Astrophysics

Lecture 04
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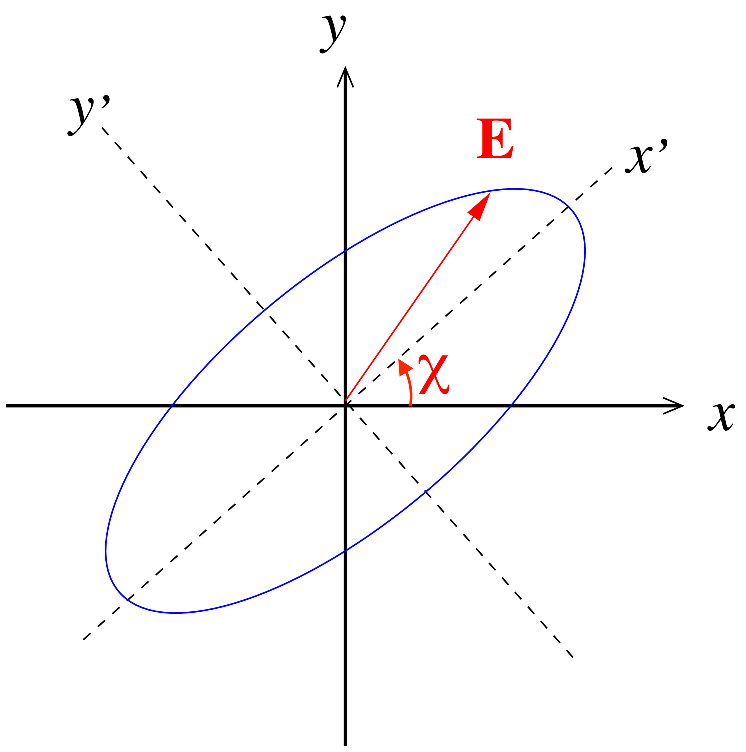
Polarization

- Let us consider a plane EM wave propagating in the $+z$ direction, and examine the electric vector at $z = 0$.

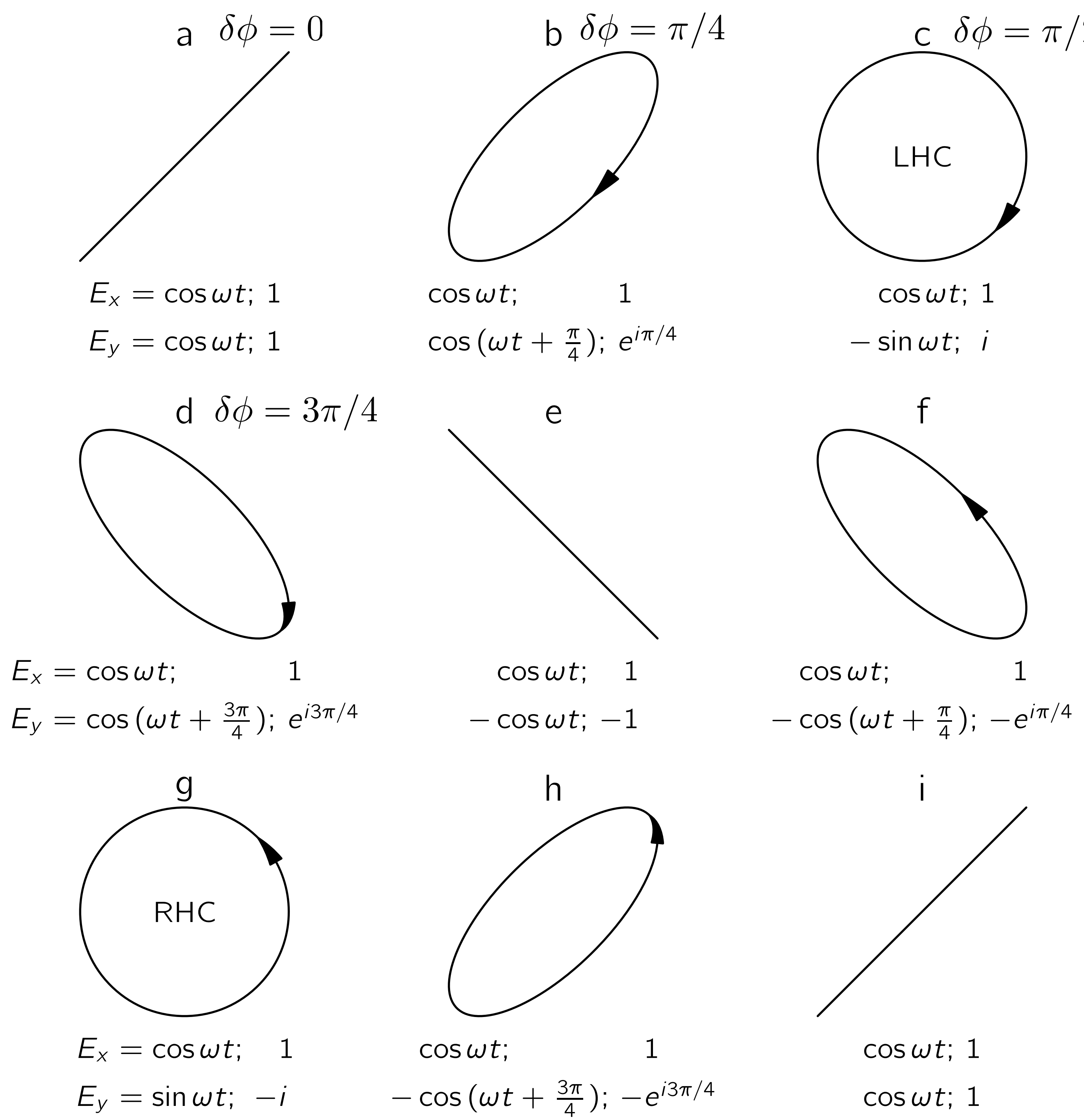
  The real part of **E** is



$$\mathbf{E} = \hat{\mathbf{x}}\mathcal{E}_1 \cos(\omega t - \phi_1) + \hat{\mathbf{y}}\mathcal{E}_2 \cos(\omega t - \phi_2)$$

Here, $\mathcal{E}_1, \ \mathcal{E}_2, \ \phi_1, \ \phi_2$ are real.

- If the x-vibration and the y-vibration are in the same phase ($\phi_1 = \phi_2$), the electric field vector oscillates on a straight line. In this case, light is linearly polarized.
- When the x-vibration and the y-vibration are not in phase, the electric field vector moves around in an ellipse.



$$E_x = \cos(\omega t)$$
$$E_y = \cos(\omega t + \delta\phi)$$

Fig. 33-2 in Vol 1 [The Feynman lectures on physics]

Stokes Parameters (for monochromatic waves)

- A convenient way to represent these quantities is by means of the **Stokes parameters for monochromatic waves**.

$$
\boxed{\begin{aligned}
I &\equiv E_1E_1^* + E_2E_2^* \\
Q &\equiv E_1E_1^* - E_2E_2^* \\
U &\equiv E_1E_2^* + E_2E_1^* \\
V &\equiv i\left(E_1E_2^* - E_2E_1^*\right)
\end{aligned}}
\begin{aligned}
&= \mathcal{E}_1^2 + \mathcal{E}_2^2 \\
&= \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \\
&= 2\mathcal{E}_1\mathcal{E}_2 \cos\left(\phi_1 - \phi_2\right) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi \\
&= -2\mathcal{E}_1\mathcal{E}_2 \sin\left(\phi_1 - \phi_2\right) = -\mathcal{E}_0^2 \sin 2\beta
\end{aligned}
\longrightarrow \quad I^2 = Q^2 + U^2 + V^2
$$

for a monochromatic wave (pure polarization)

Then, we have

$$
\boxed{\mathcal{E}_0 = \sqrt{I}, \quad \sin 2\beta = -\frac{V}{I}, \quad \tan 2\chi = \frac{U}{Q}}
$$

Pure elliptical polarization is determined solely by three parameters $(\mathcal{E}_0,\ \beta,\ \chi)$.

- Meaning of the Stokes parameters:

I : total energy flux or intensity

V : circularity parameter ($V > 0$: right-handed, $V < 0$: left-handed)

Q, U : orientation of the ellipse (or line) relative to the x-axis

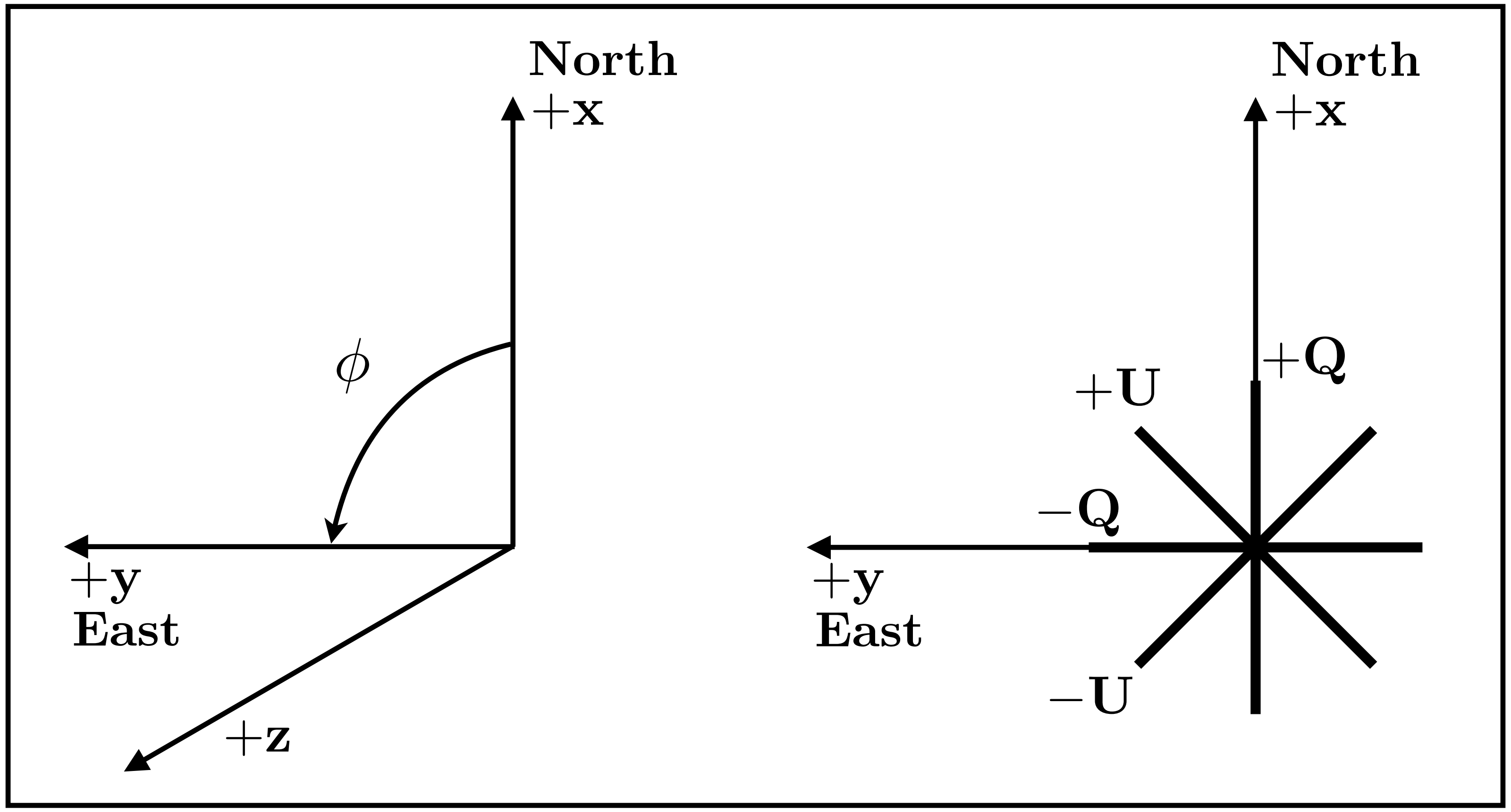
$Q \times U \neq 0, V = 0$: linear polarization

$Q = U = 0, V \neq 0$: circular polarization

$Q \times U \neq 0, V \neq 0$: elliptical polarization

The IAU definition of coordinate system

Hamaker & Bregman (1996, A&AS)



$$I = I_{0^\circ} + I_{90^\circ}$$
$$Q = I_{0^\circ} - I_{90^\circ}$$
$$U = I_{45^\circ} - I_{135^\circ}$$

Note that the phase is assumed to be $e^{-i\omega t}$.

