Astrophysics

Lecture 03
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선광일 (Kwang-Il Seon)
UST / KASI

Basic Theory of Radiation Fields

Applicability of the Radiative Transfer Theory*

- We defined specific intensity by the relation: $dE = I_\nu dA d\Omega d\nu dt$

  We should note that dA and $d\Omega$ cannot both be made arbitrarily small because of the uncertainty principle:

$$dx dp_x dy dp_y = p^2 dA d\Omega \geq h^2 \quad \rightarrow \quad dA d\Omega \geq h^2/p^2 = \lambda^2$$

  There is another limitation because of the energy uncertainty principle:

$$dE dt \geq h \quad \rightarrow \quad d\nu dt \geq 1$$

- Therefore, when the wavelength of light is larger than atomic dimensions (Bohr radius, $a_0 = 0.53$ Å), as in the optical, we cannot describe the interaction of light on the atomic scale in terms of specific intensity.
- However, we may still regard transfer theory as a valid macroscopic theory, <u>provided the absorption and emission properties are correctly calculated from microscopic theories</u> (electromagnetic or quantum theory).
- A more precise, classical treatment of the validity of rays is known as the eikonal approximation. (from German “eikonal”, which is from Greek word meaning “image”)

- Observational astronomy is primarily based on the detection of radiation emitted by astrophysical objects.

  Continuum radiation is a natural consequence of the principle that accelerating charges radiate.

  We will review and apply principles of electromagnetism to further our understanding of important astrophysical phenomena, demonstrating in the process that important radiation processes can be derived from the basic principle that accelerating charges radiate.

  We will develop the theory that describes bremsstrahlung and synchrotron radiation. A theoretical understanding of these two radiation mechanisms allows us to interpret the emission of a wide range of objects, ranging from distant radio galaxies to nearby H II regions.

Useful Mathematical Formulae

- Dirac delta function:

$$\delta(\omega - \omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega-\omega')t}dt$$

- Fourier Transform:

<u>Rybicki & Lightman</u>

$$\bar{a}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(t)e^{-i\omega t}dt$$

$$a(t) = \int_{-\infty}^{\infty} \bar{a}(\omega)e^{i\omega t}d\omega$$

Parseval's Theorem

$$\int_{-\infty}^{\infty} a(t)b^*(t)dt = 2\pi\int_{-\infty}^{\infty} \bar{a}(\omega)\bar{b}^*(\omega)d\omega$$

$$\int_{-\infty}^{\infty} |a(t)|^2 dt = 2\pi\int_{-\infty}^{\infty} |\bar{a}(\omega)|^2 d\omega$$

<u>Afken (Mathematical Methods for Physicists)</u>

$$\bar{a}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(t)e^{i\omega t}dt$$

$$a(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{a}(\omega)e^{-i\omega t}d\omega$$

$$\int_{-\infty}^{\infty} a(t)b^*(t)dt = \int_{-\infty}^{\infty} \bar{a}(\omega)\bar{b}^*(\omega)d\omega$$

$$\int_{-\infty}^{\infty} |a(t)|^2 dt = \int_{-\infty}^{\infty} |\bar{a}(\omega)|^2 d\omega$$

- Vector identities:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\nabla\times(\nabla\phi) = 0$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Electromagnetic force on a single charged particle

- **Lorentz force**: If a particle of charge q and mass m moves with velocity $\mathbf{v}$ in the presence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$, then it will experience a force:

$$\boxed{\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)}$$

(in Gaussian units, or cgs units)

- **Power** supplied by the EM fields (the rate of work done by the fields) on a particle is

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{F} &= q\mathbf{v} \cdot \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \\
\mathbf{v} \cdot m\frac{d\mathbf{v}}{dt} &= q\mathbf{v} \cdot \mathbf{E} \\
\therefore \frac{dU_{\rm mech}}{dt} &\equiv \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\mathbf{v} \cdot \mathbf{E}
\end{aligned}$$

- Note $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$, meaning that the magnetic fields do not work.

Electromagnetic force on a continuous medium

- Consider a medium with **charge density** and **current density**:

$$\rho \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i$$

$$\mathbf{j} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i$$

- **Force density** (force per unit volume):

$$\mathbf{f} = \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B}$$

- **Power density** supplied by the field (the rate of work done by the field per unit volume):

$$\frac{du_{\text{mech}}}{dt} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i \cdot \mathbf{E} = \mathbf{j} \cdot \mathbf{E}$$

Note typos in the Rybicki & Lightman's book. They use the same symbol to denote the energy density u and the total energy U within a volume.

Maxwell's equations

- **Maxwell's eqs.** (in macroscopic forms) relates fields to charge and current densities.

$$\nabla \cdot \mathbf{D} = 4\pi\rho \quad \text{Gauss's law}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{Gauss's law for magnetism (no magnetic monopoles)}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \quad \text{Maxwell-Faraday equation}$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \quad \text{Ampere-Maxwell equation}$$

D,**H** : macroscopic fields
B,**E** : microscopic fields

$\mathbf{D} = \epsilon \mathbf{E}$

$\mathbf{B} = \mu \mathbf{H}$

ϵ : dielectric constant

μ : magnetic permeability

Dielectric material (절연체): an electrical insulator that can be polarized by an applied electric field. Electric charges do not flow through the material as they do in a conductor, but only slightly shift from their average equilibrium positions causing dielectric polarization.

Permeability (투자율): the degree of magnetization of a material in response to a magnetic field.

Note $\epsilon = \mu = 1$ in the absence of dielectric or permeability media.

- **Conservation of charge**

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \left(\frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \right)$$

$$0 = \frac{4\pi}{c}\nabla \cdot \mathbf{j} + \frac{1}{c}\frac{\partial}{\partial t}(4\pi\rho) \quad \rightarrow \quad \boxed{\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0}$$

Poynting's Theorem: Electromagnetic Field Energy

- Use the Ampere's law to obtain the mechanical energy density

$$\frac{du_{\rm mech}}{dt} = \mathbf{j} \cdot \mathbf{E} = \frac{1}{4\pi} \mathbf{E} \cdot \left(c \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

- Use a vector identity and Faraday's law:

$$\begin{aligned} \mathbf{E} \cdot (\nabla \times \mathbf{H}) &= \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{H}) \\ &= -\frac{1}{c} \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) \end{aligned}$$

Then,

$$\mathbf{j} \cdot \mathbf{E} = \frac{1}{4\pi} \left(-c \nabla \cdot (\mathbf{E} \times \mathbf{H}) - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right)$$

- **Poynting's theorem** in differential form.

$$\mathbf{j} \cdot \mathbf{E} + \frac{1}{8\pi} \frac{\partial}{\partial t} \left(\epsilon E^2 + \frac{B^2}{\mu} \right) = -\nabla \cdot \left(\frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \right)$$

- **Electromagnetic field density** (field energy per unit volume) and **Poynting vector** (electromagnetic flux vector) are identified:

$$\boxed{u_{\rm field} = \frac{1}{8\pi} \left(\epsilon E^2 + \frac{B^2}{\mu} \right) = u_E + u_B \qquad \mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}}$$

- The Poynting's theorem becomes an expression of the **local conservation of energy**.

