Interstellar Medium (ISM)

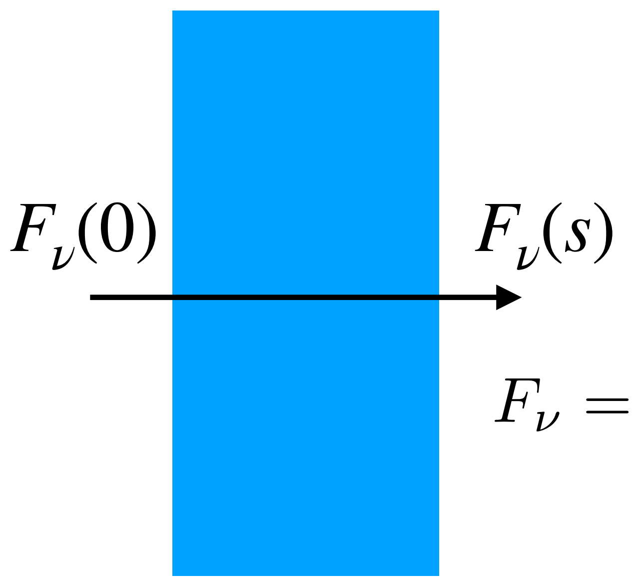
Week 6
April 10 (Monday), 2023

updated 04/09, 15:55

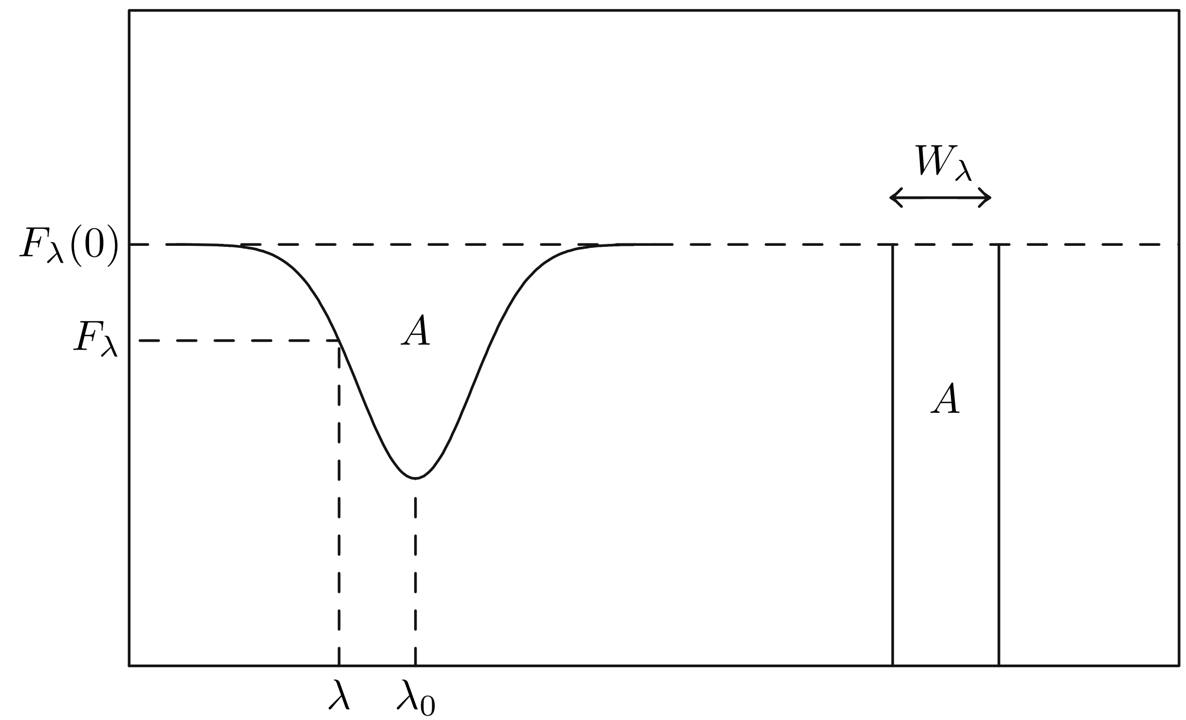
선광일 (Kwangil Seon)
KASI / UST

Equivalent Width & Curve of growth

- ***Equivalent width***
  - The spectrograph often lacks the spectral resolution to resolve the profiles of narrow lines, but can measure the total amount of “missing power” resulting from a narrow absorption line.
  - *The equivalent width is a measure of the strength of an absorption line, in terms of “missing power” in the unresolved absorption line.*



$$F_\nu = F_\nu(0)e^{-\tau_\nu}$$



- ***Curve of growth***
  - The curve of growth refers to the numerical relation between the observed equivalent width and the underlying optical depth (or the column density) of the absorber.

Equivalent Width

- Suppose that we measure the energy flux density F_ν using an aperture of solid angle $\Delta\Omega$. Then, we obtain the flux density at the observer:

$$F_\nu = F_\nu(0)e^{-\tau_\nu} + B_\nu(T_{\rm exc})\Delta\Omega(1 - e^{-\tau_\nu})$$

  Here, $F_\nu(0)$ is the flux density of the background light source.

- At optical frequencies and in the neutral medium, nearly all atoms are in their ground state. Thus, we normally have $n_u/n_\ell \ll 1$ and $B_\nu(T_{\rm exc})\Delta\Omega \ll F_\nu(0)$. Then, we can neglect the emission from the ISM.

$$\frac{h\nu}{k_{\rm B}T_{\rm exc}} = \frac{6000\ \text{Å}}{\lambda}\frac{2.4\times10^4\ {\rm K}}{T_{\rm exc}}$$

$$F_\nu = F_\nu(0)e^{-\tau_\nu}$$

- If the background spectrum is smooth, we can define the ***dimensionless equivalent width*** and the ***wavelength equivalent width*** as follows:

$$W \equiv \int \frac{d\nu}{\nu_0}\left[1 - \frac{F_\nu}{F_\nu(0)}\right] = \int \frac{d\nu}{\nu_0}\left(1 - e^{-\tau_\nu}\right)$$

$$W_\lambda \equiv \int d\lambda \left(1 - e^{-\tau_\lambda}\right) \approx \lambda_0 W$$

  - The equivalent width is the width of a straight-sided, perfectly black absorption line that has the same integrated flux deficit as the actual absorption line.

Overall Shape of the Curve of Growth

The Curve of Growth

= the relation between optical depth at line center and equivalent width

Curve of Growth for Lyα

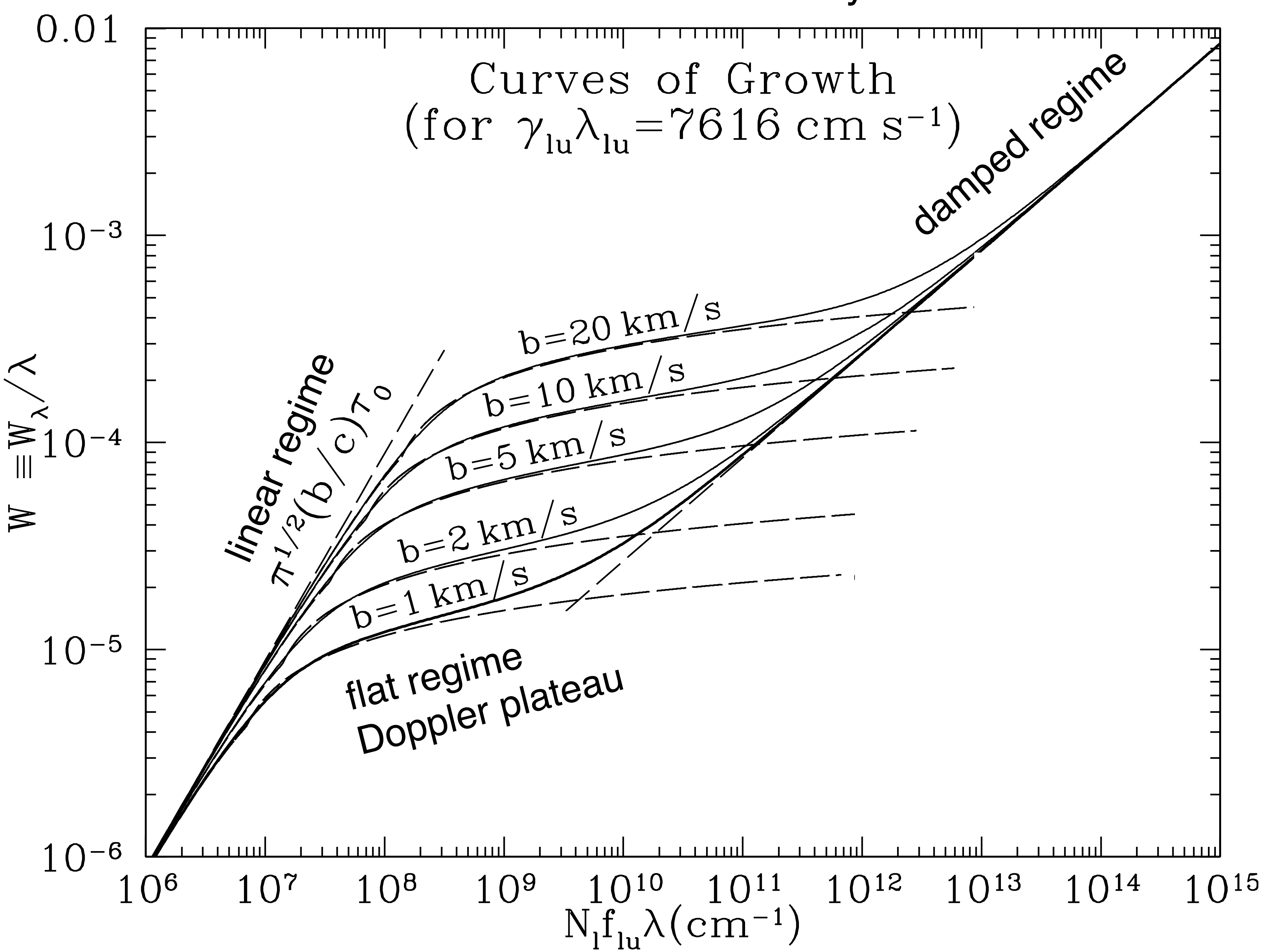


Figure 9.2 in [Draine]

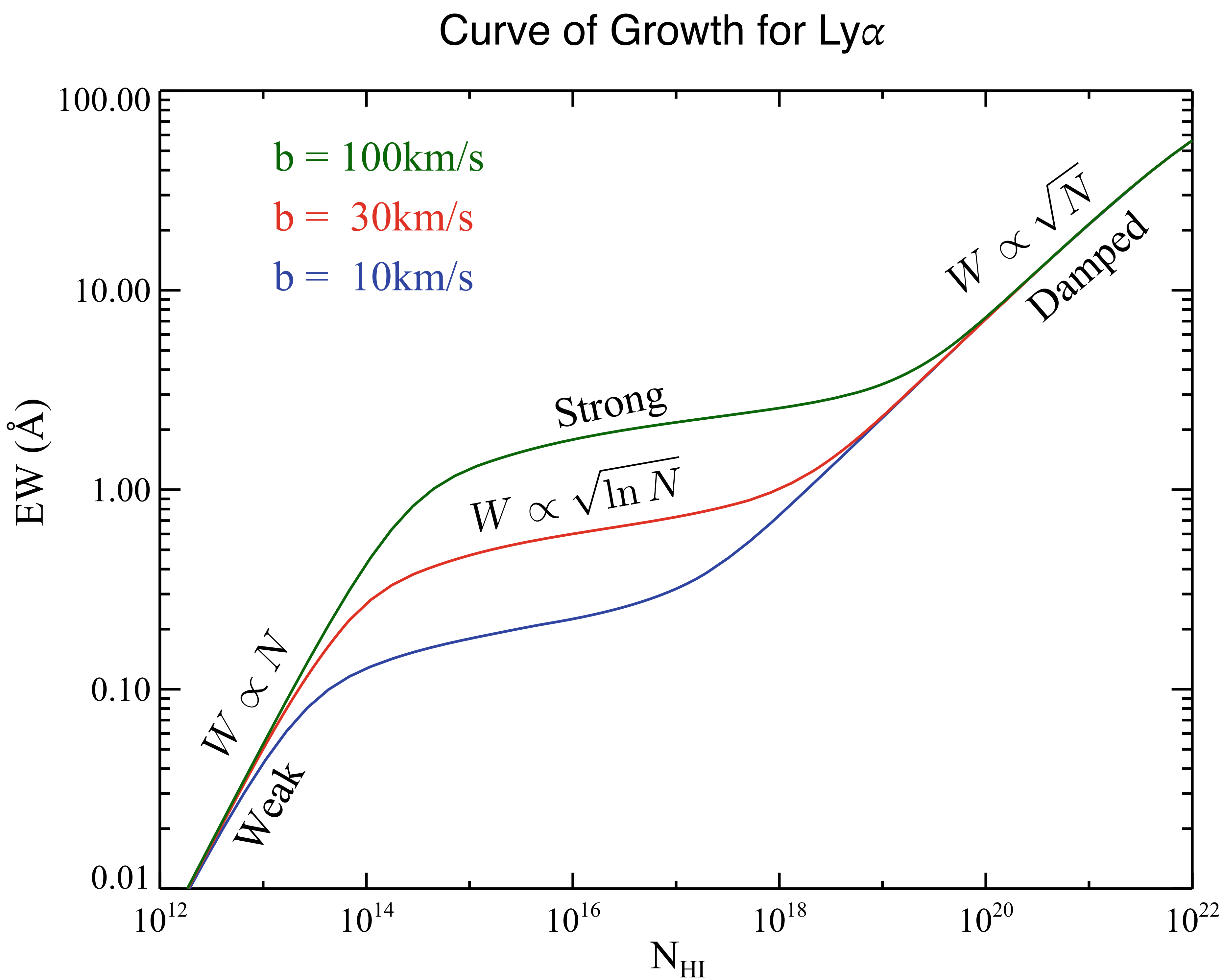


Figure 2.10 in Chap. 2 [Prochaska]
Lyman-alpha as An Astrophysical and Cosmological Tool

Detailed Analysis of The Curve of Growth

- Equivalent width: $$W = \int_{-\infty}^{\infty} \frac{d\nu}{\nu_0} \left(1 - e^{-\tau_\nu}\right) = \frac{b}{c} \int_{-\infty}^{\infty} du \left(1 - e^{-\tau_\nu}\right)$$

- ***Optically Thin Absorption,*** $\tau_0 \lesssim 1$ ***(linear regime)***

$$W = \frac{b}{c} \int_{-\infty}^{\infty} du \left(\tau_\nu - \frac{\tau_\nu^2}{2} + \cdots\right) \approx \frac{b}{c} \int_{-\infty}^{\infty} du \left(\tau_0 e^{-u^2} - \tau_0^2 \frac{e^{-2u^2}}{2} + \cdots\right)$$

$$\tau_\nu = \tau_0 H(u,a) \approx \tau_0 e^{-u^2} \quad \text{if} \quad a \ll 1$$

$$= \sqrt{\pi} \frac{b}{c} \tau_0 \left(1 - \frac{\tau_0}{2\sqrt{2}} + \cdots\right)$$

$$W \approx \sqrt{\pi} \frac{b}{c} \frac{\tau_0}{1 + \tau_0/(2\sqrt{2})} \longleftarrow 1 - x \approx \frac{1}{1+x}$$

$$= \frac{\pi e^2}{m_e c^2} N_\ell f_{\ell u} \lambda_{\ell u} \frac{1}{1 + \tau_0/(2\sqrt{2})} \longleftarrow \tau_0 = \frac{\sqrt{\pi} e^2}{m_e c} f_{\ell u} \frac{\lambda_{\ell u}}{b} N_\ell$$

$$W = 4.48 \times 10^{-6} \left(\frac{N_\ell}{10^{12}\,\mathrm{cm}^{-2}}\right) \left(\frac{f_{\ell u}}{0.4164}\right) \left(\frac{\lambda_{\ell u}}{1215.67\,\text{Å}}\right)$$

$$N_\ell = 1.84 \times 10^{12}\,\mathrm{cm}^{-2} \left(\frac{0.4164}{f_{\ell u}}\right) \left(\frac{1215.67\,\text{Å}}{\lambda_{\ell u}}\right)^2 \left(\frac{W_\lambda}{0.01\,\text{Å}}\right) \quad \text{if} \quad \tau_0 \lesssim 1$$

$$\leftarrow \quad N_\ell = W \frac{m_e c^2}{\pi e^2} \frac{1}{f_{\ell u} \lambda_{\ell u}}, \qquad W \approx \frac{W_\lambda}{\lambda_{\ell u}}$$

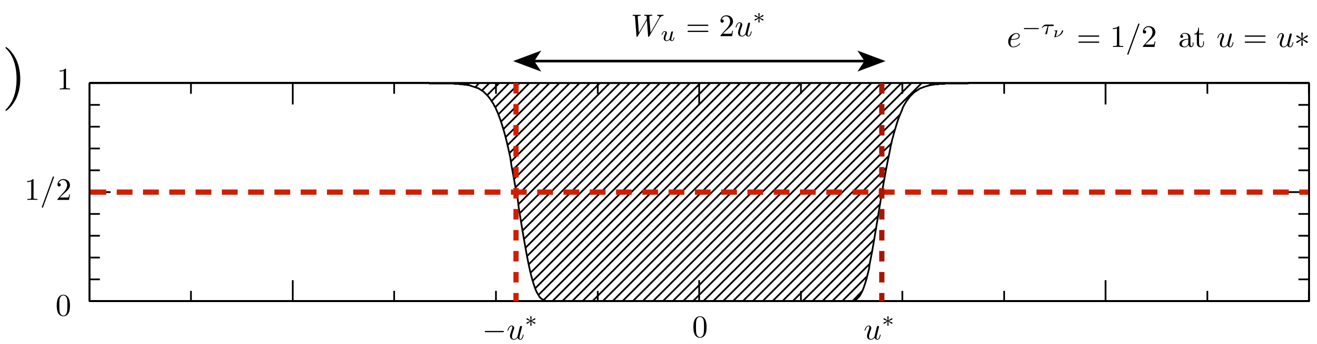
The measurement of W allows us to determine N_ℓ, provided that the line is optical thin.

- ***Flat Portion of the Curve of Growth,*** $1 < \tau_0 \lesssim \tau_{\rm damp}$
  - Now consider what happens when an absorption line is optically thick, but not so optically thick that the the broad Lorentz wing ν^{-2} provides a significant contribution to the absorption.
  - The optical depth at which the wing becomes important is called the ***damping optical depth*** $\tau_{\rm damp}$.

$$W = \frac{b}{c}\int_{-\infty}^{\infty} du \left[1 - \exp\left(-\tau_0 e^{-u^2}\right)\right]$$

  - The absorption line shape is almost "box-shaped." We assume that the term in square brackets equals "1" until a certain value u_* and then, suddenly drops to "0".
  - We define u_* to be the location at half maximum of the square brackets:

$$\exp\left(-\tau_0 e^{-u_*^2}\right) = \frac{1}{2} \;\longrightarrow\; u_*^2 = \ln\left(\tau_0/\ln 2\right)$$



  - Then, we have

$$W \approx \frac{b}{c}\int_{-u_*}^{u_*} du = \frac{b}{c}(2u_*) \longrightarrow W \approx \frac{2b}{c}\sqrt{\ln\left(\tau_0/\ln 2\right)}$$

  - Note that W is very insensitive to τ_0 (and thus N_ℓ) in this regime. Because W increases so slowly with increasing N_ℓ, this is referred to as the flat portion of the curve of growth.

- Inverting the above equation, we obtain

$$\tau_0 \approx (\ln 2) \exp\left[\left(\frac{cW}{2b}\right)^2\right]$$

$$N_\ell \approx \frac{\ln 2}{\sqrt{\pi}} \frac{m_e c}{e^2} \frac{b}{f_{\ell u}\lambda_{\ell u}} \exp\left[\left(\frac{cW}{2b}\right)^2\right]$$

$$N_\ell \approx 9.15 \times 10^{12}\,\mathrm{cm}^{-2} \left(\frac{0.4164}{f_{\ell u}}\right) \left(\frac{1215.67\,\text{Å}}{\lambda_{\ell u}}\right) \left(\frac{b}{10\,\mathrm{km\,s^{-1}}}\right)$$
$$\times \exp\left[0.0152 \left(\frac{1215.67\,\text{Å}}{\lambda_{\ell u}}\right)^2 \left(\frac{10\,\mathrm{km\,s^{-1}}}{b}\right)^2 \left(\frac{W_\lambda}{0.01\,\text{Å}}\right)^2\right]$$

- The column density at a given equivalent width depends on the temperature, and thus on the thermal broadening.

- Any error in evaluating W_λ (from misestimating the continuum flux, for instance) will propagate exponentially into an error in N_ℓ.
- Therefore, it is advised not to use the above equation unless you have a very good idea of what the equivalent width W_λ and the thermal broadening b are for the sightline in question.

