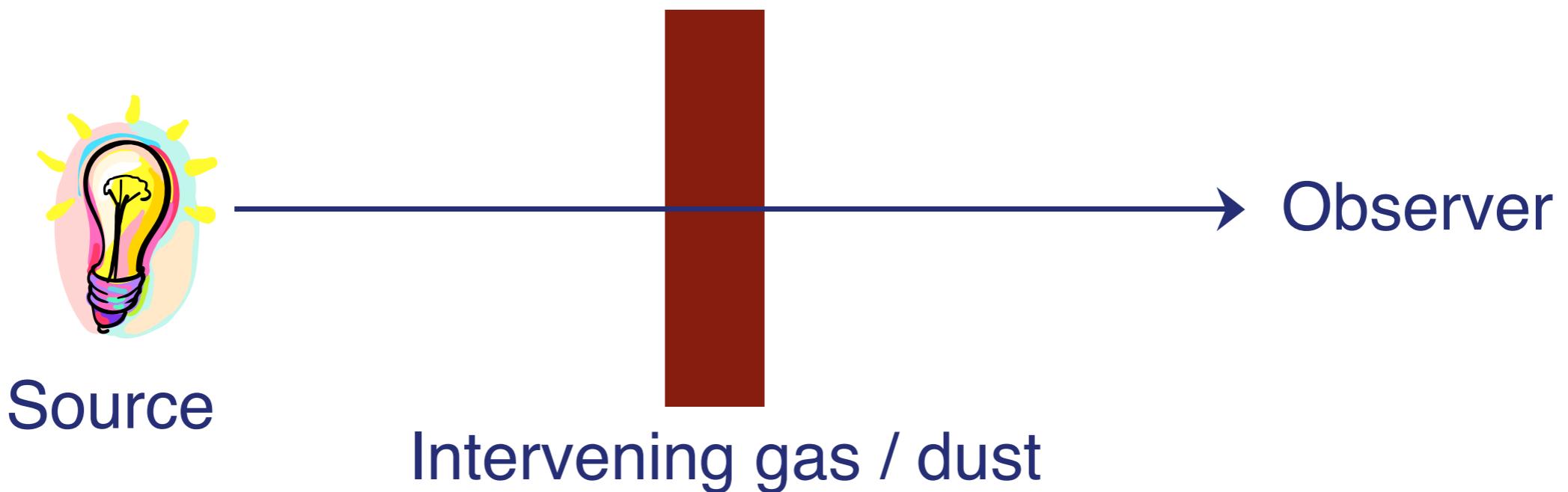
quantum mechanically: photons

- quanta: massless, spin-1 particles (boson)
- Plank:  $E = h\nu = hc/\lambda$  ( $h = 6.625 \times 10^{-27} \text{ ergs s}$ )
- Einstein:  $E^2 = (m_\gamma c^2)^2 + (pc)^2$   
 $p = E/c$

# Radiative Transfer

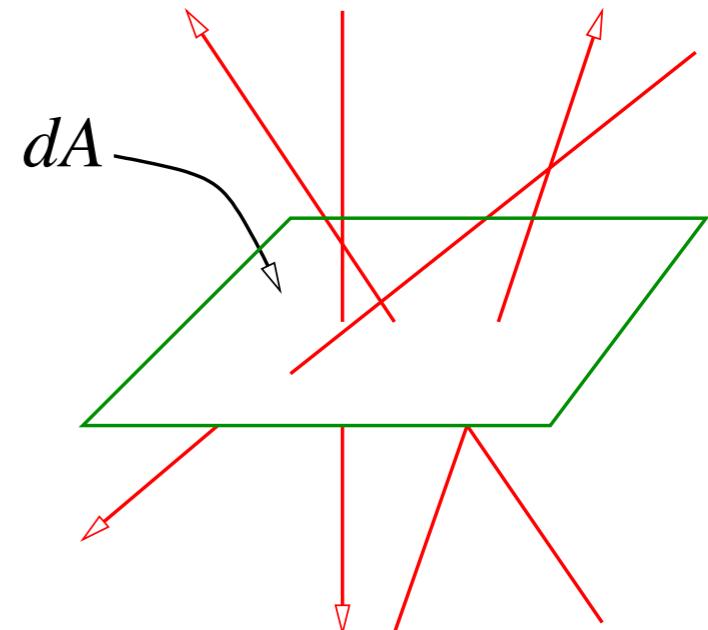
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- How is radiation affected as it propagates through intervening gas and dust media to the observer?



# Simplification & Complexity

- Simplification:
  - Astronomical objects are normally much larger than the wavelength of radiation they emit.
  - Diffraction can be neglected.
  - Light rays travel to us along straight lines.
- Complexity:
  - At one point, photons can be traveling in several different directions.
  - For instance, at the center of a star, photons are moving equally in all directions. (However, radiation from a star seen by a distant observer is moving almost exactly radially.)
  - Full specification of radiation needs to say how much radiation is moving in each direction at every point. Therefore, we are dealing with the five- or six-dimensional problem. ( $[x, y, z] + [\theta, \phi] + [t]$ )

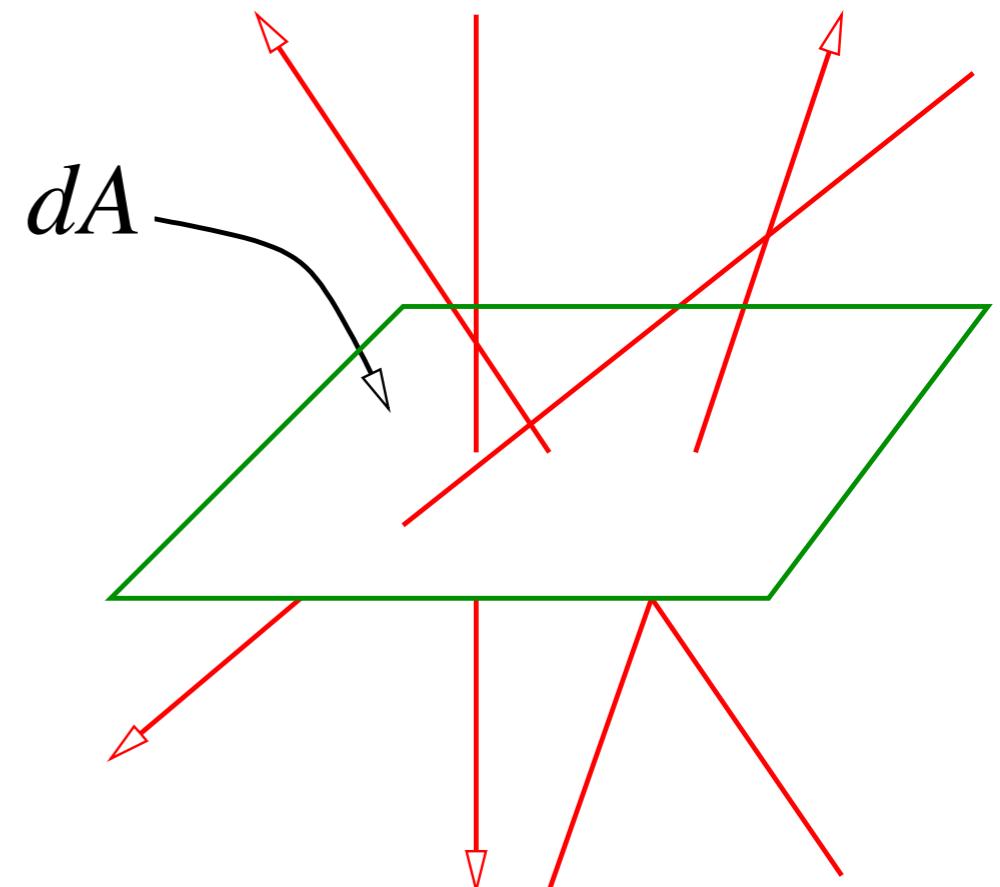


# Energy Flux

- Definition
  - Consider a small area  $dA$ , exposed to radiation for a time  $dt$ .
  - Energy flux  $F$  is defined as ***the net energy  $dE$  passing through the element of area in all directions in the time interval*** so that

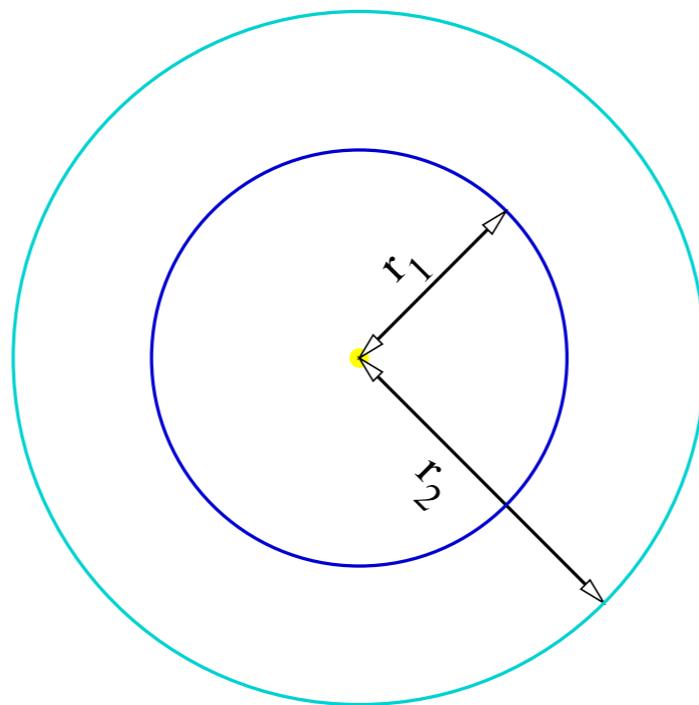
$$dE = F \times dA \times dt$$

- Note that  $F$  ***depends on the orientation of the area element  $dA$ .***
- Unit: erg cm<sup>-2</sup> s<sup>-1</sup>



# Inverse Square Law

- Flux from an isotropic radiation source, i.e., a source emitting equal amounts of energy in all directions.



- Because of energy conservation, flux through two shells around the source must be the same.

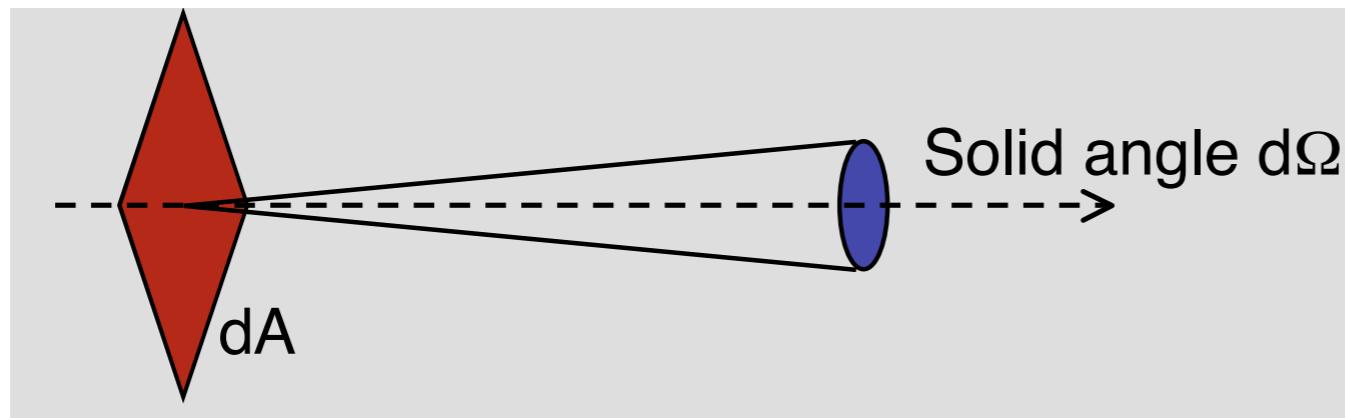
$$4\pi r_1^2 F(r_1) = 4\pi r_2^2 F(r_2)$$

- Therefore, we obtain the inverse square law.

$$F = \frac{\text{const.}}{r^2}$$

# Specific Intensity or Surface Brightness

- Recall that ***flux is a measure of the energy carried by all rays passing through a given area***
- Intensity is the energy carried along by individual rays.***



- Let  $dE_\nu$  be the amount of radiant energy which crosses the area  $dA$  in a direction  $\mathbf{k}$  within solid angle  $d\Omega$  about in a time interval  $dt$  with photon frequency between  $\nu$  and  $\nu + d\nu$ .
- The monochromatic specific intensity  $I_\nu$  is then defined by the equation.

$$dE_\nu = I_\nu(\mathbf{k}, \mathbf{x}, t) dA_{\mathbf{k}} d\Omega d\nu dt$$

- Unit:  $\text{erg s}^{-1} \text{ cm}^{-2} \text{ sr}^{-1} \text{ Hz}^{-1}$
- From the view point of an observer, the specific intensity is called ***surface brightness***.

# Relation between the flux and the specific intensity

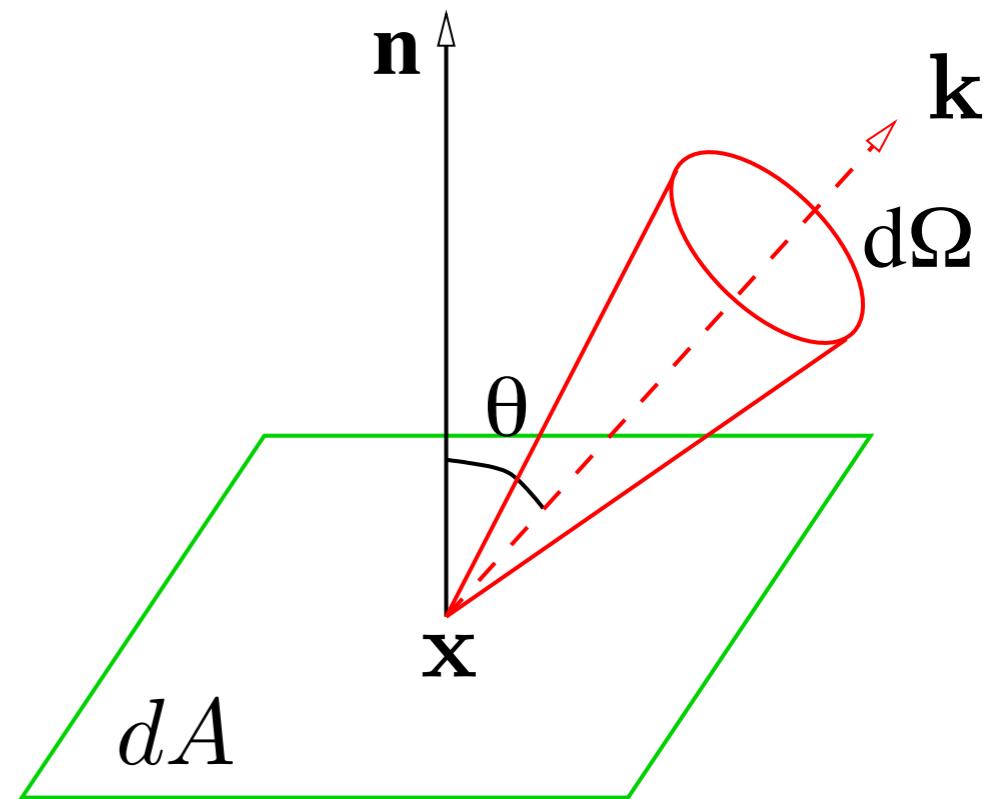
- Let's consider a small area  $dA$ , with light rays passing through it at all angles to the normal vector  $\mathbf{n}$  of the surface.
- For a ray centered about  $\mathbf{k}$ , the area normal to  $\mathbf{k}$  is

$$dA_{\mathbf{k}} = dA \cos \theta$$

- By the definition,

$$F_{\nu} dAd\nu dt = \int I_{\nu}(\mathbf{k}, \mathbf{x}, t) dA_{\mathbf{k}} d\Omega d\nu dt$$