$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{field}}) + \nabla \cdot \mathbf{S} = 0$$

- Integrating the equation over a volume element and using the divergence theorem, we obtain the **conservation of energy**:

$$\boxed{\frac{d}{dt}(U_{\text{mech}} + U_{\text{field}}) = -\int_\Sigma \mathbf{S} \cdot d\mathbf{A}}$$

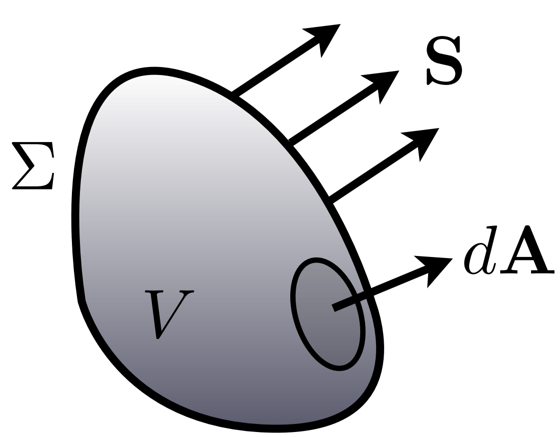
Here,

$$U_{\text{mech}} \equiv \int_V u_{\text{mech}} dV \quad \text{and} \quad U_{\text{field}} \equiv \int_V u_{\text{field}} dV$$

or

$$\int_V (\mathbf{j} \cdot \mathbf{E}) dV + \frac{d}{dt} \int_V \left(\frac{\epsilon E^2 + B^2/\mu}{8\pi} \right) dV = -\int_\Sigma \mathbf{S} \cdot d\mathbf{A}$$

Meaning: <u>the rate of change of total (mechanical + field) energy within the volume *V* is equal to the net inward flow of energy through the bounding surface Σ.</u>



divergence theorem:

$$\int_V \nabla \cdot \mathbf{S} dV = \int_\Sigma \mathbf{S} \cdot d\mathbf{A}$$

- In electrostatics and magnetostatics, we recall that

$$\mathbf{E} \propto r^{-2} \quad \text{and} \quad \mathbf{B} \propto r^{-2} \quad \text{as} \quad r \to \infty \qquad \to \quad \mathbf{S} \propto r^{-4}$$

$$\therefore \quad \int_{\Sigma} \mathbf{S} \cdot d\mathbf{A} = 0 \quad \text{as} \quad r \to \infty$$

- However, for time-varying fields, we will find that

$$\mathbf{E} \propto r^{-1} \quad \text{and} \quad \mathbf{B} \propto r^{-1} \quad \text{as} \quad r \to \infty$$

$$\therefore \quad \int_{\Sigma} \mathbf{S} \cdot d\mathbf{A} \neq 0 \quad \text{as} \quad r \to \infty$$

- This finite energy flowing outward (or inward) at large distances is called **radiation**. Those parts of **E** and **B** that decreases as r^{-1} at large distances are said to constitute the **radiation field**.

Plane Electromagnetic Waves

- In vacuum $(\rho = 0 = \mathbf{j},\ \epsilon = 1 = \mu)$, Maxwell's equations give the vector wave equations:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

$\longrightarrow$ These equations are invariant under $\begin{array}{l} \mathbf{E} \rightarrow \mathbf{B} \\ \mathbf{B} \rightarrow -\mathbf{E} \end{array}$

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times (\nabla \times \mathbf{E}) &= -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \\
\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \\
\therefore\ \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \\
\nabla \times (\nabla \times \mathbf{B}) &= \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \\
\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} \\
\therefore\ \nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} &= 0
\end{aligned}$$

$\longrightarrow$ We obtain the vector wave equations:

$$\boxed{\begin{aligned}
\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0 \\
\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} &= 0
\end{aligned}}$$

Properties of a single Fourier mode

- Consider an arbitrary Fourier mode in vacuum:

$$\mathbf{E} = E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{e}}, \qquad \mathbf{B} = B_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{b}}$$

wave vector $\mathbf{k}$

angular frequency $\omega = 2\pi\nu$

(E_0 and B_0 are complex constants.)

- Substituting into Maxwell's equations yields:

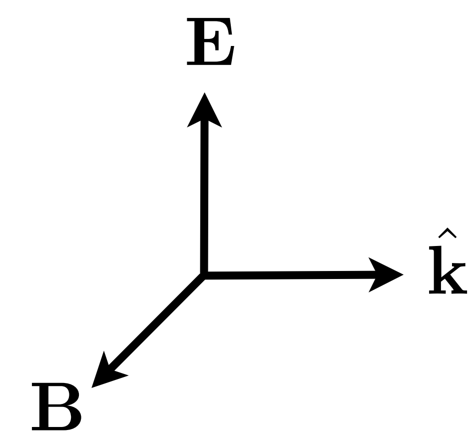
$$\nabla\cdot\mathbf{E} = 0 \quad\rightarrow\quad \mathbf{k}\cdot\mathbf{E} = 0$$

$$\nabla\cdot\mathbf{B} = 0 \quad\rightarrow\quad \mathbf{k}\cdot\mathbf{B} = 0$$

$$\left(\mathbf{k} = \hat{\mathbf{k}}\frac{\omega}{c}\right)$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \quad\rightarrow\quad \mathbf{k}\times\mathbf{E} = \frac{\omega}{c}\mathbf{B} \qquad \text{or} \quad \hat{\mathbf{k}}\times\mathbf{E} = \mathbf{B}$$

$$\nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \quad\rightarrow\quad \mathbf{k}\times\mathbf{B} = -\frac{\omega}{c}\mathbf{E} \qquad \text{or} \quad \hat{\mathbf{k}}\times\mathbf{B} = -\mathbf{E}$$



$$\hat{\mathbf{k}}\times\mathbf{E} = \mathbf{B} \quad\rightarrow\quad \hat{\mathbf{k}}\times\hat{\mathbf{e}}E_0 = \hat{\mathbf{b}}B_0$$

$$E_0\left(\hat{\mathbf{k}}\times\hat{\mathbf{e}}\right)\cdot\hat{\mathbf{b}} = B_0$$

$$E_0 = B_0$$

$$\begin{aligned} E_0 &= |\mathbf{E}|e^{i\phi_E} \\ B_0 &= |\mathbf{B}|e^{i\phi_B} \end{aligned} \quad\rightarrow\quad \phi_E = \phi_B$$

(1) EM waves are **transverse** (perpendicular to the direction of propagation).

(2) $\mathbf{E}$ and $\mathbf{B}$ are **orthogonal** to each other.

(3) $\left(\hat{\mathbf{k}},\ \hat{\mathbf{e}},\ \hat{\mathbf{b}}\right)$ form an orthogonal basis (triad).

(4) **Field amplitudes and phases are equal**: $\quad |\mathbf{B}| = |\mathbf{E}|,\quad B_0 = E_0 \quad\text{and}\quad \phi_B = \phi_E$

- Fourier transform of fields:

$$\bar{\mathbf{E}}(\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int d^3\mathbf{r}\int dt\mathbf{E}(\mathbf{r},t)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

- Inverse transformation:

$$\mathbf{E}(\mathbf{r},t) = \int d^3\mathbf{k}\int d\omega\bar{\mathbf{E}}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

- Apply the wave equation to Fourier expansion:

$$\begin{aligned}\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t^2} &= -\int d^3\mathbf{k}\int d\omega\left(k^2 - \frac{\omega^2}{c^2}\right)\mathbf{E}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \\ &= 0\end{aligned}$$

$$\therefore k = \frac{\omega}{c}$$

- Phase velocity

$$v_{\mathrm{ph}} \equiv \frac{\omega}{k} = c$$

Dispersion relation*

- We obtain the vacuum dispersion relation, phase velocity, and group velocity:

$$\omega = ck \qquad v_{\rm ph} \equiv \frac{\omega}{k} = c \qquad v_{\rm g} \equiv \frac{\partial \omega}{\partial k} = c$$

dispersion relation = a function which gives ω as a function of k.

phase velocity = the rate at which the phase of the wave propagates in space.

group velocity = the velocity with which the overall shape of the waves' amplitudes (modulation or envelope of the wave) propagates through space.