- ***Damped Portion of the Curve of Growth,*** $\tau_0 \gtrsim \tau_{\rm damp}$ ***(square-root regime)***
  - In this regime, the Doppler core of the line is totally saturated, but the "damping wing" of the Voigt profile start to contribute significantly to the equivalent width.

$$\boxed{W = \frac{b}{c}\int_{-\infty}^{\infty} du \left[1 - \exp\left(-\tau_0 \frac{a}{\sqrt{\pi}u^2}\right)\right]}$$

$$\exp\left(-\tau_0 e^{-u^2}\right) = 0 \qquad \text{if} \quad \tau_0 \gtrsim \tau_{\rm damp} \quad \text{and} \quad \mathrm{u} \lesssim \mathrm{u}_*$$
$$\exp\left(-\tau_0 \frac{a}{\sqrt{\pi}u^2}\right) = 0$$

change of variables: Let $\tau_0 \dfrac{a}{\sqrt{\pi}u^2} = \dfrac{1}{x^2} \quad \rightarrow \quad u = \left(\dfrac{\tau_0 a}{\sqrt{\pi}}\right)^{1/2} x$

$$W = \frac{b}{c}\left(\frac{\tau_0 a}{\sqrt{\pi}}\right)^{1/2} \underbrace{\int_{-\infty}^{\infty} dx \left[1 - \exp\left(-1/x^2\right)\right]}_{2\sqrt{\pi}} = \frac{b}{c}\left(\frac{\tau_0 a}{\sqrt{\pi}}\right)^{1/2} 2\sqrt{\pi}$$

$$\begin{aligned} I \equiv \int_{-\infty}^{\infty} dx \left(1 - e^{-1/x^2}\right) &= 2\int_0^{\infty} dx \left(1 - e^{-1/x^2}\right) \\ &= 2\int_0^{\infty} \frac{dy}{y^2}\left(1 - e^{-y^2}\right) \qquad \leftarrow \ y = 1/x \\ &= 2\left[-\frac{1}{y}\left(1 - e^{-y^2}\right)\right]_0^{\infty} + 2\int_0^{\infty} \frac{1}{y}\left(2ye^{-y^2}\right) dy \\ &= \lim_{y\to 0} \frac{2}{y}\left(1 - e^{-y^2}\right) + 4\int_0^{\infty} e^{-y^2} dy \\ &= \lim_{y\to 0} \frac{2}{y}\left(1 - 1 + y^2\right) + 2\int_{-\infty}^{\infty} e^{-y^2} dy = 2\sqrt{\pi} \end{aligned}$$

Therefore, we have

$$W = \frac{b}{c}\left(4\sqrt{\pi}\tau_0 a\right)^{1/2}$$

$$a = \frac{\gamma_{\ell u}}{4\pi}\frac{1}{\nu_{\ell u}(b/c)} = \frac{\gamma_{\ell u}}{4\pi}\frac{\lambda_{\ell u}}{b}$$

$$W = \sqrt{\frac{b}{c}\frac{\tau_0}{\sqrt{\pi}}\frac{\gamma_{\ell u}\lambda_{\ell u}}{c}} \quad \longrightarrow \quad \tau_0 = \sqrt{\pi}\frac{c}{b}\frac{c}{\gamma_{\ell u}\lambda_{\ell u}} W^2$$

- Approximating the absorption profile as a boxy shape (square brackets)
  - Draine's book define u_* to be the location at half maximum. (FWHM = $2u^*$)

$$\exp\left(-\tau_0 \frac{a}{\sqrt{\pi}u_*^2}\right) = \frac{1}{2} \longrightarrow u_*^2 = \frac{\tau_0}{\sqrt{\pi}} \frac{a}{\ln 2}$$

$$a = \frac{\gamma_{\ell u}}{4\pi} \frac{1}{\nu_{\ell u}(b/c)} = \frac{\gamma_{\ell u}}{4\pi} \frac{\lambda_{\ell u}}{b}$$

$$W \approx \frac{b}{c} \int_{-u_*}^{u_*} du = \frac{b}{c}(2u_*) \longrightarrow W \approx \frac{2b}{c} \sqrt{\frac{\tau_0}{\sqrt{\pi}} \frac{a}{\ln 2}}$$

$$= \frac{1}{\sqrt{\pi \ln 2}} \sqrt{\frac{b}{c} \frac{\tau_0}{\sqrt{\pi}} \frac{\gamma_{\ell u}\lambda_{\ell u}}{c}}$$

    - He note that this value is smaller than by a factor of $\sqrt{\pi \ln 2} = 1.476$. Multiplying by this factor, he obtain the same result as ours.
  - Our result is obtained by choosing the following u_*:

$$\exp\left(-\tau_0 \frac{a}{\sqrt{\pi}u_*^2}\right) = \exp\left(-\frac{1}{\pi}\right) = 0.7274$$

$$u_*^2 = \frac{\tau_0}{\sqrt{\pi}}(\pi a) = \pi \ln 2 \left(\frac{\tau_0}{\sqrt{\pi}} \frac{a}{\ln 2}\right) \longrightarrow W = \frac{b}{c}(2u_*) = \frac{b}{c}\left(4\sqrt{\pi}\tau_0 a\right)^{1/2}$$

$$N_\ell = \frac{m_e c^3}{e^2} \frac{1}{f_{\ell u} \gamma_{\ell u} \lambda_{\ell u}^2} W^2 = \frac{m_e c^3}{e^2} \frac{1}{f_{\ell u} \gamma_{\ell u} \lambda_{\ell u}^4} W_\lambda^2$$

$$N_\ell = 1.867 \times 10^{18}\,\mathrm{cm}^{-2} \left(\frac{0.4164}{f_{\ell u}}\right) \left(\frac{6.265 \times 10^8\,\mathrm{s}^{-1}}{\gamma_{\ell u}}\right) \left(\frac{1215.67\,\text{Å}}{\lambda_{\ell u}}\right)^4 \left(\frac{W_\lambda}{1\,\text{Å}}\right)^2$$

- The equivalent width is proportional to the square-root of the optical depth. The column density (optical depth) is proportional to the square of the measured equivalent width.
- Furthermore, ***the column density is independent of the thermal broadening.***

• The ***damping optical depth*** at which the transition from the flat to the damped portion of the curve of growth occurs are obtained by setting $W^{\mathrm{flat}} = W^{\mathrm{sq.-root}}$.

$$\frac{2b}{c}\sqrt{\ln\left(\tau_{\mathrm{damp}}/\ln 2\right)} = \sqrt{\frac{b}{c}\frac{\tau_{\mathrm{damp}}}{\sqrt{\pi}}\frac{\gamma_{\ell u}\lambda_{\ell u}}{c}}$$

$$\tau_{\mathrm{damp}} = 4\sqrt{\pi}\frac{b}{\gamma_{\ell u}\lambda_{\ell u}} \ln\left(\tau_{\mathrm{damp}}/\ln 2\right)$$
$$\approx 4\sqrt{\pi}\frac{b}{\gamma_{\ell u}\lambda_{\ell u}} \ln\left(\frac{4\sqrt{\pi}}{\ln 2}\frac{b}{\gamma_{\ell u}\lambda_{\ell u}}\right)$$

An approximate solution for (when $c \gg 1$)

$$y = c\ln\left(y/\ln 2\right) \rightarrow \text{ set } x = y/\ln 2 \text{ and } C = c/\ln 2$$
$$\rightarrow x = C\ln(x)$$

We use the fixed point iteration method.

$$x^{(0)} = e$$
$$x^{(1)} = C\ln x^{(0)} = C$$
$$x^{(2)} = C\ln x^{(1)} = C\ln C$$
$$x^{(3)} = C\ln x^{(2)} = C\ln C + C\ln\left(\ln C\right)$$

Then, we obtain an approximate solution:

$$x \approx x^{(2)} = C\ln C \quad \rightarrow \quad y \approx c\ln(c/\ln 2)$$

This solution is found to underestimate τ_{damp} by a factor of ~1.4 (~1.3) for b = 1 (10) km s^{-1}.

$$C \equiv 4\sqrt{\pi}\frac{b}{\gamma_{\ell u}\lambda_{\ell u}} = 93.1\left(\frac{b}{1\,\mathrm{km\,s^{-1}}}\right)\left(\frac{6.265\times 10^{8}\,\mathrm{s^{-1}}}{\gamma_{\ell u}}\right)\left(\frac{1215.67\,\mathrm{\AA}}{\lambda_{\ell u}}\right)$$

$$\begin{aligned}
\tau_{\mathrm{damp}} &\approx 93.1\left(\frac{b}{1\,\mathrm{km\,s^{-1}}}\right)\left(\frac{7616\,\mathrm{cm\,s^{-1}}}{\gamma_{\ell u}\lambda_{\ell u}}\right)\ln\left[134\left(\frac{b}{1\,\mathrm{km\,s^{-1}}}\right)\left(\frac{7616\,\mathrm{cm\,s^{-1}}}{\gamma_{\ell u}\lambda_{\ell u}}\right)\right] \\
&= 456\left(\frac{b}{1\,\mathrm{km\,s^{-1}}}\right)\left[1+0.204\ln\left(\frac{b}{1\,\mathrm{km\,s^{-1}}}\right)\right] \\
&= 635\left(\frac{b}{1.3\,\mathrm{km\,s^{-1}}}\right)\left[1+0.194\ln\left(\frac{b}{1.3\,\mathrm{km\,s^{-1}}}\right)\right] \\
&= 931\left(\frac{b}{10\,\mathrm{km\,s^{-1}}}\right)\left[1+0.139\ln\left(\frac{b}{10\,\mathrm{km\,s^{-1}}}\right)\right]
\end{aligned}$$

$$[N_\ell]_{\mathrm{damp}} \approx \frac{4m_e c}{e^2}\frac{b^2}{f_{\ell u}\gamma_{\ell u}\lambda_{\ell u}^2}\ln\left[\frac{4\sqrt{\pi}}{\ln 2}\frac{b}{\gamma_{\ell u}\lambda_{\ell u}}\right]$$

$$[N_\ell]_{\mathrm{damp}} = 1.23\times 10^{16}\,[\mathrm{cm^{-2}}]\left(\frac{0.4164}{f_{\ell u}}\right)\left(\frac{1215.67\,\mathrm{\AA}}{\lambda_{\ell u}}\right)\left(\frac{b}{10\,\mathrm{km\,s^{-1}}}\right)\left(\frac{\tau_{\mathrm{damp}}}{931}\right)$$

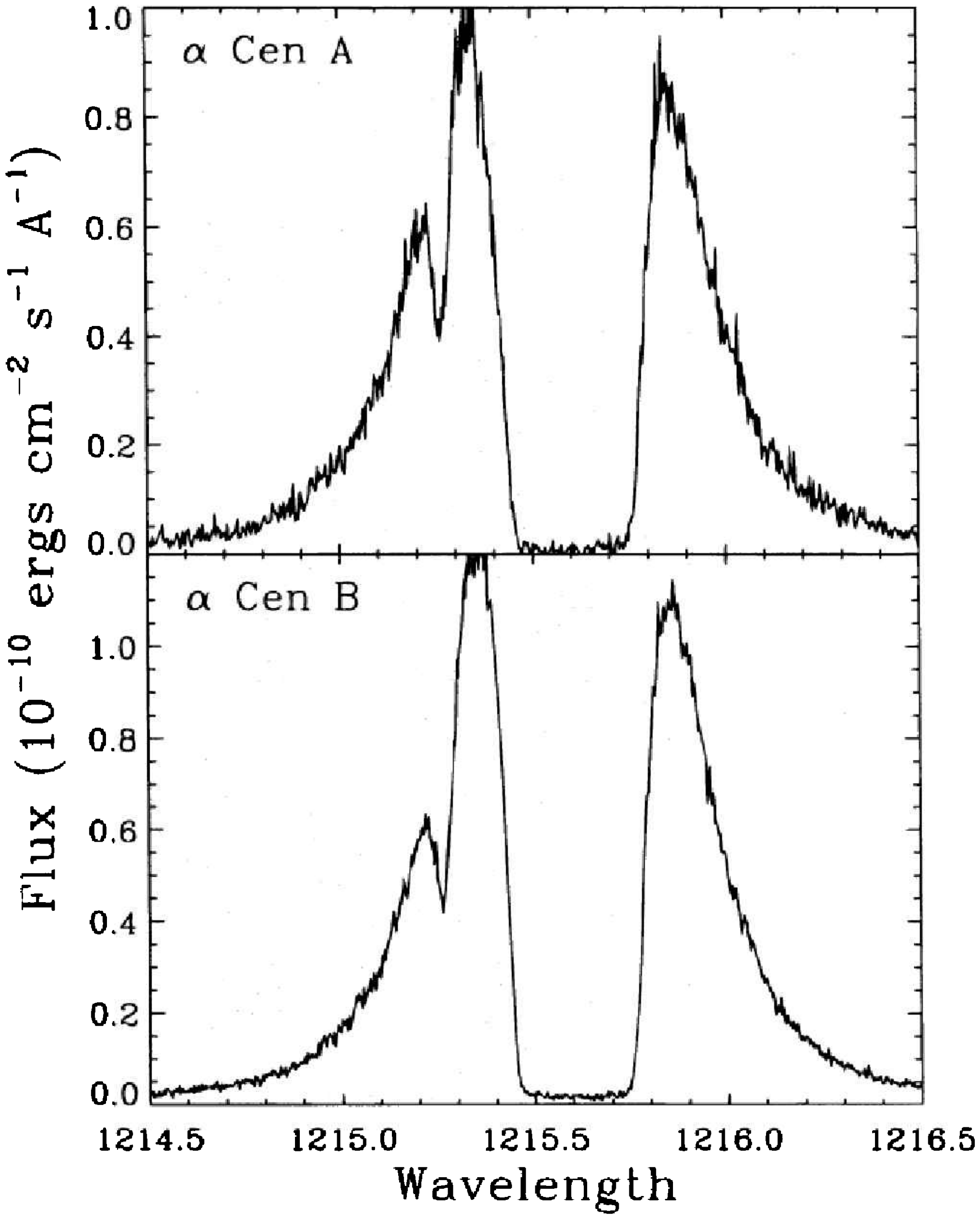
Observations: H and D Lyα

- The study of Lyα absorption line is useful to studying the cold clouds in our galaxy. However, Lyα tends to be optically thick.
- α Cen A and B (d = 1.34 pc) have broad Lyα "emission" lines from their hot chromospheres.
  - Superposed on the emission lines are optically thick absorption lines by the ISM.
  - For both α Cen A and B, $W_\lambda = 0.3\,\text{Å}$.

    $$\tau_0 = 68,000, \;\; b = 11.8\,\text{km}\,\text{s}^{-1} \;\; (T = 8300\,\text{K})$$
    $$N_\ell = 1.1 \times 10^{18}\,\text{cm}^{-2}$$

  - This represents the regime where the flat part of the curve of growth gives way to the square-root part. Hence, the column density is independent of the thermal broadening.
  - The stars are within our Local "Hot" Bubble so that the temperature is high. The column density imply that a density of 0.25 cm^{-3}.

$$n_\text{H} = N_\text{H}/d = 1.1 \times 10^{18}\,\text{cm}^{-2}/1.34\,\text{pc}$$



Lyα lines toward α Cen A (above) and α Cen B (below).

Figure 2.8 in [Ryden], (Linsky & Wood 1996)

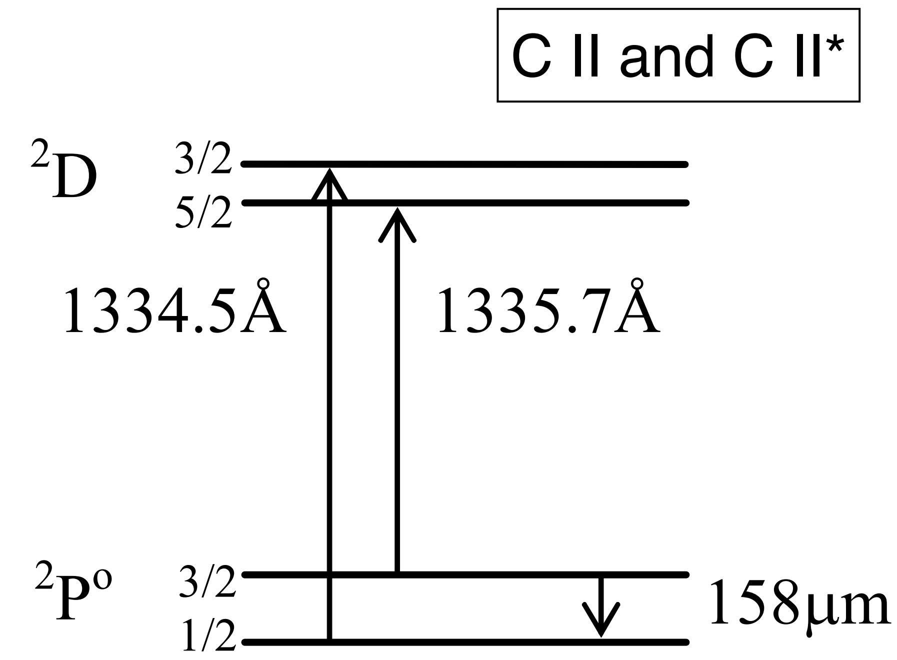
- On the left-hand slope of the Lyα emission lines, there is an optically thin absorption line. This is the absorption line of deuterium Lyα.
  - The deuterium Lyα lines are slightly blueshifted. wavelength for H Lyα = 1215.67Å, wavelength for deuterium Lyα is 1215.24Å.
  - The deuterium Lyα lines are optical thin, with $\tau_0 = 0.68$, and thus easier to interpret than the H Lyα from ordinary hydrogen. The column density of deuterium toward α Cen A and B, is $N_\ell = 6.1 \times 10^{12}\,\mathrm{cm}^{-2}$, giving a deuterium to hydrogen ratio $D/H \approx 6 \times 10^{-6}$. This is lower than the usual $D/H \approx 1.6 \times 10^{-5}$ in the Local Bubble, and is much less than the primordial value of $D/H \approx 2.5 \times 10^{-5}$.

- For stars outside the Local Bubble, at d ~ 100 pc, Lyα lines are in the square-root part of the curve of growth, with $W_\lambda \sim 10\,\text{Å},\ N_\ell \sim 2 \times 10^{20}\,\mathrm{cm}^{-2},\ n_\ell \sim 0.6\,\mathrm{cm}^{-3}$.

- We observe in the visible and UV spectrum of many stars a substantial number of atomic interstellar absorption lines.
- The coexistence of Ca^0 and Ca^+ lines also allows us to obtain the degree of ionization of the corresponding cloud. We observe that Ca^+ is much more abundant than Ca^0 (ionization potential of Ca is 6.11 eV).

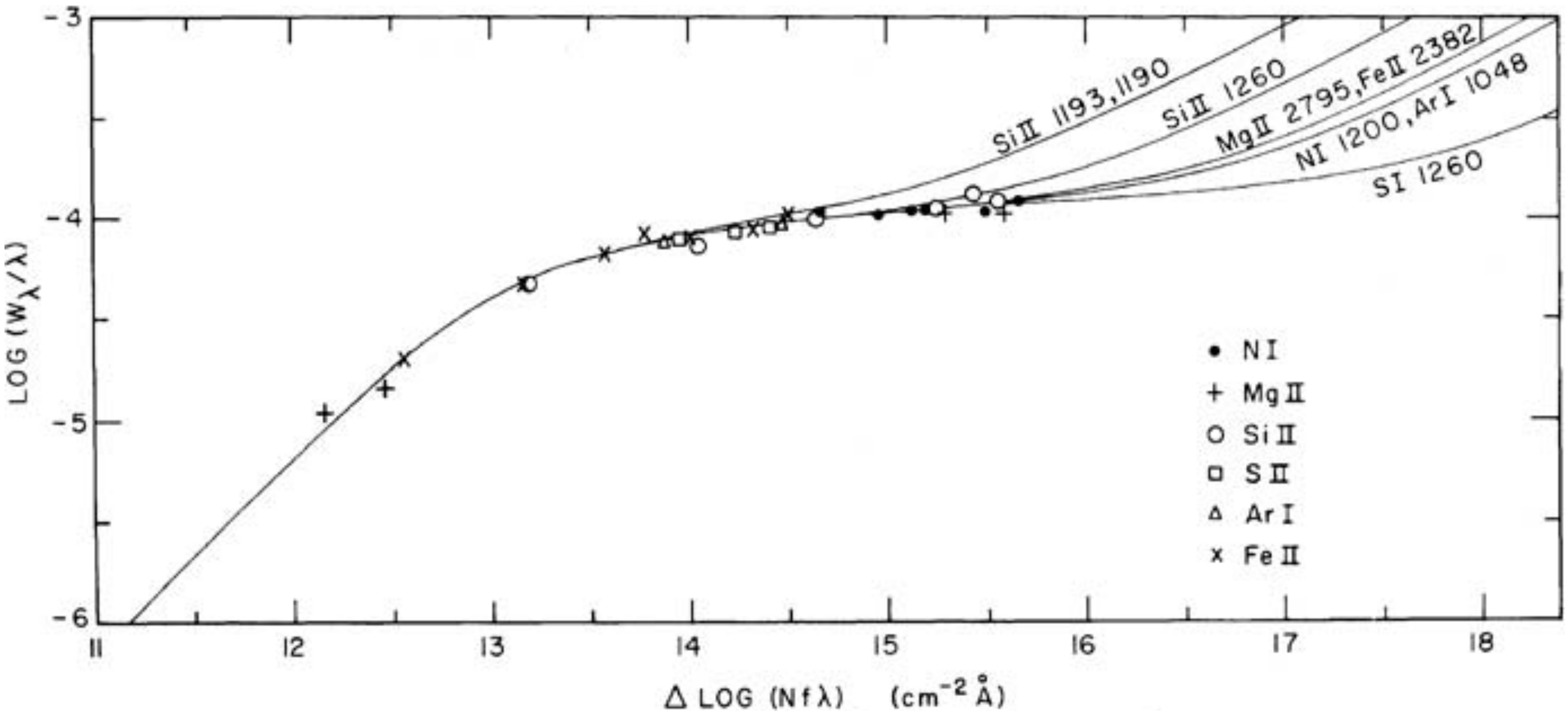
Observations: Absorption lines from fine-structure levels

- We also observe several absorption lines with wavelength close to each other, which comes from fine-structure levels of the fundamental ground state of the same atom or ion.
  - We can then directly obtain the relative populations of these levels. This gives valuable information on those physical parameters that determine their excitation, essentially the electron density.
  - For examine, C II 1334.57Å and C II* 1335.70Å lines, which unfortunately are often saturated.



  - Morton (1975)'s observations of the C II and C II* ratio showed a significant population of atoms in $^2P^o_{3/2}$. This suggested that [C II] 157.7 μm line could be a strong cooling line in H I regions. It was not until the 1980s that the Kuiper Airborne Observatory detected Far-IR [C II] emission from the ISM, as predicted by Morton.
  - We may determine the cooling rate of the diffuse CNM due to [C II] 157.7 μm by observing the C II* 1335.70Å absorption line.
  - This observation gave a cooling rate of $\sim 3.5 \times 10^{-26}\,\mathrm{erg\,s^{-1}}$ per hydrogen nucleus (Pottasch et al. 1979; Gry et al. 1992), which is in agreement with the more direct determination of Bennett et al. (1994)

- Observed curve of growth for the ISM in front of the star ζ Oph (Morton1975, ApJ)



  - As can be seen in the above figure, most observed absorption lines lie on the Doppler plateau.
  - Therefore, a better reduction technique than the use of curves of growth would be to adopt a fitting technique for the line profiles. This technique is the only one that can be used for complex line profiles.
- So, optical and UV absorption lines provide us useful information about the cold regions in the ISM. How much of each element and isotope is present? How hot is the gas? What are the integrated densities along the line of sight?

