Special Topics in Radiative Transfer

Lecture 1
2026 March 3 (Tuesday), 9AM

updated on 02/28

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References

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https://www.bartol.udel.edu/~owocki/phys633/RadProc-RybLightman.pdf

[2] Radiative Transfer in Stellar Atomospheres (R. J. Rutten)
https://robrutten.nl/rrweb/rjr-pubs/2003rtsa.book.....R.pdf

[3] Monte Carlo Methods for Radiation Transport (O. N. Vassiliev; Chapters 1 & 2)
http://ndl.ethernet.edu.et/bitstream/123456789/66046/1/3.pdf

[4] Monte Carlo radiative transfer
(Noebauer & Sim, 2019, Living Reviews in Computational Astrophysics)
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[5] A nearby galaxy perspective on interstellar dust properties and their evolution
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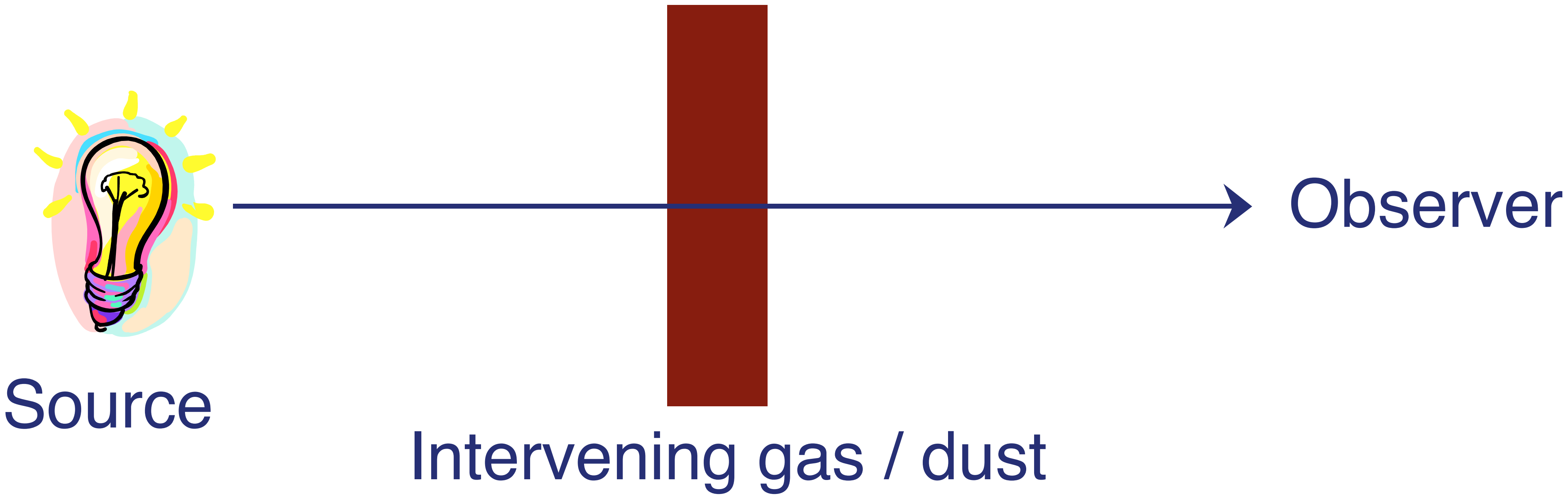
Basic Radiative Transfer