V should be multiplied by -1 if the phase of $e^{i\omega t}$ is adopted.

$$\mathbf{E} = \hat{\mathbf{x}} E_1 e^{-i\omega t} + \hat{\mathbf{y}} E_2 e^{-i\omega t}$$

rotation of axis

$$\begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \end{pmatrix} = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix}$$

$$\hat{\mathbf{x}}' = \frac{1}{\sqrt{2}} (\hat{\mathbf{x}} + \hat{\mathbf{y}})$$

$$\hat{\mathbf{y}}' = \frac{1}{\sqrt{2}} (-\hat{\mathbf{x}} + \hat{\mathbf{y}})$$

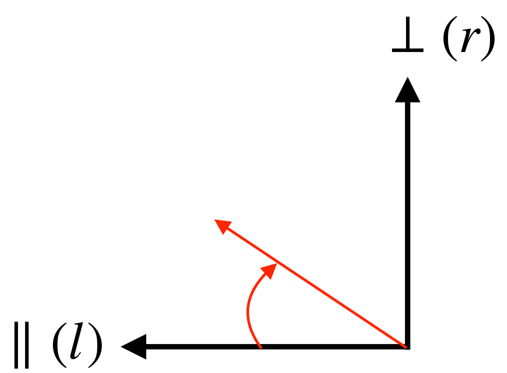
when $\chi = \pi/4$

$$I_{0^\circ} = |E_1|^2$$
$$I_{90^\circ} = |E_2|^2$$
$$I_{45^\circ} = |\mathbf{E} \cdot \hat{\mathbf{x}}'|^2$$
$$I_{135^\circ} = |\mathbf{E} \cdot \hat{\mathbf{y}}'|^2$$

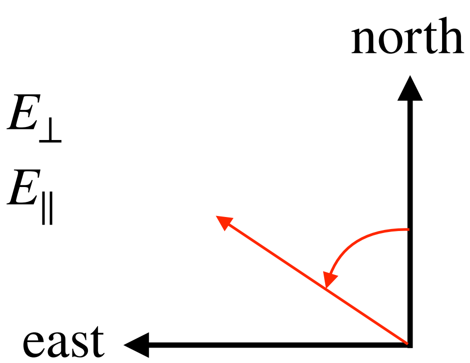
$$\begin{aligned} I_{45^\circ} &= \frac{1}{2} |E_1 e^{-i\omega t} + E_2 e^{-i\omega t}|^2 \\ &= \frac{1}{2} \left(|E_1|^2 + |E_2|^2 + E_1 E_2^* + E_1^* E_2 \right) \\ I_{135^\circ} &= \frac{1}{2} |-E_1 e^{-i\omega t} + E_2 e^{-i\omega t}|^2 \\ &= \frac{1}{2} \left(|E_1|^2 + |E_2|^2 - E_1 E_2^* - E_1^* E_2 \right) \end{aligned}$$

Differences in Definitions of Stokes vector

- BH: Bohren & Huffman (Absorption and Scattering of Light by Small Particles)
- C: Chandrasekhar (Radiative Transfer)
- IAU recommendation
- IEEE standard



$$E_n = E_\perp$$
$$E_e = E_\parallel$$



$$\boxed{\begin{aligned} I_{\mathrm{BH}} &= E_\parallel E_\parallel^* + E_\perp E_\perp^* \\ Q_{\mathrm{BH}} &= E_\parallel E_\parallel^* - E_\perp E_\perp^* \\ U_{\mathrm{BH}} &= E_\parallel E_\perp^* + E_\perp E_\parallel^* \\ V_{\mathrm{BH}} &= i\left(E_\parallel E_\perp^* - E_\perp E_\parallel^*\right) \\ V_{\mathrm{C}} &= -i\left(E_\parallel E_\perp^* - E_\perp E_\parallel^*\right) \end{aligned}}$$

$$\boxed{\begin{aligned} I_{\mathrm{IAU}} &= E_n E_n^* + E_e E_e^* \\ Q_{\mathrm{IAU}} &= E_n E_n^* - E_e E_e^* \\ U_{\mathrm{IAU}} &= E_n E_e^* + E_e E_n^* \\ V_{\mathrm{IAU}} &= i\left(E_n E_e^* - E_e E_n^*\right) \end{aligned}}$$

$$\therefore \begin{pmatrix} I_{\mathrm{IAU}} \\ Q_{\mathrm{IAU}} \\ U_{\mathrm{IAU}} \\ V_{\mathrm{IAU}} \end{pmatrix} = \begin{pmatrix} I_{\mathrm{BH}} \\ -Q_{\mathrm{BH}} \\ U_{\mathrm{BH}} \\ -V_{\mathrm{BH}} \end{pmatrix} = \begin{pmatrix} I_{\mathrm{C}} \\ -Q_{\mathrm{C}} \\ U_{\mathrm{C}} \\ V_{\mathrm{C}} \end{pmatrix}$$

Note that I and U remain unchanged, regardless of the definitions.

Conventions adopted by various authors

Peest et al. (2017, A&A, 601, A92) + α
(Typo: +/-U should read as +/-Q)

	+Q	-Q
+V	IAU (1974) Martin (1974) Tsang et al. (1985) Trippe (2014)	Chandrasekhar (1950) van de Hulst (1957) Hovenier & van der Mee (1983) Fischer et al. (1994) Code & Whitney (1995) Mishchenko et al. (1999) Gordon et al. (2001) Lucas (2003) Gorski et al. (2005)
-V	Shurcliff (1962) Bianchi et al. (1996)	Bohren & Huffman (1998) Rybicki & Lightman (1979) Mishchenko et al. (2002)

Stokes Parameters (for quasi-monochromatic waves)

- In practice we never see a single monochromatic component but rather a superposition of many components (frequencies), each with its own polarization. In general, EM waves vary over time and with wavenumber. Let's consider EM wave with **slowly varying** amplitudes and phases:

$$E_1(t) = \mathcal{E}_1(t)e^{i\phi_1(t)} \qquad E_2(t) = \mathcal{E}_2(t)e^{i\phi_2(t)}$$

- How slow is slow? **Quasi-monochromatic wave**:

  Assumption: over a time interval $\Delta t > \Delta t_c \equiv 1/\omega$, the amplitudes and phases do not change significantly. By the uncertainty relation, its frequency spread $\Delta\omega$ about the central value ω can be estimated as $\Delta\omega/\omega \approx \Delta t_c/\Delta t < 1$.

  For this reason, the wave slowly varying over a time interval $\Delta t > \Delta t_c = 1/\omega$ is called **quasi-monochromatic**, and the time Δt_c is called the **coherence time**.

- The **Stokes parameters for quasi-monochromatic waves** are defined by **the average over time**, to be consistent with the definition for monochromatic waves:

$$\boxed{\begin{aligned} I &\equiv \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle &&= \left\langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \right\rangle \\ Q &\equiv \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle &&= \left\langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \right\rangle \\ U &\equiv \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle &&= 2 \left\langle \mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) \right\rangle \\ V &\equiv i \left(\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle \right) &&= -2 \left\langle \mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) \right\rangle \end{aligned}}$$

Here, $\langle\rangle$ denotes the time average.

- With the Schwartz inequality $\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle \geq \langle E_1 E_2^* \rangle \langle E_2 E_1^* \rangle$

  we can easily verify that

$$\boxed{I^2 \geq Q^2 + U^2 + V^2}$$

  The equality holds only for a completely polarized wave.

- Most sources of EM radiation contain a large number of atoms or molecules that emit light. The orientation of the electric fields produced by these emitters may not be correlated, in which case the light is said to be **unpolarized**. For completely **unpolarized** wave, where the phase difference $\phi_1 - \phi_2$ between E_1 and E_2 maintain no permanent relation and where there is no preferred orientation in the *x*-*y* plane, so that $\langle \mathcal{E}_1^2 \rangle = \langle \mathcal{E}_2^2 \rangle$.

$$Q = U = V = 0$$

Superposition of independent waves

- Radiation will generally originate from a variety of regions different polarizations and different wave phases. Consider therefore a beam consisting of a mixture of many independent waves:

$$E_1 = \sum_k E_1^{(k)} \qquad E_2 = \sum_k E_2^{(k)} \qquad \text{where } k = 1, 2, 3, \cdots .$$

$$\left\langle E_i E_j^* \right\rangle = \sum_k \sum_l \left\langle E_i^{(k)} E_j^{(l)*} \right\rangle = \sum_k \left\langle E_i^{(k)} E_j^{(k)*} \right\rangle \qquad (i, j = 1 \text{ or } 2)$$

Because the relative phases are random, only the terms $k = l$ survive the averaging. Therefore, the **Stokes parameters have additive properties**:

$$\boxed{I = \sum_k I^{(k)}, \quad Q = \sum_k Q^{(k)}, \quad U = \sum_k U^{(k)}, \quad V = \sum_k V^{(k)}}$$

- By the superposition principle, an arbitrary wave can be decomposed of a completely unpolarized wave and a completely polarized wave.