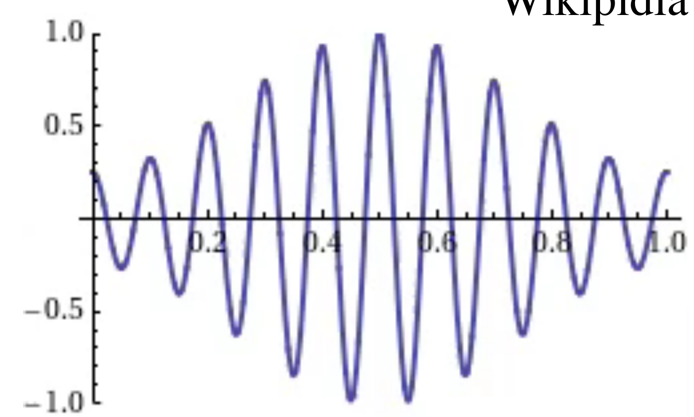
- Assume the wave packet E is almost monochromatic, so that its Fourier component is nonzero only in the vicinity of a central wavenumber k_0. Then, linearization gives:

$$\omega(k) \approx \omega_0 + (k - k_0)\frac{\partial \omega(k)}{\partial k}|_{k=k_0}$$
$$= \omega_0 + (k - k_0)\omega_0'$$

$$E(x,t) = \int dk \int d\omega \bar{E}(k,\omega) e^{i(kx-\omega t)}$$
$$\approx e^{it(\omega_0' k_0 - \omega_0)} \int dk \bar{E}(k,\omega_0) e^{ik(x-\omega_0' t)}$$

$$|E(x,t)| = |E(x - \omega_0' t, 0)|$$

Wikipidia



$\leftarrow v_{\rm ph} \quad \rightarrow v_{\rm g}$

- The envelope of the wavepacket travels at velocity $\omega_0' = (\partial \omega / \partial k)_{k=k_0}$.

- If *A(t)* and *B(t)* are two complex quantities with the same sinusoidal time dependence,

$$A(t) = \mathcal{A}e^{i\omega t} \qquad B(t) = \mathcal{B}e^{i\omega t}$$

then the time average of the product of their real parts is

$$\begin{aligned}
\langle \mathrm{Re}A(t) \cdot \mathrm{Re}B(t) \rangle &= \frac{1}{4}\left\langle \left(\mathcal{A}e^{i\omega t} + \mathcal{A}^* e^{-i\omega t}\right)\left(\mathcal{B}e^{i\omega t} + \mathcal{B}^* e^{-i\omega t}\right)\right\rangle \\
&= \frac{1}{4}\left\langle \mathcal{A}\mathcal{B}^* + \mathcal{A}^*\mathcal{B}\right\rangle \\
&= \frac{1}{2}\mathrm{Re}(\mathcal{A}\mathcal{B}^*) = \frac{1}{2}\mathrm{Re}(\mathcal{A}^*\mathcal{B})
\end{aligned}$$

- Time-averaged Poynting vector amplitude:

$$\begin{aligned}
\langle S \rangle &= \frac{c}{4\pi}\left\langle \mathrm{Re}\left(E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right)\mathrm{Re}\left(B_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right)\right\rangle \\
&= \frac{c}{8\pi}\mathrm{Re}\left(E_0 B_0^*\right) \\
&= \frac{c}{8\pi}|E_0|^2 = \frac{c}{8\pi}|B_0|^2 \qquad \leftarrow E_0 = B_0
\end{aligned}$$

Energy per unit area per unit time:

$$\frac{dW}{dAdt} = \frac{c}{8\pi}|E_0|^2$$

- Time-averaged field energy density:

$$\langle U_{\mathrm{field}} \rangle = \frac{1}{8\pi}\left\langle |\mathbf{E}|^2 + |\mathbf{B}|^2 \right\rangle = \frac{1}{16\pi}\mathrm{Re}(E_0E_0^* + B_0B_0^*) = \frac{1}{8\pi}|E_0|^2 = \frac{1}{8\pi}|B_0|^2$$

- Velocity of energy flow:

$$\langle S \rangle / \langle U_{\mathrm{field}} \rangle = c$$

Power Spectrum

- A common property of any wave theory:

  If we have a time record of the radiation field of length Δt, we can only define the spectrum to within a frequency resolution $\Delta\omega$ where

$$\boxed{\Delta\omega\Delta t > 1. \quad \text{(uncertainty relation)}}$$

- Let us consider only a component of the transverse electric field: $E(t) \equiv \hat{\mathbf{a}} \cdot \mathbf{E}(t)$
- Fourier transform and its inverse are:

$$\bar{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)e^{-i\omega t}dt, \qquad E(t) = \int_{-\infty}^{\infty} \bar{E}(\omega)e^{i\omega t}d\omega$$

  Since E(t) is real, the negative frequencies are redundant, i.e., $\bar{E}(-\omega) = \bar{E}^*(\omega)$.

- Total energy per unit area per unit time: $\dfrac{dW}{dAdt} = \dfrac{c}{4\pi}E^2(t)$ (Poynting vector)
- Total energy per unit area:

$$\frac{dW}{dA} = \int_{-\infty}^{\infty} \frac{dW}{dAdt}dt = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)dt$$

$$= \frac{c}{2}\int_{-\infty}^{\infty} |\bar{E}(\omega)|^2 d\omega = c\int_{0}^{\infty} |\bar{E}(\omega)|^2 d\omega$$

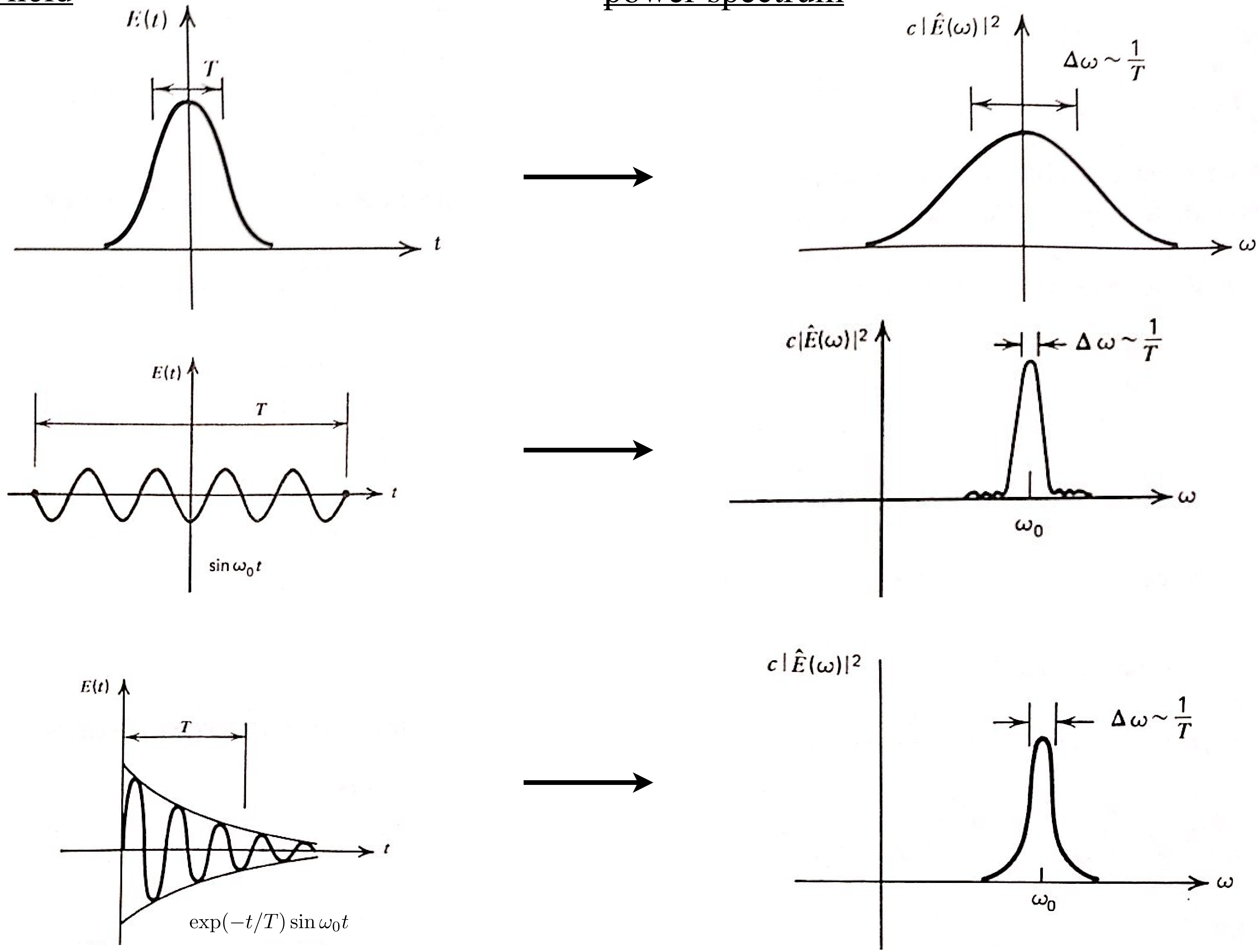
Energy per unit area per unit frequency:

$$\frac{dW}{dAd\omega} = c|\bar{E}(\omega)|^2$$

Here, we used Parseval's theorem: $\int_{-\infty}^{\infty} E^2(t)dt = 2\pi\int_{-\infty}^{\infty} |\bar{E}(\omega)|^2 d\omega$

electric field

power spectrum



Electromagnetic Potentials (from homogeneous Maxwell eqs)