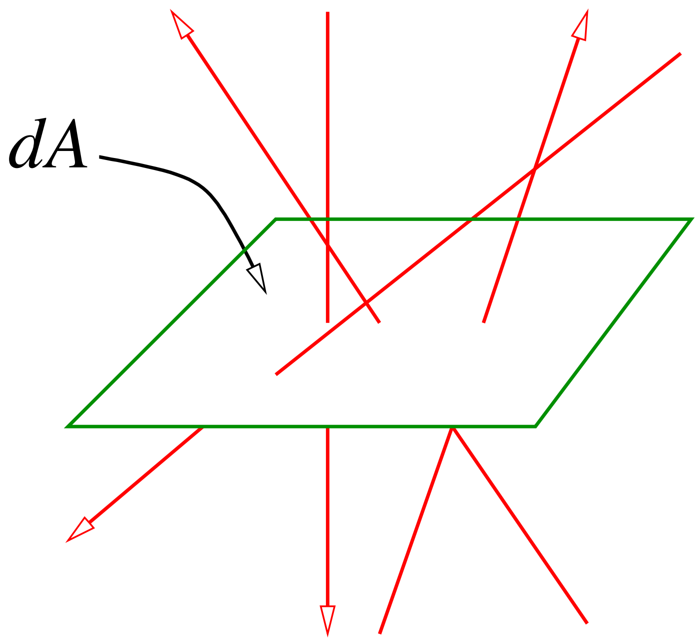
Radiative Transfer

- Radiative transfer describes how radiation changes as it travels through a medium (i.e., gas and dust).
  - A wide variety of absorption and emission processes from atom, molecules, and solid particles contribute to the radiation.



Complexities of RT

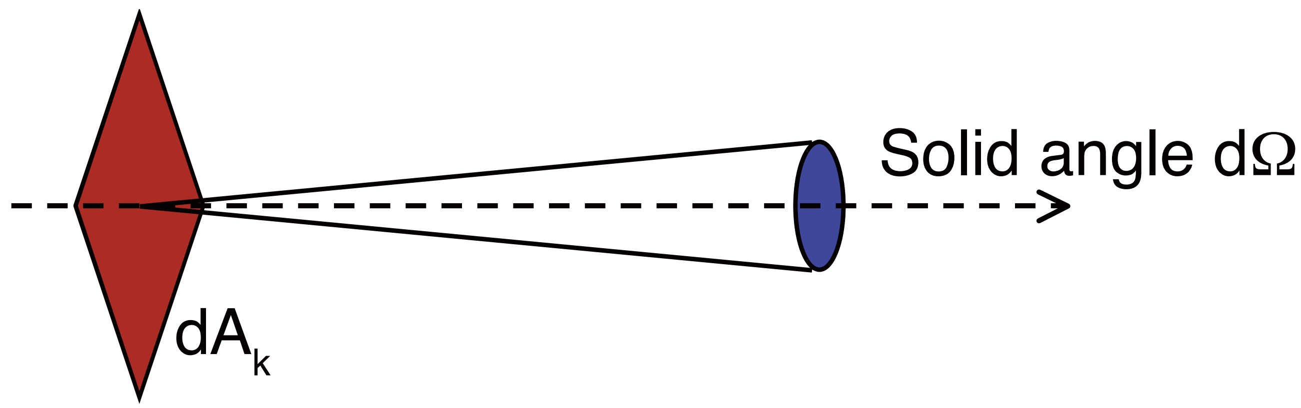
- Simplification:
  - Astronomical objects are normally much larger than the wavelength of radiation they emit.
  - Diffraction can be neglected.
  - Light rays travel to us along straight lines.



- Complexity:
  - At one point, photons can be traveling in several different directions.
  - For instance, at the center of a star, photons are moving equally in all directions. (However, radiation from a star seen by a distant observer is moving almost exactly radially.)
  - Full specification of radiation needs to say how much radiation is moving in each direction at every point. Therefore, we are dealing with the five- or six-dimensional problem. ($[x, y, z] + [\theta, \phi] + [t]$)

Intensity (Surface Brightness)

- ***Intensity is the energy carried along by individual rays.***



$\mathbf{k}$ = direction of propagation

- Let dE_ν be the amount of energy transported through the area dA_k, at location $\mathbf{x}$, perpendicular to a direction $\mathbf{k}$ over solid angle $d\Omega$ around the direction $\mathbf{k}$ in a time interval dt in the frequency band between ν and $\nu + d\nu$.
- The monochromatic specific intensity I_ν is then defined by the equation.

$$\boxed{dE_\nu = I_\nu(\mathbf{k},\, \mathbf{x},\, t)dA_{\mathbf{k}}d\Omega d\nu dt} \quad \Rightarrow \quad I_\nu(\mathbf{k}, \mathbf{x}, t) = \frac{dE_\nu}{dA_{\mathrm{k}}d\Omega d\nu dt}$$

