

# Astrophysics

Lecture 15

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# Molecular Structure

## [Introduction]

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- The Make-Up of Molecules and Solids

Atoms can be combined to form molecules or solids.

The properties of these compounds depend greatly on the way their constitutive atoms are bonded together, by their electrons.

- Molecular Bonding

Chemical bonds are the result of the overlap between the outer orbitals of two atoms whose valence shell is not full.

Despite their mutual repulsion, sharing electrons leads to a lower energy state, in which a stable bonded molecule is formed.

The atomic spacing in a molecule or a solid is of the order of a few Å.

- Valence shell

The outer shell is called the valence shell. It contains the electrons responsible for molecular bonds and shaping the optical properties of solids.

The chemical bond depends on the tendency of its atoms: (i) to share electrons; (ii) to form cations, by losing one or several electrons; or (iii) to form anions, by gaining one or several electrons.

- So far, about 200 molecules have been detected in the interstellar and circumstellar medium by direct observation of their spectra. These molecules are ones which contain up to 13 atoms.

- Molecules formed from two atoms are called diatomics while those formed from more than two atoms are described as polyatomic.
- Because molecular binding energies are relatively small, i.e., generally less than ionization energies, molecules are only found in cooler, or less active, astronomical environments.

Polyatomic molecules are only a significant component of matter at temperatures below about 4000 K. Diatomic systems can survive to somewhat higher temperatures and may be found in environments with temperatures up to about 8000 K.

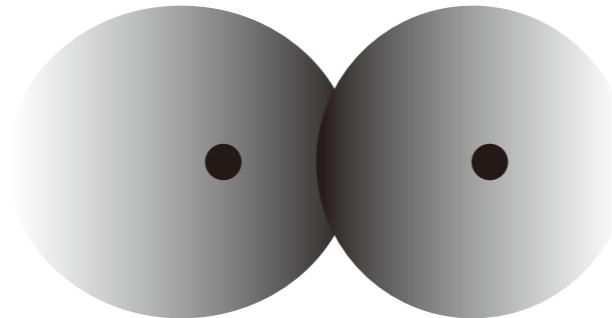
- The Structure, and hence the spectra, of molecules are more complicated than atoms in two ways:
  - There is no single charge center about which electrons move. The electronic wavefunctions therefore have lower symmetry, making them harder to calculate and harder to work with.
  - The nuclei themselves move, giving rise to both rotational and vibrational motions of the atoms within the molecule. These motions give rise to discrete spectra.

# [Born-Oppenheimer Approximation]

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- **Born-Oppenheimer approximation:**

- To a very good approximation, the motions of the electrons and nuclei could be treated separately. The electrons move much faster than the nuclei.
- *This come about because of the great difference between the masses of the electron and a typical nuclei.*
- The slowly moving nuclei only sense the electrons as a kind of smoothed-out cloud. As the nuclei move the electrons have sufficient time to adjust to adiabatically the new nuclear positions. (***It is like flies buzzing round an elephant - as the elephant moves the flies move with it.***) The nuclei then feel only an equivalent potential that depends on the internuclear distance and on the particular electronic state.

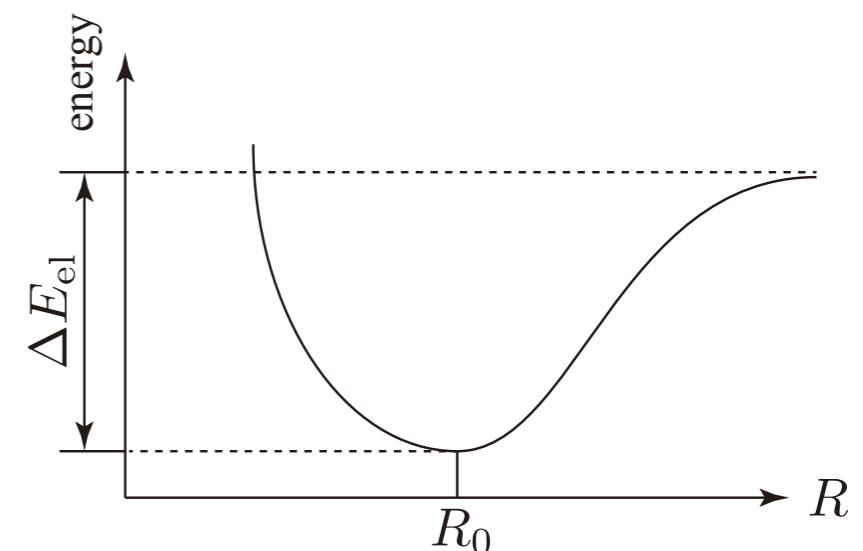


- One separates the wavefunction for the motions of electrons from the wavefunction for the motions of the nuclei. One can then consider the electronic wavefunction separately for each position of nuclei, as if the nuclei are held fixed.
- Due to very different energies of the **electronic, vibrational, and rotational states**, these interactions can be assumed to be decoupled. The separation of wavefunctions is referred to as the Born-Oppenheimer approximation. Under the Born-Oppenheimer approximation, the total wavefunction is a product of the nuclear, electronic, vibrational, and rotational wavefunctions.

$$\psi_{\text{tot}} = \psi_{\text{nuc}} \psi_{\text{el}} \psi_{\text{vib}} \psi_{\text{rot}}$$

- Order of magnitude of energy levels

- Electronic energy:



As the separation between the two atoms  $R \rightarrow 0$  (at very small  $R$ ), the overall interactions are strongly repulsive. There is repulsion due to nuclear-nuclear interaction whose potential depends on  $Z_A Z_B / R$ . There is also repulsion due to the electron-electron interactions, which also behave approximately as  $1/R$ . However, these repulsive interactions are largely cancelled by the attractive electron-nuclear interaction.

As  $R \rightarrow \infty$ , the molecule is pulled apart and it separates into atoms in a process known as dissociation. The energy of the system at dissociation is clearly just the sum of the atomic exchange.

At intermediate  $R$ , to get binding there must be some region of  $R$  where the molecular energy is less than the sum of the atomic energies. In this case, the electronic state is described as ‘attractive’ and there is a minimum in the potential energy curve.

For a diatomic molecule, a stable chemical bond can form between two atoms that approach within a distance of each other comparable to the Bohr radius  $a_0 = \hbar^2/m_e c^2$ . Then, the electron energy will be given by

$$E_{\text{elect}} \sim m_e v^2 = \frac{p^2}{m_e} \sim \frac{\hbar^2}{m_e a_0^2} \quad (\leftarrow p \times a_0 \sim \hbar, \text{ uncertainty relation}) \quad \Rightarrow \text{visible/UV (a few eV)} \\ (a \sim 1 \text{ \AA})$$

- **Vibrational energy:**

If the two nuclei are displaced from the equilibrium separation  $R_0$  by a displacement comparable to  $\xi \sim a_0$ , they will vibrate about the equilibrium position with a frequency  $\omega_{\text{vib}}$  such that the vibrational energy contained in the motion and displacements of the two nuclei (of typical mass  $M$ ) will be comparable to the depth of the electronic potential well.

$$M\omega_{\text{vib}}^2 a_0^2 \sim E_{\text{elect}}$$

where  $\omega$  = frequency of vibration.

$M$  = mass of the molecule.

Then, the vibrational energy is

$$E_{\text{vib}} \sim \hbar\omega \sim \left(\frac{m_e}{M}\right)^{1/2} \frac{\hbar^2}{m_e a^2} \sim \left(\frac{m_e}{M}\right)^{1/2} E_{\text{elect}} \quad \Rightarrow \text{Near-IR / Mid-IR}$$

- **Rotational energy:**

The nuclei can also rotate about each other. Then, the energy of rotation is

$$E_{\text{rot}} \sim \frac{L^2}{2I} \sim \frac{\hbar^2 \ell(\ell + 1)}{2I} \quad \Rightarrow \text{radio}$$

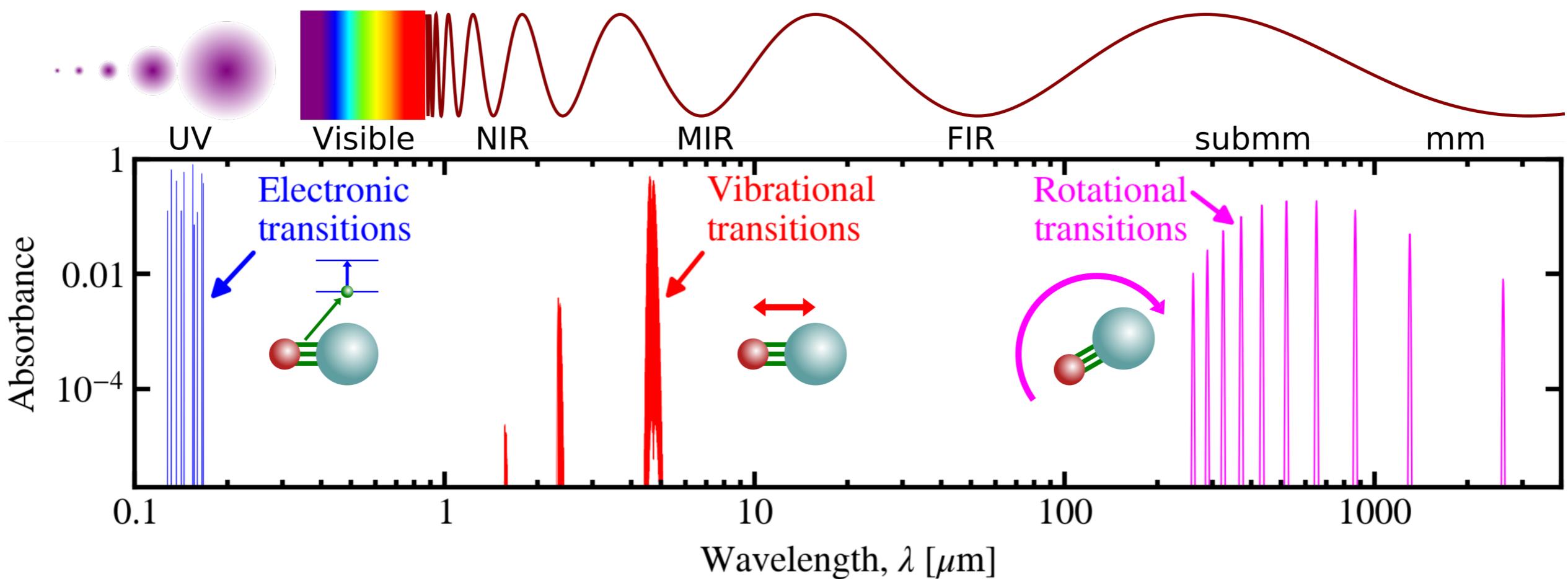
where  $I$  is the moment of inertia of the molecule:  $I = Ma_0^2$ . Therefore, we obtain