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I - \sqrt{Q^2 + U^2 + V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{Q^2 + U^2 + V^2} \\ Q \\ U \\ V \end{pmatrix}$$

- Proof of the inequality: $\boxed{I^2 = (I_{\text{pol}} + I_{\text{unpol}})^2 \geq I_{\text{pol}}^2 = Q^2 + U^2 + V^2}$

Degree of Polarization

- **Degree of polarization** for a partially polarized wave = ratio of the intensity of the polarized part to the total intensity

$$\boxed{\Pi \equiv \frac{I_{\rm pol}}{I} = \frac{\sqrt{Q^2+U^2+V^2}}{I}}$$

- In the case of partial **linear polarization ($V = 0$)**, the measurement consists of rotating a linear polarization filter until the maximum values of intensity are found. The maximum value will occur when the filter is aligned with the plane of polarization, and the minimum value will occur along in the direction perpendicular to it.

  Total value of the unpolarized intensity is shared equally between any two perpendicular directions. Therefore,

$$I_{\rm max} = \frac{1}{2} I_{\rm unpol} + I_{\rm pol} \qquad \text{where} \quad I_{\rm unpol} = I - \sqrt{Q^2+U^2}$$

$$I_{\rm min} = \frac{1}{2} I_{\rm unpol} \qquad\qquad I_{\rm pol} = \sqrt{Q^2+U^2}$$

$$\boxed{\therefore \quad \Pi_{\rm linear} = \frac{I_{\rm max} - I_{\rm min}}{I_{\rm max} + I_{\rm min}}}$$

  This equation will underestimate the true degree of polarization if circular or elliptical polarization is present.

$$\begin{aligned} I_{\rm max} &= \tfrac{1}{2}(I_{\rm unpol} + I_{\rm cir}) + I_{\rm lin} \\ I_{\rm min} &= \tfrac{1}{2}(I_{\rm unpol} + I_{\rm cir}) \end{aligned} \quad \rightarrow \quad \frac{I_{\rm max} - I_{\rm min}}{I_{\rm max} + I_{\rm min}} = \frac{I_{\rm lin}}{I} < \frac{I_{\rm pol}}{I} = \frac{I_{\rm lin} + I_{\rm cir}}{I_{\rm unpol} + I_{\rm lin} + I_{\rm cir}}$$

Stokes Parameters + Unpolarized Light

- A convenient way to solve these equations is by means of the **Stokes parameters for monochromatic waves**.

$$\boxed{\begin{aligned} I &\equiv E_1 E_1^* + E_2 E_2^* \\ Q &\equiv E_1 E_1^* - E_2 E_2^* \\ U &\equiv E_1 E_2^* + E_2 E_1^* \\ V &\equiv i\left(E_1 E_2^* - E_2 E_1^*\right) \end{aligned}}$$

$$\longrightarrow I^2 = Q^2 + U^2 + V^2$$

for a monochromatic wave
(pure polarization)

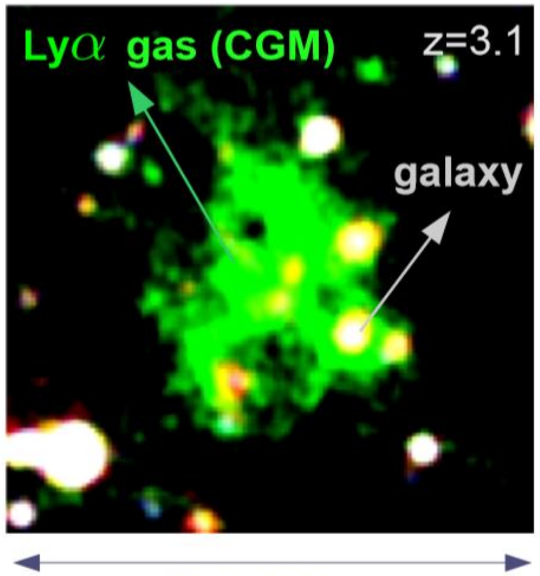
- **Unpolarized Light:**

  If the light is not absolutely monochromatic, or if the *x*- and *y*-phases are not kept perfectly together, so that the electric vector first vibrates in one direction, then in another, the polarization is constantly changing.

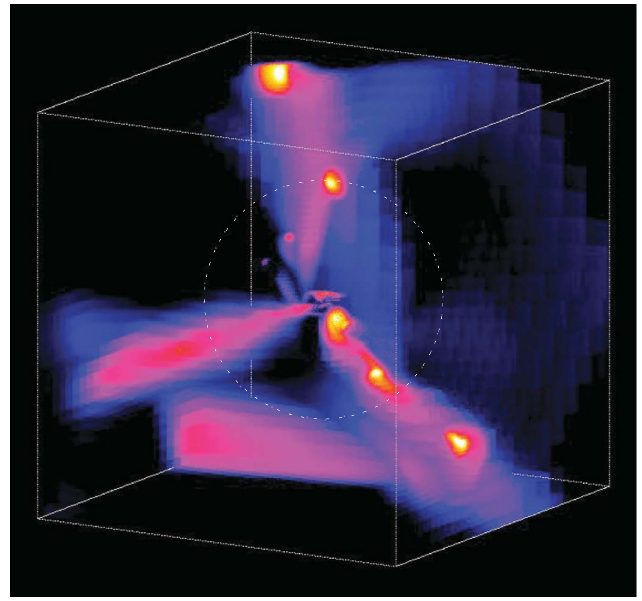
  Remember that one atom emits during 10^{-8} sec, and if one atom emits a certain polarization, and then another atom emits light with a different polarization, the polarizations will change every 10^{-8} sec. If the polarization change more rapidly than we can detect it, then all the light unpolarized, because all the effects of the polarization average out.

- Light is unpolarized if we are unable to find out whether the light is polarized or not.

Astrophysical Examples: Lyman Alpha Blobs

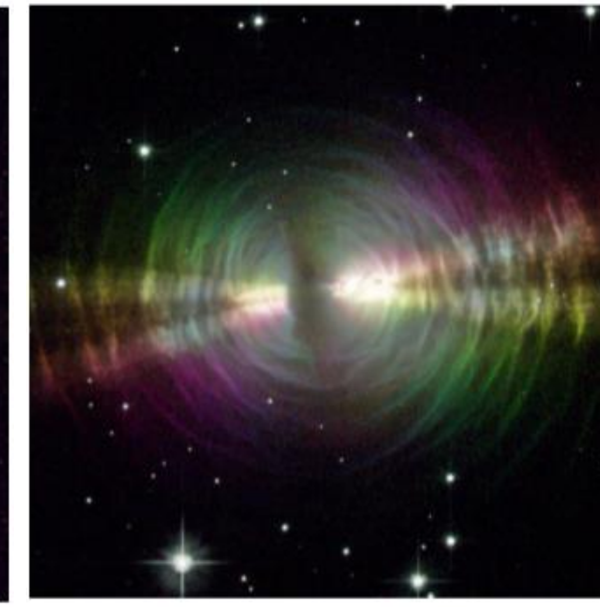


25″ ~ 200kpc
SSA22-LAB1
(Matsuda+04; Subaru)



cold gas stream
Dekel et al. (2009)

photo-ionization
by stars & AGN
(H II region in M33)



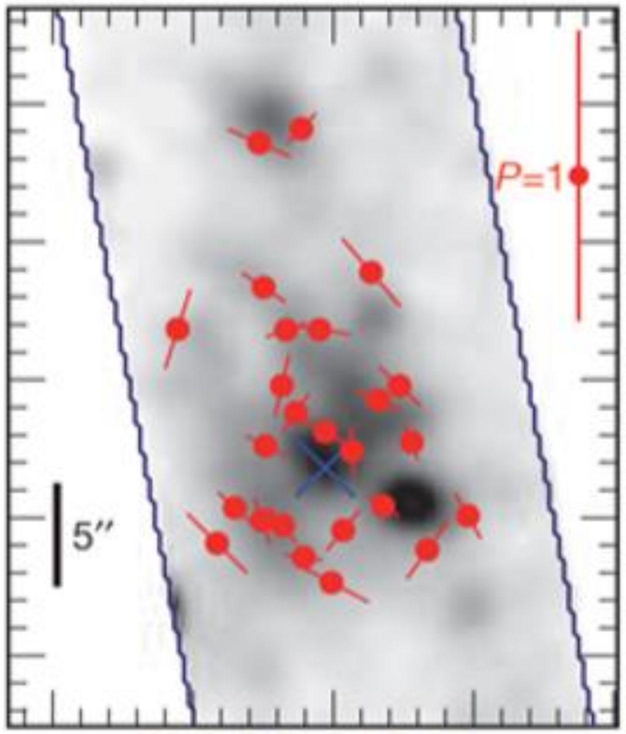
resonant scattering
(scattering by dust
in Egg Nebula)

LABs are one of the biggest objects in the Universe: gigantic clouds of hydrogen. They may evolve into the present-day groups and clusters of galaxies.

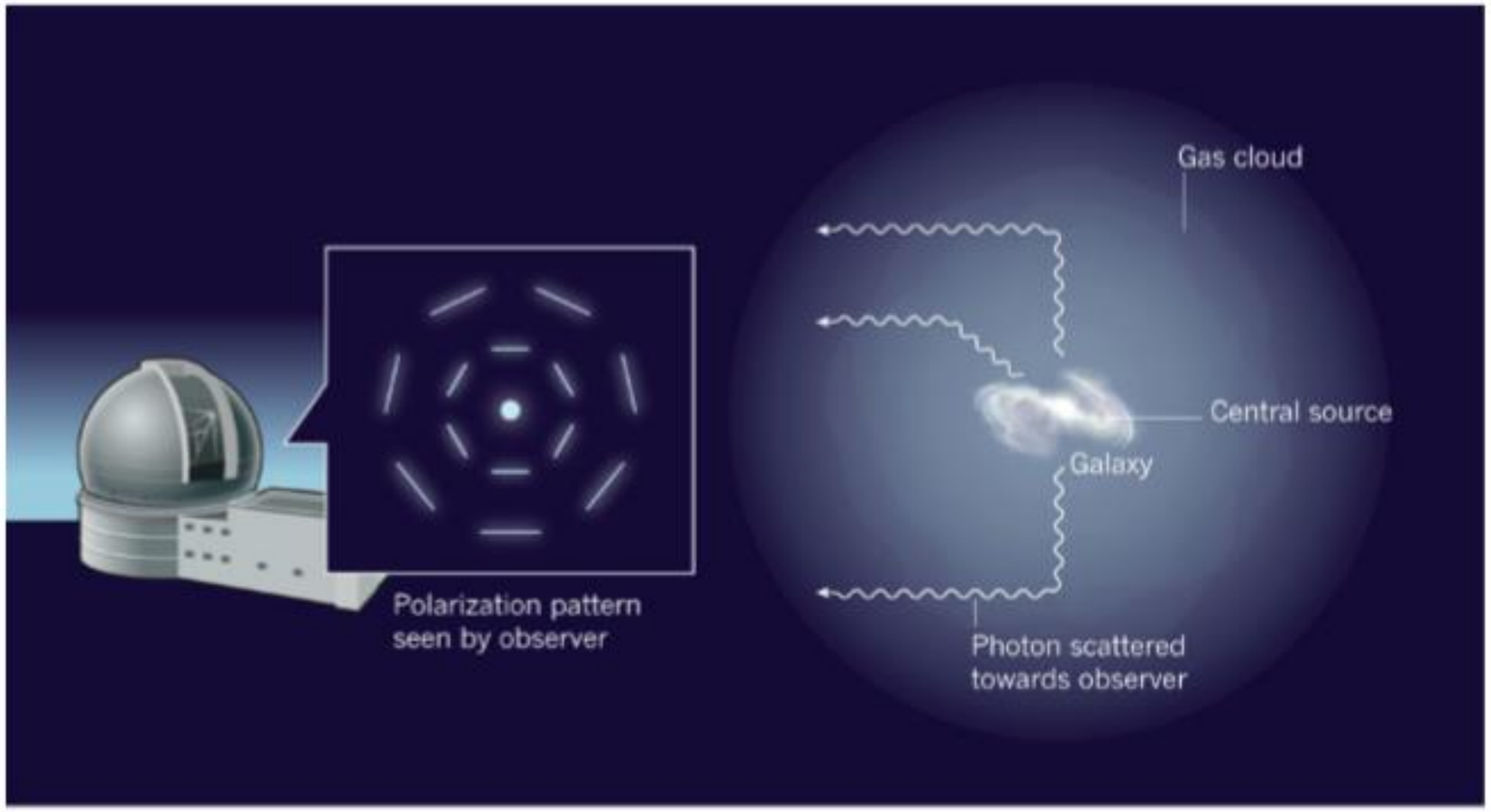
Possible Origins of the Ly-alpha Nebulosity

