

Interstellar Medium (ISM)

Week 10

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선광일 (Kwangil Seon)
KASI / UST

Dynamics of H II regions

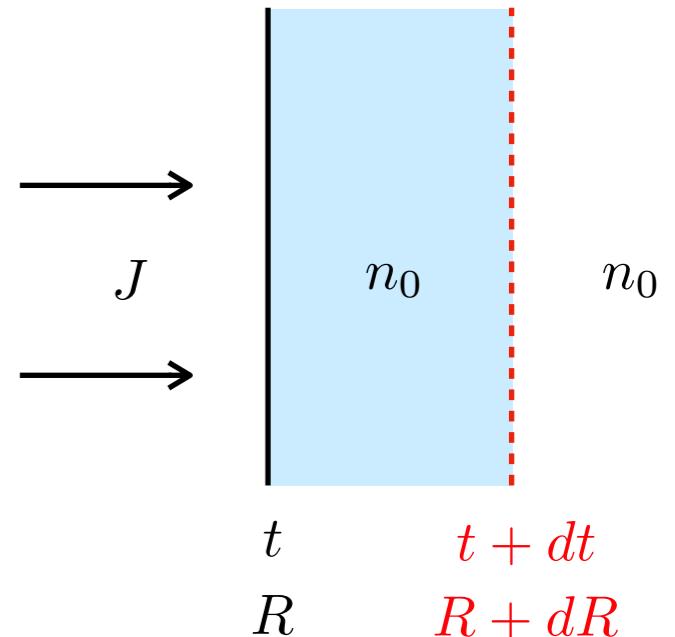
- A new O star is presumably born within clouds of relatively dense cold gas. The appearance of a source of UV photons will have two effects.
 - First, the gas surrounding the new star will become ionized. Since the mean free path of an UV photon is very short in neutral hydrogen, the photons will be absorbed in a relatively thin surrounding shell of neutral hydrogen, producing new ionization. Thus ***the ionized and neutral gases are separated by an ionization front, which moves rapidly outward*** as more and more atoms become ionized by the stream of photons.
 - Second, the process increases the gas temperature from $\sim 10^2$ K to $\sim 10^4$ K, by a factor of about a hundred. Third, the ionization process itself increases the number of gas particles, by a factor two. As a result, ***the pressure in the ionized gas is ~200 times greater than that in surrounding neutral material.*** This ionized gas cannot be confined and will expand. The ionized and neutral gas are set in motion.
 - Since the expansion velocity is likely to exceed the sound velocity in the surrounding H I region, ***a shock front may be expected to form***, moving out through the neutral gas.
 - The dynamical analysis of H II regions must consider the interactions between the ionization front and the shock front, together with the equations of motion of the gas behind the two fronts.
- This process is not the only way in which ISM is set in motion by means of interaction with stars.
 - There are effects produced by the very high speed continuous mass loss - a ***stellar wind***.
 - Many massive stars terminate their existence in a violent explosive event - a ***supernova***.

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- Basic Assumptions:
 - Any disturbances to the cloud structure produced by the formation of a star are neglected. *After a relatively short time ($< 10^5$ yr), the star reaches a static configuration* in which it can remain for a much longer time ($> 3 \times 10^6$ yr). The stellar radiant energy output rate and the spectral distribution of the radiation are more or less constant during this phase. The star then produces Lyman continuum (LyC) photons at a constant rate. Since *the star formation time scale is so short, we may take the star to be ‘switched on’ instantaneously.*
 - The gas around the star will be assumed to be at rest (in the frame of reference of the star).
 - The gas has initially assumed to be uniform in density and temperature.
 - Ionization front
 - The term “front” describes a more-or-less abrupt boundary between two regions of the ISM with very different properties.
 - An ionized nebula can be approximated as a region of highly ionized gas, separated from the surrounding neutral medium by a thin boundary region, of thickness $\lambda_{\text{mfp}} \approx 0.002$ pc. Thus, an H II region is surrounded by an ionization front.

The velocity of the ionization front

- Suppose that at time t the ionization front is located at a distance R from the star and at time $t + dt$ it is at a distance $R + dR$.

- Let n_0 = number density of the undisturbed neutral hydrogen
 - J = number of LyC photons incident normally on unit area of the ionization front per unit time.



- **Ionization balance at the ionization front:** While the ionization front moves from R to $R + dR$, the photons will ionize all the neutral atoms lying between these two positions ($R, R + dR$).
 - We assume that only one photon is needed to ionize each atom as the front moves the distance dR . In other words, *no recombination occurs within the distance interval dR* . For unit area of the ionization front, the following relation must be satisfied:

$$J \Delta Adt = n_0 \Delta AdR$$

- Then, the velocity of the ionization front (in a fixed frame of reference) is:

$$\frac{dR}{dt} = \frac{J}{n_0}$$

The initial stage of evolution of an ionized region

- Suppose that the UV source has been suddenly turned on.
- ***Ionization balance for the ionized region:*** We consider two factors:
 - *The radiation field at the ionization front is diluted because of the spherical geometry.*
 - Recombination takes place continuously inside the ionized region, and *some of the UV photons produced by the central source must go to reionize the atoms that have recombined.*
- Inside the ionized sphere, the fractional ionization is near unity. Thus, $n_e = n_p = n_0$. Using this condition, we obtain an equation for the expansion velocity of the ionization front.

$$Q_0 = (4\pi R^2) J + \left(\frac{4\pi}{3}R^3\right) \alpha_B n_e n_p$$

$$\frac{dR}{dt} = \frac{J}{n_0} \quad \Rightarrow \quad \frac{dR}{dt} = \frac{Q_0}{4\pi R^2 n_0} - \frac{1}{3} R n_0 \alpha_B$$

- Let's define the following dimensionless quantities:

$$\rho \equiv R/R_s \quad \text{where} \quad R_s \equiv \left(\frac{3}{4\pi} \frac{Q_0}{\alpha_B n_0^2} \right)^{1/3}$$

$$\tau \equiv t/t_{\text{rec}} \quad \text{where} \quad t_{\text{rec}} \equiv \frac{1}{\alpha_B n_0}$$

Then, the equation in dimensionless form is

$$\frac{d\rho}{d\tau} = \frac{1}{3} \left(\frac{1}{\rho^2} - \rho \right)$$

- ▶ The equation can be written:

$$\frac{d\rho}{d\tau} = \frac{1}{3} \left(\frac{1}{\rho^2} - \rho \right) \rightarrow \frac{d\rho^3}{d\tau} = 1 - \rho^3$$

- ▶ It's solution is

$$\rho^3 = 1 - e^{-\tau}$$

$$R(t) = R_s \left(1 - e^{-t/t_{\text{rec}}} \right)^{1/3}$$

initial condition: $R(t = 0) = 0$

$$\frac{dx}{d\tau} + x = 1$$

$$e^\tau \frac{dx}{d\tau} + e^\tau x = e^\tau$$

$$\frac{d(e^\tau x)}{d\tau} = e^\tau$$

$$\rightarrow e^\tau x = \int_0^\tau e^{\tau'} d\tau' = e^\tau - 1$$

$$x = 1 - e^{-\tau}$$

- Scale Parameters:

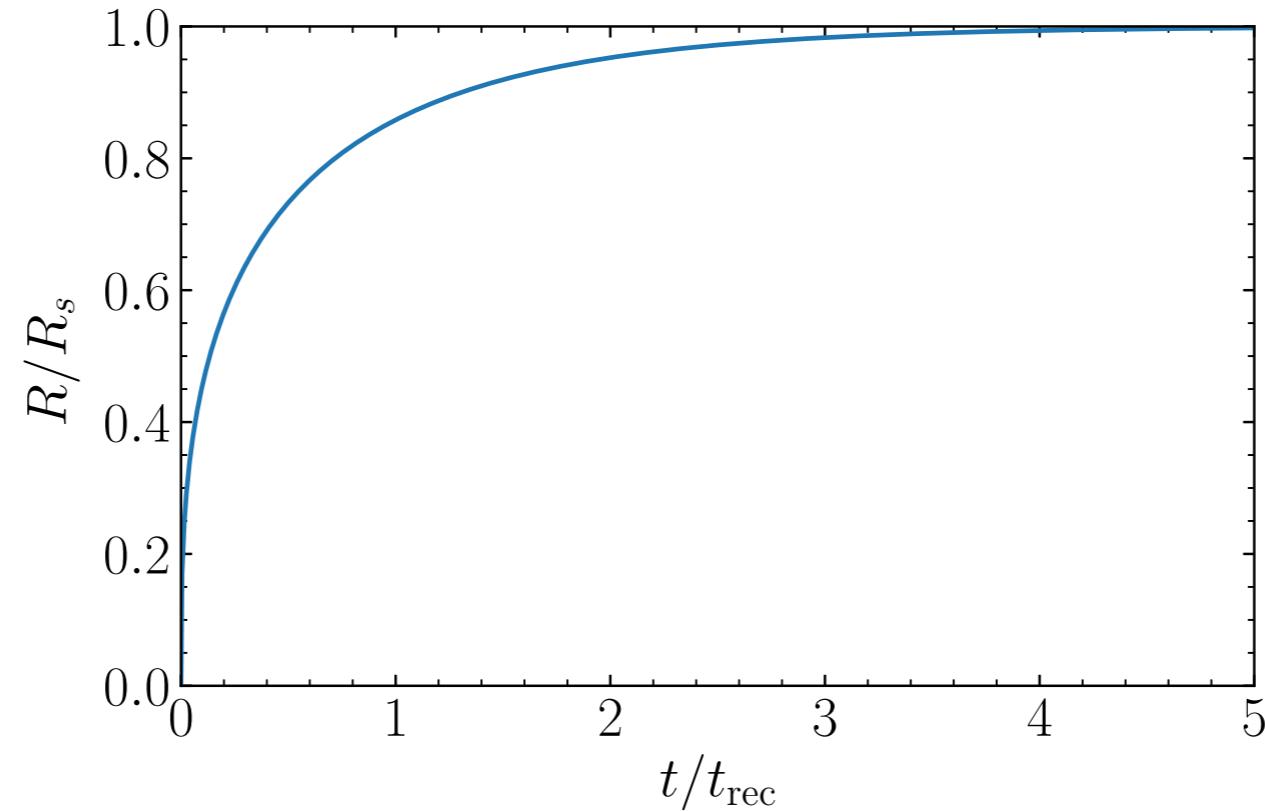
- The time scale introduced is the recombination time scale:

$$t_{\text{rec}} \equiv \frac{1}{\alpha_B n_0} \approx 4000 \text{ yr} \left(\frac{\alpha_B}{2.6 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1}} \right)^{-1} \left(\frac{n_0}{30 \text{ cm}^{-3}} \right)^{-1}$$

the length scale introduced is the Strömgren radius:

$$R_s \equiv \left(\frac{3}{4\pi} \frac{Q_0}{\alpha_B n_0^2} \right)^{1/3} \approx 7.2 \text{ pc} \left(\frac{Q_0}{10^{49} \text{ s}^{-1}} \right)^{1/3} \left(\frac{\alpha_B}{2.6 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1}} \right)^{-1/3} \left(\frac{n_0}{30 \text{ cm}^{-3}} \right)^{-2/3}$$

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- Hence, the time required to create a Strömgren sphere after turning on a hot star is an order of ~ 4000 yr. This is also the time it takes the ionized Strömgren sphere to revert to neutral gas after the central UV source has been turned off.



- At times $t \gg t_{\text{rec}} \sim 4000$ yr , the gas medium will be fully ionized with radius $R \sim R_s \sim 7$ pc, surrounded by a partially ionized boundary of thickness $\sim \lambda_{\text{mfp}} = (n_{\text{H}}\sigma_{\text{pi}})^{-1} \sim 0.002$ pc $\ll R_s$.

- We can compute the ***rate of expansion of the ionization front:***

$$\frac{dR}{dt} = \frac{R_s}{3t_{\text{rec}}} \frac{e^{-t/t_{\text{rec}}}}{(1 - e^{-t/t_{\text{rec}}})^{2/3}}$$

where the characteristic expansion velocity is

$$v_* \equiv \frac{R_s}{3t_{\text{rec}}} \simeq 560 \text{ km s}^{-1} \left(\frac{Q_0}{10^{49} \text{ s}^{-1}} \right)^{1/3} \left(\frac{n_0}{30 \text{ cm}^{-3}} \right)^{1/3}$$

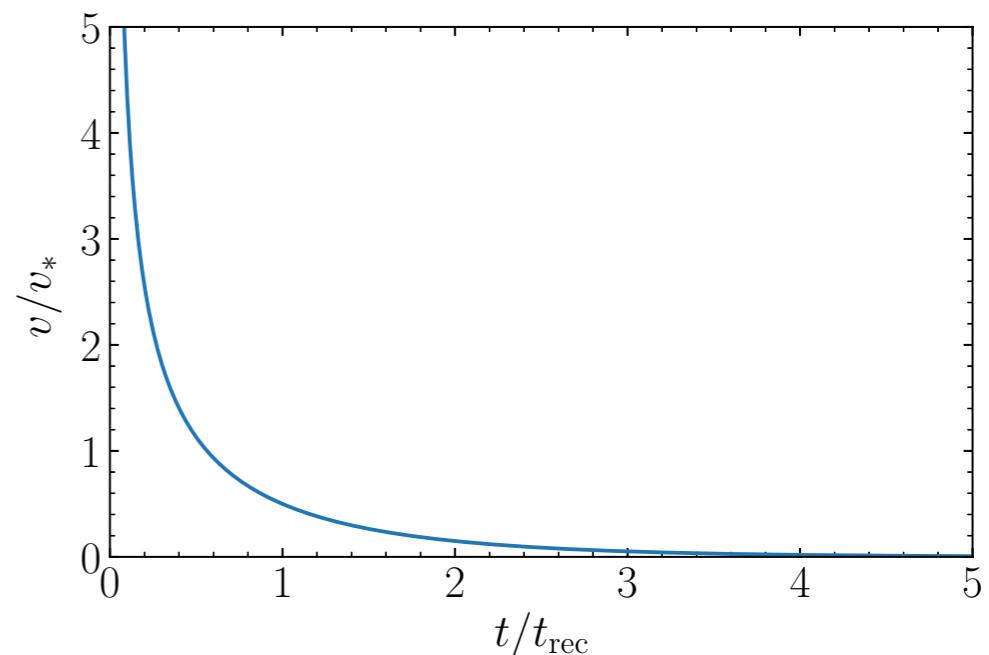
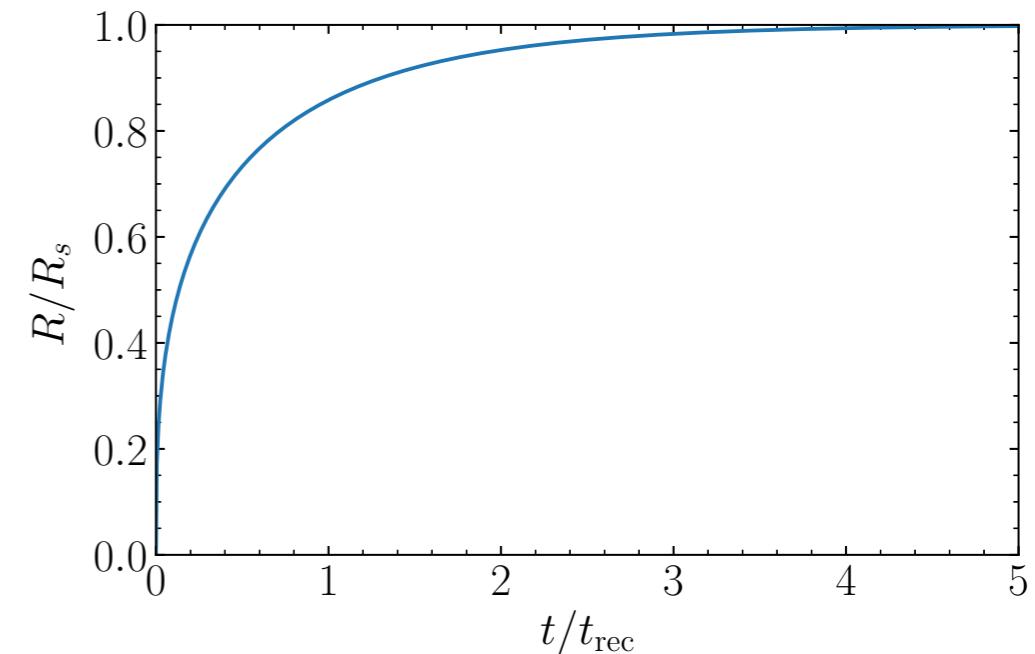
This is much larger than the sonic speed $c_s \approx 1 \text{ km s}^{-1}$ in the neutral medium as well as $c_s \approx 10 \text{ km s}^{-1}$ in the ionized medium.