$$E_{\text{rot}} \sim \left(\frac{m_e}{M}\right) E_{\text{elect}}$$

In summary,  $E = E_{\text{elect}} + E_{\text{vib}} + E_{\text{rot}}$

$$E_{\text{elect}} : E_{\text{vib}} : E_{\text{rot}} = 1 : \left(\frac{m_e}{M}\right)^{1/2} : \left(\frac{m_e}{M}\right)$$

for hydrogen       $\sim 0.02$        $\sim 0.0005$



Molecular transitions.

The different types of transitions are illustrated with the CO molecule.

[credit: Frédéric Galliano]

# Schrödinger equation for a diatomic molecule

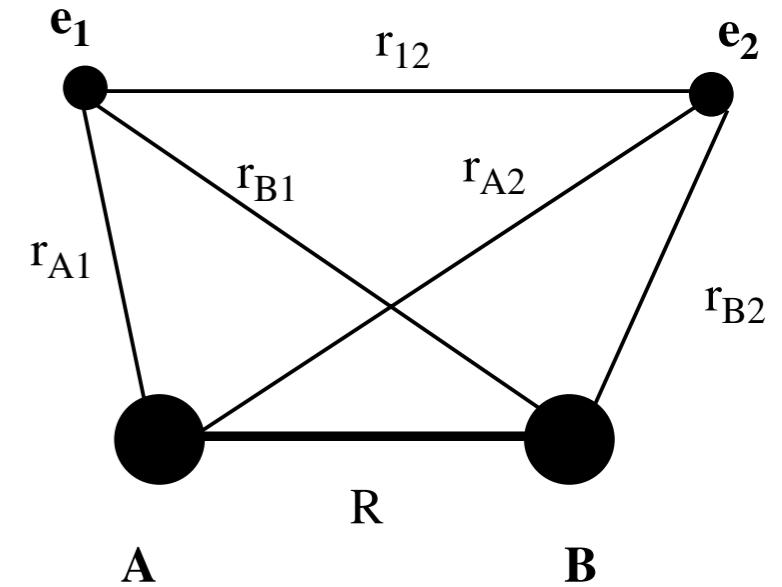
For a diatomic molecule with  $N$  electrons,

$$\left( -\frac{\hbar^2}{2M_A} \nabla_A^2 - \frac{\hbar^2}{2M_B} \nabla_B^2 - \frac{\hbar^2}{2m_e} \sum_{i=1}^N \nabla_i^2 + V_e - E \right) \Psi(\mathbf{R}_A, \mathbf{R}_B, \{\mathbf{r}_i\}) = 0$$

The first two terms are the kinetic energy operators for the motions of nuclei A and B, the third term gives the kinetic energy operator for the electrons,  $V_e$  is the potential and  $E$  is the total energy of the system.

The potential is given by the various Coulomb interactions within the molecule:

$$V_e = - \sum_{i=1}^N \frac{Z_A e^2}{r_{Ai}} - \sum_{i=1}^N \frac{Z_B e^2}{r_{Bi}} + \sum_{i=2}^N \sum_{j=1}^{i-1} \frac{e^2}{r_{ij}} + \frac{Z_A Z_B e^2}{R}$$



- (1) attraction of the electrons by nucleus A
  - (2) attraction of the electrons by nucleus B
  - (3) electron-electron repulsion
  - (4) nuclear-nuclear repulsion
- Born-Oppenheimer approximation: One can write the wave function as a product of electronic and nuclear wave functions.

$$\Psi(\mathbf{R}_A, \mathbf{R}_B, \{\mathbf{r}_i\}) = \psi_e(\{\mathbf{r}_i\}) \psi_n(\mathbf{R}_A, \mathbf{R}_B)$$

Then the equation becomes

$$\left[ \left( -\frac{\hbar^2}{2M_A} \nabla_A^2 - \frac{\hbar^2}{2M_B} \nabla_B^2 - E \right) \psi(\{\mathbf{R}_A, \mathbf{R}_B\}) + \left[ \left( -\frac{\hbar^2}{2m_e} \sum_{i=1}^N \nabla_i^2 + V_e \right) \psi(\{\mathbf{r}_i\}) \right] \psi(\{\mathbf{R}_A, \mathbf{R}_B\}) = 0 \right]$$

- In this case the **electronic wavefunction** satisfies the following equation:

$$\left( -\frac{\hbar^2}{2m_e} \sum_{i=1}^N \nabla_i^2 + V_e \right) \psi_e(\mathbf{r}_i) = E_e \psi_e(\mathbf{r}_i)$$

*This equation is solved separately for each value of the internuclear separation  $R$ .*

Then, the resulting eigenvalue  $E_e$  is the electronic energy at  $R$  and gives the electronic potential  $V(R) = E_e$  upon which the nuclei move.

The equation for the wavefunction of nuclei is obtained to be

$$\left( -\frac{\hbar^2}{2M_A} \nabla_A^2 - \frac{\hbar^2}{2M_B} \nabla_B^2 + V(R) - E \right) \psi_n(\mathbf{R}_A, \mathbf{R}_B) = 0 \quad \text{where } V(R) \equiv E_e.$$

Here,  $E$  (the eigenvalue) is the total energy of the system.

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- The Schrödinger equation for the nuclei:

$$\left( -\frac{\hbar^2}{2M_A} \nabla_A^2 - \frac{\hbar^2}{2M_B} \nabla_B^2 + V(R) - E \right) \psi_n(\mathbf{R}_A, \mathbf{R}_B) = 0 \quad \text{where } V(R) \equiv E_e.$$

The equation deals with three types of motions of the nuclei: (1) translation of the whole system, (2) vibrations, and (3) rotations. The motions can be separated into the translational motion of the center-of-mass of the system plus the internal motion of one body in a ‘central’ potential, which depends on the distance between the particles. The effective mass of this one-body problem is the reduced mass:

$$\mu = \frac{M_A M_B}{M_A + M_B}$$

The Schrödinger equation for nuclear motion, neglecting the translational motion, becomes:

$$\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + V(R) - E \right] \psi_n(\mathbf{R}) = 0$$

where  $\mathbf{R} = (R, \theta, \phi)$ .  $R$  is the internuclear separation,  $(\theta, \phi)$  is the orientation of the molecular axis relative to the laboratory  $z$ -axis.

The vibrational and rotational motion cannot be separated rigorously. However, as a good first approximation, the vibration and rotational motion may be separated.

$$\psi_n(\mathbf{R}) = \psi_{\text{vib}}(R)\psi_{\text{rot}}(\theta, \phi)$$

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Then, we obtain two equations for the rotational motion and vibrational motion:

angular equation:  $\left\{ -\frac{\hbar^2}{2\mu R^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] - E_r \right\} \psi_{\text{rot}}(\theta, \phi) = 0$

 the angular part of the Laplacian operator  $\nabla^2$

radial equation:  $\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dR^2} + V(R) - E_v \right] \psi_{\text{vib}}(R) = 0$

(1) The solution of the angular equation is:

$$\begin{aligned} \psi_{\text{rot}}(\theta, \phi) &= Y_{JM}(\theta, \phi) \\ E_r &= \frac{\hbar^2}{2\mu R^2} J(J+1) \end{aligned}$$

(2) The potential  $V(R)$  is not a simple function and thus the radial equation has no general algebraic solution. But, we can approximate  $V(R)$  about its minimum by a parabola:

$$\begin{aligned} V(R) &= V(R_e) + \frac{1}{2} \left. \frac{d^2V}{dR^2} \right|_{R=R_e} (R - R_e)^2 + \mathcal{O}((R - R_e)^3) & \xleftarrow{\frac{dV}{dR} = 0 \text{ at } R = R_e} & \frac{dV}{dR} = 0 \text{ at } R = R_e \\ &= V_0 + \frac{1}{2} k(R - R_e)^2 + \dots \end{aligned}$$