- **Cooling of the accreting gas** that is heated by galactic outflows during powerful starbursts or by the dissipation of gravitational energy as gas falls toward galaxies
- **Photoionization** by luminous AGNs, young stars, and/or the intergalactic UV background
- **Resonance scattering** of Lyα photons produced by star forming galaxies and/or AGNs hosted within the nebulae

Polarization in LABs



SSA22-LAB1
(Hayes et al. 2011)



(R. Bower 2010)

In-situ production of Lyα

➡ no or weak Lyα polarization

Resonant scattering: production of Lyα within a central source and scattering by neutral hydrogen

➡ concentric Lyα line polarization pattern

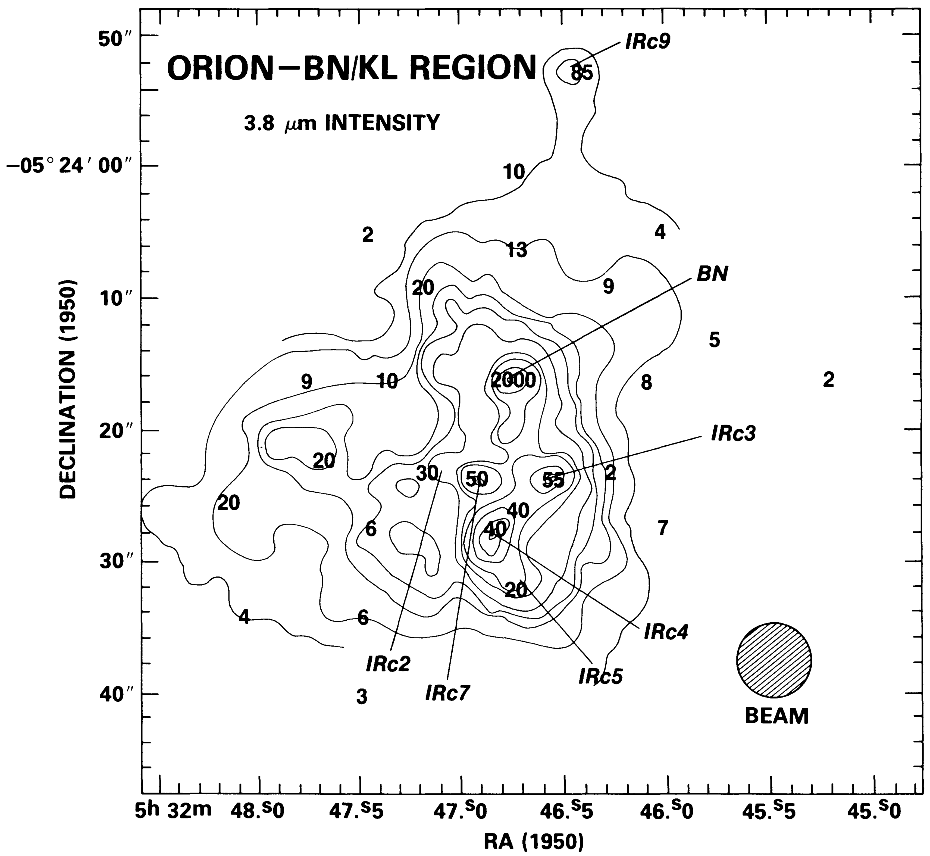
➡ polarization degree increases outwards

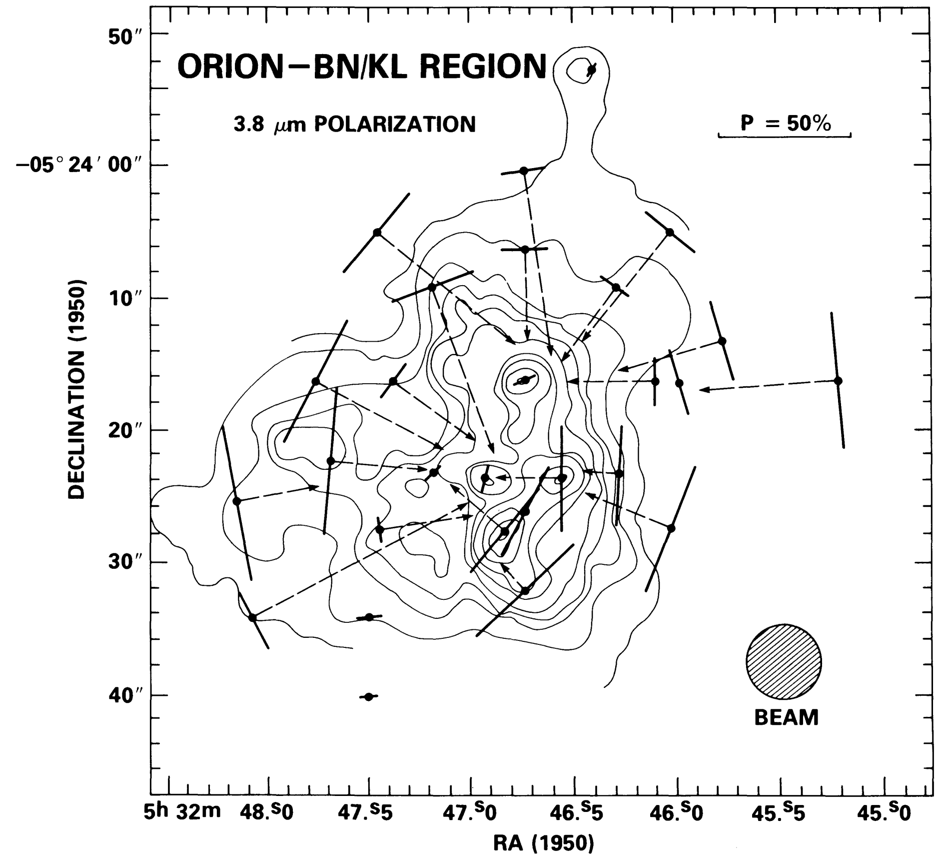
- The detection of polarized radiation is inconsistent with the in situ production of Lyα photons.

Astrophysical Applications of Polarization by Scattering

- Detection of a concentric pattern of polarization vectors in an extended region indicates that the light comes via scattering from a central point source.

Werner et al. (1983, ApJL, 265, L13)





- Left map shows the IR intensity map at 3.8 um of the Becklin-Neugebauer/Kleinmann-Low region of Orion. It is not easy to identify which bright spots correspond to locations of possible protostars.
- However, the polarization map singles out only two positions of intrinsic luminosity: IRc2 (now known to be an intense protostellar wind) and BN (suspected to be a relatively high-mass star)
- All the other bright spots (IRc3 through 7) correspond to IR reflection nebulae.

Radiation from Moving Charges 1

Mathematics: Note on the Dirac delta function.

$$\delta(\mathbf{r}-\mathbf{r}_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

$$\int f(x)\delta(x-x_0)dx = f(x_0) \ \text{ if } x_0 \text{ is not a function of } x.$$

$$\int f(x)\delta(g(x))dx = \int f(x)\delta(y)\frac{dy}{(dg/dx)} \qquad \leftarrow \qquad \begin{aligned} y &\equiv g(x') \\ dy &= (dg/dx')dx' \\ dx' &= \frac{dy}{(dg/dx')} \end{aligned}$$

$$= \sum_{x_j} \frac{f(x_j)}{dg/dx|_{x_j}}$$

where x_j are roots of the equation $y = g(x) = 0$

A single moving charge: Potentials

- Recall the retarded potentials:

$$\phi(\mathbf{r},t) = \int d^3\mathbf{r}' \int dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$
$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}' \int dt' \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

- Consider a particle of charge q that moves along a trajectory $\mathbf{r} = \mathbf{r}_0(t)$. Its velocity is then $\mathbf{u}(t) = \dot{\mathbf{r}}_0(t)$. The charge and current densities are given by

$$\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{r}_0(t)), \qquad \mathbf{j}(\mathbf{r},t) = q\mathbf{u}(t)\delta(\mathbf{r}-\mathbf{r}_0(t))$$

The δ-function has the property of localizing the charge and current. Let us calculate the retarded potentials due to this charge and current density. Using the property of the δ -function, the potentials become

$$\boxed{\begin{aligned}\phi(\mathbf{r},t) &= q\int \frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|}\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)\\ \mathbf{A}(\mathbf{r},t) &= \frac{q}{c}\int \frac{dt'\,\mathbf{u}(t')}{|\mathbf{r}-\mathbf{r}_0(t')|}\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)\end{aligned}}$$

This is now an integral over the single variable t'.

- We now introduce the notations:

$$\mathbf{R}(t') \equiv \mathbf{r} - \mathbf{r}_0(t') \quad \rightarrow \quad R(t') = |\mathbf{r} - \mathbf{r}_0(t')|$$

We then have

$$\phi(\mathbf{r}, t) = q \int \frac{dt'}{R(t')} \delta(t' - t + R(t')/c)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{c} \int \frac{\mathbf{u}(t')dt'}{R(t')} \delta(t' - t + R(t')/c)$$

These equation can be simplified further. Let us change variables:

(1) $$t'' = t' - t + R(t')/c \quad \rightarrow \quad dt'' = \left[1 + \frac{1}{c}\dot{R}(t')\right] dt'$$ (Here, t is a constant.)

