Astrophysics

Lecture 03
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Boundary conditions: Two-stream approximation

- To solve the second order differential equation, we need two boundary conditions. The boundary conditions can be provided in several ways. One way to do is to use two-stream approximation, in which the entire radiation field is represented by radiation at just two angles, i.e., $\mu = \pm\,\mu_0$:

$$\boxed{I(\tau,\mu) = I^+(\tau)\delta(\mu-\mu_0) + I^-(\tau)\delta(\mu+\mu_0)}$$

- The two terms denote the outward and inward intensities. Then, the three moments are

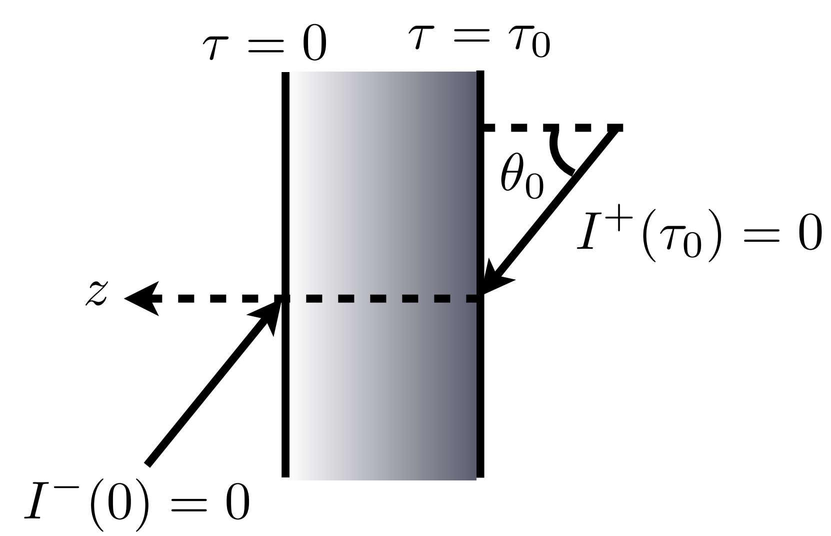
$$\begin{aligned} J &= \frac{1}{2}\left(I^+ + I^-\right) \\ H &= \frac{1}{2}\mu_0\left(I^+ - I^-\right) \\ K &= \frac{1}{2}\mu_0^2\left(I^+ + I^-\right) \end{aligned} \quad \rightarrow \text{ we obtain } \mu_0 = \frac{1}{\sqrt{3}} \text{ in order to satisty } K = \frac{1}{3}J$$

$$(\theta_0 = \cos^{-1}\mu_0 = 54.74^\circ)$$

- Using $H = \dfrac{1}{3}\dfrac{\partial J}{\partial \tau}$, we obtain: $$\boxed{I^+ = J + \frac{1}{3}\frac{\partial J}{\partial \tau}, \quad I^- = J - \frac{1}{3}\frac{\partial J}{\partial \tau}}$$

Suppose the medium extends from $\tau = 0$ to $\tau = \tau_0$ and there is no incident radiation. Then, we obtain two boundary conditions:

$$I^+(\tau_0) = 0 \ \text{ and } \ I^-(0) = 0 \ \rightarrow \ \boxed{\begin{aligned} \frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau} &= J \quad \text{at } \ \tau = 0 \\ \frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau} &= -J \ \text{ at } \ \tau = \tau_0 \end{aligned}}$$



Note the N-stream approximation

Iteration Method

- Dust radiative transfer (in UV/optical wavelengths):

$$\frac{dI(s)}{ds} = -\alpha^{\rm ext} I(s) + \alpha^{\rm sca} \int \Phi(\Omega,\Omega') I(s,\Omega') d\Omega' + j(s)$$

$$\text{or} \quad \frac{dI(\tau)}{d\tau} = -I(\tau) + a \int \Phi(\Omega,\Omega') I(\tau,\Omega') d\Omega' + \mathsf{S}(\tau) \qquad \left(d\tau \equiv \alpha^{\rm ext} ds, \ \mathsf{S}(\tau) \equiv \frac{j(\tau)}{\alpha^{\rm ext}} \right)$$

  - Let I_0 be the intensity of photons that come directly from the source, I_1 the intensity of photons that have been scattered once by dust, and I_n the intensity after *n* scatterings. Then,

$$\boxed{I(s) = \sum_{n=0}^{\infty} I_n(s)}$$

  - The intensities I_n satisfy the equations.

$$\begin{aligned} \frac{dI_0(\tau)}{d\tau} &= -I_0(\tau) + \mathsf{S}(\tau) \\ \frac{dI_n(\tau)}{d\tau} &= -I_n(\tau) + a \int \Phi(\Omega,\Omega') I_{n-1}(\tau,\Omega') d\Omega' \\ &\equiv -I_n(\tau) + \mathsf{S}_{n-1}(\tau) \qquad \left(\mathsf{S}_{n-1}(\tau) \equiv a \int \Phi(\Omega,\Omega') I_{n-1}(\tau,\Omega') d\Omega' \right) \end{aligned}$$

  - Then, the formal solutions are:

$$\rightarrow \boxed{\begin{aligned} I_0(\tau) &= e^{-\tau} I_0(0) + \int_0^{\tau} e^{-(\tau-\tau')} \mathsf{S}(\tau') d\tau' \\ I_n(\tau) &= e^{-\tau} I_n(0) + \int_0^{\tau} e^{-(\tau-\tau')} \mathsf{S}_{n-1}(\tau') d\tau' \end{aligned}}$$

(1) application to the edge-on galaxies

- The solution can be further simplified by assuming that

$$\frac{I_n}{I_{n-1}} \approx \frac{I_1}{I_0} \quad (n \geq 2)$$

- Then, the infinite series becomes

$$\boxed{I \approx I_0 \sum_{n=0}^{\infty} \left(\frac{I_1}{I_0} \right)^n = \frac{I_0}{1 - I_1/I_0}} \longleftarrow$$

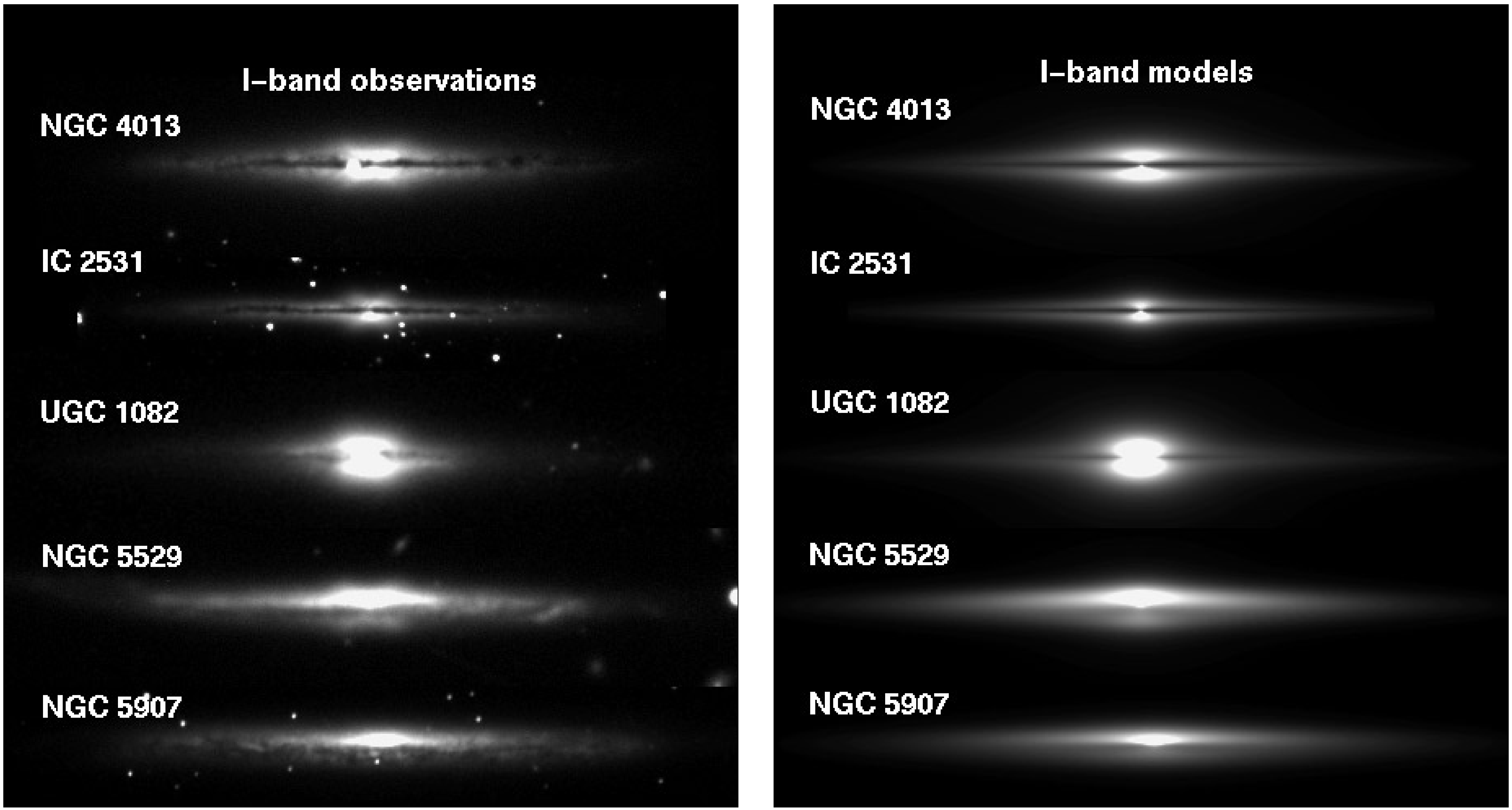
$$I = \sum_{n=0}^{\infty} I_n$$

$$\begin{aligned} I_n &= \frac{I_n}{I_{n-1}} I_{n-1} \\ &\approx \left(\frac{I_1}{I_0} \right)^n \end{aligned}$$

$$\uparrow$$

$$\sum_{n=0}^{\infty} \beta^n = \frac{1}{1-\beta} \quad \text{for} \quad \beta < 1$$

- Kylafis & Bahcall (1987) and Xilouris et al. (1997, 1998, 1999) applied this approximation to model the dust radiative transfer process in edge-on galaxies.