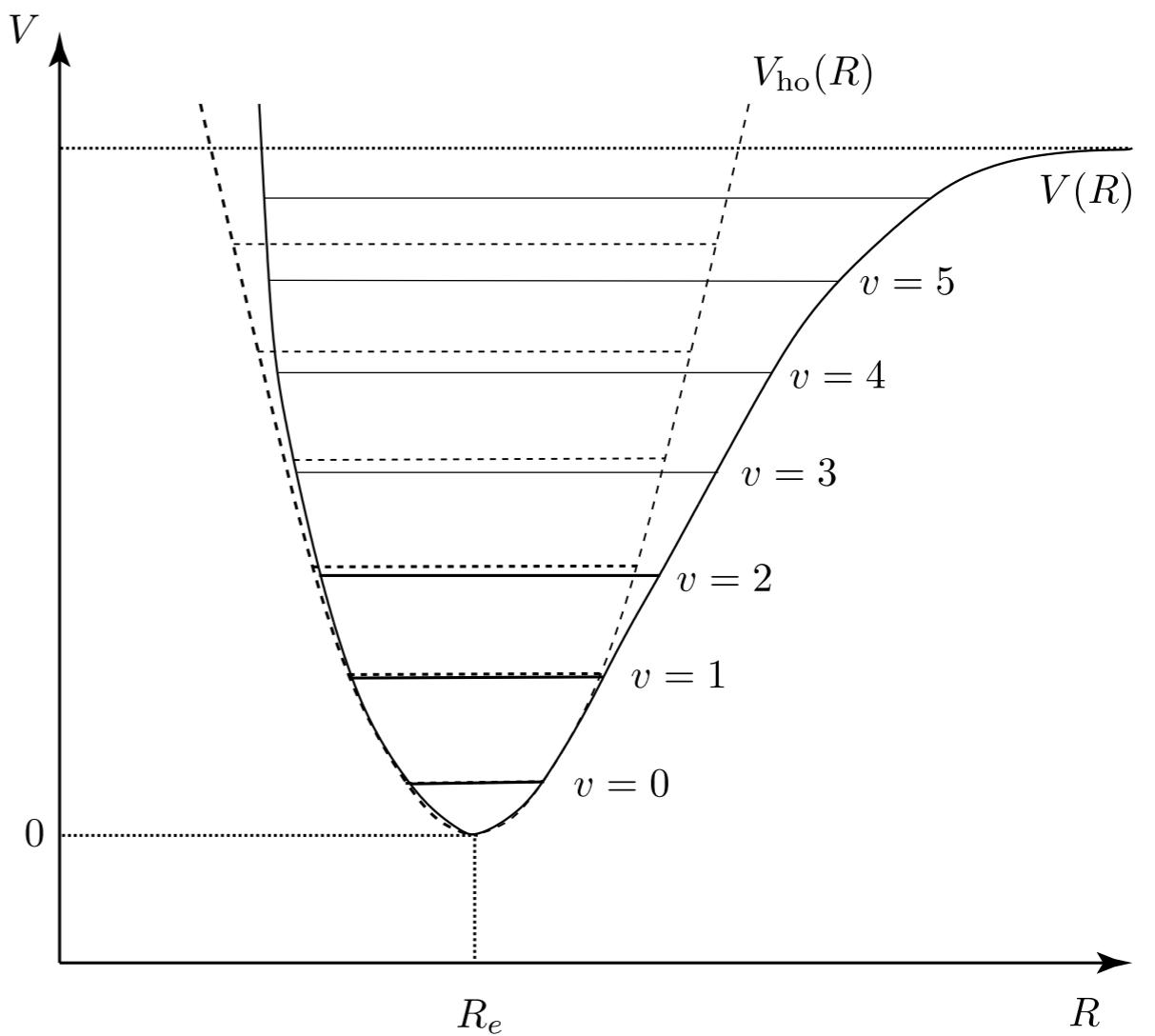
Setting the zero of energy at the minimum potential,  $V_0 = V(R_e) = 0$ , the radial equation becomes

$$\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dR^2} + \frac{1}{2}k(R - R_e)^2 - E_v \right] \psi_{\text{vib}}(R) = 0$$

This is the QM equation for the harmonic oscillator with the spring constant  $k$ . The energy levels of this equation are:

$$E_v = \hbar\omega \left( v + \frac{1}{2} \right)$$

$$\text{where } \omega = \left( \frac{k}{\mu} \right)^{1/2}, \quad v = 0, 1, 2, \dots$$



## [Labelling of Electronic States of Diatomic Molecules]

- **Heteronuclear diatomic molecules** (e.g., HD, OH, or CO): The notation of the electronic structure of a diatomic molecule is similar to the spectroscopic notation for atomic structure under *LS* coupling. Each electronic state is designated by the **term symbol**.

$$2S+1 \Lambda_{\Omega}$$

$S$  = total electronic spin

$\Lambda$  = projection of the total electronic “orbital” angular momentum along the internuclear axis  
 $(\rightarrow L_z)$

$\Omega$  = projection of the total electronic angular momentum onto the internuclear axis  
 $= |\Lambda + S_z| \quad (S_z = -S, -S + 1, \dots, S - 1, S)$

- The uppercase Greek letters are used to denote the total “orbital” angular momentum.

$$\Lambda = \Sigma, \Pi, \Delta, \dots \quad (\text{for } L_z = 0, 1, 2, \dots)$$

Recall  $S, P, D, \dots$  in the atomic spectroscopy.

- If the term symbol  $\Lambda$  is  $\Sigma$  ( $L_z = 0$ ), then additional superscript  $\pm$  is applied.

$$\pm = \begin{cases} + & \text{if symmetric under reflection through (all) planes} \\ & \text{containing the nuclei,} \\ - & \text{if antisymmetric under reflection through a plane} \\ & \text{containing the nuclei.} \end{cases}$$

Nearly all  $\Sigma$  states are + state  $\Sigma^+$ .  
One exception is O<sub>2</sub>, of which the lowest electronic state is  ${}^3\Sigma_g^-$ .

- **Homonuclear diatomic molecule:** Diatomic molecules with identical nuclei ( $\text{H}_2$ ,  $\text{N}_2$ ,  $\text{O}_2$ ,  $\text{C}_2$ ) are referred to as homonuclear. The energy levels of homonuclear diatomic molecules are designated by

$$2S+1 \Lambda_{u,g}$$

$$u, g = \begin{cases} g & (\text{"gerade"}) \text{ if symmetric under reflection through the} \\ & \text{center of mass, } \Rightarrow \text{even} \text{ (게하드)} \\ u & (\text{"ungerade"}) \text{ if antisymmetric under reflection through the} \\ & \text{center of mass. } \Rightarrow \text{odd} \text{ (운게하드)} \end{cases}$$

For the special case of  $\Sigma$  state, a superscript + or - is added.

$$2S+1 \Sigma_{u,g}^{\pm}$$

- The ***electronic states*** of diatomic molecules are also labelled with one of the following letters, appearing in front of the term symbol.

X labels the ground electronic state

A, B, C, ... label states of same spin multiplicity as the ground state

a, b, c, ... label states of different spin multiplicity to the ground state

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- Examples:
 

$S \rightarrow \Sigma$
$P \rightarrow \Pi$
$D \rightarrow \Delta$
$F \rightarrow \Phi$
$G \rightarrow \Gamma$

$^1\Sigma$  denotes a state with  $S = 0$  and  $\Lambda = 0$

$^3\Pi$  denotes a state with  $S = 1$  and  $\Lambda = 1$ , etc
  - For most (stable) diatomic, the electronic ground state is a closed shell, meaning that it is  $^1\Sigma$ .
    - Examples include  $H_2$ ,  $N_2$  and most other homonuclear diatomics. The exception is  $O_2$  which has a  $^3\Sigma$  ground state. (Each O has 4 valence electrons, and thus  $S = 1$ )
    - CO and many other heteronuclear diatomics with an even number of electrons also have  $^1\Sigma$  ground states.
    - Diatomics with an odd number of electrons usually have  $S = 1/2$ . For example,  $H_2^+$ ,  $CH^+$ , and  $CN$  all have  $^2\Sigma$  ground state.
    - $CH$ ,  $OH$  and  $NO$  all have  $\Lambda = 1$  and thus their ground states are  $^2\Pi$ . These molecules have extra lines in their spectra due to a process called  $\Lambda$ -doubling. (In this case, it is necessary to consider the coupling of the rotational motion to the spin and/or orbital angular momenta.)