- Hence, net flux in the direction of  $\mathbf{n}$  is given by integrating over all solid angles:



$$F_{\nu} = \int I_{\nu} \cos \theta d\Omega = \int_0^{2\pi} \int_0^{\pi} I_{\nu} \cos \theta \sin \theta d\theta d\phi$$

[Note] **flux** = “sum of all ray vectors” which is then projected onto a normal vector  
**intensity** = magnitude of a single ray vector

## Net flux

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Net flux in the direction  $\mathbf{n}$  is obtained by integrating the differential flux over all solid angle.

$$F_\nu = \int dF_\nu = \int I_\nu \cos \theta d\Omega$$

- The net flux is zero, if the radiation is isotropic.

$$\begin{aligned} F_\nu &= \int_0^{2\pi} \int_0^\pi I_\nu \cos \theta \sin \theta d\theta d\phi = 2\pi I_\nu \int_0^\pi \cos \theta \sin \theta d\theta \\ &= \pi I_\nu [\sin^2 \theta]_0^\pi = 0 \end{aligned}$$

## Note

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### Moments of intensity

- intensity : scalar (amplitude of the differential flux)
- differential flux : vector
- momentum flux (radiation pressure) : tensor

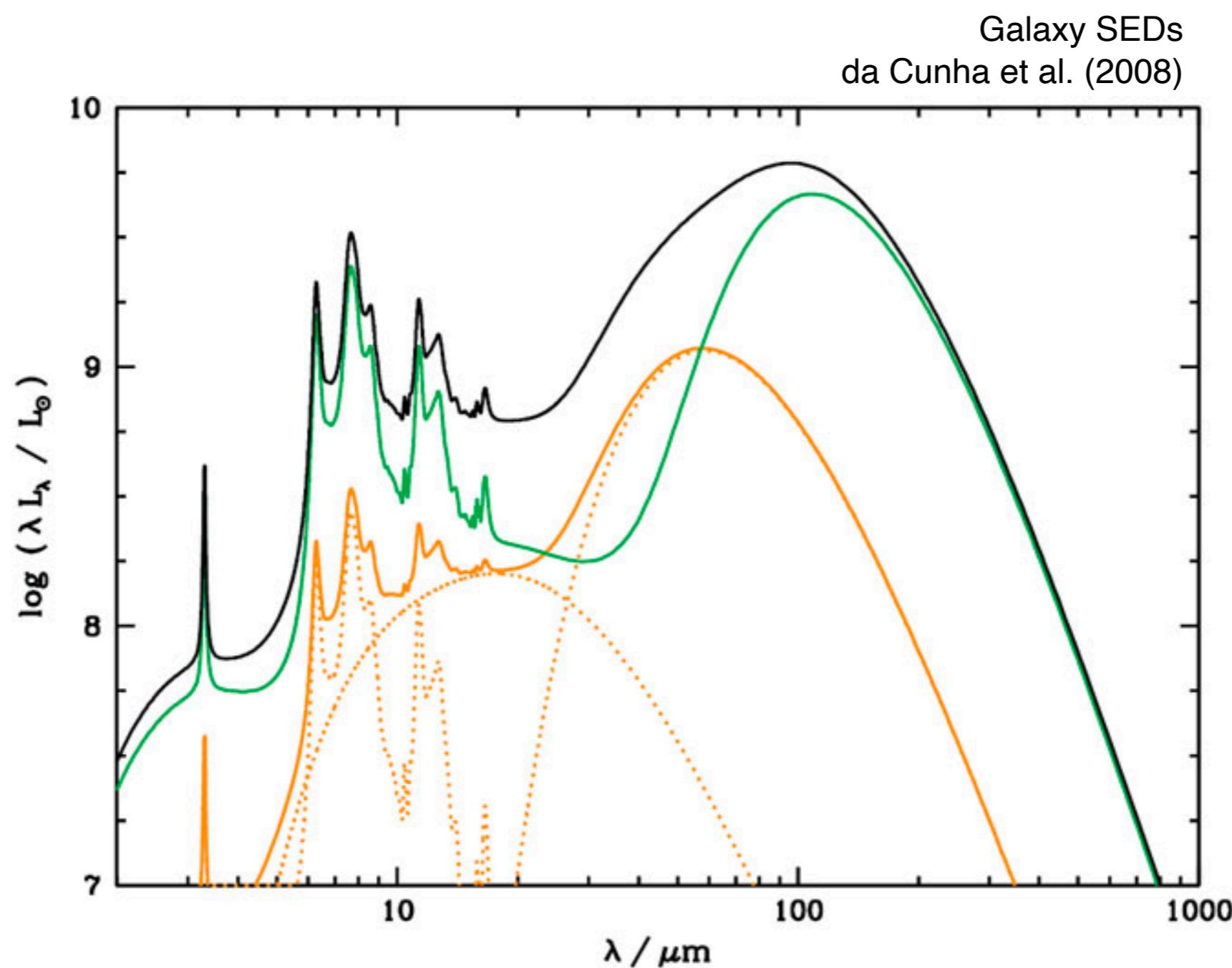
Intensity can be defined as per wavelength interval.

$$\begin{aligned} I_\nu |d\nu| &= I_\lambda |d\lambda| \\ \nu I_\nu &= \lambda I_\lambda \end{aligned} \quad \leftarrow \left| \frac{d\nu}{d\lambda} \right| = \frac{c}{\lambda^2} = \frac{\nu}{\lambda}$$

Integrated intensity is defined as the intensity over all frequencies.

$$I = \int_0^\infty I_\nu d\nu = \int_0^\infty I_\lambda d\lambda$$

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- In astrophysics, we plot the **spectral energy distribution (SED)** as  $\nu I_\nu$  versus  $\nu$  or  $\lambda I_\lambda$  versus  $\lambda$ .



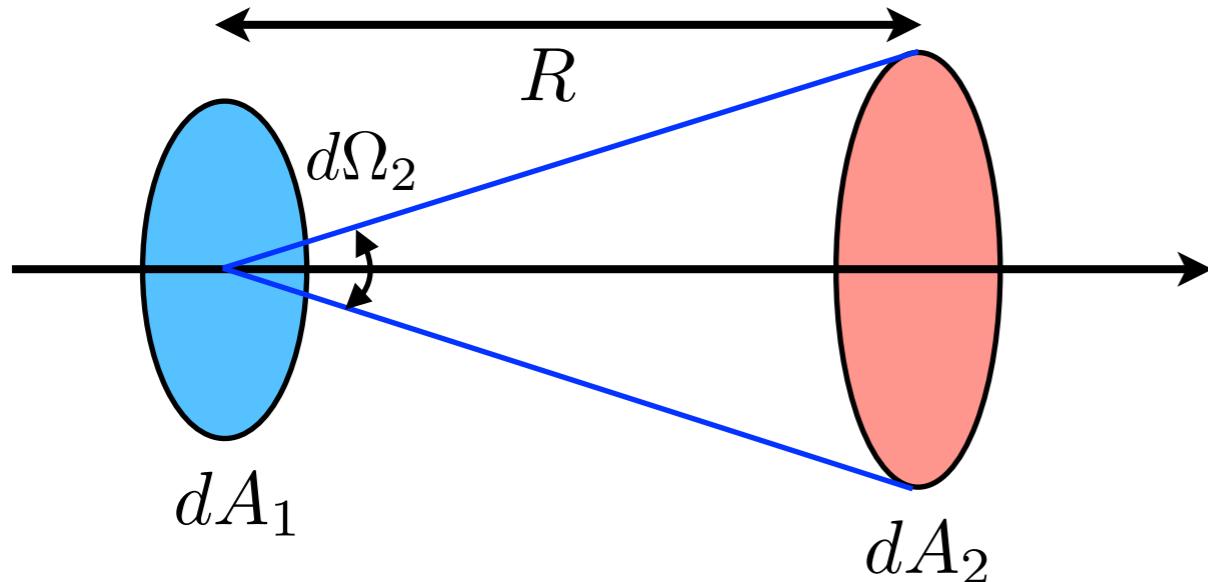
## Constancy of Intensity

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- How does specific intensity changes along a ray in free space
  - Suppose a bundle of rays and any two points along the rays and construct areas  $dA_1$  and  $dA_2$  normal to the rays at these points.
  - What are the energies carried by the rays passing through both areas?

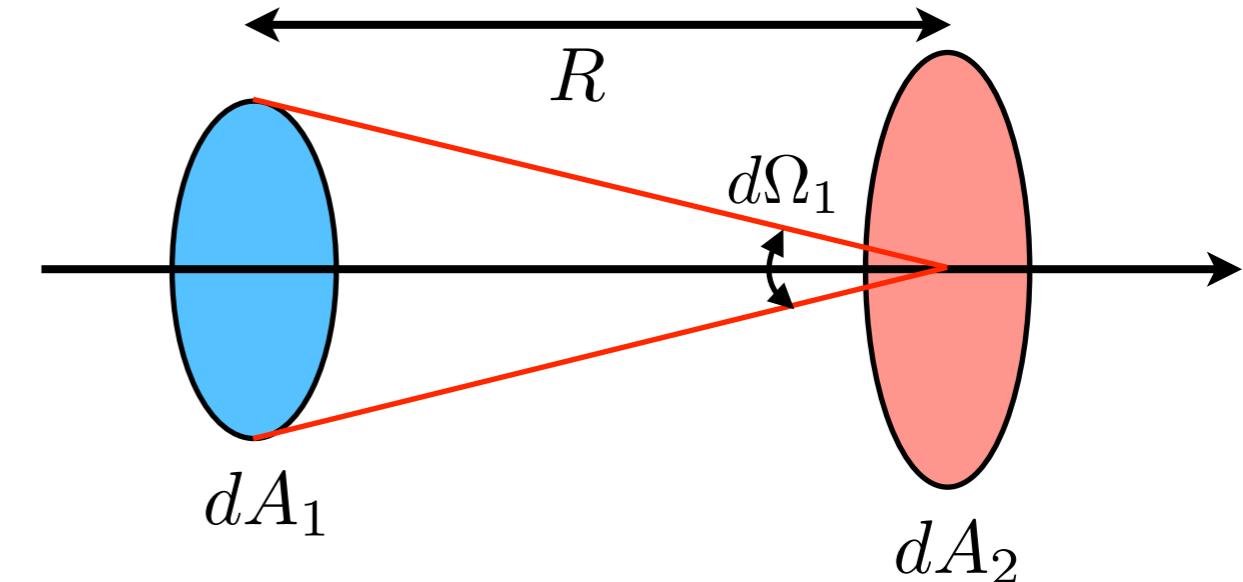
energy passing through 1

$$dE_1 = I_1 dA_1 d\Omega_2 d\nu dt$$



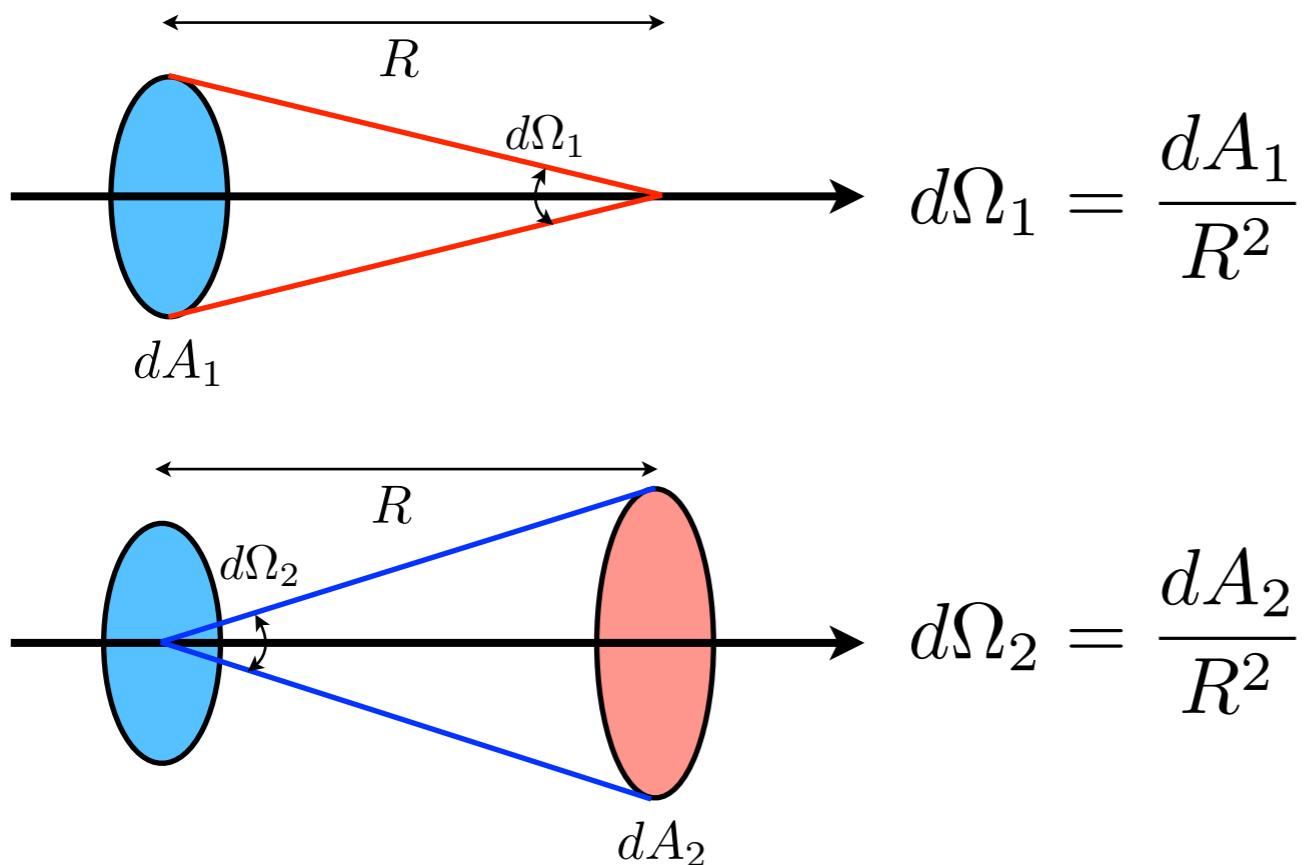
energy passing through 2

$$dE_2 = I_2 dA_2 d\Omega_1 d\nu dt$$



- Here,  $d\Omega_1$  is the solid angle subtended by  $dA_2$  at the location 1 and so forth.