- The expansion speed of the ionization front at two limits:

$$\frac{dR}{dt} \approx \frac{R_s}{3t_{\text{rec}}} \left(\frac{t}{t_{\text{rec}}} \right)^{-2/3} \quad \text{for } t \ll t_{\text{rec}}$$

$$\frac{dR}{dt} \approx \frac{R_s}{3t_{\text{rec}}} e^{-t/t_{\text{rec}}} \quad \text{for } t \gg t_{\text{rec}}$$

Note that the expansion speed diverges at $t = 0$.



- The ionization front will initially expand supersonically.

When will the ionization front expand at subsonic speeds?

$$\frac{dR}{dt} = \frac{R_s}{3t_{\text{rec}}} e^{-t/t_{\text{rec}}} \lesssim c_i \quad c_i \approx 13 \text{ km s}^{-1} \text{ sound speed in the ionized medium}$$

$$t \lesssim t_{\text{sonic}} \equiv t_{\text{rec}} \ln \left(\frac{R_s}{3t_{\text{rec}}} \frac{1}{c_n} \right) \approx 3.8t_{\text{rec}} \simeq 15,000 \text{ yr}$$

- At this time, the ionization front will have a size of:

$$R(t = t_{\text{sonic}}) = R_s (1 - e^{-3.8})^{1/3} = 0.9925 R_s$$

- The ionization front will expand at a supersonic velocity until $t \approx t_{\text{sonic}}$ ($\sim 15,000$ yr). By that time, the ionized sphere has reached a radius $R \sim 0.99R_S$ and then it starts to expand at subsonic speed.
- ***However, our analysis has ignored the pressure imbalance between the hot ionized gas inside the Stromgren sphere and the cold neutral gas outside.***
- ***After the sound crossing time $t = R_S/c_s \sim 0.5$ Myr, the gas starts to flow outward as a result of the pressure gradient that has build up.***

The final stage of evolution of an ionized region

- Although the ionized sphere approaches ionization equilibrium at $t \gtrsim t_{\text{rec}}$, it would be still far from pressure equilibrium.
 - Outside the ionized zone, it will be embedded in the cold neutral medium with a temperature $T \sim 100$ K.
 - Inside the sphere, the heating and cooling processes yield a temperature of $T \sim 10,000$ K.
 - Also, the density of particles inside the ionized sphere will double when the hydrogen is ionized.
 - Thus, *the pressure inside the sphere will be ~ 200 times higher than the pressure outside, meaning that the ionized gas will begin to expand.*
 - The ionized gas expands as long as it has a higher pressure than its surroundings. This expansion produces a shock and will cease when the hot ionized gas reaches pressure equilibrium with the surrounding cold neutral gas.
- ***The condition of final pressure equilibrium*** can be written in the form:

$$2n_f k T_i = n_0 k T_n$$

n_f = number density of the ionized hydrogen.

T_i and T_n = temperatures of the ionized and neutral gas, typically $T_i = 10^4$ K, $T_n = 10^2$ K.

- The ionized gas sphere must still absorb all the stellar UV photons. Thus,

$$Q_0 = \frac{4}{3}\pi R_f^3 n_f^2 \alpha_B$$

Here, R_f is the final radius of the ionized gas sphere. From the pressure equilibrium condition, we obtain the final size:

$$n_f = (T_n/2T_i)n_0 \approx 0.005n_0 \quad \rightarrow \quad R_f = (2T_i/T_n)^{2/3} R_{s0} \approx 34R_{s0}$$

- The ratio of the mass of gas finally ionized to that contained within the initial Strömgren sphere is:

$$\frac{M_f}{M_s} = \frac{R_f^3 n_f}{R_{s0}^3 n_0} = \frac{2T_i}{T_n} \approx 200$$

- This indicates that *the initial Strömgren sphere contains only a very small fraction of the material which, in principle, a star could ultimately ionize.*

The intermediate stage of evolution of an ionized region

- Before the pressure equilibrium is established, the gas density and temperature will be

$$n_i \approx 2n_0 > n_f \quad \text{and} \quad T_i = 10^4 \text{ K}$$

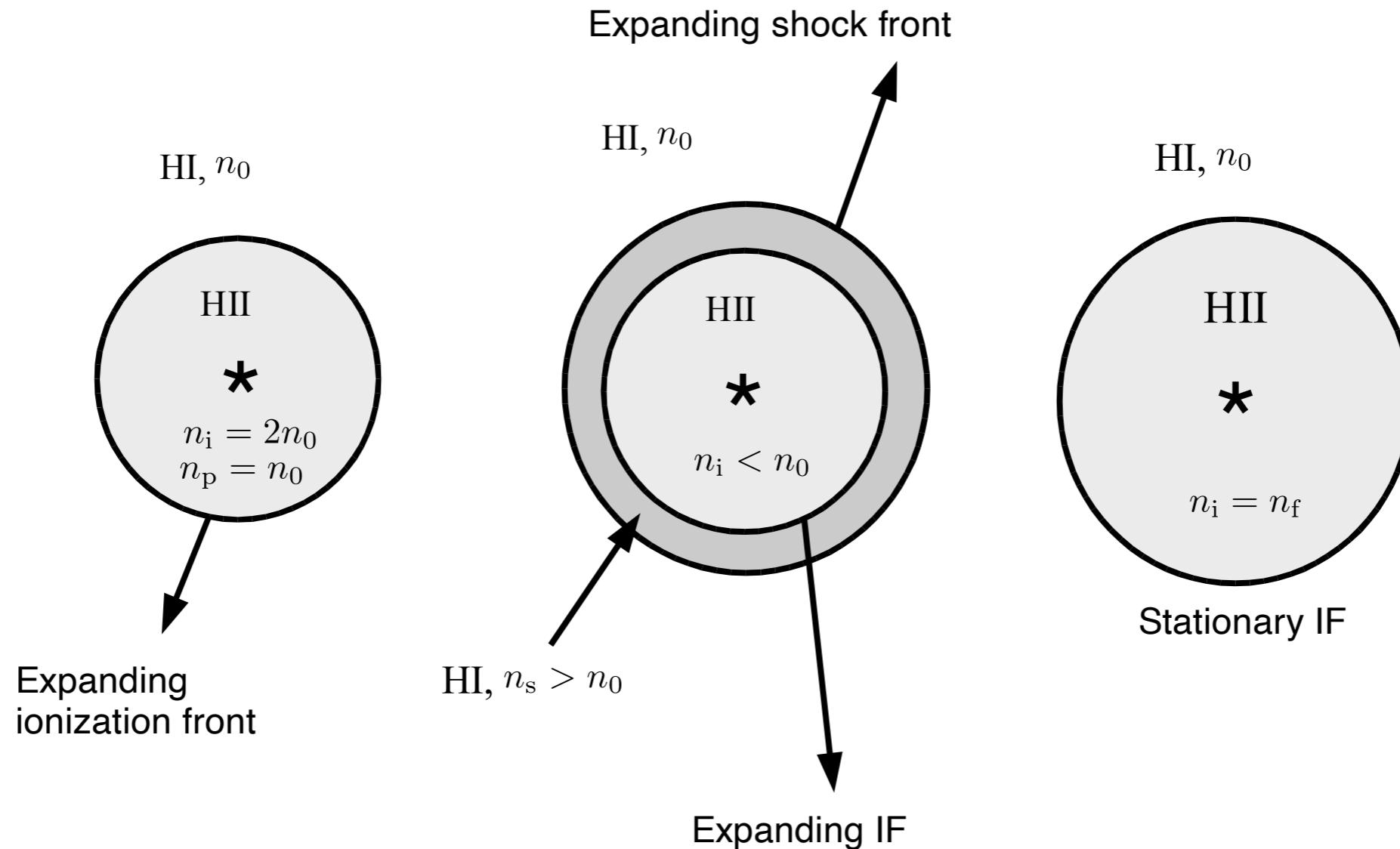
- Then the isothermal sound speeds of the ionized gas and neutral gas are, respectively:

$$c_i^2 = \frac{P_i}{\rho_i} \approx \frac{2n_0 k T_i}{n_0 m_H} \quad c_n^2 = \frac{P_n}{\rho_n} = \frac{n_0 k T_n}{n_0 m_H}$$

$$\frac{c_i}{c_n} = \left(\frac{n_i T_i}{n_0 T_n} \right)^{1/2} \approx \sqrt{200} = 14.14$$

- The sound speed of the ionized gas is much larger than that of the neutral gas.
- The ionized gas has a higher pressure and thus plays the role of a piston and pushes a shock wave into the neutral gas. *The expansion speed of the ionized gas is originally equal to about c_i , which is highly supersonic with respect to the sound speed in the neutral gas.*
- Note also that, at $t \gtrsim t_{\text{sonic}} \approx 3.8t_{\text{rec}}$, the expansion speed (c_i) of ionized gas is larger than that of the ionization front.

$\frac{dR}{dt} > c_i$ at $t \lesssim t_{\text{sonic}}$	\longrightarrow	$\frac{dR}{dt} \approx c_i$ at $t \approx t_{\text{sonic}}$
initial stage		intermediate stage



Evolutionary scheme of an expanding H II region. (a) The initial stage, (b) expansion with a shock in the neutral gas, (c) the final equilibrium state.

[Figure 7.2 Dyson]

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- Sound crossing time
 - The ionized region will likely be overpressured relative to its surroundings, in which case it will expand on the sound crossing time.
 - The isothermal sound speed in fully ionized hydrogen is

$$c_s = (2kT/m_{\text{H}})^{1/2} = 13 (T/10^4 \text{ K})^{1/2} \text{ km s}^{-1} \quad p = (n_{\text{HI}} + n_e)kT = 2n_{\text{H}}kT$$

- The time for a pressure wave to propagate a distance equal to Strömgren radius is

$$t_{\text{sound}} = \frac{R_s}{c_s} \approx 2.39 \times 10^5 \frac{Q_0/10^{49} \text{ s}^{-1}}{(n/10^2 \text{ cm}^{-3})^{2/3}} \text{ [yr]}$$

- This is about a hundred times longer than the recombination time (timescale of the expanding ionization front).

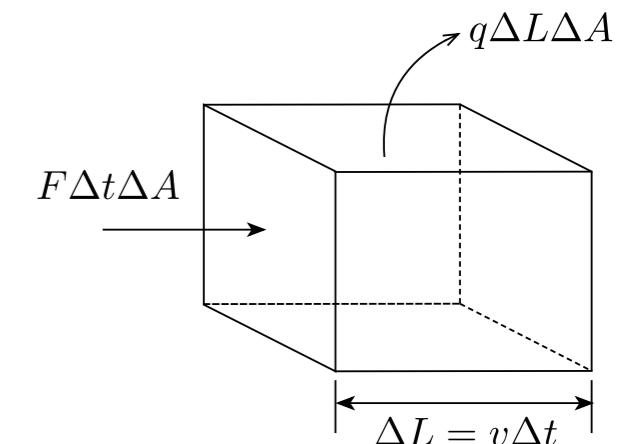
Introduction to Gas Dynamics

- Assumption for hydrodynamics:
 - particle mean free path \ll size of the region
 - We will derive the equations for conservation of mass, momentum and energy, in 1D space.

- Definition**

- Flux of a hydrodynamic quantity q (for instance, density):
 - Fluid moves a distance ΔL during a time interval Δt with a velocity v .

$$F\Delta t\Delta A = q\Delta L\Delta A \rightarrow F = qv$$

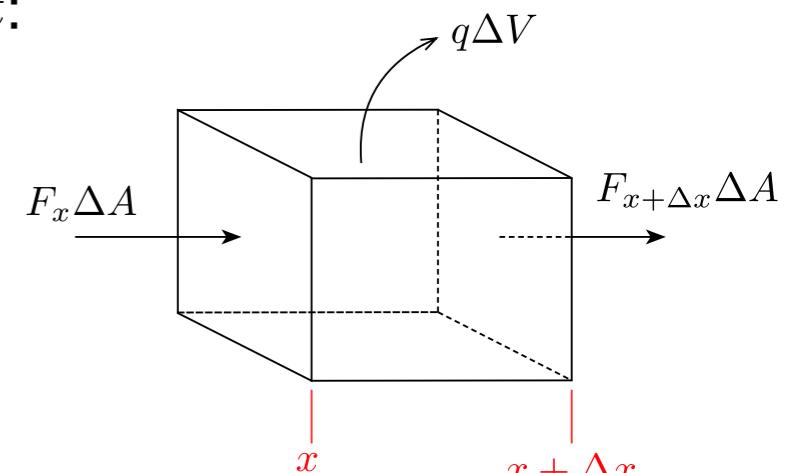


- Conservation equation for a quantity q**

- change of the quantity within a volume ΔV for a time interval Δt :
Here, Δt and Δx are independent.

$$\frac{q\Delta V|_{t+\Delta t} - q\Delta V|_t}{\Delta t} = F\Delta A|_x - F\Delta A|_{x+\Delta x}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial F}{\partial x} \rightarrow \frac{\partial q}{\partial t} = -\frac{\partial(qv)}{\partial x}$$



- Here, no sources or sinks of the quantity within ΔV were assumed. If any, the loss and gain terms should be added in the right-hand side.

Mass Conservation

- Conservation equations
 - ***Mass conservation (continuity equation)***
 - ▶ mass within a volume $dV = \rho dV$
 - ▶ no sources or sinks of material within dV
 - ▶ Consider the mass per unit area (dA), contained in the volume

$$\rho dV/dA = \rho dx \quad \longrightarrow \quad \frac{\partial}{\partial t}(\rho dx) = \overbrace{\rho u}^{\text{incoming}} - \overbrace{(\rho + d\rho)(u + du)}^{\text{outgoing}}$$

$$= -(\rho du + ud\rho + d\rho du)$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$

- ▶ Mass loss and gain terms should be added in the right-hand side, if necessary.

Momentum Conservation

- **Momentum conservation (Euler's equation)**

- ▶ momentum within dV (per unit area) = $(\rho dV)u/dA = \rho u dx$
= change of momentum due to fluid flow and gas pressure acting on the surface of dV

$$\begin{aligned}\frac{\partial}{\partial t}(\rho u dx) &= \overbrace{\rho u^2}^{\text{incoming}} - \overbrace{(\rho + d\rho)(u + du)^2}^{\text{outgoing}} + \overbrace{P}^{\text{incoming}} - \overbrace{P + dP}^{\text{outgoing}} \\ &= \rho u^2 - \left(\rho u^2 + 2\rho u du + \cancel{\rho du^2} + u^2 d\rho + \cancel{2ud\rho du} + \cancel{d\rho du^2} \right) - dP\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\rho u) &= -2\rho u \frac{\partial u}{\partial x} - u^2 \frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x} \\ &= -\frac{\partial}{\partial x}(\rho u^2) - \frac{\partial P}{\partial x}\end{aligned}$$

or

$$\begin{aligned}\frac{\partial}{\partial t}(\rho u) &= -2\rho u \frac{\partial u}{\partial x} - u^2 \frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x} \\ \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} &= -\rho u \frac{\partial u}{\partial x} - u \left(\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} \right) - \frac{\partial P}{\partial x}\end{aligned}$$

$$\frac{\partial}{\partial t}(\rho u) = -\frac{\partial}{\partial x}(\rho u^2 + P)$$

Using mass conservation, $\frac{\partial u}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$

$$\rho \frac{\partial u}{\partial t} = -\rho u \frac{\partial u}{\partial x} - \frac{\partial P}{\partial x}$$

- ▶ Further terms could be added in the right-hand side, accounting for forces due to gravity, magnetic fields, radiation field, and viscosity.