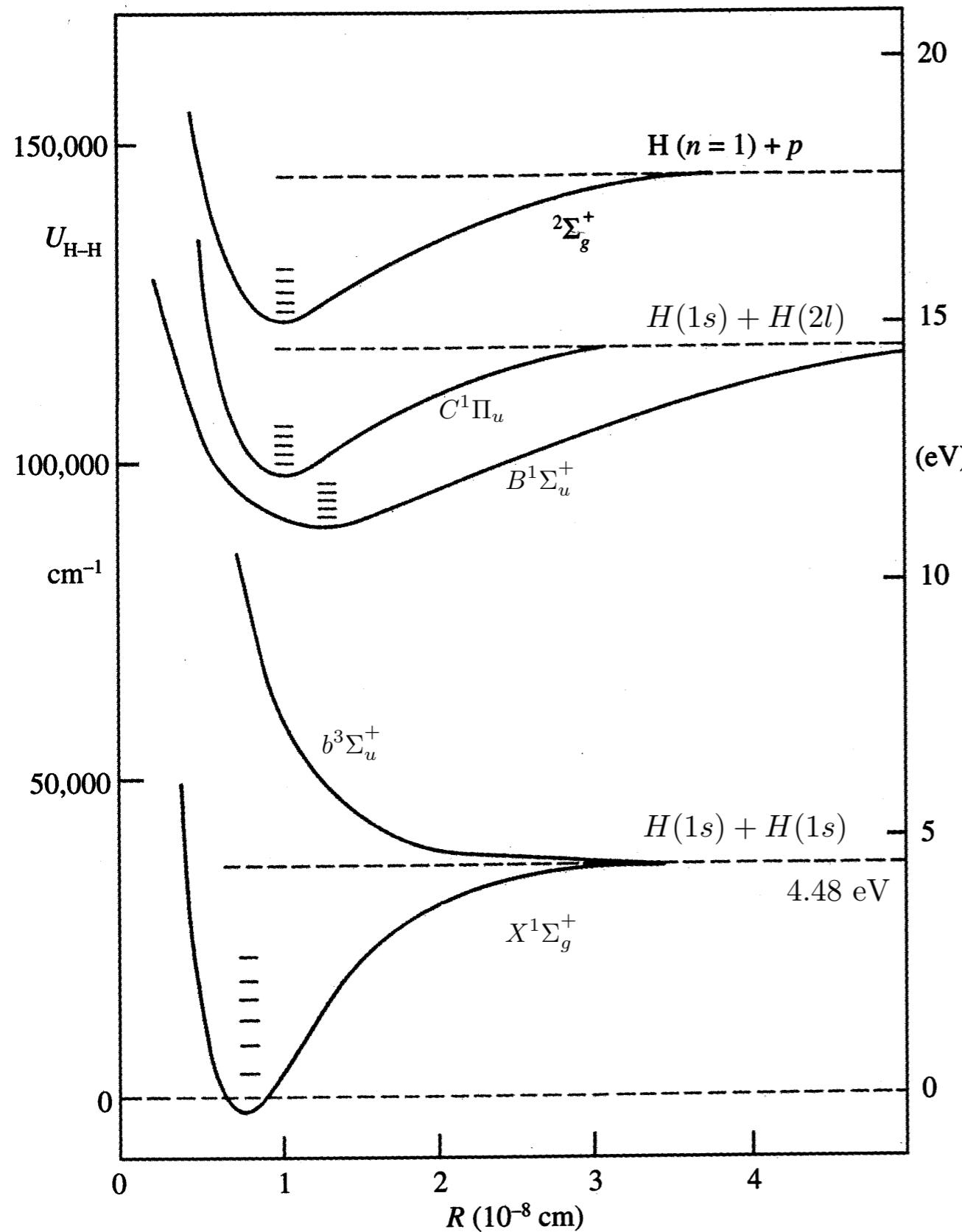
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$\Lambda =$	0	1	2	3	4	...
Orbitals	$\sigma$	$\pi$	$\delta$	$\phi$	$\gamma$	...
States	$\Sigma$	$\Pi$	$\Delta$	$\Phi$	$\Gamma$	...
Degeneracy	1	2	2	2	2	...

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Letter designations for projected total orbital angular momentum.

# [Energy levels of Molecular Hydrogen]



- The short horizontal lines in each of the bound states indicate the vibrational levels.
- The transition from the ground state  $X^1\Sigma_g^+$  to the excited states  $B^1\Sigma_u^+$  and  $C^1\Pi_u$  are called **Lyman and Werner bands**.

Werner band:  $C^1\Pi_u - X^1\Sigma_g^+$  at about 1100 Å;  
Lyman band:  $B^1\Sigma_u^+ - X^1\Sigma_g^+$  at about 1010 Å.

In principle, states are labelled alphabetically in ascending energy order. However, there are many exceptions.

The lowest triplet state of  $\text{H}_2$  is the  $b^3\Sigma_u^+$  with the  $a^3\Sigma_g^+$  lying somewhat higher.

# Other molecules

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	Ground term
H <sub>2</sub>	$^1\Sigma_g^+$
CH	$^2\Pi_{1/2,3/2}$
CH <sup>+</sup>	$^1\Sigma_0^+$
OH	$^2\Pi_{3/2,1/2}$
CN	$^2\Sigma_{1/2}^+$
CO	$^1\Sigma_0^+$
SiO	$^1\Sigma_0^+$
CS	$^1\Sigma_0^+$

The electronic ground state of H<sub>2</sub> (two electrons) has zero electronic orbital angular momentum ( $L_e = 0$ ), has zero electron spin ( $S_e = 0$ ), is symmetric under reflection through the center of mass (g), and is symmetric under reflection through planes containing the nuclei (+). The ground state is X $^1\Sigma_g^+$ .

CO has 2 *p* electrons contributed by C and 4 *p* electrons contributed by O; together these 6 *p* electrons fill the 2*p* subshell, and as a result, the ground electronic state of CO has zero electronic angular momentum and zero electronic spin:  $^1\Sigma_0^+$ , just like H<sub>2</sub>.

OH is an example of a molecule with the ground electronic state having nonzero electronic orbital angular momentum: with seven electrons, the OH ground state has  $L_{ez} = 1$  and  $S_{ez} = 1/2$ , and is therefore designated by  $^2\Pi_{1/2,3/2}$ . The electron spin and orbital angular momenta can couple to give  $J_e = 1/2$  or  $3/2$ , with energies that are separated due to spin-orbit coupling (i.e., fine-structure splitting in atoms or ions); the  $J_e = 3/2$  state has the lower energy.

## [Pure rotational & ro-vibrational transitions]

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- **Energy Levels**

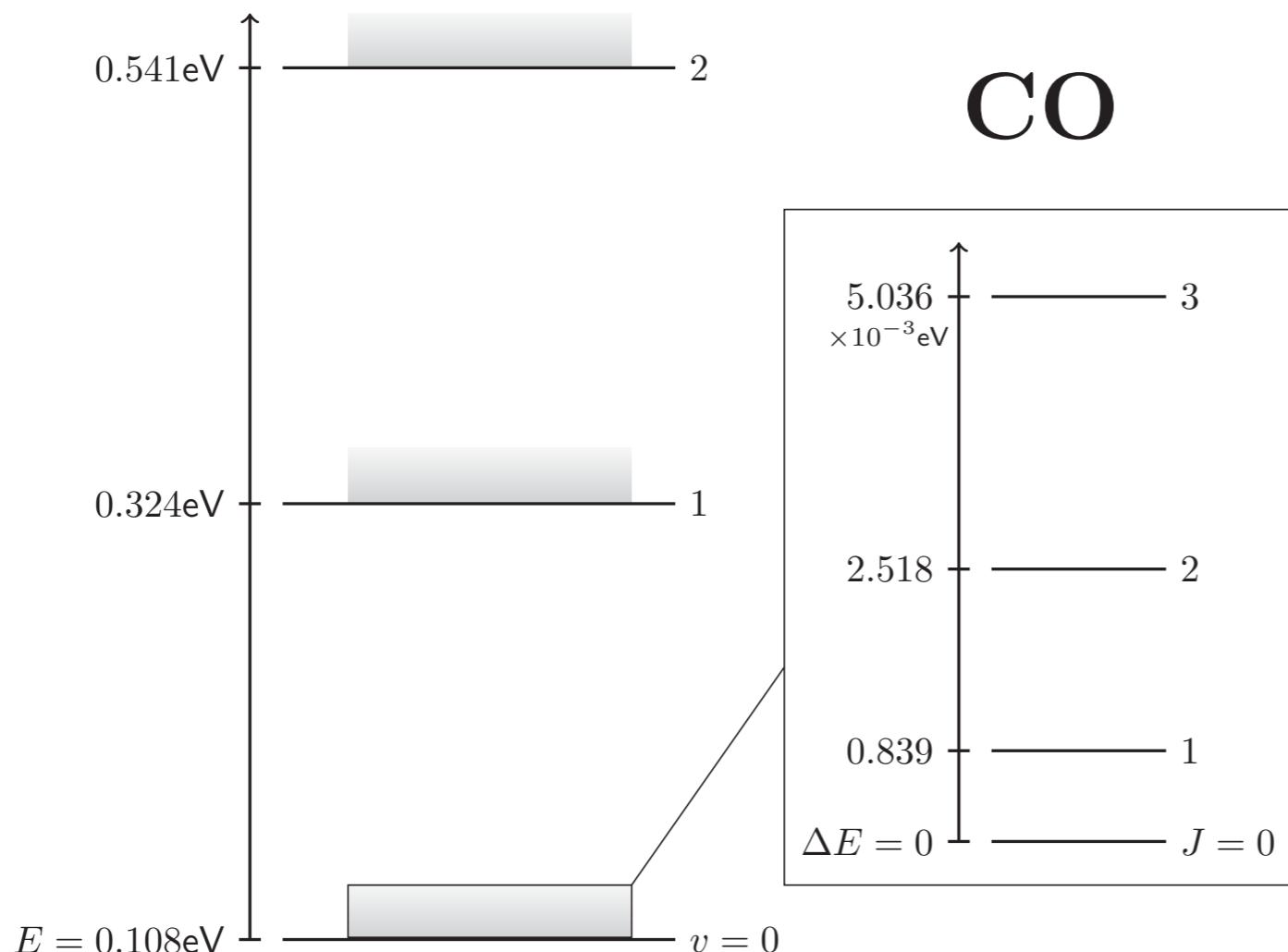
An electronic transition consists of vibrational bands, which in turn are made up of rotational transitions.

$$E_q(v, J) = V_q(r_0) + h\nu_0 \left( v + \frac{1}{2} \right) + B_v J(J+1)$$

$$\nu_0 \equiv \frac{\omega_0}{2\pi} \quad B_v = \frac{\hbar^2}{2I} \quad I = \mu r_0^2 \quad = \text{moment of inertia of the molecule.}$$

Here,  $q$  denotes an electronic state.

