Astrophysics

Lecture 10
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Synchrotron Emission from Crab Nebula

Crab Nebula ($d \sim 2$ kpc) caused by SN explosion in 1054 A.D. - composite image. Chandra X-ray [blue], HST optical [red and yellow], Spitzer infrared [purple]. X-ray image is smaller than others as extremely energetic electrons emitting X-rays radiate away their energy more quickly than lower-energy electrons emitting in optical and infrared [Credits: NASA]

- Emission as synchrotron radiation of relativistic electrons, characteristic frequency:

$$\nu_c \simeq \frac{1}{2\pi}\gamma^2 \frac{eB}{m_e c} \simeq 280 \left(\frac{B}{10^{-4}\,\mathrm{G}}\right) \gamma^2\,\mathrm{Hz}$$

  for average $B \sim 10^{-4}$ G in Crab Nebula

- **Optical Emission:**

  Optical emission (HST) at $\nu \sim 5 \times 10^{14}$ Hz requires electrons with $\gamma \sim 10^6$.

  Cooling time scale $t_{\mathrm{cool}} \sim 2500 \left(10^6/\gamma\right)$ yr $\gtrsim t_{\mathrm{age}}$ age of Nebula.

- **X-ray Emission:**

  Chandra (ACIS, 0.2-10 keV) X-ray emission $\nu \sim 10^{17}$ Hz requires $\gamma \sim 10^7$, electrons cool quicker by a factor ~ 10.

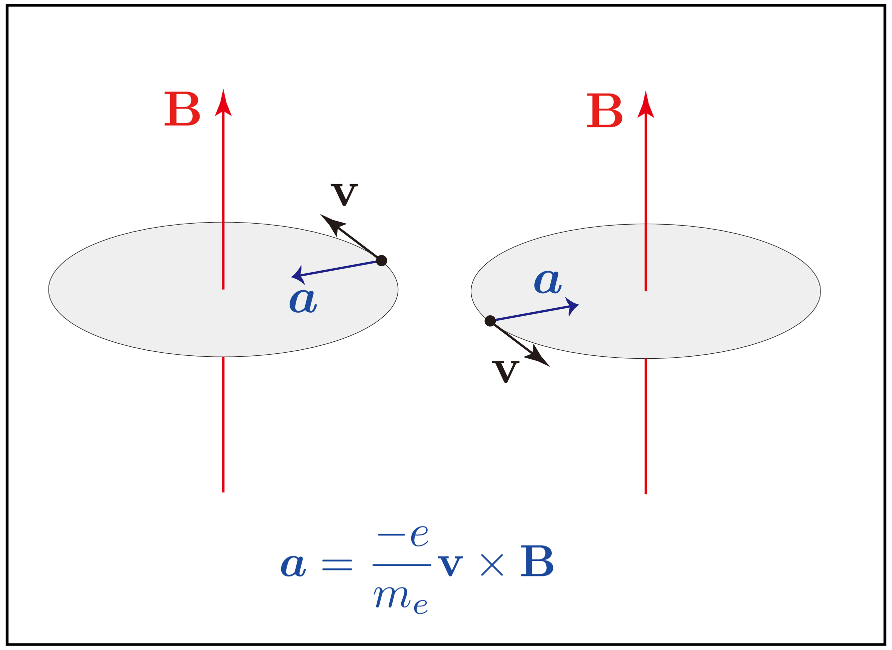
  X-ray emission spatially less extended.

  $t_{\mathrm{cool}} < t_{\mathrm{age}} \sim 950$ yr of Nebula, need continuous supply of fresh electrons.

- **Radio Emission:**

  Crab Nebula also bright in radio (NRAO, $\nu \sim 5 \times 10^9$ Hz), less energetic electrons needed, $\gamma \sim 5 \times 10^3$, size constrained by the age of Nebula

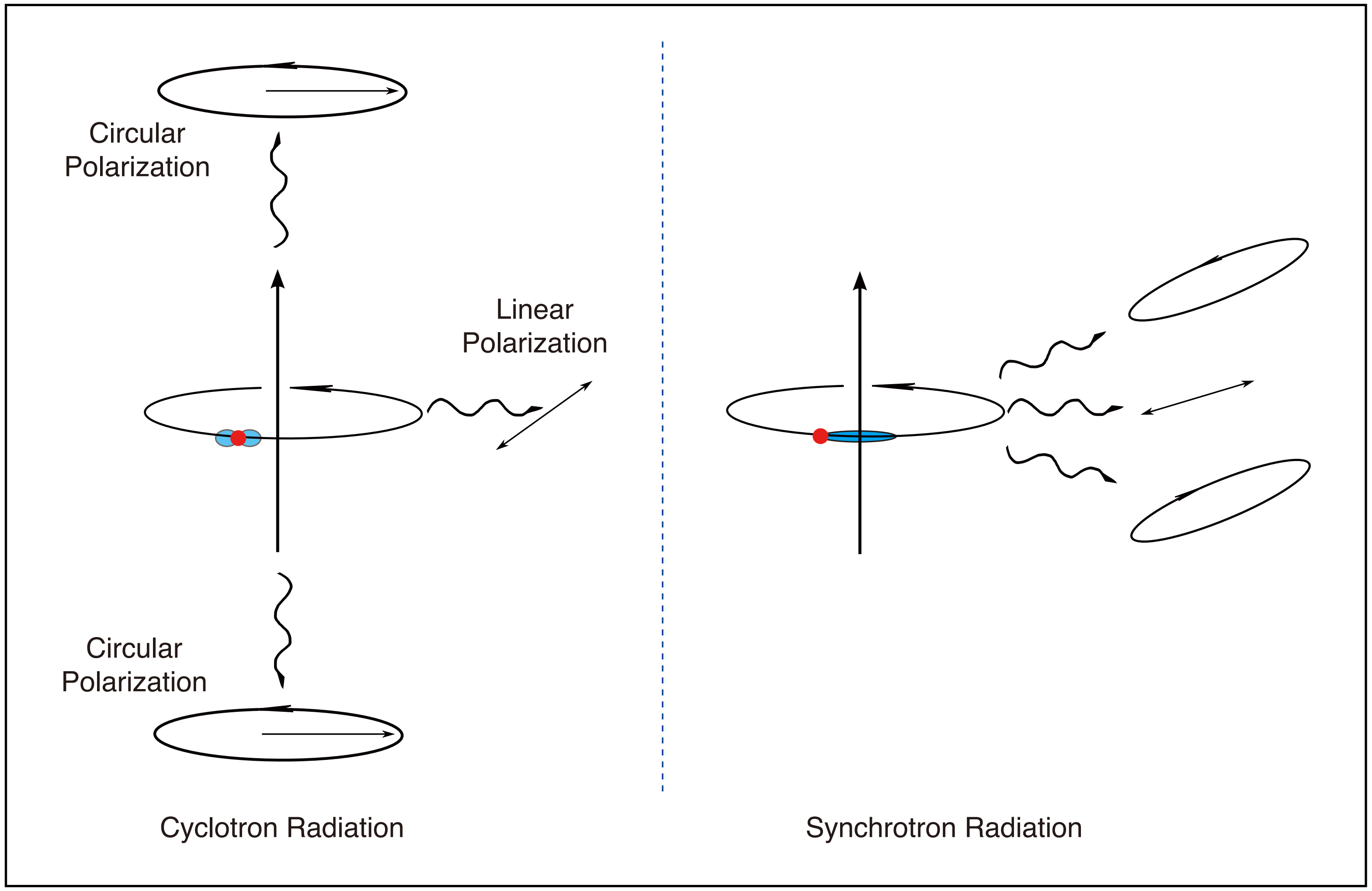
Polarization of Synchrotron Radiation



Electrons rotate counterclockwise when viewed from the positive tip of the **B** axis.