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- ▶ The following quantity is sometimes known as **Bernoulli's constant**.

$$\rho u^2 + P$$

One may use it to understand why, for example, fast winds engulfing a house causes it to **explode**, rather than **implode**, because the pressure external to the house becomes lower than its value inside it.

- ▶ Viscous force is due to “internal friction” in the fluid (resistivity of the fluid to the flow), as two adjacent fluid parcels move relative to each other.

$$\text{viscous force} \propto \frac{\partial^2 u}{\partial x^2}$$

The viscous force is usually much smaller than force due to gas pressure, but important in high-speed flows with large velocity gradients, as in accretion disks.

Ionization Front: Jump Condition

- Low density gas, like that of the ISM, can be treated as an ideal gas, with no viscosity with a pressure given by the ideal gas law:

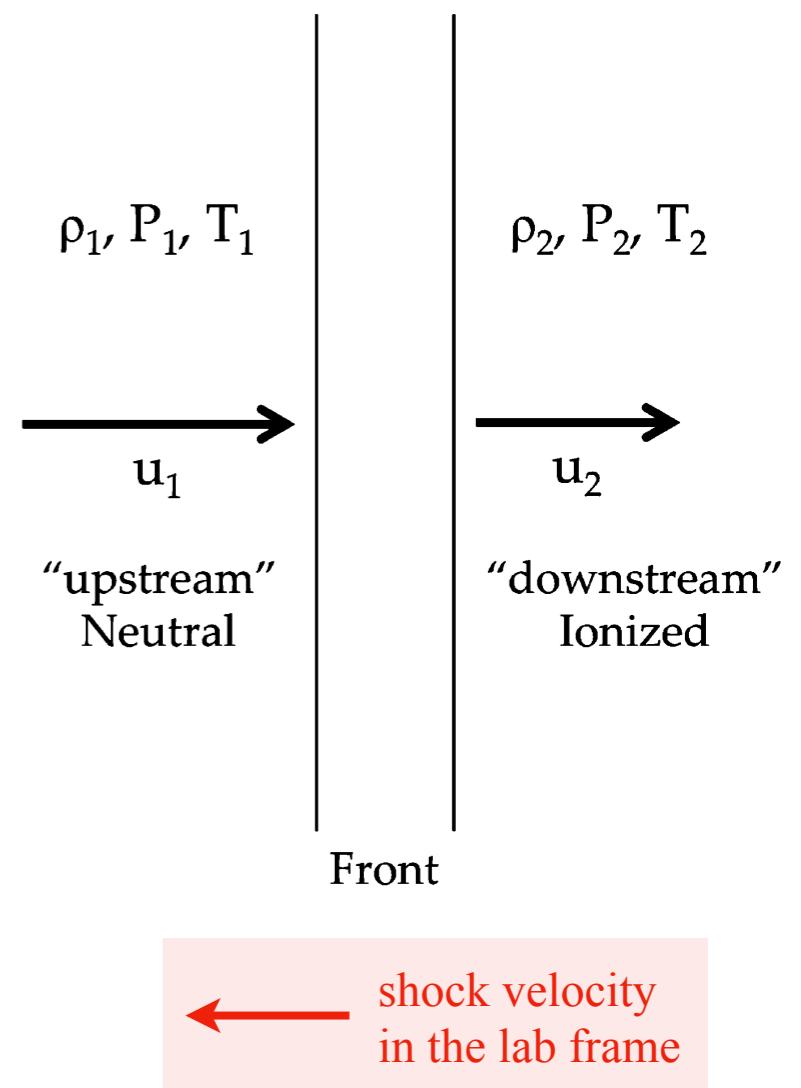
$$P = \frac{\rho k T}{m}$$

ρ = mass density, T = temperature,
 m = mean molecular mass

ρ, P, T, u = density, pressure, temperature, and bulk velocity

- Let's consider a small patch of the ionization front between the interior of an H II region and its exterior.

- If the patch is small compared to the ionization front's radius of curvature, then we can treat the ionization front as if it has **plane parallel** symmetry.
- It is convenient to use **a frame of reference in which the ionization front is stationary**; in this frame, the bulk velocity \mathbf{u}_1 of the neutral gas points toward the ionization front. The bulk velocity \mathbf{u}_2 of the ionized gas points away from the ionization front.



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- Let's consider a steady state solution.
 - We have seen that the speed of the ionization front surrounding a Strömgren sphere changes with time. However, the steady state solution gives us some intuition about the behavior of ionization fronts in general.
 - Then, the mass conservation and momentum conservation equation becomes:

$$\frac{d}{dx} (\rho u) = 0 \quad \frac{d}{dx} (\rho u^2 + P) = 0$$

- Let subscript **1** denote fluid variables in the neutral gas ahead of the I-front, and subscript **2** denotes fluid variables in the ionizing gas behind the I-front. Integrating these equations across the ionization front, we obtain:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

- *The number of H atoms flowing through the ionization front per unit area per second must equal to J , the corresponding number of ionizing photons reaching the front.*
Hence, the equation becomes

$$\rho_1 u_1 = \rho_2 u_2 = m_i J$$

Here, $u_1 = \frac{dR}{dt} = \frac{J}{n_0}$, $\rho_1 = m_i n_0$

where m_i is the mean mass of the gas per newly created positive ion ($m_i = m_H$ in a pure hydrogen gas). We may also write the equation of momentum conservation using the isothermal sound speeds:

$$\rho_1 (u_1^2 + c_1^2) = \rho_2 (u_2^2 + c_2^2)$$

$c_s^2 = \frac{P}{\rho}$ for isothermal gas
 $P = nkT$

- We will consider a hydrogen gas.

$$c_1 = \left(\frac{kT_1}{m_H} \right)^{1/2} = 0.91 \text{ km s}^{-1} \left(\frac{T_1}{100 \text{ K}} \right)^{1/2}$$

neutral hydrogen gas

$$c_2 = \left(\frac{2kT_2}{m_H} \right)^{1/2} = 12.9 \text{ km s}^{-1} \left(\frac{T_2}{10^4 \text{ K}} \right)^{1/2}$$

fully ionized gas

Here, the number density of particles is $2n_H$ in a fully-ionized hydrogen gas (downstream) and thus the factor 2 in c_2 .

- In summary, the equations are

$$\begin{aligned}\rho_1 u_1 &= \rho_2 u_2 = m_i J \\ \rho_1 (u_1^2 + c_1^2) &= \rho_2 (u_2^2 + c_2^2)\end{aligned}$$

- We assume that ρ_1 and u_1 are known, and we seek to solve for the unknown ρ_2 and u_2 . We obtain a simple quadratic equation for $x \equiv \rho_1/\rho_2 = u_2/u_1$.

$$\begin{aligned}\frac{\rho_1}{\rho_2} (u_1^2 + c_1^2) &= \left(\frac{\rho_1}{\rho_2}\right)^2 u_1^2 + c_2^2 \\ u_1^2 x^2 - (u_1^2 + c_1^2) x + c_2^2 &= 0 \quad \longrightarrow \quad x = \frac{1}{2u_1^2} \left[(u_1^2 + c_1^2) \pm \sqrt{(u_1^2 + c_1^2)^2 - 4u_1^2 c_2^2} \right]\end{aligned}$$

Then, the ratios between densities and velocities are:

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{1}{2u_1^2} \left[(u_1^2 + c_1^2) \pm \sqrt{(u_1^2 + c_1^2)^2 - 4u_1^2 c_2^2} \right]$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{1}{2c_2^2} \left[(u_1^2 + c_1^2) \mp \sqrt{(u_1^2 + c_1^2)^2 - 4u_1^2 c_2^2} \right]$$

- The roots are real if and only if

$$\begin{aligned} f(u_1) &\equiv (u_1^2 + c_1^2)^2 - 4u_1^2 c_2^2 \\ &= (u_1^2 + c_1^2 + 2u_1 c_2)(u_1^2 + c_1^2 - 2u_1 c_2) \geq 0 \end{aligned}$$

This requires:

$$\begin{aligned} u_1^2 + c_1^2 - 2u_1 c_2 &\geq 0 \\ \left[u_1 - \left(c_2 + \sqrt{c_2^2 - c_1^2} \right) \right] \left[u_1 - \left(c_2 - \sqrt{c_2^2 - c_1^2} \right) \right] &\geq 0 \end{aligned}$$

Therefore,

$$u_1 \geq u_R \equiv c_2 + \sqrt{c_2^2 - c_1^2} \quad \text{or} \quad u_1 \leq u_D \equiv c_2 - \sqrt{c_2^2 - c_1^2}$$

Note that
 $u_R > c_2 > c_1 > u_D$

We also note that

$$\begin{aligned} u_1^2 + c_1^2 + 2u_1 c_2 &= \left[u_1 + \left(c_2 + \sqrt{c_2^2 - c_1^2} \right) \right] \left[u_1 + \left(c_2 - \sqrt{c_2^2 - c_1^2} \right) \right] \\ \rightarrow \quad f(u_1) &= (u_1^2 - u_R^2)(u_1^2 - u_D^2) \end{aligned}$$

$$\begin{aligned} \frac{u_2}{u_1} &= \frac{\rho_1}{\rho_2} = \frac{1}{2u_1^2} \left[(u_1^2 + c_1^2) \pm \sqrt{(u_1^2 - u_R^2)(u_1^2 - u_D^2)} \right] \\ \frac{\rho_2}{\rho_1} &= \frac{u_1}{u_2} = \frac{1}{2c_2^2} \left[(u_1^2 + c_1^2) \mp \sqrt{(u_1^2 - u_R^2)(u_1^2 - u_D^2)} \right] \end{aligned}$$

- The rapidly propagating ionization fronts, with $u_1 \geq u_R$ are called ***R-type fronts (R stands for “rarefied” or rapid)***. The dilatory ionization fronts are called ***D-type fronts (D stands for “dense” or dilatory)***.
 - ▶ An R-type front has $u_1 \geq u_R > c_2 > c_1$, and is supersonic with respect to the neutral medium.
 - ▶ A D-type front has $u_1 \leq u_D < c_1 < c_2$, and is subsonic with respect to the neutral medium.
- For a given front propagation speed u_1 , there are two possible values of the density ratio ρ_2/ρ_1 across the ionization front as a function of the propagation speed u_1 .
 - ▶ The front that has the ***larger density contrast*** is called a ***strong*** front.
 - ▶ The front that has the ***smaller density contrast*** is called a ***weak*** front.
 - ▶ Thus, there are four types of ionization front: weak R, strong R, weak D, strong D.

$$\frac{\rho_2}{\rho_1} = \frac{1}{2c_2^2} \left[(u_1^2 + c_1^2) \pm \sqrt{(u_1^2 - u_R^2)(u_1^2 - u_D^2)} \right]$$

R-front: $u_1 \geq u_R$ weak –; strong +
 D-front: $u_1 \leq u_D$ weak +; strong –

- ▶ The solutions for $u_1 = u_R$ and $u_1 = u_D$ are called ***“R-critical”*** and ***“D-critical”***, respectively.

- Since c_2 exceeds c_1 by about one or two order of magnitude in an interstellar ionization front ($c_2 \gg c_1$),

$$u_R = c_2 + \sqrt{c_2^2 - c_1^2} \approx c_2 + c_2 \left(1 - \frac{1}{2} \frac{c_1^2}{c_2^2} - \frac{1}{8} \frac{c_1^4}{c_2^4} \right)$$

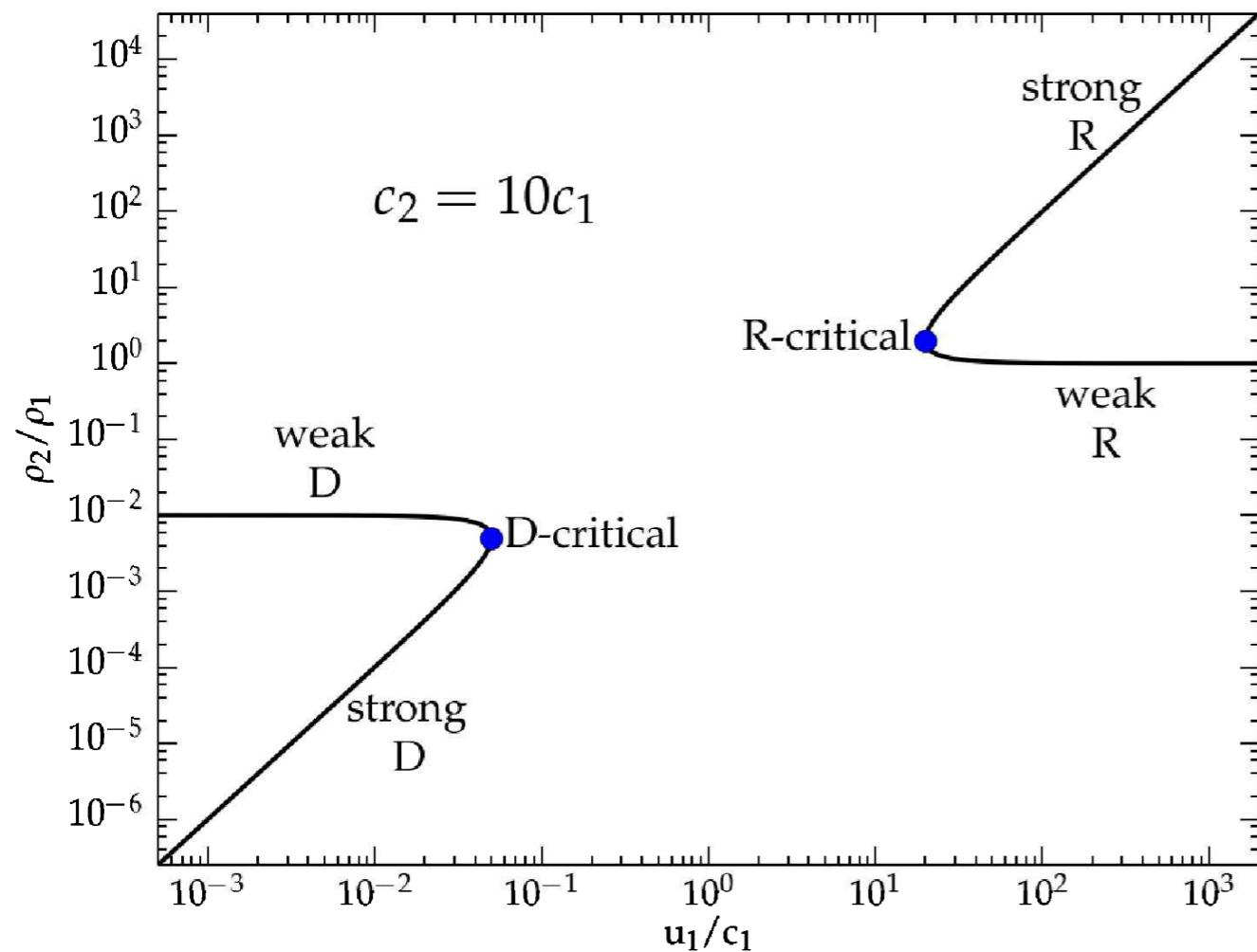
$$u_D = c_2 - \sqrt{c_2^2 - c_1^2} \approx c_2 - c_2 \left(1 - \frac{1}{2} \frac{c_1^2}{c_2^2} - \frac{1}{8} \frac{c_1^4}{c_2^4} \right)$$

$$u_R \approx 2c_2 \left(1 - \frac{1}{4} \frac{c_1^2}{c_2^2} \right) > c_2 > c_1 > u_D$$

$$u_D \approx \frac{1}{2} \frac{c_1^2}{c_2} \left(1 + \frac{1}{4} \frac{c_1^2}{c_2^2} \right) < c_1 < c_2 < u_R$$