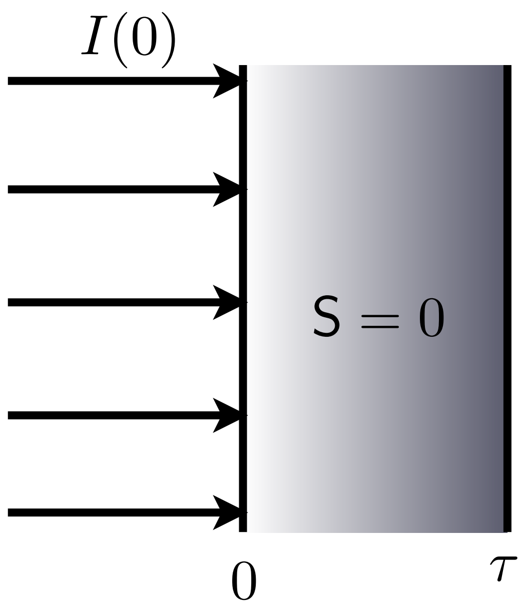


Xilouris et al. (1999)

(2) solution for the perfect forward scattering

- Assume the perfect forward-scattering:

$$\boxed{\begin{aligned}&\Phi(\Omega,\Omega') = \delta(\Omega' - \Omega)\\ &\rightarrow \quad \mathsf{S}_{n-1}(\tau) = aI_{n-1}(\tau)\end{aligned}}$$



- Then, the solution can be given by iteration:

$$I_0(\tau) = e^{-\tau} I_0(0)$$

$$\rightarrow \quad \mathsf{S}_0(\tau) = aI_0(\tau) = ae^{-\tau} I_0(0)$$

$$I_1(\tau) = e^{-\tau}\int_0^\tau e^{\tau'}\mathsf{S}_0(\tau')d\tau' = (a\tau)e^{-\tau} I_0(0)$$

$$\rightarrow \quad \mathsf{S}_1(\tau) = aI_1(\tau) = (a^2\tau)e^{-\tau} I_0(0)$$

$$I_2(\tau) = e^{-\tau}\int_0^\tau e^{\tau'}\mathsf{S}_1(\tau')d\tau' = \frac{(a\tau)}{2}e^{-\tau} I_0(0)$$

$$\rightarrow \quad \mathsf{S}_2(\tau) = aI_2(\tau) = \frac{(a^3\tau^2)}{2}e^{-\tau} I_0(0)$$

$$I_3(\tau) = e^{-\tau}\int_0^\tau e^{\tau'}\mathsf{S}_2(\tau')d\tau' = \frac{(a\tau)^3}{3\times 2}e^{-\tau} I_0(0)$$

$$\vdots$$

$$\rightarrow \quad \mathsf{S}_{n-1}(\tau) = aI_{n-1}(\tau) = \frac{(a^n\tau^{n-1})}{(n-1)!}e^{-\tau} I_0(0)$$

$$I_n(\tau) = e^{-\tau}\int_0^\tau e^{\tau'}\mathsf{S}_{n-1}(\tau')d\tau' = \frac{(a\tau)^n}{n!}e^{-\tau} I_0(0)$$

The final solutions are:

$$I^{\rm direc}(\tau) = e^{-\tau} I(0)$$

$$\begin{aligned}I^{\rm scatt}(\tau) &= \sum_{n=1}^{\infty} I_n(\tau) = \sum_{n=1}^{\infty}\frac{(a\tau)^n}{n!}e^{-\tau} I(0)\\ &= (e^{a\tau}-1)e^{-\tau} I(0)\\ &\approx a\tau e^{-\tau} I(0) \quad \text{if} \quad a\tau \ll 1\\ &\approx a\tau I(0) \qquad\ \ \text{if} \quad \tau \ll 1\end{aligned}$$

$$\begin{aligned}I^{\rm tot}(\tau) &= I^{\rm direc}(\tau) + I^{\rm scatt}(\tau)\\ &= e^{-(1-a)\tau} I(0)\\ &= e^{-\tau_{\rm abs}} I(0)\end{aligned}$$

Basic Theory of Radiation Fields

Overview

Observational astronomy is primarily based on the detection of radiation emitted by astrophysical objects.

- Continuum radiation is a natural consequence of the principle that accelerating charges radiate.

- We will review and apply **principles of electromagnetism** to improve our understanding of important astrophysical phenomena, demonstrating in the process that **important radiation processes can be derived from the fundamental principle that accelerating charges radiate**.

- We will develop the theory that describes **bremsstrahlung** and **synchrotron radiation**. A theoretical understanding of these two radiation mechanisms allows us to interpret the emission of a wide range of objects, ranging from distant radio galaxies to nearby H II regions.

Useful Mathematical Formulae

- Dirac delta function:

$$\delta(\omega - \omega_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)t}dt$$

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$$

- Fourier Transform:

Rybicki & Lightman	Afken (Mathematical Methods for Physicists)
$\bar{a}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(t)e^{-i\omega t}dt$	$\bar{a}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(t)e^{i\omega t}dt$
$a(t) = \int_{-\infty}^{\infty} \bar{a}(\omega)e^{i\omega t}d\omega$	$a(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{a}(\omega)e^{-i\omega t}d\omega$
$\int_{-\infty}^{\infty} a(t)b^*(t)dt = 2\pi\int_{-\infty}^{\infty} \bar{a}(\omega)\bar{b}^*(\omega)d\omega$	$\int_{-\infty}^{\infty} a(t)b^*(t)dt = \int_{-\infty}^{\infty} \bar{a}(\omega)\bar{b}^*(\omega)d\omega$

Parseval's Theorem

$$\int_{-\infty}^{\infty} |a(t)|^2 dt = 2\pi\int_{-\infty}^{\infty} |\bar{a}(\omega)|^2 d\omega \qquad \int_{-\infty}^{\infty} |a(t)|^2 dt = \int_{-\infty}^{\infty} |\bar{a}(\omega)|^2 d\omega$$

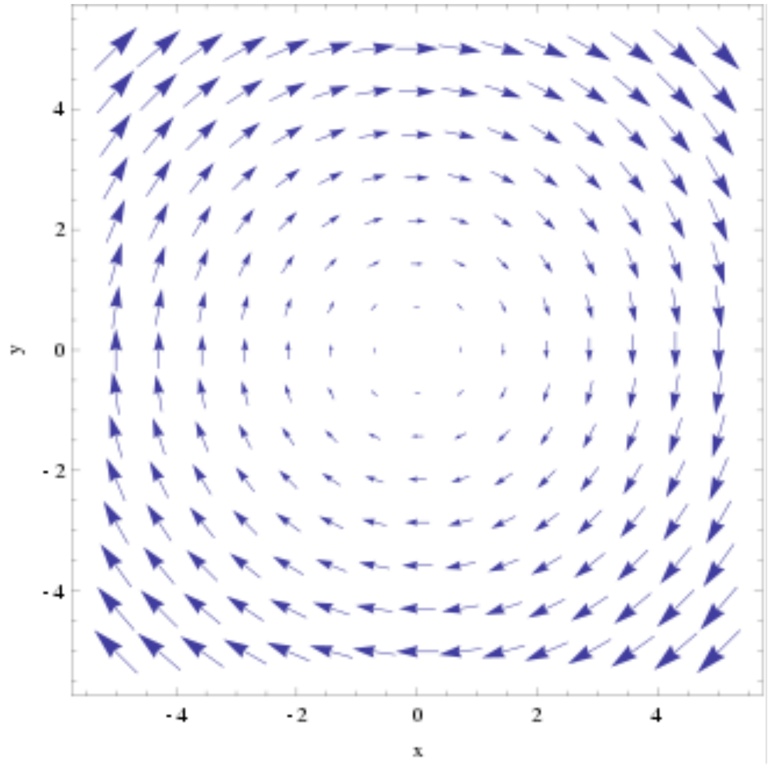
- Curl

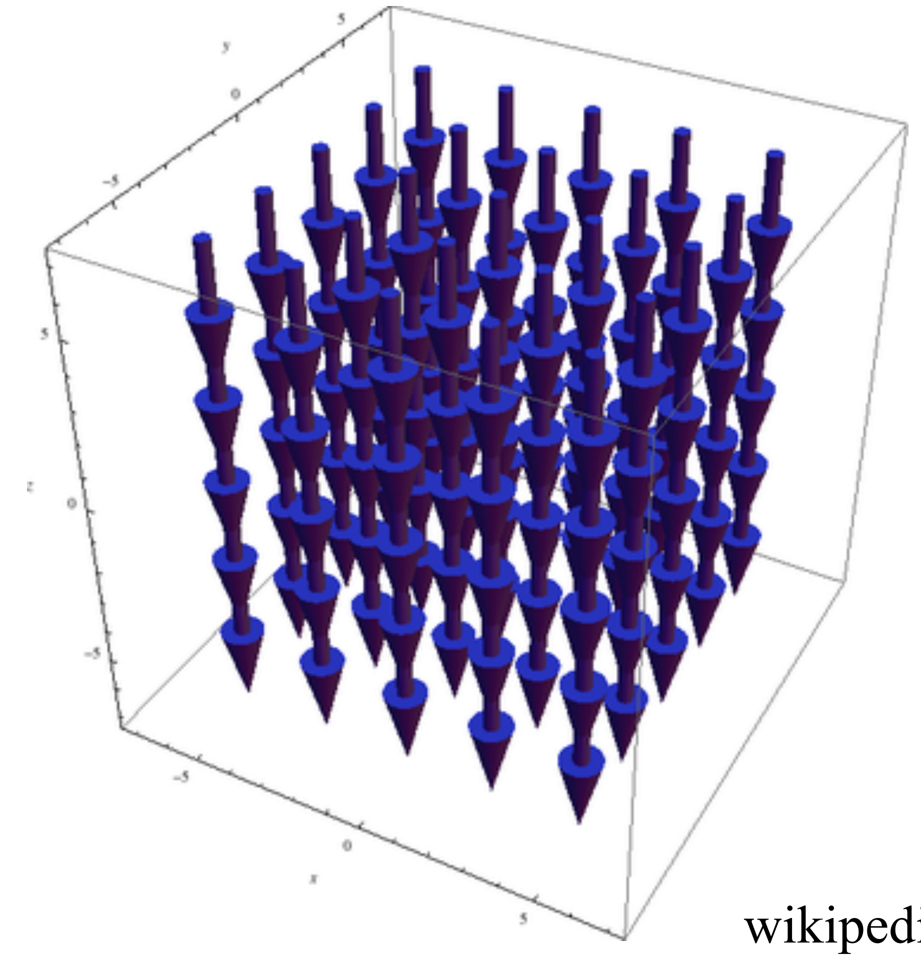
∇ is read nabla .

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\boxed{\mathbf{B} \times \mathbf{A} = (B_y A_z - B_z A_y)\,\hat{\mathbf{x}} + (B_z A_x - B_x A_z)\,\hat{\mathbf{y}} + (B_x A_y - B_y A_x)\,\hat{\mathbf{z}}}$$

$$\mathbf{A} = y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$$





$$\nabla \times A = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}y\right)\hat{\mathbf{z}}$$

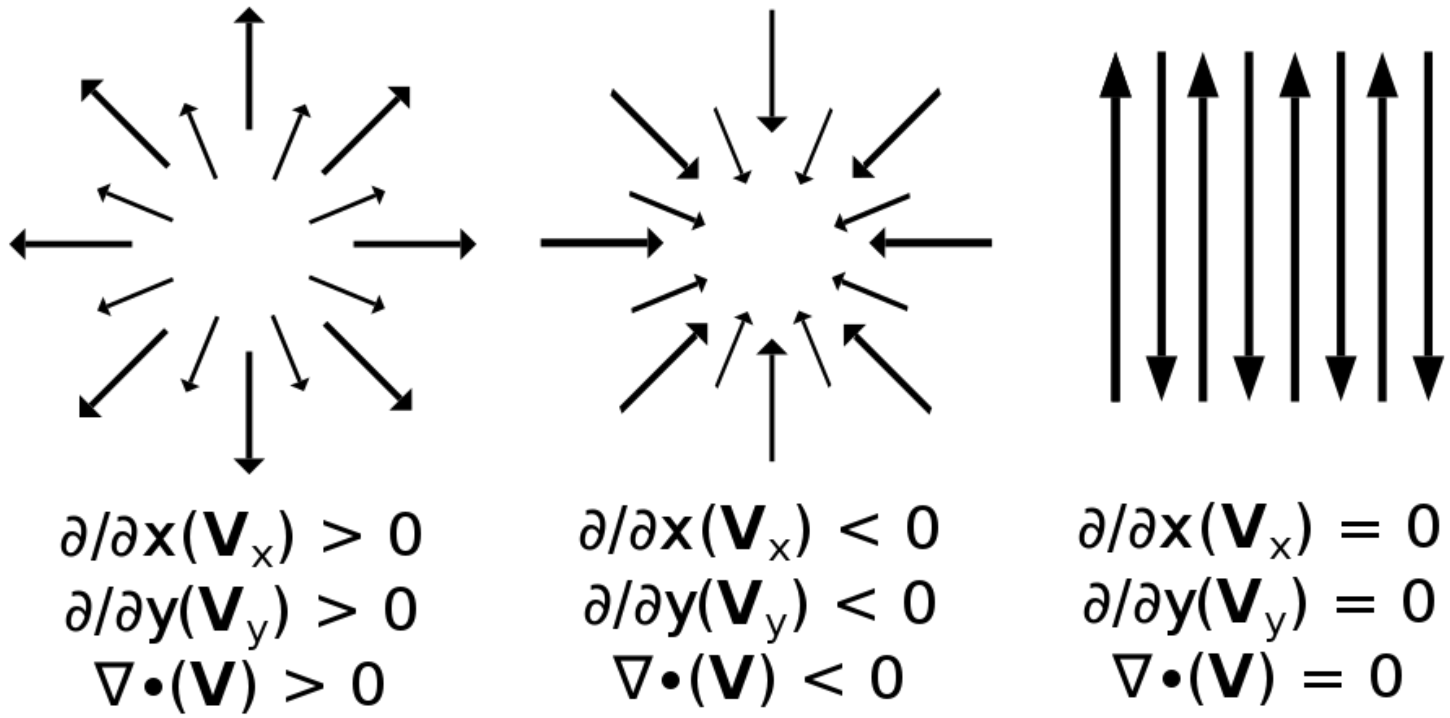
$$= -2\hat{\mathbf{z}}$$

wikipedia

- Divergence

$$\text{div }\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\boxed{\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z}$$