Observations: The Gas Phase Abundances

- The gas phase abundances of many elements relative to H have been determined on many different nightlines using interstellar absorption lines.
  - The observed gas-phase abundances vary from one sightline to another, which is presumed to reflect primarily variations in the amounts of various elements trapped in dust grains. Such removal of elements from the gas is known as ***interstellar depletion***.
  - Some elements, line Fe, are extremely under abundant in the gas phase, with gas-phase abundances that are typically only a few percent of the solar abundance.

Element	Solar system	Stars	H II	T_c[1]	ζ Oph cold	ζ Oph warm
	12 + log(X/H)			K	[X/H]	
H	12.00	12.00	12.00	–	–	–
D	7.53	–	–	–	–0.33:[2]	
He	10.99	–	10.95	–	–	–
Li	3.31	–	–	1 225	–1.58	–
B	2.88	–	–	650	–0.93	–
C	8.55	8.33	8.60	75	–0.41	–
N	7.97	7.82	7.89	120	–0.07	–
O	8.87	8.66	8.77	180	–0.39	0.00
Ne	–	–	8.03	–	–	–
Na	6.31	–	–	970	–0.95	–
Mg	7.58	7.40	–	1 340	–1.55	–0.89
Si	7.55	7.27	–	1 311	–1.31	–0.53
P	5.57	–	–	1 151	–0.50	–0.23
S	7.27	7.09	7.31	648	+0.18	–
Ar	6.56	–	–	25	–0.48	–
K	5.13	–	–	1 000	–1.09	–
Ca	6.34	6.20	–	1 518	–3.73	–
Ti	4.93	4.81	–	1 549	–3.02	–1.31
Fe	7.50	7.43	6.59	1 336	–2.27	–1.25

[1] Condensation temperature at thermal and chemical equilibrium, appropriate for the Solar nebula with an initial gas pressure of 10^{-4} bar. ($1\,\mathrm{bar} = 10^6\,\mathrm{dyn\,cm^{-2}}$)
[2] For lines of sight other than that of ζ Oph: Linsky et al. (1995) .

The gas phase abundances along two lines of sight, compared to abundances in the Solar system.

Abundances are given as $12 + \log(X/H)$, X being the chemical symbol for the element and H that of hydrogen.

The deficiencies in columns 6 and 7 are expressed as:

$$[X/H] = \log\,(X/H) - \log\,(X/H)_{\odot}$$

The data come mainly from Savage & Sembach (1996) and from Snow & Witt (1996).

Table 4.2 in The Interstellar Medium [Lequeux]

Gas-phase abundance vs. Condensation Temperature

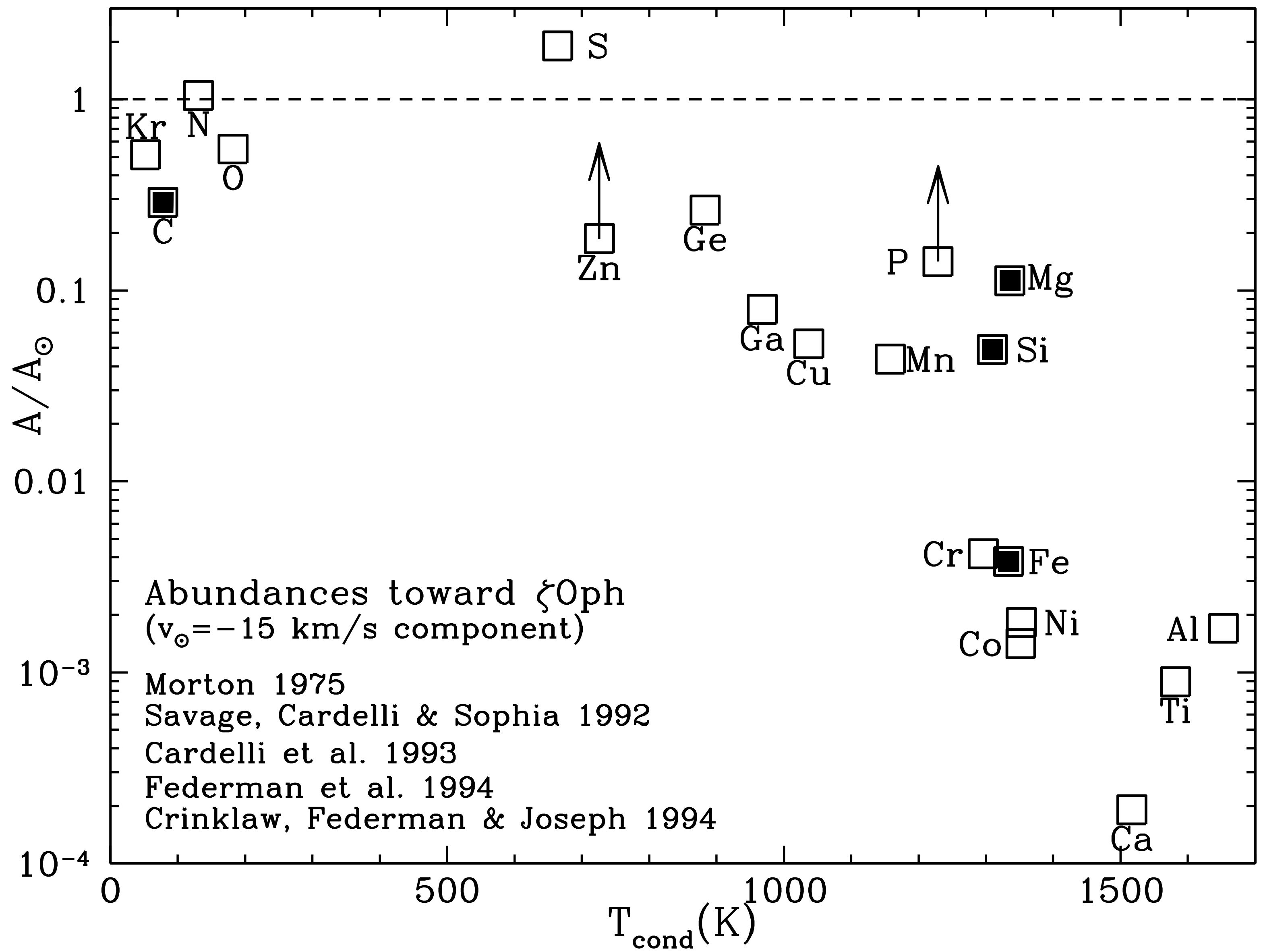


Figure 23.1 in [Draine]

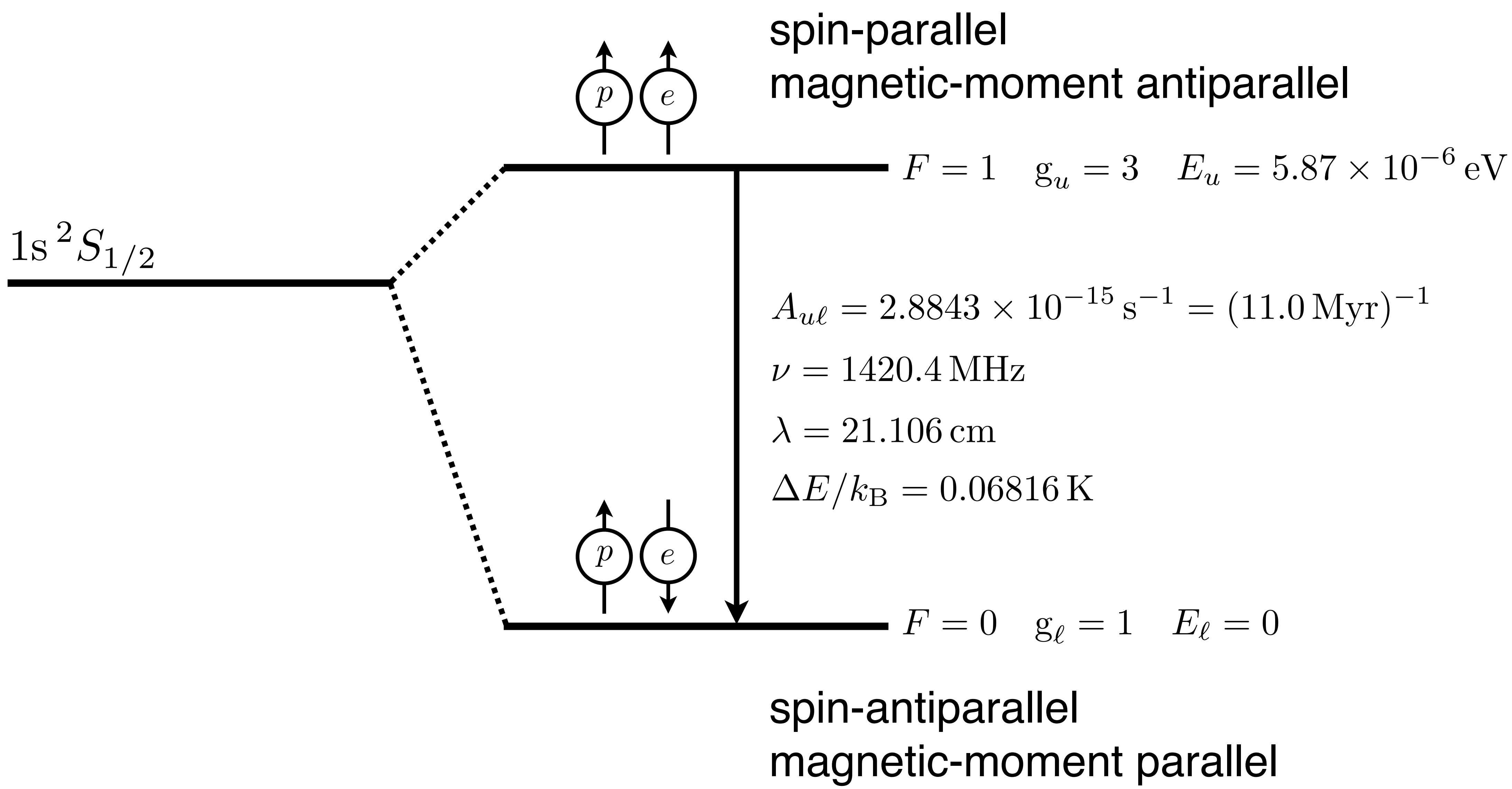
Gas-phase abundances (relative to solar) in the diffuse cloud toward ζ Ophiuchi (O9.5V star, 138 pc), plotted versus "condensation temperature". Solid symbols: major grain constituents C, Mg, Si, Fe. The apparent overabundance of S may be due to observational error, but may be arise because of S II absorption in the H II region around ζ Oph. There's a strong tendency for elements with high $T_{\rm cond}$ to be under abundant in the gas phase, presumably because most of the atoms are in solid grains.

Condensation temperature : temperature at which 50% of the element in question would be incorporated into solid material in a gas of solar abundances, at LTE at a pressure $p = 10^2$ dyn cm^{-2} (Lodders 2003).

21 cm hyperfine line

- **The CNM and WNM, taken together, provide over half the mass of the ISM.**
  - H is the most abundant element in the universe. In the CNM and WNM, most of the hydrogen is in the form of neutral atoms.
  - The Lyα line of H provides a useful probe of the properties of the CNM and WNM. However, at its wavelength the Earth's atmosphere is highly opaque, and thus observing Lyα absorption requires orbiting UV satellites. In addition, Lyα can be seen in absorption only along those lines of sight toward sources with a high UV flux.
  - To do a global survey of atomic hydrogen in the galaxy, we need some way of easily detecting radiation from hydrogen, regardless of its kinetic temperature or number density.
  - Such a way was first found in 1944, by Henk van de Hulst.
    - He attempted to find emission lines at the wavelengths ~ 1 cm to 20 m, at which the Earth's atmosphere is transparent. He then realized that the hyperfine structure line resulting from a flip of the electron spin within a hydrogen atom should have a wavelength of 21 cm.
    - This was confirmed by Harold Ewen and Edward Purcell in 1951, when they first detected 21 cm emission from the Milky Way.

Hyperfine splitting of the 1s ground state of atomic H



Note that the magnetic moment is proportional to the charge, so the electron and proton have opposite directions of the magnetic moments.

Difference between Lyα and 21 cm transitions

- The excitation energy for Lyα (E = 10.2 eV, $E/k_{\rm B}$ = 118,000 K) is much higher than the kinetic temperature of the neutral ISM.

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{118,000\,{\rm K}}{T}\right) = 1.7 \times 10^{-51} \text{ at } T = 1000\,{\rm K}$$

  - Collisional excitation is unimportant, and most hydrogen atoms are in the lower level of the Lyα transition.
  - The Lyα has a higher energy by a factor of 1.7x10^6 than the 21 cm.

- The excitation energy for 21 cm is ~ 5.9 μeV, and its equivalent temperature $E/k_{\rm B}$ = 0.068 K is much lower than the temperature of the cosmic microwave background.
  - ***Even the CMB is able to populate the upper level.***
  - ***If collisions are frequent, then the spin temperature will be solely determined by collisions, and thus will be a good tracer of the gas kinetic temperature.***
  - Thus, there is ample opportunity to populate the upper energy level of the 21 cm hyperfine transition. The level populations for the 21 cm levels, since $T_{\rm exc} \gg 0.068\,{\rm K}$ in all circumstances of the ISM.

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} e^{-h\nu/kT_{\rm exc}} = 3\,e^{-0.068\,{\rm K}/T_{\rm exc}} \simeq 3 \longrightarrow \boxed{n_u \simeq \frac{3}{4} n_{\rm H}, \; n_\ell \simeq \frac{1}{4} n_{\rm H}}$$

  - However, in many cases (in particular in WNM), the hyperfine levels may not be in excitation equilibrium. Radio astronomers use the term ***spin temperature*** for 21 cm rather than the "excitation temperature."

Emissivity and Optical Depth of H I 21 cm

- ***Emissivity***:
  - The upper level contains ~ 75% of the H I under all conditions of interest, and thus *the 21-cm emissivity is effectively independent of the spin temperature*.

$$j_\nu = n_u \frac{A_{u\ell}}{4\pi} h\nu_{u\ell}\phi_\nu \simeq \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} n_{\rm H}\phi_\nu \qquad \left(n_u \simeq \frac{3}{4} n_{\rm H}\right)$$

- ***Optical depth***

$$\kappa_\nu = n_\ell \sigma_{\ell u} - n_u \sigma_{u\ell} = n_\ell \sigma_{\ell u}\left(1 - e^{-h\nu_{u\ell}/kT_{\rm spin}}\right)$$

Because $h\nu_{u\ell}/kT_{\rm spin} \ll 1$ for all conditions of interest, the correction for stimulated emission is very important!

$$\kappa_\nu \simeq n_\ell \sigma_{\ell u} \frac{h\nu_{u\ell}}{kT_{\rm spin}} \ll n_\ell \sigma_{\ell u} \qquad \longleftarrow \qquad e^{-h\nu_{u\ell}/kT_{\rm spin}} \simeq 1 - k\nu_{u\ell}/kT_{\rm spin}$$

$$\kappa_\nu \simeq \left(\frac{1}{4} n_{\rm H}\right)\left(\frac{g_u}{g_\ell}\frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell}\phi_\nu\right)\frac{h\nu_{u\ell}}{kT_{\rm spin}} \qquad \longleftarrow \qquad \left(n_\ell \simeq \frac{1}{4} n_{\rm H}\right)$$

$$\boxed{\kappa_\nu \simeq \frac{3}{32\pi} A_{u\ell} \frac{hc\,\lambda_{u\ell}}{kT_{\rm spin}} n_{\rm H}\phi_\nu} \qquad \longleftarrow \qquad \frac{g_u}{g_\ell} = 3$$

The absorption coefficient is inversely proportional to the spin temperature.

- The damping constant of the 21 cm line profile is extremely small, and thus we can assume that *the line profile is a Gaussian*.

$$a = \frac{\gamma_{u\ell}}{4\pi}\frac{\lambda_{u\ell}}{b} = 4.844\times10^{-20}\left(\frac{\gamma_{u\ell}}{2.8843\times10^{-15}\,\mathrm{s}^{-1}}\right)\left(\frac{\lambda_{u\ell}}{21.106\,\mathrm{cm}}\right)\left(\frac{1\,\mathrm{km\,s}^{-1}}{b}\right)$$

- Hence,

$$\phi_\nu = \frac{1}{\sqrt{\pi}\Delta\nu_D}H(u,a) \simeq \frac{c}{\sqrt{\pi}\nu_{\ell u}b}e^{-u^2}$$

$$\left(u = v/b,\; b = \sqrt{2}v_{\rm rms} = \sqrt{2kT_{\rm gas}/m_{\rm H}}\right)$$

$$\tau_\nu = \kappa_\nu s = \frac{3}{32\pi}A_{u\ell}\frac{hc\,\lambda_{u\ell}}{kT_{\rm spin}}N_{\rm H\,I}\phi_\nu$$

$$= \frac{3}{32\pi}\frac{1}{\sqrt{\pi}}\frac{A_{u\ell}\lambda_{u\ell}^2}{b}\frac{hc}{kT_{\rm spin}}N_{\rm H\,I}e^{-u^2}$$

$$= 3.111\left(\frac{N_{\rm H\,I}}{10^{21}\,\mathrm{cm}^{-2}}\right)\left(\frac{100\,\mathrm{K}}{T_{\rm spin}}\right)\left(\frac{1\,\mathrm{km\,s}^{-1}}{b}\right)e^{-u^2}$$

or $$\tau_\nu = 2.201\left(\frac{N_{\rm H\,I}}{10^{21}\,\mathrm{cm}^{-2}}\right)\left(\frac{100\,\mathrm{K}}{T_{\rm spin}}\right)\left(\frac{1\,\mathrm{km\,s}^{-1}}{b/\sqrt{2}}\right)e^{-u^2}$$

$N_{\rm H\,I} \equiv \int n_{\rm H}ds$ is the column density of H I.

$\sim 10^{21}\,\mathrm{cm}^{-21}$ toward the Galactic disk.

Some lines of sight through our galaxy (at high galactic latitude) are optically thin and other lines of sight (at low galactic latitude) are optically thick at 21 cm.

- ***Self-absorption in the 21-cm line can be important*** in many sightlines in the ISM.
- ***The optical depth is inversely proportional to the spin temperature.***

- Typical optical depths of the 21-cm line:

$$\tau_0 = 0.311 \left(\frac{N_{\mathrm{H\,I}}}{10^{21}\,\mathrm{cm}^{-2}} \right) \left(\frac{100\,\mathrm{K}}{T_{\mathrm{spin}}} \right) \left(\frac{10\,\mathrm{km\,s}^{-1}}{b} \right)$$

- In the CNM, a typical spin temperature is $T_{\mathrm{spin}} \approx 50 - 100\,K$:

$$\tau_0^{\mathrm{CNM}} \approx 0.3 - 0.6$$
$$e^{-\tau_0} \approx 0.55 - 0.74$$

The CNM is in general optically thin, but show significant absorption.

- In the WNM, a typical spin temperature is $T_{\mathrm{spin}} \approx 5000 - 8000\,K$:

$$\tau_0^{\mathrm{WNM}} \approx 0.004 - 0.006$$
$$e^{-\tau_0} \approx 0.995$$

The 21-cm absorption is negligible in the WNM.

A typo in page 59 of Ryden's book: For thermal broadening b values typical of the ~~warm~~ cold neutral medium and excitation temperatures $T_{\mathrm{exc}} \sim 100\,\mathrm{K}$, lines of sight with $N_{\mathrm{HI}} > 10^{21}\,\mathrm{cm}^{-2}$ show significant absorption. (Remember from Section 2.3 that Lyman α becomes

[1] Column Density Determination - emission

- ***Radio astronomers express the line profile as a function of radial velocity rather than of frequency.*** This is logical because line broadening is only caused by the Doppler effect, and its natural width being extremely narrow since the lifetime of the upper level is only limited by collisions which is rare in the diffuse medium.