- Approximate solutions:

R-critical	$\frac{\rho_2}{\rho_1} \approx 2 \left(1 - \frac{1}{4} \frac{c_1^2}{c_2^2} \right)$	for $u_1 = u_R$	$\left[\frac{\rho_2}{\rho_1} \right]_{R-\text{critical}} \approx 2$
D-critical	$\frac{\rho_2}{\rho_1} \approx \frac{1}{2} \frac{c_1^2}{c_2^2} \left(1 + \frac{1}{4} \frac{c_1^2}{c_2^2} \right)$	for $u_1 = u_D$	$\left[\frac{\rho_2}{\rho_1} \right]_{D-\text{critical}} \ll 1$
weak R-front	$\frac{\rho_2}{\rho_1} \approx 1 + \frac{c_2^2}{u_1^2}$	for $u_1 \gg u_R$	$1 \approx \left[\frac{\rho_2}{\rho_1} \right]_{\text{weak R}} < \left[\frac{\rho_2}{\rho_1} \right]_{\text{strong R}}$
strong R-front	$\frac{\rho_2}{\rho_1} \approx \frac{u_1^2}{c_2^2} - 1$		
weak D-front	$\frac{\rho_2}{\rho_1} \approx \frac{c_1^2}{c_2^2} - \frac{u_1^2}{c_1^2}$	for $u_1 \ll u_D$	$\left[\frac{\rho_2}{\rho_1} \right]_{\text{strong D}} < \left[\frac{\rho_2}{\rho_1} \right]_{\text{weak D}} \ll 1$
strong D-front	$\frac{\rho_2}{\rho_1} \approx \frac{u_1^2}{c_1^2} \left(1 + \frac{c_2^2}{c_1^4} u_1^2 \right)$		



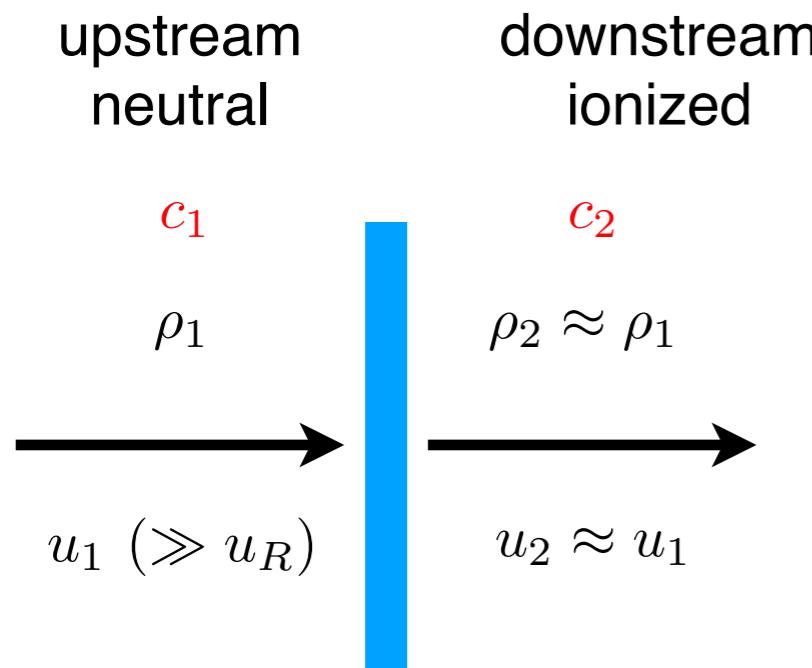
	$u_R \approx 2c_2, \quad u_D \approx \frac{1}{2} \frac{c_1^2}{c_2}$
R-critical	$\frac{\rho_2}{\rho_1} \approx 2$ for $u_1 = u_R$
D-critical	$\frac{\rho_2}{\rho_1} \approx \frac{1}{2} \frac{c_1^2}{c_2^2}$ for $u_1 = u_D$
strong R-front	$\frac{\rho_2}{\rho_1} \approx \frac{u_1^2}{c_2^2}$ for $u_1 \gg u_R$
weak R-front	$\frac{\rho_2}{\rho_1} \approx 1$
weak D-front	$\frac{\rho_2}{\rho_1} \approx \frac{c_1^2}{c_2^2} \Rightarrow \rho_1 c_1^2 \approx \rho_2 c_2^2$ for $u_1 \ll u_D$
strong D-front	$\frac{\rho_2}{\rho_1} \approx \frac{u_1^2}{c_1^2}$

Figure 4.11 [Ryden]

- We note that the four types are not all relevant to H II regions.
 - ▶ The strong R type means a lower density in the upstream (neutral gas). The strong R-type fronts are in fact unstable (Rayleigh-Taylor instability). In H II regions, the neutral gas has a higher density than the ionized gas. (or the same density at the initial stage).
 - ▶ The strong D type implies that the density in neutral gas increases forever when the ionization front slows down ($\rho_1 \rightarrow \infty$ as $u_1 \rightarrow 0$).
- **The fronts relevant to the H II regions are weak R-front and weak D-front.**

Evolution of Ionization Front

[1] Weak R front



We will assume that

$$c_1 = (kT_1/m_H)^{1/2}$$

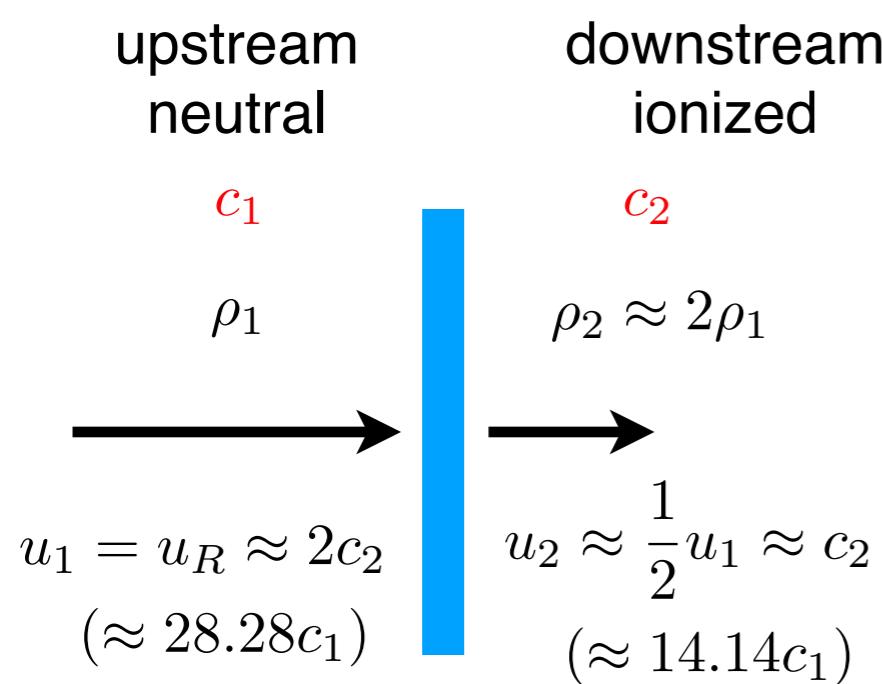
$$c_2 = (2kT_2/m_H)^{1/2} = (2T_2/T_1)^{1/2} c_1 = \sqrt{200} c_1$$

for $T_1 = 10^2 \text{ K}$, $T_2 = 10^4 \text{ K}$

(1) Weak R front:

- Initially, the photon flux J is very large. Thus, u_1 is very large, and the ionization front is initially a weak R-type front. The densities of neutral gas and ionized gas are nearly the same: $\rho_2/\rho_1 \approx 1$. (A weak R-type front compresses the gas only slightly.)
- As the ionization front expands, the flux of ionizing photons steadily decreases, and the propagation speed u_1 of the front slows down.

[2] R-critical front

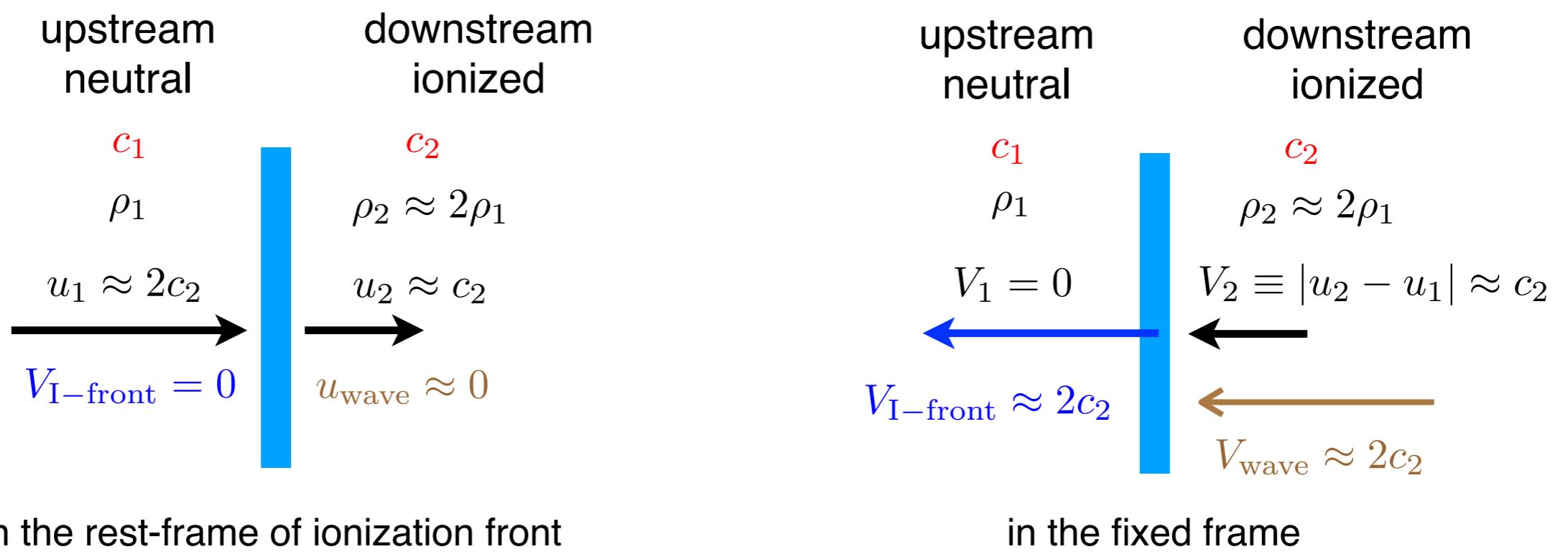


(2) R-critical front:

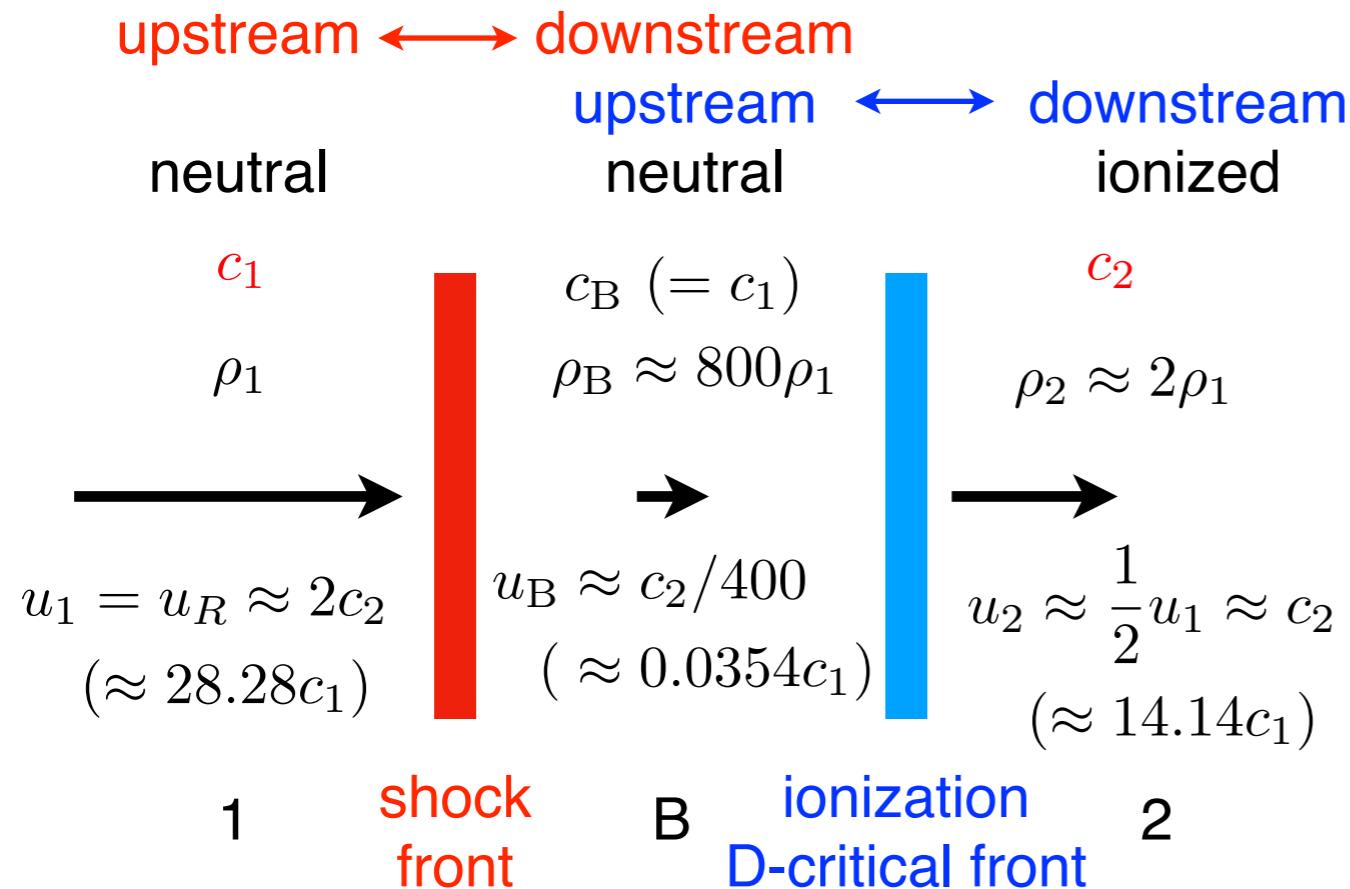
- Eventually, the speed drops to a value $u_1 = u_R \approx 2c_2$.
- At this point, the density ratio has risen to $\rho_2/\rho_1 \approx 2$.
- The speed of the ionized gas is $u_2 \approx (1/2)u_R \approx c_2$ relative to the ionization front, or $u_2 - u_1 \approx -c_2$ in a fixed frame of reference.
- As the ionization front slows down further, the R-type front can no longer exist.

● How does the evolution proceed once the ionization front becomes R-critical?

- When the R-critical condition is reached, the gas in the H II region just behind the front is moving at a speed equal to $c_2 \gg c_1$.
- This should derive a shock wave into the pre-ionization front gas. Before this point, the large pressure discrepancy between the H II region and the H I region ahead of it has no chance to act dynamically, because the ionization front races ahead with speed u_1 so much faster than a pressure wave can catch it.
- When the ionization front slows down to a speed $u_1 = u_R \approx 2c_2$, however, the pressure wave (moving at a speed c_2 on top of the speed $u_2 \approx c_2$ that the H II fluid itself moves) can catch up with the ionization front and overtake it.
- In doing so, the pressure wave will steepen into a shock wave, thereby compressing the atomic gas behind it into a denser state that the lagging ionization front then has to eat into.