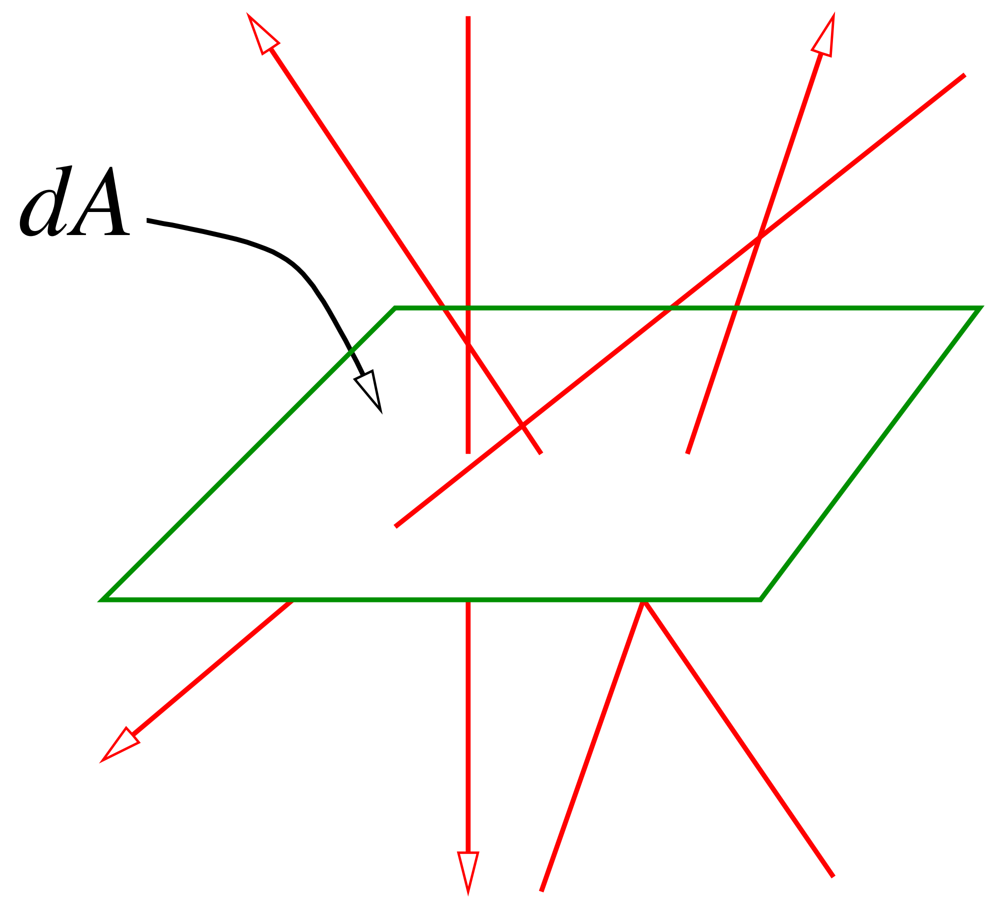
- Unit: $\mathrm{erg\ s^{-1}\ cm^{-2}\ sr^{-1}\ Hz^{-1}}$
- Intensity represents the macroscopic quantity (per infinitesimally small time interval, area, band width and solid angle) to specify the energy carried by a bundle of identical photons along a single “ray”.

- From the view point of an observer, the specific intensity is called ***surface brightness***.

Flux

- Definition
  - ***Flux is a measure of the energy carried by all rays passing through a given area***
  - Consider a small area dA, exposed to radiation for a time dt.
  - Flux F_ν is defined as ***the total (net) energy passing through a unit area in all directions within a unit time interval.***

$$\boxed{dE = F \times dA \times dt} \Rightarrow F_\nu = \frac{dE_\nu}{dA d\nu dt}$$