$$\text{gyrofrequency}: \omega_{\rm B} = \frac{eB}{\gamma m_e c}$$

$$\text{gyroradius}: r_{\rm B} = \frac{v_\perp}{\omega_{\rm B}} \ (v_\perp = v \sin\alpha)$$



[left] **Non-relativistic cyclotron motion**. When viewed in orbital plane, radiation is 100% linearly polarized with electric vector oscillating perpendicular to magnetic field **B**. Viewed from along **B**, emission is 100% circularly polarized. **Note that the electric vector depends only on the observer's direction and is independent of the pitch angle.**

[right] For **relativistic motion**, radiation is beamed into direction of motion. The emission for a single electron is effectively confined to within a small angle $1/\gamma$ of **v** (The electric field depends both on the observer's direction and pitch angle). The fourth Stokes parameter is an odd function of the angle between **n** and **v**. The number of electrons passing with an pitch angle α is the same as that with $-\alpha$. These two components of circular polarization effectively cancel almost, whereas linear polarization largely survives.

Compton Scattering

Thomson & Compton Scattering

- The simplest interaction between photons and free electrons is scattering.
  - **Thomson scattering:** When the energy of the incoming photons (as seen in the coming frame of the electron) is small with respect to the electron rest mass-energy, the process is called Thomson scattering.

$$\epsilon = \epsilon_1$$

$$\frac{d\sigma_T(\Omega)}{d\Omega} = \frac{1}{2} r_0^2 \left(1 + \cos^2\theta\right)$$

$$\sigma_T = \frac{8\pi}{3} r_0^2$$

$\epsilon =$ energy of the incident photon

$\epsilon_1 =$ energy of the scattered photon

$$r_0 = \frac{e^2}{m_e c^2}$$

Thomson scattering condition in the rest frame:

$\epsilon' \ll m_e c^2 = 0.5\text{MeV}$

  - When $\epsilon = \epsilon_1$, the scattering is called **coherent or elastic**.

  - **Compton scattering:** As the energy of the incoming photons is comparable or greater than the electron rest mass-energy, it is called Compton scattering and a quantum treatment is necessary (Klein-Nishina regime).

[**Compton Scattering**: Scattering from Electrons at Rest]

- **<u>Compton scattering</u>:**

However, a photon carries momentum $h\nu/c$ and energy $h\nu$.

Quantum effects appear in two ways.

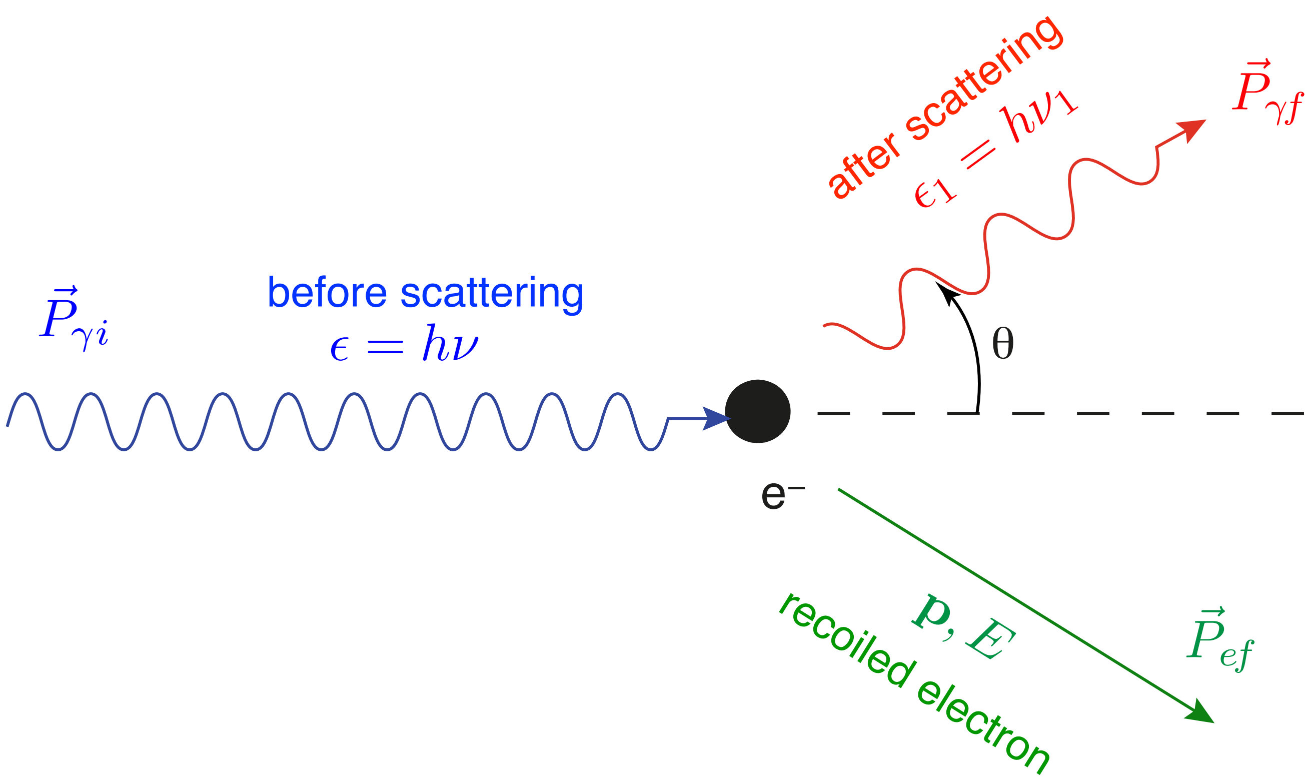
(1) The scattering will no longer be elastic ($\epsilon_1 \neq \epsilon_2$) because of the recoil of the electron.

(2) The cross sections are altered by the quantum effects.

- Conservation of momentum and energy (for the case in which **the electron is initially at rest**)

Let the initial and final four-momenta of the photon:

$$\vec{P}_{\gamma i} = (\epsilon/c)(1,\ \mathbf{n}_i),\ \vec{P}_{\gamma f} = (\epsilon_1/c)(1,\ \mathbf{n}_f)$$