[3] D-critical front



(3) D-critical front:

- As the ionization front slows down further, the R-type front can no longer exit. What happens next is that ***the R-critical ionization front splits into a pair of fronts (shock front + ionization front)***.
- **A leading shock front is followed by a D-critical ionization front.** The shock front is the boundary between two regions of gas with different density, pressure, and temperature, but no necessarily different ionization states. The shock front propagates with a supersonic speed relative to the gas in the upstream of the shock front.

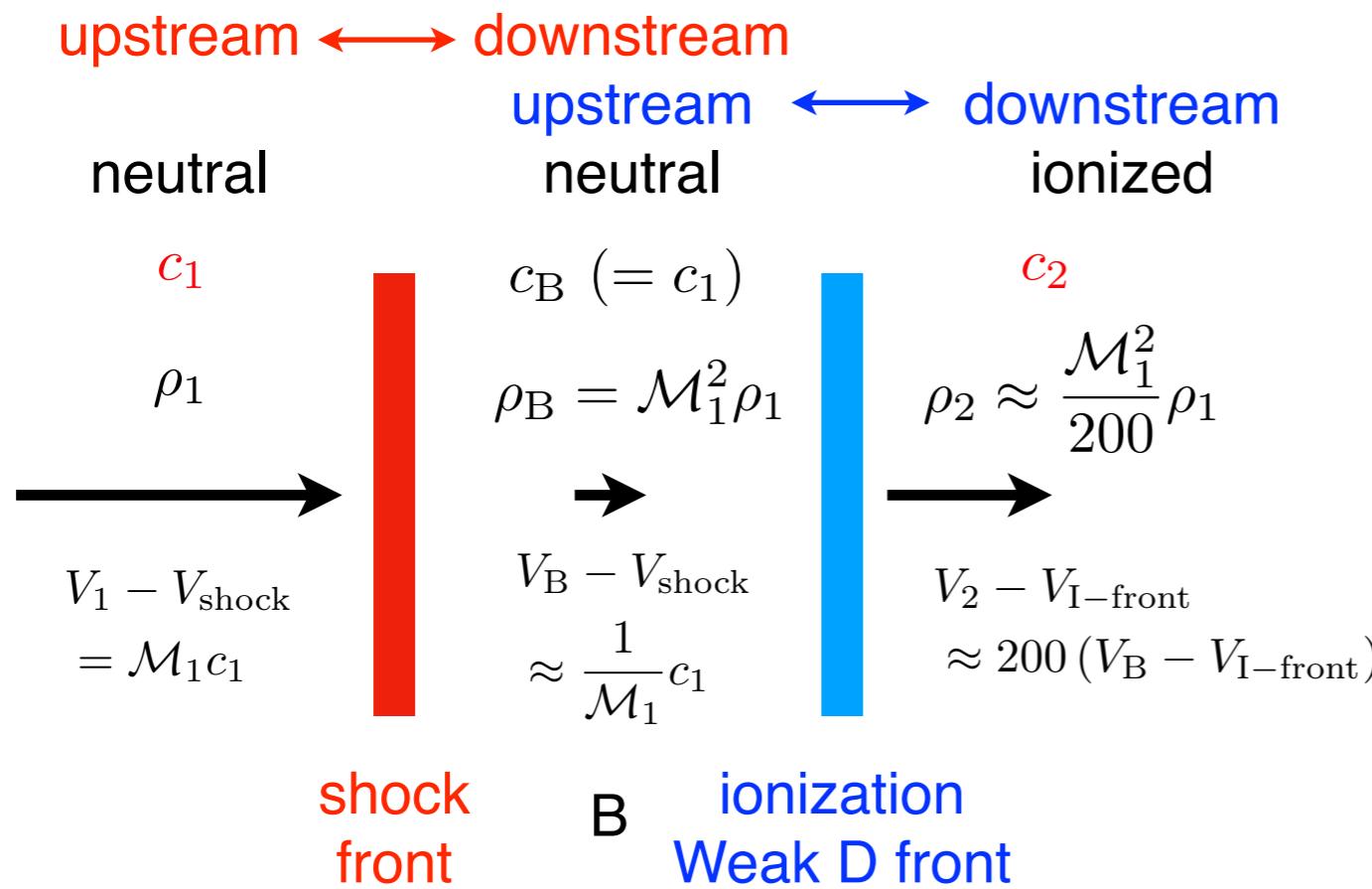
- We will assume ***an isothermal shock***. Then, the sound speed of the shocked region (B) must be $c_s = c_1$ (from the Rankin-Hugoniot jump condition). Then, using the condition for the D-critical, we obtain the density and speed of the shocked region (B):

$$\begin{aligned}\frac{\rho_2}{\rho_s} &\approx \frac{1}{2} \frac{c_1^2}{c_2^2} = \frac{1}{400} \\ \rho_s &\approx 400\rho_2 \approx 800\rho_1\end{aligned}\quad \begin{aligned}\frac{u_s}{u_2} &= \frac{\rho_2}{\rho_s} \\ u_s &\approx \frac{1}{400}u_2 \approx \frac{1}{400}c_2 \approx \frac{1}{\sqrt{800}}c_1 = 0.0354c_1\end{aligned}$$

$$\longrightarrow \begin{aligned}\rho_s &\approx 800\rho_1 \\ u_s &\approx 0.0354c_1 \\ (\mathcal{M}_1 &= \sqrt{800})\end{aligned}$$

- The shocked region (B) has a very high density, and is almost stationary relative to the ionization front. The velocities u_1, u_s, u_2 are measured in the rest-frame of I-front. The R-critical condition between 1 and 2 is still satisfied.

[4] Weak D front



(4) Weak D front:

- As the H II region expands still further, the leading shock front gradually weakens and the trailing D-critical front develops into a weak D-type front.
- Notice that the weakest of weak-D ionization fronts corresponds to the density discontinuity:

$$\frac{\rho_2}{\rho_B} = \frac{c_1^2}{c_2^2}$$

This is the condition for the static pressure equilibrium in isothermal gas,

$$\rho_2 c_2^2 = \rho_B c_1^2 \quad (P = \rho c_s^2)$$

the state that we expect for the final Strömgren sphere.

- The condition for the weak D-type front must be satisfied between the regions “B” and “2”.
- In addition to this condition, The shock jump condition should be satisfied between the regions “1” and “B”.
- However, notice that the velocities of the shock front and the ionization front can be different, in general.

$$V_{\text{shock}} \neq V_{\text{I-front}}$$

Intermediate States - expansion phase

- Assumptions:
 - The shocked gas layer is thin.
 - The ionization front follows the shock front and the expansion velocity of ionized sphere is approximately the same as the shock velocity.

$$V_{\text{I-front}} \approx V_{\text{shock}} \quad \frac{dR}{dt} = V_s$$

- Expansion:
 - The pressure behind a strong “isothermal” shock (high Mach number) is related to the shock velocity:

$$P_s = \rho_0 V_s^2 = n_0 m_H V_s^2$$

- Now assume that the pressure behind the shock wave is equal to the pressure of the ionized gas (pressure equilibrium).

$$P_i = 2n_i kT = n_i m_H c_i^2 \quad \left(c_i^2 \equiv \frac{2kT}{m_H} \right) \quad \text{for fully-ionized hydrogen gas}$$

- Then, the shock velocity is given by

$$P_s = P_i \rightarrow V_s^2 = \frac{n_i}{n_0} c_i^2 \rightarrow \frac{V_s^2}{c_i^2} = \frac{n_i}{n_0}$$

- We assume that the amount of fresh neutral gas to be ionized is very small. Then, the ionization balance for the region within R gives

$$Q_0 = \frac{4\pi}{3} R^3 n_i^2 \alpha_B \quad \rightarrow \quad R^3 = \frac{3Q_0}{4\pi n_i^2 \alpha_B} = R_s^3 \left(\frac{n_0}{n_i} \right)^2 \quad R_s = \text{Strömgren radius for the initial stage.}$$

- Combining with $\frac{V_s^2}{c_i^2} = \frac{n_i}{n_0}$, the equation for the expansion of the ionization front is

$$\begin{aligned} R^3 &= R_s^3 \left(\frac{c_i}{V_s} \right)^4 & \leftarrow \frac{dR}{dt} = V_s \\ \rho \equiv R/R_s, \quad \tau \equiv c_i t / R_s &\longrightarrow \quad \rho^3 \left(\frac{d\rho}{d\tau} \right)^4 = 1 & \rightarrow \boxed{\rho^{3/4} \frac{d\rho}{d\tau} = 1} \end{aligned}$$

- For a suitable boundary condition, we assume that the initial Strömgren sphere is set up at $\tau_0 = c_i t_0 / R_s$ (a very small fraction of the lifetime of the H II region):

$$R = R_s \text{ at } \tau = \tau_0$$

Then, the solution of the differential equation is

$$\rho = \left[1 + \frac{7}{4}(\tau - \tau_0) \right]^{4/7}$$

$$R = R_s \left(1 + \frac{7}{4} \frac{t - t_0}{R_s/c_i} \right)^{4/7}$$

-
- Expanding velocity is

$$\frac{dR}{dt} = c_i \left(1 + \frac{7}{4} \frac{t - t_0}{R_s/c_i} \right)^{-3/7}$$

- What is the time scale to reach the pressure equilibrium?

$$R(t_{\text{eq}}) = R_f \approx 34R_s$$

$$R_s \left(1 + \frac{7}{4} \frac{t_{\text{eq}}}{R_s/c_i} \right)^{4/7} \approx 34R_s$$

$$t_{\text{eq}} \approx 273 (R_s/c_i)$$

- The expanding velocity at this point is:

$$V_s = \frac{dR}{dt} = 0.71 c_i \quad \text{at} \quad t_{\text{eq}} = 273R_s/c_i$$

Timescales for typical HII region

- Let's examine the case of an O7V star with

$$Q_0 = 10^{49} \text{ s}^{-1}, \quad n_0 = 10^2 \text{ cm}^{-3}, \quad T = 10^4 \text{ K}$$

- Initial state: recombination time scale $t_{\text{rec}} = (n_0 \alpha_B)^{-1}$

$$R \approx R_s \text{ at } t = t_{\text{rec}}$$

$$R_s \approx 3 \text{ pc} (\approx 10^{19} \text{ cm})$$

$$t_{\text{rec}} \approx 1000 \text{ yr}$$

$$\begin{aligned} t &\lesssim t_{\text{rec}} \\ R(t) &= R_s \left(1 - e^{-t/t_{\text{rec}}}\right)^{1/3} \\ \frac{dR}{dt} &= \frac{R_s}{3t_{\text{rec}}} \frac{e^{-t/t_{\text{rec}}}}{(1 - e^{-t/t_{\text{rec}}})^{2/3}} \end{aligned}$$

- Expansion phase: expansion timescale $t_{\text{exp}} = R_s/c_i$

expansion velocity : $V_s \leq 0.65 c_i$ at $t \geq t_{\text{exp}}$

$$c_i \approx 10 \text{ km s}^{-1}$$

$$t_{\text{exp}} \approx 3 \times 10^5 \text{ yr} \rightarrow t_{\text{exp}} \approx 200 t_{\text{rec}}$$

$$\begin{aligned} t_{\text{rec}} &\lesssim t \lesssim t_{\text{eq}} \quad t_0 \approx t_{\text{rec}} \ll t_{\text{exp}} \\ R &= R_s \left(1 + \frac{7}{4} \frac{t - t_0}{R_s/c_i}\right)^{4/7} \\ \frac{dR}{dt} &= c_i \left(1 + \frac{7}{4} \frac{t - t_0}{R_s/c_i}\right)^{-3/7} \end{aligned}$$

- Final state: equilibrium timescale $t_{\text{eq}} \approx 273 R_s/c_i$ (from expansion phase model)

$$R = R_f \text{ at } t = t_{\text{eq}}$$

$$R_f/R_s \approx 34$$

$$t_{\text{eq}} \approx 10^8 \text{ yr} \rightarrow t_{\text{eq}} \approx 300 t_{\text{exp}}$$

Does the Stromgren sphere reach pressure equilibrium?

- Main-sequence lifetime of an ionizing star

$$t_{\text{MS}} \approx 10^{10} \left(\frac{M}{M_{\odot}} \right)^{-2} \text{ yr} \quad t_{\text{MS}} \approx 10^7 \text{ yr} \text{ for } M \approx 15M_{\odot}$$

- Size

- During the lifetime of an O star, which is less than 10 Myr, interstellar gas moving at 10 km/s will travel less than 100 pc, which is comparable with the diameter of the large H II regions.
- Thus, *before an H II region has expanded very far, its central energy source will be extinguished.*

- Time Scale:

- Main-sequence lifetime of an ionizing star is 10 times smaller than the time scale for the pressure equilibrium:

$$t_{\text{MS}} \approx 10^7 \text{ yr} \ll t_{\text{eq}} \approx 10^8 \text{ yr}$$

- *It is unlikely that the final state (pressure equilibrium) of H II region can be reached during lifetime of star.*

Gas Dynamics

- Gas Dynamics / Shock

Gas Dynamics - Energy Conservation

- ***Energy conservation***

- ▶ The first law of thermodynamics states that

Heat added in a system = Change in internal energy + Work done on surroundings

$$dQ = dU + PdV$$

- ▶ Internal energy (per particle) for ideal gas is

$U/N = \frac{3}{2}kT$ for monatomic gas (translation about 3 axes)

$U/N = \frac{5}{2}kT$ for diatomic gas (+rotation about 2 axes)

$U/N = 3kT$ for polyatomic gas (+rotation about 3 axes)

Here, N is the number of particles.

An ideal gas is a theoretical gas composed of many randomly moving point particles whose only interactions are perfectly elastic collisions (no viscosity or heat conduction).

- ▶ In general, the internal energy per particle is

$$U/N = \frac{f}{2}kT \quad (f = \text{degree of freedom})$$

At high temperature, molecules have access to an increasing number of vibrational degrees of freedom, as they start to bend and stretch.

- The ideal gas law (the equation of state) for a perfect Maxwellian distribution.

$$PV = NkT$$

$$P = \frac{N}{V}kT$$

- **Specific heat capacity** is the amount of **heat energy required to raise the temperature of a material per unit of mass**.

- ▶ specific heat capacity **at constant volume**:

$$c_V \equiv \frac{1}{M} \left(\frac{\partial Q}{\partial T} \right)_V = \frac{1}{M} \left(\frac{\partial U}{\partial T} \right)_V$$

$$c_V = \frac{f}{2} \frac{k}{m}$$

M = total mass

$m = M/N$ = mass per particle

$m = \mu m_H$

(μ = mean atomic weight per particle)

- ▶ specific heat capacity **at constant pressure**:

$$c_P \equiv \frac{1}{M} \left(\frac{\partial Q}{\partial T} \right)_P = \frac{1}{M} \left(\frac{\partial U}{\partial T} \right)_P + \frac{P}{M} \left(\frac{\partial V}{\partial T} \right)_P = \frac{1}{M} \frac{f}{2} Nk + \frac{P}{M} \frac{Nk}{P}$$

$$\therefore c_P = \frac{f+2}{2} \frac{k}{m} = c_V + \frac{k}{m}$$

- ▶ Ratio of specific heat capacities:

$$\gamma \equiv \frac{c_P}{c_V} = \frac{f+2}{f} = \frac{5}{3} \text{ for monatomic gas}$$

$$= \frac{7}{5} \text{ for diatomic gas}$$

$$= \frac{4}{3} \text{ for polyatomic gas}$$

γ is called the **adiabatic index**.

$$c_P > c_V$$

This inequality implies that when pressure is held constant, some of the added heat goes into PdV work instead of into internal energy.

