Interstellar Medium (ISM)

Week 3
2025 March 17 (Monday), 9AM

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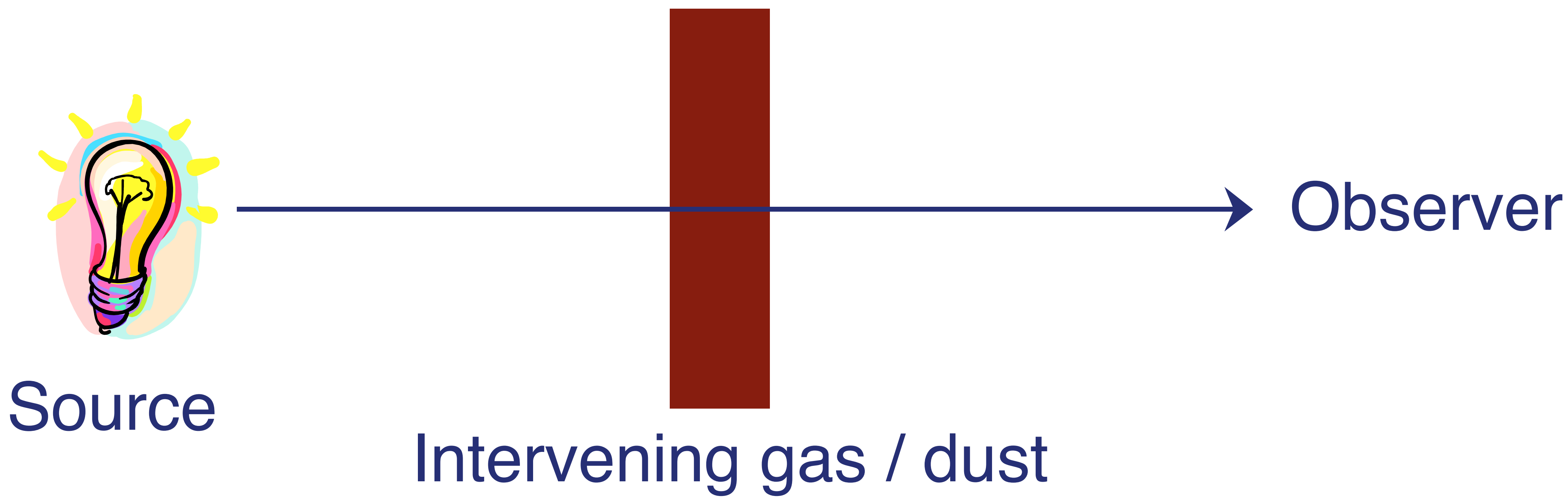
Radiative Transfer

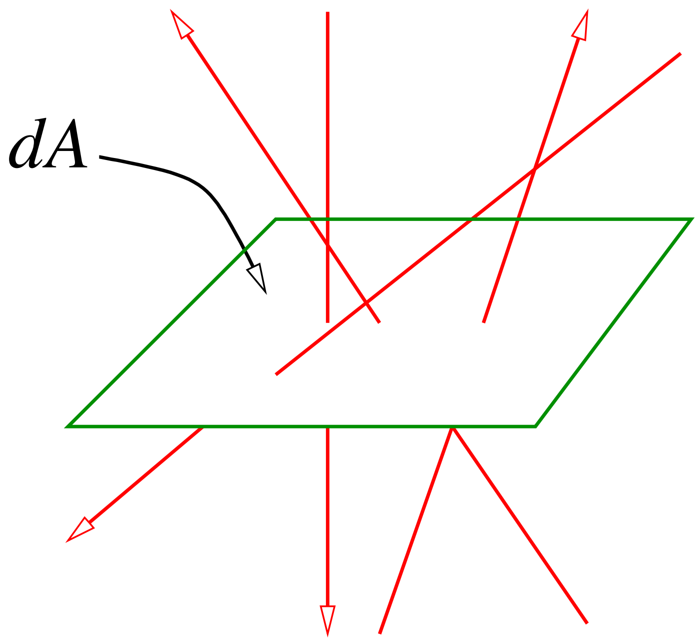
Radiative Transfer

- How is radiation affected as it propagates through intervening gas and dust media to the observer?



Simplication & Complexity

- Simplification:
  - Astronomical objects are normally much larger than the wavelength of radiation they emit.
  - Diffraction can be neglected.
  - Light rays travel to us along straight lines.



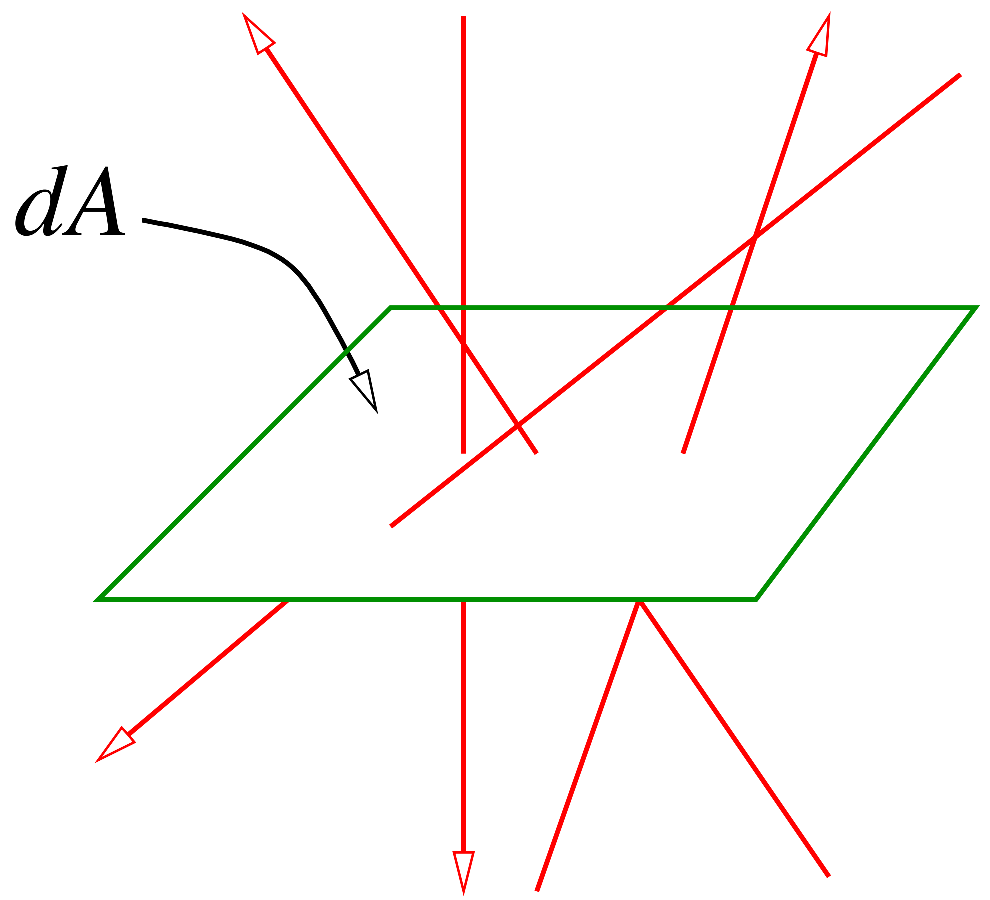
- Complexity:
  - At one point, photons can be traveling in several different directions.
  - For instance, at the center of a star, photons are moving equally in all directions. (However, radiation from a star seen by a distant observer is moving almost exactly radially.)
  - Full specification of radiation needs to say how much radiation is moving in each direction at every point. Therefore, we are dealing with the five- or six-dimensional problem. ($[x, y, z] + [\theta, \phi] + [t]$)

Energy Flux

- Definition
  - Consider a small area dA, exposed to radiation for a time dt.
  - Energy flux F is defined as ***the net energy dE passing through the element of area in all directions in the time interval*** so that

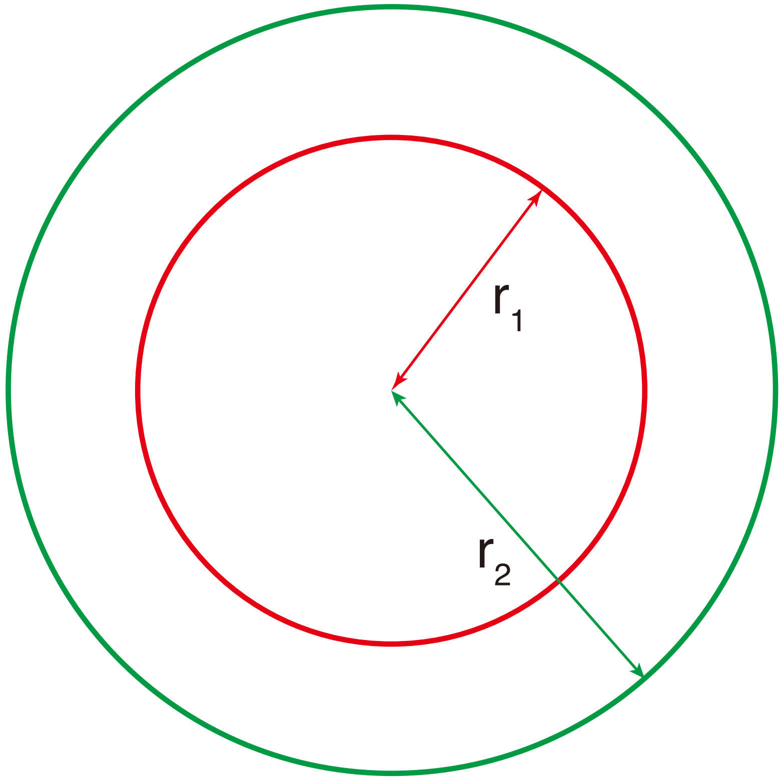
$$dE = F \times dA \times dt$$

  - Note that F ***depends on the orientation of the area element*** dA.
  - Unit: $\mathrm{erg\ cm^{-2}\ s^{-1}}$



Inverse Square Law

- Flux from an isotropic radiation source, i.e., a source emitting equal amounts of energy in all directions.



  - Because of energy conservation, flux through two shells around the source must be the same.

$$4\pi r_1^2 F(r_1) = 4\pi r_2^2 F(r_2)$$

  - Therefore, we obtain the inverse square law.

$$F = \frac{\text{const.}}{r^2}$$

Energy Flux Density

- Real detectors are sensitive to a limited range of wavelengths. We need to consider how the incident radiation is distributed over frequency.