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- Vector identities:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\nabla\times(\nabla\phi) = 0$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

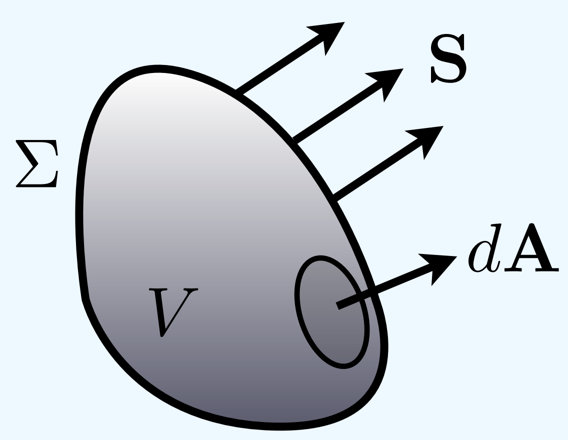
$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

- Divergence Theorem:

The divergence theorem relates the flux of a vector field through a closed surface to the divergence of the field in the volume enclose.

The surface integral of a vector field over a closed surface, which is called the "flux" through the surface, is equal to the volume integral of the divergence over the region inside the surface.

Intuitively, it states that "the sum of all sources of the field in a region gives the net flux out of the region."



$$\int_V \nabla \cdot \mathbf{S} dV = \int_\Sigma \mathbf{S} \cdot d\mathbf{A}$$

Here, $d\mathbf{A} = \mathbf{n} dA$

Electromagnetic force on a single charged particle

- **Lorentz force**: The operational definitions of the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$ are made through observations on a particle of charge q at point $\mathbf{r}$ with velocity $\mathbf{v}$, and by means of the formula for the Lorentz force. If a particle of charge q and mass m moves with velocity $\mathbf{v}$ in the presence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$, then it will experience a force:

$$\boxed{\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)}$$

(in Gaussian units, or cgs units)

- **Power** (the rate of work done by the fields on a particle) supplied by the EM fields is

$$\mathbf{v} \cdot \mathbf{F} = q\mathbf{v} \cdot \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

$$\mathbf{v} \cdot m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \cdot \mathbf{E}$$

$$\therefore \frac{dU_{\text{mech}}}{dt} \equiv \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\mathbf{v} \cdot \mathbf{E}$$

- Note $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$, meaning that the magnetic fields do not work.

Electromagnetic force on a continuous medium

- Consider a medium with **charge density** and **current density**:

$$\rho \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i$$

$$\mathbf{j} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i$$

  ΔV must be chosen much smaller than characteristic scales but much larger than the volume containing a single particle.

- **Force density** (force per unit volume) on a unit volume containing many charges:

$$\mathbf{f} = \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B}$$

- **Power density** supplied by the field (the rate of work done by the field per unit volume). This is also **the rate of change of mechanical energy per unit volume** due to the fields:

$$\frac{du_{\text{mech}}}{dt} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i \cdot \mathbf{E} \longrightarrow \boxed{\frac{du_{\text{mech}}}{dt} = \mathbf{j} \cdot \mathbf{E}}$$

Note typos in the Rybicki & Lightman's book. They use the same symbol to denote the energy density u and the total energy U within a volume.

Maxwell's equations

- **Maxwell's eqs.** (in macroscopic forms) relates fields to charge and current densities.

$\nabla \cdot \mathbf{D} = 4\pi\rho$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	Gauss's law for magnetism (no magnetic monopoles)
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	Maxwell-Faraday equation
$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	Ampere-Maxwell equation

$\mathbf{D}$,$\mathbf{H}$: macroscopic fields
$\mathbf{B}$,$\mathbf{E}$: microscopic fields

$\mathbf{D} = \epsilon\mathbf{E}$

$\mathbf{B} = \mu\mathbf{H}$

ϵ : dielectric constant

μ : magnetic permeability

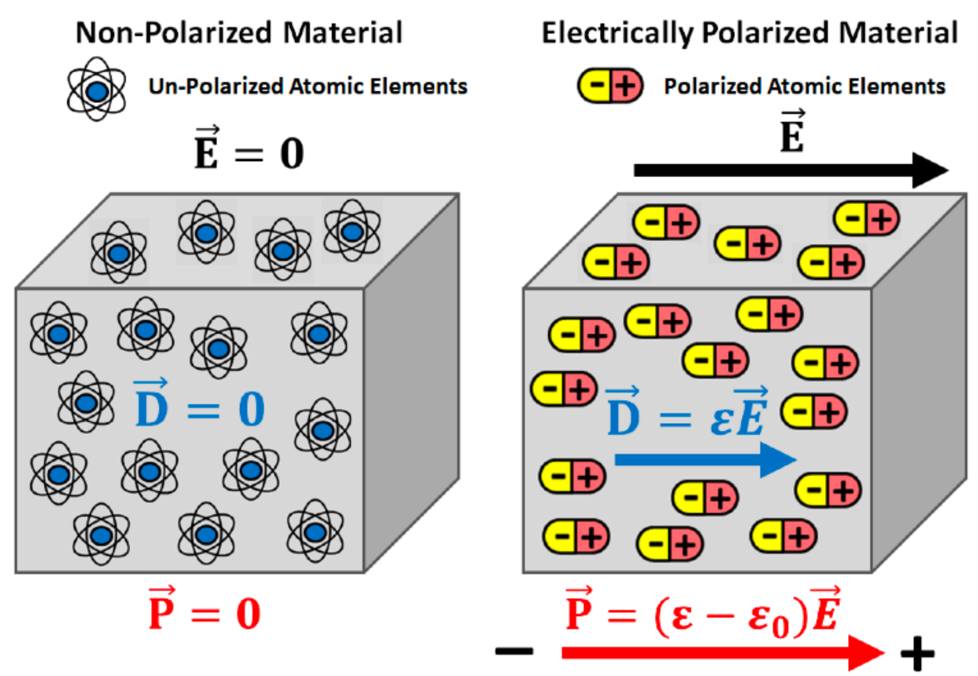
Dielectric material (절연체): an electrical insulator that can be polarized by an applied electric field. Electric charges do not flow through the material as they do in a conductor, but only slightly shift from their average equilibrium positions causing dielectric polarization.

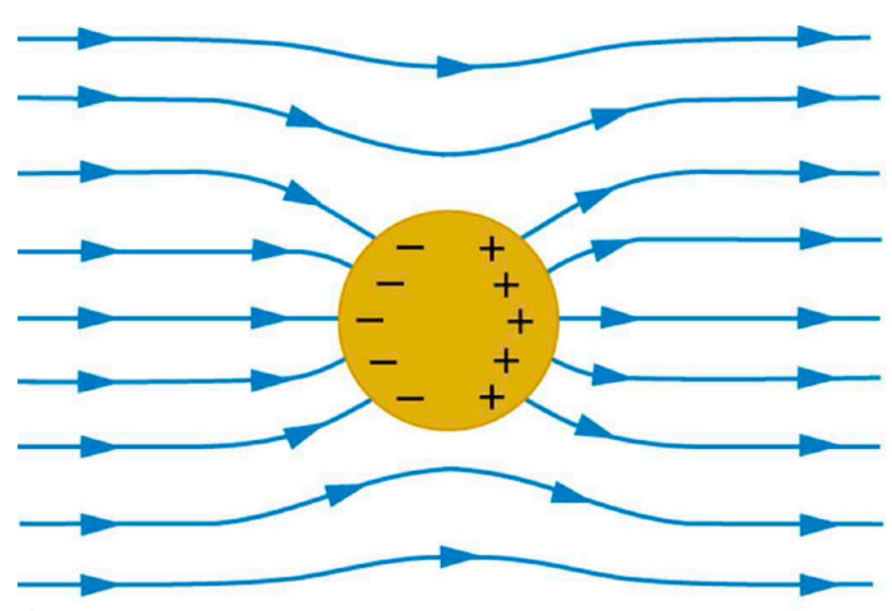
Permeability (투자율): the degree of magnetization of a material in response to a magnetic field.

Note $\epsilon = \mu = 1$ in the absence of dielectric or permeability media.

Dielectrics and Conductors

- Materials
  - ***Dielectrics***: Dielectrics are substances which do not contain free charge carriers. They are isolators and no constant current can be sustained within them. Neverthless, alternating currents produced by a time-variable electric field are possible. In these currents, the charges do no travel far from their equilibrium positions.
  - ***Conductors***: The substances having free charge carriers are called the conductors. When a piece of metal is connected at its ends to the poles of a battery, a steady current flows under the influence of an electric field. When this piece of metal is placed in a static electric field, the charges accumulate at its surface and arrange themselves in such a way that the electric field inside vanishes and then there is no internal current. However, time-varying electric fields and currents are possible.
  - In the ISM, one finds both dielectric and metallic particles, but the latter are far from being perfect conductors.





Conservation of charge

- **Conservation of charge** for a volume element:

  An immediate consequence of Maxwell's equation is conservation of charge:

Taking $\nabla \cdot$ of the $\nabla \times \mathbf{H}$ equation: $\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{H}) &= \nabla \cdot \left(\frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\right)\\ 0 &= \frac{4\pi}{c}\nabla \cdot \mathbf{j} + \frac{1}{c}\frac{\partial}{\partial t}(4\pi\rho)\end{aligned}$$

$$\rightarrow \boxed{\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0}$$

Using the divergence theorem: $$\begin{aligned}\frac{d}{dt}\int_V \rho dV &= -\int_V \nabla \cdot \mathbf{j} dV\\ &= -\int_\Sigma \mathbf{j} \cdot d\mathbf{A}\end{aligned}$$

Poynting's Theorem: Electromagnetic Field Energy

- We now give definitions of **energy density** and **energy flux** of the electromagnetic field.
- Use the Ampere's law to obtain the work done per unit volume on a particle distribution (the mechanical energy density)

$$\frac{du_{\mathrm{mech}}}{dt} = \mathbf{j} \cdot \mathbf{E} = \frac{1}{4\pi} \mathbf{E} \cdot \left(c\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

- Use a vector identity and Faraday's law:

$$\begin{aligned} \mathbf{E} \cdot (\nabla \times \mathbf{H}) &= \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{H}) \\ &= -\frac{1}{c} \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) \end{aligned}$$

Then, we obtain

$$\mathbf{j} \cdot \mathbf{E} = \frac{1}{4\pi} \left(-c\nabla \cdot (\mathbf{E} \times \mathbf{H}) - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right)$$

This equation can be simplified using the relations: $\mathbf{D} = \epsilon \mathbf{E}$

$$\mathbf{B} = \mu \mathbf{H}$$

- Now, if ϵ and μ are independent of time, then the above may be written as **Poynting's theorem** in differential form.

$$\boxed{\mathbf{j}\cdot\mathbf{E}+\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2+\frac{B^2}{\mu}\right)=-\nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\right)}$$

Recall that

$$\frac{du_{\text{mech}}}{dt}=\mathbf{j}\cdot\mathbf{E}$$

This equation can be interpreted as saying that the rate of change of **mechanical energy per unit volume** plus the rate of change of **field energy per unit volume** equals minus the divergence of **the field energy flux**.