(2)
$$R^2(t') = \mathbf{R}(t') \cdot \mathbf{R}(t')$$
$$2R(t')\dot{R}(t') = -2\mathbf{R}(t') \cdot \mathbf{u}(t') \qquad \leftarrow \dot{\mathbf{R}}(t') = -\mathbf{u}(t')$$
$$\dot{R}(t') = -\frac{\mathbf{R}(t')}{R(t')} \cdot \mathbf{u}(t')$$

(3)
$$\dot{R}(t') = -\mathbf{n}(t') \cdot \mathbf{u}(t')$$
$$dt'' = [1 - \mathbf{n}(t') \cdot \boldsymbol{\beta}(t')]\, dt'$$
where $\mathbf{n}(t') \equiv \dfrac{\mathbf{R}(t')}{R(t')}$ and $\boldsymbol{\beta} \equiv \dfrac{\mathbf{u}}{c}$

(4) $$dt'' = \kappa(t')dt'$$ where $\kappa(t') = 1 - \mathbf{n}(t') \cdot \boldsymbol{\beta}(t')$

A single moving charge: The Lienard-Wiechart Potential

Finally, we obtain

$$\phi(\mathbf{r},t) = q\int \frac{dt''}{\kappa(t')R(t')}\delta(t'')$$
$$\mathbf{A}(\mathbf{r},t) = q\int \frac{dt''\boldsymbol{\beta}(t')}{\kappa(t')R(t')}\delta(t'')$$

Now the integration over the δ-function can be performed by setting $t'' = 0$ or $t' = t_{\mathrm{ret}} \equiv t - R(t_{\mathrm{ret}})/c$.

$$\boxed{\begin{aligned}\phi(\mathbf{r},t) &= \frac{q}{\kappa(t_{\mathrm{ret}})R(t_{\mathrm{ret}})}\\ \mathbf{A}(\mathbf{r},t) &= \frac{q\boldsymbol{\beta}(t_{\mathrm{ret}})}{c\kappa(t_{\mathrm{ret}})R(t_{\mathrm{ret}})}\end{aligned}}$$

$Li\acute{e}nard - Wiechart\ potentials$
(리에나르-비헤르트)
(French-German)

These potentials are called the Lienard-Wiechart potentials.

$$\boxed{\begin{aligned}\phi(\mathbf{r},t) &= \frac{q}{\kappa(t_{\text{ret}})R(t_{\text{ret}})}\\ \mathbf{A}(\mathbf{r},t) &= \frac{q\boldsymbol{\beta}(t_{\text{ret}})}{c\kappa(t_{\text{ret}})R(t_{\text{ret}})}\end{aligned}}$$

$Liénard - Wiechart\ potentials$
(리에나르-비헤르트)
(French-German)

These potentials differ from those of static electromagnetic theory in two ways:

- **Beaming effect**:
  - First, there is the factor $\kappa(t_{\text{ret}}) = 1 - \mathbf{n}(t_{\text{ret}}) \cdot \boldsymbol{\beta}(t_{\text{ret}})$.
  - This factor becomes very important at velocities close to the speed of light, where it tends to concentrate the potentials into a narrow cone about the particle velocity. It is related to the beaming effect found in the Lorentz transformation of photon direction or propagation.
- **Retardation makes it possible for a particle to radiate**:
  - The second difference is that the quantities are all to be evaluated at the retarded time t_{ret}. The major consequence of retardation is that it makes it possible for a particle to radiate.
  - The potentials roughly decrease as $1/r$ so that differentiation to find the fields would give a $1/r^2$ decrease if this differentiation acted solely on the $1/r$ factor.
  - In addition to this, the implicit dependence of the retarded time on a position gives $1/r$ behavior in the fields. We will see that this allows radiation energy to flow to infinite distances.

A single moving charge: Electromagnetic Fields

- The differentiation of the potentials gives the electromagnetic field. The calculation is straightforward but lengthy (see 14.1 of Classical Electrodynamics, Jackson).
- Note that **E** and **B** are always perpendicular, and $\mathbf{E} = \mathbf{B}$. However, **E** is not in general perpendicular to **n**.

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

$\longrightarrow$

velocity field — acceleration field

$$\mathbf{E}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]_{\mathrm{ret}} + \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\right]_{\mathrm{ret}}$$
$$\mathbf{B}(\mathbf{r},t) = [\mathbf{n}\times\mathbf{E}(\mathbf{r},t)]_{\mathrm{ret}}$$

- The electric field appears as composed of two terms:

  (1) The first, the **velocity field**, falls off as $1/R^2$ and is just the generalization of the Coulomb law to moving particles.

  - For $u \ll c$ this becomes precisely Coulomb's law.
  - When the particle moves with constant velocity it is only this term that contributes to the fields.

  (2) The second term, the **acceleration field**, falls off as $1/R$, is proportional to the particle's acceleration and is perpendicular to **n**.

  - This electric field, together with the corresponding magnetic field, constitutes the radiation field:

Here, [ ] denotes the quantities calculated at the retarded position $\mathbf{r}(t_{\mathrm{ret}})$ and time t_{ret}.

where

$$\mathbf{u} \equiv \dot{\mathbf{r}}_0(t_{\mathrm{ret}})$$
$$\boldsymbol{\beta} \equiv \frac{\mathbf{u}(t_{\mathrm{ret}})}{c} = \frac{\dot{\mathbf{r}}_0(t_{\mathrm{ret}})}{c}$$
$$\dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{u}}(t_{\mathrm{ret}})}{c} = \frac{\ddot{\mathbf{r}}_0(t_{\mathrm{ret}})}{c}$$
$$\mathbf{R} \equiv \mathbf{r} - \mathbf{r}_0(t_{\mathrm{ret}})$$
$$\mathbf{n} \equiv \frac{\mathbf{R}}{R} = \frac{\mathbf{r}-\mathbf{r}_0(t_{\mathrm{ret}})}{|\mathbf{r}-\mathbf{r}_0(t_{\mathrm{ret}})|}$$
$$\kappa \equiv 1 - \mathbf{n}\cdot\boldsymbol{\beta}$$

"Velocity" Field

- The first term depends only on position and velocity.

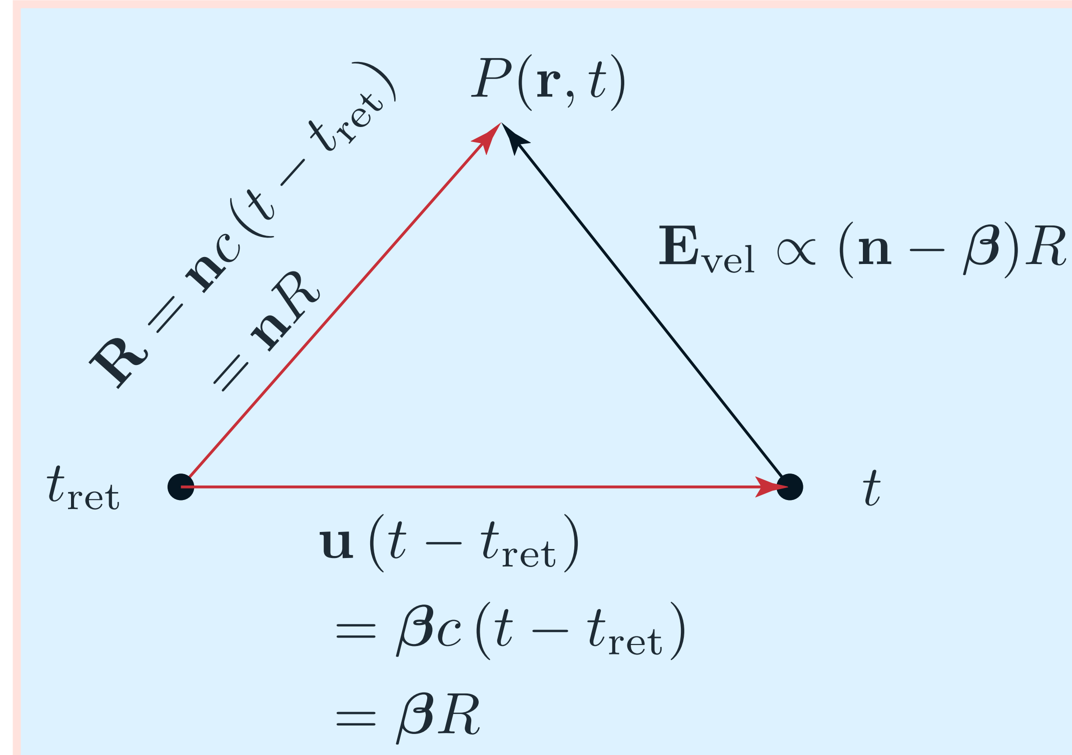
$$\mathbf{E}_{\rm vel}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]_{\rm ret} \qquad \mathbf{B}_{\rm vel}(\mathbf{r},t) = 0$$

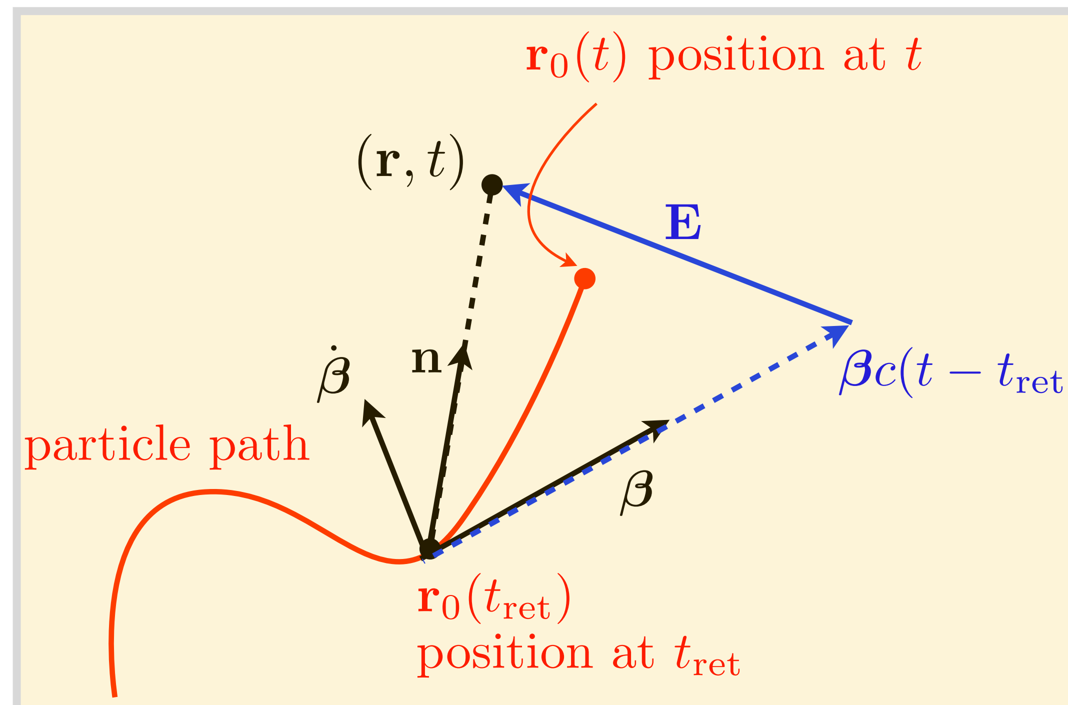
- A remarkable fact is that **the "velocity" electric field always points along the line toward the "current" position of the particle, expected when $\mathbf{u} = \mathbf{u}(t_{\rm ret})$ = constant.**

  The displacement of the photon from the retarded point $\mathbf{r}_0(t_{\rm ret})$ (point at $t_{\rm ret}$) to the field point $\mathbf{r}$ during the light travel time = $\mathbf{n}c(t-t_{\rm ret})$.