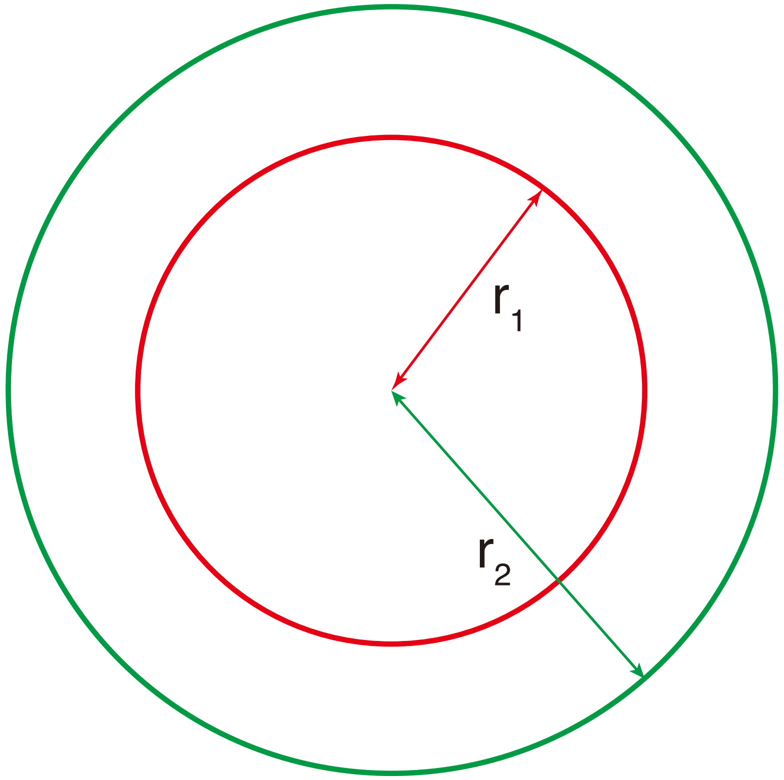


  - Note that F_ν ***depends on the orientation of the area element*** dA.
  - Unit: $\mathrm{erg\ cm^{-2}\ s^{-1}}$

- F_ν is often called the "flux density."
- Radio astronomers use a special unit to define the flux density: $1\ \mathrm{Jansky\ (Jy)} = 10^{-23}\ \mathrm{erg\ s^{-1}\ cm^{-2}\ Hz^{-1}}$

Inverse Square Law

- Flux from an isotropic radiation source, i.e., a source emitting equal amounts of energy in all directions.



  - Because of energy conservation, flux through two shells around the source must be the same.

$$4\pi r_1^2 F(r_1) = 4\pi r_2^2 F(r_2)$$

  - Therefore, we obtain the inverse square law.

$$F = \frac{\mathrm{const.}}{r^2}$$

Total Flux / Total Intensity

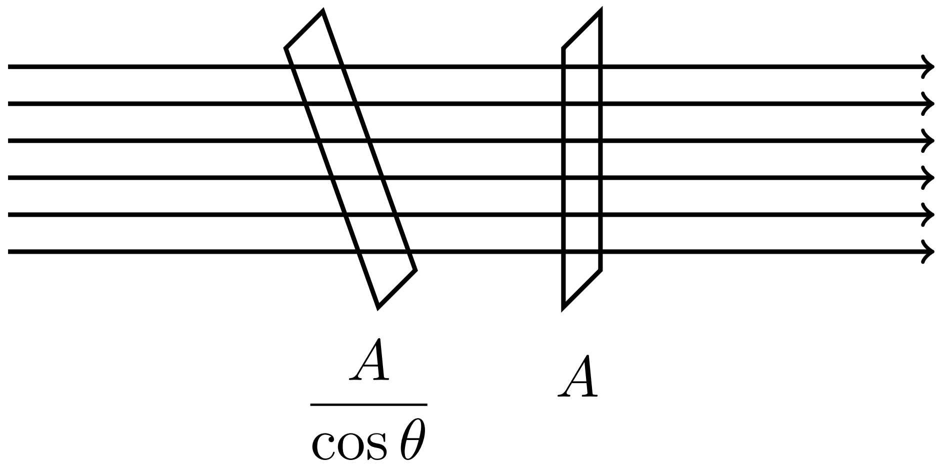
- Real detectors are sensitive to a limited range of wavelengths. We need to consider how the incident radiation is distributed over frequency.

Total flux: $F = \int F_\nu d\nu$ Integral of F_ν over all frequencies

Units: $\mathrm{erg\ s^{-1}\ cm^{-2}\ Hz^{-1}}$

Total intensity: $I = \int I_\nu d\nu$ Integral of I_ν over all frequencies

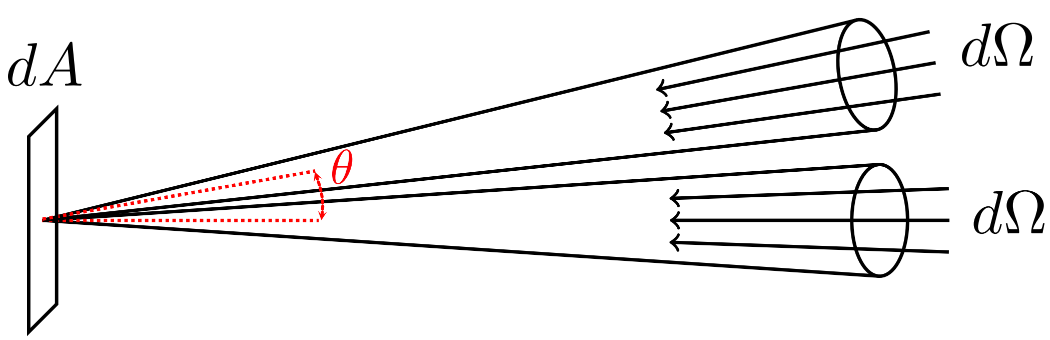
Flux vs. Intensity



Power = energy per unit time

The power delivered to the two surfaces are equal though there areas differ.