- Energy Conservation - limiting cases

- **Adiabatic flow** - negligible heat transport (Internal energy is changed only by work).

$$dQ = dU + PdV = Mc_VdT + PdV$$

$$dQ = 0$$

$$\rightarrow PdV = -Mc_VdT$$

$$PV = NkT$$

$$\rightarrow VdP + PdV = NkdT$$

We combine two equations and eliminate dT term:

$$\begin{aligned} VdP + PdV &= -\frac{Nk}{Mc_V} PdV \\ &= -\frac{k}{m c_V} PdV \end{aligned}$$



$$\begin{aligned} VdP &= -\left(1 + \frac{k}{m c_V}\right) PdV \\ &= -\frac{1}{c_V} \left(c_V + \frac{k}{m}\right) PdV \\ &= -\gamma PdV \end{aligned}$$



$$\frac{dP}{P} = -\gamma \frac{dV}{V}$$

We can rewrite this in terms of density:

$$\rho V = M$$

$$\rightarrow \rho dV + Vd\rho = 0$$

$$\rightarrow \frac{d\rho}{\rho} = -\frac{dV}{V}$$

$$\longrightarrow \frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$

In summary,

$$\frac{dP}{P} = -\gamma \frac{dV}{V}$$

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$

$$P \propto \rho^\gamma$$

$$P \propto V^{-\gamma}$$

$$\rightarrow T \propto V^{-(\gamma-1)}$$

adiabatic heating/cooling

- ▶ **Isothermal flow** - extremely efficient cooling (heat transport).

heat transport timescale << dynamic timescale

This implies the balance between heating and cooling, hence a constant temperature.

From the ideal gas law,

$$P = \frac{N}{V} kT = \rho \frac{kT}{m}$$

$$\begin{aligned} P &\propto \rho \\ P &\propto V^{-1} \end{aligned}$$

- ▶ **In general**, we have

$$\begin{aligned} P &\propto \rho^\gamma \\ P &\propto V^{-\gamma} \end{aligned}$$

$(\gamma = 1$ for isothermal gas)

A gas that has an equation of state with this power-law form is called a **polytope**, from the Greek polytropos, meaning “turning many ways” or “versatile.”

(A polytope should not be confused with a polytrope, which is the n-dimensional generalization of a 2D polygon and 3D polyhedron.)

- **Specific internal energy** of the gas (*per unit mass*):

$$\begin{aligned}\epsilon &\equiv U/M \\ U/N &= \frac{f}{2}kT\end{aligned}\longrightarrow \epsilon = \frac{f}{2} \frac{kT}{m} \quad \text{or} \quad \epsilon = \frac{1}{\gamma-1} \frac{kT}{m} = \frac{1}{\gamma-1} \frac{P}{\rho}$$

- **Total Energy (per unit volume):**

► **Internal energy per unit volume:**

$$\mathcal{E}_{\text{int}} = \rho\epsilon = \frac{1}{\gamma-1}P$$

► **Kinetic energy due to bulk motion, per unit volume:**

$$\mathcal{E}_{\text{kin}} = \rho \frac{u^2}{2}$$

► **Work on unit volume:**

$$\mathcal{E}_{\text{mech}} = \frac{PdV}{dV} = P$$

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_{\text{int}} + \mathcal{E}_{\text{kin}} + \mathcal{E}_{\text{mech}} \\ &= \rho \left(\frac{u^2}{2} + \epsilon \right) + P\end{aligned}\longrightarrow \mathcal{E} = \rho \frac{u^2}{2} + \frac{\gamma}{\gamma-1}P$$

- **Energy conservation:**

$$\frac{\partial \mathcal{E}}{\partial t} = -\frac{\partial(u\mathcal{E})}{\partial x}$$

$$\frac{\partial}{\partial t} \left(\rho \frac{u^2}{2} + \frac{\gamma}{\gamma-1}P \right) = -\frac{\partial}{\partial x} \left[u \left(\rho \frac{u^2}{2} + \frac{\gamma}{\gamma-1}P \right) \right]$$

Sound Wave

- Suppose that we are surrounded by an ideal gas with a plane parallel symmetry:
 - We consider a region where the gas has initially a uniform density, pressure, and no bulk velocity: $\rho_0, P_0, u_0 = 0$

In the uniform gas, we introduce small perturbations of the form:

$$\begin{array}{ll} \rho(x, t) = \rho_0 + \rho_1(x, t) & P_1 = P - P_0 \\ u(x, t) = u_1(x, t) & \propto (\rho_0 + \rho_1)^\gamma - \rho_0^\gamma \\ P(x, t) = P_0 + P_1(x, t) & \propto \gamma \rho_0^{\gamma-1} \rho_1 \end{array} \longrightarrow \quad \longrightarrow \quad P_1 = \frac{\gamma P_0}{\rho_0} \rho_1$$

We obtain:

$$\begin{array}{ccc} \frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} & \rightarrow & \frac{\partial \rho_1}{\partial t} = -\rho_0 \frac{\partial u_1}{\partial x} \\ \rho \frac{\partial u}{\partial t} = -\rho u \frac{\partial u}{\partial x} - \frac{\partial P}{\partial x} & & \rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial P_1}{\partial x} = -\frac{\gamma P_0}{\rho_0} \frac{\partial \rho_1}{\partial x} \end{array} \quad \downarrow \quad \rightarrow \quad \boxed{\frac{\partial^2 \rho_1}{\partial t^2} = -\frac{\gamma P_0}{\rho_0} \frac{\partial^2 \rho_1}{\partial x^2}}$$

- The resulting equation represents a **sound wave (acoustic wave)** with a constant sound speed:

$$c_s = \left(\frac{\gamma P}{\rho} \right)^{1/2} = \left(\frac{\gamma k T}{m} \right)^{1/2}$$

$$c_s \propto \rho^{(\gamma-1)/2}$$

For $\gamma > 1$ sound travels more rapidly in a denser gas.

-
- The sound speed is of the same order as the mean thermal velocity:

$$c_s = 1.2 \text{ km s}^{-1} \left(\frac{\gamma}{5/3} \right)^{1/2} \left(\frac{m}{m_p} \right)^{-1/2} \left(\frac{T}{100 \text{ K}} \right)^{1/2}$$

$(m_p = \text{proton mass})$

- **Sound crossing time:**

- ▶ sound crossing time = time it takes for a signal to cross a region of size L :

$$t_{\text{cross}} = L/c_s$$

- ▶ A small pressure gradient tends to be smoothed out within the sound crossing time. Generally, when a stationary gas is disturbed, the resultant changes in velocity, density, pressure, and temperature are communicated downstream at the sound speed.

Fast changes occurring on timescales $\ll t_{\text{cross}}$ will survive, and a shock front forms.

Slow changes occurring on timescales $\gg t_{\text{cross}}$ will be damped.

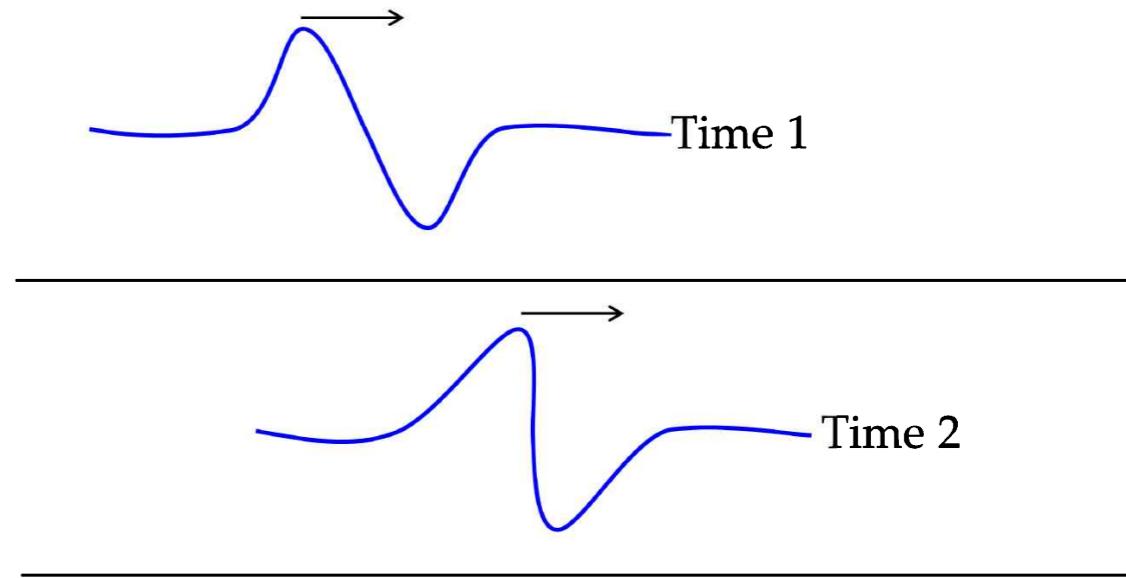
- **Mach number** = gas velocity / sound speed

$$\mathcal{M} \equiv u/c_s$$

$\mathcal{M} > 1$	supersonic
$\mathcal{M} < 1$	subsonic

Shock

- Shock
 - A low-amplitude sound wave traveling through a medium will be adiabatic; that is it will not increase the entropy of the gas through which it passes.
 - For an adiabatic process, the equation of state for the gas is
$$c_s \propto \rho^{(\gamma-1)/2}$$
- Thus, for $\gamma > 1$, sound travels more rapidly in a denser gas.
- ***For a supersonic gas, the motion itself is faster than the speed of communication, and instead of a smooth transition, the physical quantities (density, pressure, and temperature) undergo a sudden change in values over a small distance.*** This phenomenon is referred to as a **shock**.
- We define the shock front as the region over which the velocity, density, and pressure of the gas undergo sudden changes. The shock front is a layer whose thickness is comparable to the mean free path between particle collisions.
- The ordinary sound that we hear every day will not, in practice, steepen into shocks.
- However, high amplitude pressure fluctuations will rapidly steepen into shocks.



Shock Front

- Jump condition (*Rankine-Hugoniot conditions*)

- Let

ρ = mass density, T = temperature,

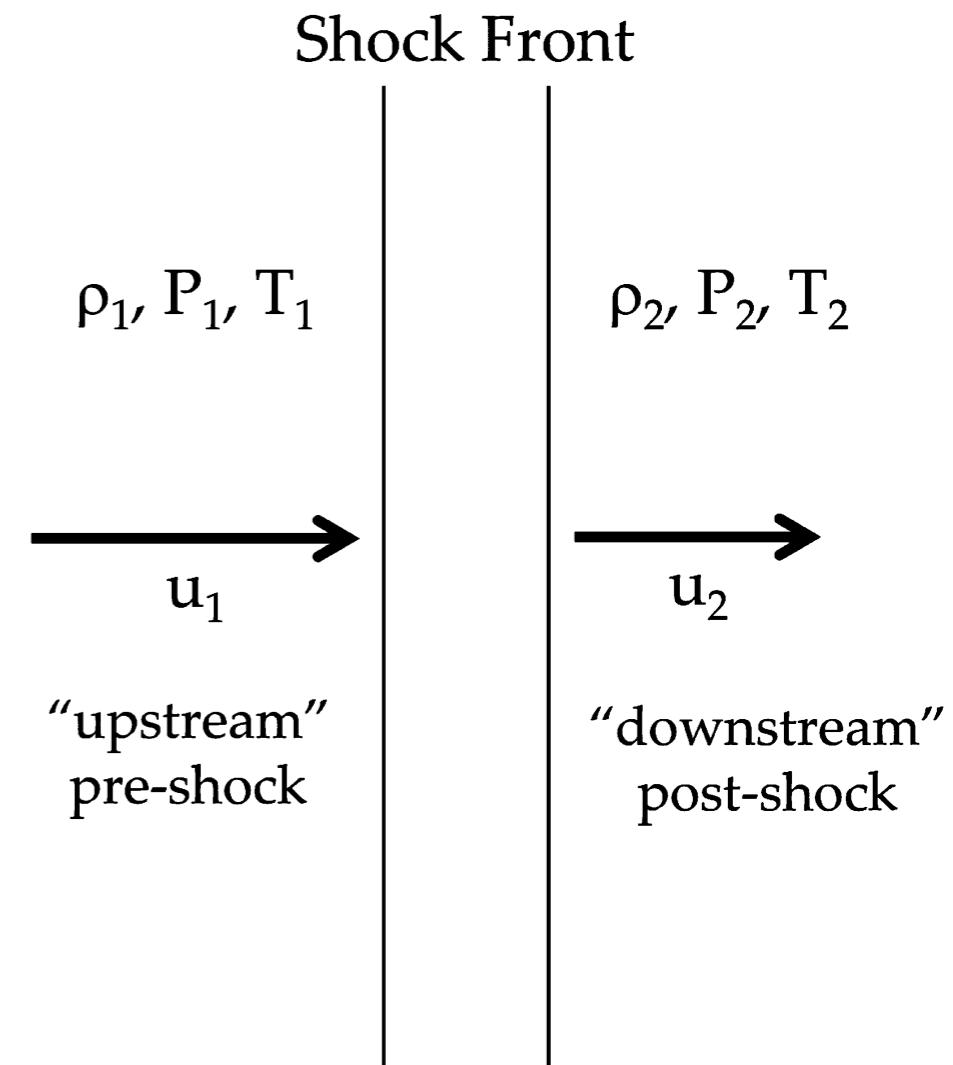
m = mean molecular mass

- If a patch is small compared to the shock front's radius of curvature, then we can treat the shock front as if it has **plane parallel** symmetry.

- ***It is convenient to use a frame of reference in which the shock front is stationary.***

- Let us consider a shock propagating with velocity V_s into a gas that is previously at rest. In the frame of reference of the shock, the gas in the pre-shock region is approaching at a velocity of $-V_s$.

- In this frame, the bulk velocity $u_1 = -V_s$ of the pre-shock (upstream) gas toward the shock front. The bulk velocity u_2 of the post-shock (downstream) gas points away from the shock front.



Plane parallel steady-state shock,
in the reference frame of the shock
front.

-
- Let's consider a steady state solution.
 - The gas properties immediately before being shocked (“1”) and immediately after being shocked (“2”) are obtained from the conservation laws:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

$$u_1 \left(\rho_1 \frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} P_1 \right) = u_2 \left(\rho_2 \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} P_2 \right)$$

Dividing the third equation with the first equation:

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}$$

In summary,

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}$$

Here, we assume that an adiabatic index is the same on both sides of the shock front.

- From the three equations, we should be able to derive the changes, ρ_2/ρ_1 , u_2/u_1 , and P_2/P_1 across the shock.