  In the same time, the particle undergoes a displacement $\boldsymbol{\beta}c(t-t_{\rm ret})$.

  The displacement between the field point and the current position of the particle is given by $(\mathbf{n}-\boldsymbol{\beta})c(t-t_{\rm ret})$ which is the direction of the velocity field.





Geometry for calculation of the radiation field at a point $(\mathbf{r}, t)$ in spacetime.

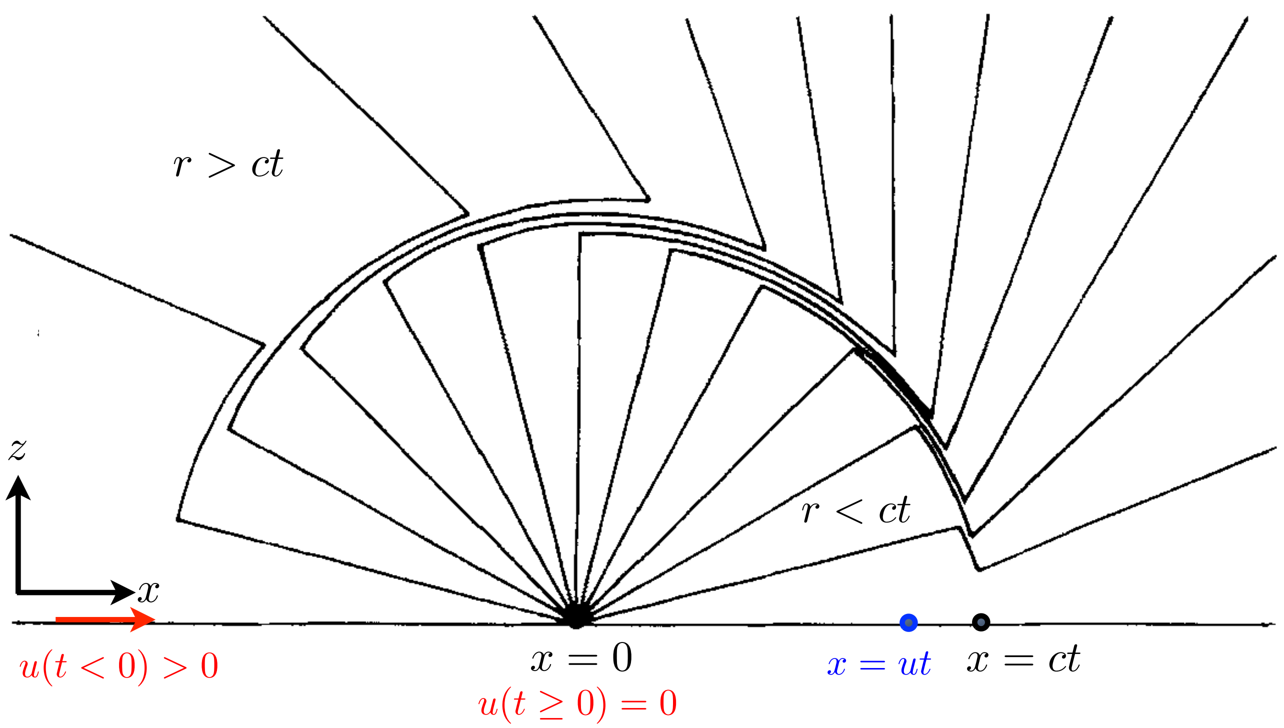
Note that, if the velocity is not a constant, the true displacement of the particle $\neq \boldsymbol{\beta}c(t-t_{\rm ret})$.

"Acceleration" (or "radiation") Field

- The second term (1) falls off as $1/R$, (2) is proportional to the particle's acceleration, and (3) is perpendicular to **n**.

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) = \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\right]_{\rm ret}$$
$$\mathbf{B}_{\rm rad}(\mathbf{r},t) = [\mathbf{n}\times\mathbf{E}_{\rm rad}]_{\rm ret}$$

- How an acceleration can give rise to a transverse field that decreases as $1/R$: Consider a particle, which originally moved with a constant velocity along the x-axis and stopped at $x = 0$ at time $t = 0$. At time t (> 0), the field outside radius ct is radial and points to the position ($x = ut$) where the particle would have been if there had been no deceleration, since no information of the deceleration has yet propagated. On the other hand, the field inside radius ct is "informed" and is radially directed to the true position ($x = 0$) of the particle.

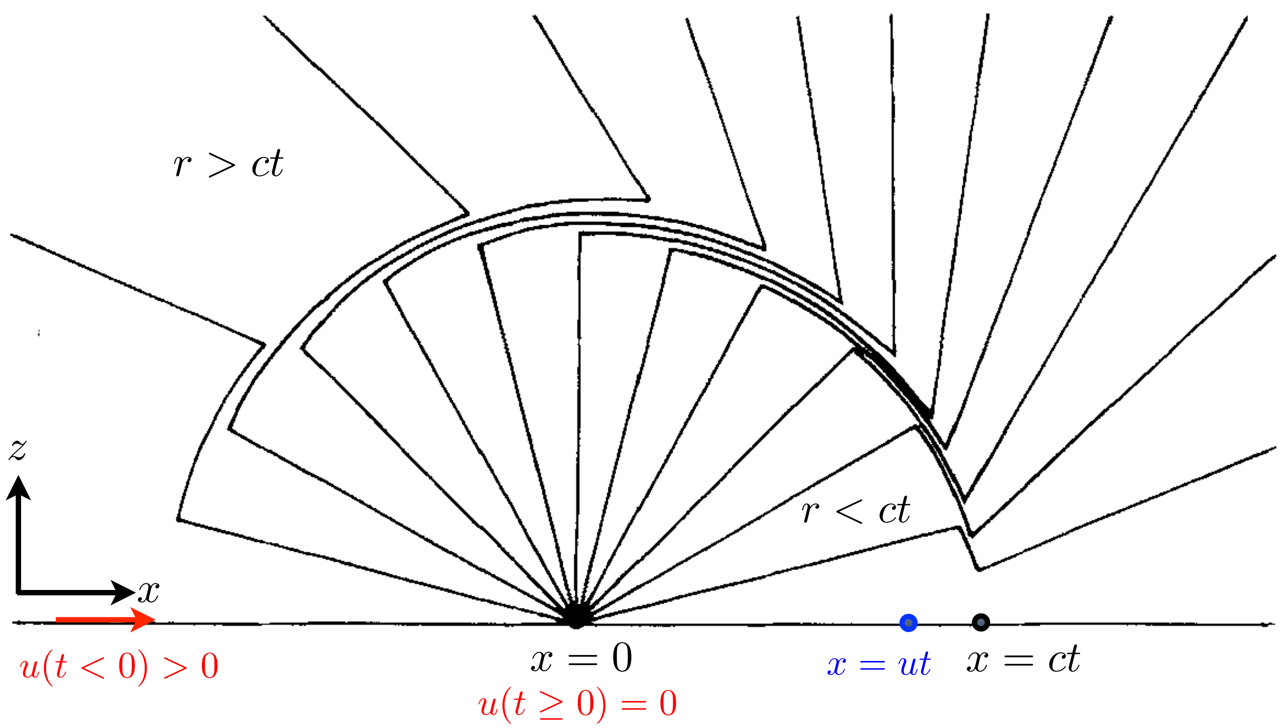


The fields at $x > ct$ were made when $t < 0$, while those at $x < ct$ were made when $t > 0$.

These two fields must be connected to be consistent with Gauss's law and flux conservation.

- The transition zone between them will propagate outward.
- The electric field in the transition (shell) zone is transverse.
- The radial thickness of the shell would be the light travel distance during the time interval over which the deceleration occurs, and thus is constant.
- However, the radius of the shell (or ring) increases as R.
- Since the total number of flux lines (in xy-plane) must be conserved, the strength of the field varies as 1/R.

$$\boxed{E(\delta x)(2\pi R) = \text{constant} \;\rightarrow\; E \propto \frac{1}{R}}$$



A single moving charge: Radiation Power*

- Power per unit frequency per unit solid angle of the radiation field of a single particle

Recall:

$$\frac{dW}{dAd\omega} = c\left|E(\hat{\omega})\right|^2$$

$$d\Omega = \frac{dA}{R^2}$$

$$\begin{aligned}\frac{dW}{d\omega d\Omega} &= \frac{R^2 dW}{d\omega dA} = R^2 c\left|\bar{E}(\omega)\right|^2 \\ &= \frac{c}{4\pi^2}\left|\int [R\mathbf{E}(t)]_{\mathrm{ret}}\, e^{i\omega t} dt\right|^2 \\ &= \frac{q^2}{4\pi^2 c}\left|\int \left[\mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\kappa^{-3}\right]_{\mathrm{ret}} e^{i\omega t} dt\right|^2\end{aligned}$$

Note: the expression in the brackets is evaluated at the retarded time $t' = t - R(t')/c$.

Now, changing variables from t to $t' = t - R(t')/c$ in the integral.

$$\begin{aligned}dt &= dt' + \frac{1}{c}\frac{dR(t')}{dt'}dt' = \kappa(t')dt' \\ &= (1 - \mathbf{n}(t')\cdot\boldsymbol{\beta}(t'))\, dt'\end{aligned}$$

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\kappa^{-2} e^{i\omega\left(t'+R(t')/c\right)} dt'\right|^2$$

We are only interested in the electric field measured at a far distance. Thus, we consider the case where $\left|\mathbf{r}_0\right| \ll |\mathbf{r}| = r$.

(1) $$R(t') = |\mathbf{r} - \mathbf{r}_0(t')| = [(\mathbf{r} - \mathbf{r_0}) \cdot (\mathbf{r} - \mathbf{r}_0)]^{1/2}$$
$$= \left[r^2 - 2(\mathbf{r} \cdot \mathbf{r}_0) + r_0^2\right]^{1/2} = r\left[1 - \frac{2(\mathbf{r} \cdot \mathbf{r}_0)}{r^2} + \frac{r_0^2}{r^2}\right]^{1/2}$$
$$\approx r\left(1 - \frac{\mathbf{r} \cdot \mathbf{r}_0}{r^2}\right)$$
$$= r - \mathbf{n} \cdot \mathbf{r}_0$$

Here, we use the following relation. We also note that $\mathbf{n}$ is now independent of t' in our approximation.

$$\mathbf{n} \equiv \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|} \approx \frac{\mathbf{r}}{r}$$

(2) $\kappa(t') = 1 - \mathbf{n}(t') \cdot \boldsymbol{\beta}(t') \approx 1 - \mathbf{n} \cdot \boldsymbol{\beta}(t')$. Here, again $\mathbf{n}$ is independent of t'.