The flux is **the power per unit area** so the tilted surface gets less flux.



Two intensities are equal.
The upper set or rays delivers less flux.
However, they delivers the same power to the area dA.

The rate that energy is delivered to a surface from light traveling around a direction θ is $I\cos\theta d\Omega$.

Relation between the flux and the intensity

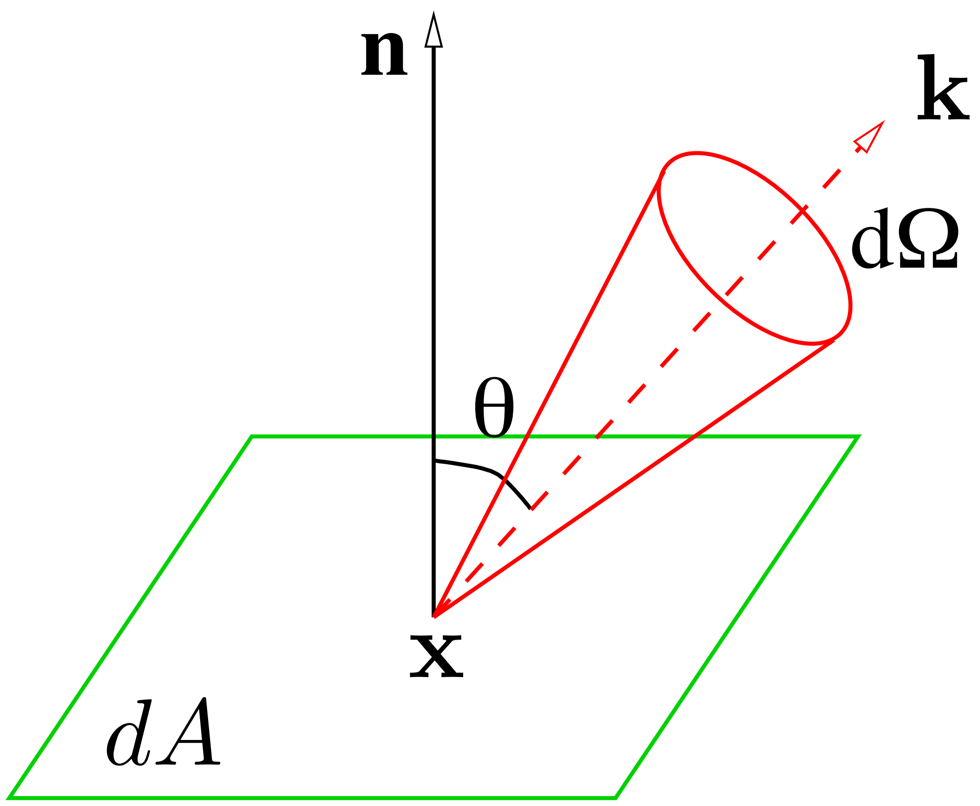
- Let's consider a small area dA, with light rays passing through it at all angles to the normal vector $\mathbf{n}$ of the surface.

- For rays centered about $\mathbf{k}$, the area normal to $\mathbf{k}$ is

$$dA_{\mathbf{k}} = dA\cos\theta$$

  - By the definition,

$$F_\nu dA d\nu dt = \int I_\nu(\mathbf{k}, \mathbf{x}, t) dA_{\mathbf{k}} d\Omega d\nu dt$$