- **Vector potential**:

(Gauss' law for magnetism)

From the vector identity $\nabla \cdot (\nabla \times A) = 0$, the equation $\nabla \cdot \mathbf{B} = 0$ allows us to define a vector potential such that

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \qquad \mathbf{A}: \ \text{vector potential}$$

Then, Maxwell-Faraday equation becomes

(Maxwell-Faraday equation) $$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

- **Scalar potential**:

From the vector identity $\nabla \times \left(\nabla \phi \right) = 0,$ this equation can be satisfied if we define a "scalar" potential such as

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi \quad \rightarrow \quad \boxed{\mathbf{E} = -\nabla \phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}} \qquad \phi: \ \text{scalar potential}$$

- **Gauge invariance**:

B will be unchanged for any transformation

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \rightarrow \quad \boxed{\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\psi} \quad \text{since } \nabla \times (\nabla\psi) = 0$$

E will also be unchanged if at the same time the scalar potential is changed by

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \quad \rightarrow \quad \boxed{\phi \rightarrow \phi' = \phi - \frac{1}{c}\frac{\partial \psi}{\partial t}}$$

EM field is invariant under the **gauge transformations.**

$$\boxed{(\phi,\ \mathbf{A}) \rightarrow \left(\phi - \frac{1}{c}\frac{\partial\psi}{\partial t},\ \mathbf{A} + \nabla\psi\right)}$$

The gauge transformations give us mathematical flexibility for solving various EM problems. They are useful because they do not change the underlying physics.

A particularly useful gauge for dealing with radiation is given by the ***Lorentz condition***.

Lorentz Gauge (Lorentz condition)

- Using the potential, the inhomogeneous Maxwell's equations can be written as

$$\nabla \cdot \mathbf{E} = 4\pi\rho \;\rightarrow\; \nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -4\pi\rho$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} \;\rightarrow\; \nabla \times (\nabla \times \mathbf{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = \frac{4\pi}{c}\mathbf{j}$$

$$\rightarrow\; -\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right) = \frac{4\pi}{c}\mathbf{j}$$

$$\mathrm{Note:}\; \nabla \times (\nabla \times \mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

- The Lorentz gauge is the most important gauge in the EM theory, defined by:

$$\boxed{\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0}$$

Note that we can alway choose a function ψ such as:

$$\nabla \cdot \mathbf{A}' + \frac{1}{c}\frac{\partial \phi'}{\partial t} = \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} + \left(\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}\right) = 0$$

- Then, with the Lorentz gauge, the above equations become:

$$\boxed{\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{j} \end{aligned}}$$

Note: Column gauge is $\nabla \cdot \mathbf{A} = 0$

Retarded potentials

- The solutions to the above equations are (called the **retarded potentials**, see Jackson):

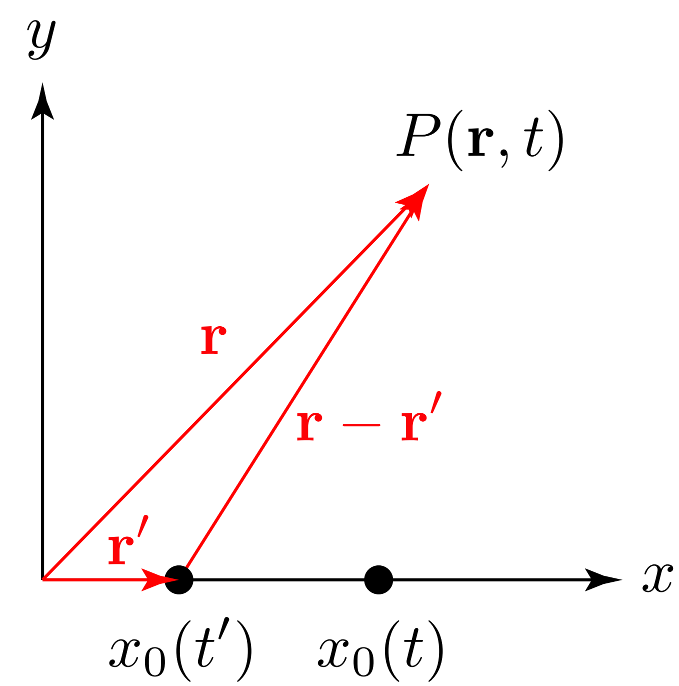
$$\phi(\mathbf{r},t) = \int_V d^3\mathbf{r}' \int dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int_V d^3\mathbf{r}' \int dt' \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)$$

$$\longrightarrow \phi(\mathbf{r},t) = \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}, \qquad \mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}$$

where

$$t' \equiv t_{\rm ret} = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$



$$x_0(t) - x_0(t') = c(t-t')$$

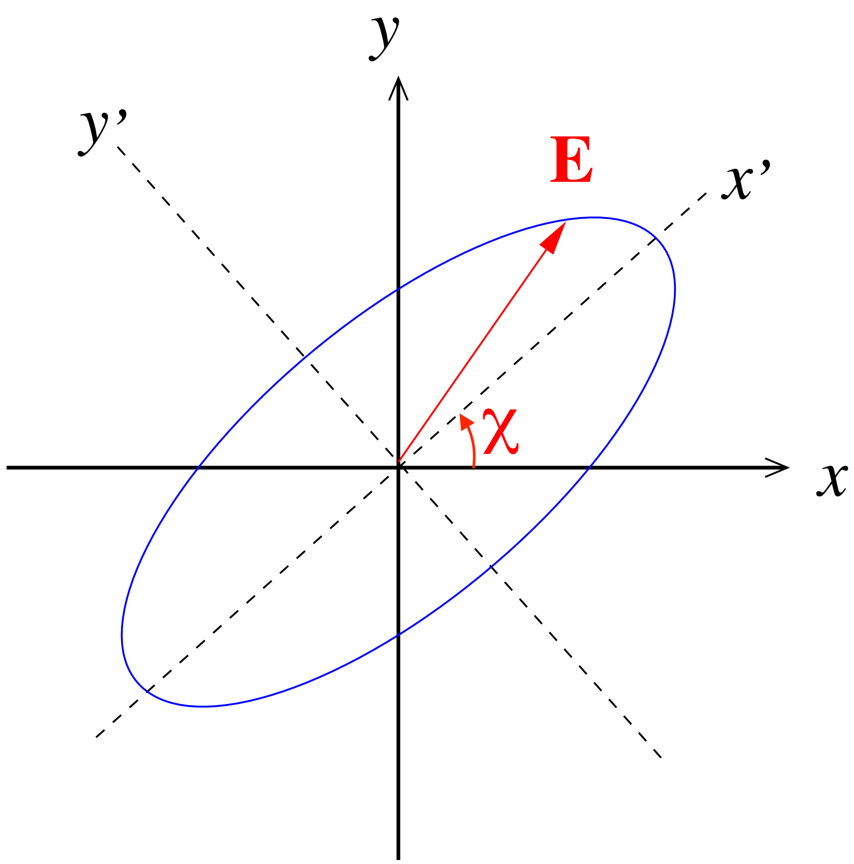
- The **retarded time** refers to conditions at the point $\mathbf{r}'$ that existed at a time earlier than *t* by just the time required for light to travel from $\mathbf{r}'$ to $\mathbf{r}$.
- The potentials responds to the changes after "retarded time" delay.