Total energy flux: $F = \int F_\nu d\nu$ Integral of F_ν over all frequencies

$\downarrow$

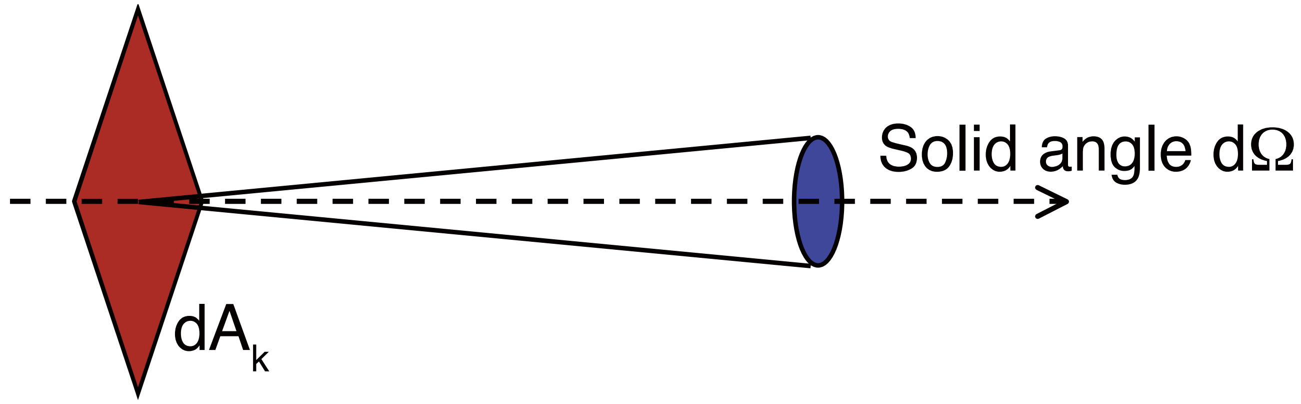
Units: $\mathrm{erg\ s^{-1}\ cm^{-2}\ Hz^{-1}}$

- F_ν is often called the "flux density."

- Radio astronomers use a special unit to define the flux density:
$1\ \mathrm{Jansky\ (Jy)} = 10^{-23}\ \mathrm{erg\ s^{-1}\ cm^{-2}\ Hz^{-1}}$

Specific Intensity or Surface Brightness

- Recall that ***flux is a measure of the energy carried by all rays passing through a given area***
- ***Intensity is the energy carried along by individual rays.***



$\mathbf{k}$ = direction of propagation

- Let dE_ν be the amount of radiant energy which crosses the area dA in a direction $\mathbf{k}$ within solid angle $d\Omega$ about in a time interval dt with photon frequency between ν and $\nu + d\nu$.
- The monochromatic specific intensity I_ν is then defined by the equation.

$$dE_\nu = I_\nu(\mathbf{k}, \mathbf{x}, t)dA_\mathbf{k}d\Omega d\nu dt$$

- Unit: $\mathrm{erg\ s^{-1}\ cm^{-2}\ sr^{-1}\ Hz^{-1}}$

- From the view point of an observer, the specific intensity is called ***surface brightness***.

Relation between the flux and the specific intensity

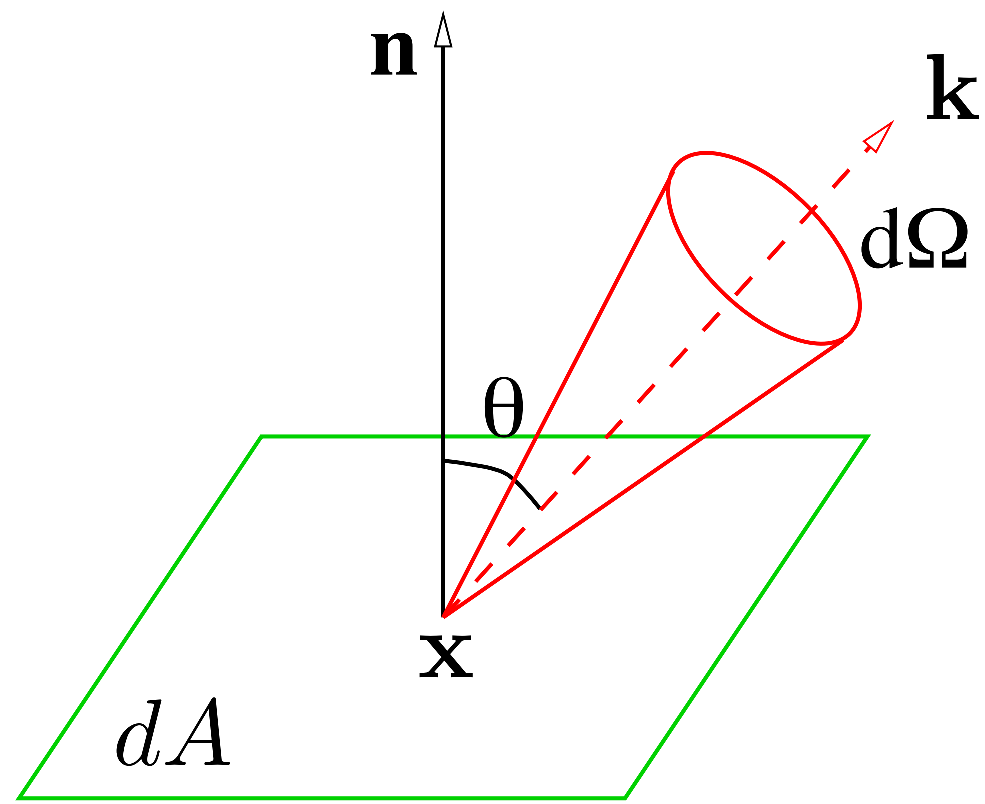
- Let's consider a small area dA, with light rays passing through it at all angles to the normal vector $\mathbf{n}$ of the surface.

- For rays centered about $\mathbf{k}$, the area normal to $\mathbf{k}$ is

$$dA_{\mathbf{k}} = dA\cos\theta$$

  - By the definition,

$$F_\nu dA d\nu dt = \int I_\nu(\mathbf{k}, \mathbf{x}, t) dA_{\mathbf{k}} d\Omega d\nu dt$$



- Hence, net flux in the direction of $\mathbf{n}$ is given by integrating over all solid angles:

$$\boxed{F_\nu = \int I_\nu \cos\theta d\Omega = \int_0^{2\pi}\int_0^{\pi} I_\nu \cos\theta \sin\theta d\theta d\phi}$$

[Note] **flux** **= "sum of all ray vectors" which is then projected onto a normal vector**
intensity = magnitude of a single ray vector

Note

- Intensity can be defined as per wavelength interval.

$$I_\nu |d\nu| = I_\lambda |d\lambda| \qquad \leftarrow \left|\frac{d\nu}{d\lambda}\right| = \frac{c}{\lambda^2} = \frac{\nu}{\lambda} \qquad \longleftarrow \qquad \nu = \frac{c}{\lambda}$$

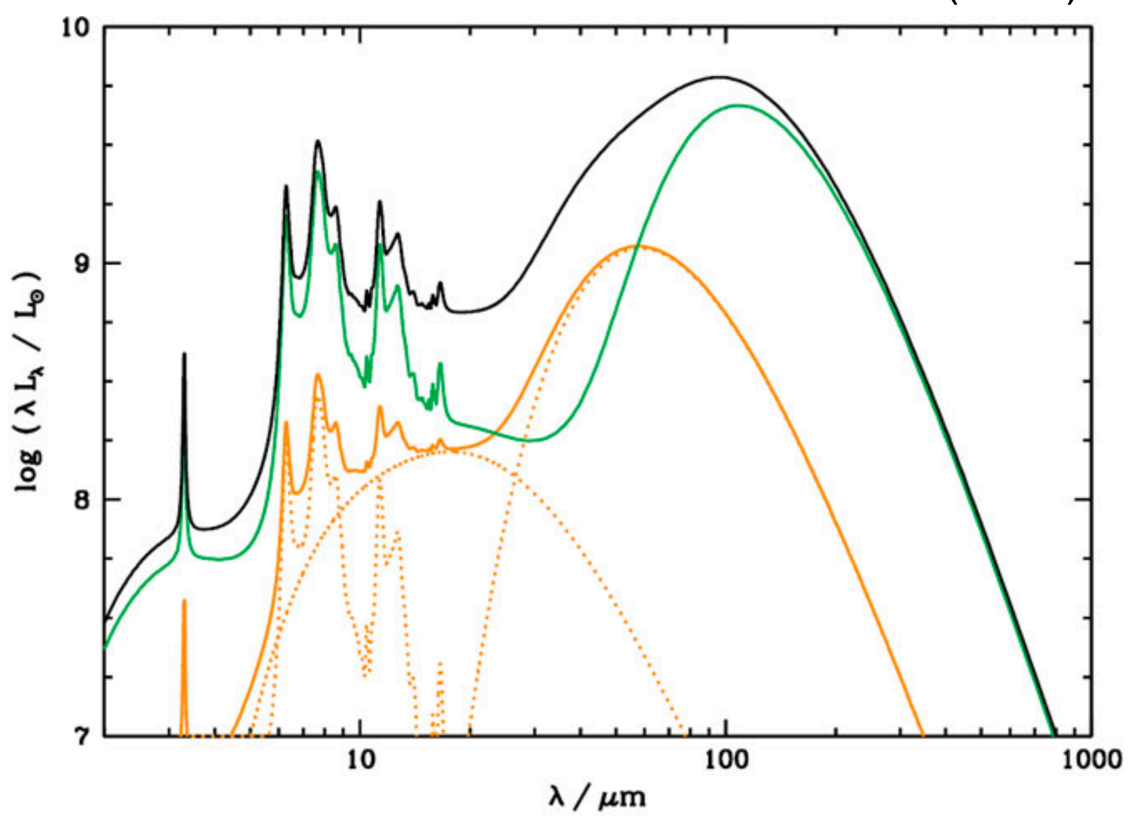
$$\nu I_\nu = \lambda I_\lambda$$

- Integrated intensity is defined as the intensity over all frequencies.

$$I = \int_0^\infty I_\nu d\nu = \int_0^\infty I_\lambda d\lambda$$

- In astrophysics, we plot the ***spectral energy distribution (SED)*** as νI_ν versus ν or λI_λ versus λ.