The initial and final momenta of the electron are:

$$\vec{P}_{ei} = (mc,\ \mathbf{0}),\ \vec{P}_{ef} = (E/c,\ \mathbf{p})$$

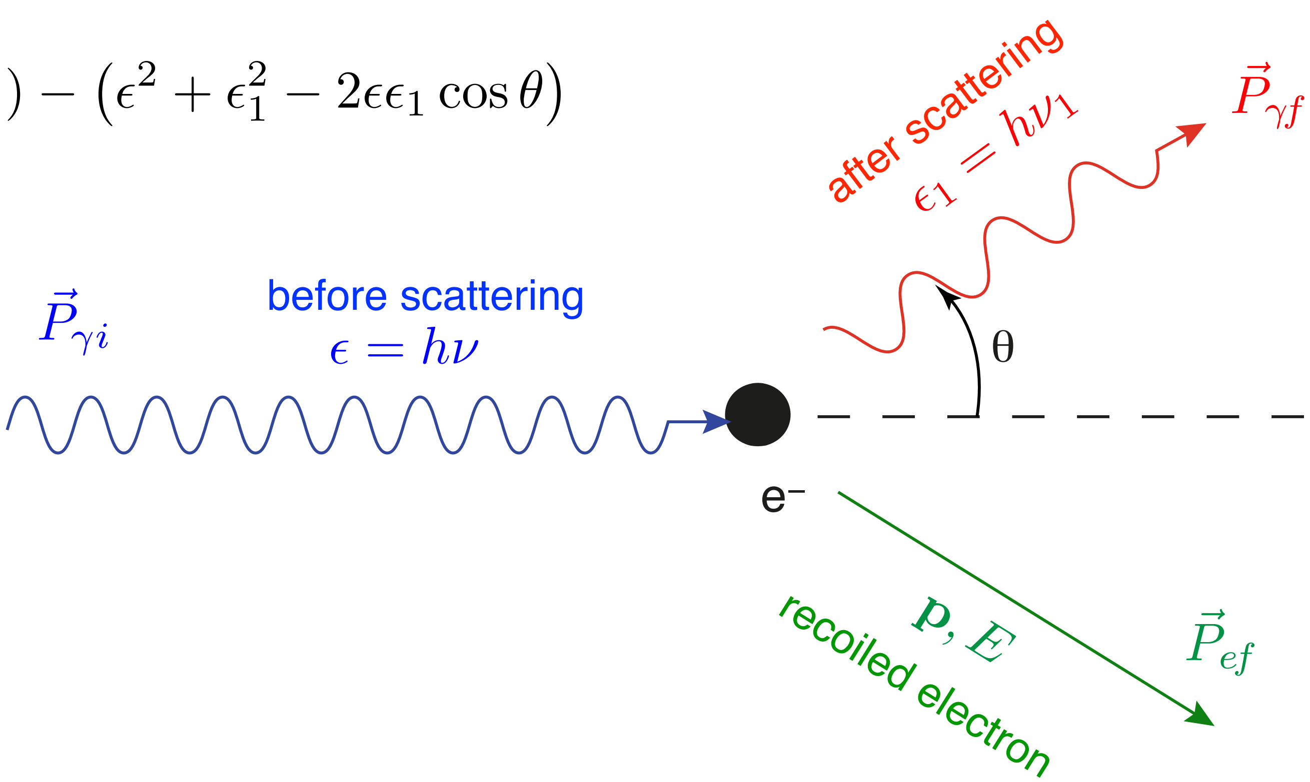
Then, the conservation of momentum and energy is expressed by

$$\vec{P}_{ei} + \vec{P}_{\gamma i} = \vec{P}_{ef} + \vec{P}_{\gamma f}$$

Here, e and γ denote the electron and photon, respectively. i and f represent the initial and final states.

- Rearranging terms and squaring gives $\left|\vec{P}_{ef}\right|^2 = \left|\vec{P}_{ei} + \vec{P}_{\gamma i} - \vec{P}_{\gamma f}\right|^2$

$$\left|\vec{P}_{ef}\right|^2 c^2 = \left|\vec{P}_{ei} + \vec{P}_{\gamma i} - \vec{P}_{\gamma f}\right|^2 c^2$$
$$E^2 - |\mathbf{p}|^2 c^2 = \left(mc^2 + \epsilon - \epsilon_1\right)^2 - |\epsilon \mathbf{n}_i - \epsilon_1 \mathbf{n}_f|^2$$
$$(mc^2)^2 = (mc^2)^2 + \epsilon^2 + \epsilon_1^2 - 2\epsilon\epsilon_1 + 2mc^2(\epsilon - \epsilon_1) - \left(\epsilon^2 + \epsilon_1^2 - 2\epsilon\epsilon_1 \cos\theta\right)$$
$$0 = mc^2\epsilon - \epsilon_1\left(\epsilon + mc^2 - \epsilon\cos\theta\right)$$



$$\epsilon_1 = \frac{\epsilon}{1 + \frac{\epsilon}{mc^2}(1 - \cos\theta)}$$

In terms of wavelength, $\lambda_1 - \lambda = \frac{h}{mc}(1 - \cos\theta)$

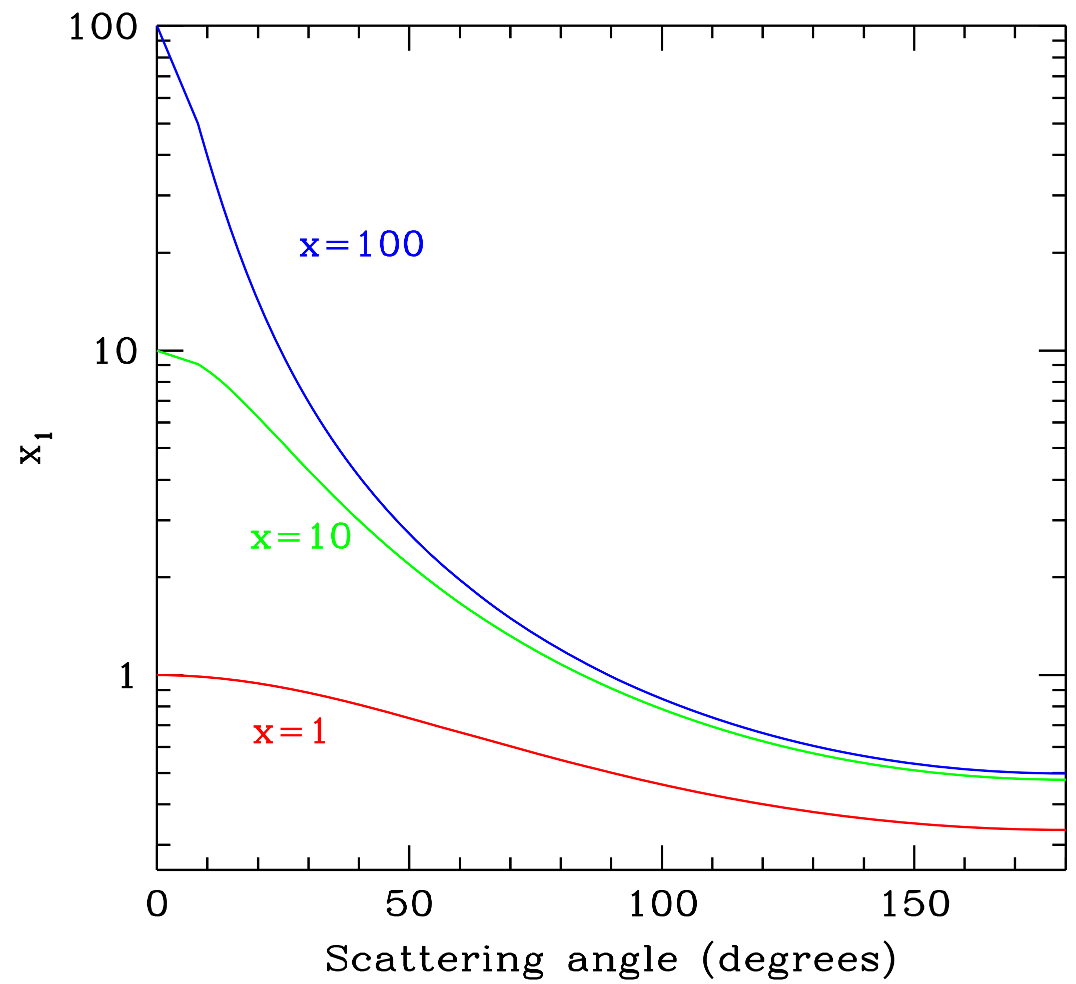
Compton wavelength: $\lambda_c \equiv \frac{h}{mc} = 0.02426\text{Å}$ for electrons