## : Radiative Transfer Equation in free space



conservation of energy:  
Because energy is conserved,

$$dE_1 = dE_2 \rightarrow I_1 = I_2$$

- Conclusion (***the constancy of intensity***):  $I_1 = I_2$ 
  - the specific intensity remains the same as radiation propagates through free space.
- We receive the same specific intensity at the telescope as is emitted at the source.
  - Imagine looking at a uniformly lit wall and walking toward it. As you get closer, a field-of-view with fixed angular size will see a progressively smaller region of the wall, but this is exactly balanced by the inverse square law describing the spreading of the light rays from the wall.

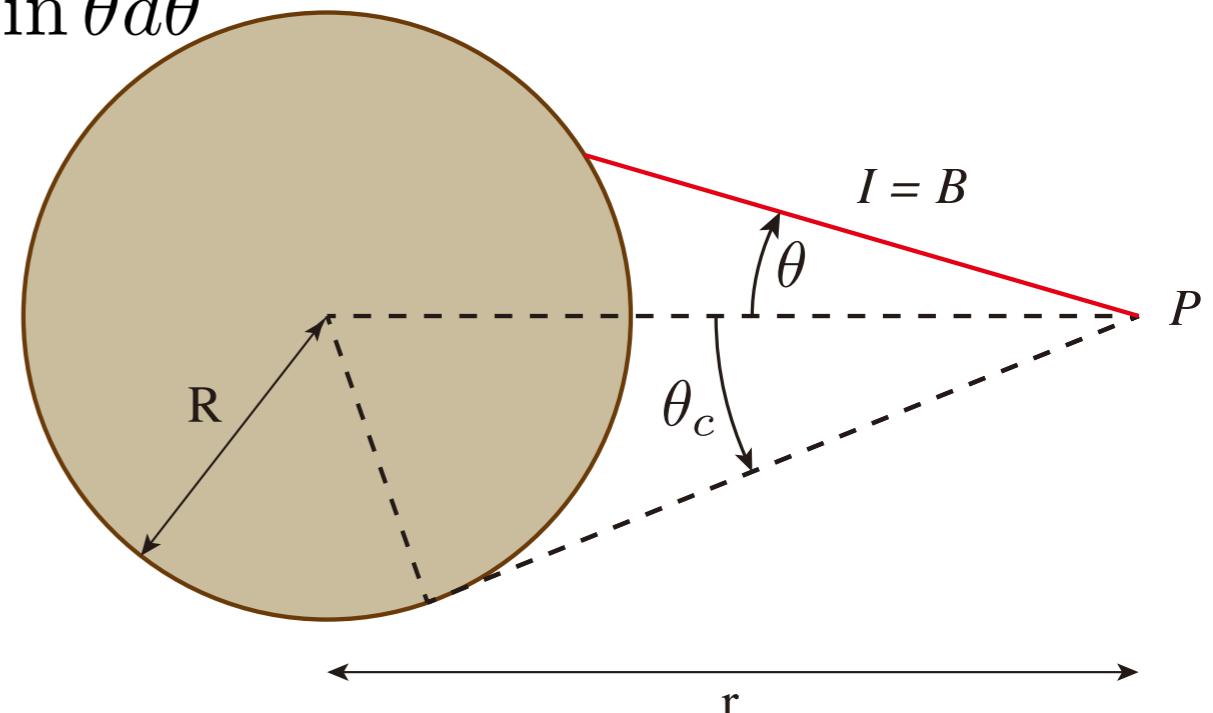
# Inverse Square Law for a Uniformly Bright Sphere

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Let's calculate the flux at  $P$  from a sphere of uniform brightness  $B$

$$\begin{aligned} F &= \int I \cos \theta d\Omega = B \int_0^\pi d\phi \int_0^{\theta_c} \cos \theta \sin \theta d\theta \\ &= \pi B (1 - \cos^2 \theta_c) = \pi B \sin^2 \theta_c \end{aligned}$$

$$\sin \theta_c = \frac{R}{r} \rightarrow F = \pi B \left( \frac{R}{r} \right)^2$$



Therefore, there is no conflict between the constancy of intensity and the inverse square law.

Note

- The flux at a surface of uniform brightness  $B$  is  $F = \pi B$ .
- For stellar atmosphere, the astrophysical flux is defined by  $F/\pi$ .

# Luminosity

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- To determine the energy per unit time, we integrate flux over area.
  - Monochromatic luminosity: Considering a sphere centered on a source with radius  $R$ , the monochromatic luminosity is

$$\begin{aligned} L_\nu &= R^2 \int d\Omega F_\nu \\ &= 4\pi R^2 F_\nu \quad \text{for an isotropic source} \end{aligned}$$

- The bolometric luminosity is

$$L_{\text{bol}} = \int L_\nu d\nu = \int L_\lambda d\lambda = 4\pi R^2 \int F_\nu d\nu$$

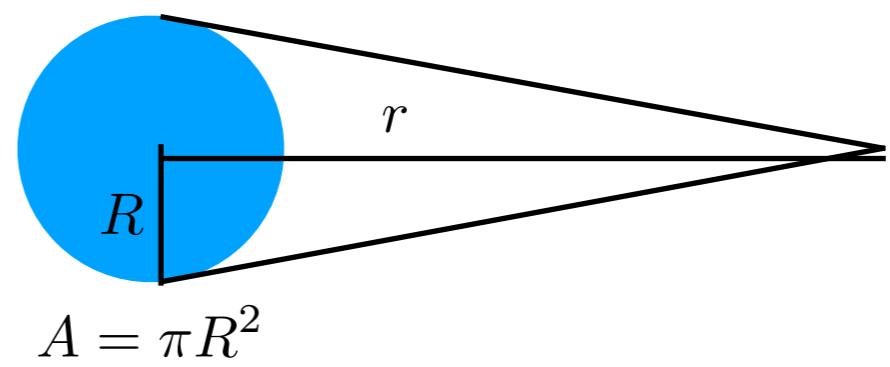
- Flux and Luminosity of an extended source

$$F = \pi I \sin^2 \theta_c = \pi I \left( \frac{R}{r} \right)^2$$

$$F = I \frac{A}{r^2}$$

$$= I \Omega_{\text{source}}$$

$$L = (4\pi r^2)F$$



## (Specific) Energy Density

Consider a bundle of rays passing through a volume element  $dV$  in a direction  $\Omega$ .

Then, the **energy density per unit solid angle** is defined by

$$dE = u_\nu(\Omega)dVd\Omega d\nu$$

Since radiation travels at velocity  $c$ ,

$$dV = dA(cdt)$$

the definition of the intensity

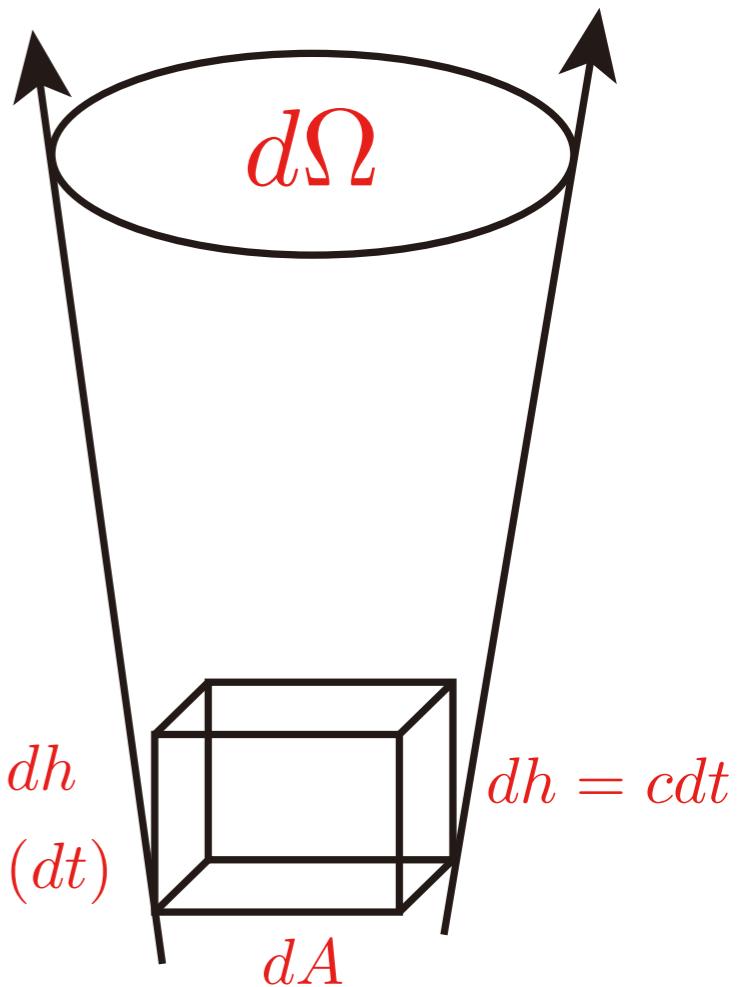
$$dE = I_\nu dA dt d\Omega d\nu$$

Therefore,

$$u_\nu(\Omega) = I_\nu(\Omega)/c$$

Volume

$$\begin{aligned} dV &= dA \times dh \\ &= c(dA)(dt) \end{aligned}$$



# Energy Density and Mean Intensity

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Integrating over all solid angle, we obtain

$$u_\nu = \int u_\nu(\Omega) d\Omega = \frac{1}{c} \int I_\nu d\Omega$$

Mean intensity is defined by

$$J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$$

Then, the energy density is

$$u_\nu = \frac{4\pi}{c} J_\nu$$

Total energy density is obtained by integrating over all frequencies.

$$u = \int u_\nu d\nu = \frac{4\pi}{c} \int J_\nu d\nu$$

## Radiation pressure

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Photons carry momentum as well as energy. If they strike a surface they transfer momentum to it. The momentum transfer per unit time represents a force on the surface, and the force per unit area on the surface is a pressure.

Thus, photons can exert a pressure on the surface of its container. The concept of pressure is valid within the gas even if there is no physical surface.

# Momentum Flux: Radiation Pressure (due to absorption)

- momentum of a photon:  $p = E/c$

- force:  $F = \frac{\Delta p}{\Delta t} = \frac{E}{c\Delta t}$

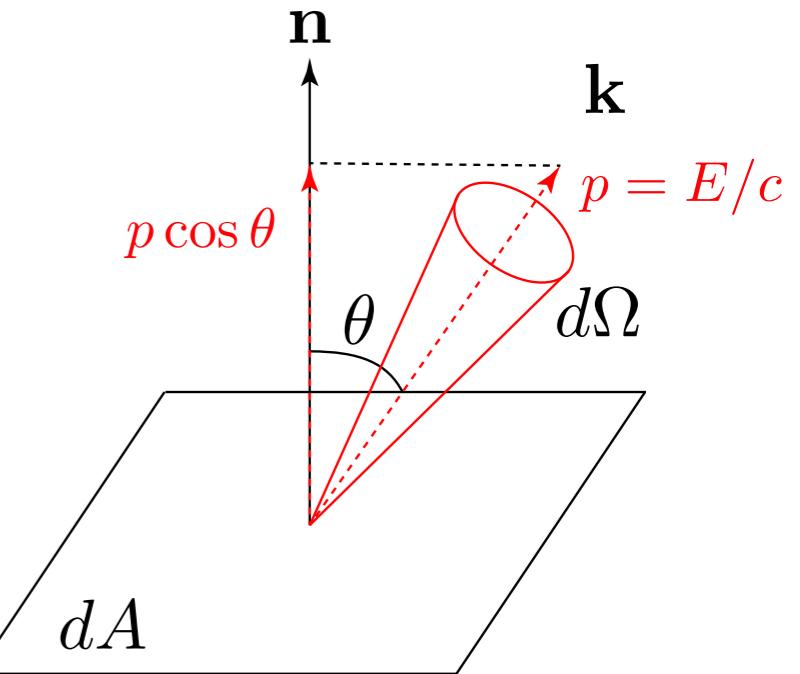
- radiation pressure = force per unit area

- radiation pressure due to energy flux

propagating within solid angle  $d\Omega$  and with  
frequency between  $(\nu, \nu + d\nu)$ :

$$p_\nu d\Omega d\nu = \frac{\Delta F_\nu}{\Delta A} = \frac{1}{\Delta A} \frac{\Delta E_\nu/c}{\Delta t} \cos \theta \quad \Longleftarrow \quad \begin{aligned} \Delta E_\nu &= I_\nu \Delta A_k \Delta t \Delta \Omega \\ \Delta A_k &= \Delta A \cos \theta \end{aligned}$$

Integrating over solid angle,



The first cosine factor is due to the area normal to  $\mathbf{k}$  and the second one is due to the projection of the differential flux vector to the normal vector  $\mathbf{n}$ .

$$P_\nu = \frac{1}{c} \int I_\nu \cos^2 \theta d\Omega = \frac{1}{6} u_\nu \quad \text{for isotropic radiation field}$$

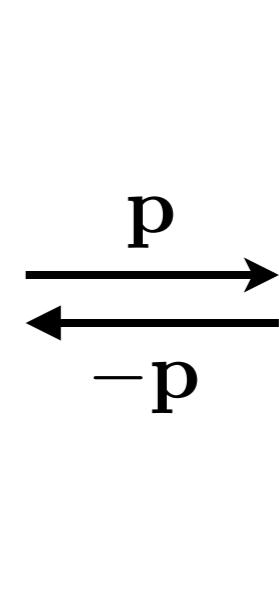
## Radiation Pressure (due to reflection)

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Consider a reflecting enclosure containing an **isotropic** radiation field.

Each photon transfers twice its normal component of momentum on reflection. Thus, we have

$$\begin{aligned}
 p_\nu &= \frac{2}{c} \int I_\nu \cos^2 \theta d\Omega \\
 &= \frac{2}{c} J_\nu \int \cos^2 \theta d\Omega \\
 &= \frac{4\pi}{c} J_\nu \int_0^1 \mu^2 d\mu \\
 &= \frac{1}{3} u_\nu
 \end{aligned}$$



$$\Delta p = 2p$$

The angular integration yields  $P_\nu = \frac{1}{3} u_\nu$

# Radiation Pressure (due to scattering)

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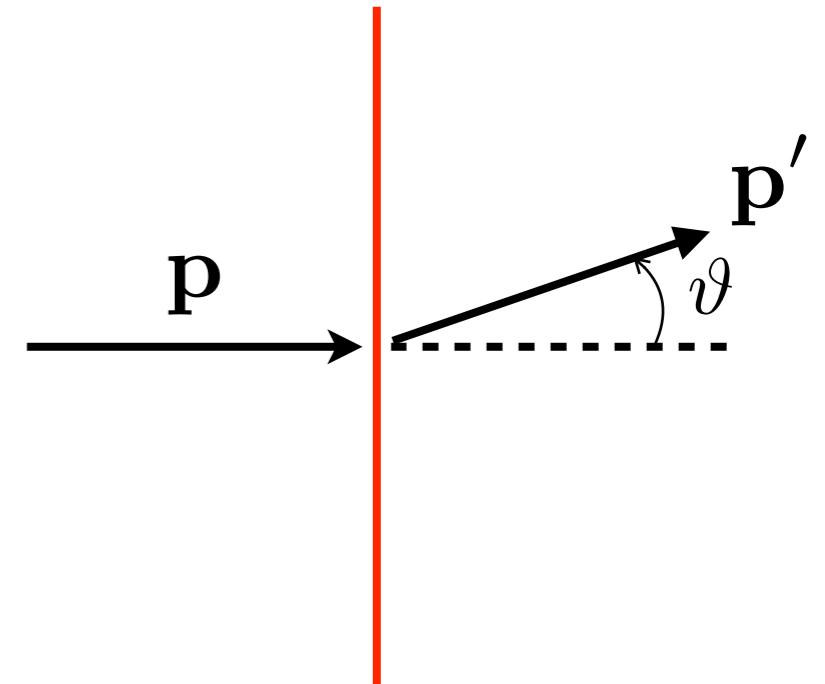
Consider a scattering of an isotropic radiation field.