Galaxy SEDs
da Cunha et al. (2008)



How does specific intensity changes along a ray in free space

- Consider a bundle of rays and any two points along the rays. Construct areas dA_1 and dA_2 normal to the rays at these points.
  - Consider the energy carried by the rays passing through both areas. Because energy is conserved,

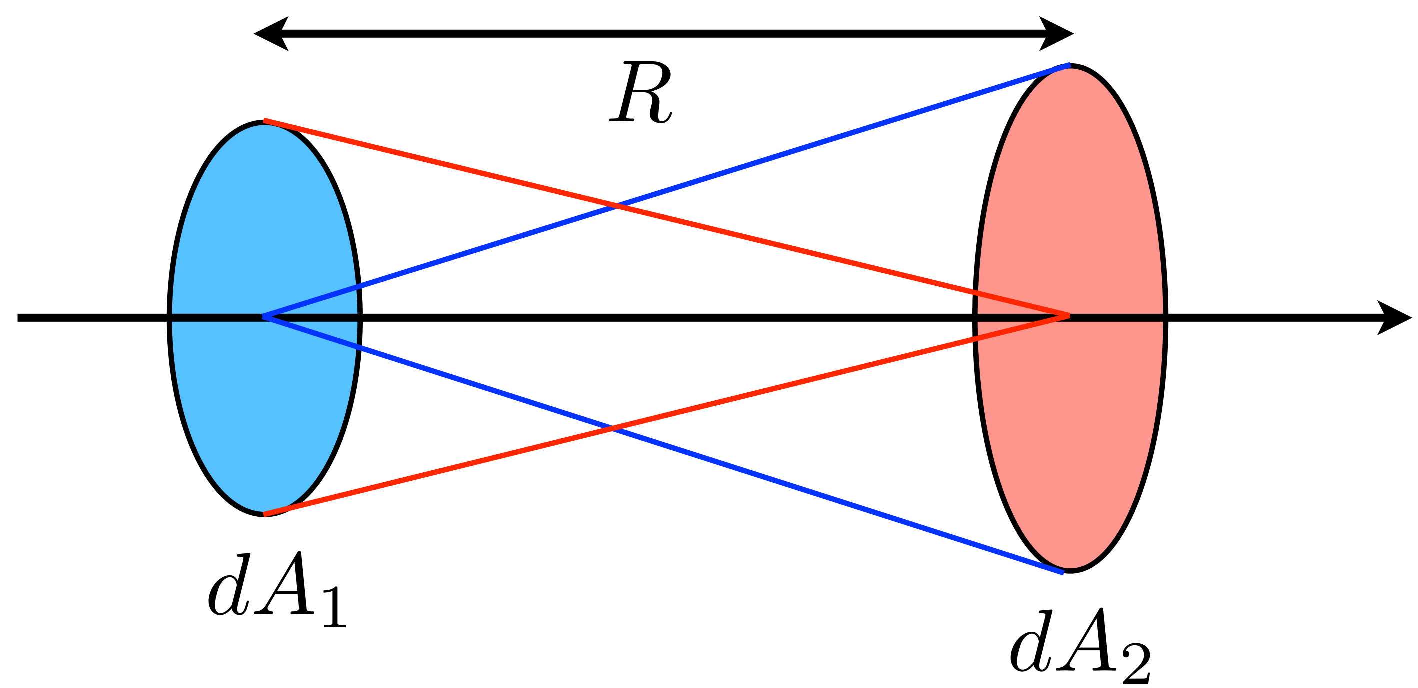
$$dE_1 = I_1 dA_1 d\Omega_1 d\nu dt = dE_2 = I_2 dA_2 d\Omega_2 d\nu dt$$

  - Here, $d\Omega_1$ is the solid angle subtended by dA_2 at the location 1 and so forth.

$$d\Omega_1 = \frac{dA_2}{R^2}$$

$$d\Omega_2 = \frac{dA_1}{R^2}$$

$$\rightarrow \quad \boxed{I_1 = I_2}$$



- Conclusion (the constancy of intensity):
  - the specific intensity remains the same as radiation propagates through free space.

$$I_1 = I_2$$

  - If we measure the distance along a ray by variable s, we can express the result equivalently in differential form:

$$\boxed{\frac{dI}{ds} = 0}$$

- We receive the same specific intensity at the telescope as is emitted at the source.
  - Imagine looking at a uniformly lit wall and walking toward it. As you get closer, a field-of-view with fixed angular size will see a progressively smaller region of the wall, but this is exactly balanced by the inverse square law describing the spreading of the light rays from the wall.

Specific Energy Density

- Consider a bundle of rays passing through a volume element dV in a direction Ω.
  - Then, the energy density per unit solid angle is defined by

$$dE = u_\nu(\Omega) dV d\Omega d\nu$$

  - Since radiation travels at velocity c, the volume element is

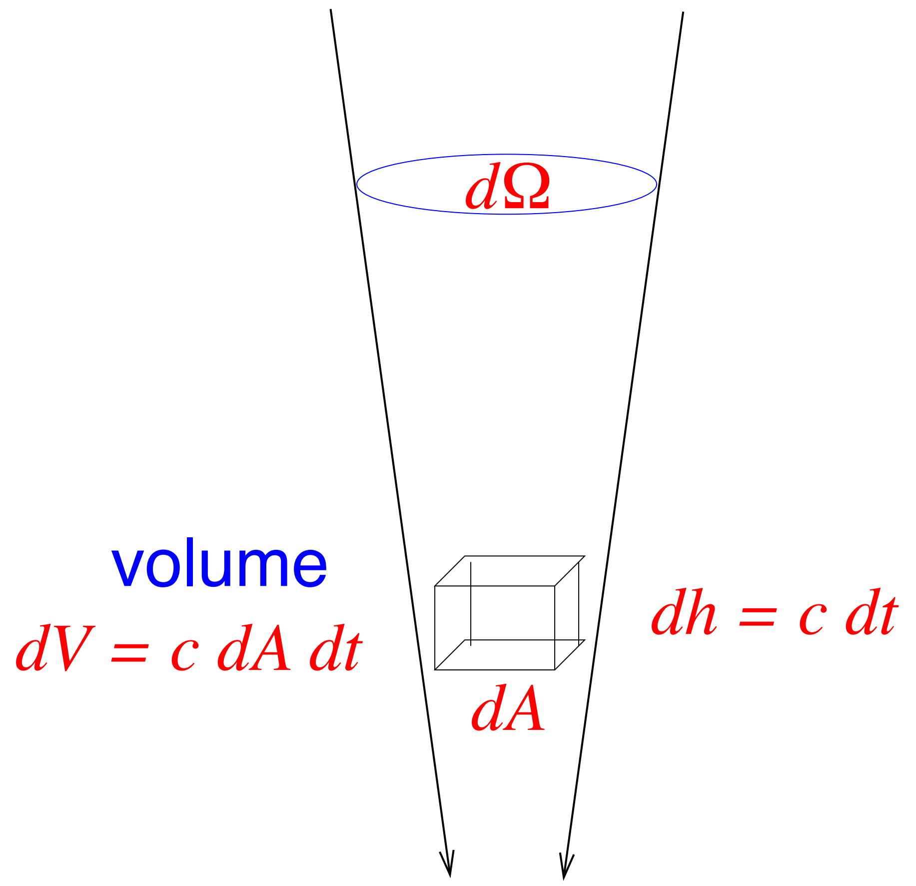
$$dV = dA(cdt)$$

  - According to the definition of the intensity,

$$dE = I_\nu dA dt d\Omega d\nu$$

  - Then, we have

$$\boxed{u_\nu(\Omega) = I_\nu(\Omega)/c}$$



Energy Density and Mean Intensity

- Integrating over all solid angle, we obtain

$$u_\nu = \int u_\nu(\Omega) d\Omega = \frac{1}{c} \int I_\nu d\Omega$$

- ***Mean intensity*** is defined by

$$\boxed{J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega}$$

- Then, the energy density is

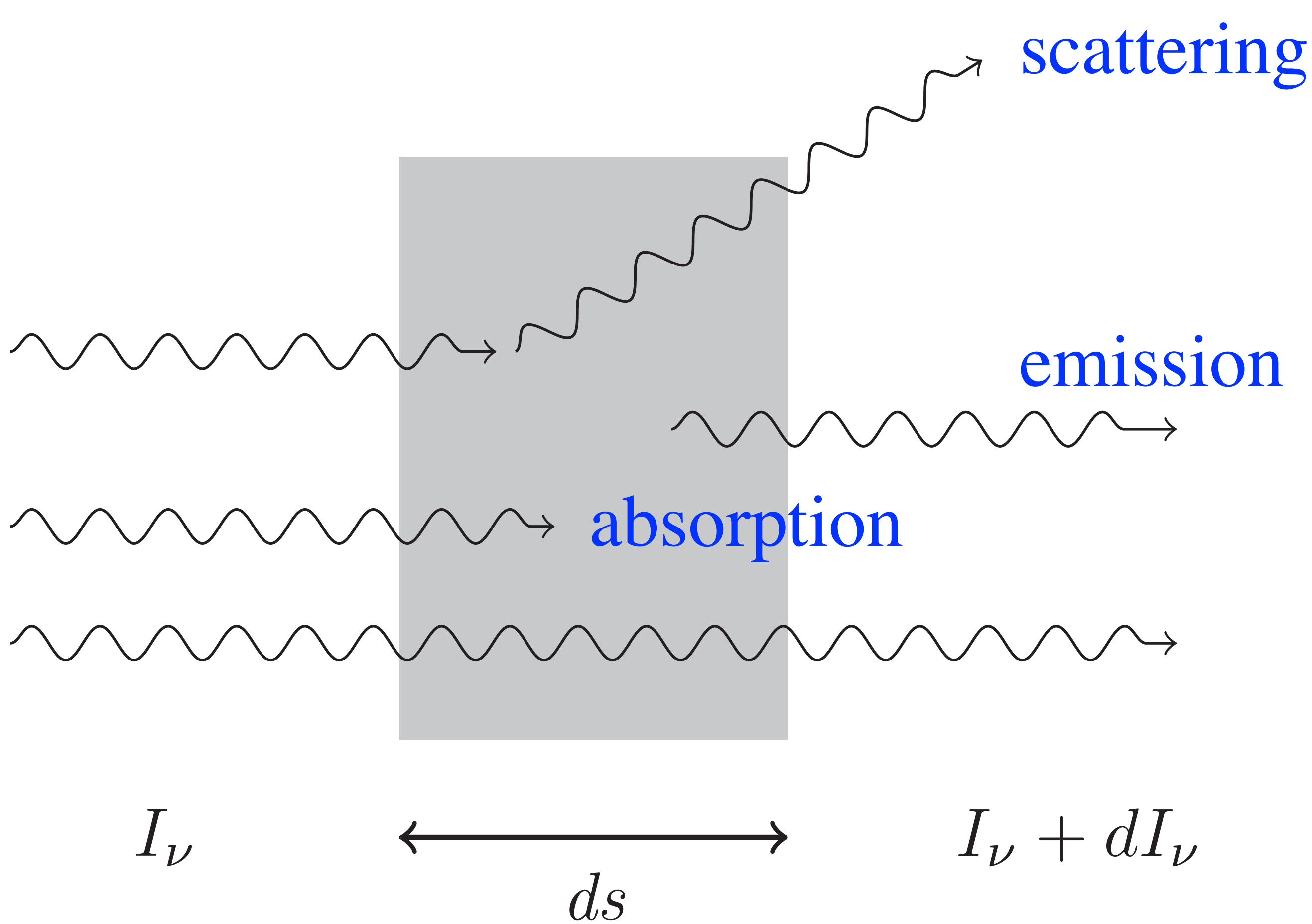
$$\boxed{u_\nu = \frac{4\pi}{c} J_\nu}$$

- Total energy density is obtained by integrating over all frequencies.

$$u = \int u_\nu d\nu = \frac{4\pi}{c} \int J_\nu d\nu$$

Radiative Transfer Equation in reality

- In reality, as a ray passes through matter, energy may be added, subtracted, or scattered from it by emission, absorption, or scattering.
  - The intensity will not in general remain constant.
  - These interactions are described by the ***radiative transfer equation***.