<u>There is a wavelength change of the order of λ_c upon scattering.</u>

For long wavelengths $\lambda \gg \lambda_c$ (i.e., $h\nu \ll mc^2$), the scattering is closely elastic.

$$\frac{\epsilon_1}{m_e c^2} = \frac{\epsilon/m_e c^2}{1 + (\epsilon/m_e c^2)\,(1 - \cos\theta)}$$

$$x = \frac{\epsilon}{m_e c^2}$$
$$x_1 = \frac{\epsilon_1}{m_e c^2}$$

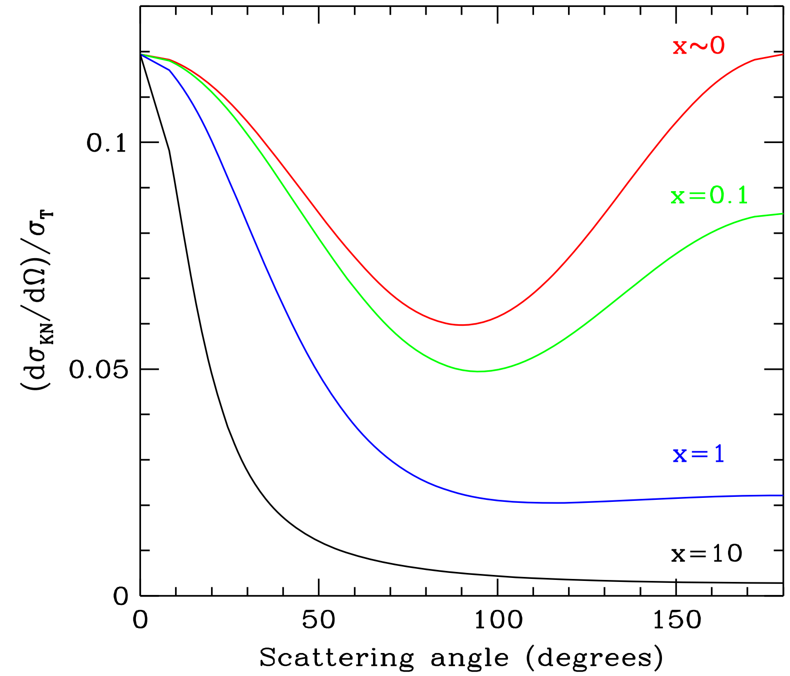


Scattered photons energies as a function of the scattering angle, for different incoming photon energies.

Note that, for $x \gg 1$ and for large scattering angle, the scattered photon energies becomes $x_1 \sim 1/2$, independent of the initial photon energy x.

- **Klein-Nishina formula** (the differential cross section for unpolarized radiation, QED)

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{16\pi}\frac{\epsilon_1^2}{\epsilon^2}\left(\frac{\epsilon}{\epsilon_1} + \frac{\epsilon}{\epsilon_1} - \sin^2\theta\right)$$



Note that the scattering becomes preferentially forward as the energy of the photon increases

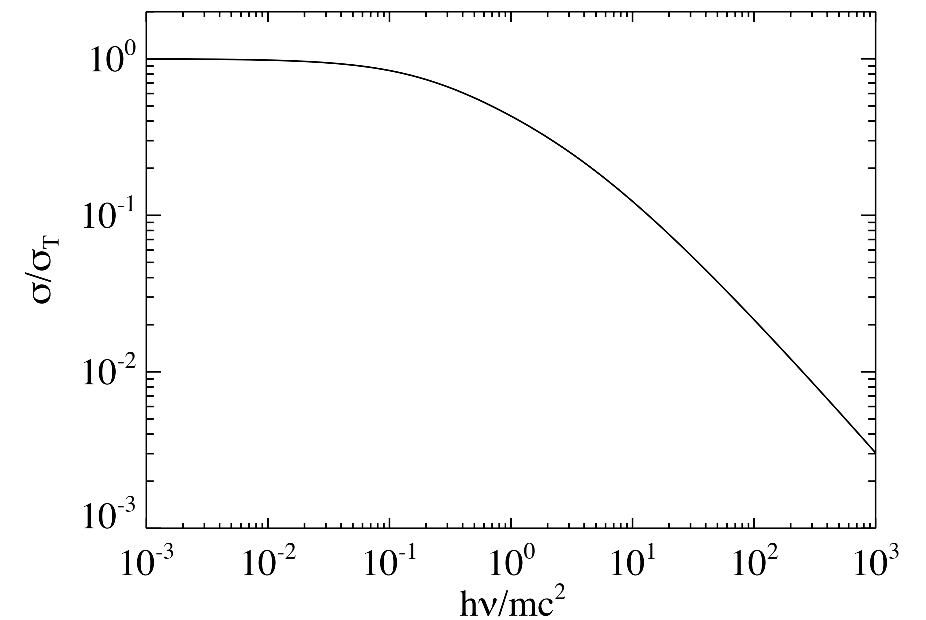
$$x = \frac{h\nu}{m_e c^2}$$

Total cross section:

$$\sigma = 2\pi\int_{-1}^{1}\frac{d\sigma}{d\Omega}d\cos\theta$$

$$= \frac{3\sigma_T}{4}\left[\frac{1+x}{x^3}\left\{\frac{2x(1+x)}{1+2x} - \ln(1+2x)\right\} + \frac{\ln(1+2x)}{2x} - \frac{1+3x}{(1+2x)^2}\right]$$

where $x \equiv \dfrac{h\nu}{mc^2}$



$(m_e c^2 = 511\text{ keV})$

Compton scattering becomes less efficient at high energies.

Approximations:

- nonrelativisitic regime: $\sigma \approx \sigma_T\left(1 - 2x + \frac{26x^2}{5} + \cdots\right), \quad x \ll 1$

- extreme relativisitic regime: $\sigma \approx \frac{3}{8}\sigma_T\frac{1}{x}\left(\ln 2x + \frac{1}{2}\right), \quad x \gg 1$