- **Pure rotational spectrum:** In the lowest vibrational and electronic states, it is possible to have transitions solely among the rotational states. Such transitions give rise to a pure rotational spectrum.
- **Rotational-vibration spectrum:** Because the energies required to excite vibrational modes are much larger than those required to excite rotation, it is unlikely to have a pure vibrational spectrum in analogy to the pure rotational spectrum. The transitions then yield a rotation-vibrational spectrum, in which both the vibrational state and the rotational state can change together.



The rotational and vibrational energy levels for CO. The left side show the vibrational energy for each level  $\nu$ . The rotational transitions are illustrated by the gray shading at each level. The rotational energies are about 100 times smaller than the vibrational and the inset on the right hand side shows a zoomed-in region of the J-ladder.

## [Selection Rules]

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- Electric-dipole selection rules for electronic transitions in a diatomic molecule.
  - (1)  $\Lambda = 0, \pm 1$ , e.g.,  $\Sigma - \Sigma$ ,  $\Pi - \Sigma$ ,  $\Delta - \Pi$ , etc.
  - (2)  $\Delta S = 0$
  - (3)  $\Delta\Omega$  (or  $\Delta J_z$ ) = 0,  $\pm 1$
  - (4)  $\Sigma^+ - \Sigma^+$ ,  $\Sigma^- - \Sigma^-$ , but not  $\Sigma^+ - \Sigma^-$
  - (5)  $g \longleftrightarrow u$
  - $\Delta\Lambda = 0, \pm 1$ , e.g.,  $\Sigma - \Sigma$ ,  $\Pi - \Sigma$ ,  $\Delta - \Pi$ , etc.
  - $\Delta S = 0$
  - $\Delta\Omega = 0, \pm 1$
  - $\Sigma^+ - \Sigma^+$ ,  $\Sigma^- - \Sigma^-$ , but not  $\Sigma^+ - \Sigma^-$
  - $g \leftrightarrow u$

- Electric-dipole selection rule for ro-vibrational transitions:

$\Delta v = \text{any}$

$\Delta J = 0, \pm 1$  not  $J = 0 \leftrightarrow 0$

- But, note that  $H_2$  has no electric-dipole for rotational transition.

The electric-quadrupole are allowed for  $\Delta J = \pm 2$  within the ground electronic state.

## O, P, Q, R, and S transitions

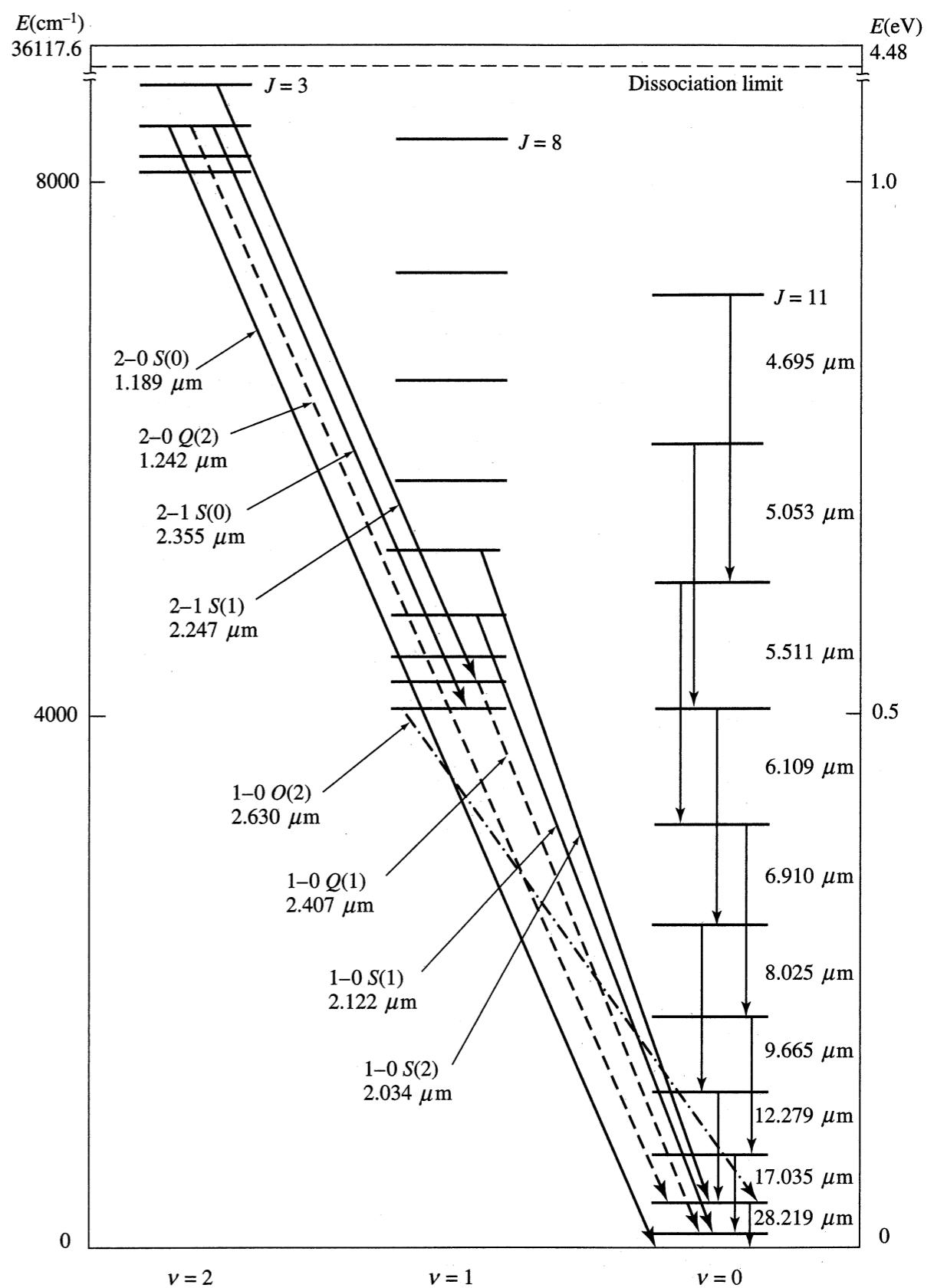
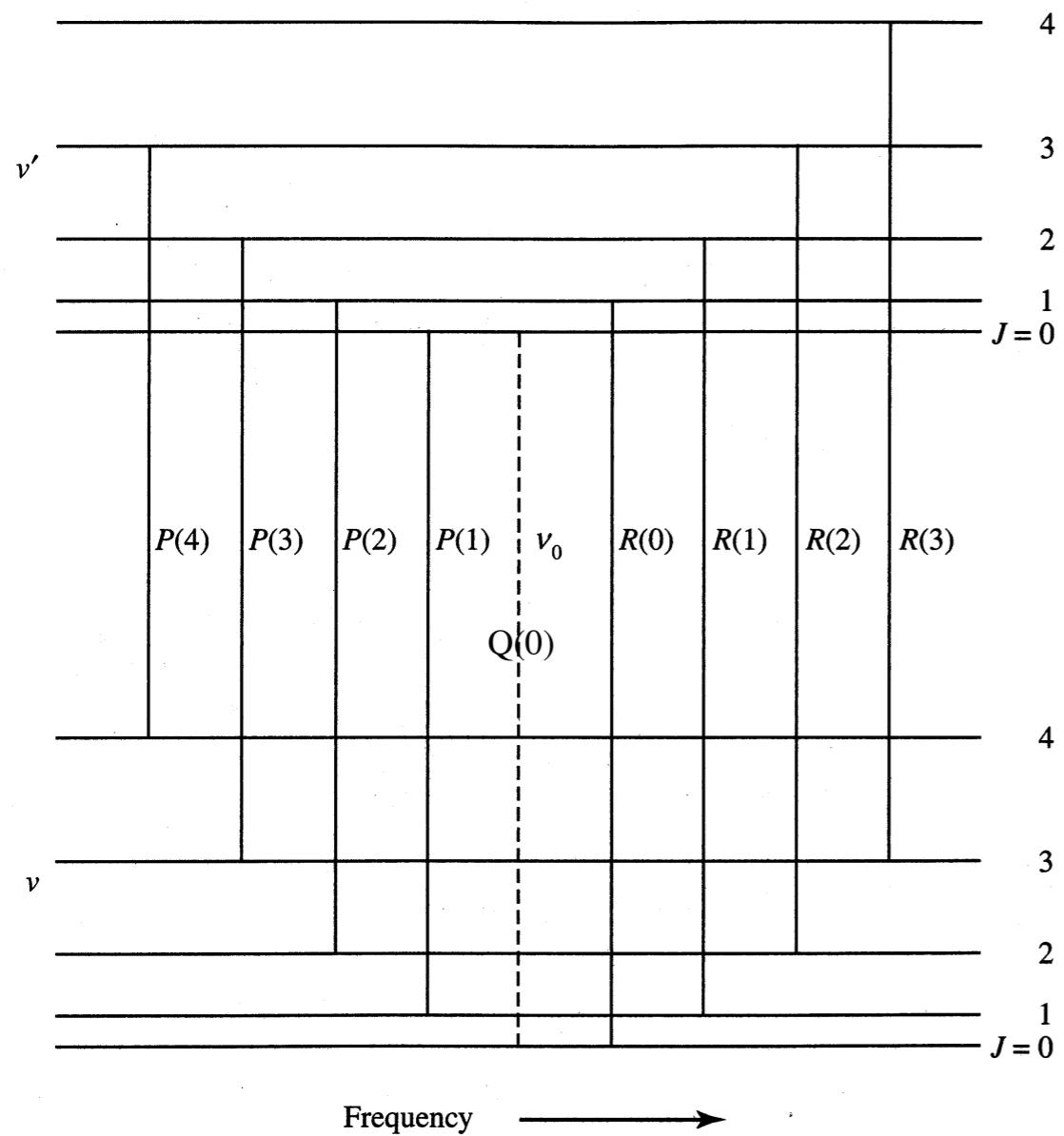
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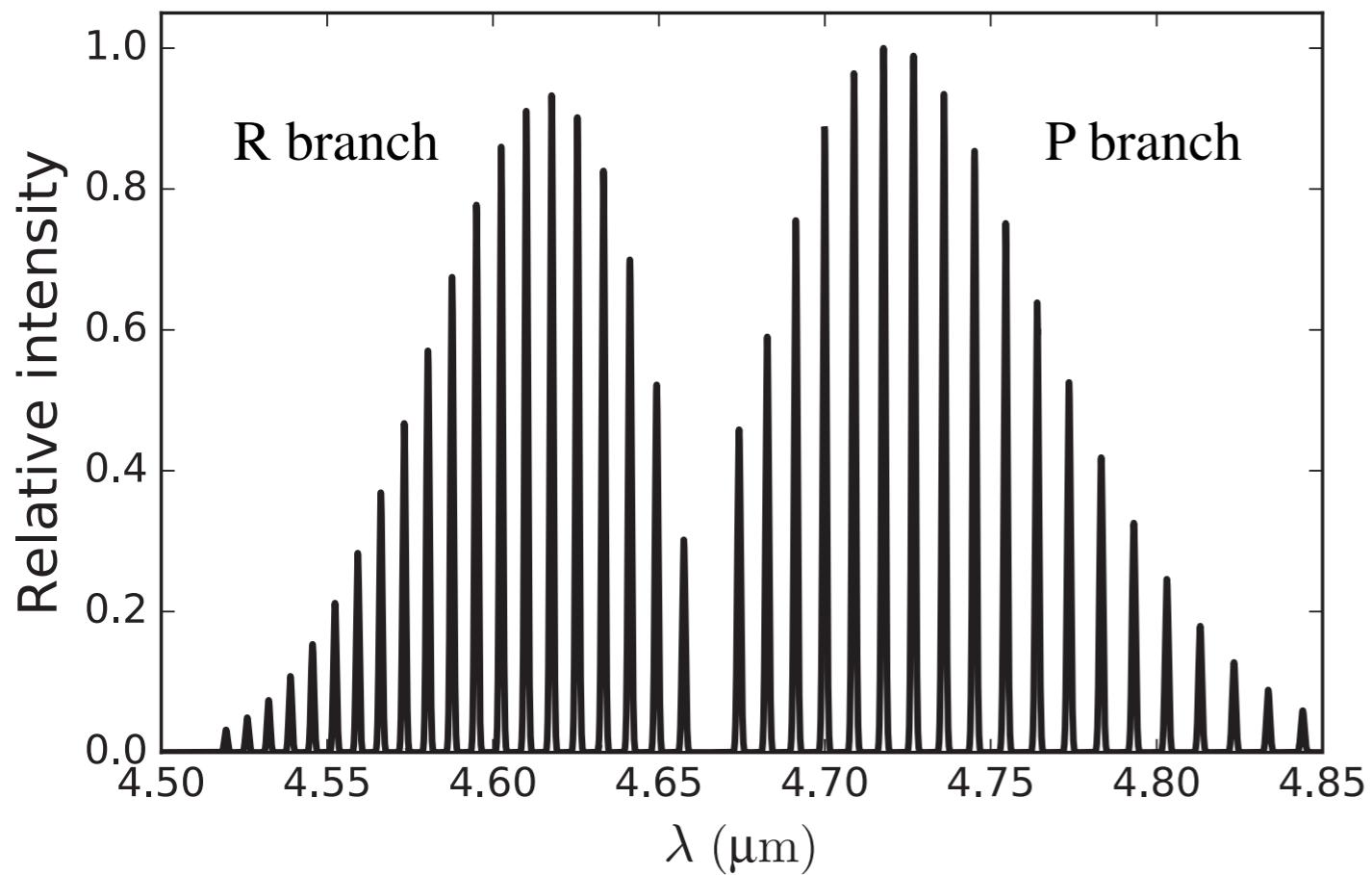
- The rotational levels of diatomic molecules are specified by a single vibrational quantum number  $v$  and rotational quantum number  $J$ .
  - Transitions will change  $J$  by either  $0, \pm 1, \pm 2$ .**
  - It is customary to identify transitions by specifying the upper and lower electronic states, upper and lower vibrational states, and one of the follows:  $O(J_\ell)$ ,  $P(J_\ell)$ ,  $Q(J_\ell)$ ,  $R(J_\ell)$ ,  $S(J_\ell)$
  - The usage of the symbols are shown in the following table.