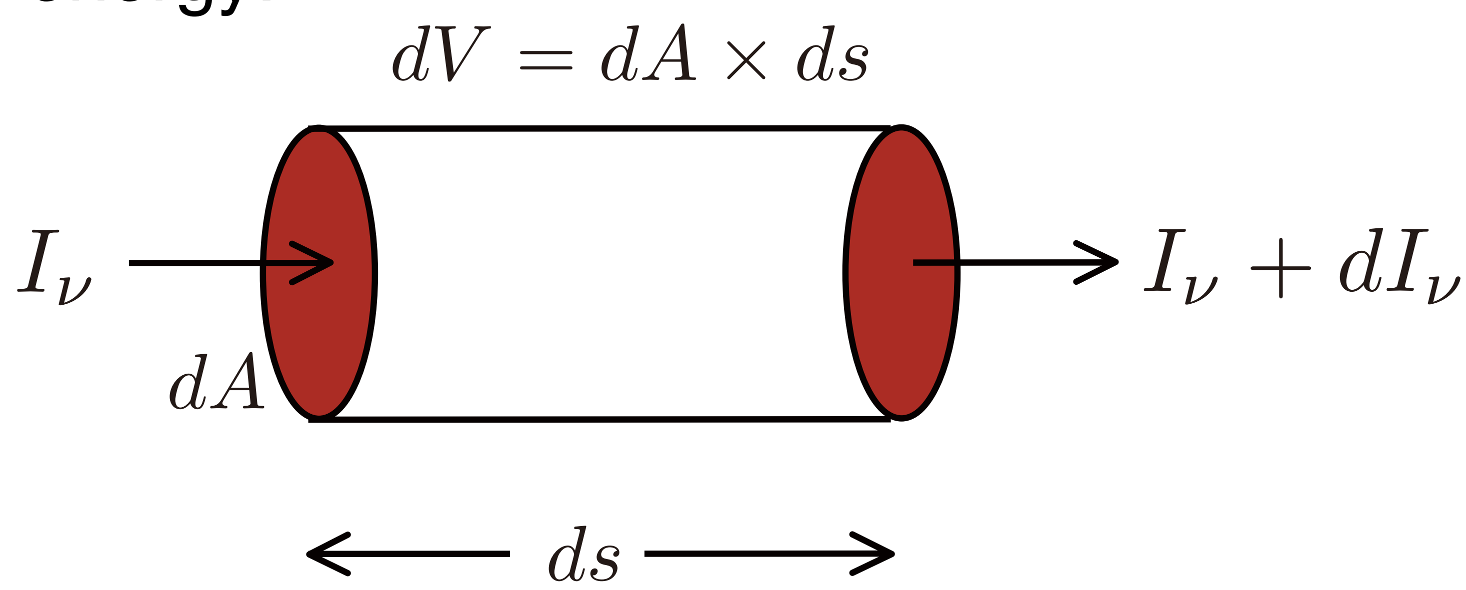
loss: scattering & absorption

gain: emission

[J. P. Williams]
Introduction to the Interstellar Medium

Emission

- If the radiation travels through a medium which is itself emitting radiation, that will add to the energy:



- Spontaneous ***"emission coefficient" or "emissivity"*** j_ν is the amount of energy emitted per unit time, per unit solid angle, per unit frequency, and per unit volume:

$$dE = j_\nu dV d\Omega dt d\nu \quad (j_\nu : \mathrm{erg\ cm^{-3}\ s^{-1}\ sr^{-1}\ Hz^{-1}})$$

- In going a distance ds, a beam of cross section dA travels through a volume $dV = dAds$. Thus the intensity added to the beam is by ds is

$$dI_\nu = j_\nu ds \longleftarrow dE = (dI_\nu) dA d\Omega dt d\nu$$

- Therefore, the equation of radiative transfer for pure emission becomes:

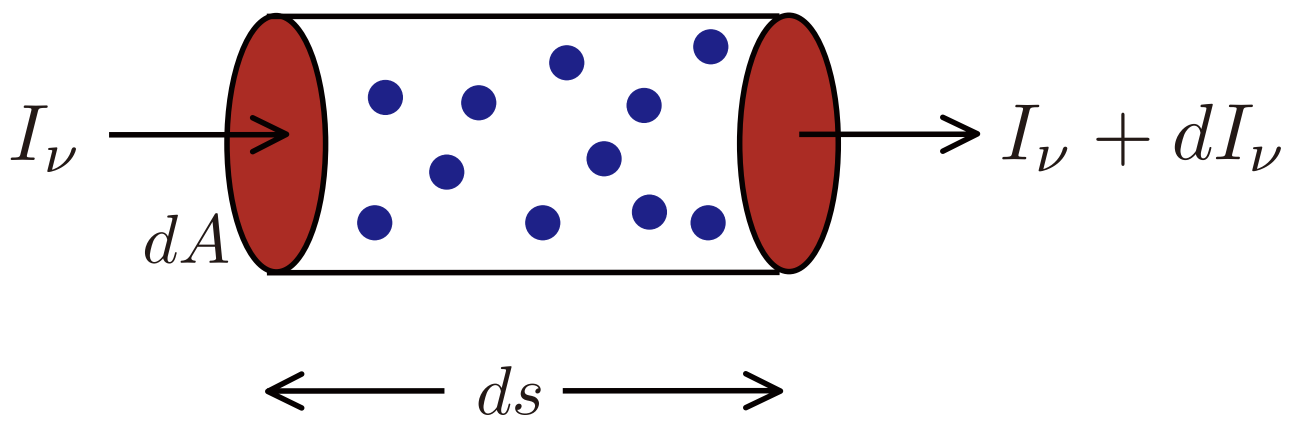
$$\boxed{\frac{dI_\nu}{ds} = j_\nu}$$

  - If we know what j_ν is, we can integrate this equation to find the change in specific intensity as radiation propagates through the medium:

$$I_\nu(s) = I_\nu(0) + \int_0^s j_\nu(s')ds'$$

Absorption

- If the radiation travels through a medium which absorbs radiation, the energy in the beam will be reduced:



  - Let n denote the number density of absorbers (particles per unit volume).
  - Assume that each absorber has a cross-sectional area of σ_ν (in units of cm^2).
  - If a beam travels through ds, total area of absorbers is

$$\text{number of absorbers } \times \text{cross} - \text{section} = n \times dA \times ds \times \sigma_\nu$$

Fraction of radiation absorbed = fraction of area blocked:

$$\frac{dI_\nu}{I_\nu} = -\frac{ndAds\sigma_\nu}{dA} = -n\sigma_\nu ds$$

$$dI_\nu = -n\sigma_\nu I_\nu ds \equiv -\kappa_\nu I_\nu ds \longrightarrow \boxed{\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu}$$

- ***Absorption coefficient*** is defined as $\boxed{\kappa_\nu \equiv n\sigma_\nu \ \text{(units: cm}^{-1}\text{)},}$ meaning the ***total cross-sectional area per unit volume***.

- If we include the effect of stimulated emission in the absorption coefficient, it may be referred to as the ***attenuation coefficient***. (as in Draine's book)

- Rearranging the previous equation, we obtain the equation of radiative transfer for pure absorption:

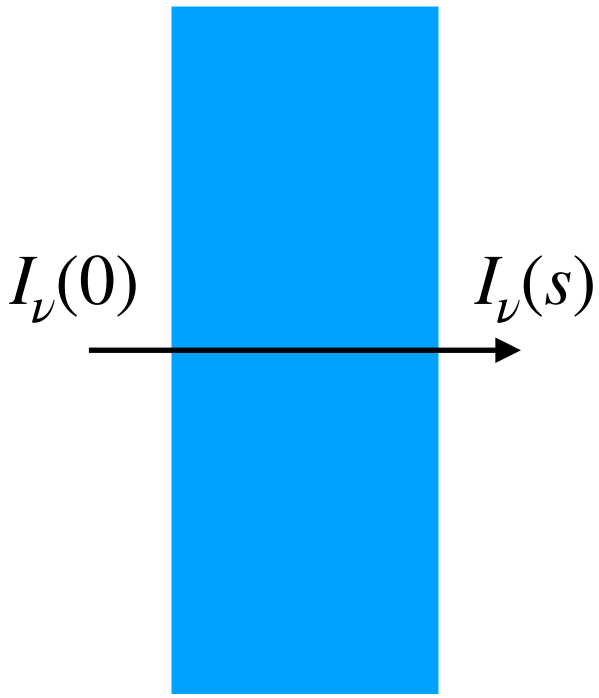
$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu$$

  - The amount of reduced energy depends on how much radiation we already have.
  - Integrate to find how radiation changes along path:

$$\int_0^s \frac{dI_\nu}{I_\nu} = -\int_0^s \kappa_\nu(s')ds'$$

$$\left[\ln I_\nu\right]_0^s = -\int_0^s \kappa_\nu(s')ds'$$

$$I_\nu(s) = I_\nu(0)\exp\left[-\int_0^s \kappa_\nu(s')ds'\right]$$



- If the absorption coefficient is a constant (example: a uniform density gas of ionized hydrogen), then we obtain

$$I_\nu(s) = I_\nu(0)e^{-\kappa_\nu s}$$

specific intensity after distance s

initial intensity at $s = 0$.

radiation exponentially absorbed with distance

- ***Optical depth***:
  - Imagine radiation traveling into a cloud of absorbing gas, exponential defines a scale over which radiation is attenuated.
  - When $\int_0^s \kappa_\nu(s')ds' = 1$, the intensity will be reduced to $1/e$ of its original value.