It is convenient to use a dimensionless number, the Mach number of the upstream:

$$\mathcal{M}_1 = u_1/c_1, \quad c_1^2 = \frac{\gamma P_1}{\rho_1} \quad \rightarrow \quad P_1 = \frac{u_1^2 \rho_1}{\gamma \mathcal{M}_1^2}$$

(1) To find the equation for densities:

$$\begin{aligned} \rho_1 u_1^2 + P_1 &= \rho_2 u_2^2 + P_2 \\ \rho_1 u_1 = \rho_2 u_2 \text{ and } P_1 = \frac{u_1^2 \rho_1}{\gamma \mathcal{M}_1^2} &\rightarrow \rho_1 u_1^2 + u_1^2 \frac{\rho_1}{\gamma \mathcal{M}_1^2} = \frac{(\rho_1 u_1)^2}{\rho_2} + P_2 \\ &\rightarrow P_2 = \rho_1 u_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right) \end{aligned}$$

Inserting these relations into the energy conservation equation:

$$\begin{aligned} \frac{u_1^2}{2} + \frac{\gamma}{\gamma-1} \frac{P_1}{\rho_1} &= \frac{u_2^2}{2} + \frac{\gamma}{\gamma-1} \frac{P_2}{\rho_2} \\ \rightarrow \frac{u_1^2}{2} + \frac{1}{\gamma-1} \frac{u_1^2}{\mathcal{M}_1^2} &= \frac{1}{2} \left(\frac{\rho_1 u_1}{\rho_2} \right)^2 + \frac{\gamma}{\gamma-1} \frac{\rho_1 u_1^2}{\rho_2} \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right) \\ \rightarrow \frac{1}{2} + \frac{1}{\gamma-1} \frac{1}{\mathcal{M}_1^2} &= \frac{1}{2} \left(\frac{\rho_1}{\rho_2} \right)^2 + \frac{\gamma}{\gamma-1} \frac{\rho_1}{\rho_2} \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right) \end{aligned}$$



$$ax^2 + bx - c = 0$$

where $x = \frac{\rho_1}{\rho_2}$

$$a = \frac{1}{2} - \frac{\gamma}{\gamma-1}$$

$$b = \frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_1^2}$$

$$c = \frac{1}{2} + \frac{1}{(\gamma-1)\mathcal{M}_1^2}$$

$$x = \frac{b^2 \pm \sqrt{b^2 + 4ac}}{2a}$$

$$\frac{\rho_1}{\rho_2} = \frac{-\left[\frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_0^2}\right] \pm \frac{\mathcal{M}_1^2 - 1}{\mathcal{M}_1^2(\gamma-1)}}{1 - \frac{2\gamma}{\gamma-1}}$$

→

$$\frac{\rho_1}{\rho_2} = 1 \quad \text{or} \quad \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2}$$

$$\therefore \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}$$

(2) Now, we obtain the equation for pressures:

Divide the following equation

$$P_2 = \rho_1 u_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right)$$

with this

$$P_1 = \frac{u_1^2 \rho_1}{\gamma \mathcal{M}_1^2}$$



$$\frac{P_2}{P_1} = \gamma \mathcal{M}_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{\rho_1}{\rho_2} \right)$$

$$= \gamma \mathcal{M}_1^2 \left(1 + \frac{1}{\gamma \mathcal{M}_1^2} - \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2} \right)$$

$$\therefore \frac{P_2}{P_1} = \frac{2\gamma \mathcal{M}_1^2 - (\gamma-1)}{\gamma+1}$$

(3) Using the ideal gas law:

$$P = \frac{\rho k T}{m} \quad \rightarrow \quad \frac{T_2}{T_1} = \frac{\rho_1}{\rho_2} \frac{P_2}{P_1}$$

Using the equations for densities and pressures:

$$\therefore \frac{T_2}{T_1} = \frac{[(\gamma-1)\mathcal{M}_1^2 + 2][2\gamma \mathcal{M}_1^2 - (\gamma-1)]}{(\gamma+1)^2 \mathcal{M}_1^2}$$

In summary, we obtain the jump conditions:

$$\begin{aligned}\frac{\rho_2}{\rho_1} &= \frac{(\gamma + 1)\mathcal{M}_1^2}{(\gamma - 1)\mathcal{M}_1^2 + 2} = \frac{u_1}{u_2} \\ \frac{P_2}{P_1} &= \frac{2\gamma\mathcal{M}_1^2 - (\gamma - 1)}{\gamma + 1} \\ \frac{T_2}{T_1} &= \frac{[(\gamma - 1)\mathcal{M}_1^2 + 2][2\gamma\mathcal{M}_1^2 - (\gamma - 1)]}{(\gamma + 1)^2\mathcal{M}_1^2}\end{aligned}$$

In the **lab frame**, let V_s = shock velocity, v_1, v_2 = gas velocities in upstream (pre-shock) and downstream (post-shock), respectively ($v_1 = 0$) .

Using $u_1 = -V_s$ and $u_2 = v_2 - V_s$, we have

$$\frac{-V_s}{v_2 - V_s} = \frac{(\gamma + 1)\mathcal{M}_1^2}{(\gamma - 1)\mathcal{M}_1^2 + 2}$$

Downstream velocity in the lab frame:

$$v_2 = \frac{2(\mathcal{M}_1^2 - 1)}{(\gamma + 1)\mathcal{M}_1^2} V_s$$

Note a typo in Equation (16.12) of Kwok's book.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)\mathcal{M}_1^2}{(\gamma - 1)\mathcal{M}_1^2 + 2} = \frac{u_1}{u_2}$$

$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma - 1)}{\gamma + 1}$$

$$\frac{T_2}{T_1} = \frac{[(\gamma - 1)\mathcal{M}_1^2 + 2][2\gamma\mathcal{M}_1^2 - (\gamma - 1)]}{(\gamma + 1)^2\mathcal{M}_1^2}$$

For a strong shock: $\mathcal{M}_1 \gg 1$

$$P_2 \approx \frac{2\gamma\mathcal{M}_1^2}{\gamma + 1} P_1 \xrightarrow{P_1 = c_1^2 \frac{\rho_1}{\gamma}} \frac{2\gamma(u_1/c_1)^2}{\gamma + 1} c_1^2 \frac{\rho_1}{\gamma}$$

$$T_2 \approx \frac{2(\gamma - 1)\gamma}{(\gamma + 1)^2} \mathcal{M}_1^2 T_1 = \frac{2(\gamma - 1)\gamma}{(\gamma + 1)^2} \left(\frac{u_1}{c_1}\right)^2 T_1$$

speed of the downstream
in the laboratory frame:

$$\frac{\rho_2}{\rho_1} \simeq \frac{\gamma + 1}{\gamma - 1}$$

$$\frac{u_2}{u_1} \simeq \frac{\gamma - 1}{\gamma + 1}$$

$$P_2 \simeq \frac{2}{\gamma + 1} \rho_1 u_1^2$$

$$T_2 \simeq \frac{2(\gamma - 1)}{(\gamma + 1)^2} \frac{m}{k} u_1^2$$

$$v_2 \simeq \frac{2}{(\gamma + 1)} V_s$$

monatomic
gas: $\gamma = 5/3$

$$\frac{\rho_2}{\rho_1} \simeq 4$$

$$\frac{u_2}{u_1} \simeq \frac{1}{4}$$

$$P_2 \simeq \frac{3}{4} \rho_1 u_1^2$$

$$T_2 \simeq \frac{3}{16} \frac{m}{k} u_1^2$$

$$v_2 \simeq \frac{3}{4} V_s$$

For an isothermal shock: $\gamma = 1$

speed of the downstream
in the laboratory frame:

$$\frac{\rho_2}{\rho_1} = \mathcal{M}_1^2 = \frac{u_1}{u_2}$$

$$P_2 = \mathcal{M}_1^2 P_1 = \rho_1 u_1^2$$

$$T_2 = T_1$$

$$v_2 = \left(1 - \frac{1}{\mathcal{M}_1^2}\right) V_s$$

$$u_1 u_2 = c_1^2$$

$$c_2 = c_1$$

- Consider a strong shock
 - **No matter how strong the shock is, the gas can only be compressed by a factor of at most 4:**

$$\frac{\rho_2}{\rho_1} \approx 4 \quad \text{for } \gamma = 5/3$$

(monatomic gas)

$$P_2 \approx \frac{3}{4} \rho_1 u_1^2$$

$$T_2 \approx \frac{3}{16} \frac{m}{k} u_1^2$$

Note that the mean molecular mass (mass per particle) is

$$m = \frac{1.4m_{\text{H}}}{1.1} = 1.273m_{\text{H}} \quad \text{for neutral gas}$$

$$m = \frac{1.4m_{\text{H}}}{2.3} = 0.609m_{\text{H}} \quad \text{for ionized gas}$$

$n \simeq 2.3n_{\text{H}}$

for ionized gas,
one electron from an ionized hydrogen
two electrons from a doubly-ionized helium.

- In the lab frame, V_s = shock velocity, v_1 , v_2 = gas velocities in upstream and downstream, respectively.

$$u_1 = v_1 - V_s = -V_s \quad (v_1 = 0)$$

$$u_2 = v_2 - V_s$$

Then, the post-shock velocity is

$$\frac{u_2}{u_1} = \frac{v_s - V_s}{-V_s} = \frac{1}{4} \quad \Rightarrow \quad v_2 = \frac{3}{4} V_s$$

Hence, **the post-shock moves in the same direction as the shock front with a velocity of 3/4 of the shock velocity.**

- Then, the post-shock pressure, temperature, specific internal energy, and specific kinetic energy are, respectively,

$$P_2 = \frac{3}{4} \rho_1 V_s^2$$

$$T_2 = \frac{3m}{16k} V_s^2$$

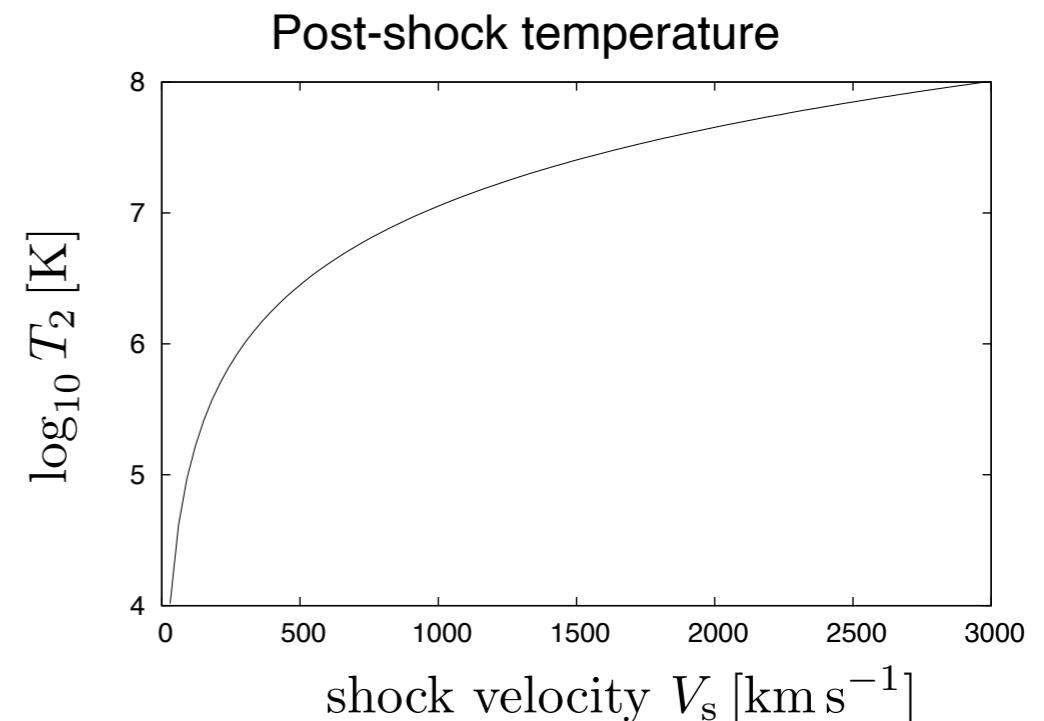
for $\gamma = 5/3$

$$\frac{\mathcal{E}_{\text{int},2}}{\rho_2} = \epsilon_2 = \frac{3}{2} \frac{P_2}{\rho_2} = \frac{3}{2} \frac{(3/4)\rho_1 V_s^2}{4\rho_1}$$

$$\frac{\mathcal{E}_{\text{kin},2}}{\rho_2} = \frac{1}{2} v_2^2$$

$$\frac{\mathcal{E}_{\text{int},2}}{\rho_2} = \frac{9}{32} V_s^2$$

$$\frac{\mathcal{E}_{\text{kin},2}}{\rho_2} = \frac{9}{32} V_s^2$$



- A strong shock can produce very high pressures and temperatures. An interstellar shock front with propagation speed $V_s \sim 1000 \text{ km s}^{-1}$ (typical for a supernova shock wave) produces shock heated gas with

$$T_2 \approx 1.38 \times 10^7 \text{ K} \left(\frac{m}{0.609m_H} \right) \left(\frac{V_s}{1000 \text{ km s}^{-1}} \right)^2$$

or $T_2 \approx 1.38 \times 10^5 \text{ K} \left(\frac{m}{0.609m_H} \right) \left(\frac{V_s}{100 \text{ km s}^{-1}} \right)^2$

assuming the shocks gas is fully ionized hydrogen.

- In general, shock fronts convert supersonic gas into subsonic gas in the shock's frame of reference. Shocks increase density, pressure, and temperature, and decrease bulk velocity relative to the shock front. *Shocks act as entropy generators.*

Hot Ionized Medium

- Hot Gas Cooling
- Supernova Remnant
- Local Hot Bubble

General Properties of the HIM

- Hot Ionized Medium, coronal gas
 - About half the volume of the ISM in our Galaxy is occupied by the HIM.
 - Temperature $\sim 10^6$ K.
 - Typical ion number density $n \sim 0.004 \text{ cm}^{-3}$
 - It provides only $\sim 0.2\%$ of the mass of the ISM, despite being the largest contributor to its volume.
 - The HIM is hot because it has been heated by shock fronts that result from supernova explosions.
 - ***We live in the “Local Bubble”, which is ~ 100 pc in size. The Local Bubble is thought to have been blown by a supernova that went off ~ 10 Myr ago.***

See https://scivis.mpi-a.de/projects/dustribution/Eos_Cloud/interactive.html and
https://scivis.mpi-a.de/projects/dustribution/Eos_Cloud/video.mp4

Collisional Ionization Equilibrium

- CIE
 - CIE assumes that the plasma is in a steady state, and that *collisional ionization, charge exchange, radiative recombination, and dielectronic recombination* are the only processes altering the ionization balance.
 - ▶ Note that the reverse process to collisional ionization is a three-body recombination, which is unlikely to occur.
 - **The ionization fractions for each element depend only on the gas temperature,** with no dependence on the gas density.
- Ionization fraction
 - For hydrogen, the balance equation is : ionization rate = recombination rate

$$n_e n(\mathrm{H}^0) k_{\mathrm{ci}, \mathrm{H}} = n_e n(\mathrm{H}^+) \alpha_{\mathrm{A}, \mathrm{H}} \quad n(\mathrm{H}^0) + n(\mathrm{H}^+) = n(\mathrm{H})$$

- The rate coefficients for collisional ionization and radiative recombination are:

$$k_{\mathrm{ci}, \mathrm{H}} = 5.849 \times 10^{-9} T_4^{1/2} e^{-15.782/T_4} \text{ [cm}^3 \text{s}^{-1}\text{]}$$

$$\begin{aligned} \alpha_{\mathrm{A}, \mathrm{H}} &= 4.13 \times 10^{-13} T_4^{-0.7131 - 0.0115 \ln T_4} \text{ [cm}^3 \text{s}^{-1}\text{]} \quad \text{for } 30 \text{ K} < T < 3 \times 10^4 \text{ K} \\ &= 5 \times 10^{-16} T_7^{-1.5} \quad \text{for } T > 10^6 \text{ K} \end{aligned} \quad [\text{from Draine}]$$