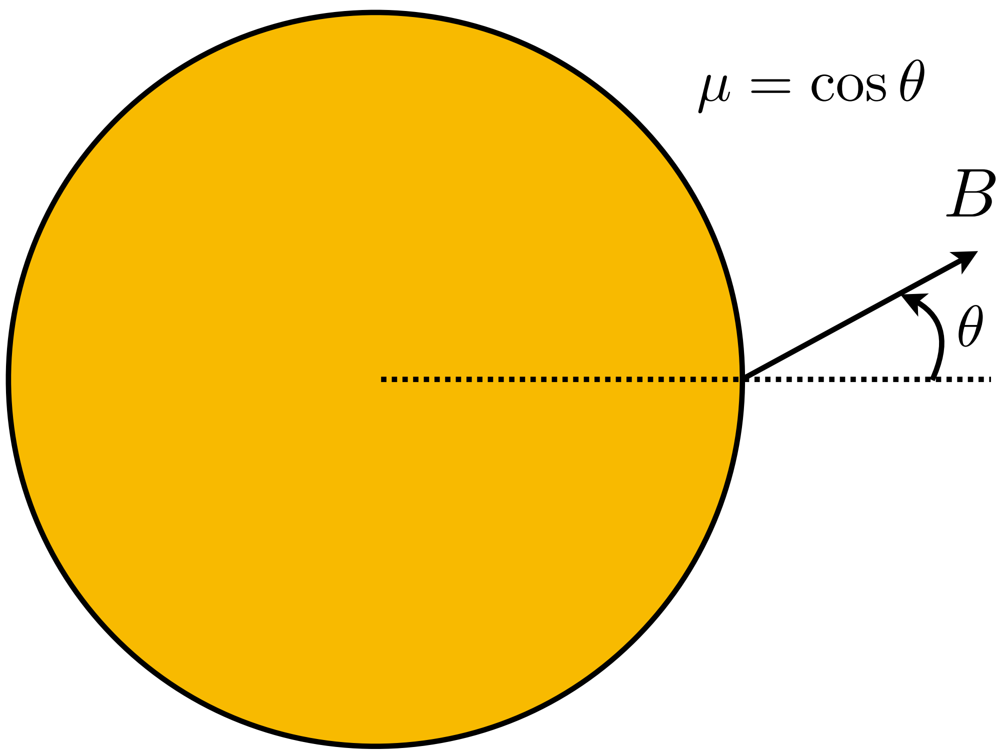
- Hence, net flux in the direction of $\mathbf{n}$ is given by integrating over all solid angles:

$$F_\nu = \int I_\nu \cos\theta d\Omega = \int_0^{2\pi}\int_0^{\pi} I_\nu \cos\theta \sin\theta d\theta d\phi = \int_{-1}^{1}\int_0^{\pi} I_\nu \mu d\mu d\phi$$

[Note] **flux** **= sum of all ray vectors projected onto a normal vector**
intensity **= absolute value of a single ray vector**

Flux from the surface of a uniformly bright sphere

- Let's calculate the flux at P on a sphere of uniform brightness B



$$F = \int B\cos\theta d\Omega = \int_0^1 \int_0^{2\pi} B\mu d\mu d\phi$$

$$F = \pi B$$

The total luminosity from the sphere is then

$$L = (4\pi R^2)F = (4\pi R^2)\pi B$$

- In stellar atmosphere, the **astrophysical flux** is defined by F/π .