- We define the optical depth τ_ν as:

$$\tau_\nu(s) = \int_0^s \kappa_\nu(s')ds' \quad \text{or} \quad d\tau_\nu = \kappa_\nu ds \qquad \longrightarrow \qquad I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu}$$

- A medium is said to be ***optically thick*** at a frequency ν if the optical depth for a typical path through the medium satisfies:

$$\tau_\nu(s) > 1$$

- The medium is ***optically thin*** if, instead:

$$\tau_\nu(s) < 1$$

- An optically thin medium is one which a typical photon of frequency ν can pass through without being (significantly) absorbed.

Mean Free Path

- From the exponential absorption law, the ***probability of a photon absorbed*** between optical depths τ_ν and $\tau_\nu + d\tau_\nu$ is

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu}$$

probability = $\boxed{\dfrac{|dI_\nu|}{I_\nu(0)} = \left|\dfrac{dI_\nu}{\tau_\nu}\right| d\tau_\nu = e^{-\tau_\nu} d\tau_\nu}$ $\rightarrow$ $\boxed{P(\tau_\nu) = e^{-\tau_\nu}}$

= **probability density function** for the absorption at an optical depth τ_ν

- The mean optical depth traveled is thus equal to unity:

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu P(\tau_\nu) d\tau_\nu = \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1$$

- ***The mean free path is defined as the average distance a photon can travel through an absorbing material until it is absorbed.*** In a homogeneous medium, the mean free path is determined by

$$\langle \tau_\nu \rangle = \kappa_\nu \ell_\nu = 1 \quad \rightarrow \quad \boxed{\ell_\nu = \frac{1}{\kappa_\nu} = \frac{1}{n\sigma_\nu}}$$

- A local mean path at a point in an inhomogeneous material can be also defined.

- The probability of a photon being absorbed within an optical depth τ_ν is

$$\int_0^{\tau_\nu} P(\tau'_\nu) d\tau'_\nu = 1 - e^{-\tau_\nu}$$

Radiative Transfer Equation

- ***Radiative transfer equation*** with both absorption and emission is

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu$$

absorption emission

  - We can rewrite the radiative transfer equation using the optical depth as a measure of \`distance' rather than s:

$$\frac{dI_\nu}{\kappa_\nu ds} = -I_\nu + \frac{j_\nu}{\kappa_\nu}$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

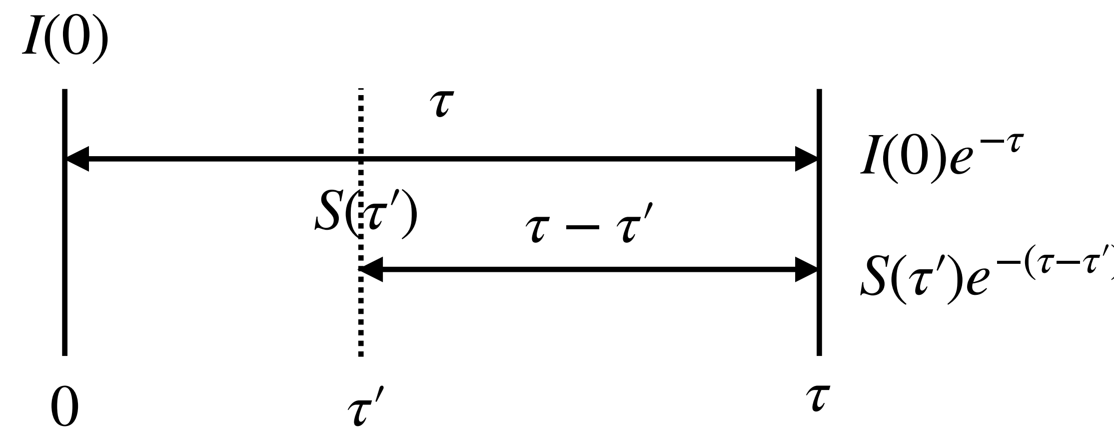
  - where $S_\nu \equiv j_\nu / \kappa_\nu$ ***is called the source function***. This is an alternative and sometimes more convenient way to write the equation.

Formal Solution of the RT equation

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

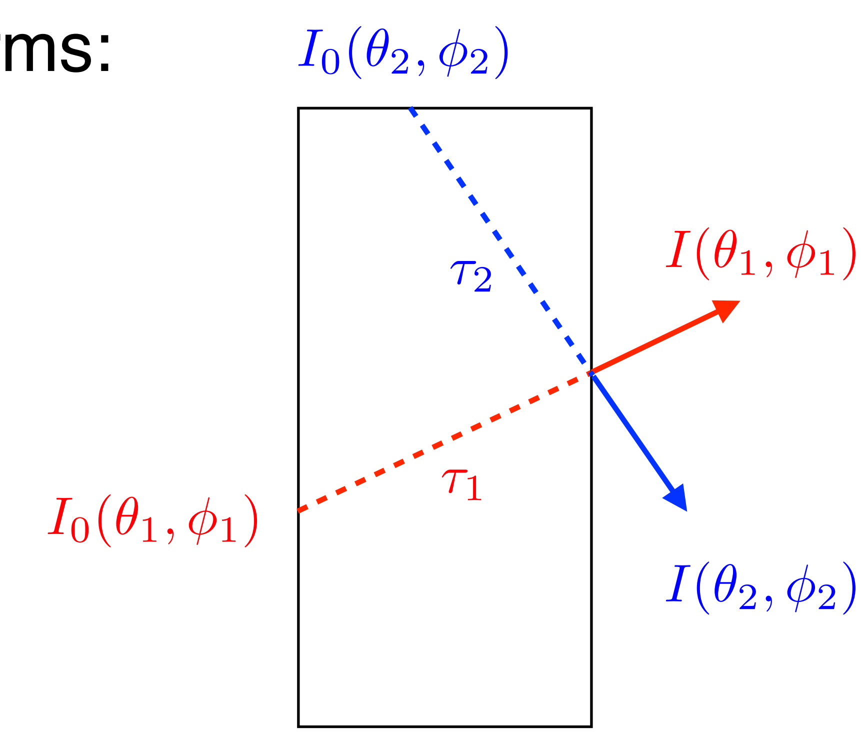
$$e^{\tau_\nu} \frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

$$\frac{d}{d\tau_\nu} \left(e^{\tau_\nu} I_\nu \right) = e_\nu^\tau S_\nu$$



$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu$$

- The solution is easily interpreted as the sum of two terms:
  - the initial intensity diminished by absorption
  - the integrated source diminished by absorption.
- For a constant source function, the solution becomes



$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu} \left(I_\nu(0) - S_\nu \right) \end{aligned}$$

Relaxation

- "Relaxation"

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$I_\nu > S_\nu \quad \rightarrow \quad \frac{dI_\nu}{d\tau_\nu} < 0, \quad \text{then } I_\nu \text{ tends to decrease along the ray}$$

$$I_\nu < S_\nu \quad \rightarrow \quad \frac{dI_\nu}{d\tau_\nu} > 0, \quad \text{then } I_\nu \text{ tends to increase along the ray}$$

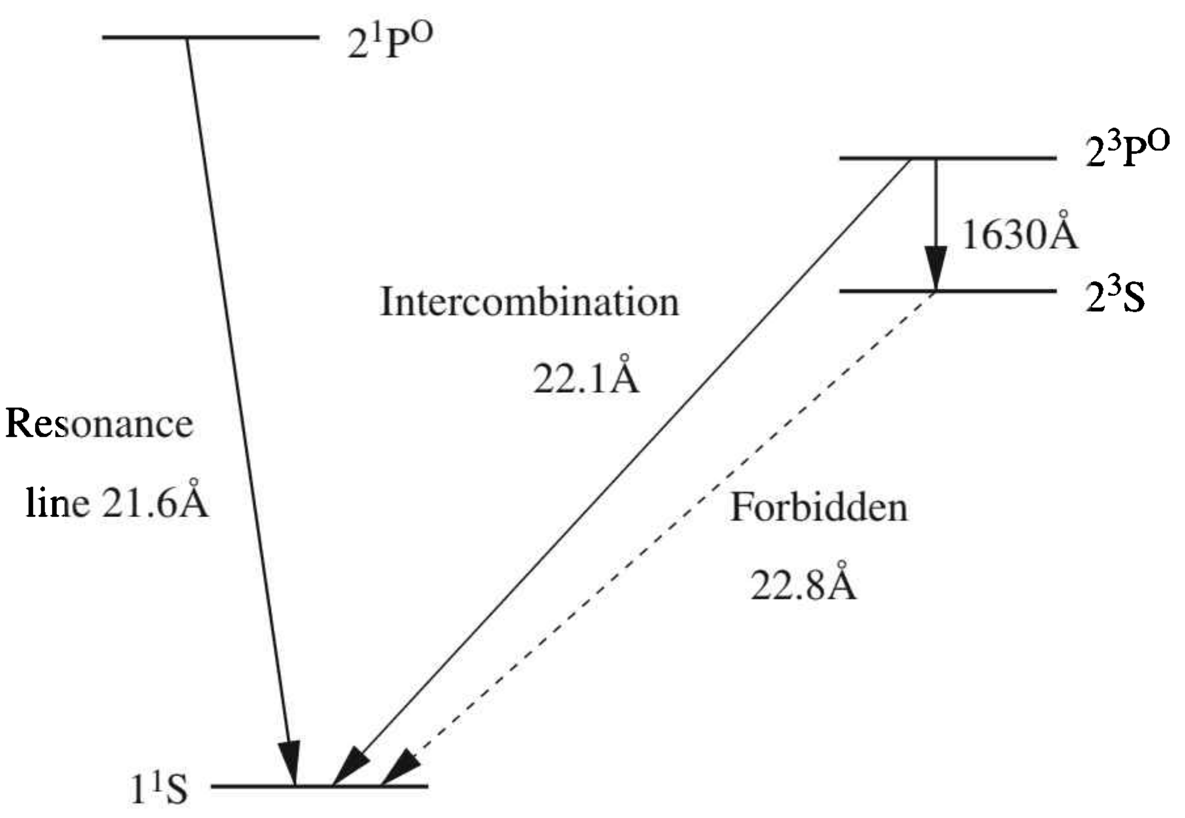
- ***The source function is the quantity that the specific intensity tries to approach***, and does approach if given sufficient optical depth.

$$\text{As } \ \tau_\nu \rightarrow \infty, \quad I_\nu \rightarrow S_\nu$$

Homework (due date: 04/10)

[Q3] Atomic Spectroscopy

1. What is the ground-state configuration, term and level of the beryllium atom, Be? One of the outer electrons in Be is promoted to the 3rd orbital. What terms and levels can this configuration have?
2. Symbols for particular levels of three different atoms are written as $^1D_1$, $^0D_{3/2}$ and $^3P_{3/2}$. Explain in each case why the symbol must be wrong.
3. Give the spectroscopic terms arising from the following configurations, using L-S coupling. Include parity and J values. Give your arguments in detail for deriving these results.
   - (a) $2s^2$
   - (b) 2p3s
   - (c) 3p4p
4. The lithium atom, Li, has three electrons. Consider the following configurations of Li: (a) $1s^2 2p$, (b) $1s2s3s$, (c) $1s2p3p$. By considering the configuration only, state which of the three sets of transitions between the configuration (a), (b) and (c) are allowed and forbidden transitions?
5. The right figure shows the term diagram for helium-like oxygen, O VII, showing transitions from the $1s2l$ states. Explain why 22.1Å line is an intercombination line and why 22.8Å is a forbidden line.