(3) We note that $e^{i\omega\left(t' + R(t')/c\right)} = e^{i\omega r/c} e^{i\omega\left(t' - \mathbf{n}\cdot\mathbf{r}_0(t')/c\right)}$ and $\left|e^{i\omega r/c}\right| = 1$

Then we obtain
$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \mathbf{n} \times \left\{(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right\} \kappa^{-2} e^{i\omega\left(t' - \mathbf{n}\cdot\mathbf{r}_0(t')/c\right)} dt'\right|^2$$

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\kappa^{-2}e^{i\omega\left(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c\right)}dt'\right|^2$$

We can integrate the above equation by parts to obtain an expression without $\dot{\boldsymbol{\beta}}$. We first note the following relation (which is proved in the next slide).

$$\kappa^{-2}\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\} = \frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right]$$

With the rule of integration by parts $\int f'g\,dt = fg - \int fg'\,dt$, we obtain

This term vanishes under the assumption of a finite wave train.

$$\begin{aligned}
&\int \mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-2}e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}dt' \\
&= \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\kappa^{-1}e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}\Big|_{-\infty}^{\infty} - \int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\kappa^{-1}\{i\omega(1-\mathbf{n}\cdot\dot{\mathbf{r}}_0(t')/c)\}e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}dt' \\
&= -i\omega\int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\kappa^{-1}(1-\mathbf{n}\cdot\boldsymbol{\beta})e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}dt' \\
&= -i\omega\int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}dt' \qquad \leftarrow \quad \kappa = 1-\mathbf{n}\cdot\boldsymbol{\beta}
\end{aligned}$$

$$\boxed{\frac{dW}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\exp\left[i\omega\left(t'-\frac{\mathbf{n}\cdot\mathbf{r}_0(t')}{c}\right)\right]dt'\right|^2}$$

This formula will be used later.

- Proof of the relation:

$$\kappa^{-2}\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\} = \frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right]$$

note the vector identity: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{C}\cdot\mathbf{A}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$$\frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right] = \kappa^{-2}\left[-\frac{d\kappa}{dt'}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) + \kappa\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right]$$

$$\text{Here, use the relations : } \kappa = 1-\mathbf{n}\cdot\boldsymbol{\beta}, \quad \frac{d\kappa}{dt'} = -\mathbf{n}\cdot\dot{\boldsymbol{\beta}}$$

$$\begin{aligned}
\frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right] &= \kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\} + (1-\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\}\right] \\
&= \kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\} + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}) - (\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\}\right] \\
&= \kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta}) - \boldsymbol{\beta}\} + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}) - (\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}}) - \dot{\boldsymbol{\beta}}\}\right] \\
&= \kappa^{-2}\left[-(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta} + (\mathbf{n}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}} + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right] \\
&= \kappa^{-2}\left[-\mathbf{n}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}) + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right] \\
&= \kappa^{-2}\left[\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]
\end{aligned}$$

Radiation from Nonrelativistic Particles

- Using the above formulae we can discuss many radiation processes. The previous formulae is fully relativistic. However, for the moment, we will discuss nonrelativistic particles:

$$\beta = \frac{u}{c} \ll 1$$

- Order of magnitude comparison of the two fields:

$$E_{\rm rad} \approx \frac{q}{c} \frac{\dot{\beta}}{\kappa^3 R}, \quad E_{\rm vel} \approx \frac{q}{\kappa^3 R^2} \quad \rightarrow \quad \frac{E_{\rm rad}}{E_{\rm vel}} \approx \frac{R\dot{u}}{c^2}$$

If we focus on a particular Fourier component of frequency ν or the particle has a characteristic frequency of oscillation $\nu \sim 1/T$, then $\dot{u} = u\nu$, and the above equation becomes:

$$\frac{E_{\rm rad}}{E_{\rm vel}} \sim \frac{Ru\nu}{c^2} = \frac{u}{c}\frac{R}{\lambda}$$

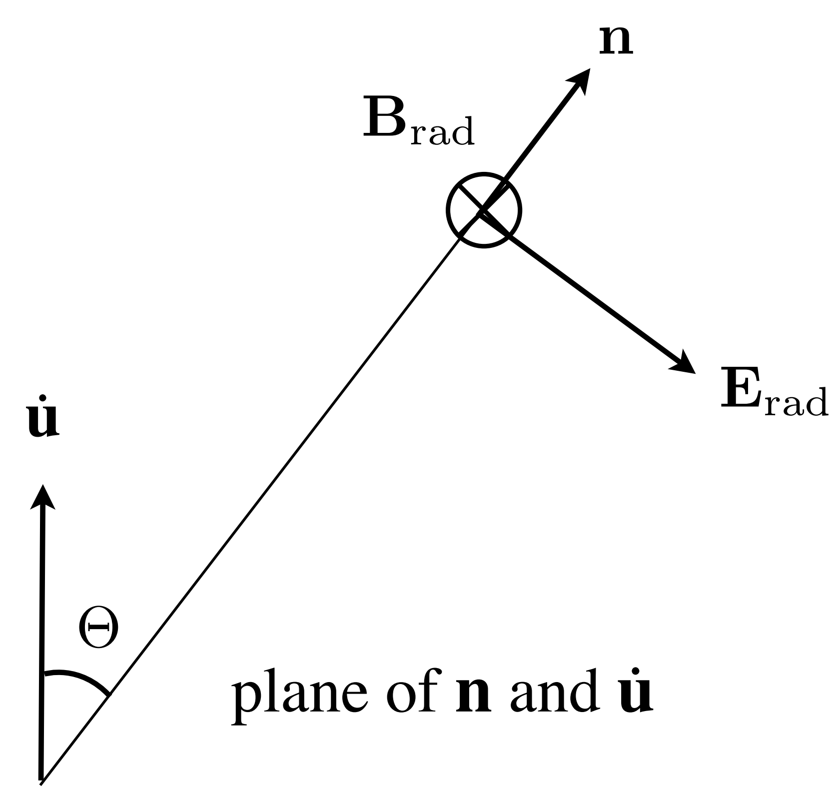
For field points inside the "near zone", $R \lesssim \lambda$, the velocity field is stronger than the radiation field by a factor $c/u = 1/\beta$.

For field points sufficiently far in the "far zone", $R \gg \lambda(c/u) = \lambda/\beta$, the radiation field dominates and increases its domination linearly with R. In astronomy, we are only interested in the "far zone". Therefore, let's consider only the radiation field.

Larmor's Formula

- When $\beta \ll 1$, the EM fields can be simplified to

$$\mathbf{E}_{\rm rad} = \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\right]_{\rm ret}$$
$$\approx \left[\frac{q}{Rc^2}\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}})\right]_{\rm ret}$$
$$\mathbf{B}_{\rm rad} = [\mathbf{n}\times\mathbf{E}_{\rm rad}]_{\rm ret}$$



The $E_{\rm rad}$ field is in the plane of ($\mathbf{n}$, $\dot{\mathbf{u}}$).

- As shown in the figure, $\mathbf{E}_{\rm rad}$ is in the plane defined by $\mathbf{n}$ and $\dot{\mathbf{u}}$.

  Note that $\mathbf{n}$ and $\mathbf{n}\times\dot{\mathbf{u}}$ are perpendicular and $\left|\mathbf{n}\times\dot{\mathbf{u}}\right| = \left|\dot{u}\right|\sin\Theta$, where Θ is the angle between $\mathbf{n}$ and $\dot{\mathbf{u}}$.

  Therefore, the magnitudes of $\mathbf{E}_{\rm rad}$ and $\mathbf{B}_{\rm rad}$ are

$$\therefore\ |\mathbf{E}_{\rm rad}| = |\mathbf{B}_{\rm rad}| = \frac{q\dot{u}}{Rc^2}\sin\Theta$$

Also, note that

$$\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}}) = \mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{u}})-\dot{\mathbf{u}}$$
$$\{\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}})\}^2 = (\mathbf{n}\cdot\dot{\mathbf{u}})^2+(\dot{\mathbf{u}})^2-2(\mathbf{n}\cdot\dot{\mathbf{u}})^2$$
$$= \dot{u}^2\cos^2\Theta+\dot{u}^2-2\dot{u}^2\cos^2\Theta$$
$$= \dot{u}^2(1-\cos^2\Theta)$$
$$= \dot{u}^2\sin^2\Theta$$

- The Poynting vector is in direction of **n** and has a magnitude.

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}E_{\mathrm{rad}}^2\mathbf{n}$$

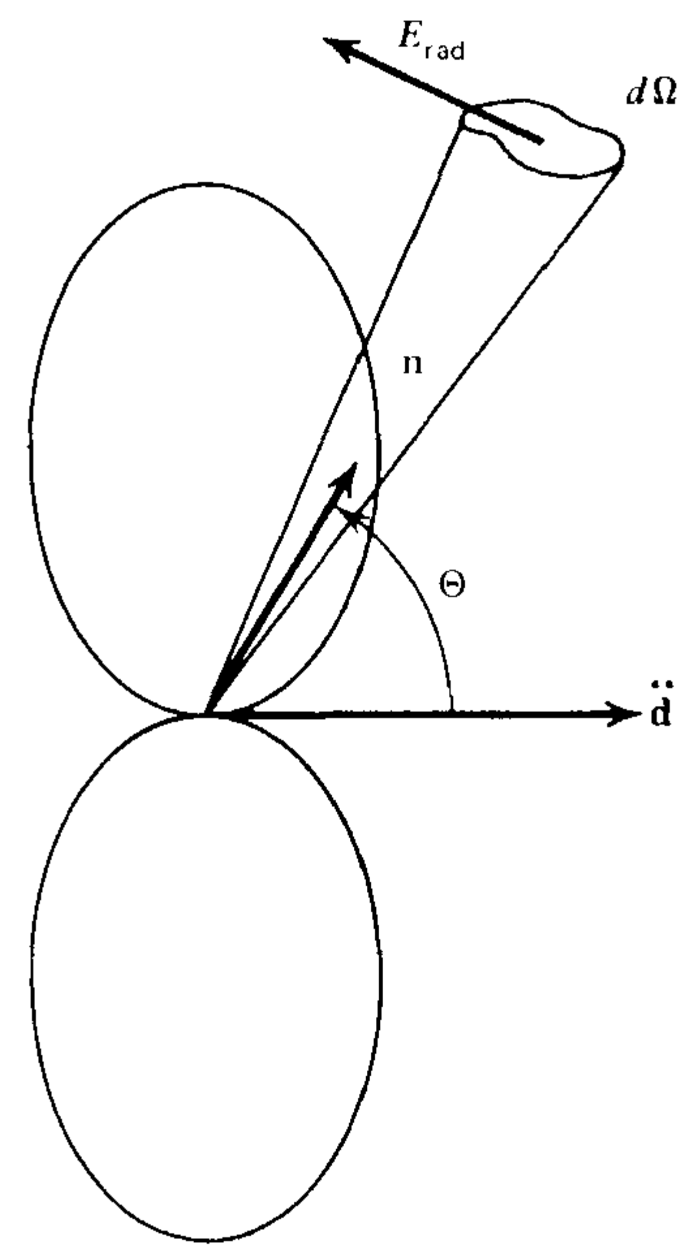
$$S = \frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^4}\sin^2\Theta \equiv \frac{dW}{dtdA} \quad (\mathrm{erg\ s^{-1}\ cm^{-2}})$$

This is an outward flow of energy (per unit time and per unit area), along the direction **n**.