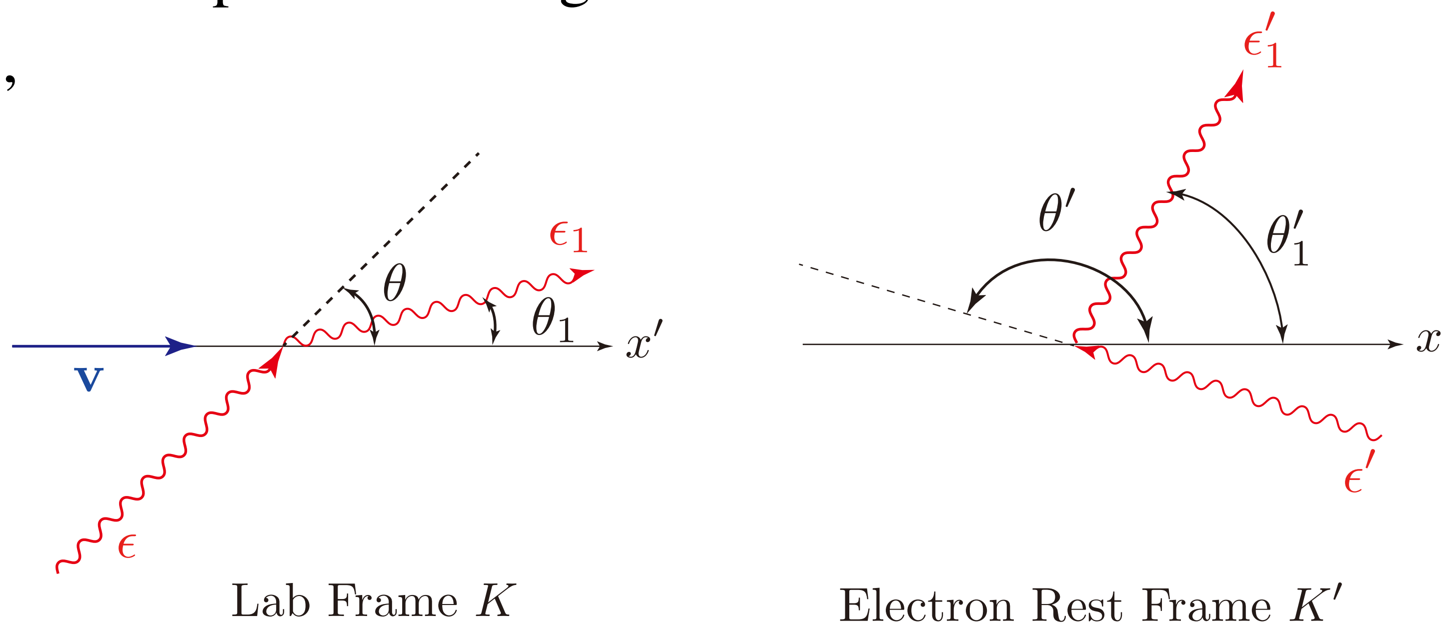
[**Inverse Compton Scattering**: Scattering from Electrons in Motion]

- **Inverse Compton Scattering**: Whenever the moving electron has sufficient kinetic energy compared to the photon, net energy may be transferred from the electron to the photon.
- What is the energy of photon after the inverse Compton scattering?

(1) In the frame K' comoving with electron, the incoming photon energy is

$$\boxed{\epsilon' = \epsilon\gamma(1-\beta\cos\theta)}$$

Here, θ is the angle between the electron velocity and the photon direction in the lab frame.



(2) In the electron rest frame, we assume the Thomson regime so that no change in the photon energy.

$$\epsilon_1' = \frac{\epsilon'}{1+\frac{\epsilon'}{mc^2}(1-\cos\Theta')}$$
$$\approx \epsilon'\left[1-\frac{\epsilon'}{mc^2}(1-\cos\Theta')\right] \quad (\text{if} \ \ \epsilon' \ll mc^2)$$
$$\approx \epsilon' \quad (\text{Thomson scattering condition})$$

$\leftarrow$

$\mathbf{n}'$ = direction vector of incident photon in the electron rest frame

$\mathbf{n}_1'$ = direction vector of scattered photon in the electron rest frame

$$\mathbf{n}' = (\sin\theta'\cos\phi', \ \ \sin\theta'\sin\phi', \ \ \cos\theta')$$
$$\mathbf{n}_1' = (\sin\theta_1'\cos\phi_1', \ \ \sin\theta_1'\sin\phi_1', \ \ \cos\theta_1')$$
$$\cos\Theta' \equiv \mathbf{n}'\cdot\mathbf{n}_1'$$
$$= \cos\theta_1'\cos\theta' + \sin\theta'\sin\theta_1'\cos(\phi'-\phi_1')$$

(Θ' = scattering angle in the electron rest frame)

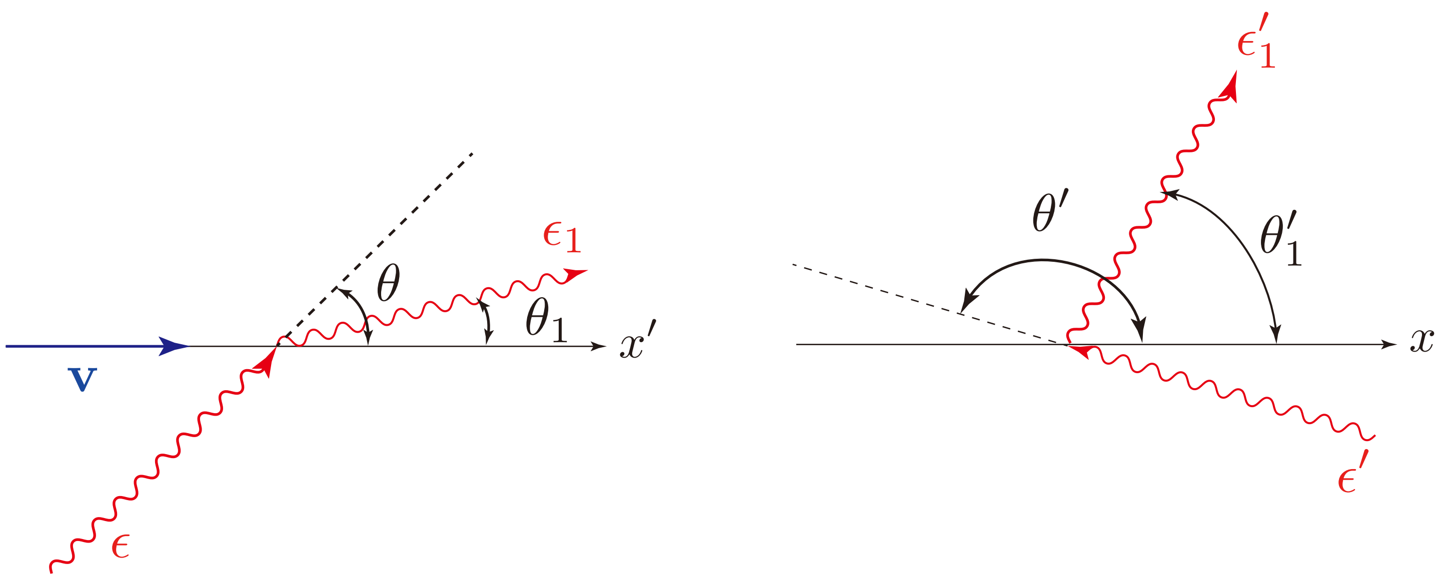
Thomson scattering condition in the rest frame:

$\epsilon' \ll m_e c^2 = 0.5\text{MeV}$

(3) Going back to the lab frame, the energy of the scattered photon is

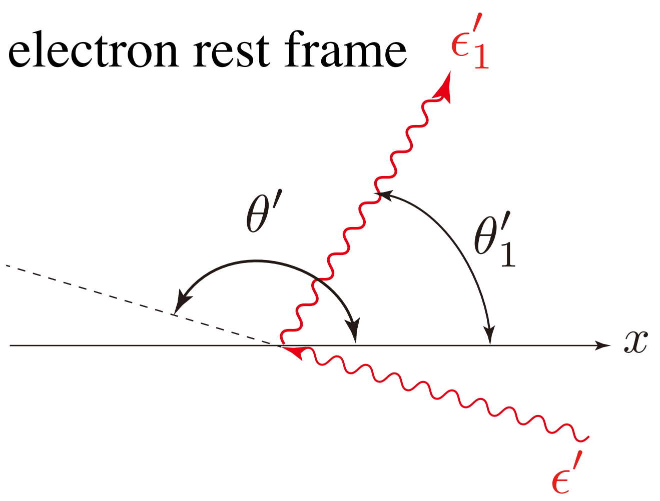
$$\boxed{\begin{aligned}\epsilon_1 &= \epsilon_1'\gamma\,(1+\beta\cos\theta_1') \\ &\approx \epsilon'\gamma\,(1+\beta\cos\theta_1') \\ &= \epsilon\gamma^2\,(1+\beta\cos\theta_1')\,(1-\beta\cos\theta)\end{aligned}}\quad\begin{aligned} & \\ &\leftarrow\ \epsilon_1' \approx \epsilon'\ \text{(Thomson scattering)} \\ &\leftarrow\ \epsilon' = \epsilon\gamma\,(1-\beta\cos\theta)\end{aligned}$$

Here, θ_1' is the scattered angle of the photon in the electron rest frame.