- We first define the column density per velocity interval.

$$\frac{dN_{\rm HI}}{dv} = N_{\rm HI}\phi_v = N_{\rm HI}\frac{1}{\lambda_{u\ell}}\phi_\nu \qquad \phi_\nu = \phi_v\left|\frac{dv}{d\nu}\right| = \phi_v\frac{c}{\nu_{u\ell}} = \lambda_{u\ell}\phi_v$$

- The column density can be written:

$$\tau_\nu = \frac{3}{32\pi}A_{u\ell}\frac{hc\,\lambda_{u\ell}}{kT_{\rm spin}}N_{\rm H\,I}\phi_\nu \longrightarrow \tau(v) = \frac{3}{32\pi}A_{u\ell}\frac{hc\,\lambda_{u\ell}^2}{kT_{\rm spin}(v)}\frac{dN_{\rm HI}}{dv}$$

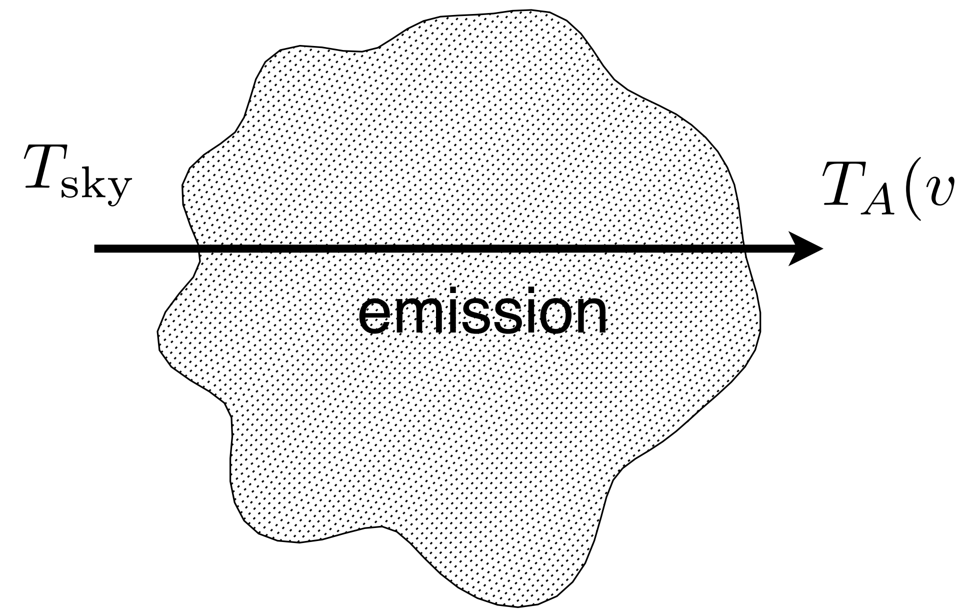
$$\begin{aligned}\frac{dN_{\rm HI}}{dv} &= \frac{32\pi}{3}\frac{k}{A_{u\ell}hc\lambda_{u\ell}^2}T_{\rm spin}(v)\tau(v)\\ &= 1.813\times10^{18}\frac{T_{\rm spin}(v)\tau(v)}{\rm K}\quad\left[\frac{\rm cm^{-2}}{\rm km\,s^{-1}}\right]\end{aligned} \qquad N_{\rm HI} = \int dv\frac{dN_{\rm HI}}{dv}$$

- This indicates that ***we need to know not only the optical depth but also the spin temperature to evaluate the column density***. However, *in an optically thin limit, we will show that the dependency on the spin temperature is removed.*

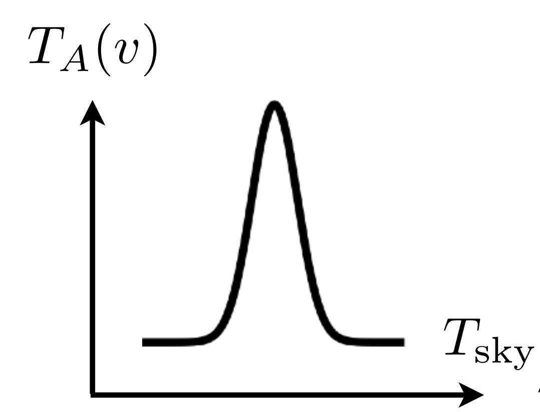
- ***Optically thin case:*** Suppose we are looking through an optically thin layer of neutral hydrogen toward a ***"dark sky"***, which is fainter than the hydrogen cloud, with an antenna temperature $T_{\rm sky}$.
  - In the optically thin limit, the RT equation becomes

$$\begin{aligned} T_A(v) &= T_{\rm sky} e^{-\tau_v} + T_{\rm spin}(v)\left(1 - e^{-\tau_v}\right) \\ &= T_{\rm sky} + \left(T_{\rm spin}(v) - T_{\rm sky}\right)\left(1 - e^{-\tau_v}\right) \\ &\approx T_{\rm sky} + T_{\rm spin}(v)\tau_v \quad \leftarrow \tau_\nu \ll 1, \;\; T_{\rm sky} \ll T_{\rm spin}(v) \end{aligned}$$

$$\tau(v) \approx \frac{T_A(v) - T_{\rm sky}}{T_{\rm spin}(v)}$$



We measure the antenna temperature of the dark sky ($T_{\rm sky}$) from the continuum at frequencies well above and below the 21-cm emission feature.



  - The column density per unit velocity interval is

$$\begin{aligned} \frac{dN_{\rm HI}}{dv} &\approx \frac{32\pi}{3}\frac{k}{A_{u\ell}hc\lambda_{u\ell}^2}\left[T_A(v) - T_{\rm sky}\right] \\ &= 1.813 \times 10^{18}\,\frac{T_A(v) - T_{\rm sky}}{\rm K}\;\left[\frac{\rm cm^{-2}}{\rm km\,s^{-1}}\right] \end{aligned}$$

$$N_{\rm HI} = \int dv \frac{dN_{\rm HI}}{dv}$$

  - Therefore, *the intensity integrated over the line profile gives us the total H I column density without need to know $T_{\rm spin}$, provided that self-absorption is not important*.

- ***Alternative approach:***
  - If we now neglect absorption, then

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu \approx j_\nu \longrightarrow I_\nu = I_\nu(0) + \int j_\nu ds = I_\nu(0) + \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} \phi_\nu N_{\rm HI}$$

  - Now suppose that $I_\nu(0)$ is known independently. We can then integrate the intensity over the line

$$\int [I_\nu - I_\nu(0)]\, d\nu = \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} N_{\rm HI}$$

  - This can be expressed in terms of antenna temperature T_A and relative velocity $v = [(\nu - \nu_{u\ell})/\nu_{u\ell}]\, c$

$$\begin{aligned}\int [T_A - T_A(0)]\, dv &= \int \frac{c^2}{2k\nu^2} [I_\nu - I_\nu(0)] \frac{c}{\nu_{u\ell}} d\nu \\ &\approx \frac{c^3}{2k\nu_{u\ell}^3} \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} N_{\rm HI} \\ &= C_0^{-1} N_{\rm HI}\end{aligned}$$

$$\begin{aligned}C_0 &\equiv \frac{32\pi}{3} \frac{k}{hc\lambda_{u\ell}^2 A_{u\ell}} \\ &= 1.813 \times 10^{18} \left[\frac{\rm cm^{-2}}{\rm K\,km\,s^{-1}}\right]\end{aligned}$$

$$C_0^{-1} = 5.516 \times 10^{-19} \left[\frac{\rm K\,km\,s^{-1}}{\rm cm^{-2}}\right]$$

- We, then, obtain the same equation as before:

$$N_{\mathrm{HI}} \approx C_0 \int [T_A - T_A(0)]\, dv$$
$$= 1.813 \times 10^{18} \int \frac{T_A - T_A(0)}{\mathrm{K\,km\,s^{-1}}} dv \ \ [\mathrm{cm}^{-2}]$$

- Here, we did not use the relation between the optical depth and column density.
  - In the first method, we assumed that $\tau_\nu \ll 1$ and $I_\nu(0) \ll S_\nu$:

$$\begin{aligned} I_\nu &= I_\nu(0)e^{-\tau_\nu} + S_\nu \left(1 - e^{-\tau_\nu}\right) \\ &= I_\nu(0) + [S_\nu - I_\nu(0)] \left(1 - e^{-\tau_\nu}\right) \\ &\approx I_\nu(0) + S_\nu \tau_\nu \end{aligned}$$

  - In the second method, we completely ignored the absorption.

$$\begin{aligned} \frac{dI_\nu}{ds} &= -\kappa_\nu I_\nu + j_\nu \\ &\approx j_\nu \end{aligned}$$

This may be a zeroth order approximation.

Integrating the equation, we obtain the same result as in the first method.

$$I_\nu \approx I_\nu(0) + S_\nu \tau_\nu$$

[2] Spin Temperature Determination

- To derive the spin temperature, **we need to combine the emission observation with an absorption observation,** which is so called ***"emission-absorption" method***.

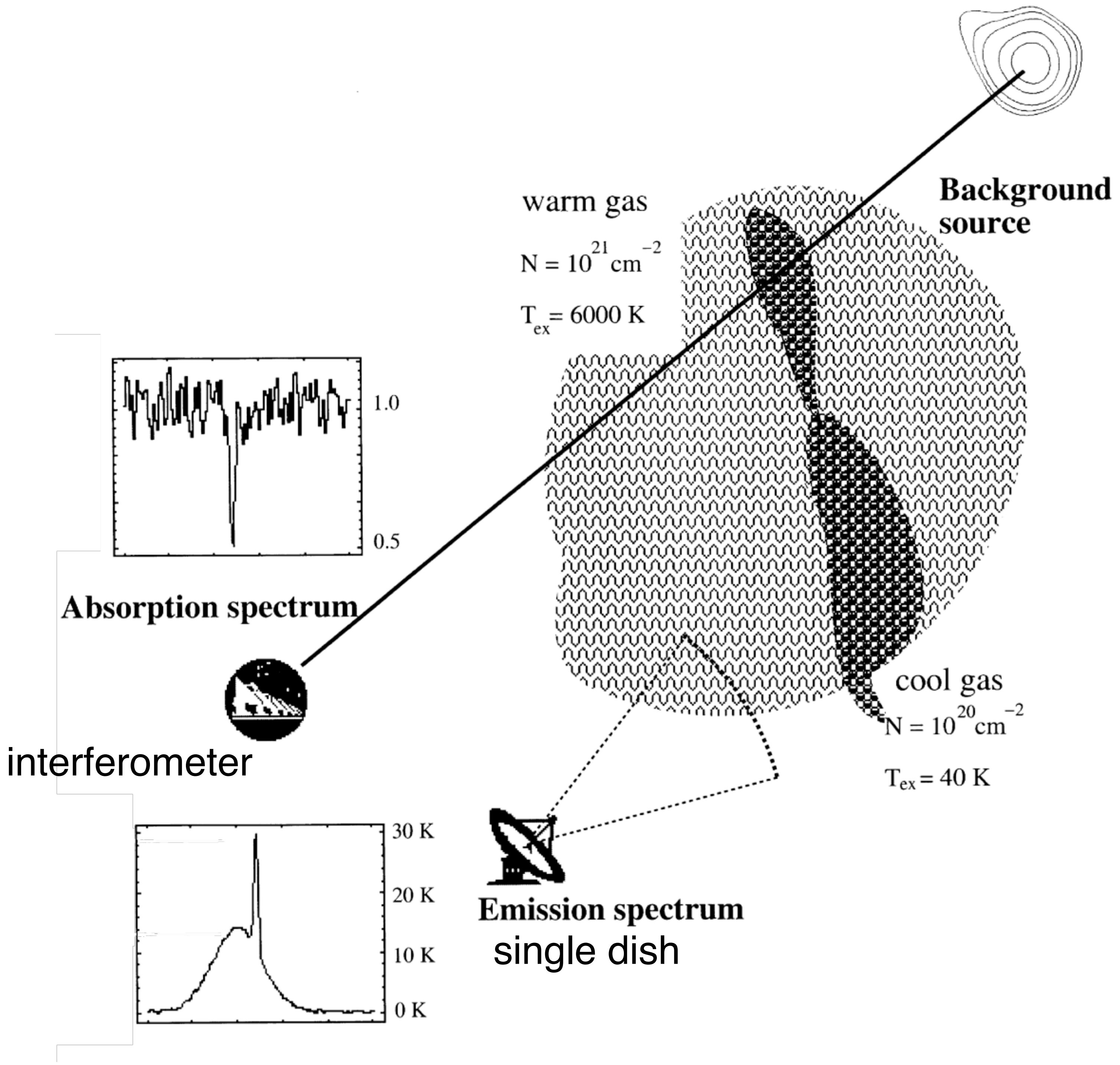
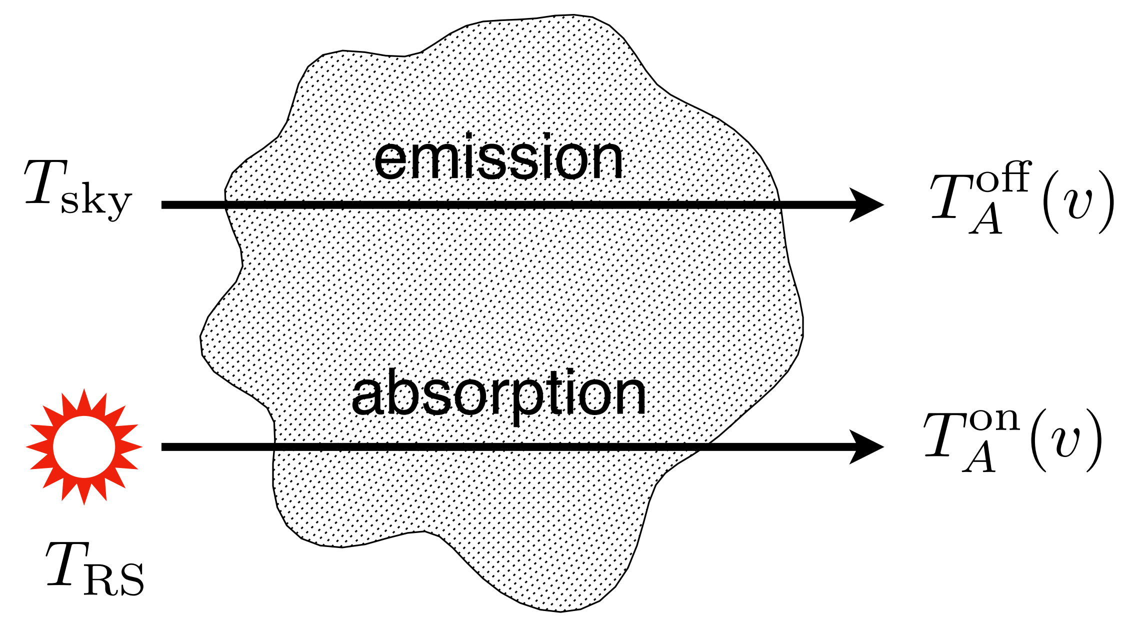


Figure 15 in Dickey et al. (2000, ApJ)

- In cases where we have a ***"bright" background radio source*** with a continuum spectrum (a typical radio-loud quasar or an active galactic nucleus, or a radio galaxy), we can study both emission and absorption by the foreground ISM in our galaxy by comparing ***"on-source" and "off-source" observations***.

- The spectra measured on the blank sky and on the radio source are, respectively,

$$T_A^{\rm on}(v) = T_{\rm RS} e^{-\tau_v} + T_{\rm spin}(v)\left(1 - e^{-\tau_v}\right)$$
$$T_A^{\rm off}(v) = T_{\rm sky} e^{-\tau_v} + T_{\rm spin}(v)\left(1 - e^{-\tau_v}\right)$$



- These two equation can be solved for the two unknowns, $\tau(v)$ and $T_{\rm spin}(v)$.

$$\tau(v) = \ln\left[\frac{T_{\rm RS} - T_{\rm sky}}{T_A^{\rm on}(v) - T_A^{\rm off}(v)}\right]$$

$$T_{\rm spin}(v) = \frac{T_A^{\rm off}(v) T_{\rm RS} - T_A^{\rm on}(v) T_{\rm sky}}{(T_{\rm RS} - T_{\rm sky}) - \left[T_A^{\rm on}(v) - T_A^{\rm off}(v)\right]}$$

We can also derive the column density from these two quantities for an optically thick cloud.

The solution gives, in general, the spin temperature as a function of velocity.

- We usually consider a case where ***the radio source is "much" brighter than the spin temperature of the intervening hydrogen cloud***.
  - The RT equations for the "on-source" and "off-source" measurements can be written:

$$\text{assumptions}: T_{\rm RS} \gg T_{\rm spin}$$

$$T_A^{\rm on} = T_{\rm RS} - T_{\rm RS}\left(1-e^{-\tau}\right) + T_{\rm spin}\left(1-e^{-\tau}\right) \approx T_{\rm RS}e^{-\tau}$$

$$(1)\quad T_A^{\rm on}(v) = T_{\rm RS}e^{-\tau_v} + T_{\rm spin}(v)\left(1-e^{-\tau_v}\right)$$

$$(2)\quad T_A^{\rm off}(v) = T_{\rm sky}e^{-\tau_v} + T_{\rm spin}(v)\left(1-e^{-\tau_v}\right)$$

$$\longrightarrow$$

$$T_A^{\rm on}(v) \approx T_{\rm RS}\, e^{-\tau_v}$$

$$T_A^{\rm off}(v) = T_{\rm sky} + \left(T_{\rm spin} - T_{\rm sky}\right)\left(1-e^{-\tau_v}\right)$$

$$(1)\quad \frac{T_{\rm RS} - T_A^{\rm on}(v)}{T_{\rm RS}} \approx 1 - e^{-\tau_v}$$

$$(2)\quad \Delta T_A^{\rm off}(v) = \Delta T_{\rm spin}(v)\left(1-e^{-\tau_v}\right)$$

$$\text{or } T_A^{\rm off}(v) \approx T_{\rm spin}(v)\left(1-e^{-\tau_v}\right)$$

Here, $\Delta T_A^{\rm off}(v) \equiv T_A^{\rm off}(v) - T_{\rm sky} \approx T_A^{\rm off}$

$$\Delta T_{\rm spin}(v) \equiv T_{\rm spin}(v) - T_{\rm sky} \approx T_{\rm spin}$$

$$\left(T_{\rm sky} \approx 3\,{\rm K}\right)$$

- ***Equivalent Width:***
  - Using the absorption spectrum from the “on-source” observation, we can “approximately” obtain the “velocity equivalent width.”

$$W_v = \int dv \left(1 - e^{-\tau_v}\right)$$
$$\approx \int dv \left[\frac{T_{\rm RS} - T_A^{\rm on}(v)}{T_{\rm RS}}\right]$$

$$\text{Note}: W_v = c \int \frac{d\nu}{\nu_{u\ell}} \left(1 - e^{-\tau_\nu}\right) = cW$$

For a weak absorption line, W_v is an upper limit.

- ***Spin Temperature:***
  - Combining the two equations (1) and (2) in the previous page, we can obtain two spin temperatures. The first one is the line-of-sight average spin temperature, and the second the spin temperature in a velocity channel.

$$\langle T_{\rm spin}\rangle \approx \frac{\int T_A^{\rm off}(v) dv}{\int \left(1 - e^{-\tau_v}\right) dv}$$

$$T_{\rm spin}(v) = \frac{T_A^{\rm off}(v)}{\left(1 - e^{-\tau_v}\right)}$$

$\longleftarrow$

assuming $T_{\rm spin} =$ constant.

$$\int dv \left[T_A^{\rm off}(v)\right] = T_{\rm spin} \int dv \left(1 - e^{-\tau_v}\right)$$

For a weak absorption line, $T_{\rm spin}$ is an lower limit.

- ***In an optically thin limit,***
  - We know the relation between the antenna temperature and column density:

$$N_{\rm HI} \approx C_0 \int T_A^{\rm off}(v) dv \qquad\qquad \frac{dN_{\rm HI}}{dv} \approx C_0 T_A^{\rm off}(v)$$

  - Then, we can express the spin temperature in terms of column density and equivalent width (absorption profile):

$$\langle T_{\rm spin} \rangle = \frac{1}{W_v} \int T_A^{\rm off}(v) dv = \frac{C_0^{-1}}{W_v} N_{\rm HI}$$

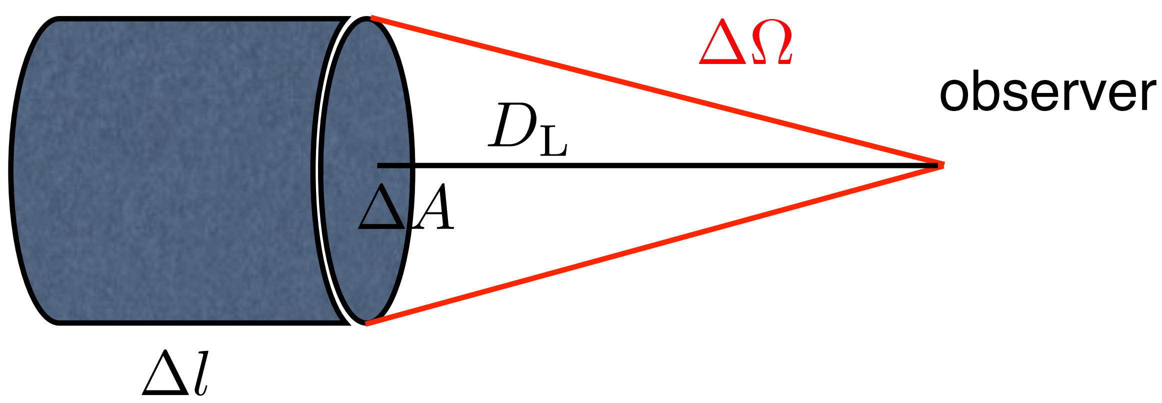
$$\boxed{\langle T_{\rm spin} \rangle \approx 0.5516 \, \frac{N_{\rm HI}/10^{18}\,{\rm cm}^{-2}}{W_v/{\rm km\,s}^{-1}} \; [{\rm K}]}$$

$$T_{\rm spin}(v) = \frac{C_0^{-1}}{(1 - e^{-\tau_v})} \frac{dN_{\rm HI}}{dv}$$

$$C_0^{-1} = 5.516 \times 10^{-19} \; \left[\frac{{\rm K\,km\,s}^{-1}}{{\rm cm}^{-2}} \right]$$

H I mass of an External Galaxy

- With the assumption that the emitting regions are optically thin, the total mass $M_{\rm HI}$ of H I in an external galaxy can be determined from the observed flux in the 21-cm line:



$$F_\nu = \text{observed flux density}$$

$$F_{\rm obs} = \int F_\nu d\nu_{\rm obs} = I\Delta\Omega \qquad \leftarrow \cos\theta \approx 1$$

$$I = \int I_\nu d\nu_{\rm obs} = \frac{3}{16\pi} A_{u\ell} h\nu_{u\ell} N_{\rm HI}$$

- Here, $D_{\rm L}$ is the luminosity distance to the galaxy.