- **Radiation patter:** The energy emitted per unit time into solid angle $d\Omega$ about **n** can be obtained by multiplying the Poynting vector by R^2.

$$\frac{dW}{dtd\Omega} = R^2\frac{dW}{dtdA} = R^2 S$$
$$= \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta$$

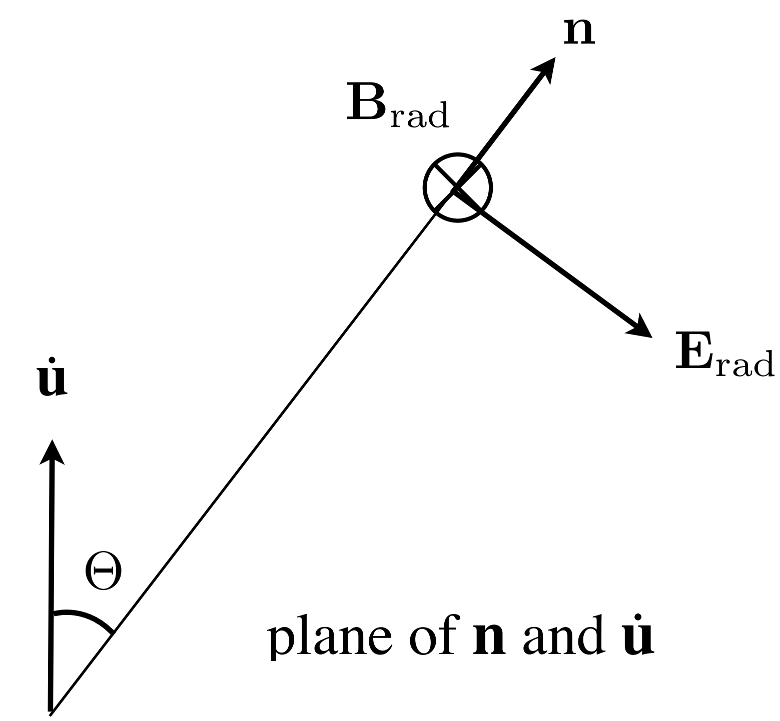
$$\boxed{\frac{dP}{d\Omega} = \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta}$$



- Total power emitted into all angles:

$$\int_{-1}^{1}(1-\mu^2)d\mu = \frac{4}{3}$$

$$P = \frac{dW}{dt} = \int d\Omega \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta = \frac{q^2\dot{u}^2}{2c^3}\int_{-1}^{1}(1-\mu^2)d\mu$$



$$\frac{dP}{d\Omega} = \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta$$

$$P = \frac{2q^2\dot{u}^2}{3c^3}$$

This is the Larmor's Formula for emission from a single accelerated charge q.

The emission from a single accelerated charge has the following properties:

1. The Power emitted is proportional to the square of the charge and the square of the acceleration.
2. We have the characteristic dipole pattern $\sin^2\Theta$: no radiation is emitted along the direction of acceleration, and the maximum is emitted perpendicular to acceleration. (see the figure in the previous slide)
3. The instantaneous direction of $\mathbf{E}_{\rm rad}$ is determined by $\dot{\mathbf{u}}$ and $\mathbf{n}$. If the particle accelerates along a line, the radiation will be 100% linearly polarized in the plane of $\dot{\mathbf{u}}$ and $\mathbf{n}$.

Dipole Approximation (the radiation from many particles)

- Consider many particles with positions $\mathbf{r}_i$, velocities $\mathbf{u}_i$, and charges q_i $(i = 1,2,3,\cdots,N)$. The radiation field at large distances can be found by adding together the $\mathbf{E}_{\rm rad}$ from each particle.
- However, the radiation field equations refer to conditions at retarded time, and the retarded times will differ for each particle. Therefore, we must keep track of the phase relations between the particles.

  There are situations in which it is possible to ignore this difficulty:

  Let L = typical size of the system
  τ = typical time scale for variations within the system

  If $\tau \gg L/c$ (light-travel-time), the differences in retarded time across the source are negligible.

  Note that τ can represent the time scale over which significant changes in the radiation field, and this in turn determines typical characteristic frequency of the emitted radiation. ***This condition is equivalent to the condition for the characteristic frequency (or characteristic wavelength) :***

$$\nu \approx \frac{1}{\tau} \ll \frac{c}{L} \quad \text{or} \quad \lambda = \frac{c}{\nu} \gg L$$

  In other words, ***the differences in retarded times can be ignored when the system size is much smaller than the characteristic wavelength.***

- We may also characterize τ as the time a particle takes to change its motion substantially.

  Let ℓ be a characteristic scale of the particle's orbit and u be a typical velocity, then $\tau \sim \ell/u$. The above condition $\tau \gg L/c$ then imply $u/c = \ell/(\tau c) \ll \ell/L$

  But since $\ell < L$, **the condition for dipole approximation is simply equivalent to the nonrelativistic condition:**

$$u \ll c$$

  With the above conditions met we can use the nonrelativistic form of the radiation fields:

$$\mathbf{E}_{\text{rad}} = \sum_i \frac{q_i}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}_i)}{R_i}$$

- Let R_0 be the distance from some point in the system to the field point. Then, $R_i = R_0 + \ell_i \approx R_0$ as $R_0 \gg \ell_i$. Finally, we have

$$\mathbf{E}_{\text{rad}} \approx \frac{1}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \sum_i q_i \dot{\mathbf{u}}_i)}{R_0} \quad \rightarrow \quad \mathbf{E}_{\text{rad}} = \frac{\mathbf{n} \times \left(\mathbf{n} \times \ddot{\mathbf{d}}\right)}{c^2 R_0}$$

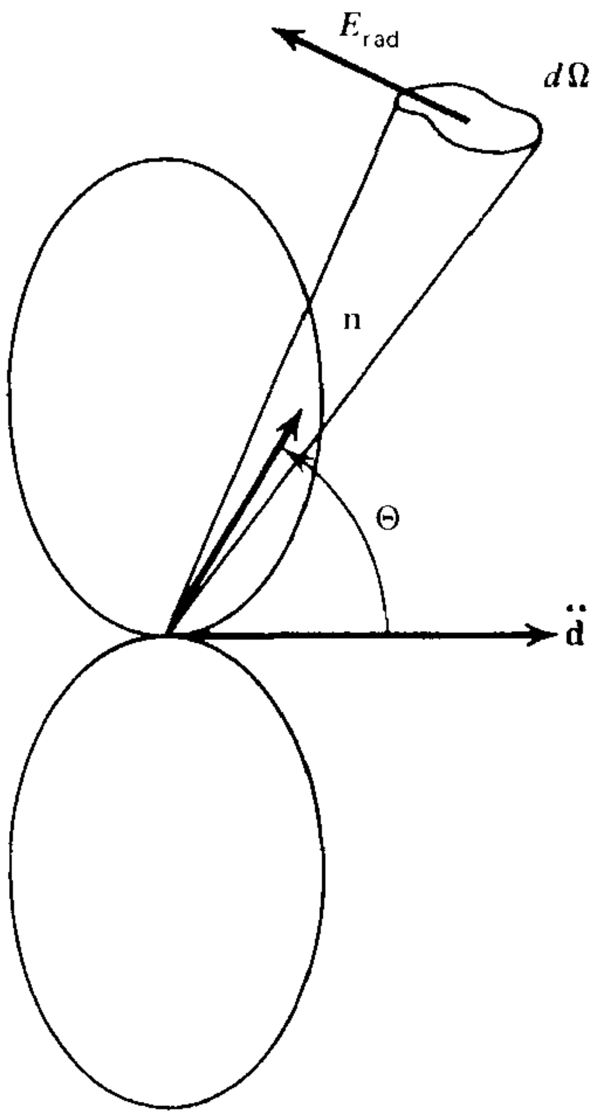
  where the electric dipole moment is defined as $\mathbf{d} = \sum_i q_i \mathbf{r}_i$

  Note that *the right-hand side of the above equations must still be evaluated at a retarded time*, but using any point within the region, say, the position used to define R_0.

- As before, for a single particle, we find the generalized formulas for the radiation patter and the total power, which are called the dipole approximation:

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\sin^2\Theta, \quad P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}$$

Note that the instantaneous polarization of **E** lines in the plane of $\ddot{\mathbf{d}}$ and **n**.



- **Spectrum of radiation in the dipole approximation**:

For simplicity we assume that **d** always lies in a single direction. Then, the magnitude of the electric field is given by

$$E(t) = \ddot{d}(t)\frac{\sin\Theta}{c^2 R_0}$$

where $d(t)$ is the magnitude of the dipole moment.

Fourier transform of $d(t)$ is defined as $d(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\bar{d}(\omega)d\omega$

Then, $\ddot{d}(t) = -\int_{-\infty}^{\infty}\omega^2 e^{-i\omega t}\bar{d}(\omega)d\omega$

$$\bar{E}(\omega) = -\frac{1}{c^2 R_0}\omega^2\bar{d}(\omega)\sin\Theta$$

- The energy per unit solid angle per frequency range in the dipole approximation is given by

$$\begin{aligned} \frac{dW}{d\omega d\Omega} &= R_0^2 \frac{dW}{d\omega dA} \\ \frac{dW}{d\omega dA} &= c\left|\bar{E}(\omega)\right|^2 \end{aligned} \longrightarrow \quad \frac{dW}{d\omega d\Omega} = \frac{\omega^4}{c^3}\left|\bar{d}(\omega)\right|^2 \sin^2\Theta$$

The total energy per frequency range is

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\left|\bar{d}(\omega)\right|^2$$

- The above formulas describe an interesting property of dipole radiation, namely, that the spectrum of the emitted radiation is related directly to the frequencies of oscillation of the dipole moment. However, this property is not true for particles with relativistic velocities.

- It is also worthwhile to note **the dependence of** $\omega^4 \propto \lambda^{-4}$ in the power spectrum.

Homeworks (due date: 10/12)

[Q5]

(1) Read the following document about the Schwartz inequality.

https://mathworld.wolfram.com/SchwarzsInequality.html

Schwartz inequality: $\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle \geq \langle E_1 E_2^* \rangle \langle E_2 E_1^* \rangle$

(2) Using the above Schwartz inequality, show that

$$I^2 \geq Q^2 + U^2 + V^2$$

from the definition:

$$\begin{aligned} I &\equiv \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle \\ Q &\equiv \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle \\ U &\equiv \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle \\ V &\equiv i \left(\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle \right) \end{aligned}$$