Let's assume an isotropic scattering phase function:

$$\int \mathcal{P}(\vartheta) \sin \vartheta d\vartheta = 1 \rightarrow \mathcal{P}(\vartheta) = \frac{1}{2} \quad (0 \leq \vartheta \leq 2\pi)$$

Here,  $\vartheta$  is a scattering angle

$$\begin{aligned} P_\nu &= \frac{1}{c} \int I_\nu \left[ \int (1 - \cos \vartheta) \mathcal{P}(\vartheta) \sin \vartheta d\vartheta \right] \cos^2 \theta d\Omega \\ &= \frac{1}{c} \int I_\nu \cos^2 \theta d\Omega \end{aligned}$$



$$\Delta p = p - p \cos \vartheta$$

The angular integration yields  $P_\nu = \frac{1}{6} u_\nu$

# Radiative Transfer Equation in free space

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- Recall ***the constancy of intensity***:
  - the specific intensity remains the same as radiation propagates through free space.

$$I_1 = I_2$$

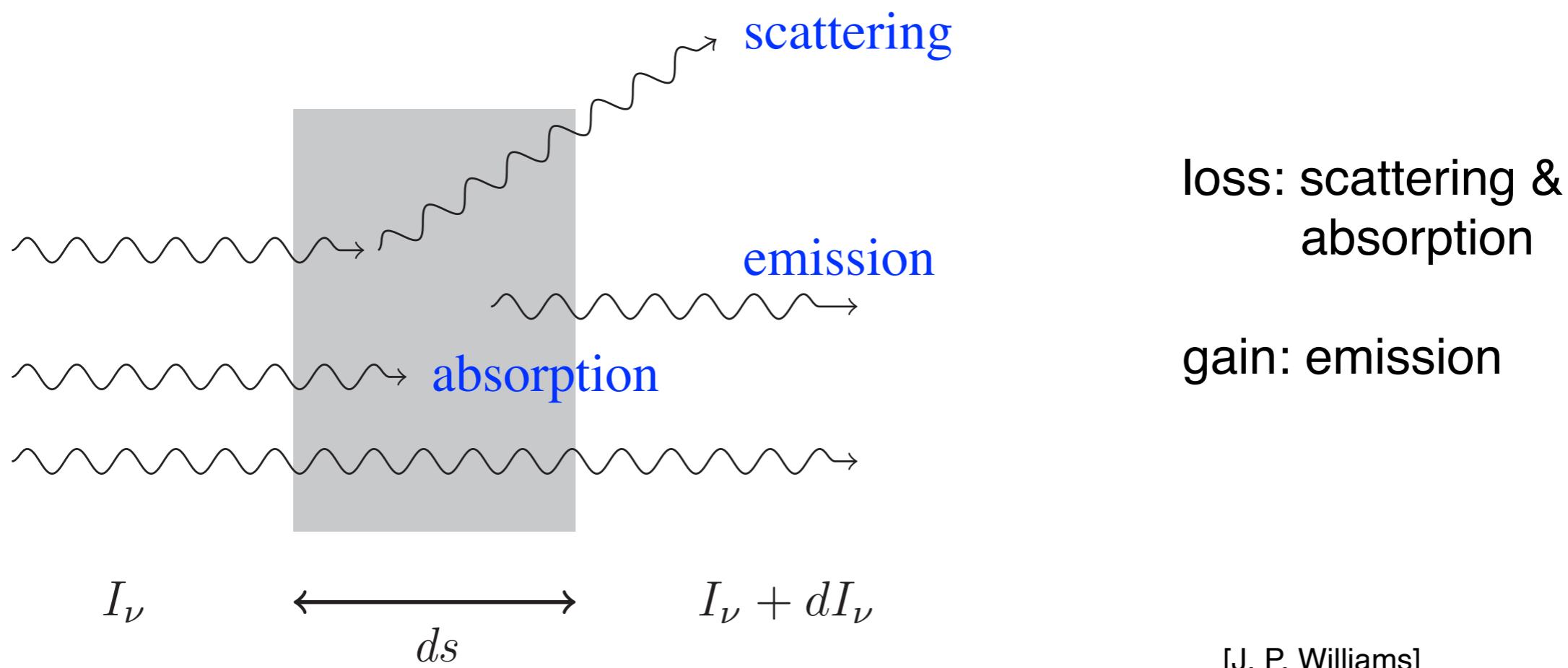
- Radiative Transfer Equation in Free Space: If we measure the distance along a ray by variable  $s$ , we can express the result equivalently in differential form:

$$\frac{dI}{ds} = 0$$

radiative transfer equation in free space

# Radiative Transfer Equation in reality

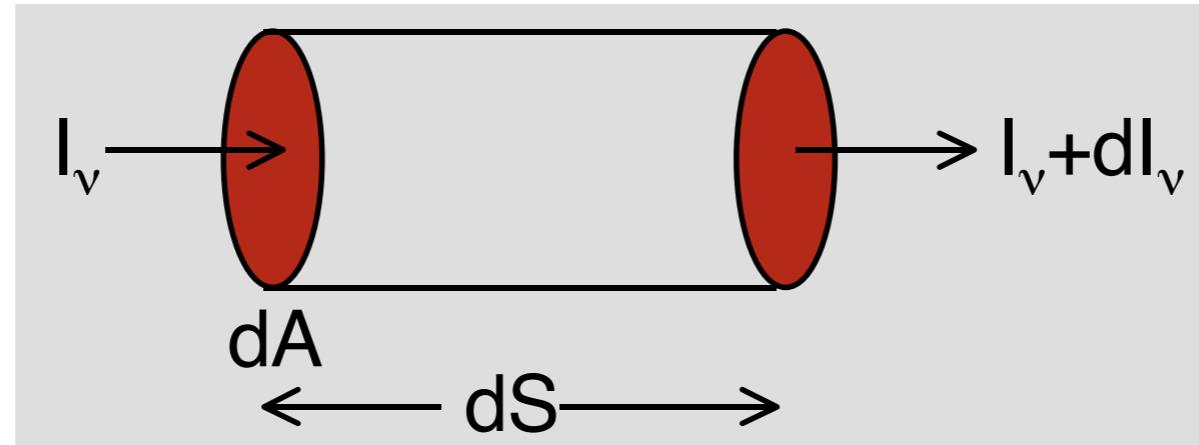
- In reality, as a ray passes through matter, energy may be added, subtracted, or scattered from it by emission, absorption, or scattering.
- The intensity will not in general remain constant.
- These interactions are described by the ***radiative transfer equation***.



# Emission

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- If the radiation travels through a medium which is itself emitting radiation, that will add to the energy:



- Spontaneous “**emission coefficient**” or “**emissivity**”  $j_\nu$  is the amount of energy emitted per unit time, per unit solid angle, per unit frequency, and per unit volume:

$$dE = j_\nu dV d\Omega dt d\nu \quad (j_\nu : \text{erg cm}^{-3} \text{ s}^{-1} \text{ sr}^{-1} \text{ Hz}^{-1})$$

- In going a distance  $ds$ , a beam of cross section  $dA$  travels through a volume  $dV = dA ds$ . Thus the intensity added to the beam is by  $ds$  is

$$dI_\nu = j_\nu ds \qquad \longleftrightarrow \qquad dE = (dI_\nu) dA d\Omega dt d\nu$$

- Therefore, the equation of radiative transfer for pure emission becomes:

$$\frac{dI_\nu}{ds} = j_\nu$$

- If we know what  $j_\nu$  is, we can integrate this equation to find the change in specific intensity as radiation propagates through the medium:

$$I_\nu(s) = I_\nu(0) + \int_0^s j_\nu(s') ds'$$

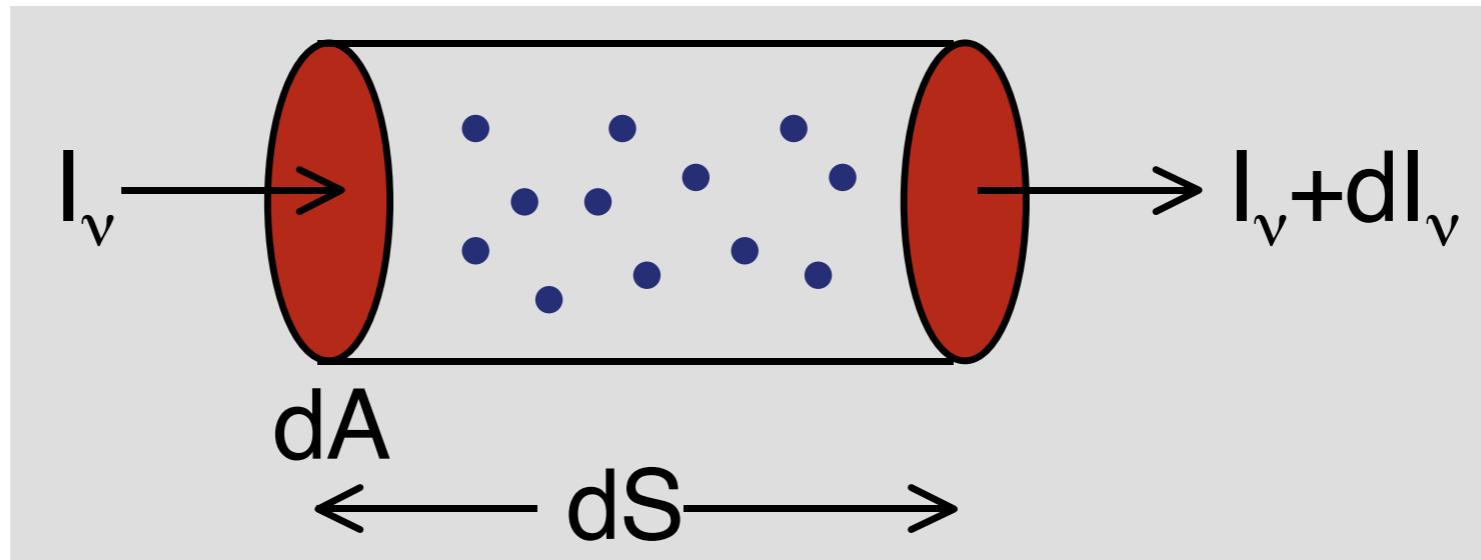
- (angle integrated) **emissivity**  $\epsilon_\nu$  = the energy emitted spontaneously per unit frequency per unit time per unit mass. For isotropic emission,

$$dE = \epsilon_\nu \rho dV dt d\nu \frac{d\Omega}{4\pi}$$

- Then, we obtain  $j_\nu = \frac{\epsilon_\nu \rho}{4\pi}$  or  $\int j_\nu d\Omega = \epsilon_\nu \rho$

# Absorption

- If the radiation travels through a medium which absorbs radiation, the energy in the beam will be reduced:



- Let  $n$  denote the number density of absorbers (particles per unit volume).
- Assume that each absorber has a cross-sectional area of  $\sigma_\nu$  (in units of  $\text{cm}^2$ ).
- number of absorbers =  $ndAds$
- If a beam travels through  $ds$ , total area of absorbers is

$$\text{number of absorbers} \times \text{cross section} = (n \times dA \times ds) \times \sigma_\nu$$

Fraction of radiation absorbed = fraction of area blocked:

$$\frac{dI_\nu}{I_\nu} = - \frac{ndAds\sigma_\nu}{dA} = - n\sigma_\nu ds \quad \longrightarrow \quad \frac{dI_\nu}{ds} = - \alpha_\nu I_\nu$$

$$dI_\nu = - n\sigma_\nu I_\nu ds \equiv - \alpha_\nu I_\nu ds$$

- **Absorption coefficient** is defined as  $\alpha_\nu \equiv n\sigma_\nu$  (units:  $\text{cm}^{-1}$ ), meaning the ***total cross-sectional area per unit volume***.

$$\alpha_\nu = n\sigma_\nu \quad [\text{cm}^{-1}]$$

$$= \rho\kappa_\nu$$

where  $\rho$  is the mass density and  $\kappa_\nu$  is called the **mass absorption coefficient** or the opacity coefficient.

- If we include the effect of stimulated emission in the absorption coefficient, it may be referred to as the **attenuation coefficient**. (as in Draine's book)

# The Radiative Transfer Equation

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Without scattering term,

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

Including scattering term, we obtain an integrodifferential equation.

$$\Omega \cdot \nabla I_\nu = -\alpha_\nu^{\text{ext}} I_\nu + j_\nu + \alpha_\nu^{\text{sca}} \int \phi_\nu(\Omega, \Omega') I_\nu(\Omega') d\Omega'$$

- scattering coefficient  $\alpha_\nu^{\text{sca}}$  ( $\text{cm}^{-1}$ )
- scattering phase function  $\int \phi_\nu(\Omega, \Omega') d\Omega = 1$
- for isotropic scattering

$$\phi_\nu(\Omega, \Omega') = \frac{1}{4\pi}$$

Stimulated emission:

- We consider “absorption” to include both “true absorption” and stimulated emission, because both are proportional to the intensity of the incoming beam (unlike spontaneous emission).

## Emission Only

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For pure emission,  $\alpha_\nu = 0$

$$\frac{dI_\nu}{ds} = j_\nu \quad \rightarrow \quad I_\nu(s) = I_\nu(s_0) + \int_{s_0}^s j_\nu(s') ds'$$

- The brightness increase is equal to the emission coefficient integrated along the line of sight.

# Absorption Only

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- Pure absorption:  $j_\nu = 0$
- Rearranging the previous equation, we obtain the equation of radiative transfer for pure absorption:

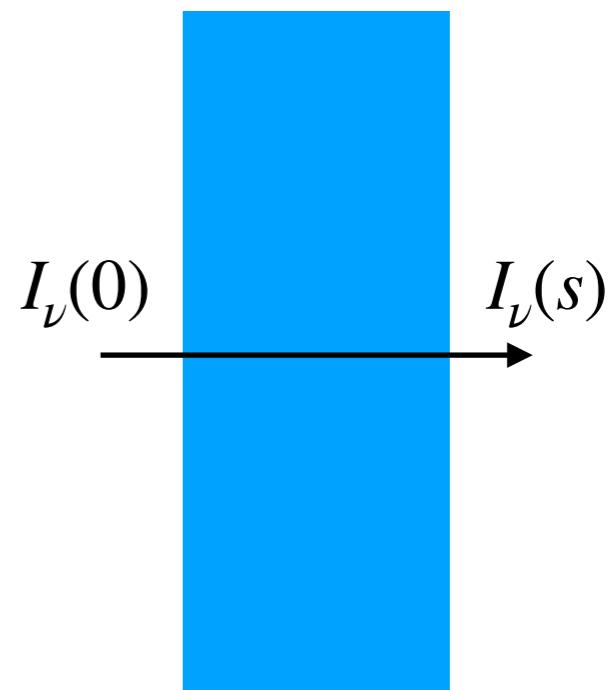
$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

- The amount of reduced energy depends on how much radiation we already have.
- Integrate to find how radiation changes along path:

$$\int_0^s \frac{dI_\nu}{I_\nu} = - \int_0^s \alpha_\nu(s') ds'$$

$$[\ln I_\nu]_0^s = - \int_0^s \alpha_\nu(s') ds'$$

$$I_\nu(s) = I_\nu(0) \exp \left[ - \int_0^s \alpha_\nu(s') ds' \right]$$



- The brightness decreases along the ray by the exponential of the absorption coefficient integrated along the line of sight.