Lab Frame K　　Electron Rest Frame K'

$$\epsilon_1 = \epsilon\gamma^2 \left(1+\beta\cos\theta_1'\right)\left(1-\beta\cos\theta\right)$$
$$= \epsilon\frac{1-\beta\cos\theta}{1-\beta\cos\theta_1} \qquad \leftarrow \quad \cos\theta_1' = \frac{\cos\theta_1 - \beta}{1-\beta\cos\theta_1} \ \text{(aberration)}$$



- Let's assume isotropic distribution of photons.

  In the electron rest frame, most photons will be incident toward the electron, i.e., $\pi - \theta' \lesssim 1/\gamma$.

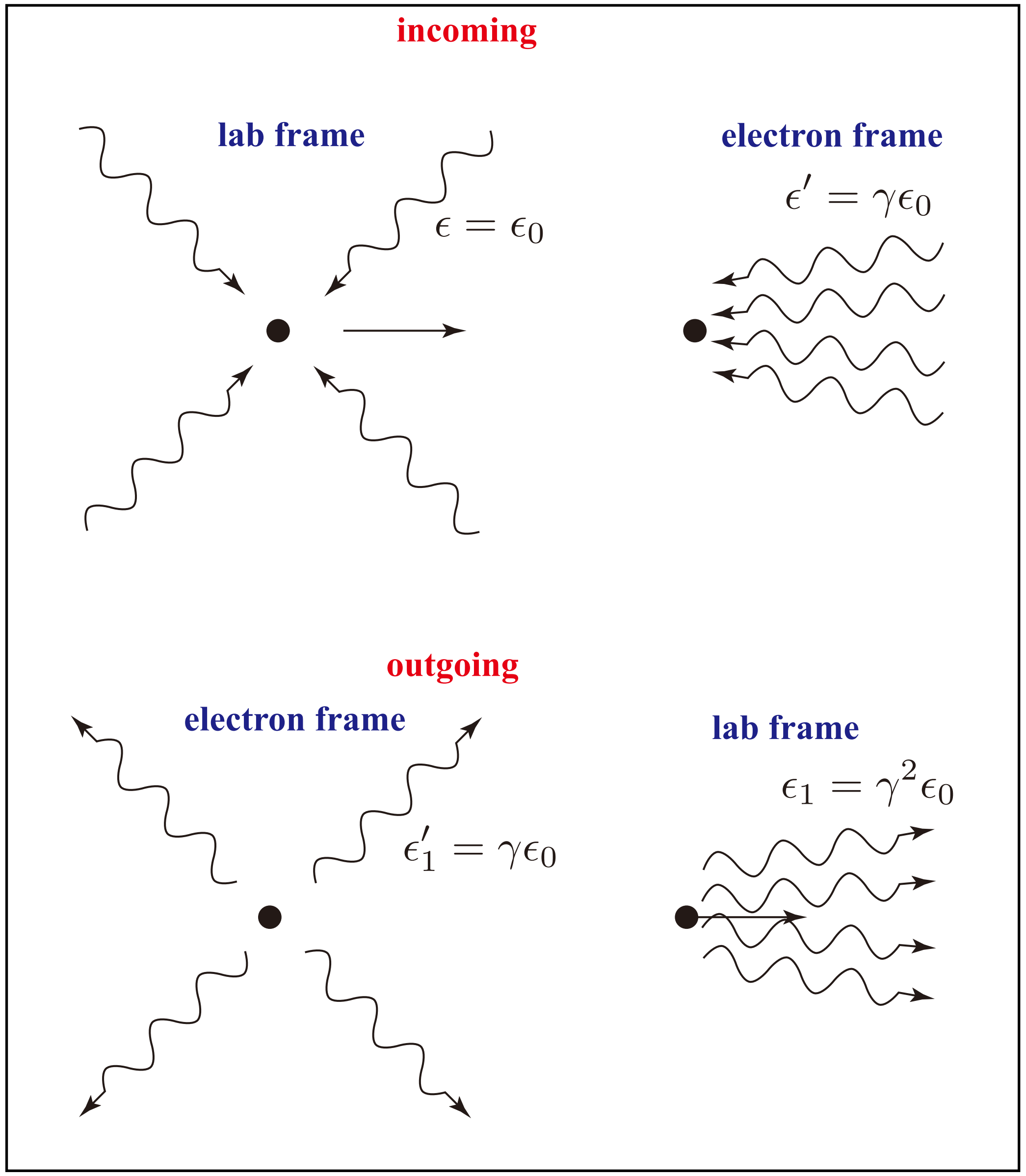
  Thomson scattering is symmetric with respect to forward and backward scattering. Therefore, in the lab frame, the scattered photon will be mostly concentrated within a narrow angle: $\theta_1 \lesssim 1/\gamma$. Assuming that $\cos\theta_1 \approx \beta$, we obtain

$$\epsilon_1 \approx \epsilon\frac{1-\beta\cos\theta}{1-\beta^2} = \left(1-\beta\cos\theta\right)\gamma^2\epsilon \quad \rightarrow \boxed{\langle\epsilon_1\rangle \approx \gamma^2\epsilon}$$

(because $\langle\cos\theta\rangle = 0$)

See the following slides for a precise formula.

$$\boxed{\langle\epsilon_1\rangle = \gamma^2\left(1+\frac{\beta^2}{3}\right)\epsilon}$$



- The energies of the photon before scattering, in the electron rest frame, and after the scattering in the lab frame are in the approximate ratios

The inverse Compton scattering converts a low-energy photon to a high-energy photon by a factor of order γ^2

$$\boxed{\epsilon : \epsilon' : \epsilon_1 \approx 1 : \gamma : \gamma^2}$$

[Inverse Compton Power for Single Scattering]

- Assumptions:

  (1) Isotropic distributions of photons and electrons.

  (2) The change in energy of the photon in the rest frame is negligible.

  (Thomson scattering is applicable in the electron's rest frame). $\epsilon_1' \approx \epsilon'$

- **Total power scattered in the electron's rest frame**:

$$\frac{dE_1'}{dt'} = c\sigma_T \int \epsilon_1' n_\epsilon' d\epsilon'$$

where $n_\epsilon' d\epsilon'$ is the number density of incident photons.

- Recall: $\dfrac{dE_1}{dt} = \dfrac{dE_1'}{dt'}$ since energy and time transforms in the same way.

  $d^3\mathbf{p} = \gamma d^3\mathbf{p}'$ transforms in the same way as energy.