$$J_u \rightarrow J_\ell$$

Designation	$(J_u - J_\ell)$	Note
$O(J_\ell)$	-2	Electric quadrupole transition
$P(J_\ell)$	-1	Electric dipole transition
$Q(J_\ell)$	0	Electric dipole or electric quadrupole; $Q(0)$ is forbidden
$R(J_\ell)$	+1	Electric dipole transition
$S(J_\ell)$	+2	Electric quadrupole transition

- For instance, a transition from the the  $v_\ell = 0, J_\ell = 1$  level of the ground electronic state to the  $v_u = 5, J_u = 2$  level of the first electronic excited state would be written to be B-X 5-0  $R(1)$





Model spectrum of ro-vibrational lines for CO  $\nu = 1 - 0$ , illustrating the two branches corresponding to a positive or negative change in  $J$  and a central gap at  $\Delta J = 0$ .

The R branch corresponds to a higher energy jump,  $J \rightarrow J - 1$ , and lies at shorter wavelengths. The P branch is a smaller energy jump,  $J \rightarrow J + 1$ , and is at longer wavelengths.

The envelope shape arises from the population level distribution that is small at low levels due to the degeneracy,  $g_J = 2J + 1$ , and at high levels due to the Boltzmann exponential,  $E^{E/kT_{\text{ex}}}$ . The difference between the relative intensity of the P and R branches is due to different value of the Einstein A coefficient.

## [Hyperfine Splitting & Ortho-H<sub>2</sub> and Para-H<sub>2</sub>]

- **Hyperfine splitting:** If one or more nuclei have nonzero nuclear spin and  $J_{ez} \neq 0$ , then there will be an interaction between the nuclear magnetic moment and the magnetic field generated by the electrons, resulting in “hyperfine splitting.” The energy will depend on the orientation of the nuclear angular momentum relative to the axis.
- **Ortho-H<sub>2</sub> and Para-H<sub>2</sub>**

In the case of H<sub>2</sub>, the electronic wave function is required to be antisymmetric under exchange of the two electrons.

The two protons, just like electrons, are identical fermions, and therefore, the Pauli exclusion principle antisymmetric requirement also applies to exchange of the two protons. The protons are spin 1/2 particles - the two protons together can have total spin 1 (parallel) or total spin 0 (antiparallel).

The consequence of the antisymmetry requirement is that

If the protons have spin 0, the rotational quantum number  $J$  must be even.  $\Rightarrow$  para-H<sub>2</sub> (even  $J$ )

(an antisymmetric nuclear spin wave function ( $I = 0$ ) and a symmetric spatial wave function involving with even values of the rotational quantum number  $J$ )

If the protons have spin 1, the rotational quantum number  $J$  must be odd.  $\Rightarrow$  ortho-H<sub>2</sub> (odd  $J$ )

(a symmetric nuclear spin wave function ( $I = 1$ ) and an antisymmetric spatial wave function involving odd values of the rotational quantum number  $J$ )

Because the nuclear spins are only weakly coupled to the electromagnetic field, ortho-H<sub>2</sub> and para-H<sub>2</sub> behave as almost distinct species.

## H<sub>2</sub> has no permanent electric dipole moment.

- The vibrational states and the rotational states radiate very weakly, via the time-variation of the electric quadrupole moment and the molecule vibrates or rotates.
- Because the nuclear spin state does not change, the ro-vibrational radiative transitions of H<sub>2</sub> must have  $\Delta J = 0$  or  $\Delta J = \pm 2$ , i.e., ortho  $\rightarrow$  ortho or para  $\rightarrow$  para  
(not  $J = 0 \leftrightarrow J = 0$ )

The vibration-rotation emission spectrum of H<sub>2</sub> therefore consists of electric quadrupole transitions. Therefore, the H<sub>2</sub> emission lines are faint and hard to detect. The downward transitions are identified by

$$\begin{aligned} v_u - v_\ell \ S(J_\ell) & \quad \text{if } J_\ell = J_u - 2 \ , \\ v_u - v_\ell \ Q(J_\ell) & \quad \text{if } J_\ell = J_u \ , \\ v_u - v_\ell \ O(J_\ell) & \quad \text{if } J_\ell = J_u + 2 \ . \end{aligned}$$

For example, 1-0 S(1) refers to the transition  $(v = 1, J = 3) \rightarrow (v = 0, J = 1)$ .