- If the absorption coefficient is a constant (example: a uniform density gas of ionized hydrogen), then we obtain

$$I_\nu(s) = I_\nu(0)e^{-\kappa_\nu s}$$

specific intensity after distance  $s$

initial intensity at  $s = 0.$

radiation exponentially absorbed with distance

- ***Optical depth:***
  - Imagine radiation traveling into a cloud of absorbing gas, exponential defines a scale over which radiation is attenuated.
  - When  $\int_0^s \kappa_\nu(s')ds' = 1$ , the intensity will be reduced to  $1/e$  of its original value.

- We define the optical depth  $\tau_\nu$  as:

$$\tau_\nu(s) = \int_0^s \kappa_\nu(s')ds' \quad \text{or} \quad d\tau_\nu = \kappa_\nu ds$$

- A medium is said to be **optically thick** at a frequency  $\nu$  if the optical depth for a typical path through the medium satisfies:

$$\tau_\nu(s) > 1$$

- The medium is **optically thin** if, instead:

$$\tau_\nu(s) < 1$$

- An optically thin medium is one which a typical photon of frequency  $\nu$  can pass through without being (significantly) absorbed.

## Source Function

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Source function:

$$S_\nu = \frac{j_\nu}{\alpha_\nu}$$

- The radiative transfer equation can now be written

$$\frac{dI_\nu}{\alpha_\nu ds} = -I_\nu + \frac{j_\nu}{\alpha_\nu}$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

- This is an alternative and sometimes more convenient way to write the equation.

# Mean Free Path

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- From the exponential absorption law, the **probability of a photon absorbed** between optical depths  $\tau_\nu$  and  $\tau_\nu + d\tau_\nu$  is

$$|dI_\nu| = \left| \frac{dI_\nu}{d\tau_\nu} \right| d\tau_\nu \quad \& \quad |dI_\nu| \propto P(\tau_\nu) d\tau_\nu \quad \rightarrow \quad P(\tau_\nu) = e^{-\tau_\nu}$$

= probability density function for the absorption at an optical depth  $\tau_\nu$ .

- The mean optical depth traveled is thus equal to unity:

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu P(\tau_\nu) d\tau_\nu = \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1$$

- The mean free path is defined as the average distance a photon can travel through an absorbing material until it is absorbed.** In a homogeneous medium, the mean free path is determined by

$$\langle \tau_\nu \rangle = \kappa_\nu \ell_\nu = 1 \quad \rightarrow \quad \ell_\nu = \frac{1}{\kappa_\nu} = \frac{1}{n\sigma_\nu}$$

- A local mean path at a point in an inhomogeneous material can be also defined.

# Formal Solution

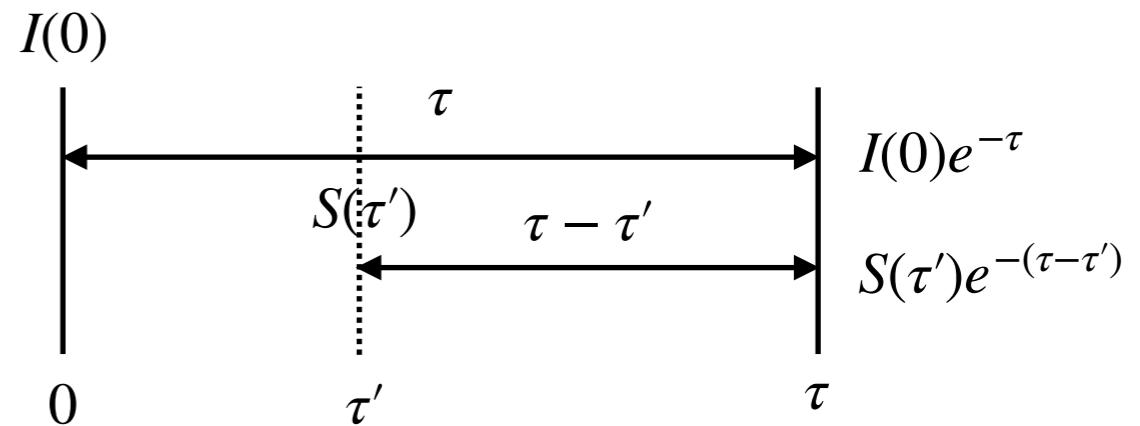
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$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$e^{\tau_\nu} \frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

$$\frac{d}{d\tau_\nu} (e^{\tau_\nu} I_\nu) = e^\tau S_\nu$$

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu$$



The solution is easily interpreted as the sum of two terms:

- the initial intensity diminished by absorption
- the integrated source diminished by absorption.

# Relaxation

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For a constant source function, the solution becomes

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu} (I_\nu(0) - S_\nu) \end{aligned}$$

## “Relaxation”

- $I_\nu > S_\nu \rightarrow \frac{dI_\nu}{d\tau_\nu} < 0$ , then  $I_\nu$  tends to decrease along the ray
- $I_\nu < S_\nu \rightarrow \frac{dI_\nu}{d\tau_\nu} > 0$ , then  $I_\nu$  tends to increase along the ray
- The source function is the quantity that the specific intensity tries to approach, and does approach if given sufficient optical depth.

As  $\tau_\nu \rightarrow \infty$ ,  $I_\nu \rightarrow S_\nu$

# Radiation Force

---

- radiation flux vector in direction  $\mathbf{n}$  :

$$\mathbf{F}_\nu = \int I_\nu \mathbf{n} d\Omega$$

- the vector momentum per unit area per unit time per unit path length absorbed by the medium is

$$\mathbf{F} = \frac{1}{c} \int \alpha_\nu \mathbf{F}_\nu d\nu \quad \leftarrow \quad n\sigma_\nu dA ds \frac{\mathbf{F}_\nu}{c}$$

- This is the force per unit volume imparted onto the medium by the radiation field. The force per unit mass of material is given by

$$\mathbf{f} = \frac{\mathbf{F}}{\rho} = \frac{1}{c} \int \kappa_\nu \mathbf{F}_\nu d\nu$$

-

# Thermal equilibrium

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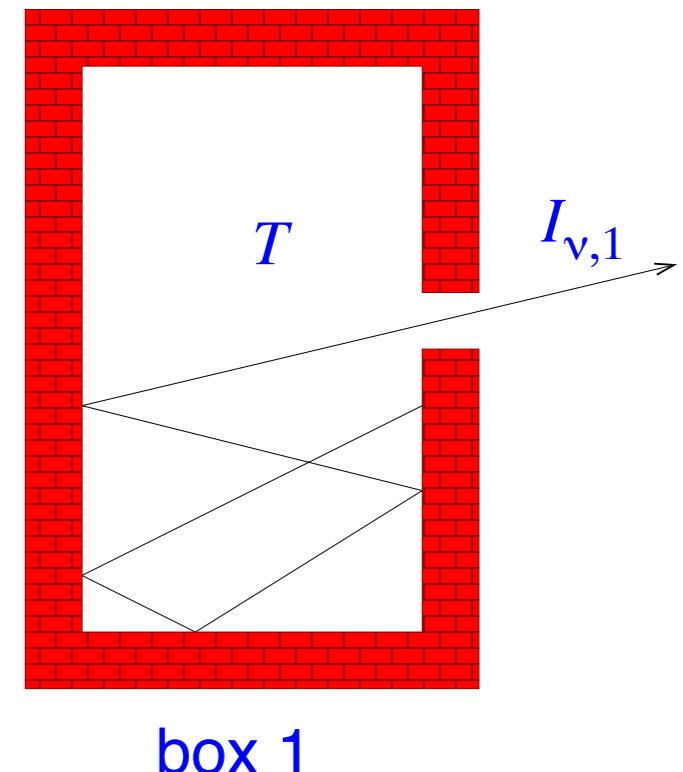
- In general, equilibrium means a state of balance.
- Thermal Equilibrium
  - ***Thermal equilibrium refers to steady states of temperature, which defines the average energy of material or photons.***
  - (for ideal gas,  $E_{\text{avg}} = \frac{3}{2}k_{\text{B}}T$ )
  - In a state of (complete) ***thermodynamic equilibrium (TE)***, no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system. ***In TE, matter and radiation are in equilibrium at the same temperature T.***
  - When the material is (locally) in thermodynamic equilibrium, and only the radiation field is allowed to depart from its TE, we refer to the state of the system as being in ***local thermodynamic equilibrium (LTE)***
  - In other words, if the **material is (locally) in thermodynamic equilibrium** at a well-defined temperature  $T$ , ***it is said to be in local thermodynamic equilibrium (LTE) even if it is not in equilibrium with the radiation field.***

# Blackbody

---

- Imagine a container bounded by opaque walls with a very small hole.

- ***Photons will be scattered and absorbed many times, (and eventually trapped and completely absorbed in the box).*** Under such conditions, the particles and photons continually share their kinetic energies. In perfect thermal equilibrium, ***the average particle kinetic energy will equal to the average photon energy, and a unique temperature T can be defined.***



- A **blackbody** is an idealized physical body that absorbs all incident radiation regardless of frequency or angle of incidence (i.e., perfect absorber). The above cavity can be regarded to be a blackbody.
- Radiation from a blackbody in thermal equilibrium is called the **blackbody radiation**.

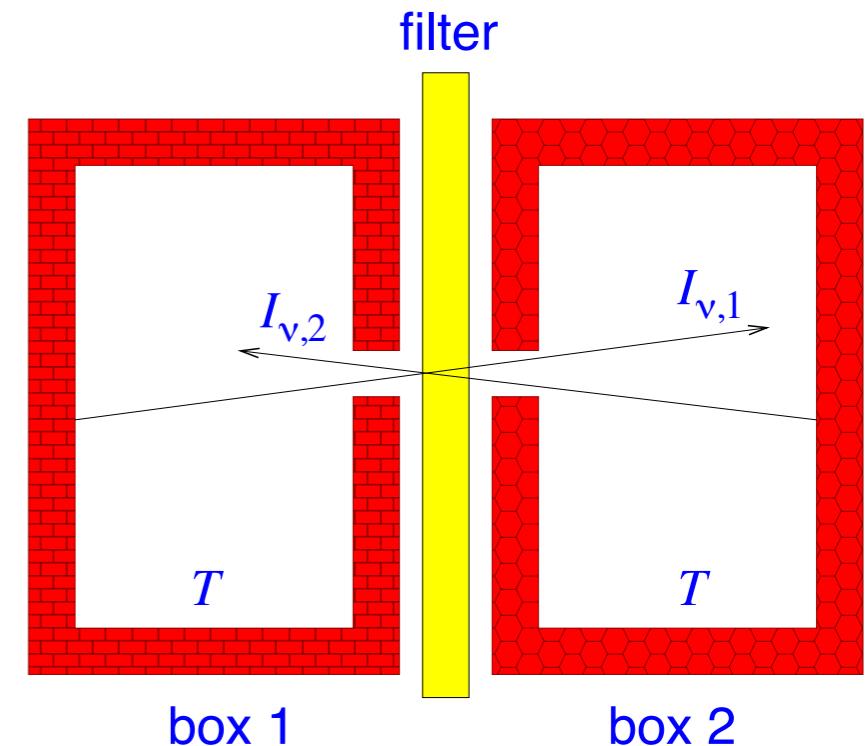
# Blackbody radiation is the universal function.

---

- Now, consider another cavity (box 2), also at the same temperature, but made of different material or shape and connect two cavities with a filter transparent only in the narrow frequency range  $\nu$  and  $\nu + d\nu$ .

- In equilibrium at  $T$ , radiation should transfer no net energy from one cavity to the other. Otherwise, one cavity will cool down and the other heats up; this violates the second law of thermodynamics.
- Therefore, the intensity or spectrum that passes through the holes should be a universal function of  $T$  and should be isotropic.
- The intensity and spectrum of the radiation emerging from the hole should be independent of the wall material (e.g., wood, copper, or concrete, etc) and any absorbing material that may be inside the cavity.

- The universal function is called the Planck function  $B_\nu(T)$ .
- This is the blackbody radiation.



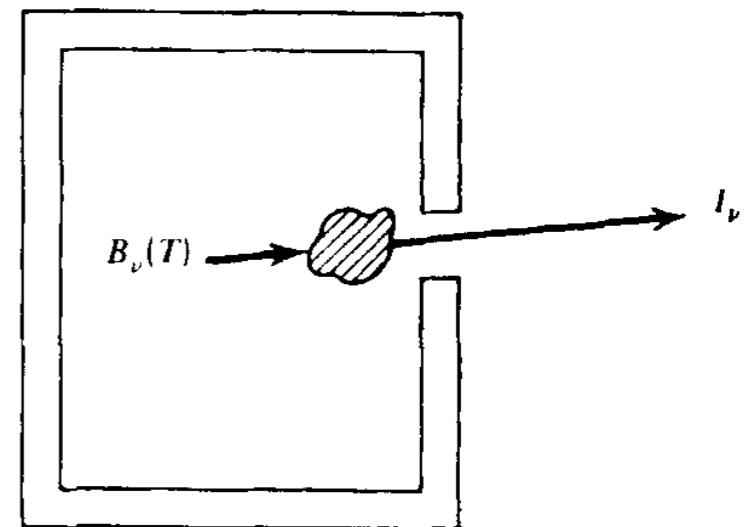
## Kirchhoff's Law in TE

- Consider an element of some thermally emitting material at temperature  $T$ .
- Put this into a blackbody enclosure at the same temperature.
- Let the source function of the material be  $S_\nu$ .

$$\text{if } S_\nu > B_\nu \rightarrow I_\nu > B_\nu$$

$$\text{if } S_\nu < B_\nu \rightarrow I_\nu < B_\nu$$

$$\therefore S_\nu = B_\nu$$



- But, the presence of the material cannot alter the radiation, since the new configuration is also a blackbody enclosure at  $T$ .
- Kirchhoff's Law: in TE, the ratio of the emission coefficient to the absorption coefficient is a function of temperature and frequency only.

$$j_\nu = \alpha_\nu B_\nu(T) \rightarrow \text{Kirchhoff's Law}$$

Note :  $j_\nu = B_\nu(T)$  if  $\alpha_\nu = 1$  (perfect absorber, i.e., blackbody)

- 
- In (full) thermodynamic equilibrium at temperature  $T$ , by definition, we know that

$$\frac{dI_\nu}{ds} = 0 \quad \text{and} \quad I_\nu = B_\nu(T)$$

We also note that

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

- Then, we can obtain ***the Kirchhoff's law for a system in TE:***

$$\frac{j_\nu}{\alpha_\nu} = B_\nu(T), \quad j_\nu = \alpha_\nu B_\nu(T)$$

- This is remarkable because it connects the properties  $j_\nu(T)$  and  $\kappa_\nu(T)$  of any kind of matter to the single universal spectrum  $B_\nu(T)$ .