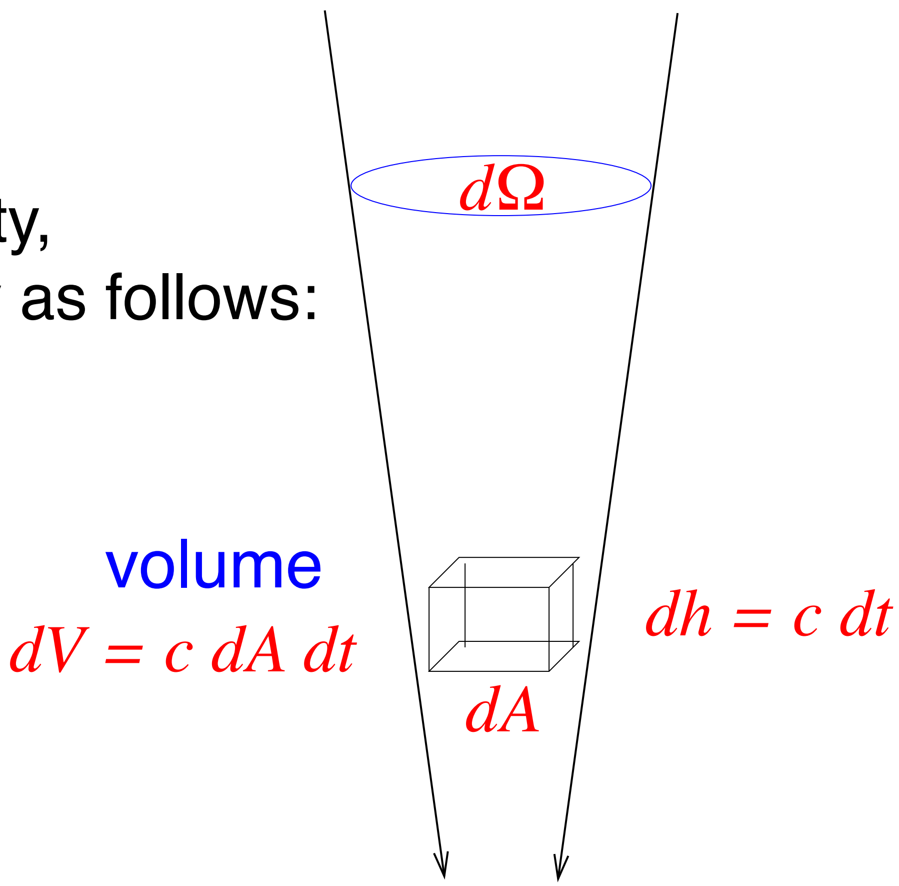
Radiation Energy Density

- Consider a bundle of rays passing through a volume element dV in a direction Ω.
  - Then, the energy density per unit solid angle is defined by

$$dE = u_\nu(\Omega) dV d\Omega d\nu$$

Since radiation travels at speed c, the volume element swept out by the rays during the time interval dt is

$$dV = dA(cdt)$$



  - Alternatively, according to the definition of the intensity, the energy can be expressed in terms of the intensity as follows:

$$dE = I_\nu dA dt d\Omega d\nu$$

  - Comparing the two expressions, we have

$$\boxed{u_\nu(\Omega) = I_\nu(\Omega)/c}$$

Energy Density and Mean Intensity

- Integrating over all solid angle, we obtain

$$u_\nu = \int u_\nu(\Omega) d\Omega = \frac{1}{c} \int I_\nu d\Omega$$

- ***Mean intensity*** is defined by

$$\boxed{J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega}$$

- Then, the energy density is

$$\boxed{u_\nu = \frac{4\pi}{c} J_\nu}$$

- Total energy density is obtained by integrating over all frequencies.

$$u = \int u_\nu d\nu = \frac{4\pi}{c} \int J_\nu d\nu$$

Luminosity

- To determine the energy per unit time, we integrate flux over area.
  - ***Monochromatic luminosity***: Considering a sphere centered on a source with radius *R*, the monochromatic luminosity is

$$L_\nu = R^2 \int d\Omega F_\nu$$
$$= 4\pi R^2 F_\nu \qquad \text{for an isotropic source}$$

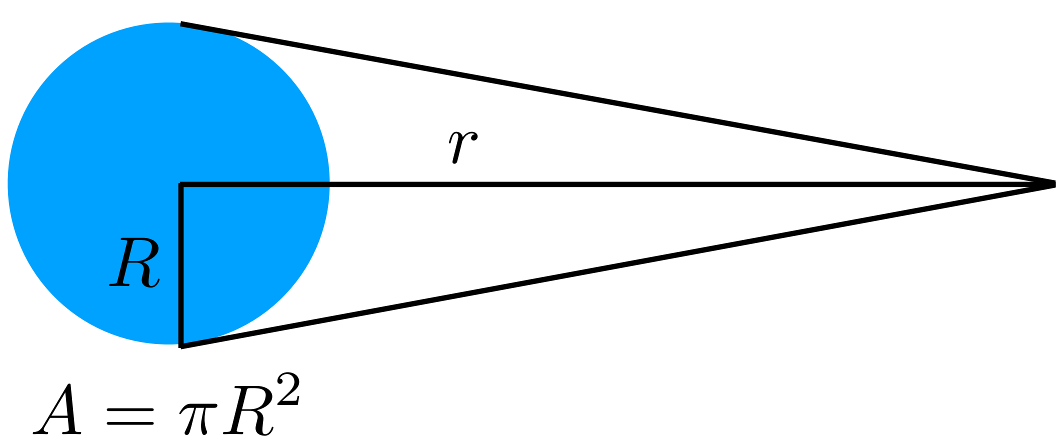
  - The ***bolometric luminosity*** is

$$L_{\rm bol} = \int L_\nu d\nu = \int L_\lambda d\lambda = 4\pi R^2 \int F_\nu d\nu$$