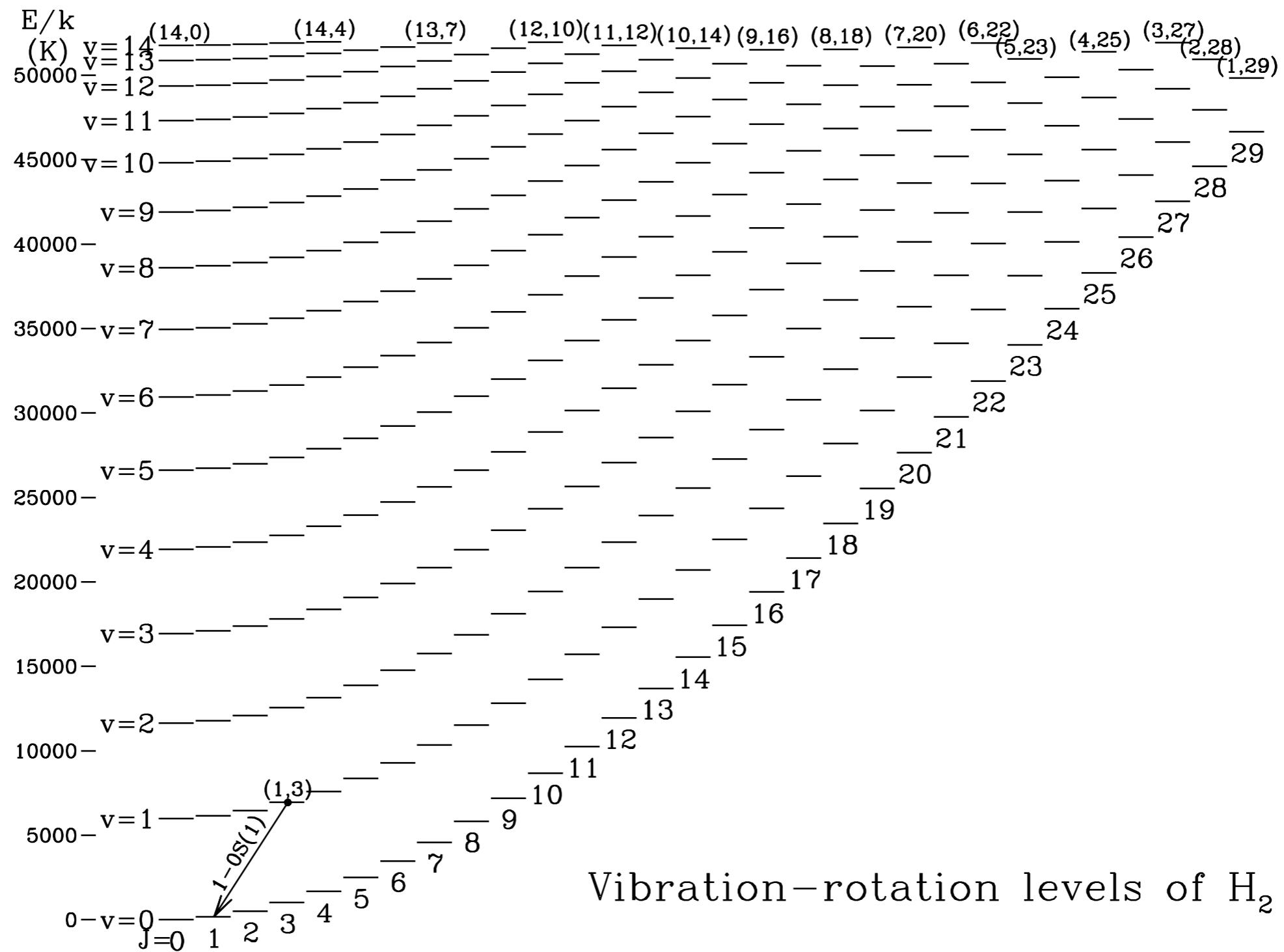
- Spin-exchange collisions with H<sup>0</sup> or H<sup>+</sup>, and a process in which H<sub>2</sub> is captured on a grain surface, can cause an ortho-para conversion.
- The statistical weight of an ortho-H<sub>2</sub> rotational level  $J$  is  $3(2J+1)$ . [because  $S_{\text{nucleus}} = 1$ ]

For a para-H<sub>2</sub> it is  $(2J+1)$ . [because  $S_{\text{nucleus}} = 0$ ]

# Ortho/Para Ratios

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- The ortho state of a molecule is defined as having the larger statistical spin weights and para as having the smaller weight.
- Since the typical energy separation between the ortho and para states of a molecule is comparable to the gas and dust temperature in the ISM and much smaller than the energy released in formation reactions, it is expected that the abundance ratio between the two states will reflect the equilibrium values at high temperatures, that is, the ratio of their statistical weight.
  - ▶ Since  $g = 2I + 1$ , the usual ortho to para ratio is  $(2x1+1)/(2x0+1) = 3$  for spin 1/2 systems such as H<sub>2</sub>.
- If a molecule cannot be converted from ortho to para (or vice versa) by radiative or collisional processes, the two states can effectively be considered as two separate molecules. In this case, the ortho to para ratio at the time of molecule formation will be preserved.



# Interstellar Molecules

- Interstellar Molecules

- Interstellar molecules were first discovered in the late 1930s through the identification of optical lines seen in absorption against background starlight with electronic transitions of molecules.
- The molecules first detected were CN ( $B^2\Sigma^+ - X^2\Sigma^+$  at  $3876.84\text{\AA}$ ), CH ( $A^2\Delta - X^2\Pi$  at  $4300.30\text{\AA}$ ) and  $\text{CH}^+$  ( $A^1\Pi - X^1\Pi^+$  at  $4232.54\text{\AA}$ )
- Over 200 interstellar molecules have been detected.

Interstellar molecules listed by number of atoms

Diatomeric	Triatomic	Four atoms	Five atoms	Six atoms	Seven atoms	Eight atoms
$\text{H}_2$	$\text{C}_3$	$c\text{-C}_3\text{H}$	$\text{C}_5$	$\text{C}_5\text{H}$	$\text{C}_6\text{H}$	$\text{CH}_3\text{C}_3\text{N}$
$\text{AlF}$	$\text{C}_2\text{H}$	$l\text{-C}_3\text{H}$	$\text{C}_4\text{H}$	$l\text{-H}_2\text{C}_4$	$\text{CH}_2\text{CHCN}$	$\text{HCOOCH}_3$
$\text{AlCl}$	$\text{C}_2\text{O}$	$\text{C}_3\text{N}$	$\text{C}_4\text{Si}$	$\text{C}_2\text{H}_4$	$\text{CH}_3\text{C}_2\text{H}$	$\text{CH}_3\text{COOH}(?)$
$\text{C}_2$	$\text{C}_2\text{S}$	$\text{C}_3\text{O}$	$l\text{-C}_3\text{H}_2$	$\text{CH}_3\text{CN}$	$\text{HC}_5\text{N}$	$\text{C}_7\text{H}$
$\text{CH}$	$\text{CH}_2$	$\text{C}_3\text{S}$	$c\text{-C}_3\text{H}_2$	$\text{CH}_3\text{NC}$	$\text{HCOCH}_3$	$\text{H}_2\text{C}_6$
$\text{CH}^+$	$\text{HCN}$	$\text{C}_2\text{H}_2$	$\text{CH}_2\text{CN}$	$\text{CH}_3\text{OH}$	$\text{NH}_2\text{CH}_3$	$\text{CH}_2\text{OHCHO}$
$\text{CN}$	$\text{HCO}$	$\text{CH}_2\text{D}^+(?)$	$\text{CH}_4$	$\text{CH}_3\text{SH}$	$c\text{-C}_2\text{H}_4\text{O}$	$\text{CH}_2\text{CHCHO}$
$\text{CO}$	$\text{HCO}^+$	$\text{HCCN}$	$\text{HC}_3\text{N}$	$\text{HC}_3\text{NH}^+$		$\text{CH}_2\text{CHOH}$
$\text{CO}^+$	$\text{HCS}^+$	$\text{HCNH}^+$	$\text{HC}_2\text{NC}$	$\text{HC}_2\text{CHO}$		
$\text{CP}$	$\text{HOC}^+$	$\text{HNCO}$	$\text{HCOOH}$	$\text{NH}_2\text{CHO}$		
$\text{CSi}$	$\text{H}_2\text{O}$	$\text{HNCS}$	$\text{H}_2\text{CHN}$	$\text{C}_5\text{N}$		
$\text{HCl}$	$\text{H}_2\text{S}$	$\text{HOCO}^+$	$\text{H}_2\text{C}_2\text{O}$	$\text{HC}_4\text{N}$		
$\text{KCl}$	$\text{HNC}$	$\text{H}_2\text{CO}$	$\text{H}_2\text{NCN}$			
$\text{NH}$	$\text{HNO}$	$\text{H}_2\text{CN}$	$\text{HNC}_3$			
$\text{NO}$	$\text{MgCN}$	$\text{H}_2\text{CS}$	$\text{SiH}_4$			
$\text{NS}$	$\text{MgNC}$	$\text{H}_3\text{O}^+$	$\text{H}_2\text{COH}^+$			
$\text{NaCl}$	$\text{N}_2\text{H}^+$	$\text{NH}_3$				
$\text{OH}$	$\text{N}_2\text{O}$	$\text{SiC}_3$				
$\text{PN}$	$\text{NaCN}$	$\text{C}_4$				
$\text{SO}$	$\text{OCS}$		Nine atoms	Ten atoms	Eleven atoms	Twelve atoms
$\text{SO}^+$	$\text{SO}_2$		$\text{CH}_3\text{C}_4\text{H}$	$\text{CH}_3\text{C}_5\text{N}(?)$	$\text{HC}_9\text{N}$	$\text{CH}_3\text{OC}_2\text{H}_5$
$\text{SiN}$	$c\text{-SiC}_2$		$\text{CH}_3\text{CH}_2\text{CN}$	$(\text{CH}_3)_2\text{CO}$		$\text{HC}_{11}\text{N}$
$\text{SiO}$	$\text{CO}_2$		$(\text{CH}_3)_2\text{O}$	$\text{NH}_2\text{CH}_2\text{COOH}$		
$\text{SiS}$	$\text{NH}_2$		$\text{CH}_3\text{CH}_2\text{OH}$	$\text{CH}_3\text{CH}_2\text{CHO}$		
$\text{CS}$	$\text{H}_3^+$			$\text{HC}_7\text{N}$		
$\text{HF}$	$\text{SiCN}$			$\text{C}_8\text{H}$		
$\text{SH}$	$\text{AlNC}$					
$\text{FeO}(?)$	$\text{SiNC}$					

Table from A. Wootten ([www.cv.nrao.edu/~awootten/allmols.html](http://www.cv.nrao.edu/~awootten/allmols.html)).