The luminosity distance is defined by the relationship between flux and luminosity:

$$d_L = \frac{L}{4\pi F}$$

$$n_{\rm H}\Delta l = N_{\rm HI}$$

$$\Delta A = D_{\rm L}^2 \Delta\Omega = D_{\rm L}^2 \frac{F_{\rm obs}}{I}$$

$$\begin{aligned} M_{\rm HI} &= m_{\rm H} n_{\rm H} \Delta V = m_{\rm H} n_{\rm H} \Delta l \Delta A \\ &= m_{\rm H} N_{\rm HI} D_{\rm L}^2 \frac{F_{\rm obs}}{I} \\ &= m_{\rm H} N_{\rm HI} D_{\rm L}^2 \frac{F_{\rm obs}}{(3/16\pi) A_{u\ell} h\nu_{u\ell} N_{\rm HI}} \end{aligned}$$

$$\begin{aligned} \therefore\ M_{\rm HI} &= \frac{16\pi}{3} \frac{m_{\rm H}}{A_{u\ell} h\nu_{u\ell}} D_{\rm L}^2 F_{\rm obs} \\ &= 4.945 \times 10^7 M_\odot \left(\frac{D_{\rm L}}{\rm Mpc}\right)^2 \left(\frac{F_{\rm obs}}{\rm Jy\ MHz}\right) \end{aligned}$$

- If the redshift of the galaxy is z :

$$\nu_{\rm obs} = \nu/(1+z)$$
$$d\nu_{\rm obs} = \frac{\nu_{u\ell}}{(1+z)}\frac{dv}{c}$$

$$\begin{aligned} M_{\rm HI} &= \frac{16\pi}{3}\frac{m_{\rm H}}{A_{u\ell}h\nu_{u\ell}}D_{\rm L}^2\int F_\nu d\nu_{\rm obs} \\ &= \frac{16\pi}{3}\frac{m_{\rm H}}{A_{u\ell}h\nu_{u\ell}}D_{\rm L}^2\frac{\nu_{u\ell}}{c}\frac{1}{1+z}\int F_\nu dv \\ &= \frac{16\pi m_{\rm H}}{3A_{u\ell}hc}D_{\rm L}^2(1+z)^{-1}\int F_\nu dv \\ &= 2.343\times 10^5\, M_\odot (1+z)^{-1}\left(\frac{D_{\rm L}}{\rm Mpc}\right)^2\frac{\int F_\nu dv}{\rm Jy\,km\,s^{-1}} \end{aligned}$$

- Radio astronomers often report the integrated flux in “Jy km s^{-1}.”

Observations: Example 1

- All-sky map of H I 21-cm line intensity from the LAB survey (Kalberla et al. 2005), with an angular resolution ~ 0.6 deg.
  - Scale gives $\log_{10}$ N(HI) [cm^{-2}]. The LMC and SMC are visible, with a connecting H I "bridge".
  - The map was obtained by assuming the optically thin case.

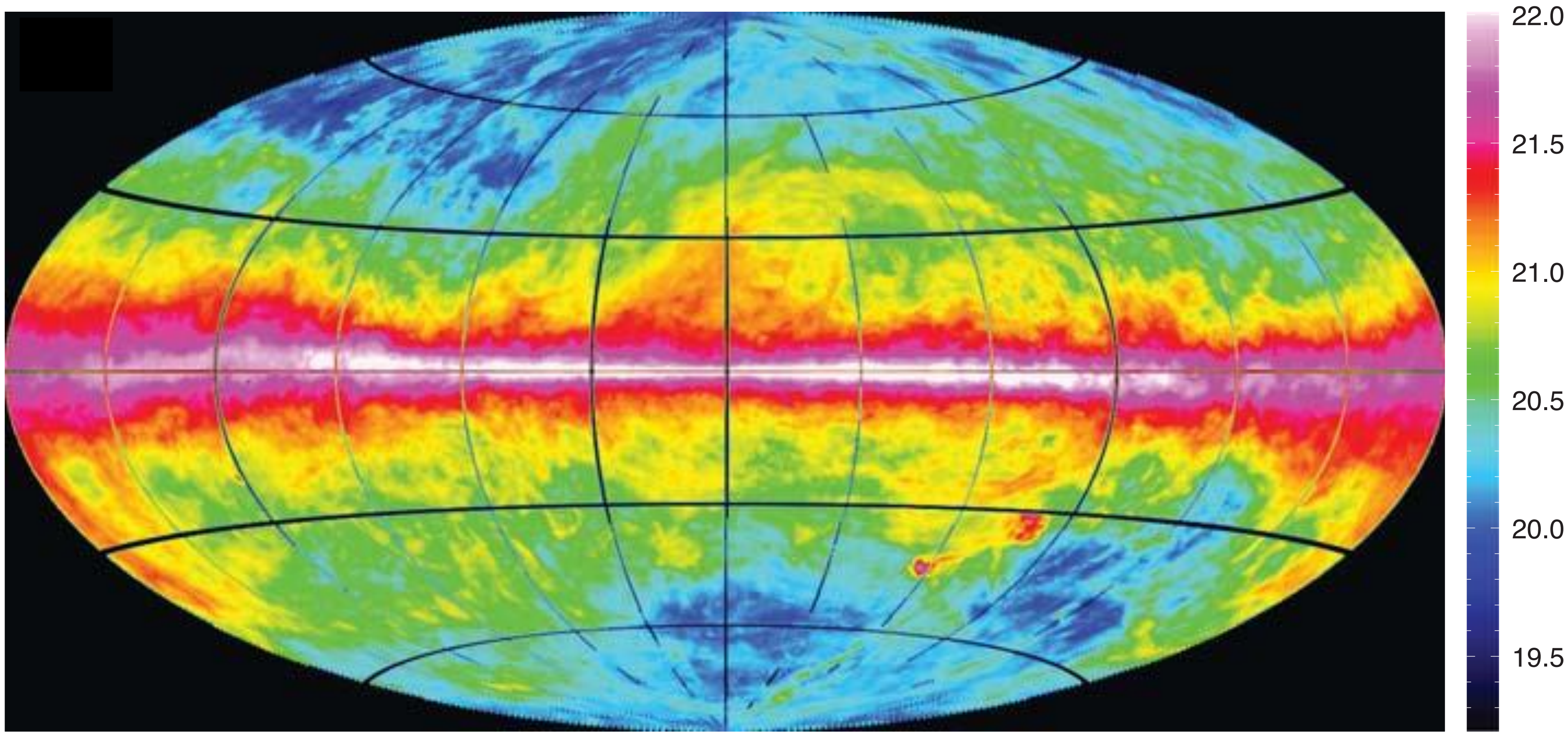
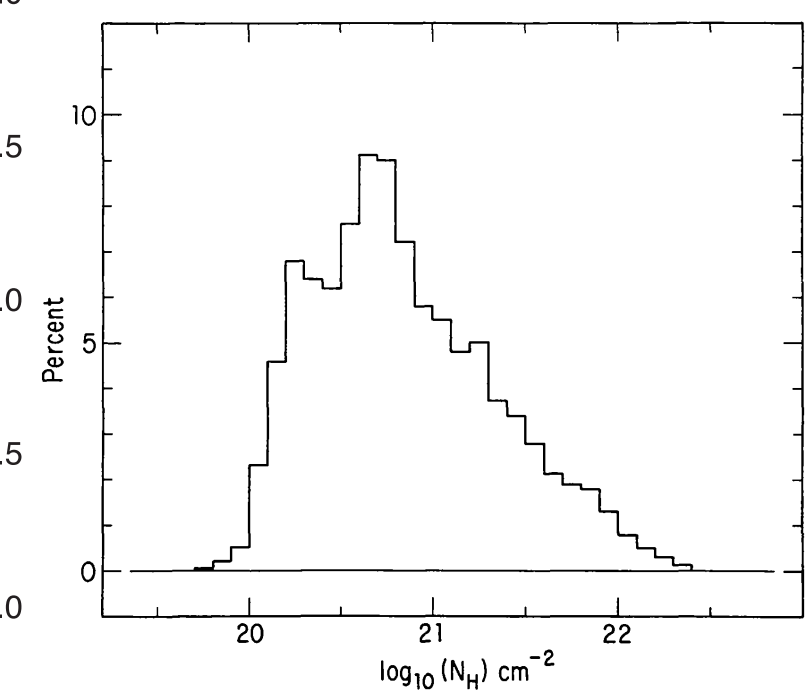


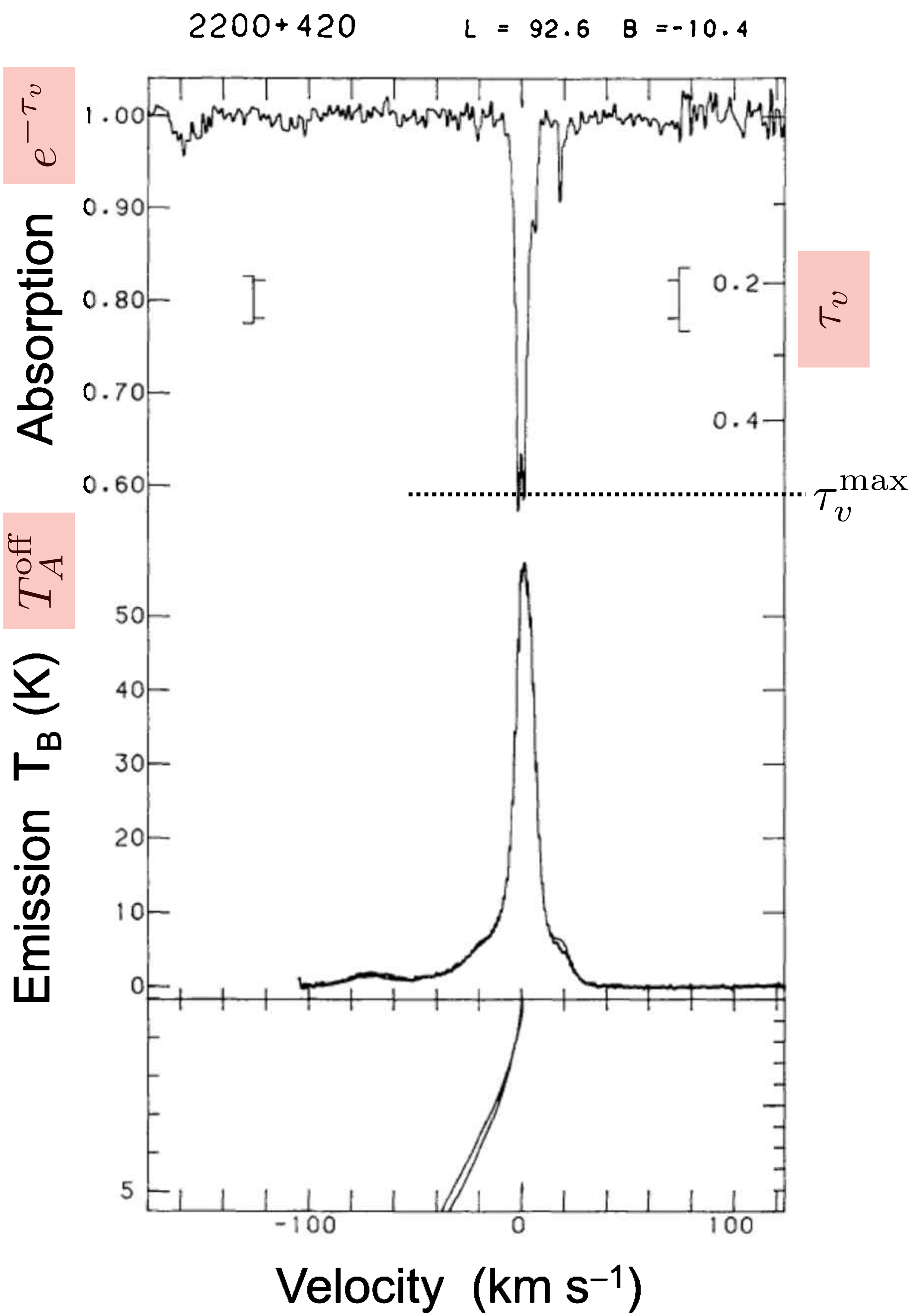
Plate 3 in [Draine]



The percentage of the sky covered by H I at a given N_H.

Figure 4 in Dickey & Lockman (1990, ARA&A)

Observations: Example 2



H I 21-cm absorption and emission along the line of sight towards BL Lacertae

[Dickey et al. 1983; Figure 3.3 in Ryden]

- Maximum optical depth :

$$\tau_v \sim 0.5 \qquad \text{optically thin}$$

- Equivalent width of the absorption line :

$$W_v = 7\,\mathrm{km\,s^{-1}}$$

- Integrated line intensity of the emission line :

$$\int T_A^{\rm off}(v)dv \approx 930\,\mathrm{K\,km\,s^{-1}}$$

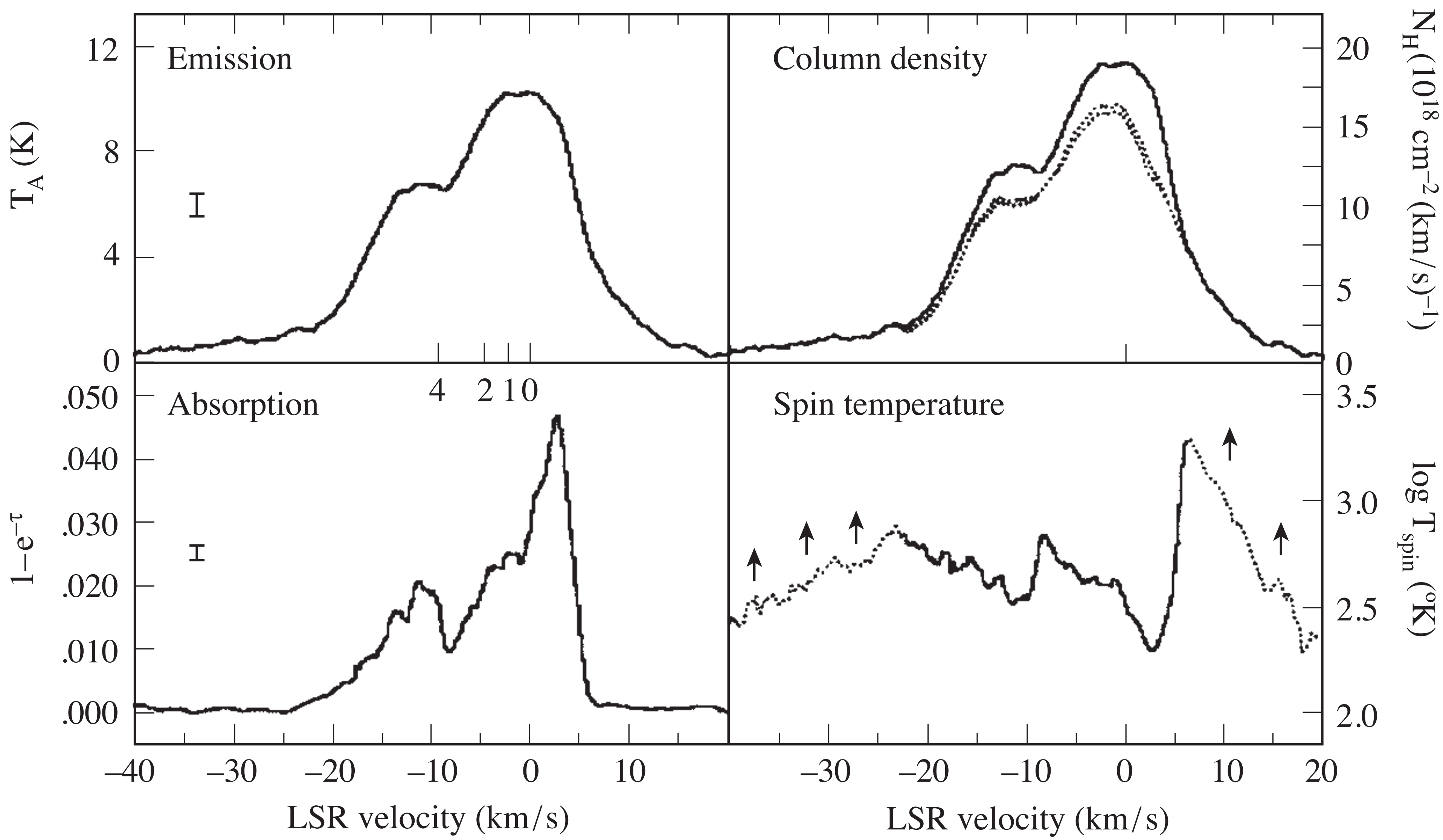
- Column density from the emission line :

$$N_{\rm HI} \approx 1.69 \times 10^{21}\,\mathrm{cm^{-2}}$$

- Now, the spin temperature is

$$T_{\rm spin} = \frac{\int T_A^{\rm off}(v)dv}{W_v} = \frac{930\,\mathrm{K\,km\,s^{-1}}}{7\,\mathrm{km\,s^{-1}}}$$
$$\approx 133\,\mathrm{K}.$$

Observations: Example 3



Left panels: Observed H I emission (off the quasar 3C48) and absorption (toward 3C48, at $\ell = 134°$, $b = -28.7°$). Lower right: spin temperature $T_{\rm spin}(v)$ as a function of LSR velocity. Tick marks labeled 0, 1, 2, and 4 on abscissa of left panels show the LSR velocity expected for gas at a distance of 0,1,2,4 kpc (for an assumed Galactic rotation curve). Upper right: $dN(\rm H\,I)/dv$ for different assumptions regarding the relative (foreground/background) locations of cold absorbing gas and warm gas seen only in emission. From Dickey et al. (1978).

[Figure 29.1 in Draine]

Define a point in space that is moving on a perfectly circular orbit around the center of the Galaxy at the Sun's galactocentric distance. We measure all velocities of astronomical objects relative to this point, which is known as the LSR (local standard of rest).

The Sun's orbital speed is 202-241 km/s (Majewski 2008, IAUS, 248)

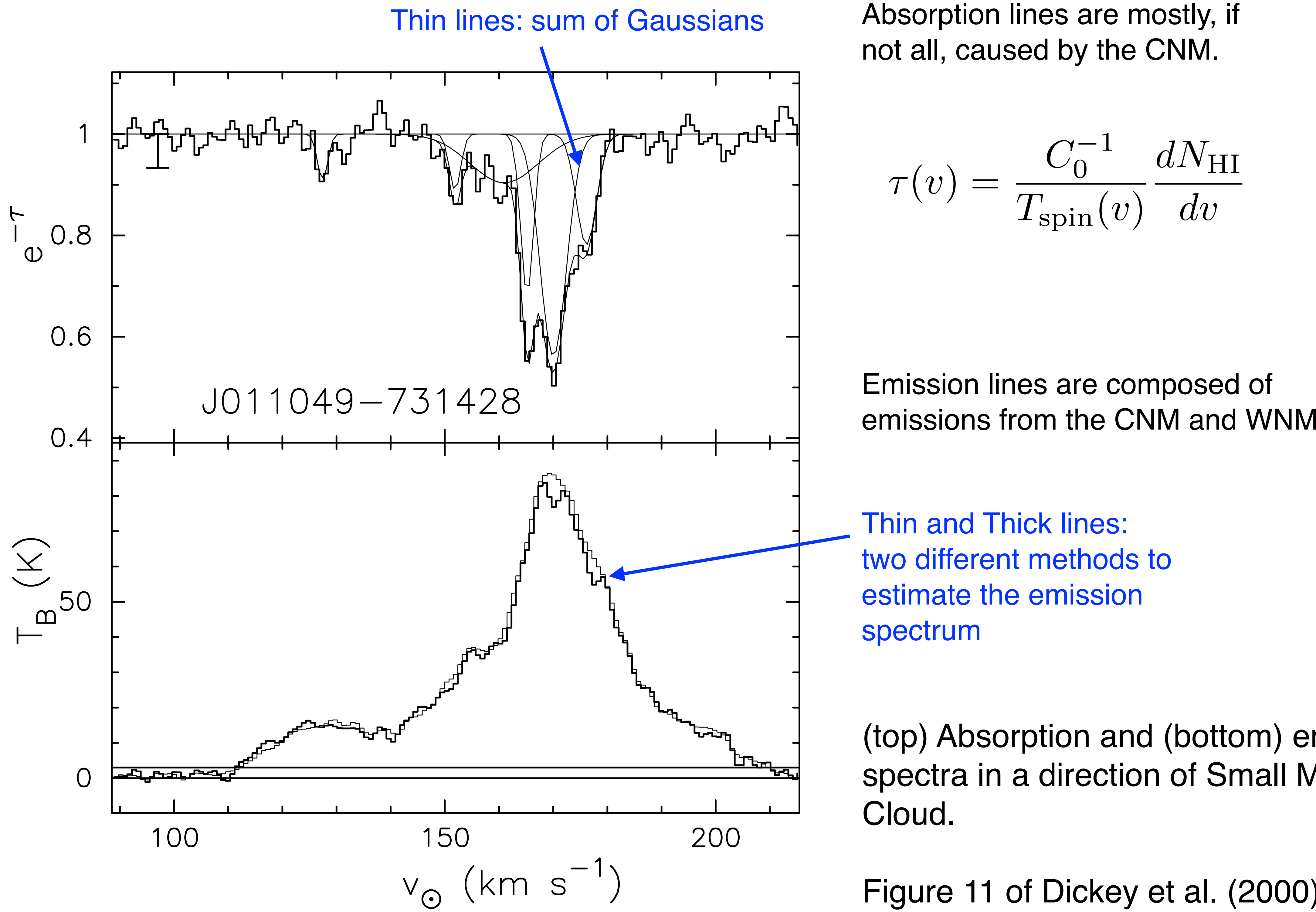
Observations: CNM & WNM

- The overall shape of the emission line profiles in our Galaxy is mainly determined by the large-scale distribution and kinematics of H I.

- Absorption lines are always narrower than emission lines.
  - Some velocities have H I emission but no detectable H I absorption.
  - Difference between emission and absorption spectrum:
    - Absorption spectra can usually be decomposed into Gaussian components.
    - However, emission spectra do not look like the superposition of a few Gaussians.
    - This is because the absorption lines arise only in regions of cool gas, which are more distinct along the line of sight, and thus have narrower intrinsic line widths than the gas that contributes to H I emission.
- The difference between emission and absorption results mainly from variation in the spin temperature of the H I along the line of sight.
  - Recall that the optical depth is inversely proportional to the spin temperature, indicating the difficulties in observing absorption spectra from the warm neutral medium, which has a temperature larger than 1000 K.

$$\tau(v) = \frac{C_0^{-1}}{T_{\rm spin}(v)} \frac{dN_{\rm HI}}{dv}$$

- Absorption and Emission spectra



Absorption lines are mostly, if not all, caused by the CNM.