Accordingly, we set the electromagnetic field energy per unit volume equal to

Electromagnetic field energy density

$$\boxed{u_{\text{field}}=\frac{1}{8\pi}\left(\epsilon E^2+\frac{B^2}{\mu}\right)=u_E+u_B}$$

and the **electromagnetic flux vector, or Poynting vector**, equal to

$$\boxed{\mathbf{S}=\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}=\frac{c}{4\pi\mu}\mathbf{E}\times\mathbf{B}}$$

- The Poynting's theorem becomes an expression of the **local conservation of energy**.

$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{field}}) + \nabla \cdot \mathbf{S} = 0$$

- Integrating the equation over a volume element and using the divergence theorem, we obtain the **conservation of energy**:

$$\boxed{\frac{d}{dt}(U_{\text{mech}} + U_{\text{field}}) = -\int_{\Sigma} \mathbf{S} \cdot d\mathbf{A}}$$

Here, $\boxed{U_{\text{mech}} \equiv \int_V u_{\text{mech}} dV \quad \text{and} \quad U_{\text{field}} \equiv \int_V u_{\text{field}} dV}$

or

$$\int_V (\mathbf{j} \cdot \mathbf{E}) dV + \frac{d}{dt} \int_V \left(\frac{\epsilon E^2 + B^2/\mu}{8\pi} \right) dV = -\int_{\Sigma} \mathbf{S} \cdot d\mathbf{A}$$

Meaning: ***the rate of change of total (mechanical + field) energy within the volume V is equal to the net inward flow of energy through the bounding surface Σ.***

- In electrostatics and magnetostatics, we recall that

$$\mathbf{E} \propto r^{-2} \text{ and } \mathbf{B} \propto r^{-2} \text{ as } r \to \infty \qquad \to \quad \mathbf{S} \propto r^{-4}$$

$$\therefore \int_{\Sigma} \mathbf{S} \cdot d\mathbf{A} = 0 \text{ as } r \to \infty$$

- However, for time-varying fields, we will find later that

$$\mathbf{E} \propto r^{-1} \text{ and } \mathbf{B} \propto r^{-1} \text{ as } r \to \infty$$

$$\therefore \int_{\Sigma} \mathbf{S} \cdot d\mathbf{A} \neq 0 \text{ as } r \to \infty$$

- This finite energy flowing outward (or inward) at large distances is called **radiation**. Those parts of **E** and **B** that decreases as r^{-1} at large distances are said to constitute the **radiation field**.

Electromagnetic Waves

- In vacuum $(\rho = 0 = \mathbf{j},\ \epsilon = 1 = \mu)$, Maxwell's equations give the vector wave equations:

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$\longrightarrow$ These equations are invariant under $\begin{array}{l} \mathbf{E} \to \mathbf{B} \\ \mathbf{B} \to -\mathbf{E} \end{array}$

Third equation: $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$

Taking the curl, we obtain $\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\therefore\ \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Fourth equation: $\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$

$$\nabla \times (\nabla \times \mathbf{B}) = \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{E})$$

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$\therefore\ \nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$\longrightarrow$ We obtain the vector wave equations for $\mathbf{E}$ and $\mathbf{B}$:

$$\boxed{\begin{aligned} \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0 \\ \nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} &= 0 \end{aligned}}$$

Plane Electromagnetic Waves

- Fourier components:

$$\mathbf{E}(\mathbf{r},t) = \int d^3\mathbf{k} \int d\omega \bar{\mathbf{E}}(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

angular frequency

$\omega = 2\pi\nu$

Here, $\mathbf{k} = k\hat{\mathbf{k}}$ and ω are the "wave vector" and frequency, respectively. The Fourier components represent waves traveling in the $\hat{\mathbf{k}}$ direction, since surfaces of constant phase with time travel in the $\hat{\mathbf{k}}$ direction.

Fourier transformation:

$$\bar{\mathbf{E}}(\mathbf{k},\omega) = \frac{1}{(2\pi)^4} \int d^3\mathbf{r} \int dt \mathbf{E}(\mathbf{r},t) e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

- Apply the wave equation to Fourier expansion:

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = -\int d^3\mathbf{k} \int d\omega \left(k^2 - \frac{\omega^2}{c^2}\right) \bar{\mathbf{E}}(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = 0$$

For each Fourier component $\bar{\mathbf{E}}$, we obtain the relation between the wave number and frequency.

$$\boxed{k = \frac{\omega}{c}}$$

Dispersion relation*

- We obtain the vacuum dispersion relation, phase velocity, and group velocity:

$$\omega = ck \qquad v_{\rm ph} \equiv \frac{\omega}{k} = c \qquad v_{\rm g} \equiv \frac{\partial \omega}{\partial k} = c$$

  dispersion relation = a function which gives ω as a function of k.

  phase velocity = the rate at which the phase of the wave propagates in space.

  group velocity = the velocity with which the overall shape of the waves' amplitudes (modulation or envelope of the wave) propagates through space.

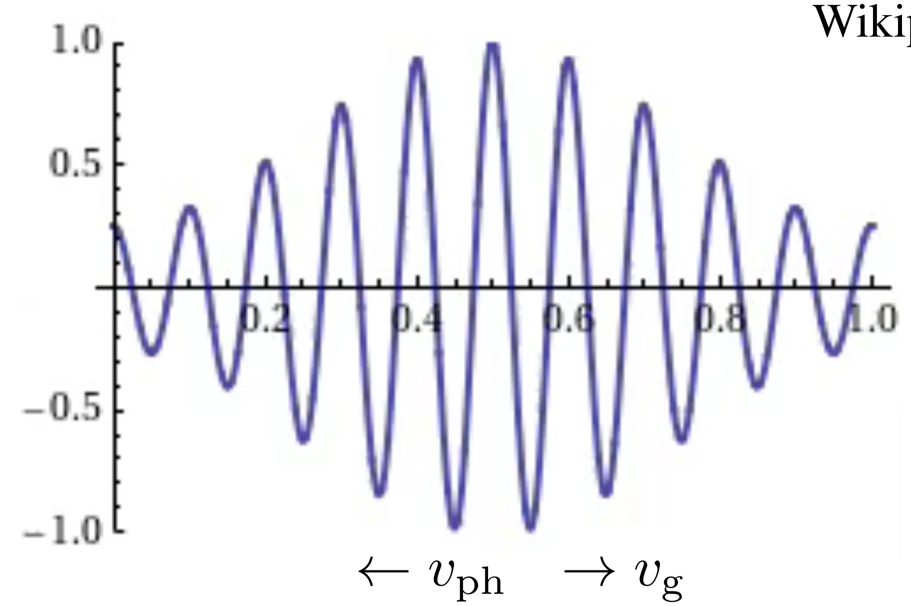
- Assume the wave packet E is almost monochromatic, so that its Fourier component is nonzero only in the vicinity of a central wavenumber k_0. Then, linearization gives:

$$\begin{aligned} \omega(k) &\approx \omega_0 + (k-k_0)\frac{\partial \omega(k)}{\partial k}|_{k=k_0} \\ &= \omega_0 + (k-k_0)\omega_0' \end{aligned}$$

$$\begin{aligned} E(x,t) &= \int dk \int d\omega \bar{E}(k,\omega) e^{i(kx-\omega t)} \\ &\approx e^{it(\omega_0' k_0 - \omega_0)} \int dk \bar{E}(k,\omega_0) e^{ik(x-\omega_0' t)} \end{aligned}$$

$$\boxed{|E(x,t)| = |E(x-\omega_0' t, 0)|}$$

Wikipidia



- The envelope of the wavepacket travels at velocity $\omega_0' = (\partial\omega/\partial k)_{k=k_0}$.

Properties of a single Fourier mode

- Consider a single Fourier mode in vacuum:

$$\boxed{\mathbf{E} = E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{e}}, \qquad \mathbf{B} = B_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{b}}}$$

wave vector $\mathbf{k}$

angular frequency $\omega = 2\pi\nu$

(E_0 and B_0 are complex constants.)

- Substituting into Maxwell's equations yields:

$$\nabla\cdot\mathbf{E} = 0 \quad\rightarrow\quad \mathbf{k}\cdot\mathbf{E} = 0$$

$$\nabla\cdot\mathbf{B} = 0 \quad\rightarrow\quad \mathbf{k}\cdot\mathbf{B} = 0$$

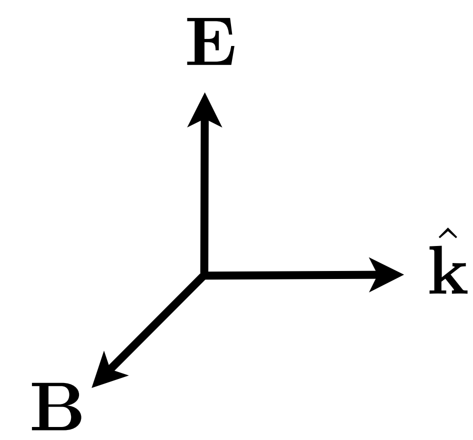
$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \quad\rightarrow\quad \mathbf{k}\times\mathbf{E} = \frac{\omega}{c}\mathbf{B}$$

$$\nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \quad\rightarrow\quad \mathbf{k}\times\mathbf{B} = -\frac{\omega}{c}\mathbf{E}$$

$$\left(\mathbf{k} = \hat{\mathbf{k}}\frac{\omega}{c}\right)$$

or $\boxed{\hat{\mathbf{k}}\times\mathbf{E} = \mathbf{B}}$

or $\boxed{\hat{\mathbf{k}}\times\mathbf{B} = -\mathbf{E}}$



$$\hat{\mathbf{k}}\times\mathbf{E} = \mathbf{B} \quad\rightarrow\quad \hat{\mathbf{k}}\times\hat{\mathbf{e}}E_0 = \hat{\mathbf{b}}B_0$$

$$E_0\left(\hat{\mathbf{k}}\times\hat{\mathbf{e}}\right)\cdot\hat{\mathbf{b}} = B_0$$

$$E_0 = B_0$$

$$E_0 = |\mathbf{E}|e^{i\phi_E}$$
$$B_0 = |\mathbf{B}|e^{i\phi_B} \quad\rightarrow\quad \boxed{\phi_E = \phi_B}$$

(1) EM waves are **transverse** (perpendicular to the direction of propagation).

(2) **E** and **B** are **orthogonal** to each other.

(3) $\left(\hat{\mathbf{k}},\ \hat{\mathbf{e}},\ \hat{\mathbf{b}}\right)$ form an orthogonal basis (triad).

(4) **Field amplitudes and phases are equal**: $|\mathbf{B}| = |\mathbf{E}|, \quad B_0 = E_0 \quad\text{and}\quad \phi_B = \phi_E$

Energy Flux and Energy Density in Vacuum

- We can now compute the energy flux and energy density of these plane waves. Since **E** and **B** both vary sinusoidally in time, the Poynting vector and the energy density actually fluctuate; however, we take a time average, since this is in most cases what is measured.

- If *A(t)* and *B(t)* are two complex quantities with the same sinusoidal time dependence,

$$A(t) = \mathcal{A}e^{i\omega t} \qquad B(t) = \mathcal{B}e^{i\omega t}$$

$$x = a + ib$$

$$a = \frac{x + x^*}{2}$$

then the time average of the product of their real parts is

$$\begin{aligned} \langle \mathrm{Re}A(t) \cdot \mathrm{Re}B(t) \rangle &= \frac{1}{4} \left\langle \left(\mathcal{A}e^{i\omega t} + \mathcal{A}^* e^{-i\omega t} \right) \left(\mathcal{B}e^{i\omega t} + \mathcal{B}^* e^{-i\omega t} \right) \right\rangle \\ &= \frac{1}{4} \langle \mathcal{A}\mathcal{B}^* + \mathcal{A}^*\mathcal{B} \rangle \\ &= \frac{1}{2}\mathrm{Re}(\mathcal{A}\mathcal{B}^*) = \frac{1}{2}\mathrm{Re}(\mathcal{A}^*\mathcal{B}) \end{aligned}$$

Here, we have used * to denote complex conjugation.