[Q4] Radiative Transfer

Suppose a sphere of gas with a radius R emits a forbidden line, and its emissivity is constant over the sphere, i.e., $j(x, y, z) = j_0$ (erg cm^{-3} s^{-1} sr^{-1}). An observer is located at a distance d from the sphere's center. The observer measures the intensity of the emission toward a direction with an angle θ, as shown in Figure (a). Without the loss of generality, we can assume that the direction vector of the line of sight is in the xy plane. The line of sight intersects the outer boundary of the sphere at two points, a and b.

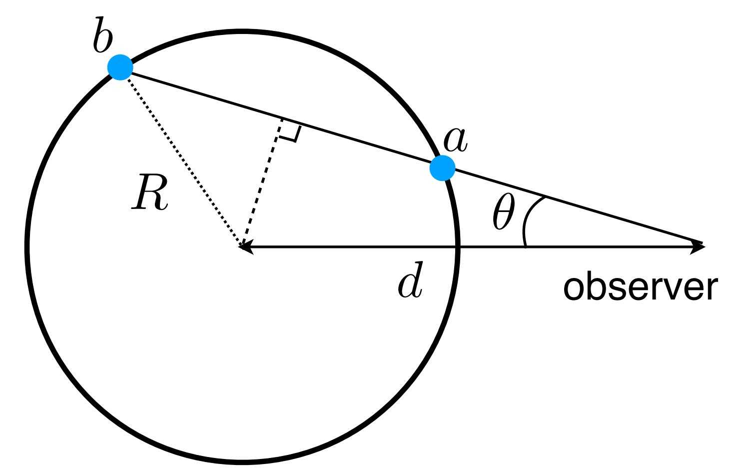


Figure (a)

(1). If we make the assumption that there is an absence of dust, and the emission line is forbidden, then, there will be no absorption. In this case, show that the intensity at the angle θ is given by:

$I(\theta) = j_0 \int_a^b ds = j_0 s_\theta$, where s_θ is the distance between the points a and b.

(2). Show that the distance is $s_\theta = 2\sqrt{R^2 - d^2 \sin^2\theta}$ and therefore the intensity at angle θ is $I(\theta) = 2j_0\sqrt{R^2 - d^2 \sin^2\theta}$.

(3). Now, assume the case where $d \to \infty$ and $\theta \to 0$ (see Figure (b)). Then, the intensity at θ can be approximated as the intensity of a parallel ray that passes through the sphere at an impact parameter defined by $p = d\tan\theta$. In this limit, show that $p \approx d\sin\theta$ and $I(p) = 2j_0\sqrt{R^2 - p^2}$.

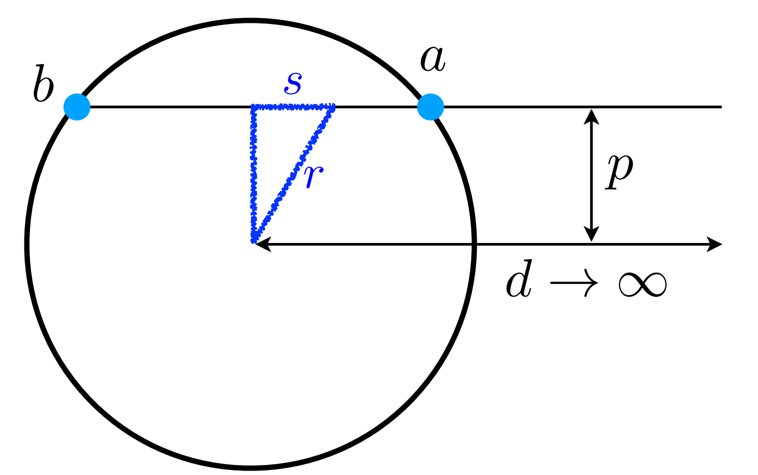


Figure (b)

Assume that the emissivity is a function of the radial distance from the sphere's center r, i.e., $j(r)$.

(4). In the limit of $d \to \infty$ (Figure (b)), show that $I(p) = \int_a^b j(r)ds = 2\int_p^R j(r)\frac{rdr}{\sqrt{r^2 - p^2}}$.

Note that this integral is called the Abel transform of $j(r)$.
(https://en.wikipedia.org/wiki/Abel_transform)

(5). Now, use the equation of (4) and derive the same result $\left(I(p) = 2j_0\sqrt{R^2 - p^2}\right)$ as in (3) if $j(r) = j_0 =$ constant.

Thermal equilibrium

- In general, equilibrium means a state of balance.
  - ***Thermal equilibrium refers to steady states of temperature, which defines the average energy of material or photons.***
  - In a state of (complete) ***thermodynamic equilibrium (TE)***, no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system. ***In TE, matter and radiation are in equilibrium at the same temperature T.***
  - If the material is (locally) in thermodynamic equilibrium at a well-defined temperature T, ***it is said to be in local thermodynamic equilibrium (LTE) even if it is not in equilibrium with the radiation field.***
- ***Note that thermal equilibrium differ from thermodynamic equilibrium.***

The state of LTE

- Macroscopically, LTE is characterized by the following three equilibrium distributions:
  - **Maxwellian velocity distribution** of particles, written here in terms of distribution for the absolute values of velocity,

$$f(v)dv = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) 4\pi v^2 dv$$

    where m is the particle mass and k the Boltzmann constant.
  - **Boltzmann excitation equation**,

$$\frac{n_i}{N_I} = \frac{g_i}{U_I} e^{-E_i/kT}$$

    where n_i is the population of level i, g_i is its statistical weight, and E_i is the level energy, measured from the ground state; N_I and U_I are the total number density and the partition function of the ionization state I to which level i belongs, respectively.
  - **Saha ionization equation**,

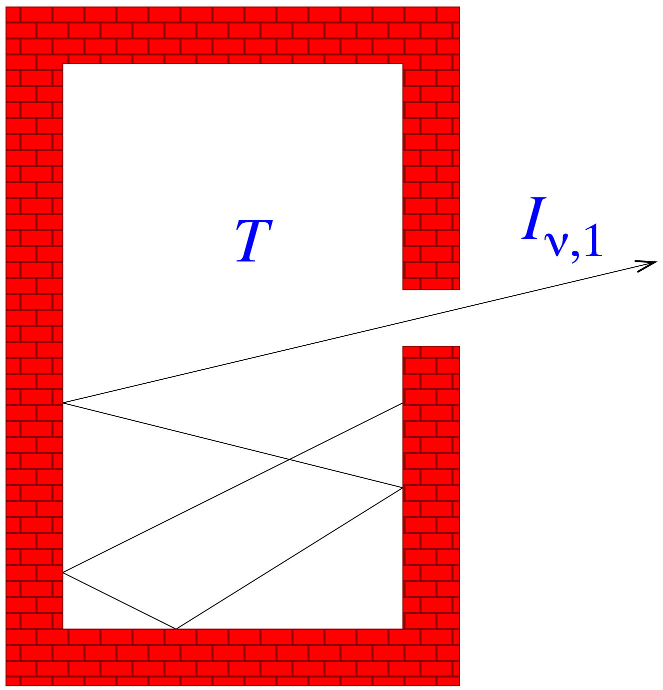
$$\frac{N_I}{N_{I+1}} = n_e \frac{U_I}{U_{I+1}} \left(\frac{h^2}{2\pi m_e kT}\right)^{3/2} e^{\chi_I/kT}$$

    where χ_I is the ionization potential of ion I.

- Microscopically, LTE holds if all atomic processes are in detailed balance, i.e., if the number of processes $A \rightarrow B$ is exactly balanced by the number of inverse processes $B \rightarrow A$.

Blackbody

- Imagine a container bounded by opaque walls with a very small hole.
  - ***Photons will be scattered and absorbed many times, (and eventually trapped and completely absorbed in the box).*** Under such conditions, the particles and photons continually share their kinetic energies. In perfect thermal equilibrium, ***the average particle kinetic energy will be equal to the average photon energy, and a unique temperature T can be defined.***



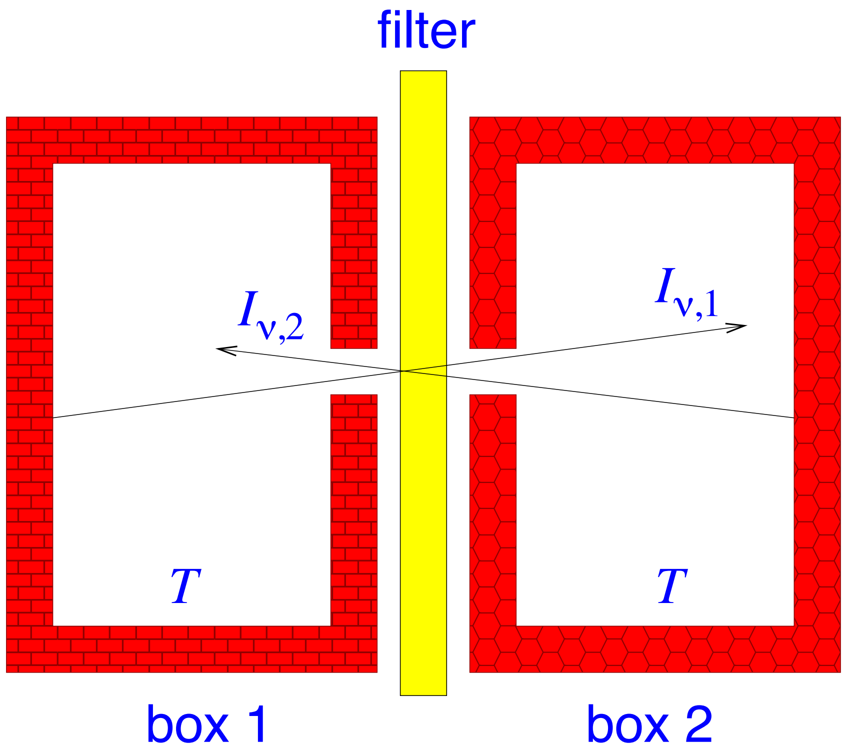
box 1

  - A **blackbody** is an idealized physical body that absorbs all incident radiation regardless of frequency or angle of incidence (i.e., perfect absorber). The above cavity can be regarded to be a blackbody.