Polarization

Polarization

- Let us consider a plane EM wave propagating in the +z direction, and examine the electric vector at z = 0. Because the electric field is transverse, the electric field can be expressed as

$$\mathbf{E} = \hat{\mathbf{x}} E_1 e^{i(kz-\omega t)} + \hat{\mathbf{y}} E_2 e^{i(kz-\omega t)}$$
$$= \hat{\mathbf{x}} E_1 e^{-i\omega t} + \hat{\mathbf{y}} E_2 e^{-i\omega t} \quad \text{at z} = 0$$



Complex amplitudes can be expressed as

$$E_1 = \mathcal{E}_1 e^{i\phi_1} \qquad E_2 = \mathcal{E}_2 e^{i\phi_2} \quad \text{where } \mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2 \text{ are real.}$$

Then, the real part of **E** is

$$\mathbf{E} = \hat{\mathbf{x}} \mathcal{E}_1 \cos(\omega t - \phi_1) + \hat{\mathbf{y}} \mathcal{E}_2 \cos(\omega t - \phi_2)$$

As a function of time, the tip of **E** will trace out an ellipse, meaning that the general wave is ***elliptically polarized***.

- In general, the principal axes of this ellipse will have a tilt angle χ w.r.t. to x-y axes. We define the zero of time so that **E** lies along the x' direction at t = 0.

$$\mathbf{E} = \hat{\mathbf{x}}' \mathcal{E}_1' \cos\omega t + \hat{\mathbf{y}}' \mathcal{E}_2' \sin\omega t$$

- We can satisfy the late part of the above equation by defining an **ellipticity angle**:

$$\mathcal{E}_1' = \mathcal{E}_0 \cos\beta \qquad \mathcal{E}_2' = -\mathcal{E}_0 \sin\beta \quad \text{where} \quad -\pi/2 \le \beta \le \pi/2 \quad (\text{or} \quad \mathcal{E}_2' = \mathcal{E}_0 \sin\beta',\ \ \beta' = -\beta)$$

- With the relations

$$\begin{aligned} &\hat{\mathbf{x}}\mathcal{E}_1 \cos(\omega t - \phi_1) + \hat{\mathbf{y}}\mathcal{E}_2 \cos(\omega t - \phi_2) \\ &= \hat{\mathbf{x}}'\mathcal{E}_0 \cos\beta \cos\omega t - \hat{\mathbf{y}}'\mathcal{E}_0 \sin\beta \sin\omega t \end{aligned} \quad + \quad \begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \end{pmatrix} = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix}$$

we obtain the relations:

Given $\mathcal{E}_1, \phi_1, \mathcal{E}_2, \phi_2,$ we can solve for $\mathcal{E}_0, \beta, \chi,$

$$\begin{aligned} \mathcal{E}_1 \cos\phi_1 &= \mathcal{E}_0 \cos\beta \cos\chi \\ \mathcal{E}_1 \sin\phi_1 &= \mathcal{E}_0 \sin\beta \sin\chi \\ \mathcal{E}_2 \cos\phi_2 &= \mathcal{E}_0 \cos\beta \sin\chi \\ \mathcal{E}_2 \sin\phi_2 &= -\mathcal{E}_0 \sin\beta \cos\chi \end{aligned} \quad \longrightarrow \quad \begin{aligned} \mathcal{E}_1^2 + \mathcal{E}_2^2 &= \mathcal{E}_0^2 \\ \mathcal{E}_1^2 - \mathcal{E}_2^2 &= \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \\ 2\mathcal{E}_1\mathcal{E}_2 \cos\delta\phi &= \mathcal{E}_0^2 \cos 2\beta \sin 2\chi \\ 2\mathcal{E}_1\mathcal{E}_2 \sin\delta\phi &= \mathcal{E}_0^2 \sin 2\beta \end{aligned}$$

(where $\delta\phi \equiv \phi_1 - \phi_2$)

Polarization

- Taking time average of the $|\mathbf{E}|^2$, we obtain:

$$\langle|\mathbf{E}|^2\rangle = \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_1'^2 + \mathcal{E}_2'^2 = \text{constant} \equiv \mathcal{E}_0^2$$

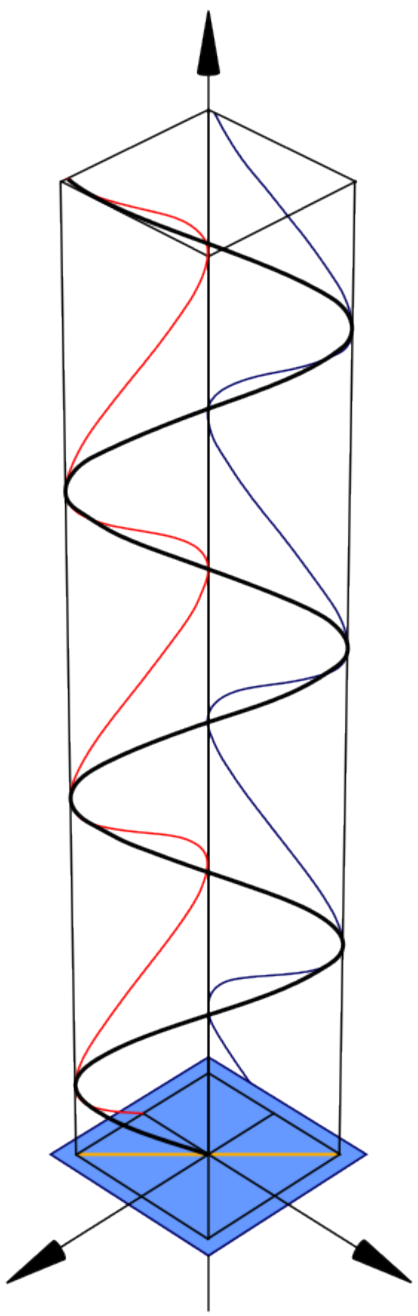
- **Polarization:**

$\beta = \pm\pi/4$: circularly polarized

$\beta = 0 \text{ or } \pm\pi/2$: linearly polarized

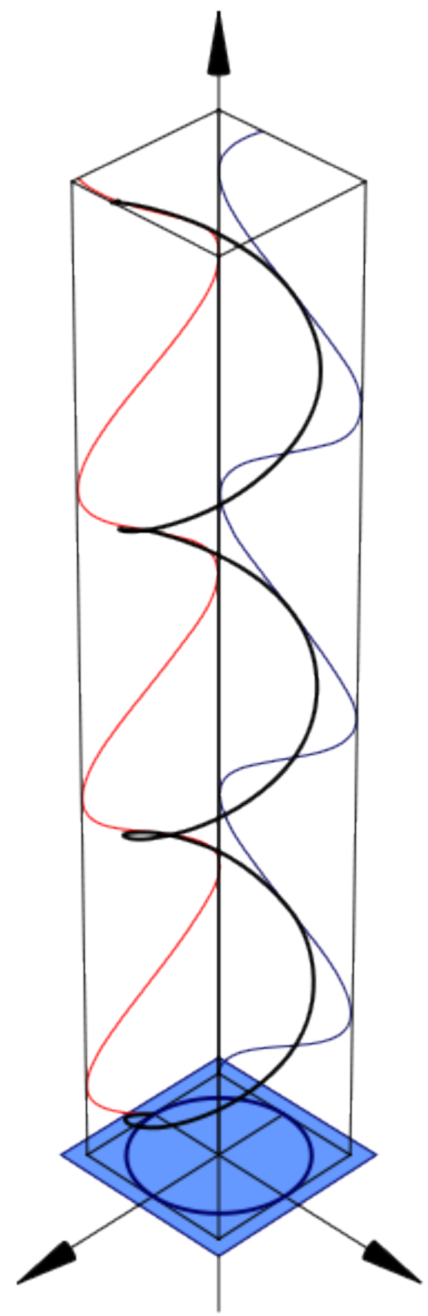
	Right-Handed Polarization	Left-Handed Polarization
Helicity	+ (positive)	- (negative)
Rotation at a fixed position	Counterclockwise	Clockwise
Screw at a fixed time	Left-Handed Screw	Right-Handed Screw
β	$-\pi/2 < \beta < 0$	$0 < \beta < \pi/2$
$\delta\phi\ (\equiv \phi_1 - \phi_2)$	$-\pi/2 < \delta\phi < 0$	$0 < \delta\phi < \pi/2$
Stokes V	V > 0	V < 0