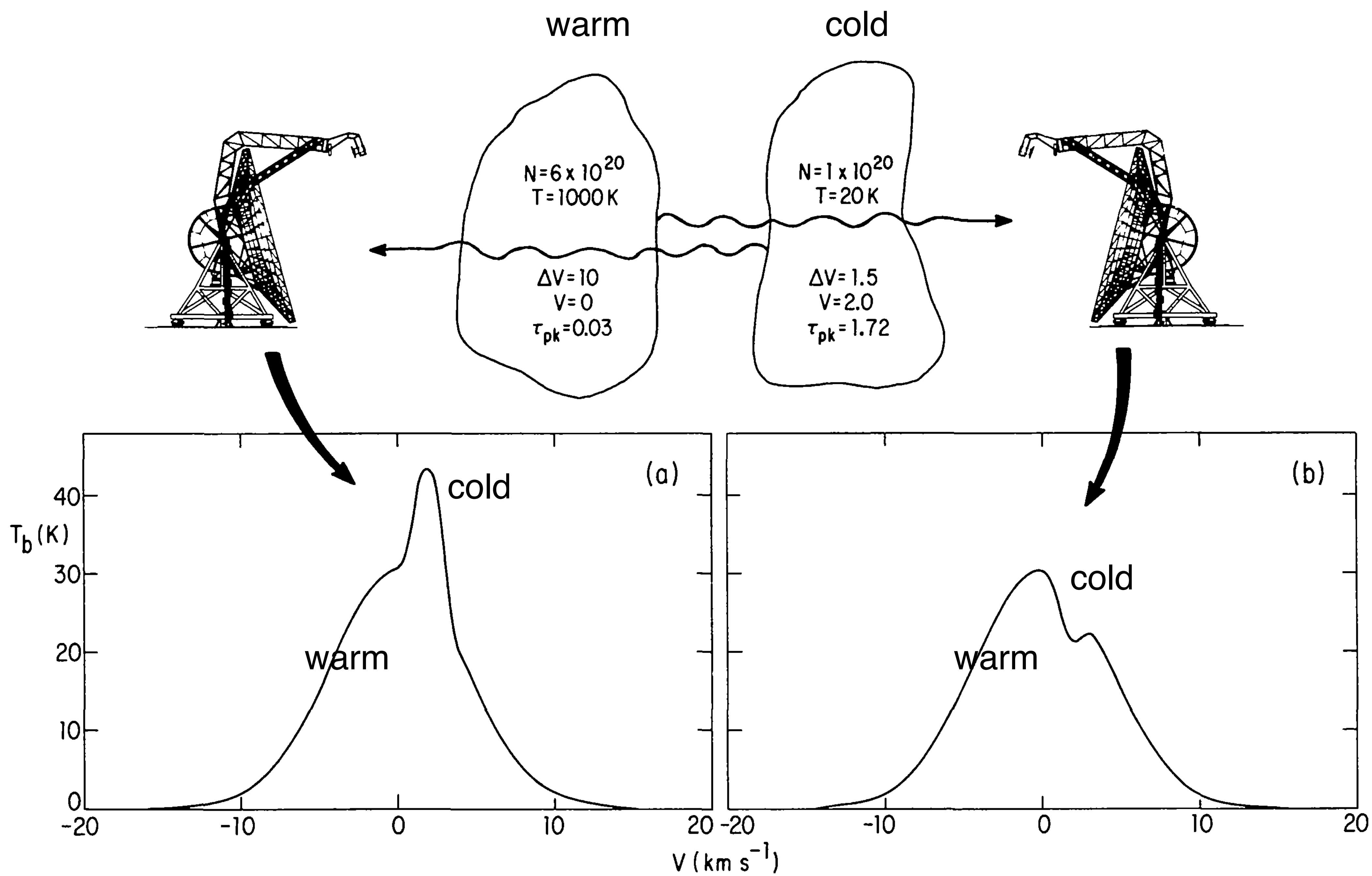
$$\tau(v) = \frac{C_0^{-1}}{T_{\rm spin}(v)} \frac{dN_{\rm HI}}{dv}$$

Emission lines are composed of emissions from the CNM and WNM.

(top) Absorption and (bottom) emission spectra in a direction of Small Magellanic Cloud.

Figure 11 of Dickey et al. (2000)

Emission spectrum depending on the relative location of WNM and CNM



Schematic of the geometry of 21-cm self-absorption.
The structure of an emission profile depends on the relative location of warm and cold clouds as viewed by the observer.

Figure 1 of Dickey & Lockman (1990)

A rough estimation of the fraction of gas in the cold phase

- The interstellar atomic hydrogen is in at least two thermal phases.
  - We assume that the warm gas temperature is large enough that no absorption is seen from the warm phase.
  - We further assume a value for the cold-phase temperature, for instance, $T_c \approx 55\,\mathrm{K}$.
  - Then, the fraction of gas in the cold phase is

$$\boxed{f_c \equiv \frac{N_c}{N_w + N_c} \approx \frac{T_c}{\langle T_{\rm spin} \rangle}}$$

$$T_c = \frac{C_0^{-1}}{W_v^c} N_c$$

$$\langle T_{\rm spin} \rangle = \frac{C_0^{-1}}{W_v} N_{\rm HI}, \ \text{where } W_v = W_v^c + W_c^w \approx W_v^c \text{ and } N_{\rm HI} = N_w + N_c$$

The warm gas and cool gas are mixd so that $T_{\rm spin}$ is a weighted averaged, whether computed for individual velocity channels or line-of-sight integrals.

  - This gives us a rough estimation of the cold-phase H I fraction for galaxies.

COOL-PHASE H I FRACTIONS FOR GALAXIES

Galaxy	Sample Size	$\langle T_s \rangle$ (K)	f_c (T_c = 55 K)
SMC	28	440	0.13
LMC	49	170	0.33
M31	16	150	0.37
M33	7	370	0.15
Milky Way	19	250	0.22

Table 3 of Dickey et al. (2000, ApJ, 536, 756)

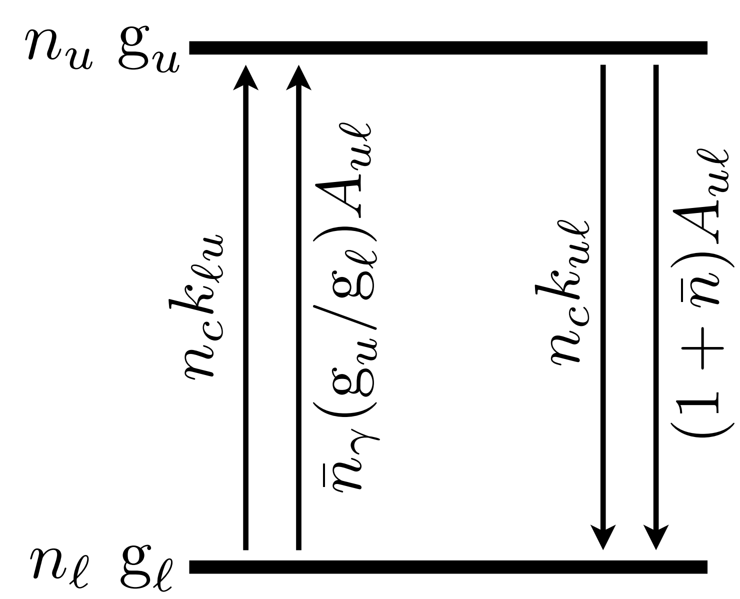
Observations: CNM + WNM in our Galaxy

- The most recent emission-absorption surveys (Heiles & Troland 2003) support the idea that, in the solar neighborhood (i.e., within ~ 500 pc of the Sun), interstellar H I is found primarily in two distinct phases: the CNM and the WNM.
  - About 40% of the H I (by mass) is in the CNM, with a median spin temperature $T_{\rm spin}$ ~ 70 K. The remaining 60% of the H I is in the WNM phase, which appears to have a volume filling factor ~ 50% near the disk midplane.
  - Because warm H I absorbs very weakly, for some of the WNM material it is only possible to determine a lower bound on $T_{\rm spin}$. Heiles & Troland (2003) conclude that > 48% of the WNM has $500 < T_{\rm spin} < 5000\,{\rm K}$, at these temperatures the gas is expected to be thermally unstable.

- Murray et al. (2014) detected a widespread warm neutral medium component with excitation temperature $\langle T_{\rm spin} \rangle = 7000^{+1800}_{-1200}\,{\rm K}$.
  - This temperature lies above theoretical predictions based on collisional excitation alone, implying that Lyα scattering, the most probable additional source of excitation, is more important in the ISM than previously assumed.
- Murray et al. (2018) found that the WNM makes up 52% of the total H I mass.
  - Following spectral modeling, they detect a stacked residual absorption feature corresponding to WNM with $T_{\rm spin} \sim 10^4\,{\rm K}$.

Excitation temperature = Gas kinetic temperature ? (Level Population)

- In some cases, it is sufficient to consider only the ground state and the first excited state.
  - Consider collisional excitation and de-excitation by some species (e.g., electrons) with density n_c, and suppose that radiation with the energy density u_ν .
  - The population of the excited state must satisfy:



$$\frac{dn_u}{dt} = n_\ell \left[n_c k_{\ell u} + \bar{n}_\gamma \frac{g_u}{g_\ell} A_{u\ell} \right] - n_u \left[n_c k_{u\ell} + (1+\bar{n}_\gamma) A_{u\ell} \right]$$

Here, photon occupation number

$$\left(\bar{n}_\gamma \equiv \frac{c^3}{8\pi h \nu^3} u_\nu \right)$$

  - The steady-state solution with radiation and collision present is

$$\frac{n_u}{n_\ell} = \frac{n_c k_{\ell u} + \bar{n}_\gamma (g_u/g_\ell) A_{u\ell}}{n_c k_{u\ell} + (1+\bar{n}_\gamma) A_{u\ell}}$$

Using this equation, we can calculate the excitation temperature between the two levels.

Here, by the principle of detailed balance, the upward collisional rate coefficient is given in term of the downward rate coefficient by

$$k_{\ell u} = \frac{g_u}{g_\ell} k_{u\ell} e^{-E_{u\ell}/kT_{\rm gas}} \qquad (T_{\rm gas} = \text{gas kinetic energy})$$

- It is instructive to examine the population equation in various limits:
  - In the limit of $n_c \rightarrow \infty$ and no or weak radiation field $\bar{n}_\gamma = 0$:

$$\frac{n_u}{n_\ell} = \frac{n_c k_{\ell u}}{n_c k_{u\ell} + A_{u\ell}} = \frac{k_{\ell u}}{k_{u\ell}} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} e^{-E_{u\ell}/kT_{\rm gas}} \qquad \therefore\ T_{\rm exc} = T_{\rm gas}$$

  - If $n_c = 0$ and the radiation field has a brightness temperature of $T_b = T_{\rm rad}$ at the frequency $\nu = E_{u\ell}/h$:

$$\frac{n_u}{n_\ell} = \frac{\bar{n}_\gamma(\mathrm{g}_u/\mathrm{g}_\ell)}{(1+\bar{n}_\gamma)} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} e^{-E_{u\ell}/kT_{\rm rad}} \qquad \therefore\ T_{\rm exc} = T_{\rm rad}$$

$$\bar{n}_\gamma = \left(e^{E_{u\ell}/kT_{\rm rad}} - 1\right)^{-1} \implies 1 + \bar{n}_\gamma = \bar{n}_\gamma\, e^{E_{u\ell}/kT_{\rm rad}}$$

  - If the radiation and gas has the same temperature $T_{\rm rad} = T_{\rm gas}$, then we can show that

$$\frac{n_u}{n_\ell} = \frac{n_c k_{\ell u} + \bar{n}_\gamma(\mathrm{g}_u/\mathrm{g}_\ell)A_{u\ell}}{n_c k_{u\ell} + (1+\bar{n}_\gamma)A_{u\ell}} = \frac{\mathrm{g}_u}{\mathrm{g}_\ell} e^{-E_{u\ell}/kT_{\rm gas}} \qquad \therefore\ T_{\rm exc} = T_{\rm rad} = T_{\rm gas}$$

  - In general, $T_{\rm exc} \neq T_{\rm gas}$ and $T_{\rm exc} \neq T_{\rm rad}$. What is the excitation temperature in the ISM?

Critical Density

- Interplay between the radiation and collisions determines the excitation temperature. Which one will be more important?
- For a collision partner c, we define the critical density $n_{\mathrm{crit},\,u}$ for an excited state u to be the density for which collisional de-excitation equals radiative de-excitation, including stimulated emission:

$$n_{\mathrm{crit},u}(c) \equiv \frac{\sum_{\ell<u}\left[1+(n_\gamma)_{u\ell}\right] A_{u\ell}}{\sum_{\ell<u} k_{u\ell}(c)}$$

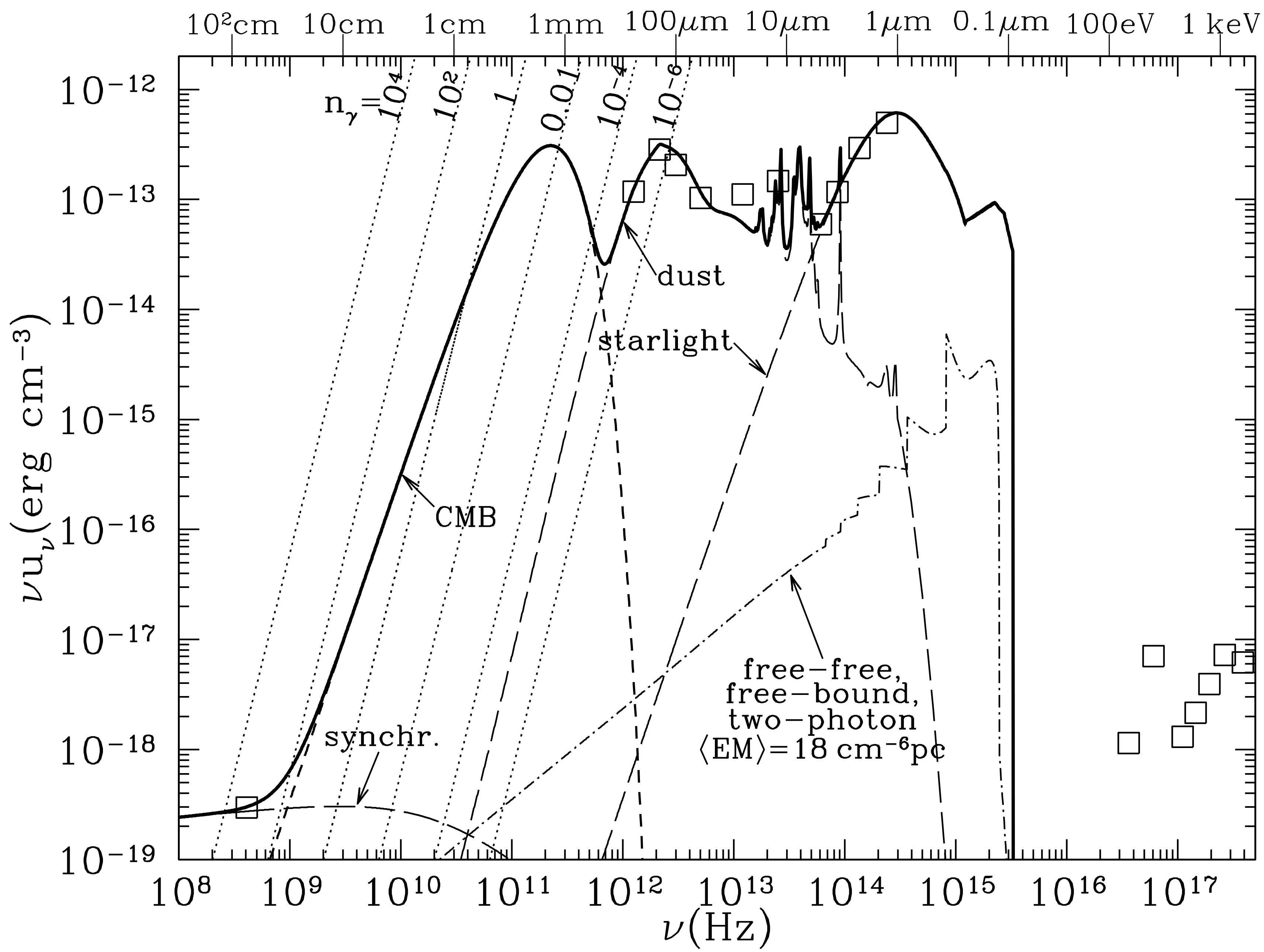
- Note that this definition applies to multilevel systems, but each excited level u may have a different critical density. The critical density depends on the intensity of ambient radiation. For many transitions, the correction by the stimulated emission is unimportant, but for 21-cm line, it is important.

Critical densities for fine-structure excitation [Table 17.1 in Draine, revised for both H and e-, errata]

Ion	ℓ	u	E_ℓ/k (K)	E_u/k (K)	$\lambda_{u\ell}$ (μm)	$n_{\mathrm{crit},u}(\mathrm{H})$ $T=100$ K (cm^{-3})	$n_{\mathrm{crit},u}(\mathrm{H})$ $T=5000$ K (cm^{-3})	$n_{\mathrm{crit},u}(e^-)$ $T=100$ K (cm^{-3})	$n_{\mathrm{crit},u}(e^-)$ $T=5000$ K (cm^{-3})
C II	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	91.21	157.74	2.7×10^3	1.5×10^3	6.8	40.
C I	$^3\mathrm{P}_0$	$^3\mathrm{P}_1$	0	23.60	609.7	620	170	76.	6.4
	$^3\mathrm{P}_1$	$^3\mathrm{P}_2$	23.60	62.44	370.37	720	150	75.	6.3
O I	$^3\mathrm{P}_2$	$^3\mathrm{P}_1$	0	227.71	63.185	2.5×10^5	4.9×10^4	1.8×10^5	4.8×10^4
	$^3\mathrm{P}_1$	$^3\mathrm{P}_0$	227.71	326.57	145.53	2.4×10^4	8.6×10^3	2.3×10^4	5.8×10^3
Si II	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	413.28	34.814	2.5×10^5	1.2×10^5	140.	1.5×10^3
Si I	$^3\mathrm{P}_0$	$^3\mathrm{P}_1$	0	110.95	129.68	4.8×10^4	2.8×10^4	2.9×10^4	830.
	$^3\mathrm{P}_1$	$^3\mathrm{P}_2$	110.95	321.07	68.473	9.9×10^4	3.6×10^4	4.4×10^4	1.9×10^3

Interstellar Radiation Fields

To estimate the critical density, we need to know the radiation field strength.



Interstellar continuum radiation field in an H I cloud in the solar neighborhood (see text). Spectral lines are not included. Solid line is the sum of all components for $h\nu \leq 13.6\,\mathrm{eV}$. Squares show the measured sky brightness at 408 MHz (Haslam et al. 1982), the all-sky measurements by COBE-DIRBE in 10 broad bands from 240 μm to 1.25 μm (Arendt et al. 1998), and all-sky measurements by ROSAT between 150 eV and 2 keV (Snowden 2005, private communication). Dotted lines are contours of constant photon occupation number n_γ.

[Figure 12.1 in Draine]

H I Spin Temperature

- ***Collionsional rate coefficients***:
  - Collision with other H atoms

(Allison & Dalgarno 1969; Zygelman 2005)

$$k_{10}(\mathrm{H}) \approx \begin{cases} 1.19\times10^{-10} T_2^{0.74-0.20\ln T_2}\,\mathrm{cm^3\,s^{-1}} & (20\,\mathrm{K} < T < 300\,\mathrm{K}) \\ 2.24\times10^{-10} T_2^{0.207} \mathrm{e}^{-0.876/T_2}\,\mathrm{cm^3\,s^{-1}} & (300\,\mathrm{K} < T < 10^3\,\mathrm{K}) \end{cases}$$

$$k_{01}(\mathrm{H}) \approx 3k_{10}(\mathrm{H})e^{-0.0682\,\mathrm{K}/T}$$

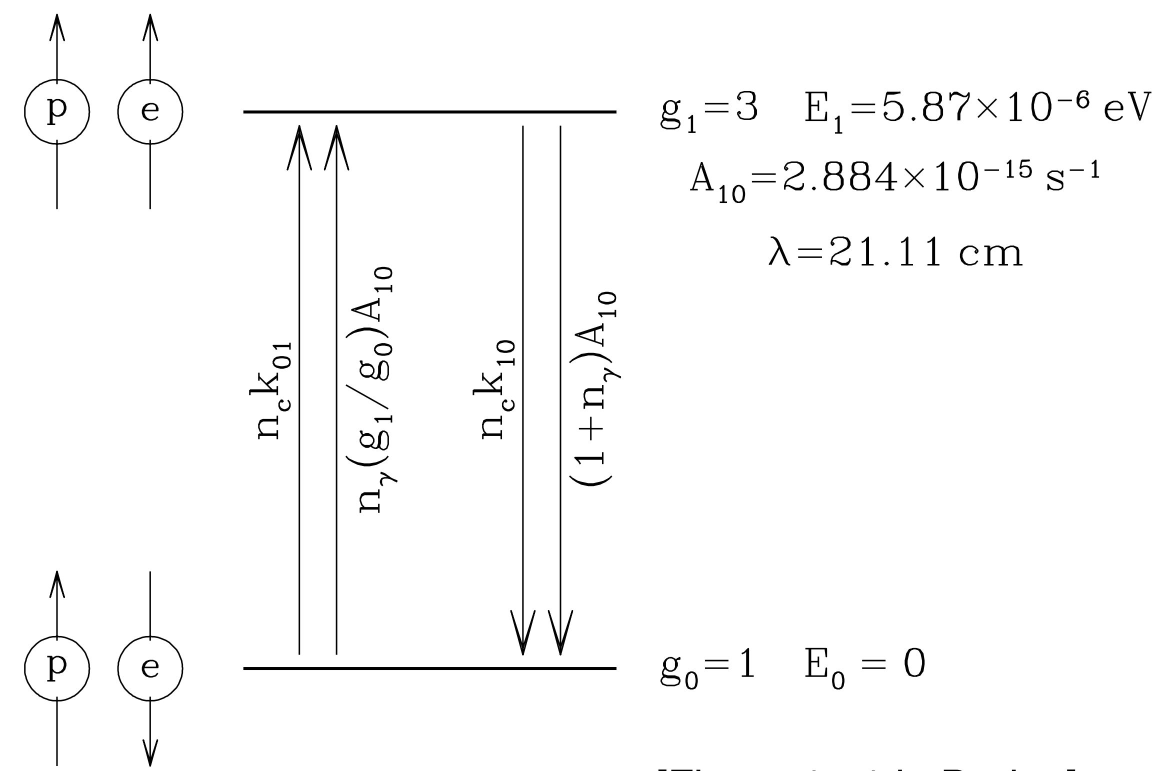
  - Collision with electrons

(Furlanetto & Furlanetto 2007)

$$k_{10}(e^-) \approx 2.26\times10^{-9}\,(T/100\,\mathrm{K})^{0.5}\,\mathrm{cm^3\,s^{-1}} \quad (1 \lesssim T \lesssim 500\,\mathrm{K})$$

$$k_{01}(e^-) \approx 3k_{10}(e^-)e^{-0.0682\,\mathrm{K}/T}$$

    - This is a factor ~ 10 larger than that for H atoms. However, ***electrons will be minor importance in regions with a fractional ionization*** $x_e \lesssim 0.03$, such as the CNM and WNM.