[Table 7.1, Kowk]

- Given the ubiquity of hydrogen in the ISM, and the inability of helium to form chemical bonds, we expect molecular gas in the ISM to consist primarily of H<sub>2</sub>.
  - A hydrogen molecule, with the dissociation energy  $D_0 = 4.52 \text{ eV}$ , is not very tightly bound. An UV photon can photo dissociate it.
  - In a gas with temperature  $T > D_0/k \sim 50,000 \text{ K}$ , collisions with other gas particles can collisionally dissociate it. Thus, we expect molecular hydrogen to survive for long periods of time only in cold regions of the ISM that are shielded from UV radiation.
  - Hydrogen has the lowest, reduced mass of any molecule,  $\mu = m_{\text{H}}/2$ , hence, hydrogen molecules have a particularly high fundamental frequency of vibration compared to other diatomic molecules.

Properties of some diatomic molecules [Table 7.1, Ryden]

Molecule	$D_0$ [eV]	$r_0$ Å	$B_0$ [meV]	$\hbar\omega_0$ [eV]	$\mu_0$ [debye]
H <sub>2</sub>	4.52	0.74	7.36	0.516	0.000
CO	11.1	1.13	0.24	0.269	0.110
CH	3.51	1.12	1.76	0.339	1.406
OH	4.39	0.97	2.30	0.443	1.668
CN	7.57	1.17	0.23	0.253	0.557

$\mu_0$  = permanent dipole moment

1 debye =  $10^{-18} \text{ statC cm}$

$D_0$  = dissociation energy

$r_0$  = speration

$$B_0 \equiv \frac{\hbar^2}{2I}$$

$\omega_0 = \sqrt{k/\mu}$  fundamental frequency of vibration

# CO

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- For any molecule to undergo a pure rotation transition, it must have a permanent dipole moment,  $\mu$ . This means that for any molecule to have a dipole-allowed rotational spectrum it must have an asymmetric charge distribution which gives rise to a permanent dipole moment.

Heteronuclear diatomics posses a permanent dipole moment but homonuclears, such as H<sub>2</sub>, do not.

- CO

Carbon monoxide, CO, is a particularly important species for astronomical observations. CO is the most stable diatomic molecule.

It has a dissociation energy  $D_0$  of 11.1 eV, which is more than double the  $D_0$  value found for most other diatomic molecules. As a result, in astronomical environments where molecules form, C and O usually combine to form CO, which is very stable and long-lived.

The wavelengths of the first few rotational transitions are 1-0 at  $\lambda = 2.60$  mm, 2-1 at 1.30 mm, and 3-2 at 0.87mm.

The J = 1-0 transition of CO is the second most important spectral line in radio astronomy after the hydrogen 21 cm line.

CO is widely distributed in the interstellar medium and maps of the CO J = 1-0 transition are a standard tool for investigating the ISM.

One reason for this is that cold H<sub>2</sub> is very difficult to observe directly because its pure rotational transitions are not only very weak but lie in the near-infrared where ground-based observations are not possible. The abundance of CO is therefore often used to estimate the total amount of molecular gas present in a given environment. It is generally assumed that the number density of CO is approximately 10<sup>-4</sup> of that of H<sub>2</sub>.

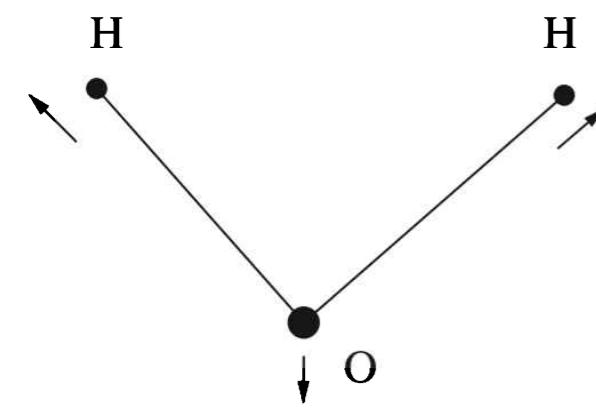
$$n(\text{CO}) \approx 10^{-4}n(\text{H}_2)$$

If, as often happens, the CO 1-0 line is optically thick, one can use higher transitions such as the CO 2-1 line instead. Another option to avoid the effects of optical thickness is to observe an isotopologue <sup>13</sup>CO, which is present with much lower densities and whose transitions are therefore much less optically thick.

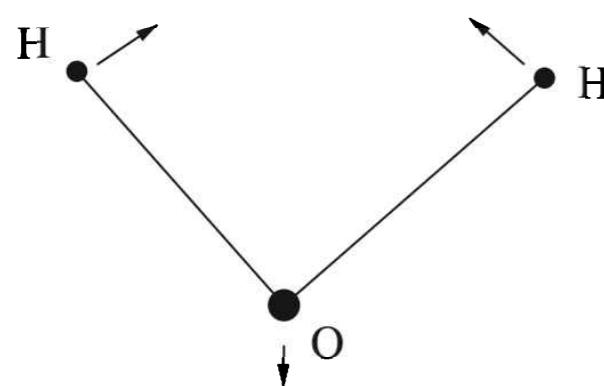
An isotopologue is a molecule that consists of at least one less abundant isotope of its constituent elements. They have the same transitions at nearby frequencies with similar decay and excitation rates. The main difference is in their abundance and observations of the rarer species help diagnose conditions in dense regions where lines from the primary species are optically thick.

# Vibrations in Polyatomic Molecules

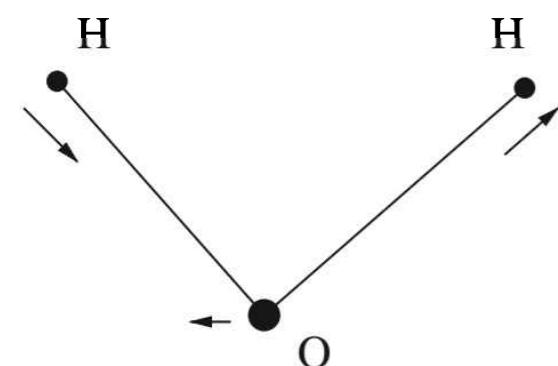
$\text{H}_2\text{O}$



Symmetric stretch,  $v_1$   
 $\omega_1 = 3667 \text{ cm}^{-1}$



Bend,  $v_2$   
 $\omega_2 = 1595 \text{ cm}^{-1}$



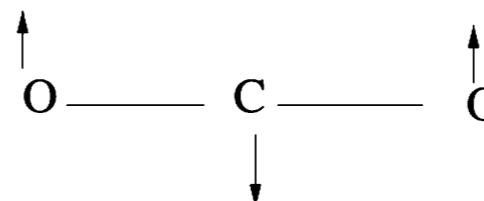
Asymmetric stretch,  $v_3$   
 $\omega_3 = 3676 \text{ cm}^{-1}$

The three vibrational modes of the water molecule

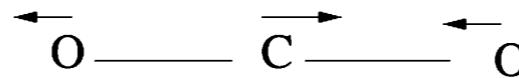
$\text{CO}_2$



Symmetric stretch,  $v_1$   
 $\omega_1 = 1388 \text{ cm}^{-1}$



Bend (degenerate),  $v_2$   
 $\omega_2 = 667 \text{ cm}^{-1}$



Asymmetric stretch,  $v_3$   
 $\omega_3 = 2349 \text{ cm}^{-1}$

The vibrational modes of carbon dioxide; note that the bending mode is doubly degenerate as the motion can occur in the plane of the page, as drawn, or identically, perpendicular to the plane of the page.

# Excitation Temperature

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- The excitation temperature for a given transition is defined as:

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} e^{-h\nu_{ul}/kT_{\text{exc}}}$$

For pure rotational transitions, the excitation temperature is often called the **rotation temperature**.

For vibrational transitions, it is called the **vibrational temperature**.

This nomenclature is analogous to the “spin temperature” defined for the H I 21-cm hyperfine transition.

# Maser Emissions

- Microwave amplification stimulated emission of radiation (maser) action is observed from at least 36 molecules including SiO, OH and water, usually at IR and microwave frequencies.
- The population inversion necessary to cause maser action can be created by a number of mechanisms including optical pumping, radiation trapping in certain long-lived levels and selective collisional excitation of the masing molecule.
- In the righthand side figure, the levels are vibration-rotation levels of the electronic ground state. If level A is excited in some collision process, such as scattering with  $H_2$ , and level B is not, then the population of level A can be greater than the population of level B. The situation gives a non-thermal population and can lead to maser action.