- Thus the time-averaged Poynting vector amplitude is

$$\begin{aligned}\langle S\rangle &= \frac{c}{4\pi}\left\langle \mathrm{Re}\left(E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right)\mathrm{Re}\left(B_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right)\right\rangle \\ &= \frac{c}{8\pi}\mathrm{Re}\left(E_0 B_0^*\right) \\ &= \frac{c}{8\pi}|E_0|^2 = \frac{c}{8\pi}|B_0|^2 \qquad \leftarrow E_0 = B_0\end{aligned}$$

Energy per unit area per unit time:

$$\left\langle \frac{dW}{dAdt}\right\rangle = \langle S\rangle = \frac{c}{8\pi}\left|E_0\right|^2$$

Note that: $\frac{1}{2\pi}\int_0^{2\pi}\cos^2(x)dx = \frac{1}{2}$

- Similarly, the time-averaged field energy density:

$$\langle u_{\mathrm{field}}\rangle = \frac{1}{8\pi}\left\langle |\mathbf{E}|^2 + |\mathbf{B}|^2\right\rangle = \frac{1}{16\pi}\left(E_0 E_0^* + B_0 B_0^*\right)$$

$$\therefore \boxed{\langle u_{\mathrm{field}}\rangle = \frac{1}{8\pi}|E_0|^2 = \frac{1}{8\pi}|B_0|^2}$$

- Therefore, the velocity of energy flow is

$$\boxed{\frac{\langle S\rangle}{\langle u_{\mathrm{field}}\rangle} = c} \qquad \longleftarrow \qquad \text{cf. } J = \frac{c}{4\pi}u$$

Radiation Spectrum

- **The spectrum of radiation depends on the time variation of the electric field** (we can ignore the magnetic field, since it mimics the electric field).
- A consequence is that one cannot give a meaning to the spectrum of radiation at a precise instant of time, knowing only the electric field at one point.

electric spectrum: $$\bar{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)e^{-i\omega t}dt$$

$E(t_1)$ at an instant of time (t_1) does not give the spectrum $\bar{E}(\omega)$.

Instead, one must talk about the spectrum of a train of waves, or of the radiation at a point during a sufficiently long time interval Δt.

- If we have such a time record of the radiation field of length Δt, we still can only define the spectrum to within a frequency resolution $\Delta\omega$ where

$\Delta\omega\Delta t > 1$ (uncertainty relation)

This uncertainty relation is not necessarily quantum in nature, but is ***a common property of any wave theory of light.***

Power Spectrum

- Let us assume, for mathematical simplicity, that the radiation is in the form of a finite pulse (i.e., $\mathbf{E}(t)$ is assumed to vanish sufficiently rapidly for $t \to \pm\infty$). Also, let us consider only one of two independent components of the transverse electric field, say

$$E(t) \equiv \hat{\mathbf{x}} \cdot \mathbf{E}(t)$$

- With these assumptions we may express $E(t)$ in terms of a Fourier integral. The Fourier transform and its inverse are:

$$\bar{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)e^{-i\omega t}dt, \qquad E(t) = \int_{-\infty}^{\infty} \bar{E}(\omega)e^{i\omega t}d\omega$$

The function $\bar{E}(\omega)$ is complex; however, since E(t) is real we can write

$$\bar{E}(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt = \bar{E}^*(\omega)$$

$\longleftarrow$ From the condition: $E(t) = E(t)^*$

so that the negative frequencies are redundant (can be eliminated).

- Contained in $\bar{E}(\omega)$ is all the information about the frequency behavior of $E(t)$.

  To obtain the frequency information about the energy we write the energy per unit time per unit area in terms of the Poynting vector:

$$\frac{dW}{dA\,dt} = \frac{c}{4\pi} E^2(t) \qquad \text{(Poynting vector at an instant of time)}$$

  The total energy per unit area in the pulse is

$$\frac{dW}{dA} = \int_{-\infty}^{\infty} \frac{dW}{dA\,dt} dt = \frac{c}{4\pi} \int_{-\infty}^{\infty} E^2(t) dt$$

  From Parseval's theorem for Fourier transforms, we know that

$$\int_{-\infty}^{\infty} E^2(t) dt = 2\pi \int_{-\infty}^{\infty} |\bar{E}(\omega)|^2 d\omega \underset{\uparrow}{=} 4\pi \int_{0}^{\infty} |\bar{E}(\omega)|^2 d\omega$$

$$\text{from } \bar{E}(-\omega) = \bar{E}^*(\omega) \text{ and } \bar{E}^*(-\omega) = \bar{E}(\omega)$$

  Thus we have the result

  Energy per unit area
$$\boxed{\frac{dW}{dA} = c \int_{0}^{\infty} |\bar{E}(\omega)|^2 d\omega}$$

- Finally, we may identify **the energy per unit area per unit frequency**:

$$\frac{dW}{dA} = c\int_0^\infty |\bar{E}(\omega)|^2 d\omega \quad \longrightarrow \quad \boxed{\frac{dW}{dA d\omega} = c|\bar{E}^2(\omega)|^2}$$

It should be noted that this is the **total energy per area per frequency range in the entire pulse.**

We have not written "per unit time." To write both dt and $d\omega$ would violate the uncertainty relation. However, if the pulse repeats on an average time scale T, then we may ***formally*** write

$$\frac{dW}{dA d\omega dt} \equiv \frac{1}{T}\frac{dW}{dA d\omega} = \frac{c}{T}|\bar{E}^2(\omega)|^2$$

If a very long signal has more or less the same properties over its entire length (property of time stationarity) then we expect that the result will be independent of T for large T, and we may write

$$\boxed{\frac{dW}{dA d\omega dt} = \lim_{T\to\infty}\frac{c}{T}|\bar{E}^2(\omega)|^2}$$

- If the properties of $E(t)$ vary with time, then one expects that the spectrum as determined by analyzing a portion of length T will depend on just what portion is analyzed

  In that case efficacy of the concept of local spectrum depends on whether the changes of character of $E(t)$ occur on a time scale long enough that one can still define a length T in which a suitable frequency resolution $\Delta\omega \sim 1/T$ can be obtained.

  If this condition is not met, a local spectrum is not useful, and one must consider the spectrum of the entire pulse as the basic entity.

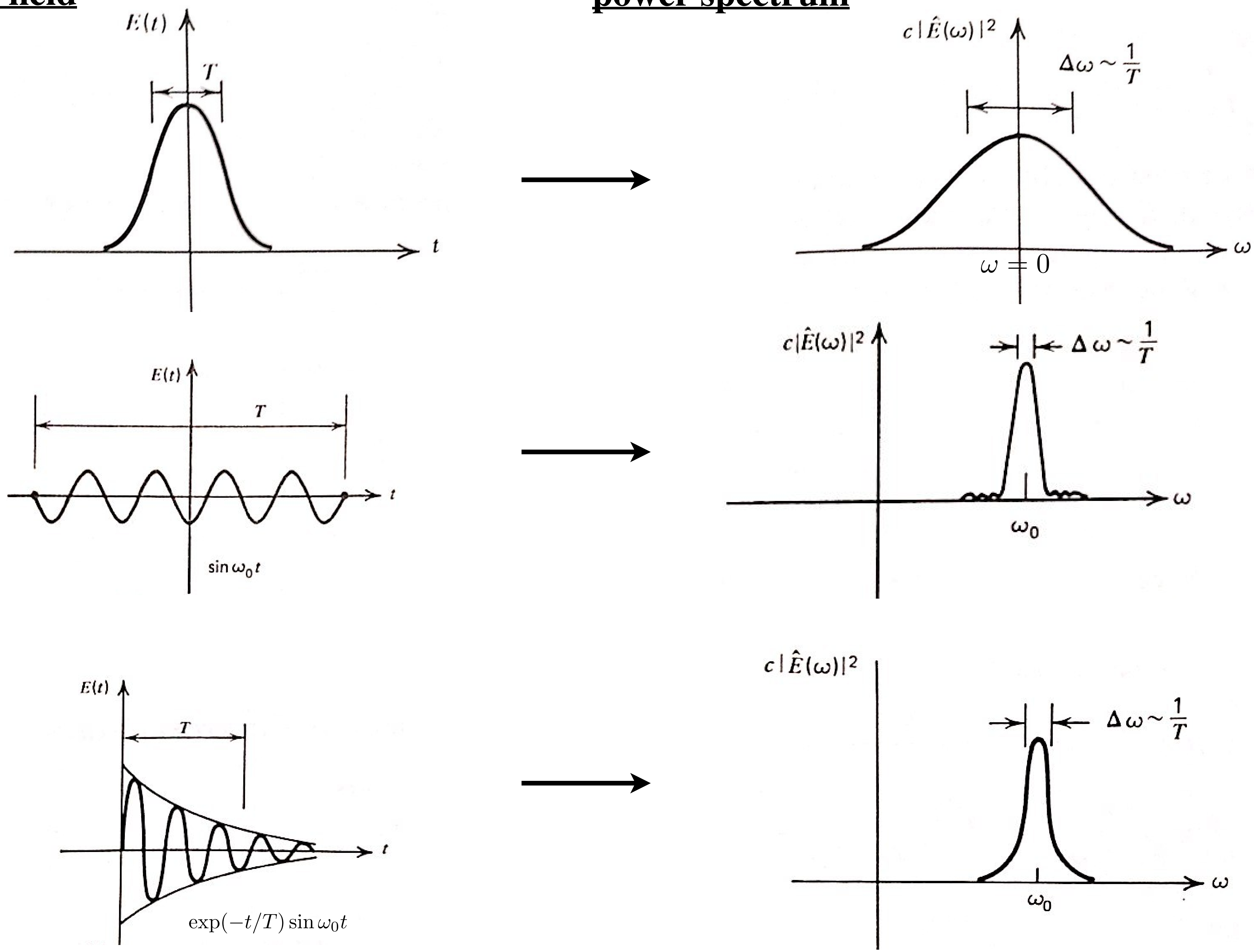
- Let us consider now some typical pulse shapes and their corresponding spectra in the next slide.

  Note that $|E(\omega)|^2$ is always symmetric about the origin ($\omega = 0$) and only the values for positive ω need concern us.

  Some general rules are:

  - ✦ The time extent of the pulse T determines the width of the finest features in the spectrum by means of $\Delta\omega \sim 1/T$.
  - ✦ The existence of a sinusoidal time dependence within the pulse shape causes the spectrum to be concentrated near $\omega \sim \omega_0$.

electric field **power spectrum**



Electromagnetic Potentials

- It can be shown that the $\mathbf{E}$ and $\mathbf{B}$ may be expressed completely in terms of a scalar potential $\phi(\mathbf{r},\ t)$ and a vector potential $\mathbf{A}(\mathbf{r},\ t)$.

1. One scalar plus one vector is simpler than two vectors.
2. The equations determining ϕ and $\mathbf{A}$ are quite a bit simpler than Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$.
3. The relativistic formulation of electromagnetic theory is simpler in terms of the potentials than in terms of the electric and magnetic field.

Electromagnetic Potentials (from homogeneous Maxwell eqs)

- **Vector potential**:

(Gauss' law for magnetism)

From the vector identity $\nabla \cdot (\nabla \times A) = 0$, the equation $\nabla \cdot \mathbf{B} = 0$ allows us to define a vector potential such that

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \qquad \mathbf{A}: \text{ vector potential}$$

Then, Maxwell-Faraday equation becomes

(Maxwell-Faraday equation) $$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

- **Scalar potential**:

From the vector identity $\nabla \times (\nabla \phi) = 0$, this equation can be satisfied if we define a "scalar" potential such as

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi \quad \rightarrow \quad \boxed{\mathbf{E} = -\nabla \phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}} \qquad \phi: \text{ scalar potential}$$

Two of Maxwell's equation have already been satisfied identically by virtue of the definitions of the potentials.