Figures from Wikipedia



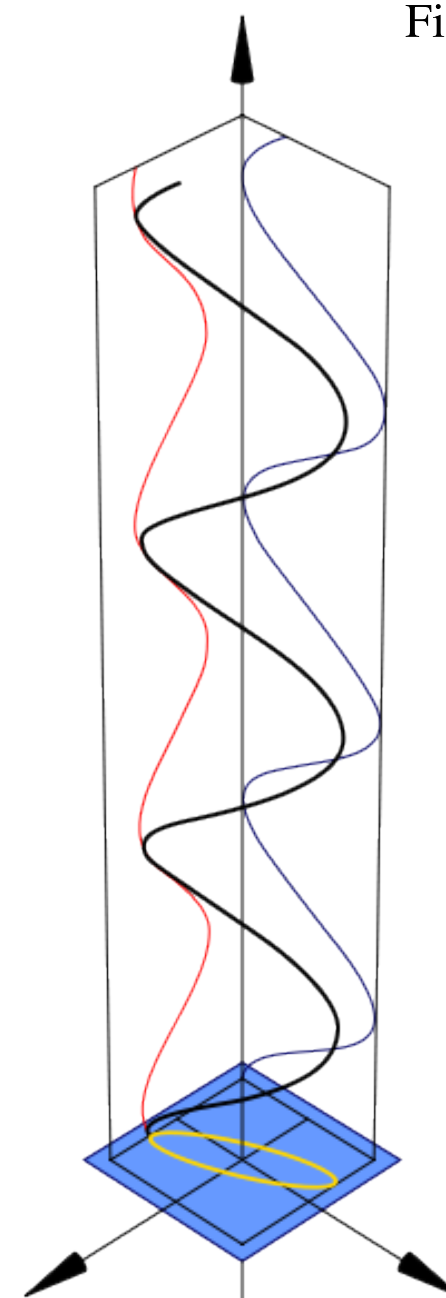
Linear

$$|\phi_1 - \phi_2| = 0$$
$$|\beta| = 0, \pi/2$$
$$\mathcal{E}_1/\mathcal{E}_2 = \text{const.}$$



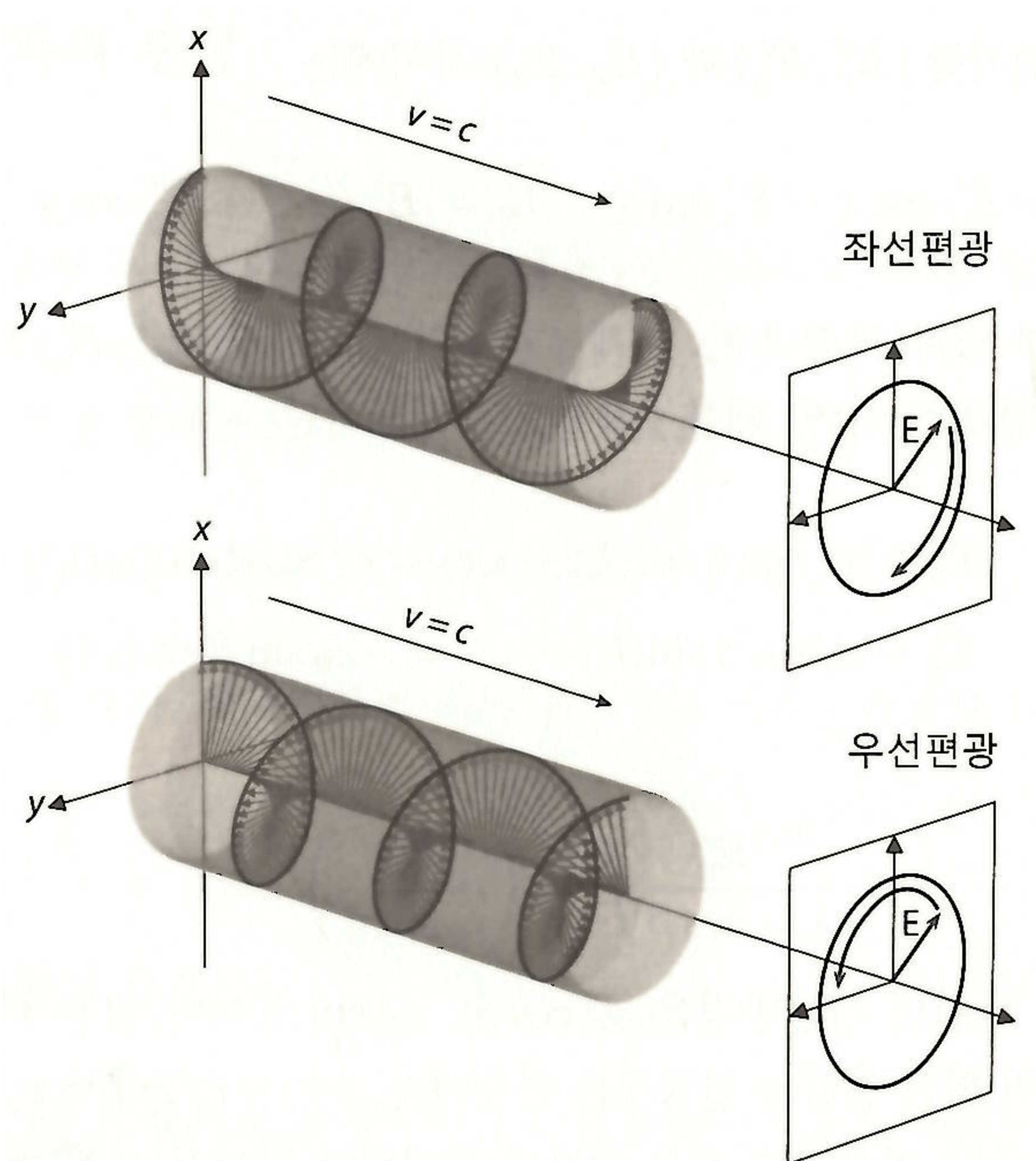
Circular

$$|\phi_1 - \phi_2| = \pi/2$$
$$|\beta| = \pi/4$$
$$|\mathcal{E}_1/\mathcal{E}_2| = 1$$



Elliptical

$$|\phi_1 - \phi_2| \neq 0, \pi/2$$
$$|\beta| \neq 0, \pi/4, \pi/4$$
$$\mathcal{E}_1/\mathcal{E}_2 \neq \pm 1$$



IEEE (1969) standard
IAU (1974) recommandation

	RHP	LHP
Helicity	+ (positive)	- (negative)
Rotation at a fixed position	Counterclockwise	Clockwise
Screw at a fixed time	Left-Handed Screw	Right-Handed Screw
β	$-\pi/2 < \beta < 0$	$0 < \beta < \pi/2$
$\delta\phi \ (\equiv \phi_1 - \phi_2)$	$-\pi/2 < \delta\phi < 0$	$0 < \delta\phi < \pi/2$
Stokes V	V > 0	V < 0

from 천체물리학 (구본철, 김웅태)

Stokes Parameters (for monochromatic waves)

- A convenient way to solve these equations is by means of the **Stokes parameters for monochromatic waves**.

$$\begin{aligned} I &\equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2 \\ Q &\equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \\ U &\equiv 2\mathcal{E}_1\mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi \\ V &\equiv -2\mathcal{E}_1\mathcal{E}_2 \sin(\phi_1 - \phi_2) = -\mathcal{E}_0^2 \sin 2\beta \end{aligned} \longrightarrow \quad I^2 = Q^2 + U^2 + V^2$$

for a monochromatic wave (pure polarization)

Then, we have

$$\mathcal{E}_0 = \sqrt{I}, \;\; \sin 2\phi = -\frac{V}{I}, \;\; \tan 2\chi = \frac{U}{Q}$$

Pure elliptical polarization is determined sole by three parameters $(\mathcal{E}_0, \; \beta, \; \chi)$.