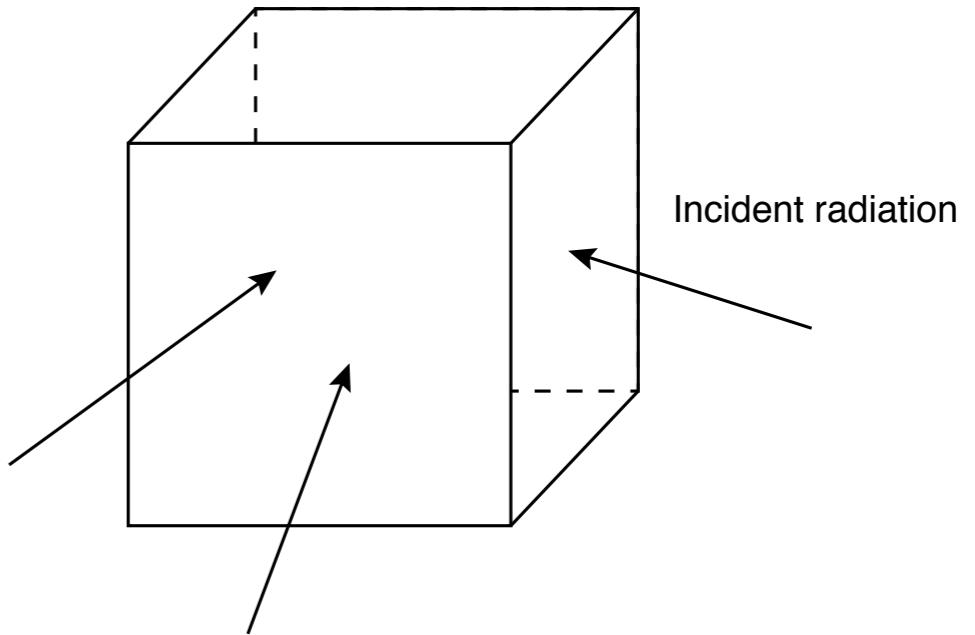
# Application of Kirchhoff's Law: Dust Emission

---

- Consider a dusty cloud with a volume  $V$  through which external radiation passes.

- (1) The total absorbed energy should be balanced by the energy emitted by dust grains.

$$\int L_\nu^{\text{abs}} d\nu = \int L_\nu^{\text{em}} d\nu$$



- (2) The energy emitted at frequency  $\nu$  can be expressed in terms of the emission coefficient:

$$L_\nu^{\text{em}} = 4\pi V j_\nu^{\text{em}}$$

- (3) Using Kirchhoff's law ( $j_\nu^{\text{em}} = \rho \kappa_\nu^{\text{abs}} B_\nu(T)$ ), we can calculate the temperature of dust:

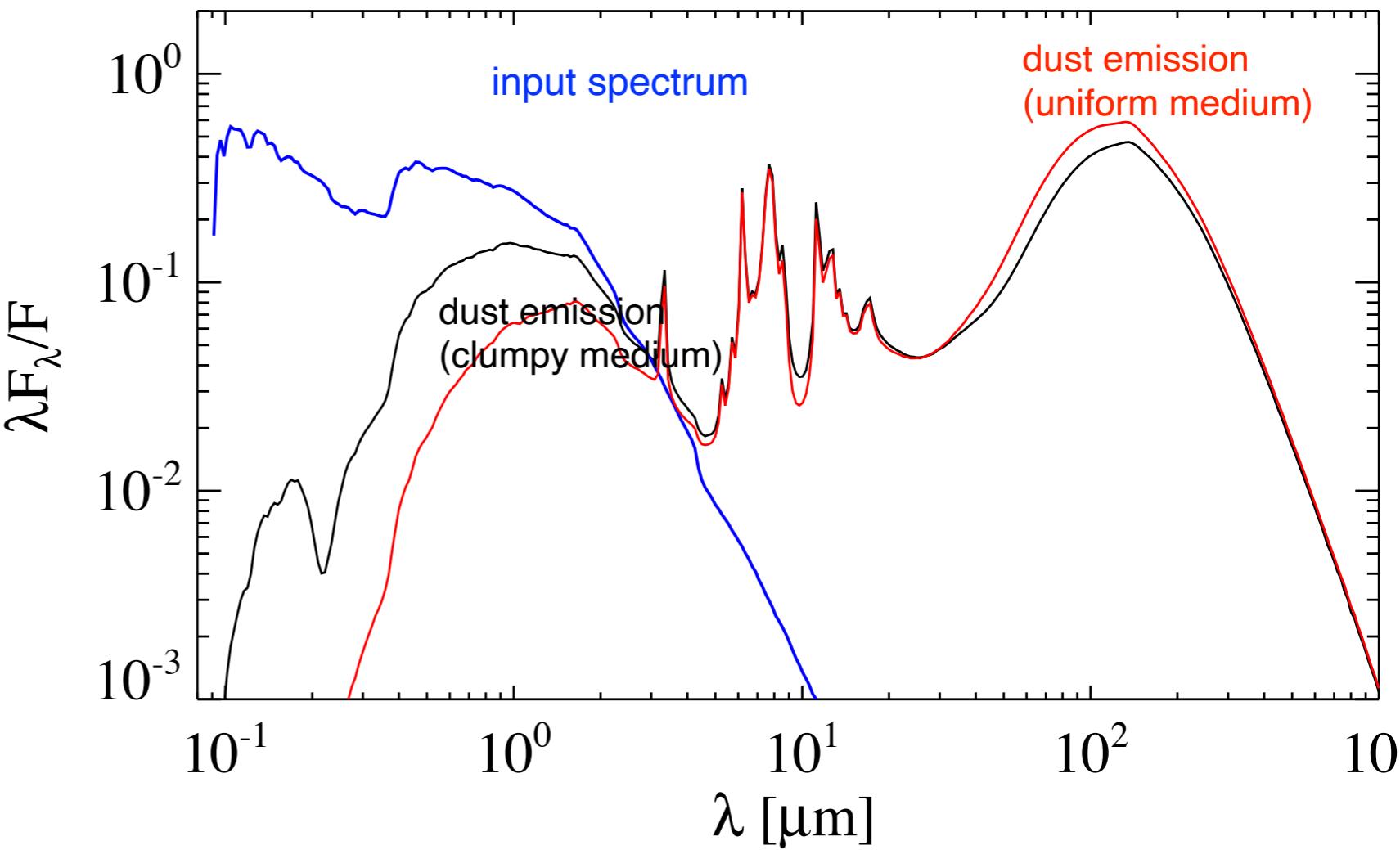
$$\int L_\nu^{\text{abs}} d\nu = 4\pi\rho V \int \kappa_\nu^{\text{abs}} B_\nu(T) d\nu$$

- (4) The emission spectrum is then obtained by

$$L_\nu^{\text{em}} = 4\pi\rho V \kappa_\nu^{\text{abs}} B_\nu(T)$$

- Spectral Energy Distribution

- Blue spectrum is the input stellar spectrum from a typical spiral galaxy.
- Dust emission was calculated using the Kirchhoff's law.
- This shows a typical SED shape of galaxies.



Please please note that  $\kappa_\nu$  is not the emissivity (nor emission coefficient), as often wrongly referred to in the literature of the external galaxies community.

$$\kappa_\nu = \kappa_0 \nu^\beta$$

$$\kappa_\lambda = \kappa_0 \lambda^{-\beta}$$

for dust absorption  
in Far-IR wavelengths

$$j_\lambda = \kappa_0 \lambda^{-\beta} B_\lambda(T)$$

is referred to as the modified blackbody.

## Kirchhoff's Law in LTE

---

- Recall that Kirchhoff's law was derived for a system in thermodynamic equilibrium.
- ***Kirchhoff's law applies not only in TE but also in LTE:***
  - Recall that  $B_\nu(T)$  is independent of the properties of the radiating /absorbing material.
  - In contrast, both  $j_\nu(T)$  and  $\kappa_\nu(T)$  depend only on the materials in the cavity and on the temperature of that material; they do not depend on the ambient radiation field or its spectrum.
  - Therefore, the Kirchhoff's law should be true even for the case of LTE.
  - ***In LTE, the ratio of the emission coefficient to the absorption coefficient is a function of temperature and frequency only.***
  - This generalized version of Kirchhoff's law is an exceptionally valuable tool for calculating the emission coefficient from the absorption coefficient or vice versa.

# Implications of Kirchhoff's Law

---

- A good absorber is a good emitter, and a poor absorber is a poor emitter. (In other words, a good reflector must be a poor absorber, and thus a poor emitter.)

$$j_\nu = \alpha_\nu B_\nu(T) \rightarrow j_\nu \text{ increases as } \alpha_\nu \text{ increases}$$

- It is not possible to thermally radiate more energy than a blackbody, at equilibrium.

$$(dI)_{\text{emiss}} = j_\nu ds = \alpha_\nu \frac{j_\nu}{\alpha_\nu} ds \leq B_\nu(T) \text{ because } \alpha_\nu ds = \frac{|dI_\nu|}{I_\nu}^{\text{abs}} \leq 1$$

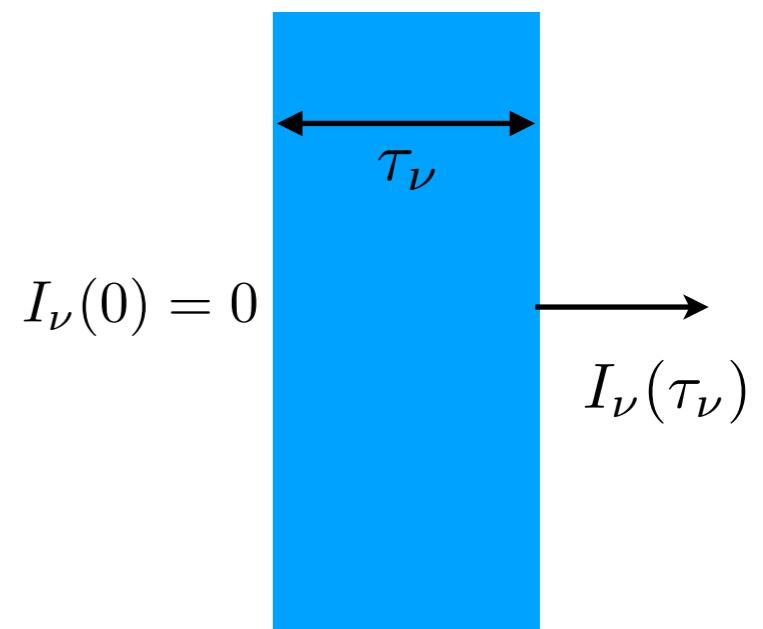
- The radiative transfer equation in LTE can be rewritten:  $\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T)$

- Remark:**

- **Blackbody radiation** means  $I_\nu = B_\nu(T)$ . An object for which the intensity is the Planck function is emitting blackbody radiation.
- **Thermal radiation is defined to be radiation emitted by “matter” in LTE.** Thermal radiation means  $S_\nu = B_\nu(T)$ . An object for which the source function is the Planck function is emitting thermal radiation.
- **Thermal radiation becomes blackbody radiation only for optically thick media.**

- To see the difference between thermal and blackbody radiation,
  - consider a slab of material with optical depth  $\tau_\nu$  that is producing thermal radiation.
  - If no light is falling on the back side of the slab, the intensity that is measured on the front side of the slab is

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu}) \\ &= B_\nu (1 - e^{-\tau_\nu}) \end{aligned}$$



- If the slab is optical thick at frequency  $\nu$  ( $\tau_\nu \gg 1$ ), then

$$I_\nu \approx B_\nu$$

- If the slab is optically thin ( $\tau_\nu \ll 1$ ), then

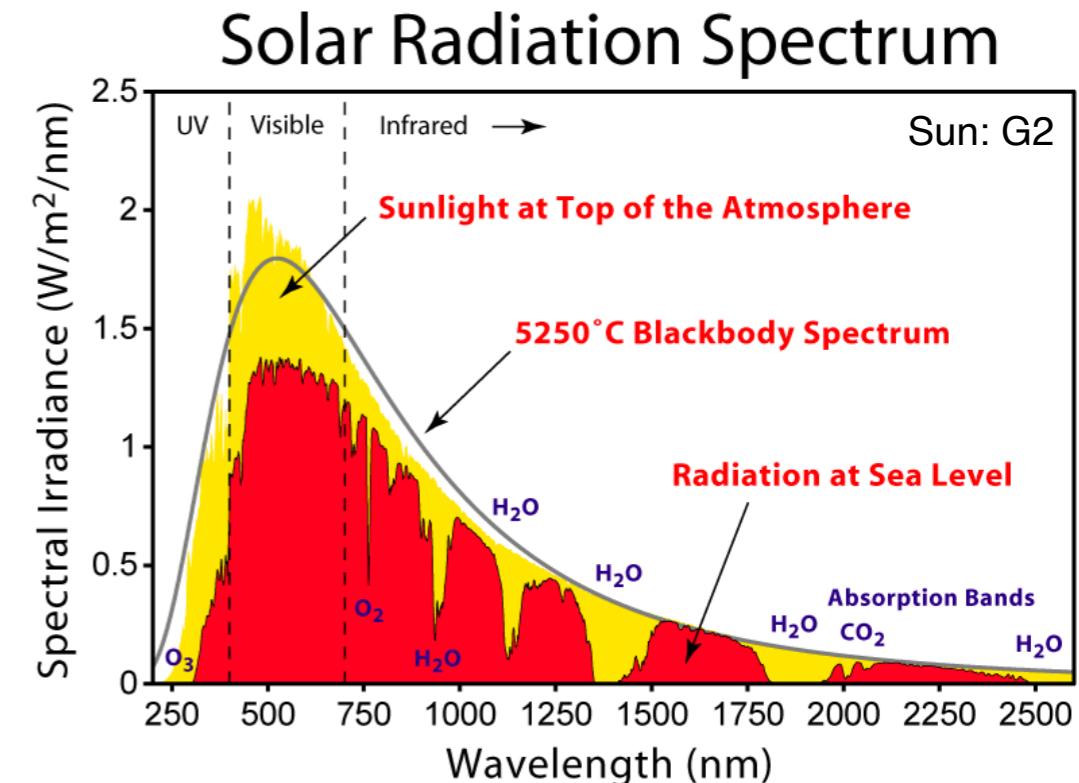
$$I_\nu \approx \tau_\nu B_\nu \ll B_\nu$$

This indicates that the radiation, although thermal, will not be blackbody.