  $n_p \equiv \dfrac{dN}{d\mathcal{V}} \left(= \dfrac{d^6 N}{d^3\mathbf{x} d^3\mathbf{p}} \right)$ is a Lorentz invariant. (density in the phase space)

$$n_p d^3\mathbf{p} = n_\epsilon d\epsilon$$
$$n_\epsilon d\epsilon = \gamma n_\epsilon' d\epsilon'$$

The number densities of incident photons, represented in terms of momentum and energy, transforms in the same way as energy.

$$\boxed{\therefore \quad \frac{n_\epsilon d\epsilon}{\epsilon} = \frac{n_\epsilon' d\epsilon'}{\epsilon'}}$$

- Thus we have the results

$\epsilon'_1 \approx \epsilon'$ **Thomson scattering assumption in the rest frame**

$$\frac{dE_1}{dt} = \frac{dE'_1}{dt'} = c\sigma_T \int \epsilon'_1 n'_\epsilon d\epsilon' = c\sigma_T \int \epsilon'^2 \frac{n'_\epsilon d\epsilon'}{\epsilon'} = c\sigma_T \int \epsilon'^2 \frac{n_\epsilon d\epsilon}{\epsilon}$$

$$= c\sigma_T \gamma^2 \int (1 - \beta\cos\theta)^2 \, \epsilon n_\epsilon d\epsilon \qquad \leftarrow \quad \epsilon' = \epsilon\gamma(1 - \beta\cos\theta)$$

For an isotropic distribution of photons, integrating over θ, we have

$$\left\langle (1 - \beta\cos\theta)^2 \right\rangle = 1 + \frac{1}{3}\beta^2 \qquad \leftarrow \quad \langle \cos\theta \rangle = 0, \quad \left\langle \cos^2\theta \right\rangle = 1/3$$

Therefore, we obtain the **total power scattered in the lab frame**:

$$\frac{dE_1}{dt} = c\sigma_T \gamma^2 \left(1 + \frac{1}{3}\beta^2\right) U_{\rm ph}$$

where $U_{\rm ph} \equiv \int \epsilon n_\epsilon d\epsilon$ is the initial photon energy density.

Note that **the rate of decrease of the total initial photon energy** is

$$\frac{dE_1^{\rm loss}}{dt} = -c\sigma_T \int \epsilon n_\epsilon d\epsilon = -c\sigma_T U_{\rm ph}$$

The incident power and scattered power in the lab frame can be represented in term of the rate of scattering (per unit time) and the initial and final photon energies.

$$\left|\frac{dE_1^{\rm loss}}{dt}\right| = \epsilon \frac{dN_{\rm scatt}}{dt}$$

$$\frac{dE_1}{dt} = \langle \epsilon_1 \rangle \frac{dN_{\rm scatt}}{dt}$$

Then, the mean photon energy after scattering will be

$$\langle \epsilon_1 \rangle = \gamma^2 \left(1 + \frac{1}{3}\beta^2\right) \epsilon$$

- Thus, **the net power lost by the electron, and converted into increased radiation**, is

$$P_{\rm compt} \equiv \frac{dE_1}{dt} - \left|\frac{dE_1^{\rm loss}}{dt}\right| = c\sigma_T U_{\rm ph}\left[\gamma^2\left(1+\frac{1}{3}\beta^2\right)-1\right]$$

$$\boxed{\therefore \quad P_{\rm compt} = \frac{4}{3}c\sigma_T\gamma^2\beta^2 U_{\rm ph}} \longleftarrow \qquad \gamma^2 - 1 = \gamma^2\beta^2$$

- **When the energy transfer in the electron rest frame is not neglected**, the power is given by

$$P_{\rm compt} = \frac{4}{3}c\sigma_T\gamma^2\beta^2 U_{\rm ph}\left[1 - \frac{63}{10}\frac{\gamma\left\langle\epsilon^2\right\rangle}{mc^2\left\langle\epsilon\right\rangle}\right] \qquad \text{(cf. Blumenthal \& Gould, 1970)}$$

Note that the above equation allows energy to be either given or taken from the photons.

- Recall that the formula for the synchrotron power emitted by each electron is

$$P_{\rm synch} = \frac{4}{3}\sigma_T c\gamma^2\beta^2 U_{\rm B}$$

Therefore, $\boxed{\dfrac{P_{\rm synch}}{P_{\rm compt}} = \dfrac{U_{\rm B}}{U_{\rm ph}}}$

The radiation losses due to synchrotron emission and to inverse Compton effect are in the same ratio as the magnetic field energy density and photon energy density.

- Let $N(\gamma)\mathrm{d}\gamma$ be the number of electrons per unit volume. Then, the total Compton power per unit volume is

$$P_{\rm tot} = \int P_{\rm compt} N(\gamma) d\gamma$$

(1) Power-law distribution of relativistic electrons ($\beta \sim 1$)

$$N(\gamma) = \begin{cases} C\gamma^{-p}, & \gamma_{\rm min} \le \gamma \le \gamma_{\rm max} \\ 0, & \text{otherwise} \end{cases} \longrightarrow \quad P_{\rm tot} = \frac{4}{3}\sigma_T c U_{\rm ph} C(3-p)^{-1}\left(\gamma_{\rm min}^{3-p} - \gamma_{\rm max}^{3-p}\right)$$

(2) **Thermal distribution** of nonrelativistic electrons ($\gamma \sim 1$) of number density n_e.

$$\langle\beta^2\rangle = \langle v^2/c^2\rangle = 3kT/mc^2 \longrightarrow \quad P_{\rm tot} = \left(\frac{4kT}{mc^2}\right)\sigma_T c n_e U_{\rm ph}$$

$$\gamma \approx 1$$

Note that $\dfrac{4kT}{mc^2}$ is the fractional photon energy gain because $\dfrac{dE^{\rm loss}}{dt} = -c\sigma_{\rm T}U_{\rm ph}$.

[Spectrum of single-scattered photons by power-law electrons]

- **We will first show that a power-law photon distribution is produced from a power-law electron distribution.**

The photon energy increases after a single scattering by a factor proportional to γ^2.

$$\boxed{\langle \epsilon_1 \rangle = \gamma^2 \left(1 + \frac{1}{3}\beta^2\right) \epsilon = \frac{4}{3}\gamma^2 \epsilon \quad \text{as} \quad \beta \to 1}$$

Therefore, the (energy) spectrum scattered power per unit volume per unit energy due to electrons with a power-law distribution is

$$x \equiv \frac{4}{3}\epsilon\gamma^2$$

$$dx = \frac{2x d\gamma}{\gamma}$$

$$\longrightarrow$$

$$\begin{aligned}
\frac{dE}{dV\,dt\,d\epsilon_1} &= \int N(\gamma)\overset{\text{energy}}{\epsilon_1}\delta\left(\epsilon_1 - (4/3)\epsilon\gamma^2\right) d\gamma \\
&\propto \int_{\gamma_1}^{\gamma_2} \epsilon_1 \gamma^{-p} \delta\left(\epsilon_1 - x\right) \frac{d\gamma}{dx} dx \\
&\propto \int_{\gamma_1}^{\gamma_2} \epsilon_1 \gamma^{-p} \delta\left(\epsilon_1 - x\right) \frac{\gamma}{2x} dx \\
&\propto \int_{\gamma_1}^{\gamma_2} \epsilon_1 x^{-(p-1)/2-1} \delta\left(\epsilon_1 - x\right) dx \\
&\propto \epsilon_1^{-(p-1)/2}
\end{aligned}$$

$$\boxed{\frac{dE}{dV\,dt\,d\epsilon_1} \propto \epsilon_1^{-(p-1)/2}}$$

The resulting spectrum has the same power-law slope as that of the synchrotron.