- Meaning of the Stokes parameters:

I : total energy flux or intensity

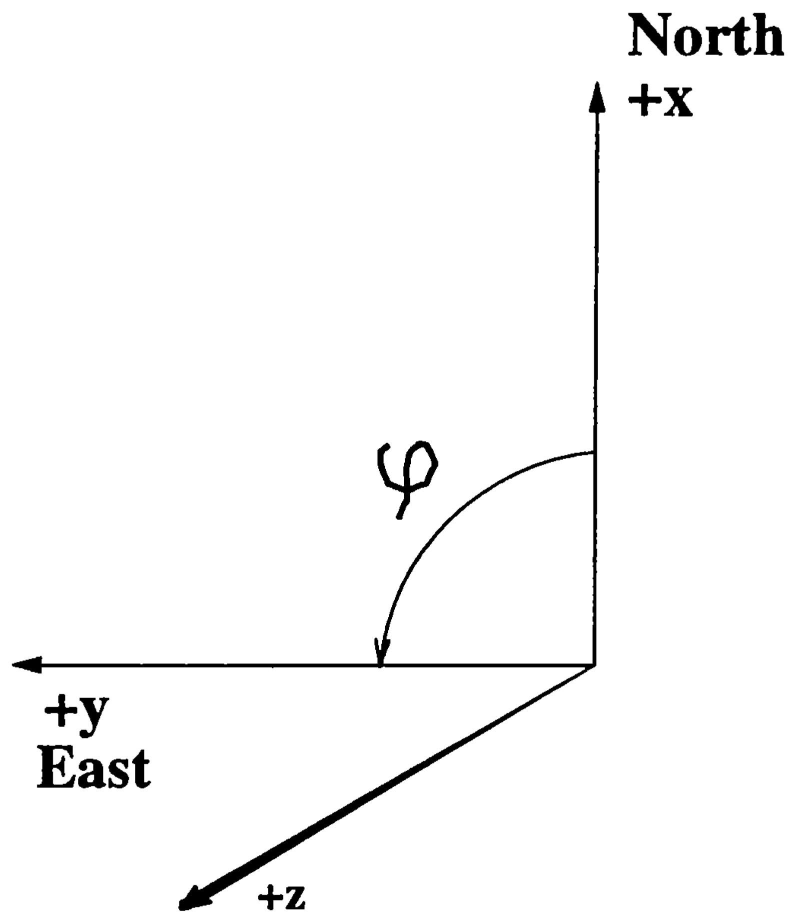
V : circularity parameter ($V > 0$: right-handed, $V < 0$: left-handed)

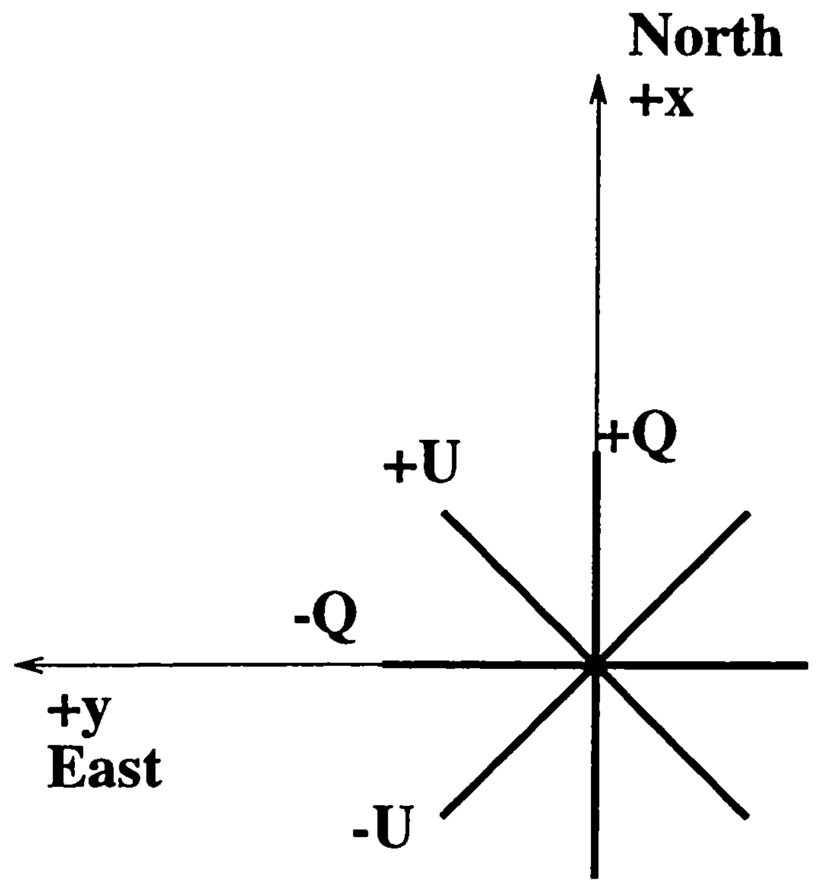
Q, U : orientation of the ellipse (or line) relative to the *x*-axis

$Q \times U \neq 0, V = 0$	: linear polarization
$Q = U = 0, V \neq 0$	: circular polarization
$Q \times U \neq 0, V \neq 0$	: elliptical polarization

The IAU definition of coordinate system

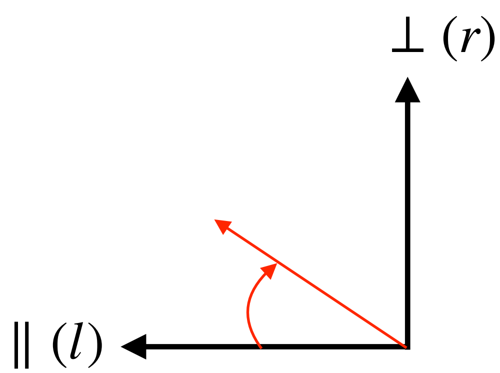
from Hamaker & Bregman (1996, A&AS)





Differences in Definition of Stokes vector

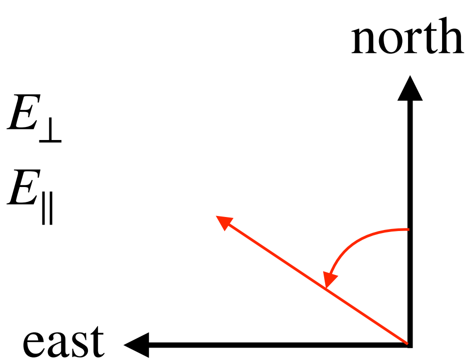
- Bohren & Huffman (Absorption and Scattering of Light by Small Particles)
- Chandrasekhar (Radiative Transfer)
- IAU recommendation
- IEEE standard



$$
\begin{aligned}
I_{\mathrm{BH}} &= E_{\parallel}E_{\parallel}^{*} + E_{\perp}E_{\perp}^{*} \\
Q_{\mathrm{BH}} &= E_{\parallel}E_{\parallel}^{*} - E_{\perp}E_{\perp}^{*} \\
U_{\mathrm{BH}} &= E_{\parallel}E_{\perp}^{*} + E_{\perp}E_{\parallel}^{*} \\
V_{\mathrm{BH}} &= i\left(E_{\parallel}E_{\perp}^{*} - E_{\perp}E_{\parallel}^{*}\right)
\end{aligned}
$$