  - Radiation from a blackbody in thermal equilibrium is called the **blackbody radiation**.

Blackbody radiation is the universal function.

- ***Now, consider another cavity (box 2), also at the same temperature***, but made of different material or shape and connect two cavities with a filter transparent only in the narrow frequency range ν and $\nu + d\nu$.
  - ***In equilibrium at T, radiation should transfer no net energy from one cavity to the other.*** Otherwise, one cavity will cool down and the other heats up; this violates the second law of thermodynamics.
  - Therefore, the intensity or spectrum that passes through the holes should be a universal function of *T* and should be isotropic.
  - The intensity and spectrum of the radiation emerging from the hole should be independent of the wall material (e.g., wood, copper, or concrete, etc) and any absorbing material that may be inside the cavity.
  - The universal function is called the Planck function $B_\nu(T)$.
  - This is the blackbody radiation.



Kirchhoff's Law in TE

- In (full) thermodynamic equilibrium at temperature T, by definition, we know that

$$\frac{dI_\nu}{ds} = 0 \quad \text{and} \quad I_\nu = B_\nu(T)$$

  - We also note that

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu$$

  - Then, we can obtain ***the Kirchhoff's law for a system in TE***:

$$\frac{j_\nu(T)}{\kappa_\nu(T)} = B_\nu(T)$$

  - This is remarkable because it connects the properties $j_\nu(T)$ and $\kappa_\nu(T)$ of any kind of matter to the single universal spectrum $B_\nu(T)$.

Kirchhoff's Law in LTE

- Recall that Kirchhoff's law was derived for a system in thermodynamic equilibrium.

- ***Kirchhoff's law applies not only in TE but also in LTE***:
  - Recall that $B_\nu(T)$ is independent of the properties of the radiating /absorbing material.
  - In contrast, both $j_\nu(T)$ and $\kappa_\nu(T)$ depend only on the materials in the cavity and on the temperature of that material; they do not depend on the ambient radiation field or its spectrum.
  - Therefore, the Kirchhoff's law should be true even for the case of LTE.
  - ***In LTE, the ratio of the emission coefficient to the absorption coefficient is a function of temperature and frequency only.***
  - This generalized version of Kirchhoff's law is an exceptionally valuable tool for calculating the emission coefficient from the absorption coefficient or vice versa.

Implications of Kirchhoff's Law

- A good absorber is a good emitter, and a poor absorber is a poor emitter. (In other words, a good reflector must be a poor absorber, and thus a poor emitter.)

$$j_\nu = \kappa_\nu B_\nu(T) \quad \rightarrow \quad j_\nu \text{ increases as } \kappa_\nu \text{ increases}$$

  - It is not possible to thermally radiate more energy than a blackbody, at equilibrium.

$$j_\nu < B_\nu(T) \;\; \text{because} \;\; \kappa_\nu < 1$$

  - The radiative transfer equation in LTE can be rewritten:

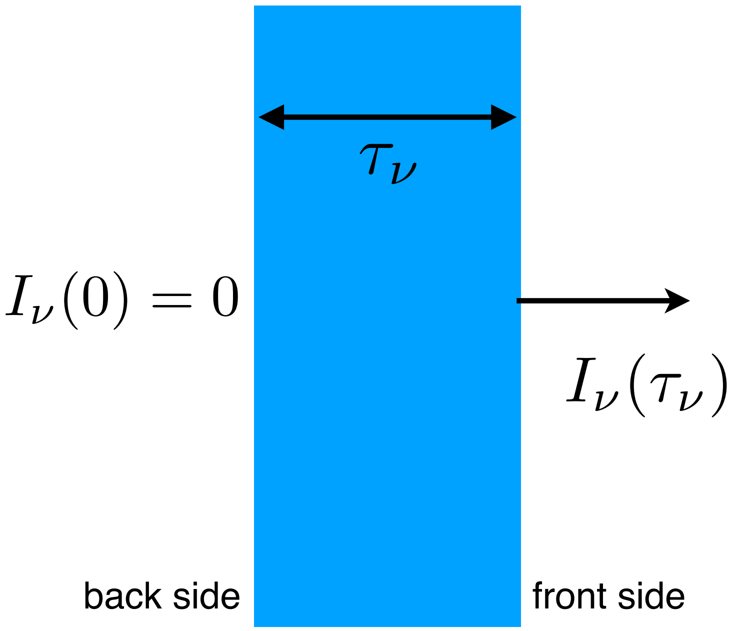
$$\boxed{\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T)}$$

- ***Blackbody radiation vs. Thermal radiation***
  - ***Blackbody radiation*** means $I_\nu = B_\nu(T)$. An object for which the intensity is the Planck function is emitting blackbody radiation.
  - ***Thermal radiation is defined to be radiation emitted by "matter" in LTE***. Thermal radiation means $S_\nu = B_\nu(T)$. An object for which the source function is the Planck function is emitting thermal radiation.
  - ***Thermal radiation becomes blackbody radiation only for optically thick media.***

Blackbody Radiation vs. Thermal Radiation

- To see the difference between thermal and blackbody radiation,
  - Consider a slab of material with optical depth τ_ν that is producing thermal radiation.
  - If no light is falling on the back side of the slab, the intensity that is measured on the front side of the slab is

$$I_\nu(0) = 0,\ S_\nu = B_\nu \longrightarrow \quad I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right) = B_\nu\left(1 - e^{-\tau_\nu}\right)$$



  - If the slab is optical thick at frequency ν $(\tau_\nu \gg 1)$, then

$$\boxed{I_\nu = B_\nu \quad \text{as } \tau_\nu \to \infty}$$

  - If the slab is optically thin $(\tau_\nu \ll 1)$, then

$$\boxed{I_\nu \approx \tau_\nu B_\nu \ll B_\nu \quad \text{as } \tau_\nu \ll 1}$$

This indicates that the radiation, although it is thermal, will not be blackbody radiation.

Thermal radiation becomes blackbody radiation only for optical thick media.

Derivation of the Planck Spectrum (Quantum Mechanics)

- There is no perfect blackbody.
  - However, the cosmic microwave background comes quite close; stars can sometimes be usefully approximated as blackbodies.
  - By the end of the 19th century, the blackbody spectrum was fairly well known empirically, from laboratory studies. In 1900, Max Planck, using his idea of quantized energies, derived the blackbody spectrum.

- How to calculate the blackbody spectrum?
  - Intensity spectrum is related to the energy density:

$$J_\nu = \frac{c}{4\pi} u_\nu$$

  - Energy density =

Number density of photon states x   Average energy of each state

  - Number density of photon states =

number of states per solid angle per volume per frequency

- (1) Number density of photon state:
  - Consider a photon propagating in direction $\mathbf{n}$ inside a box with dimensions L_x, L_y, L_z in x, y, z directions.
  - wave vector: $\mathbf{k} = \frac{2\pi}{\lambda}\mathbf{n} = \frac{2\pi\nu}{c}\mathbf{n}$
  - If each dimension of the box is much longer than a wavelength, the photon can be represented by standing wave in the box.
  - Number of nodes in each direction: $n_x = L_x/\lambda = k_x L_x/2\pi$
  - Number of node changes in a wave number interval (if $n_i \gg 1$):

$$\Delta n_x = \frac{L_x \Delta k_x}{2\pi}$$

  - Number of states in 3D wave vector element $\Delta k_x \Delta k_y \Delta k_z = d^3k$:

$$\Delta N = \Delta n_x \Delta n_y \Delta n_z = 2\frac{L_x L_y L_z d^3k}{(2\pi)^3} = 2\frac{V d^3k}{(2\pi)^3}$$

two independent polarizations
(two spin angular momentum = +/-1)

- the density of states (**number of states per solid angle per volume per frequency**):

$$d^3k = k^2 dk d\Omega = \frac{(2\pi)^3 \nu^2 d\nu d\Omega}{c^3}$$

$$\rightarrow \quad \rho_s = \frac{dN}{V d\nu d\Omega} = \frac{2\nu^2}{c^3}$$

- (2) Average energy of each state:
  - Each state may contain n photons of energy $h\nu$.
  - The energy of the state is $E_n = nh\nu$.
  - The probability of a state of energy E_n is proportional to $e^{-\beta E_n}$, where $\beta = \left(k_\mathrm{B} T\right)^{-1}$ and k_B is the Boltzmann's constant. (from statistical mechanics)
  - Therefore, the average energy is:

$$\langle E \rangle = \frac{\Sigma_{n=0}^{\infty} E_n e^{-\beta E_n}}{\Sigma_{n=0}^{\infty} e^{-\beta E_n}} = -\frac{\partial}{\partial \beta} \ln \left(\Sigma_{n=0}^{\infty} e^{-\beta E_n} \right)$$

$$\Sigma_{n=0}^{\infty} e^{-\beta E_n} = \Sigma_{n=0}^{\infty} \left(e^{-\beta h\nu}\right)^n = \left(1 - e^{-\beta h\nu}\right)^{-1}$$

$$\langle E \rangle = \frac{h\nu e^{-\beta h\nu}}{1 - e^{-\beta h\nu}} = \frac{h\nu}{\exp(h\nu/k_{\mathrm{B}}T) - 1}$$

- Energy density, integrated over all solid angle:

$$u_\nu = 4\pi\rho_s \langle E \rangle = \frac{8\pi h\nu^3/c^3}{\exp\left(h\nu/k_{\mathrm{B}}T\right) - 1}$$

- Planck Law:

$$B_\nu d\nu = B_\lambda d\lambda$$

$$J_\nu = \frac{c}{4\pi} u_\nu \quad \rightarrow \quad \boxed{B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/k_{\mathrm{B}}T) - 1} \text{ or } B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda k_{\mathrm{B}}T) - 1}}$$

See “Fundamentals of Statistical and Thermal Physics” (Federick Reif) or “Astrophysical Concepts” (Harwit) for more details.