$$\frac{n_1}{n_0} = \frac{n_c k_{01} + \bar{n}_\gamma (\mathrm{g}_1/\mathrm{g}_0) A_{10}}{n_c k_{10} + (1+\bar{n}_\gamma) A_{10}}$$

[Figure 17.1 in Draine]

- Radiation Field strength
  - The radiation field near 21 cm is dominated by the cosmic microwave background plus Galactic synchrotron emission. The antenna temperature is

$$T_A \approx T_{\rm CMB} + T_{\rm syn} = 2.73\,{\rm K} + 1.04\,{\rm K} = 3.77\,{\rm K}$$

  - Photon occupation number:

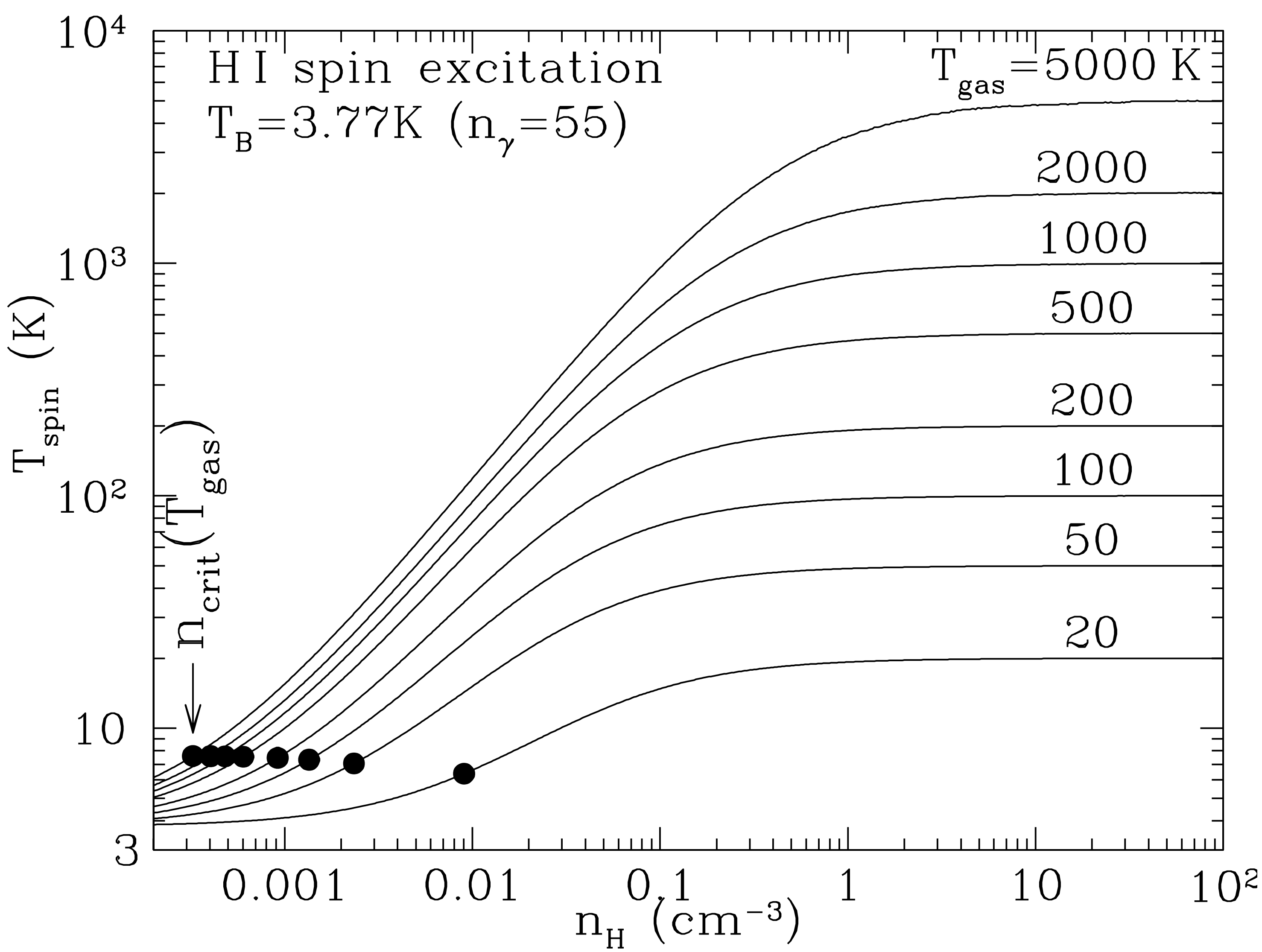
$$\bar{n}_\gamma = \left(e^{h\nu/kT_{\rm rad}} - 1\right)^{-1} \approx \frac{kT_A}{h\nu} \approx \frac{3.77\,{\rm K}}{0.0682\,{\rm K}} \approx 55$$

  - The critical density is then

$$\begin{aligned} n_{\rm crit}(H) &= \frac{(1+\bar{n}_\gamma)A_{10}}{k_{10}} \\ &\approx 1.4 \times 10^{-3}\,{\rm cm}^{-3} \quad \text{at } T \sim 100\,{\rm K} \end{aligned}$$

$$\begin{aligned} n_{\rm crit} &\approx 0.02\,{\rm cm}^{-3} \qquad \text{at } T \sim 10\,{\rm K} \\ &\approx 5 \times 10^{-4}\,{\rm cm}^{-3} \ \text{at } T \sim 1000\,{\rm K} \end{aligned}$$

- H I spin temperature as a function of density $n_{\rm H}$, including only 21 cm continuum radiation and collisions with H atoms. Lyα scattering is not included.
  - Filled circles show $n_{\rm crit}(\rm H)$ for each temperature.
  - It is important to note that one requires $n \gg n_{\rm crit}$ in order to have $T_{\rm spin}$ within, say, 10% of $T_{\rm gas}$, particularly at high temperatures.



[Fig. 17.2 in Draine]

Note that Ryden states that "in the CNM and WNM, we expect the hyperfine levels of atomic hydrogen to be collisionally excited, and to have a spin temperature close to the gas temperature." based on that $n_{\rm crit}$ ~ 6x10^{-4} cm^{-3} at T ~ 1000 K.

The collisional excitation is strong enough, only in the CNM, to bring the spin temperature close to the gas kinetic temperature.

However, this is not true in the WNM. In the WNM, the WF effect can thermalize the 21-cm spin temperature to the gas kinetic temperature.

In the CNM, the 21-cm spin temperature is a good tracer of the gas kinetic temperature. *This is not true for other levels in other atoms.*

C II Fine Structure Excitation

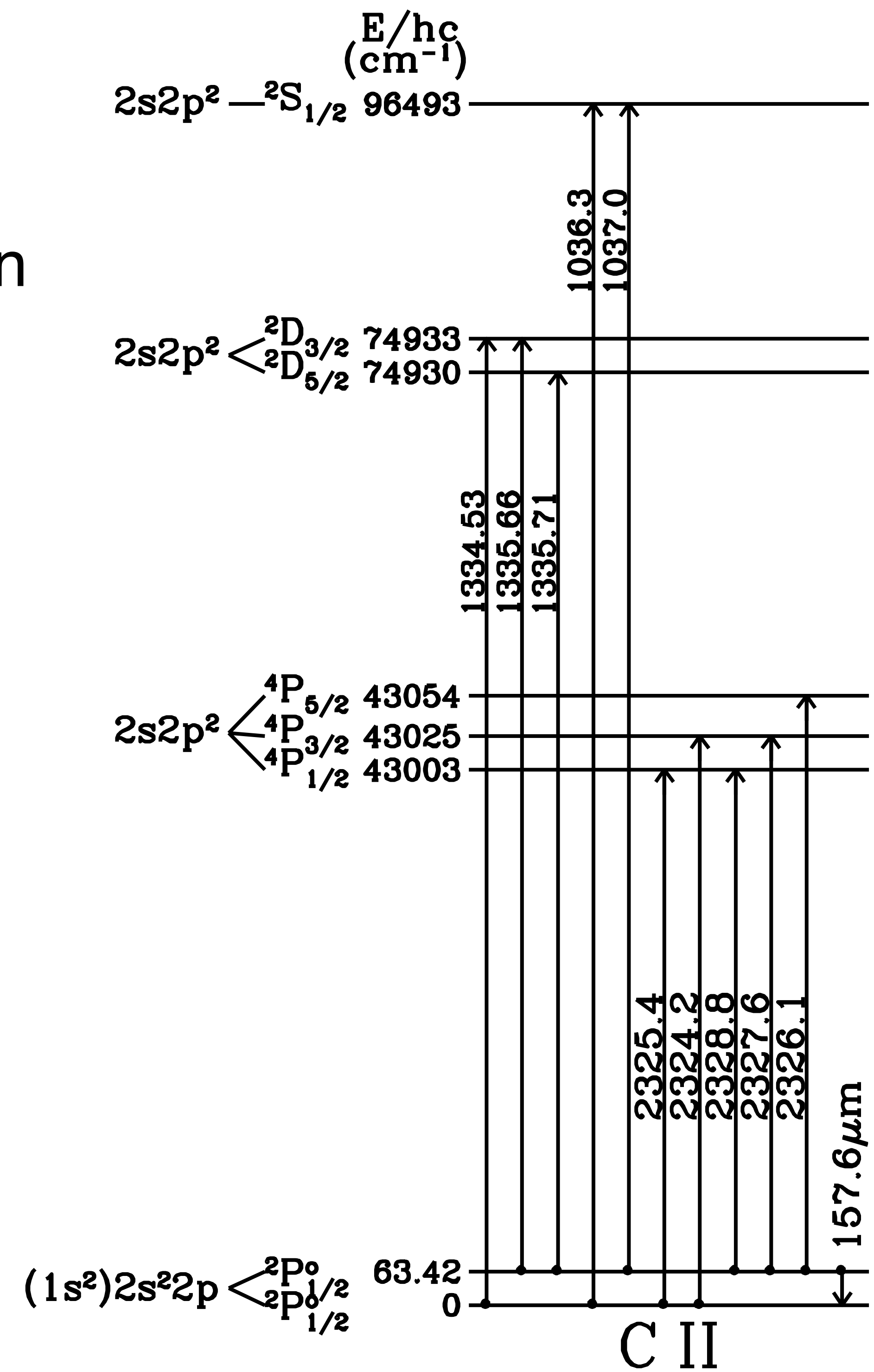
- The ground electronic state $1s^2 2s^2 2p\ ^2P^o$ of C^+ contains two fine-structure levels.

- The electronically excited states have an excitation energy that is much higher than the kinetic temperature of the CNM.

$$2235\,\text{Å} \rightarrow E_{u\ell} = 0.56\,\text{eV} \rightarrow T = 6440\,\text{K}$$

- We may, therefore, consider the two fine-structure levels in the ground electronic state to be a two level atom.

- Will the populations of these two levels be thermalized in the ISM?



- Rate coefficients for collisional de-excitation:

$$\left\langle \Omega \left({}^2P^o_{1/2}, {}^2P^o_{3/2} \right) \right\rangle \approx 2.1$$

$$k_{10}(e^-) \approx 4.53 \times 10^{-8}\, T_4^{-1/2}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$$

$$k_{10}(\mathrm{H}) \approx 7.58 \times 10^{-10}\, T_2^{0.1281+0.0087\ln T_2}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$$

(Barinovs et al. 2005)

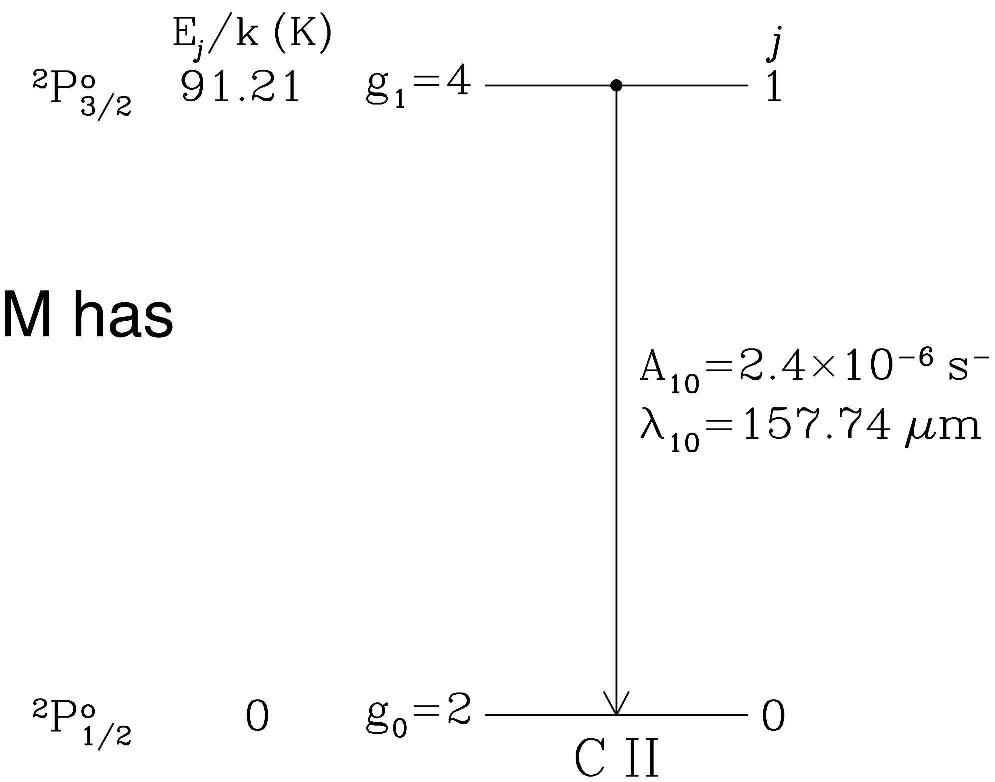
- At $\lambda = 158\,\mu\mathrm{m}$, the continuum background in the ISM has

$$\bar{n}_\gamma \approx 10^{-5} \ll 1 \longrightarrow n_{\mathrm{crit}} \simeq \frac{A_{10}}{k_{10}}$$

- Critical densities:

$$n_{\mathrm{crit}}(e^-) \approx 53\, T_4^{1/2}\,\mathrm{cm}^{-3}$$

$$n_{\mathrm{crit}}(\mathrm{H}) \approx 3.2 \times 10^3\, T_2^{-0.1281-0.0087\ln T_2}\,\mathrm{cm}^{-3}$$



[Figure 17.3 in Draine]

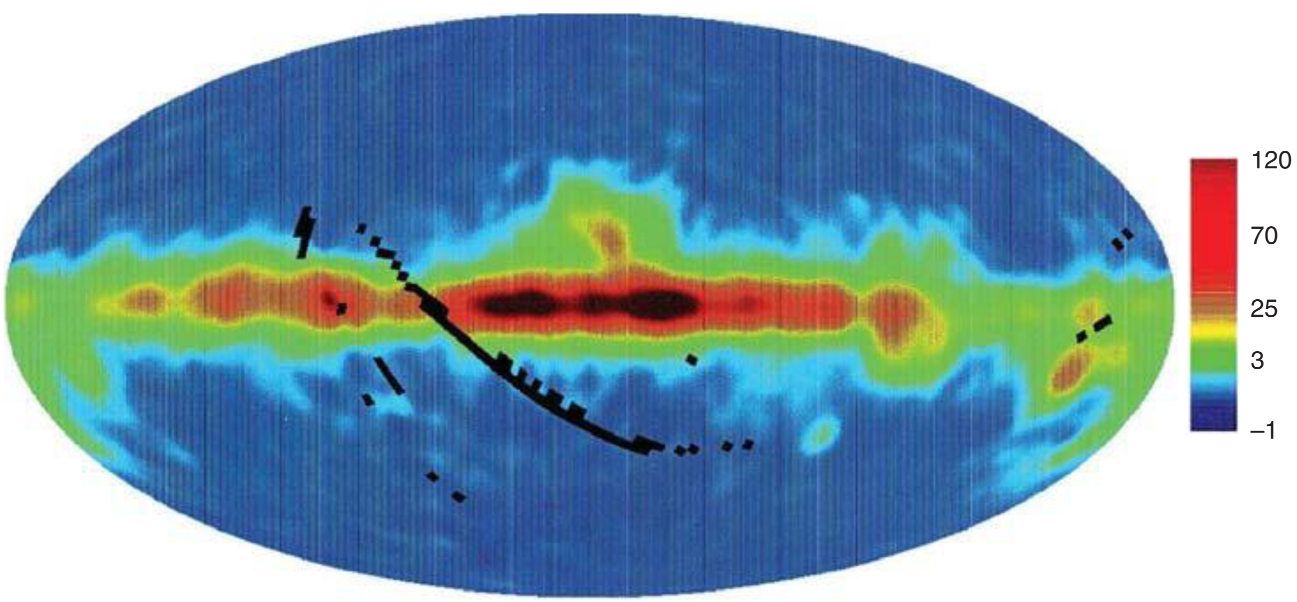
- The critical densities are much higher than the typical densities in both the CNM and WNM. Thus, **the C II fine-structure levels will be sub-thermally excited.**

$$\frac{n_1}{n_0} = \frac{n_c k_{01}}{n_c k_{10} + A_{10}} \simeq n_c \frac{k_{01}}{A_{10}} = \frac{k_{01}}{k_{10}} \frac{n_c}{n_{\mathrm{crit}}} = 2e^{-91.21\,\mathrm{K}/T_{\mathrm{gas}}} \frac{n_{\mathrm{H}}}{n_{\mathrm{crit}}} \quad \rightarrow \quad = 2e^{-91.21\,\mathrm{K}/T_{\mathrm{exc}}}$$

because $n_c \ll n_{\mathrm{crit}}$

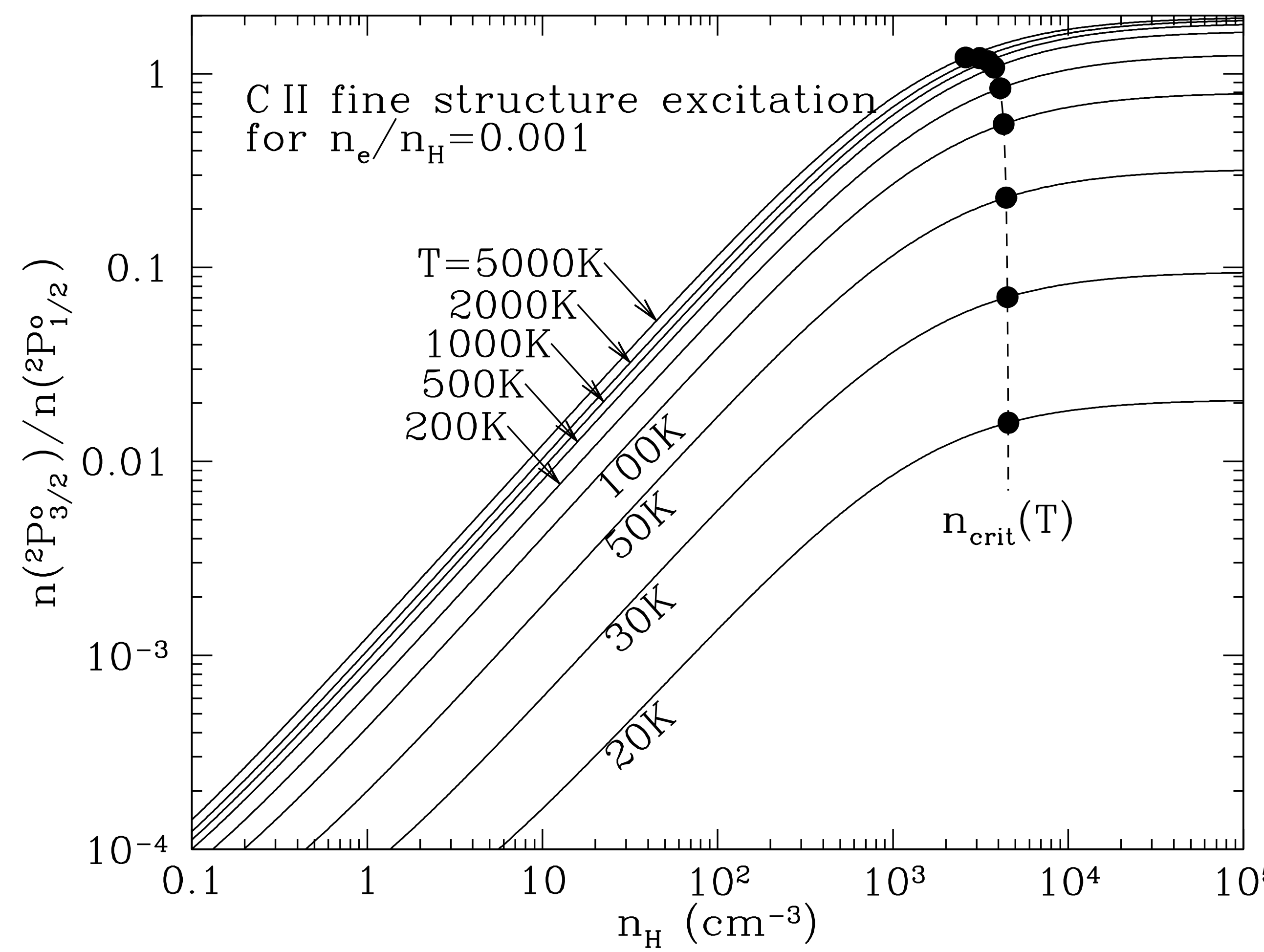
$$\frac{T_{\mathrm{exc}}}{T_{\mathrm{gas}}} \simeq \frac{1}{1 + (T_{\mathrm{gas}}/91.21\,\mathrm{K})\ln(n_{\mathrm{crit}}/n_{\mathrm{H}})} < 1$$

- The C II fine-structure levels will be sub-thermally excited. Collisional excitations of the upper level ${}^2P^o_{3/2}$ will usually be followed by radiative decays, removing energy from the gas.
- The [C II] 158 µm transition is the principal cooling transition for the diffuse gas in star-forming galaxies.