$$V_{\mathrm{C}} = -i\left(E_{\parallel}E_{\perp}^{*} - E_{\perp}E_{\parallel}^{*}\right)$$

$$
\begin{aligned}
E_n &= E_{\perp} \\
E_e &= E_{\parallel}
\end{aligned}
$$



$$
\begin{aligned}
I_{\mathrm{IAU}} &= E_nE_n^{*} + E_eE_e^{*} \\
Q_{\mathrm{IAU}} &= E_nE_n^{*} - E_eE_e^{*} \\
U_{\mathrm{IAU}} &= E_nE_e^{*} + E_eE_n^{*} \\
V_{\mathrm{IAU}} &= i\left(E_nE_e^{*} - E_eE_n^{*}\right)
\end{aligned}
$$

$$\therefore \begin{pmatrix} I_{\mathrm{IAU}} \\ Q_{\mathrm{IAU}} \\ U_{\mathrm{IAU}} \\ V_{\mathrm{IAU}} \end{pmatrix} = \begin{pmatrix} I_{\mathrm{BH}} \\ -Q_{\mathrm{BH}} \\ U_{\mathrm{BH}} \\ -V_{\mathrm{BH}} \end{pmatrix} = \begin{pmatrix} I_{\mathrm{C}} \\ -Q_{\mathrm{C}} \\ U_{\mathrm{C}} \\ V_{\mathrm{C}} \end{pmatrix}$$

Conventions adopted by various authors

Peest et al. (2017, A&A, 601, A92) + α
(Typo: +/-U should read as +/-Q)

	+Q	-Q
+V	IAU (1974) Martin (1974) Tsang et al. (1985) Trippe (2014)	Chandrasekhar (1950) van de Hulst (1957) Hovenier & van der Mee (1983) Fischer et al. (1994) Code & Whitney (1995) Mishchenko et al. (1999) Gordon et al. (2001) Lucas (2003) Gorski et al. (2005)
-V	Shurcliff (1962) Bianchi et al. (1996)	Bohren & Huffman (1998) Rybicki & Lightman (1979) Mishchenko et al. (2002)

Stokes Parameters (for quasi-monochromatic waves)

- In general, EM waves vary over time and with wavenumber. Clearly, then, the practical measurement of EM waves involves taking a time average over a time interval.
- Consider EM wave with **slowly varying** amplitudes and phases:

$$E_1(t) = \mathcal{E}_1(t)e^{i\phi_1(t)} \qquad E_2(t) = \mathcal{E}_2(t)e^{i\phi_2(t)}$$

- How slow is slow? **Quasi-monochromatic wave**:

  Assumption: over a time interval $\Delta t > \Delta t_c \equiv 1/\omega$, the amplitudes and phases do not change significantly. By the uncertainty relation, its frequency spread $\Delta\omega$ about the value ω can be estimated as $\Delta\omega/\omega \approx \Delta t_c/\Delta t < 1$.

  For this reason, the wave slowly varying over a time interval $\Delta t > \Delta t_c = 1/\omega$ is called **quasi-monochromatic**, and the time Δt_c is called the **coherence time**.

- The **Stokes parameters for quasi-monochromatic waves** are defined by the following average over time, to be consistent with the definition for monochromatic waves:

$$\boxed{\begin{aligned}
I &\equiv \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle = \left\langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \right\rangle \\
Q &\equiv \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle = \left\langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \right\rangle \\
U &\equiv \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle = 2 \left\langle \mathcal{E}_1 \mathcal{E}_2 \cos\left(\phi_1 - \phi_2\right) \right\rangle \\
V &\equiv i \left(\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle \right) = -2 \left\langle \mathcal{E}_1 \mathcal{E}_2 \sin\left(\phi_1 - \phi_2\right) \right\rangle
\end{aligned}}$$

- With the Schwartz inequality $\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle \geq \langle E_1 E_2^* \rangle \langle E_2 E_1^* \rangle$

  we can easily verify that

$$\boxed{I^2 \geq Q^2 + U^2 + V^2}$$

  The equality holds only for a completely polarized wave.

- Most sources of EM radiation a large number of atoms or molecules that emit light. The orientation of the electric fields produced by these emitters may not be correlated, in which case the light is said to be **unpolarized**. For completely **unpolarized** wave, where the phase difference $\phi_1 - \phi_2$ between E_1 and E_2 maintain no permanent relation and where there is no preferred orientation in the *x*-*y* plane, so that $\langle \mathcal{E}_1^2 \rangle = \langle \mathcal{E}_2^2 \rangle$.

$$Q = U = V = 0$$

Proof of the inequality:
Homework:
(1) Derive the Schwartz inequality.
(2) Show that $I^2 \geq Q^2 + U^2 + V^2$

Superposition of independent waves

- Radiation will generally originate from a variety of regions different polarizations and different wave phases. Consider therefore a beam consisting of a mixture of many independent waves:

$$E_1 = \sum_k E_1^{(k)} \qquad E_2 = \sum_k E_2^{(k)} \qquad \text{where } k = 1, 2, 3, \cdots .$$

$$\left\langle E_i E_j^* \right\rangle = \sum_k \sum_l \left\langle E_i^{(k)} E_j^{(l)*} \right\rangle = \sum_k \left\langle E_i^{(k)} E_j^{(k)*} \right\rangle \qquad (i, j = 1 \text{ or } 2)$$

Because the relative phases are random, only term $k = l$ survive the averaging. Therefore, the **Stokes parameters have additive properties**:

$$\boxed{I = \sum_k I^{(k)}, \quad Q = \sum_k Q^{(k)}, \quad U = \sum_k U^{(k)}, \quad V = \sum_k V^{(k)}}$$

- By the superposition principle, an arbitrary wave can be decomposed of a completely unpolarized wave and a completely polarized wave.

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I - \sqrt{Q^2 + U^2 + V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{Q^2 + U^2 + V^2} \\ Q \\ U \\ V \end{pmatrix}$$

- Proof of the inequality: $I^2 = (I_{\rm pol} + I_{\rm unpol})^2 \geq I_{\rm pol}^2 = Q^2 + U^2 + V^2$

- **Degree of polarization** for a partially polarized wave = ratio of the intensity of the polarized part to the total intensity

$$\boxed{\Pi \equiv \frac{I_{\rm pol}}{I} = \frac{\sqrt{Q^2+U^2+V^2}}{I}}$$

- In the case of partial **linear polarization (*V* = 0)**, the measurement consists of rotating a linear polarization filter until the maximum values of intensity are found. The maximum value will occur when the filter is aligned with the plane of polarization, and the minimum value will occur along in the direction perpendicular to it.

  Total value of the unpolarized intensity is shared equally between any two perpendicular directions. Therefore,

$$I_{\rm max} = \frac{1}{2}I_{\rm unpol} + I_{\rm pol} \quad \text{where} \quad I_{\rm unpol} = I - \sqrt{Q^2+U^2}$$

$$I_{\rm min} = \frac{1}{2}I_{\rm unpol} \qquad\qquad I_{\rm pol} = \sqrt{Q^2+U^2}$$

$$\boxed{\therefore \quad \Pi_{\rm linear} = \frac{I_{\rm max}-I_{\rm min}}{I_{\rm max}+I_{\rm min}}}$$

  This equation will underestimate the true degree of polarization if circular or elliptical polarization is present.

$$\begin{aligned} I_{\rm max} &= \tfrac{1}{2}(I_{\rm unpol}+I_{\rm cir}) + I_{\rm lin} \\ I_{\rm min} &= \tfrac{1}{2}(I_{\rm unpol}+I_{\rm cir}) \end{aligned} \quad \rightarrow \quad \frac{I_{\rm max}-I_{\rm min}}{I_{\rm max}+I_{\rm min}} = \frac{I_{\rm lin}}{I} < \frac{I_{\rm pol}}{I} = \frac{I_{\rm lin}+I_{\rm cir}}{I_{\rm unpol}+I_{\rm lin}+I_{\rm cir}}$$