Spectrum of blackbody radiation

- The frequency dependence of blackbody radiation is given by the ***Planck function***:

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp\left(h\nu/k_{\rm B}T\right) - 1} \quad \text{or} \quad B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp\left(hc/\lambda k_{\rm B}T\right) - 1}$$

$$h = 6.63 \times 10^{-27} \text{ erg s (Planck's constant)}$$

$$k_{\rm B} = 1.38 \times 10^{-16} \text{ erg K}^{-1} \text{ (Boltzmann's constant)}$$

- ***Energy density:***

$$u_\nu(T) = \frac{4\pi}{c} B_\nu(T) = \frac{8\pi h\nu^3/c^3}{\exp\left(h\nu/k_{\rm B}T\right) - 1}$$

Note that the textbook Ryden's "Interstellar and Intergalactic Medium" use the symbol $\varepsilon_\nu(T)$ to denote the energy density.

- Photon occupation number:
  - The photon occupation number is dimensionless, and is simply **the average number of photons per mode per polarization**.

$$n_\gamma(\nu) = \frac{1}{4\pi\rho_s}\frac{u_\nu}{h\nu} = \frac{\langle E\rangle}{h\nu}$$

$$\rho_s = \frac{2\nu^2}{c^3}$$

$$\boxed{n_\gamma(\nu) = \frac{c^2}{2h\nu^3}I_\nu}$$

  - If the radiation field is a blackbody, the photon occupation number is given by

$$\boxed{n_\gamma(\nu;\, T) = \frac{1}{\exp\left(h\nu/k_{\rm B}T\right) - 1}}$$

Bose-Einstein statistics

Stefan-Boltzmann Law

- Emergent flux is proportional to T^4.

$$F = \pi \int_0^\infty B_\nu(T) d\nu = \pi B(T) \longleftarrow B(T) = \int_0^\infty B_\nu(T) d\nu = \frac{ac}{4\pi} T^4 = \frac{\sigma_{\rm SB}}{\pi} T_4$$

$$F = \sigma_{\rm SB} T^4$$

$$\text{Stephan} - \text{Boltzmann constant}: \sigma_{\rm SB} = \frac{2\pi^5 k_{\rm B}^4}{15 c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^2 \text{ s}^{-1} \text{ K}^{-4} \text{ sr}^{-1}$$

- Total energy density ***(another form of the Stefan-Boltzmann law)***

$$u = \frac{4\pi}{c} \int_0^\infty B_\nu(T) d\nu = \frac{4\pi}{c} B(T) \qquad u(T) = \left(\frac{T}{3400 \text{ K}} \right)^4 \text{ erg cm}^{-3}$$

$$u = aT^4$$

$$\text{radiation constant}: a \equiv \frac{4\sigma_{\rm SB}}{c} = 7.57 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$$

Rayleigh-Jeans Law & Wien Law

Rayleigh-Jeans Law (low-energy limit)

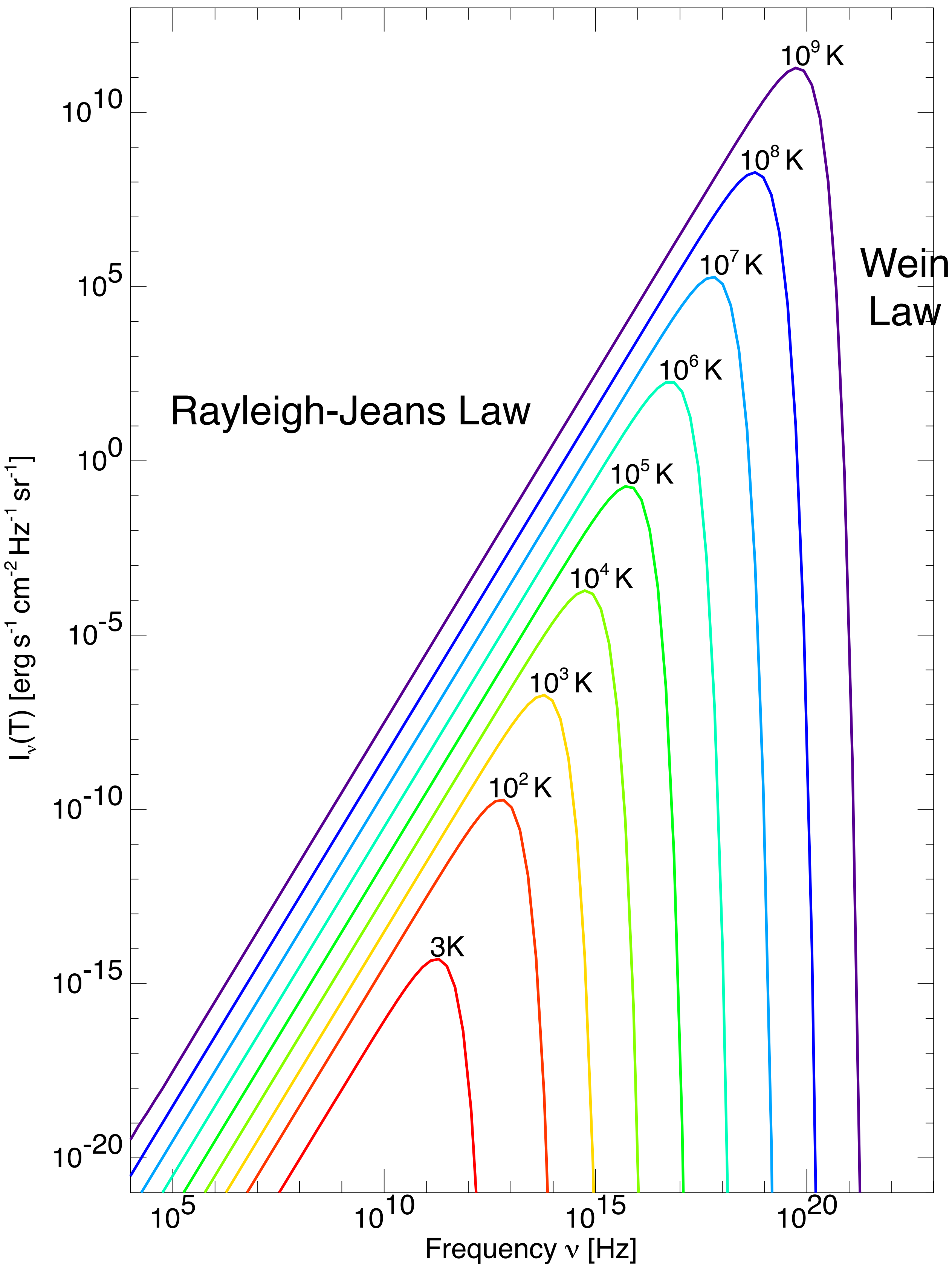
$$h\nu \ll k_{\rm B}T \quad \left(\nu \ll 2 \times 10^{10}\,(T/1\,{\rm K})\,{\rm Hz}\right)$$

$$\boxed{I_\nu^{RJ}(T) = \frac{2\nu^2}{c^2} k_{\rm B}T}$$

Wien Law (high-energy limit)

$$h\nu \gg k_{\rm B}T$$

$$\boxed{I_\nu^{W}(T) = \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{k_{\rm B}T}\right)}$$



Characteristic Temperatures

- ***Brightness Temperature:***
  - The brightness temperature is defined to be the temperature such that a blackbody at that temperature would have specific intensity:

$$I_\nu = B_\nu(T_b) \quad \rightarrow \quad T_b(\nu) = \frac{h\nu/k_\mathrm{B}}{\ln\left[1 + 2h\nu^3/(c^2 I_\nu)\right]}$$

- ***Antenna Temperature:***
  - Radio astronomers are used to working at very low frequencies. They define the antenna temperature as being the brightness temperature in the Rayleigh-Jeans tail.

$$I_\nu = \frac{2\nu^2}{c^2} k_\mathrm{B} T_b \quad \rightarrow \quad T_A \equiv \frac{c^2}{2k_\mathrm{B}\nu^2} I_\nu$$

  - Radiative transfer equation in the RJ limit:
    - If the matter has its energy levels populated according to an excitation temperature $T_\mathrm{exc} \gg h\nu/k_\mathrm{B}$, then the source function is given by $S_\nu(T_\mathrm{exc}) = \left(2\nu^2/c^2\right) k_\mathrm{B} T_\mathrm{exc}$ from the generalized Kirchhoff's law.
    - Then, RT equation becomes

$$\boxed{\frac{dT_A}{d\tau_\nu} = -T_A + T_\mathrm{exc} \quad \text{if } h\nu \ll k_\mathrm{B} T_\mathrm{exc}}$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

$$T_A = T_A(0)e^{-\tau_\nu} + T_\mathrm{exc}(1 - e^{-\tau_\nu}) \quad \text{if } T_\mathrm{exc} \text{ is constant.}$$

- ***Color Temperature:***
  - By fitting the spectrum to a blackbody curve without regarding to vertical scale (absolute intensity scale), a color temperature T_c is obtained.
  - The color temperature correctly gives the temperature of a blackbody source of unknown absolute scale.
- ***Effective Temperature:***
  - The effective temperature of a source is obtained by equating the actual flux F to the flux of a blackbody at temperature $T_{\rm eff}$.

$$F = \int \int I_\nu \cos\theta d\nu d\Omega = \sigma T_{\rm eff}^4$$

Stefan-Boltzmann law

- ***Excitation Temperature:***
  - The excitation temperature of level u relative to level ℓ is defined by

Boltzmann distribution

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{E_{u\ell}}{k_{\rm B} T_{\rm exc}}\right) \quad \rightarrow \quad T_{\rm exc} \equiv \frac{E_{u\ell}/k_{\rm B}}{\ln\left(\frac{n_\ell/g_\ell}{n_u/g_u}\right)} \quad (E_{u\ell} \equiv E_u - E_\ell)$$

  - Radio astronomers studying the 21 cm line sometimes use the term "spin temperature" $T_{\rm spin}$ for excitation temperature.