All-sky map of [C II] 158 µm emission, made by Far InfraRed Absolute Spectrophotometer (FIRAS) on the COsmic Background Explorer (COBE) satellite (Fixsen et al. 1999).

[Plate 3 in Draine]



[Fig. 17.4 in Draine]

Equation for the 21-cm Spin Temperature

- We have derived the equation for the level populations in the presence of collision and radiation. Now, we will derive an intuitive equation for the spin temperature of the 21-cm line.

$$\frac{n_1}{n_0} = \frac{n_c k_{01} + \bar{n}_\gamma (\mathrm{g}_1/\mathrm{g}_0) A_{10}}{n_c k_{10} + (1 + \bar{n}_\gamma) A_{10}}$$

- Let's define the temperature corresponding to the 21-cm transition.

$$T_* = E_{10}/k = 0.0682\,\mathrm{K}$$

- The temperatures of radiation and gas will be much higher than this:

$$T_{\mathrm{gas}} \approx 10 - 10^4\,\mathrm{K} \gg T_*, \quad T_{\mathrm{rad}} = 3.77\,\mathrm{K} \gg T_*, \quad T_{\mathrm{spin}} \gg T_*$$

- The population ratio can be written in terms of the excitation (spin) temperature:

$$\frac{n_1}{n_0} = \frac{\mathrm{g}_1}{\mathrm{g}_0} e^{-T_*/T_{\mathrm{spin}}} \simeq \frac{\mathrm{g}_1}{\mathrm{g}_0} \left(1 - \frac{T_*}{T_{\mathrm{spin}}} \right)$$

- Similarly,

$$k_{01} = \frac{\mathrm{g}_1}{\mathrm{g}_0} k_{10} e^{-T_*/T_{\mathrm{gas}}} \simeq \frac{\mathrm{g}_1}{\mathrm{g}_0} k_{10} \left(1 - \frac{T_*}{T_{\mathrm{gas}}} \right)$$

$$\bar{n}_\gamma = \frac{1}{e^{T_*/T_{\mathrm{rad}}} - 1} \simeq \frac{T_{\mathrm{rad}}}{T_*}$$

- Substituting these into the population equation, we obtain

$$1-\frac{T_*}{T_{\rm spin}}=\frac{n_c k_{10}\left(1-T_*/T_{\rm gas}\right)+\left(T_{\rm rad}/T_*\right)A_{10}}{n_c k_{10}+\left(1+T_{\rm rad}/T_*\right)A_{10}}$$

- Finally, we obtain the following equation:

$$T_{\rm spin}=\frac{T_*+T_{\rm rad}+y_c T_{\rm gas}}{1+y_c} \qquad \Longleftarrow \qquad y_c\equiv\frac{T_*}{T_{\rm gas}}\frac{n_c k_{10}}{A_{10}}$$

- Ignoring T_* term, we obtain an intuitive equation for the spin temperature.

$$T_{\rm spin}=\frac{T_{\rm rad}+y_c T_{\rm gas}}{1+y_c} \qquad \Longleftarrow \qquad y_c\equiv\frac{T_*}{T_{\rm gas}}\frac{n_c k_{10}}{A_{10}}$$

This equation was first derived by G. Field (1958).

- This equation describes the spin temperature as ***a weighted mean of the radiation and gas temperatures with weights of 1 and y_c***.

- From the equation, we can show that

$$\begin{aligned} T_{\rm spin} &\simeq T_{\rm rad} \quad \text{if} \quad y_c \ll 1 \\ T_{\rm spin} &\simeq T_{\rm gas} \quad \text{if} \quad y_c \gg 1 \end{aligned}$$

- A new critical density of the colliding particle may be defined:

$$y_c = 1 \quad \Longrightarrow \quad n^*_{\rm crit} = \frac{T_{\rm gas}}{T_*}\frac{A_{10}}{k_{10}}$$

- Now, compare this density with the previous definition of the critical density.

$$\begin{aligned} n_{\rm crit} &\equiv \frac{[1+(n_\gamma)_{10}]\,A_{10}}{k_{10}} \\ &= \left[1+\frac{1}{e^{h\nu_{10}/kT_{\rm rad}}-1}\right]\frac{A_{10}}{k_{10}} \\ &\simeq \left(1+\frac{T_{\rm rad}}{T_*}\right)\frac{A_{10}}{k_{10}} \end{aligned}$$

$$\frac{n^*_{\rm crit}}{n_{\rm crit}} \approx \frac{T_{\rm gas}}{T_{\rm rad}}$$

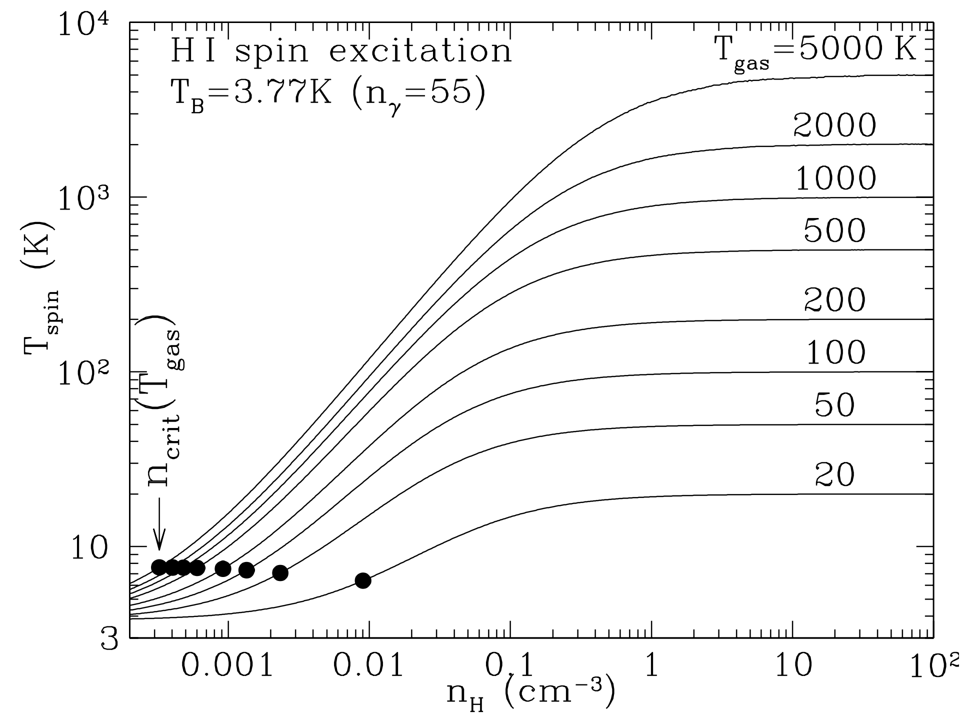
Detectability of Hydrogen in a Low Density Medium

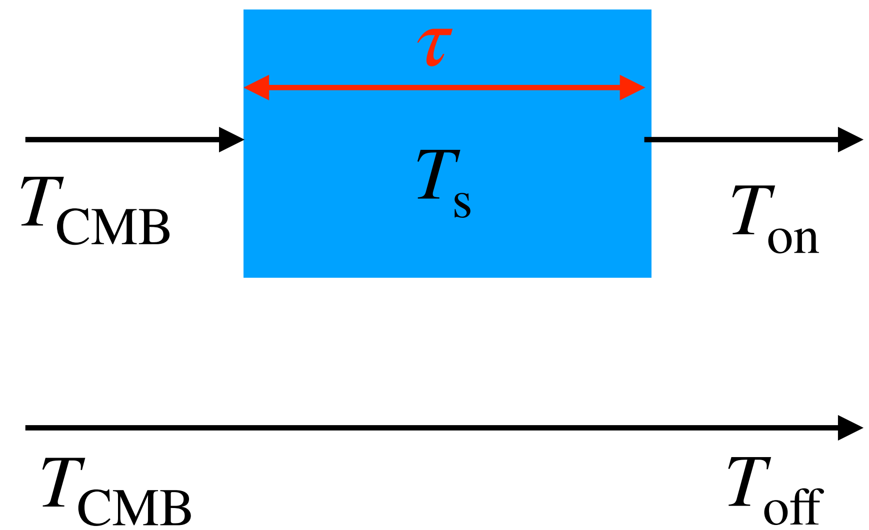
- In a very low density medium (WNM, CGM, IGM), the particle collisions are very rare ($n_{\rm HI} \ll n_{\rm crit}$).
- The radiative transition due to the CMB photons will control the relative population between the hyperfine structures.
    - This indicates $T_{\rm s} = T_{\rm CMB}$.
    - The RT equation in the Rayleigh-Jeans regime can be written in terms of temperature:

$$T_{\rm on} = T_{\rm CMB} e^{-\tau} + T_{\rm s}(1 - e^{-\tau}) = T_{\rm CMB}$$
$$T_{\rm off} = T_{\rm CMB}$$
$$T_{\rm on} - T_{\rm off} = 0$$

    - Then, we have $T_{\rm on} = T_{\rm off} = T_{\rm CMB}$.
    - ***Neither emission nor absorption feature from the hydrogen gas is detectable.***
- ***We need something that can make*** $T_{\rm s} \neq T_{\rm CMB}$.

[Fig. 17.2 in Draine]





The Wouthuysen-Field effect: The Third Mechanism controlling the Spin Temperature

- Wouthuysen (1952, AJ, 57, 31)

"Wouthuysen" is pronounced as roughly "Vowt-how-sen." (바우타이슨)

Wouthuysen, S. A. On the excitation mechanism of the 21-cm (radio-frequency) interstellar hydrogen emission line.

The mechanism proposed here is a radiative one: as a consequence of absorption and re-emission of Lyman-α resonance radiation, a redistribution over the two hyperfine-structure components of the ground level will take place. Under the assumption—here certainly permitted—that induced emissions can be negelcted, it can easily be shown that the relative distribution of the two levels in question, under stationary conditions, will depend solely on the shape of the radiation spectrum in the Lα region, and not on the absolute intensity.

The shape of the spectrum of resonance radiation, quasi-imprisoned in a large gas cloud, could only be determined by a careful study of the "scattering" process (absorption and re-emission) in a cloud of definite shape and dimensions. The spectrum will turn out to depend upon the localization in the cloud.

Some features can be inferred from more general considerations. Take a gas in a large container, with perfectly reflecting walls. Let the gas be in equilibrium at temperature T, together with Planck radiation of that same temperature. The scattering processes will not affect the radiation spectrum. One can infer from this fact that the photons, after an infinite number of scattering processes on gas atoms with kinetic temperature T, will obtain a statistical distribution over the spectrum proportional to the Planck-radiation spectrum of temperature T. After a finite but large number of scattering processes the Planck shape will be produced in a region around the initial frequency.

Photons reaching a point far inside an interstellar gas cloud, with a frequency near the Lα resonance frequency, will have suffered on the average a tremendous number of collisions. Hence in that region, which is wider the larger the optical depth of the cloud is for the Lyman radiation, the Planck spectrum corresponding to the gas-kinetic temperature will be established as far as the shape is concerned. Because, however, the relative occupation of the two hyperfine-structure components of the ground state depends only upon the shape of the spectrum near the Lα frequency, this occupation will be the one corresponding to equilibrium at the gas temperature.

The conclusion is that the resonance radiation provides a long-range interaction between gas atoms, which forces the internal (spin-)degree of freedom into thermal equilibrium with the thermal motion of the atoms.

Institute for Theoretical Physics of the City University, Amsterdam.

From a thermodynamic argument, Wouthuysen speculated the followings:

A tremendous number of scattering will establish the Planck-like spectrum, at the Lyα line center, corresponding to the gas-kinetic temperature.

The Lyα radiation is coupled with the hyperfine state of the hydrogen atom.

In the end, ***the 21cm spin temperature will become equal to the kinetic temperature of the hydrogen gas.***

Mechanisms that controls the spin temperature

- The spin temperature (T_s) is determined by three mechanisms.

(1) Direct Radiative Transitions by the background radiation field (Cosmic Microwave Background or Galactic Synchrotron)

$$I_\nu = \frac{2k_B T_R}{\lambda^2}$$

(Rayleigh-Jeans Law)

$$T_R = \text{brightness temperature}$$
$$= 2.73\,\text{K or } 3.77\,\text{K}$$

(2) Collisional Transitions (collision with other hydrogen and electron)

$$T_K = \text{gas kinetic temperature}$$

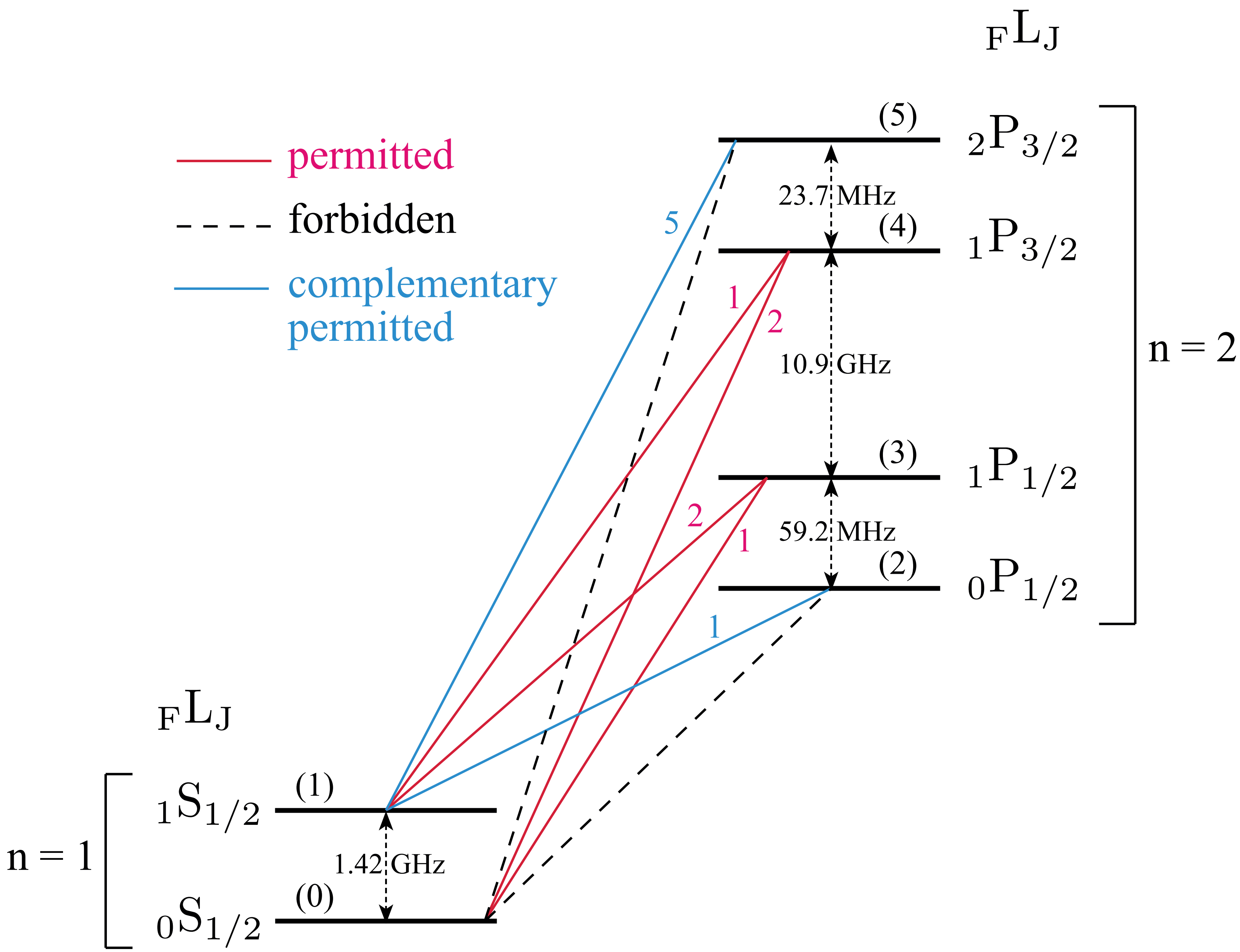
(3) Lyα pumping: Indirect Radiative Transitions involving intermediate levels caused by Lyα resonance scattering

$$T_\alpha = \text{color temperature}$$

$$J(\nu) \propto \exp\left(-\frac{h\nu}{k_B T_\alpha}\right)$$

Indirect Level Population by Lyα Scattering

The WF effect is a mechanism that the resonance scattering of Lyα photons indirectly control the relative populations between the hyperfine levels in the ground state (n = 1) via transitions involving the n = 2 state as an intermediate state.



Homework (due date: 04/24)

[Q7]

- Suppose that we observe a radio-bright QSO and detect absorption lines from Milky Way gas in its spectra. The 21 cm line is seen in optically-thin absorption with a profile with FWHM(H I) = 10 km s^{-1}. We also have high-resolution observations of the Na I doublet lines referred to as D_1 (5898Å) and D_2 (5892Å) in absorption. The Na I D_2 5892Å line width is FWHM(Na I D_2) = 5 km s^{-1}. The line profiles are the result of a combination of thermal broadening plus turbulence with a Gaussian velocity distribution with one-dimensional velocity dispersion $\sigma_{v,\,\mathrm{turb}}$.

  You will want to employ the following theorem: If the turbulence has a Gaussian velocity distribution, the overall velocity distribution function of atoms of mass *M* will be Gaussian, with one-dimensional velocity dispersion:

$$v_{\mathrm{rms}}^2 = \sigma_v^2 = \sigma_{v,\,\mathrm{turb}}^2 + \frac{kT}{M}$$

- If the Na I D_2 line is optically thin, estimate the kinetic temperature *T* and $\sigma_{v,\,\mathrm{turb}}$.

  Note hat for a Gaussian function, $\mathrm{FWHM} = 2\sqrt{2\ln 2}\sigma$.

[Q8]

A dwarf galaxy at a distance D_L = 15 Mpc is emitting in the 21-cm line of atomic hydrogen. The observed 21-cm line flux is $F = 1 \times 10^{-8}\,\mathrm{erg\,cm^{-2}\,s^{-1}}$.

If the emitting gas is assumed to be optically thin, and there is no absorption by intervening gas, estimate the mass of H I in the dwarf galaxy. Express your answer in solar mass $M_\odot$.

The Einstein *A* coefficient for the 21-cm line is $A_{u\ell} = 2.88 \times 10^{-15}\,\mathrm{s^{-1}}$.

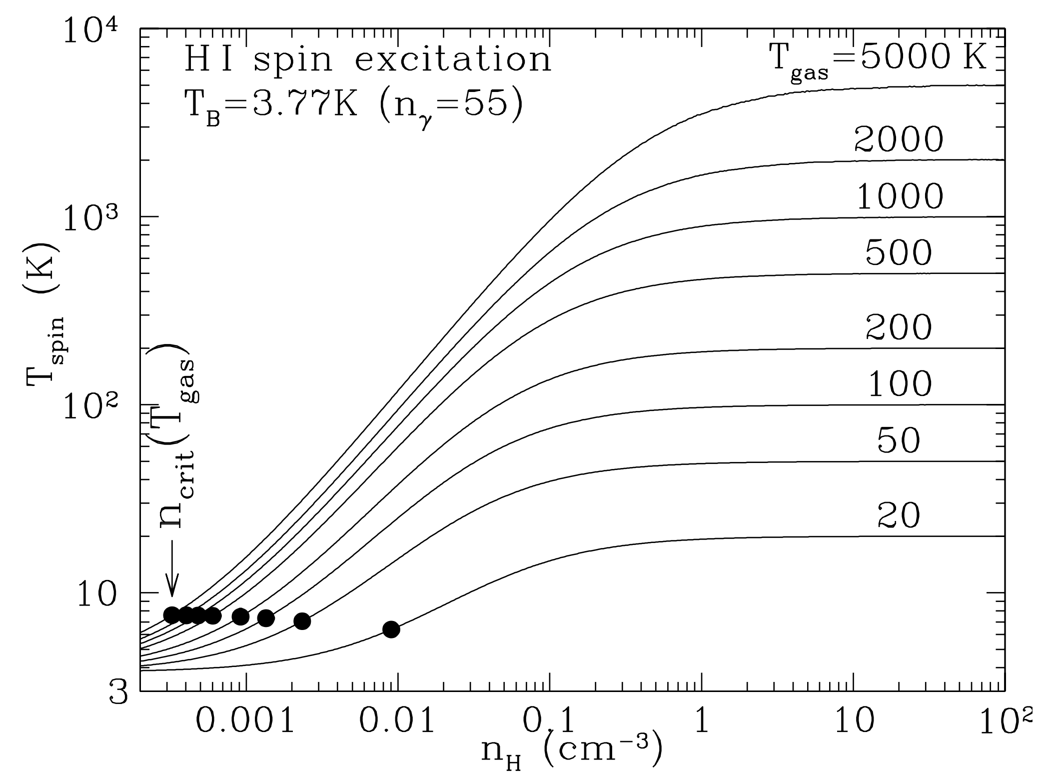
[Q9]

(1) If we consider only the background radiation field and collisions with hydrogen, the spin temperature of the 21-cm transition is given by

Eq(a): $$T_{\rm spin} = \frac{T_{\rm rad} + y_c T_{\rm gas}}{1 + y_c} \quad \text{where} \quad y_c \equiv \frac{T_*}{T_{\rm gas}} \frac{n_c k_{10}}{A_{10}}$$

- Using the above equation, make a plot similar to the right side figure. (Extrapolate the approximate formula for k_{10} down below 20 K and up above 10^3 K.)
- Denote the two critical densities, for each gas temperature, defined by

Eq(b): $$n^*_{\rm crit} = \frac{T_{\rm gas}}{T_*} \frac{A_{10}}{k_{10}} \quad \text{and} \quad n_{\rm crit} = \frac{(1 + n_\gamma) A_{10}}{k_{10}}$$



(2) Discuss whether Eq(a) for the spin temperature for the 21-cm transition can be applied to the [C II] 158μm line or not.

Explain why the equation cannot be applied?