Gauge invariance

- **Gauge invariance**:

B will be unchanged for any transformation

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \rightarrow \quad \boxed{\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\psi} \quad \text{since } \nabla \times (\nabla\psi) = 0$$

E will also be unchanged if at the same time the scalar potential is changed by

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \quad \rightarrow \quad \boxed{\phi \rightarrow \phi' = \phi - \frac{1}{c}\frac{\partial \psi}{\partial t}}$$

EM field is invariant under the **gauge transformations.**

$$\boxed{(\phi,\ \mathbf{A}) \rightarrow \left(\phi - \frac{1}{c}\frac{\partial \psi}{\partial t},\ \mathbf{A} + \nabla\psi\right)}$$

The gauge transformations give us mathematical flexibility for solving various EM problems. They are useful because they do not change the underlying physics.

A particularly useful gauge for dealing with radiation is given by the ***Lorentz condition***.

Lorentz Gauge (Lorentz condition)

- Using the potential, the inhomogeneous Maxwell's equations can be written as

$$\nabla\cdot\mathbf{E} = 4\pi\rho \;\rightarrow\; \nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -4\pi\rho$$

$$\nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} \;\rightarrow\; \nabla\times(\nabla\times\mathbf{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) = \frac{4\pi}{c}\mathbf{j}$$

$$\rightarrow\; -\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} + \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = \frac{4\pi}{c}\mathbf{j}$$

$$\text{Note}: \nabla\times(\nabla\times\mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A})$$

- The Lorentz gauge is the most important gauge in the EM theory, defined by:

$$\boxed{\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0}$$

Note that we can alway choose a function ψ such as:

$$\nabla\cdot\mathbf{A}' + \frac{1}{c}\frac{\partial\phi'}{\partial t} = \nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} + \left(\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}\right) = 0$$

- Then, with the Lorentz gauge, the above equations become:

$$\boxed{\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{j} \end{aligned}}$$

Note: Column gauge is $\nabla\cdot\mathbf{A} = 0$

Retarded potentials

- The solutions to the above equations are (called the **retarded potentials**, see Jackson):

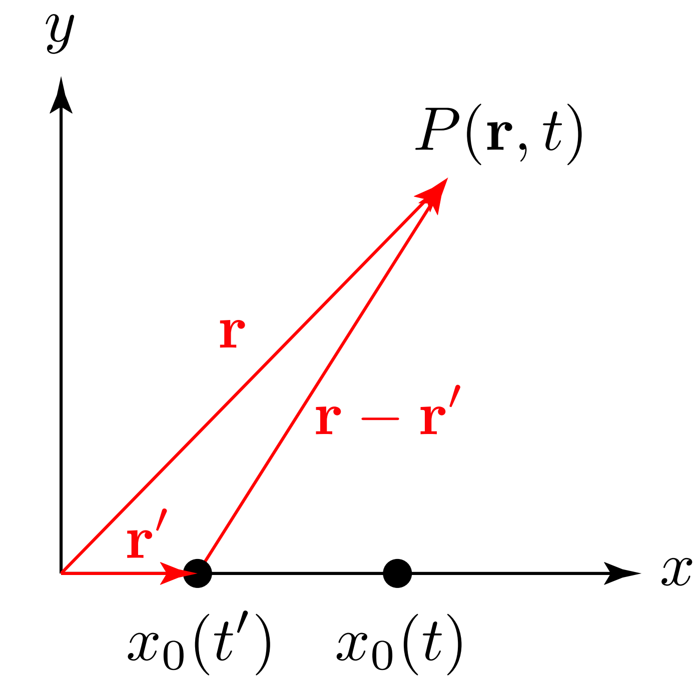
$$\phi(\mathbf{r},t) = \int_V d^3\mathbf{r}' \int dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int_V d^3\mathbf{r}' \int dt' \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)$$

$$\longrightarrow \phi(\mathbf{r},t) = \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}, \qquad \mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}$$

where

$$t' \equiv t_{\rm ret} = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$



$$x_0(t) - x_0(t') = c(t-t')$$

- The **retarded time** refers to conditions at the point $\mathbf{r}'$ that existed at a time earlier than t by just the time required for light to travel from $\mathbf{r}'$ to $\mathbf{r}$.

  Information at point $\mathbf{r}'$ propagates at the speed of light, so that the potentials a point $\mathbf{r}$ can only be affected by conditions at point $\mathbf{r}'$ at such a retarded time.

  The potentials responds to the changes after “retarded time” delay.

Applicability of the Radiative Transfer Theory*

- We defined specific intensity by the relation: $dE = I_\nu dA d\Omega d\nu dt$

  We should note that dA **and** $d\Omega$ **cannot both be made arbitrarily small** because of the uncertainty principle:

  uncertainty principle: $dx dp_x \geq h$ $\left(p = \frac{E}{c} = \frac{1}{c}\frac{hc}{\lambda} \right)$

$$dx dp_x dy dp_y = p^2 dA d\Omega \geq h^2 \quad \rightarrow \quad dA d\Omega \geq h^2/p^2 = \lambda^2$$

  Therefore, when the wavelength of light is larger than atomic dimensions (Bohr radius, $a_0 = 0.53$ Å), as in the optical, we cannot describe the interaction of light on the atomic scale in terms of specific intensity.

  - However, we may still regard transfer theory as a valid macroscopic theory, provided the absorption and emission properties are correctly calculated from microscopic theories (electromagnetic or quantum theory).
  - A more precise, classical treatment of the validity of rays is known as the eikonal approximation. (from German "eikonal", which is from Greek word meaning "image")

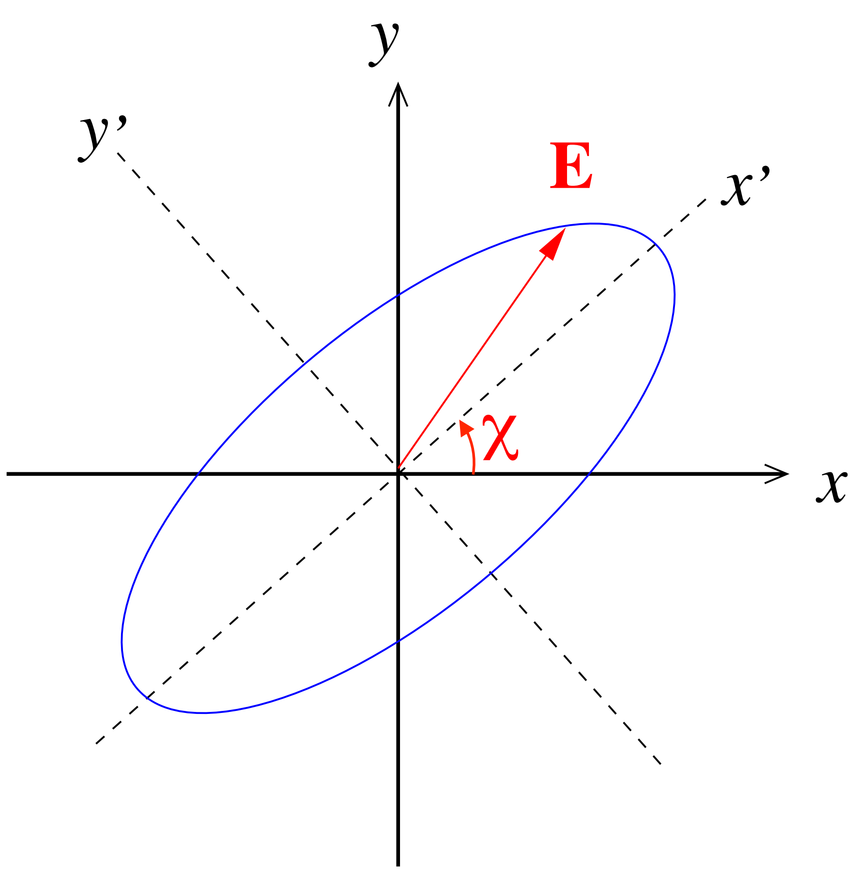
- There is another limitation because of the energy uncertainty principle:

$$dE dt > h \quad \rightarrow \quad d\nu dt > 1$$

Polarization

Polarization

- Let us consider a plane EM wave propagating in the +z direction, and examine the electric vector at $z = 0$. Because the electric field is transverse, the electric field can be expressed as



$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{x}}E_1 e^{i(kz-\omega t)} + \hat{\mathbf{y}}E_2 e^{i(kz-\omega t)} \\ &= \hat{\mathbf{x}}E_1 e^{-i\omega t} + \hat{\mathbf{y}}E_2 e^{-i\omega t} \quad \text{at z} = 0\end{aligned}$$

Complex amplitudes can be expressed as

$$E_1 = \mathcal{E}_1 e^{i\phi_1} \qquad E_2 = \mathcal{E}_2 e^{i\phi_2} \quad \text{where } \mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2 \text{ are real.}$$

Then, the real part of **E** is

$$\mathbf{E} = \hat{\mathbf{x}}\mathcal{E}_1 \cos(\omega t - \phi_1) + \hat{\mathbf{y}}\mathcal{E}_2 \cos(\omega t - \phi_2)$$

As a function of time, the tip of **E** will trace out an ellipse, meaning that the general wave is **elliptically polarized**.

- In general, the principal axes of this ellipse will have a tilt angle χ with respect to x-y axes. We define the zero of time so that **E** lies along the x' direction at $t = 0$.

$$\mathbf{E} = \hat{\mathbf{x}}'\mathcal{E}_1' \cos\omega t + \hat{\mathbf{y}}'\mathcal{E}_2' \sin\omega t$$

- We can satisfy the latter part of the equation by defining an **ellipticity angle**:

$$\mathcal{E}_1' = \mathcal{E}_0 \cos\beta \qquad \mathcal{E}_2' = -\mathcal{E}_0 \sin\beta \quad \text{where} \quad -\pi/2 \le \beta \le \pi/2 \quad (\text{or} \quad \mathcal{E}_2' = \mathcal{E}_0 \sin\beta',\ \beta' = -\beta)$$

- With the relations

$$\begin{aligned} &\hat{\mathbf{x}}\mathcal{E}_1 \cos(\omega t - \phi_1) + \hat{\mathbf{y}}\mathcal{E}_2 \cos(\omega t - \phi_2) \\ &= \hat{\mathbf{x}}'\mathcal{E}_0 \cos\beta \cos\omega t - \hat{\mathbf{y}}'\mathcal{E}_0 \sin\beta \sin\omega t \end{aligned} \quad + \quad \begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \end{pmatrix} = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix}$$

we obtain the relations:

Given $\mathcal{E}_1, \phi_1, \mathcal{E}_2, \phi_2,$ we can solve for $\mathcal{E}_0, \beta, \chi,$

$$\begin{aligned} \mathcal{E}_1 \cos\phi_1 &= \mathcal{E}_0 \cos\beta \cos\chi \\ \mathcal{E}_1 \sin\phi_1 &= \mathcal{E}_0 \sin\beta \sin\chi \\ \mathcal{E}_2 \cos\phi_2 &= \mathcal{E}_0 \cos\beta \sin\chi \\ \mathcal{E}_2 \sin\phi_2 &= -\mathcal{E}_0 \sin\beta \cos\chi \end{aligned} \quad \longrightarrow \quad \begin{aligned} \mathcal{E}_1^2 + \mathcal{E}_2^2 &= \mathcal{E}_0^2 \\ \mathcal{E}_1^2 - \mathcal{E}_2^2 &= \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \\ 2\mathcal{E}_1\mathcal{E}_2 \cos\delta\phi &= \mathcal{E}_0^2 \cos 2\beta \sin 2\chi \\ 2\mathcal{E}_1\mathcal{E}_2 \sin\delta\phi &= \mathcal{E}_0^2 \sin 2\beta \end{aligned}$$