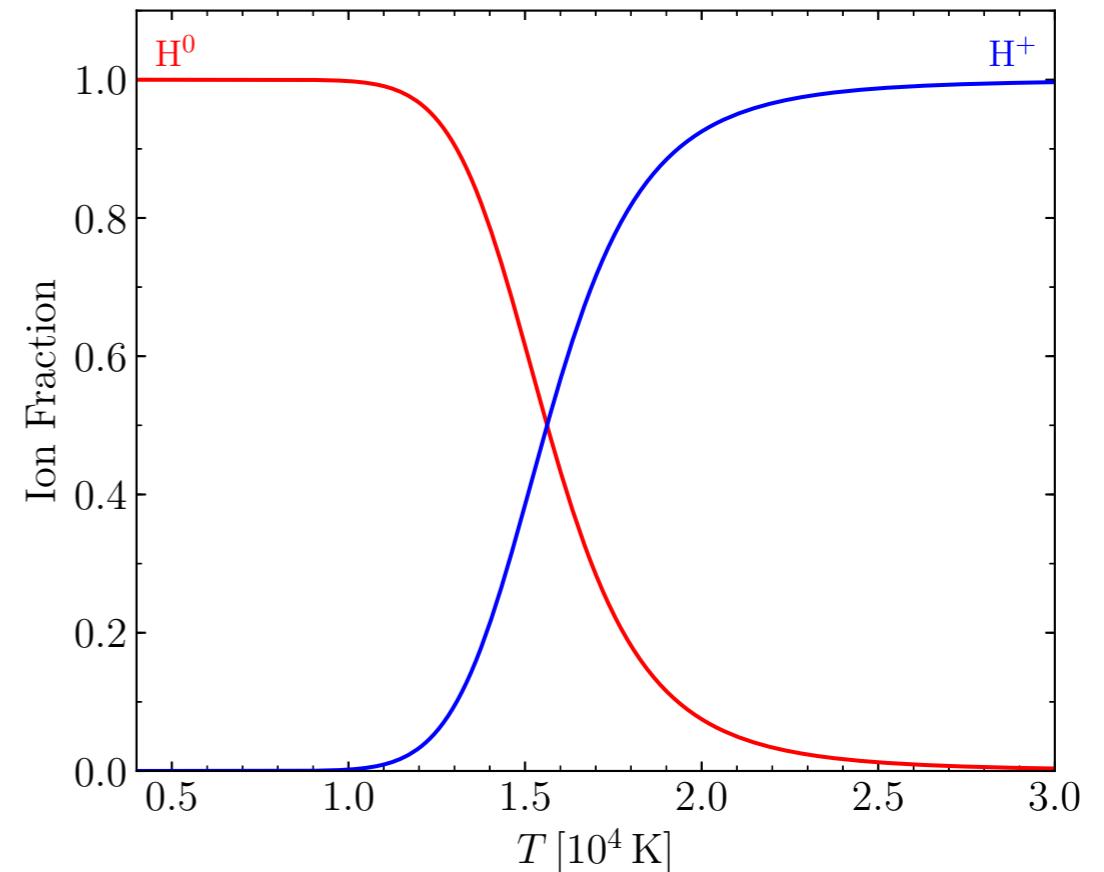
$$\alpha_{\mathrm{A}, \mathrm{H}} = 1.269 \times 10^{-13} \text{ [cm}^3 \text{s}^{-1}\text{]} \frac{x^{1.503}}{(1 + (x/0.522)^{0.47})^{1.923}} \quad \text{where } x = 2 \times 157807 \text{ K}/T \quad [\text{Hui \& Gendin 1997, MNRAS}]$$

- The ionization fraction is

$$\begin{aligned} x &\equiv \frac{n(\text{H}^+)}{n(\text{H}^0) + n(\text{H}^+)} \\ &= \frac{k_{\text{ci},\text{H}}}{k_{\text{ci},\text{H}} + \alpha_{\text{A},\text{H}}} \end{aligned}$$

- The ion fractions are

$$\begin{aligned} x &\approx 0.002 \quad \text{at } T = 10^4 \text{ K} \\ 1 - x &\approx 3 \times 10^{-7} \quad \text{at } T = 10^6 \text{ K} \end{aligned}$$



H II regions with $T = 10^4$ K are photoionized by UV photons from hot stars.

Hydrogen gas with $T = 10^6$ K is almost entirely collisionally ionized.

- For Helium, the balance equations are:

$$n(\text{He}^+) \alpha_{10} = n(\text{He}^0) k_{01}$$

$$n(\text{He}^+) k_{12} = n(\text{He}^{2+}) \alpha_{21}$$

$$n(\text{He}) = n(\text{He}^0) + n(\text{He}^+) + n(\text{He}^{2+})$$

Here, ij indicates $X^{i+} \rightarrow X^{j+}$.

- The rate coefficients are

$$k_{01} = 2.39 \times 10^{-11} T^{1/2} e^{-285,335/T}$$

from R. Cen (1992, ApJS)

$$k_{12} = 5.68 \times 10^{-12} T^{1/2} e^{-631,515/T}$$

$$\alpha_{10} = 1.50 \times 10^{-10} T^{-0.6353} \text{ radiative recombination}$$

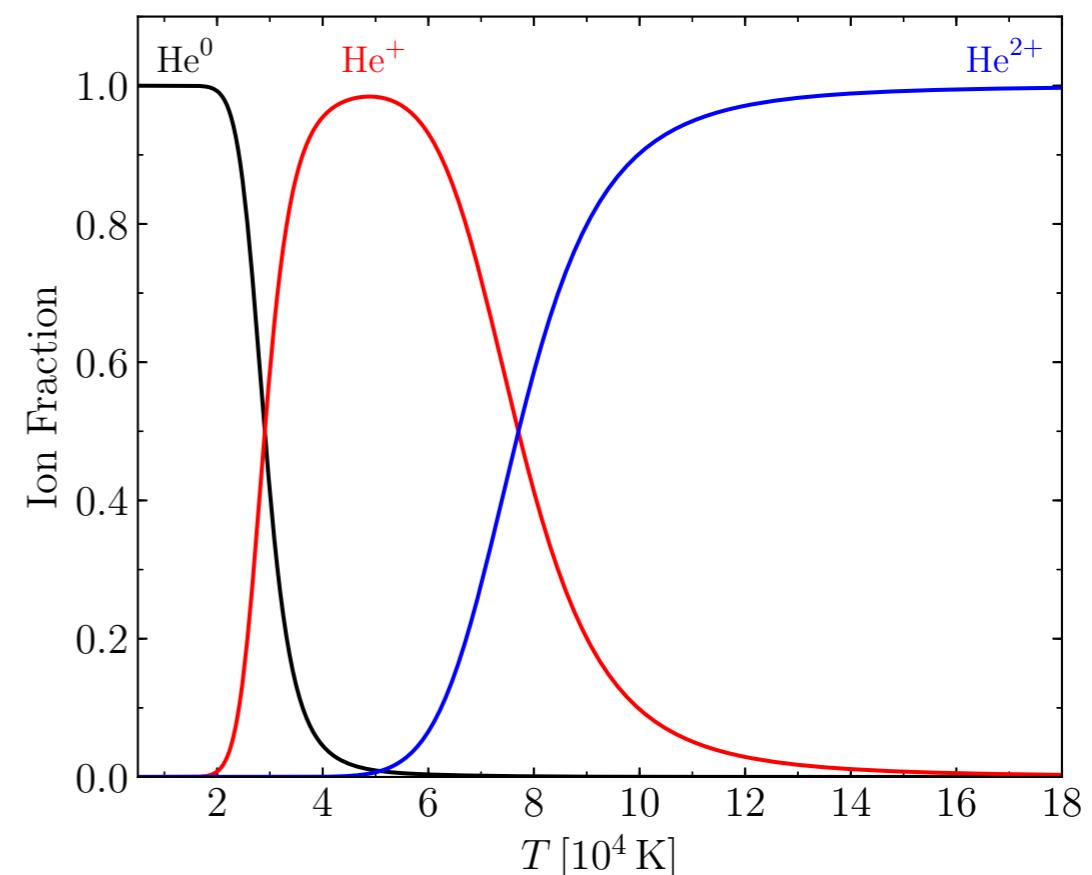
$$+ 1.9 \times 10^{-3} T^{-1.5} e^{-470,000/T} (1 + 0.3e^{-94,000/T}) \text{ dielectronic recombination (but not significant)}$$

$$\alpha_{21} = 3.36 \times 10^{-10} T^{-1/2} T_3^{-0.2} / (1 + T_6^{0.7})$$

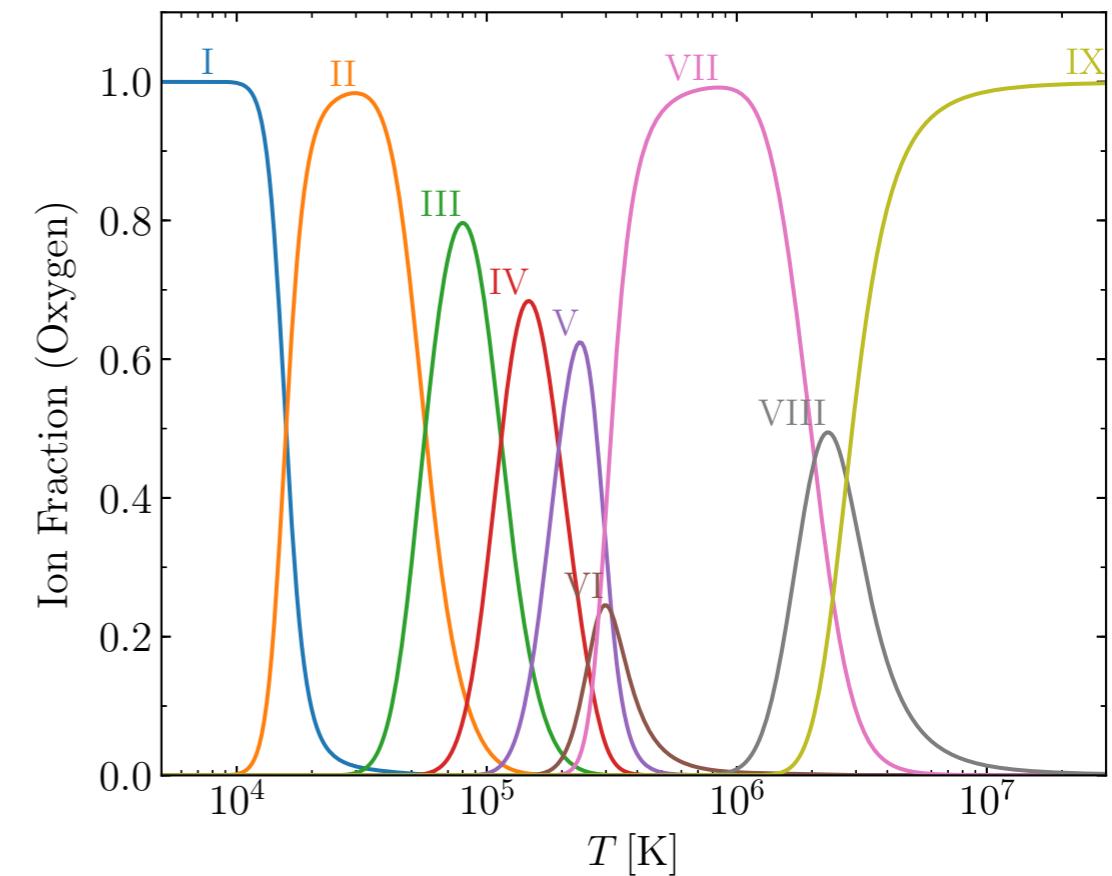
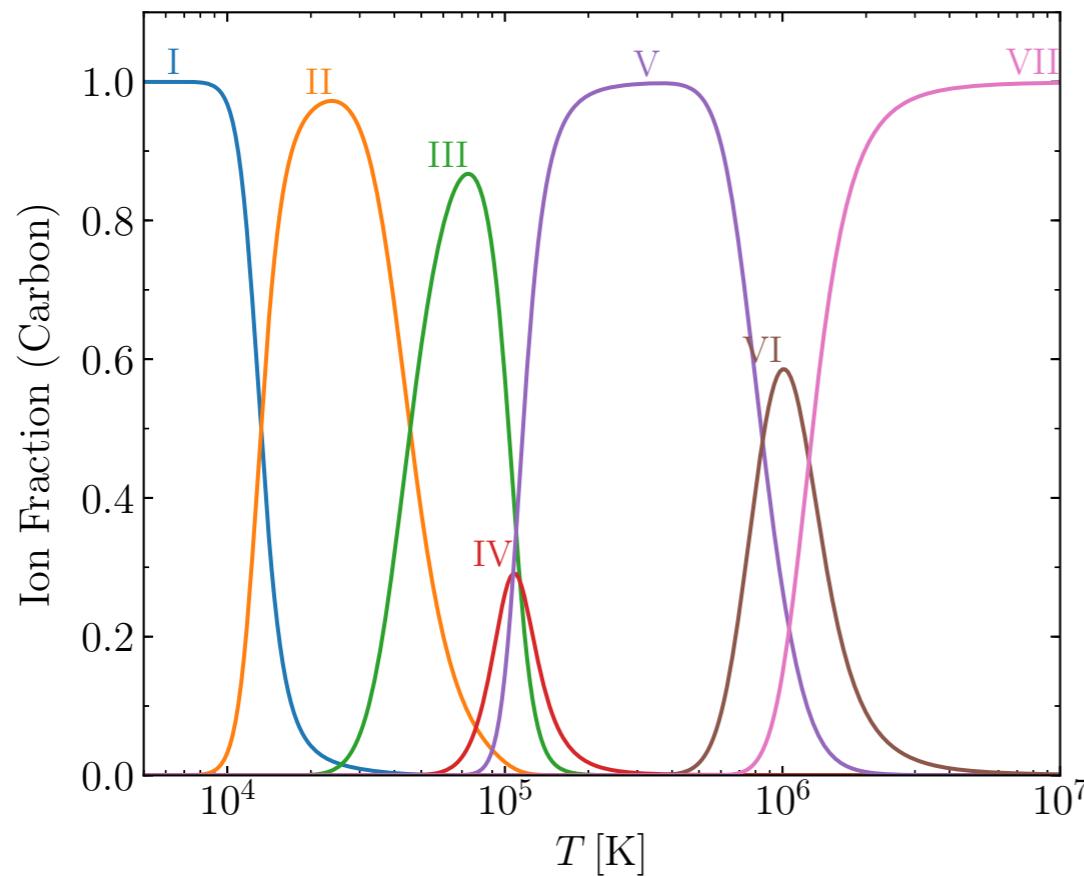
- Using the above rate coefficients, the ionization fractions can be estimated as follows:

$$x \equiv \frac{n(\text{He}^+)}{n(\text{He})} = \frac{1}{1 + \alpha_{10}/k_{01} + k_{12}/\alpha_{21}}$$

$$y \equiv \frac{n(\text{He}^{2+})}{n(\text{He})} = \frac{k_{12}}{\alpha_{21}} x$$



- Heavy Elements
 - ▶ The calculation is usually done numerically, for instance, using CHIANTI
[CHIANTI: https://www.chiantidatabase.org/](https://www.chiantidatabase.org/)
 - ▶ For instance, the ion fractions of carbon and oxygen as a function of temperature are:



- At $T \sim 10^6$ K, we expect a mix of C V, C VI, and C VII.
- At $T \sim 4 \times 10^6$ K and higher, almost all the carbon will be in the form of fully ionized C VII.

The results were calculated using CHIANTI.

Homework (due date: 05/26)

[Q13]

The “cooling time” $\tau_{\text{cool}} \equiv |d \ln T / dt|^{-1}$. Suppose the power radiated per unit volume Λ can be approximated by

$$\Lambda \approx A n_{\text{H}} n_e \left[T_6^{-0.7} + 0.021 T_6^{1/2} \right]$$

for gas of cosmic abundances, where $A = 1.1 \times 10^{-22} \text{ erg cm}^3 \text{ s}^{-1}$, and $T_6 \equiv T / 10^6 \text{ K}$. Assume the gas to have $n_{\text{He}} = 0.1 n_{\text{H}}$, with both H and He fully ionized.

Compute the cooling time (at constant pressure) due to radiative cooling

- (a) in a supernova remnant at $T = 10^7 \text{ K}$, $n_{\text{H}} = 10^{-2} \text{ cm}^{-3}$.
- (b) for intergalactic gas within a dense galaxy cluster (the “intracluster medium”) with $T = 10^8 \text{ K}$, $n_{\text{H}} = 10^{-3} \text{ cm}^{-3}$.

[Q14]

Consider a strong shock wave propagating into a medium that was initially at rest. Assume the gas to be monatomic ($\gamma = 5/3$). Consider the material just behind the shock front. The gas has an energy density u_{thermal} from random thermal motions, and an energy density u_{flow} from the bulk motion of the shocked gas. If cooling is negligible, calculate the ratio $u_{\text{flow}}/u_{\text{thermal}}$ in the frame of reference where the shock front is stationary.