[Repeated Scattering: The Compton y Parameter]

- We restrict our considerations to situations in which the Thomson limit applies: $\gamma\epsilon \ll mc^2$
- **Compton y parameter**, to determine whether a photon will significantly change its energy in traversing the medium:

$$y \equiv \left(\begin{array}{l} \text{average fractional} \\ \text{energy change per} \\ \text{scattering} \end{array} \right) \times \left(\begin{array}{l} \text{mean number of} \\ \text{scatterings} \end{array} \right)$$

When $y \gtrsim 1$, the total photon energy and spectrum will be significantly altered; whereas for $y \ll 1$, the total energy is not much changed.

- **Average fractional energy change per scattering** (for a thermal distribution of electrons)

(a) Consider first the nonrelativistic limit.

$$\epsilon_1' \approx \epsilon' \left[1 - \frac{\epsilon'}{mc^2} (1 - \cos\Theta) \right] \quad \rightarrow \quad \left\langle \frac{\Delta\epsilon'}{\epsilon'} \right\rangle \equiv \left\langle \frac{\epsilon_1' - \epsilon'}{\epsilon'} \right\rangle = -\frac{\epsilon'}{mc^2} \quad \text{: angle average}$$

In the lab frame to lowest order, this must be of the form

$$\left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle = -\frac{\epsilon}{mc^2} + \alpha \frac{kT}{mc^2}$$

To calculate α, **imagine that the photons and electrons are in complete equilibrium but interact only through scattering**.

Assume that the photon density is sufficiently small that stimulated processes can be neglected. Then, we obtain the Wien's law for the photon distribution:

$$n_\epsilon = K\epsilon^2 \exp\left(-\frac{\epsilon}{kT}\right)$$

We have the averages
$$\langle\epsilon\rangle \equiv \int \epsilon n_\epsilon d\epsilon \bigg/ \int n_\epsilon d\epsilon = 3kT$$
$$\left\langle\epsilon^2\right\rangle \equiv \int \epsilon^2 n_\epsilon d\epsilon \bigg/ \int n_\epsilon d\epsilon = 12(kT)^2$$

For this case, no net energy can be transferred from photons to electrons, so

$$\Delta\epsilon = 0 = -\frac{\left\langle\epsilon^2\right\rangle}{mc^2} + \alpha\frac{kT}{mc^2}\langle\epsilon\rangle = \frac{3kT}{mc^2}(\alpha-4)kT \quad \rightarrow \quad \alpha = 4$$

Thus for nonrelativisitic electrons in thermal equilibrium, the energy transfer per scattering is

$$\boxed{(\Delta\epsilon)_{\rm NR} = \frac{\epsilon}{mc^2}(4kT-\epsilon)}$$

Note that if the electrons have high enough temperature relative to incident photons, the photons gain energy. This is the inverse Compton scattering.

If $\epsilon > 4kT$, on the other hand, energy is transferred from photons to electrons.

(b) In the ultrarelativistic limit $(\gamma \gg 1,\ \beta \approx 1)$, ignoring the energy transfer in the electron rest frame,

$$\frac{P_{\rm compt}}{\left|dE_1^{\rm loss}/dt\right|} = \frac{4/3\sigma_T c\gamma^2\beta^2 U_{\rm ph}}{\sigma_T c U_{\rm ph}} = \frac{4}{3}\gamma^2\beta^2 \quad \rightarrow \quad (\Delta\epsilon)_{\rm R} \approx \frac{4}{3}\gamma^2\epsilon$$

For a thermal distribution of ultrarelativistic electrons,

$$\langle\gamma^2\rangle = \frac{\langle\epsilon^2\rangle}{(mc^2)^2} = 12\left(\frac{kT}{mc^2}\right)^2 \longrightarrow \boxed{(\Delta\epsilon)_{\rm R} \approx 16\epsilon\left(\frac{kT}{mc^2}\right)^2}$$

- **Mean number of scatterings**,

  Recall that, for a pure scattering medium,

$$\begin{pmatrix}\text{mean number of} \\ \text{scatterings}\end{pmatrix} \approx {\rm Max}(\tau_{\rm es},\ \tau_{\rm es}^2)$$

where $\tau_{\rm es} \sim \rho\kappa_{\rm es}R$

$$\kappa_{\rm es} = \frac{\sigma_T}{m_p} = 0.40\ {\rm cm}^2\ {\rm g}^{-1} \text{ for ionized hydrogen}$$

$$R = \text{ size of the finite medium}$$

- **Compton y parameter**:

$$\boxed{y_{\rm NR} = \frac{4kT}{mc^2}{\rm Max}(\tau_{\rm es},\ \tau_{\rm es}^2)} \qquad \boxed{y_{\rm R} = 16\epsilon\left(\frac{kT}{mc^2}\right)^2{\rm Max}(\tau_{\rm es},\ \tau_{\rm es}^2)}$$

[Repeated Scattering: Spectra and Power]

- We have already shown that a power-law spectrum may be a natural consequence of a power-law distribution of electrons.
- **We will show that a power-law photon distribution can also be produced from repeated scattering off a nonpower-law electron distribution.**

Let A = the mean amplification of photon energy per scattering

$$A \equiv \frac{\epsilon_1}{\epsilon} \sim \frac{4}{3}\langle\gamma^2\rangle$$
$$= 16\left(\frac{kT}{mc^2}\right)^2$$

for thermal electron distribution

mean photon energy = ϵ_i

(number) intensity = $I(\epsilon_i)$ at ϵ_i

After k scattering, the photon energy will be $\epsilon_k \sim \epsilon_i A^k$.

For an optically thin scattering medium ($\tau_{\rm es} < 1$), the probability of a photon undergoing k scattering before escaping the medium is $p_k(\tau_{\rm es}) \sim \tau_{\rm es}^k$.

$$\tau^{\ln(\epsilon_k/\epsilon_i)/\ln A} = x$$
$$[\ln(\epsilon_k/\epsilon_i)/\ln A]\ln\tau = \ln x$$
$$\ln(\epsilon_k/\epsilon_i)^{\ln\tau/\ln A} = \ln x$$
$$(\epsilon_k/\epsilon_i)^{\ln\tau/\ln A} = x$$

<u>The emergent intensity at energy ϵ_k is given by</u>

$$I(\epsilon_k) \sim I(\epsilon_i)\tau_{\rm es}^k \sim I(\epsilon_i)\tau_{\rm es}^{\ln(\epsilon_k/\epsilon_i)/\ln A} = I(\epsilon_i)\left(\frac{\epsilon_k}{\epsilon_i}\right)^{\ln\tau_{\rm es}/\ln A}$$

$$\therefore\ I(\epsilon_k) \sim I(\epsilon_i)\left(\frac{\epsilon_k}{\epsilon_i}\right)^{-\alpha} \quad \text{where} \quad \alpha \equiv \frac{-\ln\tau_{\rm es}}{\ln A}$$

$\longrightarrow$ power-law shape

- Total Compton power in the output spectrum is given by

$$P \propto \int I(\epsilon_k) d\epsilon_k = I(\epsilon_i)\epsilon_i \left[\int x^{-\alpha} dx\right]$$

The factor in square brackets is approximately **the factor by which the initial power** $I(\epsilon_i)\epsilon_i$ **is amplified** in energy.

Clearly, this amplification will be important if $\alpha \ll 1$. Therefore, **energy amplification of a soft photon input spectrum is important when**

$$\alpha = \frac{-\ln \tau_{\rm es}}{\ln A} \lesssim 1 \quad \rightarrow \quad \ln(\tau_{\rm es} A) \gtrsim 0$$

$$\rightarrow \quad y = A\tau_{\rm es} \sim 16 \left(\frac{kT}{mc^2}\right)^2 \tau_{\rm es} \gtrsim 1$$