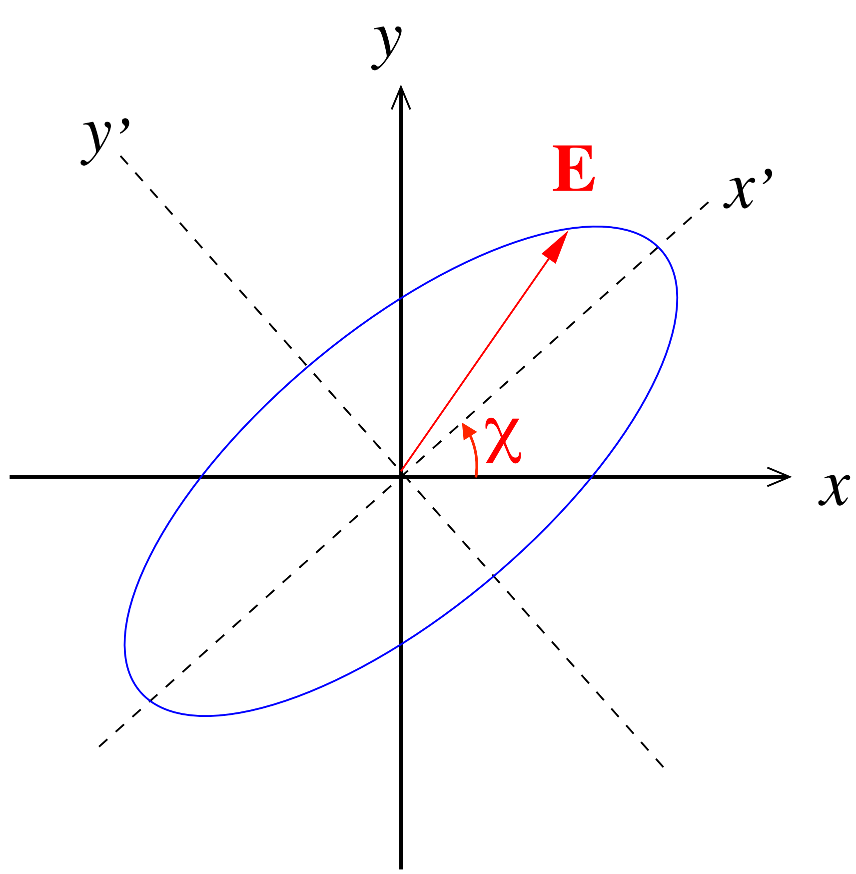
(where $\delta\phi \equiv \phi_1 - \phi_2$)

- Taking time average of the $|\mathbf{E}|^2$, we obtain:

$$\langle|\mathbf{E}|^2\rangle = \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_1'^2 + \mathcal{E}_2'^2 = \text{constant} \equiv \mathcal{E}_0^2$$

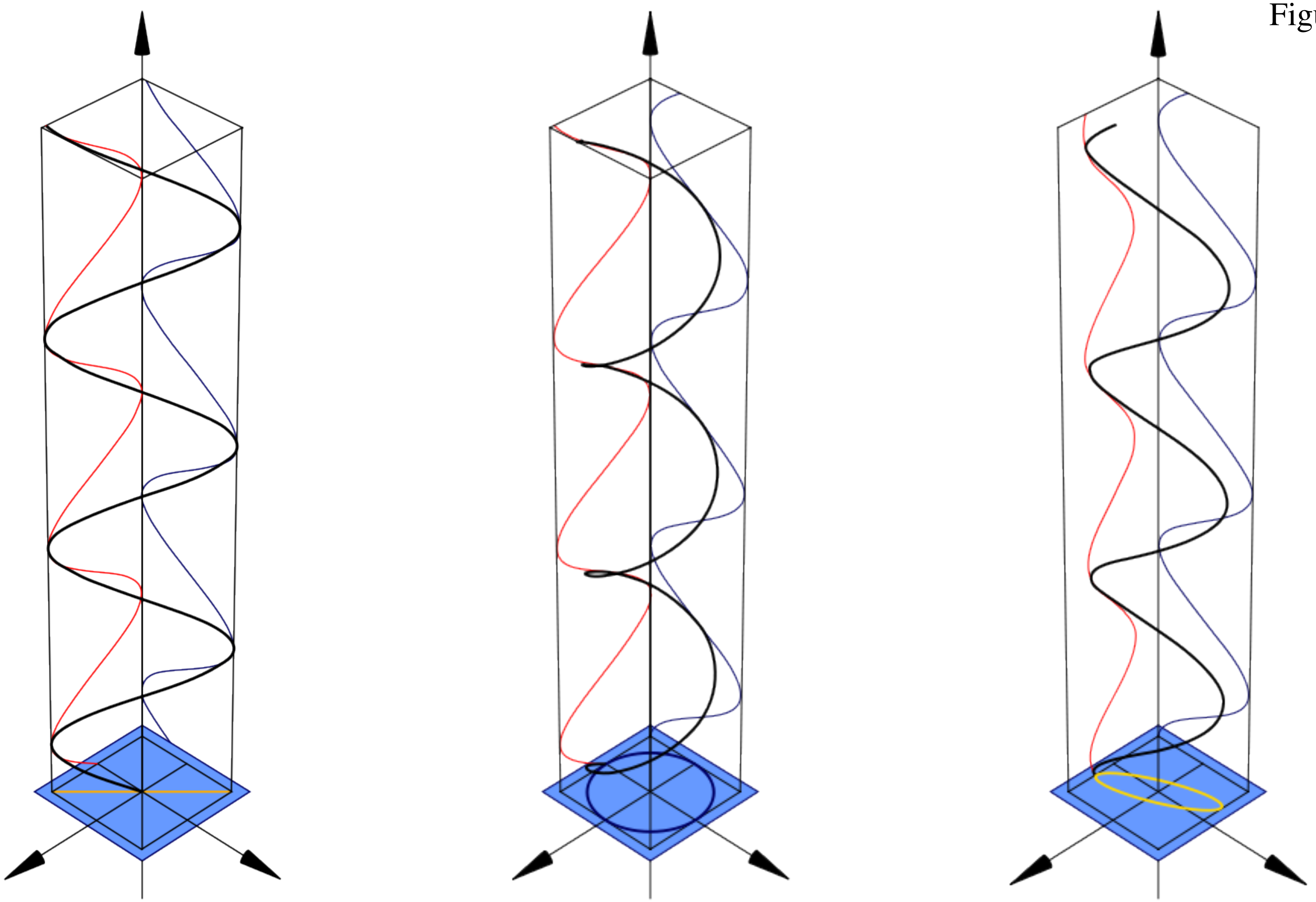
- **Polarization:**

$\beta = \pm\pi/4$	: circularly polarized	$\longrightarrow$	$\lvert\mathcal{E}_1'\rvert = \lvert\mathcal{E}_1'\rvert$
$\beta = 0$ or $\pm\pi/2$	: linearly polarized	$\longrightarrow$	$\lvert\mathcal{E}_2'\rvert = 0$ or $\lvert\mathcal{E}_1'\rvert = 0$

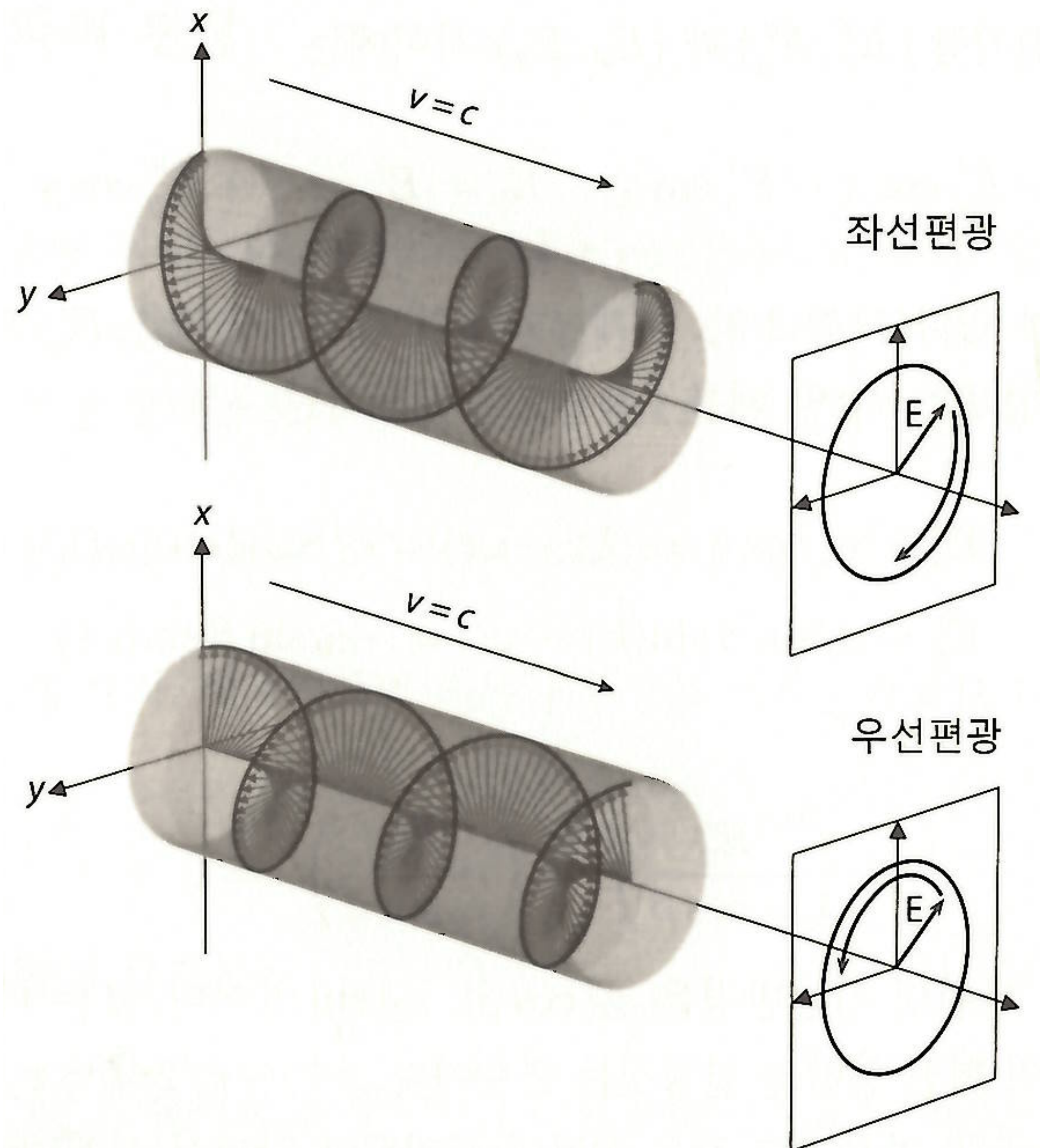


$$\mathbf{E} = \hat{\mathbf{x}}'\mathcal{E}_1' \cos\omega t + \hat{\mathbf{y}}'\mathcal{E}_2' \sin\omega t$$

Figures from Wikipedia



Linear	Circular	Elliptical
$\lvert\phi_1 - \phi_2\rvert = 0$	$\lvert\phi_1 - \phi_2\rvert = \pi/2$	$\lvert\phi_1 - \phi_2\rvert \neq 0, \pi/2$
$\lvert\beta\rvert = 0,\ \pi/2$	$\lvert\beta\rvert = \pi/4$	$\lvert\beta\rvert \neq 0, \pi/4, \pi/2$
$\mathcal{E}_1 = 0$ or $\mathcal{E}_2 = 0$	$\lvert\mathcal{E}_1/\mathcal{E}_2\rvert = 1$	$\mathcal{E}_1/\mathcal{E}_2 \neq \pm 1$



from 천체물리학 (구본철, 김웅태)

IEEE (1969) standard
IAU (1974) recommandation

	RHP	LHP
Helicity	+ (positive)	- (negative)
Rotation at a fixed position	Counterclockwise	Clockwise
Screw at a fixed time	Left-Handed Screw	Right-Handed Screw
β	$-\pi/2 < \beta < 0$	$0 < \beta < \pi/2$
$\delta\phi\ (\equiv \phi_1 - \phi_2)$	$-\pi/2 < \delta\phi < 0$	$0 < \delta\phi < \pi/2$
Stokes V	V > 0	V < 0

IEEE (1969) standard
IAU (1974) recommandation

	Right-Handed Polarization	**Left-Handed Polarization**
Helicity	+ (positive)	- (negative)
Rotation at a fixed position	Counterclockwise	Clockwise
Screw at a fixed time	Left-Handed Screw	Right-Handed Screw
β	$-\pi/2 < \beta < 0$	$0 < \beta < \pi/2$
$\delta\phi \ (\equiv \phi_1 - \phi_2)$	$-\pi/2 < \delta\phi < 0$	$0 < \delta\phi < \pi/2$
Stokes V	V > 0	V < 0

Stokes Parameters (for monochromatic waves)

- A convenient way to represent these quantities is by means of the **Stokes parameters for monochromatic waves**.

$$
\begin{aligned}
I &\equiv E_1 E_1^* + E_2 E_2^* = \mathcal{E}_1^2 + \mathcal{E}_2^2 \\
Q &\equiv E_1 E_1^* - E_2 E_2^* = \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \\
U &\equiv E_1 E_2^* + E_2 E_1^* = 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi \\
V &\equiv i\,(E_1 E_2^* - E_2 E_1^*) = -2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) = -\mathcal{E}_0^2 \sin 2\beta
\end{aligned}
\longrightarrow \quad I^2 = Q^2 + U^2 + V^2
$$

for a monochromatic wave (pure polarization)

Then, we have

$$\boxed{\mathcal{E}_0 = \sqrt{I}, \quad \sin 2\beta = -\frac{V}{I}, \quad \tan 2\chi = \frac{U}{Q}}$$

Pure elliptical polarization is determined sole by three parameters $(\mathcal{E}_0,\ \beta,\ \chi)$.