***Thermal radiation becomes blackbody radiation only for optical thick media.***

# Spectrum of Blackbody Radiation

- There is no perfect blackbody.
  - However, the cosmic microwave background comes quite close; stars can sometimes be usefully approximated as blackbodies.
  - By the end of the 19th century, the blackbody spectrum was fairly well known empirically, from laboratory studies. In 1900, Max Planck, using his idea of quantized energies, derived the blackbody spectrum.



[https://pages.uoregon.edu/imamura/321/122/lecture-3/stellar\\_spectra.html](https://pages.uoregon.edu/imamura/321/122/lecture-3/stellar_spectra.html)

- The frequency dependence of blackbody radiation is given by the **Planck function**:

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/k_B T) - 1} \quad \text{or} \quad B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda k_B T) - 1}$$

$h = 6.63 \times 10^{-27}$  erg s (Planck's constant)

$k_B = 1.38 \times 10^{-16}$  erg K<sup>-1</sup> (Boltzmann's constant)

# Thermodynamics

---

- First Law of Thermodynamics: heat is energy in transit.

$$dQ = dU + pdV$$

where Q is heat and U is total energy.

- Second Law of Thermodynamics: heat is entropy.

$$dS = \frac{dQ}{T}$$

where S is entropy.

See “Fundamentals of Statistical and Thermal Physics”  
(Frederick Reif)

# Thermodynamics of Blackbody Radiation

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$$\begin{aligned}
 dS &= \frac{dU}{T} + p \frac{dV}{T} = \frac{1}{T} d(uV) + \frac{1}{3} \frac{u}{T} dV \\
 &= \frac{V}{T} \frac{du}{dT} dT + \frac{u}{T} dV + \frac{1}{3} \frac{u}{T} dV \\
 &= \frac{V}{T} \frac{du}{dT} dT + \frac{4}{3} \frac{u}{T} dV
 \end{aligned}$$

$$\begin{aligned}
 \rightarrow \quad \frac{\partial S}{\partial T} &= \frac{V}{T} \frac{du}{dT} \quad \text{and} \quad \frac{\partial S}{\partial V} = \frac{4u}{3T} \\
 \rightarrow \quad \frac{\partial^2 S}{\partial T \partial V} &= \frac{1}{T} \frac{u}{T} = -\frac{4u}{3T^2} + \frac{4}{3T} \frac{du}{dT} \\
 \therefore \frac{du}{dT} &= \frac{4u}{T} \quad \text{or} \quad \frac{du}{u} = 4 \frac{dT}{T}
 \end{aligned}$$

- Stefan-Boltzmann law:

Stenfan – Boltzmann law :  $u(T) = aT^4 \leftarrow \log u = 4 \log T + \log a$

$$u(T) = \left( \frac{T}{3400 \text{ K}} \right)^4 \text{ erg cm}^{-3}$$

- total energy density:

$$u = \frac{4\pi}{c} \int B_\nu(T) d\nu = \frac{4\pi}{c} B(T)$$

- the integrated Planck function

$$B(T) = \int B_\nu(T) d\nu = \frac{ac}{4\pi} T^4 = \frac{\sigma}{\pi} T^4$$

- emergent flux (another form of the Stefan-Boltzmann law)

$$\begin{aligned} F &= \int F_\nu d\nu = \pi \int B_\nu d\nu = \pi B(T) \\ &= \sigma T^4 \end{aligned}$$

# Entropy of Blackbody Radiation

---

- Entropy:

$$dS = \frac{V}{T} 4aT^3 dT + \frac{4a}{3} T^3 dV \rightarrow S = \frac{4}{3} aT^3 V$$

- Entropy density:

$$s = S/V = \frac{4}{3} T^3$$

- The law of adiabatic expansion for blackbody radiation:

$$T_{\text{ad}} \propto V^{-1/3}$$

$$p_{\text{ad}} \propto T_{\text{ad}}^4 \propto V^{-4/3}$$

Thus, we have the adiabatic index for blackbody radiation:

$$\gamma = \frac{4}{3} \leftarrow pV^\gamma = \text{constant}$$

# The Planck Spectrum

---

## Density of photon state:

- Consider a photon propagating in direction  $\mathbf{n}$  inside a box with dimensions  $L_x, L_y, L_z$  in x, y, z directions.
- wave vector:  $\mathbf{k} = \frac{2\pi}{\lambda} \mathbf{n} = \frac{2\pi\nu}{c} \mathbf{n}$
- If each dimension of the box is much longer than a wavelength, the photon can be represented by standing wave in the box.
- number of nodes in each direction:  $n_x = k_x L_x / 2\pi$
- number of node changes in a wave number interval (if  $n_i \gg 1$ ):

$$\Delta n_x = \frac{L_x \Delta k_x}{2\pi}$$

- number of stats in 3D wave vector element  $\Delta k_x \Delta k_y \Delta k_z = d^3 k$ :

$$\Delta N = \Delta n_x \Delta n_y \Delta n_z = 2 \frac{L_x L_y L_z d^3 k}{(2\pi)^3} = 2 \frac{V d^3 k}{(2\pi)^3}$$



two independent polarizations

- 
- the density of states (number of states per solid angle per volume per frequency):

$$d^3k = k^2 dk d\Omega = \frac{(2\pi)^3 \nu^2 d\nu d\Omega}{c^3}$$

$$\rightarrow \rho_s = \frac{dN}{V d\nu d\Omega} = \frac{2\nu^2}{c^3}$$

### Average energy of each state:

- Each state may contain  $n$  photons of energy  $h\nu$ . The energy of the state is  $E_n = nh\nu$
- The probability of a state of energy  $E_n$  is proportional to  $e^{-\beta E_n}$ , where  $\beta = (k_B T)^{-1}$  and  $k_B$  is the Boltzmann's constant.
- Therefore, the average energy is:

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = -\frac{\partial}{\partial \beta} \ln \left( \sum_{n=0}^{\infty} e^{-\beta E_n} \right)$$

$$\sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} (e^{-\beta h\nu})^n = (1 - e^{-\beta h\nu})^{-1}$$

$$\langle E \rangle = \frac{h\nu e^{-\beta h\nu}}{1 - e^{-\beta h\nu}} = \frac{h\nu}{\exp(h\nu/k_B T) - 1}$$

**Average number of photons (occupation number):**

$$n_\nu = \langle E \rangle / h\nu = \frac{1}{\exp(h\nu/k_B T) - 1} \rightarrow \text{Bose-Einstein statistics}$$

**Energy density:**

$$u_\nu(\Omega) = \rho_s \langle E \rangle = \frac{2h\nu^3/c^3}{\exp(h\nu/k_B T) - 1}$$

**Planck Law:**

$$B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/k_B T) - 1} \text{ or } B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda k_B T) - 1}$$

# Stefan-Boltzmann constant & Riemann zeta function

---

$$\begin{aligned}
 \text{Bose integral : } I_n &= \int_0^\infty dx \frac{x^n}{e^x - 1} = \int_0^\infty dx x^n \sum_{i=0}^\infty e^{-(i+1)x} \\
 &= \sum_{i=0}^\infty \frac{1}{(i+1)^{n+1}} \int_0^\infty dy y^n e^{-y} \quad (y \equiv (i+1)x) \\
 &= \zeta(n+1)\Gamma(n+1)
 \end{aligned}$$

$$\begin{aligned}
 \int_0^\infty B_\nu(T) d\nu &= (2h/c^2)(k_B T/h)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \\
 &= \frac{2k_B^4 T^4}{c^2 h^3} \zeta(4)\Gamma(4) = \frac{2k_B^4 T^4}{c^2 h^3} \frac{\pi^4}{90} 6 \\
 &= \frac{2\pi^4 k_B^4}{15 c^2 h^3} T^4
 \end{aligned}$$

$$\begin{aligned}
 \therefore \sigma &= \frac{2\pi^5 k_B^5}{15 c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{ deg}^{-4} \text{ s}^{-1} \\
 a &= \frac{8\pi^5 k_B^4}{15 c^3 h^3} = 7.56 \times 10^{-15} \text{ erg cm}^{-3} \text{ deg}^{-4}
 \end{aligned}$$

# Rayleigh-Jeans Law & Wien Law

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## Rayleigh-Jeans Law

$$h\nu \ll k_B T \ (\nu \ll 2 \times 10^{10} \text{Hz}(T/1\text{K})) \rightarrow I_\nu^{RJ}(T) = \frac{2\nu^2}{c^2} k_B T$$

- Originally derived by assuming the classical equipartition energy

$$\langle E \rangle = 2 \times (1/2)k_B T$$

- ultraviolet catastrophe: if the equation is applied to all frequencies, the total amount of energy would diverge.

$$\int \nu^2 d\nu \rightarrow \infty$$

## Wien Law

$$h\nu \gg k_B T \rightarrow I_\nu^W(T) = \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{k_B T}\right)$$

# Wien Displacement Law

---

- Frequency at which the peak occurs:

$$\frac{\partial B_\nu}{\partial \nu} \Big|_{\nu=\nu_{\max}} = 0 \quad \rightarrow \quad x = 3(1 - e^{-x}), \text{ where } x = h\nu_{\max}/k_B T$$

$$h\nu_{\max} = 2.82k_B T \quad \text{or} \quad \frac{\nu_{\max}}{T} = 5.88 \times 10^{10} \text{ Hz deg}^{-1}$$

- Wavelength at which the peak occurs:

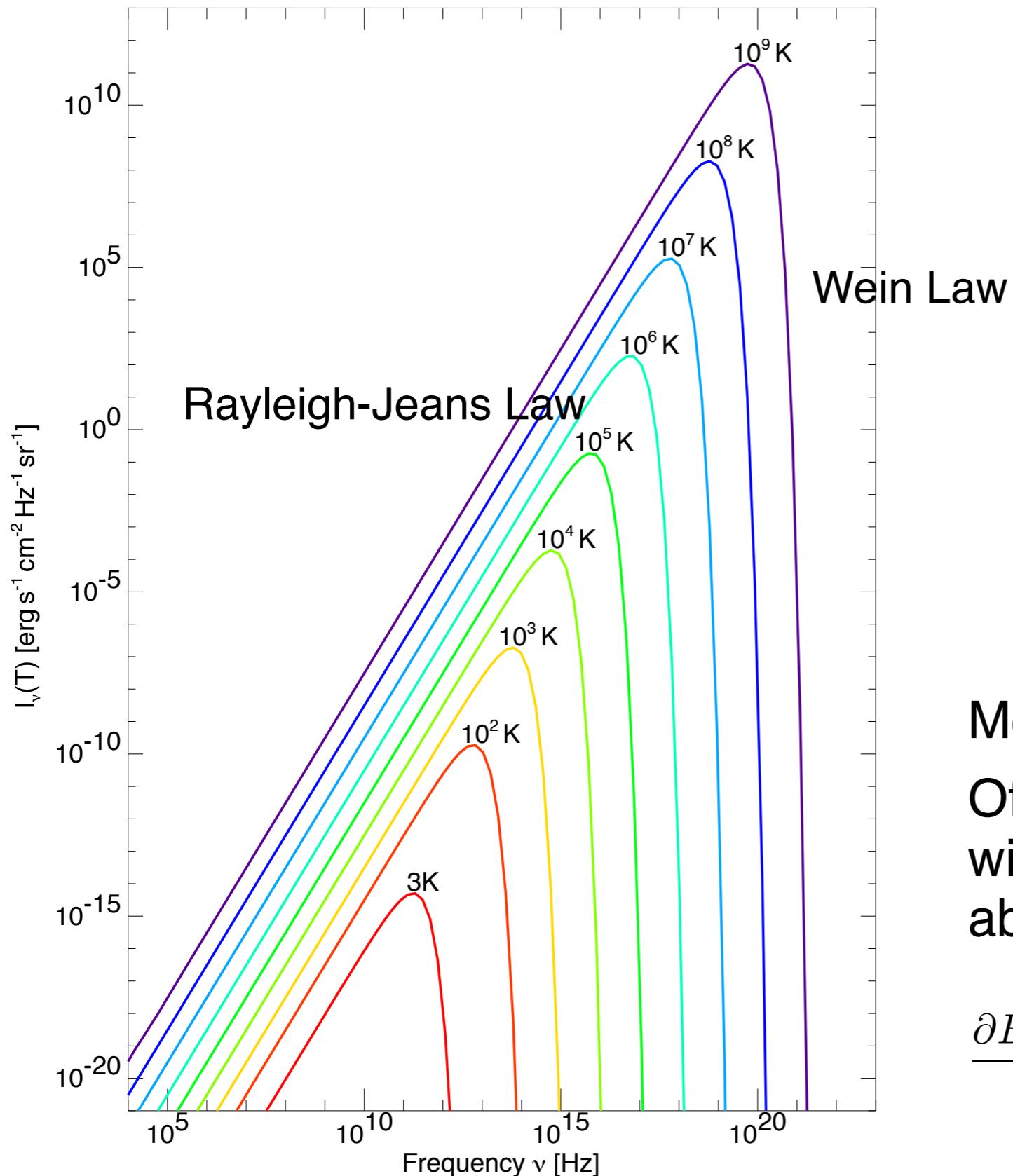
$$\frac{\partial B_\lambda}{\partial \lambda} \Big|_{\lambda=\lambda_{\max}} = 0 \quad \rightarrow \quad y = 5(1 - e^{-y}), \text{ where } y = hc/(\lambda_{\max} k_B T)$$

$$y = 4.97 \quad \text{and} \quad \lambda_{\max} T = 0.290 \text{ cm deg}$$

- Note that  $\nu_{\max} \neq c/\lambda_{\max}$

# Monotonicity with Temperature

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Wein Law

Rayleigh-Jeans Law

**Monotonicity:**  
Of two blackbody curves, the one with higher temperature lies entirely above the other.

$$\frac{\partial B_\nu(T)}{\partial T} = \frac{2h^2\nu^4}{c^2k_B T^2} \frac{\exp(h\nu/k_T)}{\left[\exp(h\nu/k_B T) - 1\right]^2} > 0$$

# Characteristic Temperatures

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## **Brightness Temperature:**

$$I_\nu = B_\nu(T_b) \rightarrow T_b(\nu) = \frac{h\nu/k_B}{\ln [1 + 2h\nu^3/(c^2 I_\nu)]}$$

- The definition is used especially in radio astronomy, where the RJ law is usually applicable. In the RJ limit,

$$T_b = \frac{c^2}{2\nu^2 k_B} I_\nu$$

- radiative transfer equation in the RJ limit:

$$\frac{dT_b}{d\tau_\nu} = -T_b + T \quad (T = \text{the temperature of the material})$$

$$T_b = T_b(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu}) \quad \text{if } T \text{ is constant.}$$

- In the Wien region, the concept is not so useful.