- Meaning of the Stokes parameters:

I : total energy flux or intensity

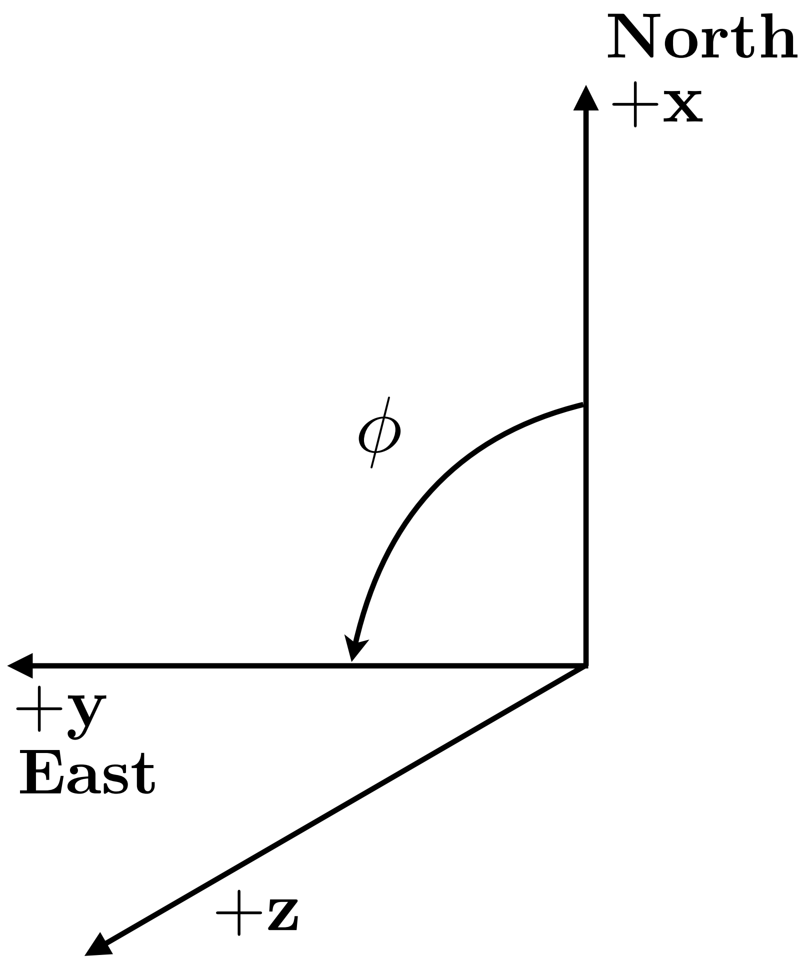
V : circularity parameter ($V > 0$: right-handed, $V < 0$: left-handed)

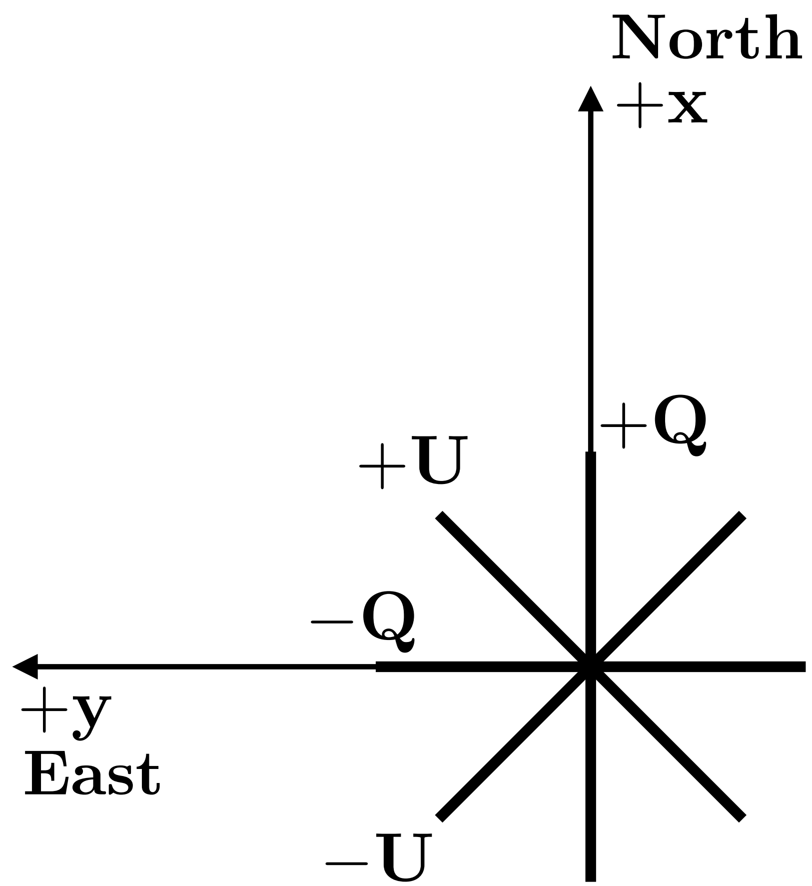
Q, U : orientation of the ellipse (or line) relative to the x-axis

$Q \times U \neq 0, V = 0$	: linear polarization
$Q = U = 0, V \neq 0$	: circular polarization
$Q \times U \neq 0, V \neq 0$	: elliptical polarization

The IAU definition of coordinate system

from Hamaker & Bregman (1996, A&AS)





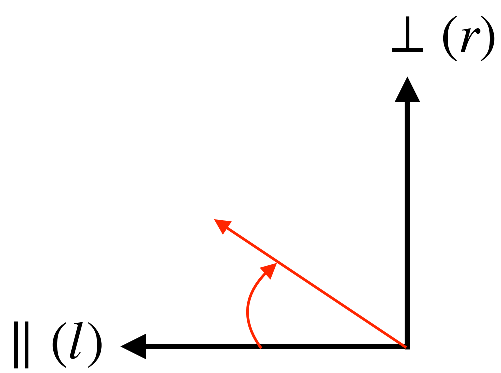
$$I = I_{0^\circ} + I_{90^\circ}$$

$$Q = I_{0^\circ} - I_{90^\circ}$$

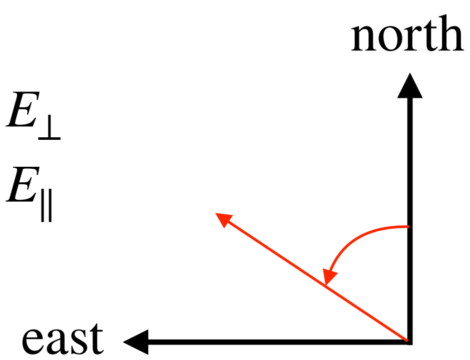
$$U = I_{45^\circ} - I_{135^\circ}$$

Differences in Definitions of Stokes vector

- Bohren & Huffman (Absorption and Scattering of Light by Small Particles)
- Chandrasekhar (Radiative Transfer)
- IAU recommendation
- IEEE standard



$$E_n = E_\perp$$
$$E_e = E_\parallel$$



$$I_{\mathrm{BH}} = E_\parallel E_\parallel^* + E_\perp E_\perp^*$$
$$Q_{\mathrm{BH}} = E_\parallel E_\parallel^* - E_\perp E_\perp^*$$
$$U_{\mathrm{BH}} = E_\parallel E_\perp^* + E_\perp E_\parallel^*$$
$$V_{\mathrm{BH}} = i\left(E_\parallel E_\perp^* - E_\perp E_\parallel^*\right)$$
$$V_{\mathrm{C}} = -\,i\left(E_\parallel E_\perp^* - E_\perp E_\parallel^*\right)$$

$$I_{\mathrm{IAU}} = E_n E_n^* + E_e E_e^*$$
$$Q_{\mathrm{IAU}} = E_n E_n^* - E_e E_e^*$$
$$U_{\mathrm{IAU}} = E_n E_e^* + E_e E_n^*$$
$$V_{\mathrm{IAU}} = i\left(E_n E_e^* - E_e E_n^*\right)$$

$$\therefore \begin{pmatrix} I_{\mathrm{IAU}} \\ Q_{\mathrm{IAU}} \\ U_{\mathrm{IAU}} \\ V_{\mathrm{IAU}} \end{pmatrix} = \begin{pmatrix} I_{\mathrm{BH}} \\ -Q_{\mathrm{BH}} \\ U_{\mathrm{BH}} \\ -V_{\mathrm{BH}} \end{pmatrix} = \begin{pmatrix} I_{\mathrm{C}} \\ -Q_{\mathrm{C}} \\ U_{\mathrm{C}} \\ V_{\mathrm{C}} \end{pmatrix}$$

Conventions adopted by various authors

Peest et al. (2017, A&A, 601, A92) + α
(Typo: +/-U should read as +/-Q)

	+Q	-Q
+V	IAU (1974) Martin (1974) Tsang et al. (1985) Trippe (2014)	Chandrasekhar (1950) van de Hulst (1957) Hovenier & van der Mee (1983) Fischer et al. (1994) Code & Whitney (1995) Mishchenko et al. (1999) Gordon et al. (2001) Lucas (2003) Gorski et al. (2005)
-V	Shurcliff (1962) Bianchi et al. (1996)	Bohren & Huffman (1998) Rybicki & Lightman (1979) Mishchenko et al. (2002)

Homeworks (due date: 10/04)

[Q4] We obtained the following solution for a slab geometry in the case of “perfect” forward scattering.

$$
\boxed{
\begin{aligned}
I_{\mathrm{direc}} &= e^{-\tau_{\mathrm{ext}}} I(0) \\
I_{\mathrm{scatt}} &= \left(e^{a\tau_{\mathrm{ext}}} - 1\right) e^{-\tau_{\mathrm{ext}}} I(0) \\
I_{\mathrm{tot}} &= e^{-\tau_{\mathrm{abs}}} I(0)
\end{aligned}
}
$$

Here,

total extinction optical depth: $\tau_{\mathrm{ext}} = \tau_{\mathrm{abs}} + \tau_{\mathrm{scatt}}$

scattering albedo: $a = \tau_{\mathrm{scatt}}/\tau_{\mathrm{ext}}$

Explain the physical meaning of the equations, in particular, the difference between the first equation and the third equation.

In other words, can you derive the third equation based only on the physical reasoning, without solving the RT equation?