# Characteristic Temperatures

---

- **Antenna Temperature:**

- Radio astronomers are used to working at very low frequencies. They define the antenna temperature as being the brightness temperature in the Rayleigh-Jeans tail.

$$I_\nu = \frac{2\nu^2}{c^2} k_B T_b \rightarrow T_A \equiv \frac{c^2}{2k_B \nu^2} I_\nu$$

- Radiative transfer equation in the RJ limit:
  - ▶ If the matter is in LTE and has its energy levels populated according to an excitation temperature  $T_{\text{exc}} \gg h\nu/k_B$ , then the source function is given by

$$S_\nu(T_{\text{exc}}) = (2\nu^2/c^2)k_B T_{\text{exc}}$$

- ▶ Then, RT equation becomes

$$\frac{dT_A}{d\tau_\nu} = -T_A + T_{\text{exc}} \quad \text{if } h\nu \ll k_B T_{\text{exc}}$$

$$T_A = T_A(0)e^{-\tau_\nu} + T_{\text{exc}}(1 - e^{-\tau_\nu}) \quad \text{if } T_{\text{exc}} \text{ is constant.}$$

- 
- **Color Temperature:**
    - By fitting the spectrum to a blackbody curve without regarding to vertical scale (absolute intensity scale), a color temperature  $T_c$  is obtained.
    - The color temperature correctly gives the temperature of a blackbody source of unknown absolute scale.
  - **Effective Temperature:**
    - The effective temperature of a source is obtained by equating the actual flux  $F$  to the flux of a blackbody at temperature  $T_{\text{eff}}$ .

$$F = \int \int I_\nu \cos \theta d\nu d\Omega = \sigma T_{\text{eff}}^4$$

- **Excitation Temperature:**
  - The excitation temperature of level  $u$  relative to level  $\ell$  is defined by

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{E_{u\ell}}{k_B T_{\text{exc}}}\right) \rightarrow T_{\text{exc}} \equiv \frac{E_{u\ell}/k_B}{\ln\left(\frac{n_\ell/g_\ell}{n_u/g_u}\right)} \quad (E_{u\ell} \equiv E_u - E_\ell)$$

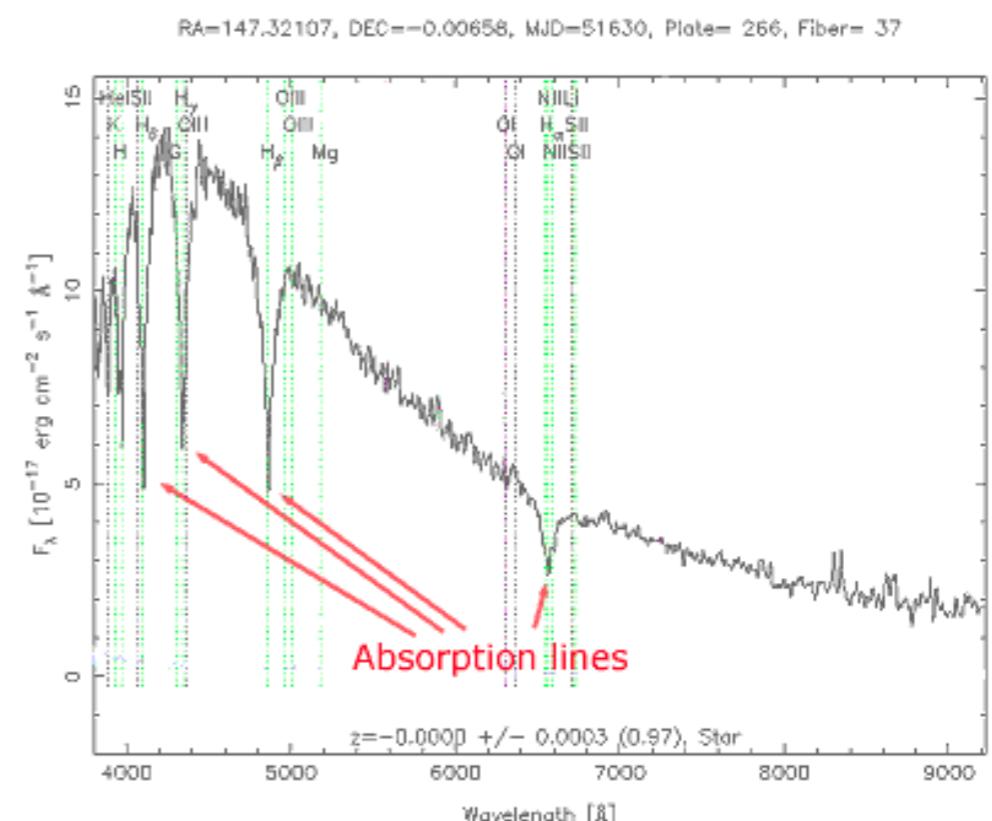
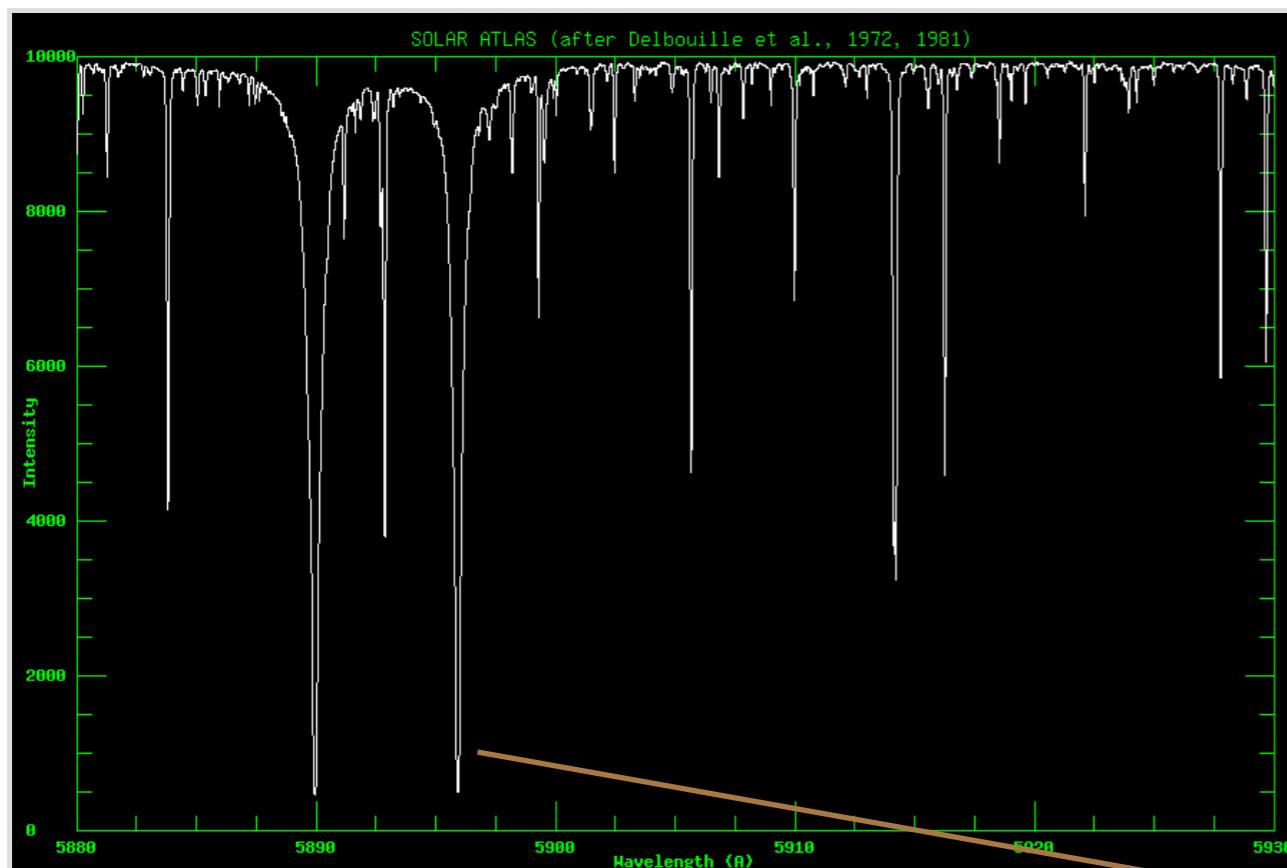
- Radio astronomers studying the 21 cm line sometimes use the term “**spin temperature**”  $T_{\text{spin}}$  for excitation temperature.

# [Homework] Absorption line and emission line spectra

- Temperature of the Solar photosphere is  $\sim 6000$  K. Lots of spectral lines of different elements are observed.

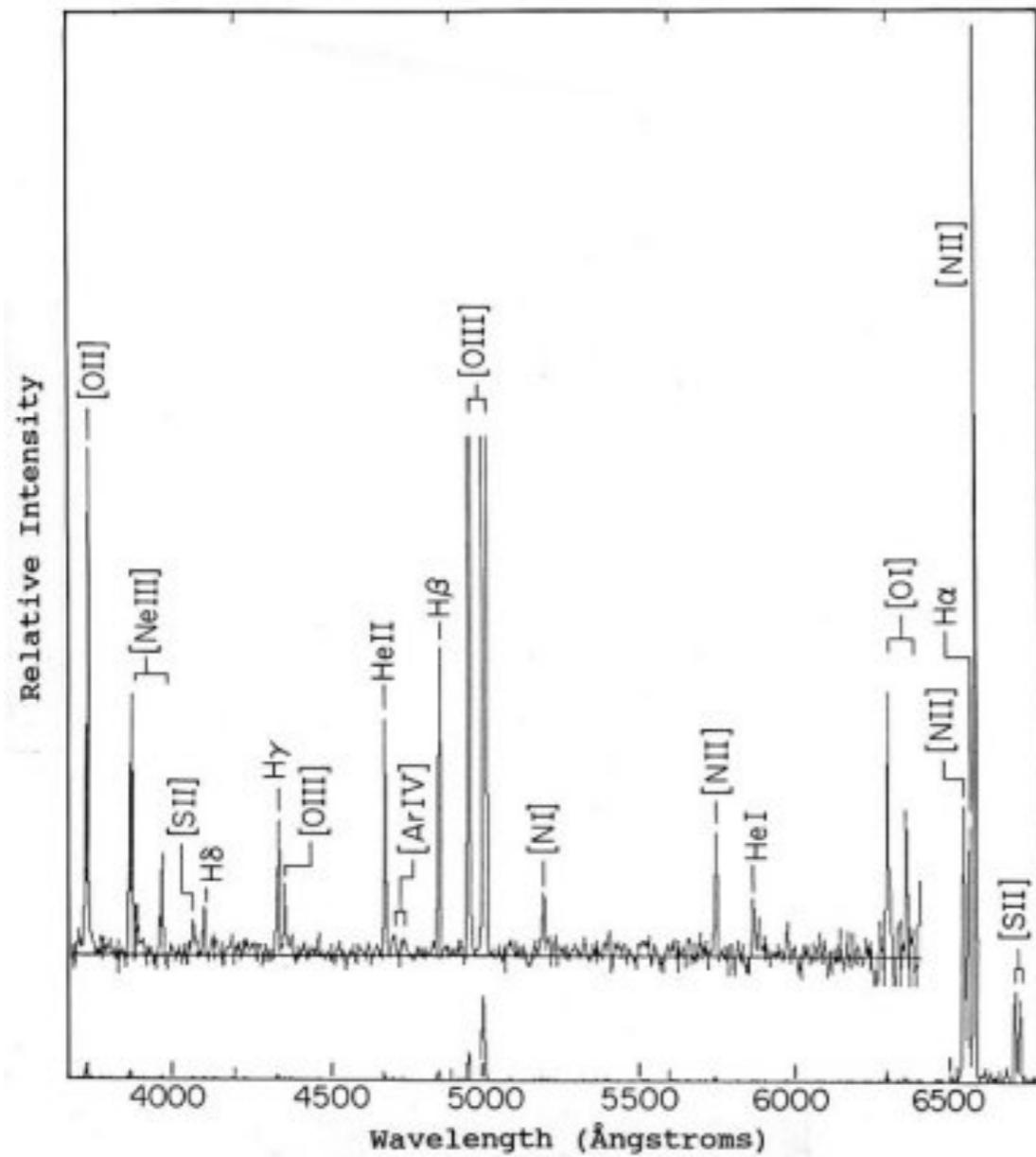


- Optical spectrum of stars is an absorption line spectrum*** - see dark absorption lines superimposed on a bright continuum.



Two strong absorption lines are Na I D lines due to sodium.

- However, ***emission nebulae typically show emission line spectra:***  
(Spectral lines are stronger than the continuum.)



Spectrum from an emission nebula

#### Homework:

- Explain why this difference happens?
- Deadline: Sept. 21 (Wed.)

# Hint for homework

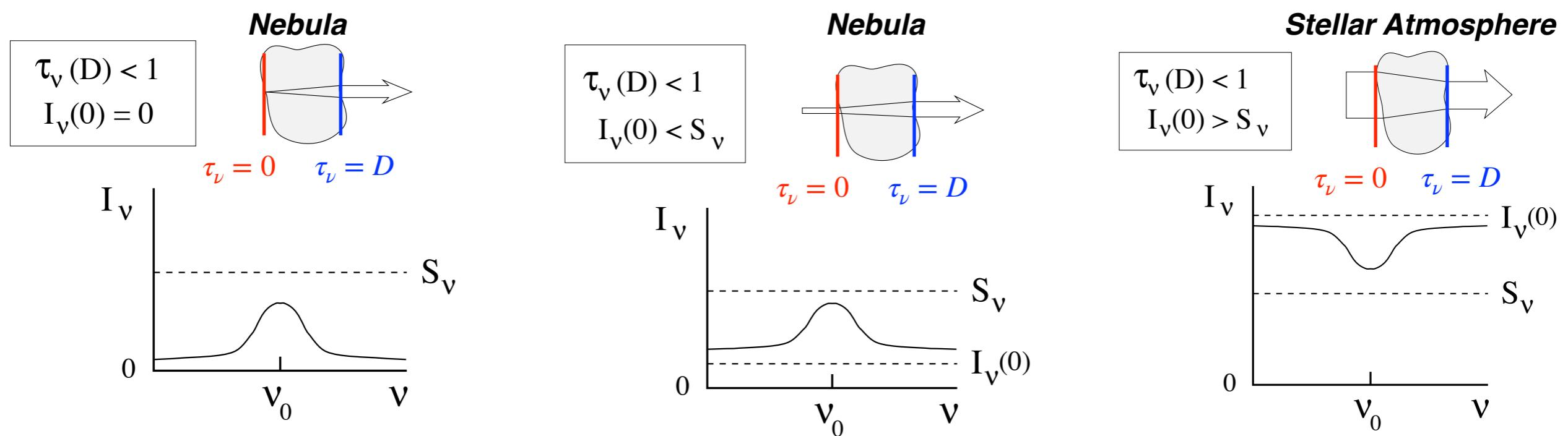
- Recall the solution for RT equation when the source function is constant.

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

- Assume optically thin regions,  $\tau_\nu \ll 1$  and show that the above equation becomes

$$I_\nu(D) \approx I_\nu(0) + \tau_\nu(D)(S_\nu - I_\nu(0)) \quad \text{at } \tau_\nu = D$$

- See the following three figures, and explain why some objects show absorption line spectra, but some show emission line spectra.
- Note that  $T(\tau_0 = 0) > T(\tau_0 = D)$  for the case of the stellar atmosphere.



## [Homework2] Eddington Luminosity

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- Home work:
  - (1) Thomson scattering is an elastic scattering of a photon by a charged particle (i.e., electrons) in the non-relativistic regime.
  - (2) Read the following website:  
<http://www-ppl.s.chiba-u.jp/lecture/radiation/node2.html>  
Explain in your own language what the Eddington Luminosity is.