- Flux and Luminosity of an extended source

$$F = \pi I \left(\frac{R}{r}\right)^2 = I \frac{A}{r^2}$$
$$= I\Omega_{\rm source}$$

$$L = (4\pi r^2)F = (4\pi r^2) I\Omega_{\rm source}$$



Note — SED

- Specific intensity can be defined as per wavelength interval.

$$I_\nu |d\nu| = I_\lambda |d\lambda| \qquad \leftarrow \left|\frac{d\nu}{d\lambda}\right| = \frac{c}{\lambda^2} = \frac{\nu}{\lambda} \qquad \longleftarrow \qquad \nu = \frac{c}{\lambda}$$

$$\nu I_\nu = \lambda I_\lambda$$

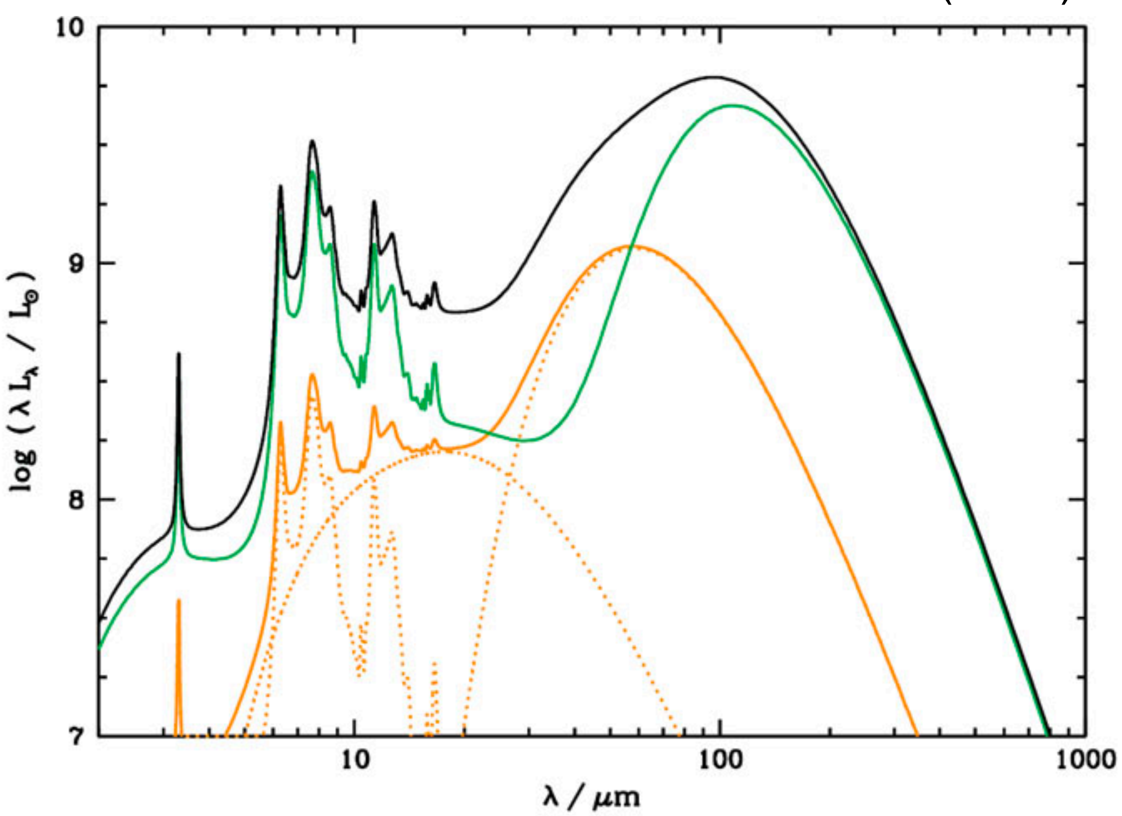
$$I_\nu \neq I_\lambda$$

- Total (integrated) intensity is defined as the intensity over all frequencies.

$$I = \int_0^\infty I_\nu d\nu = \int_0^\infty I_\lambda d\lambda$$

- In astrophysics, we plot the ***spectral energy distribution (SED)*** as νI_ν versus ν or λI_λ versus λ.

Galaxy SEDs
da Cunha et al. (2008)



< Radiative Transfer Equation > — in free space

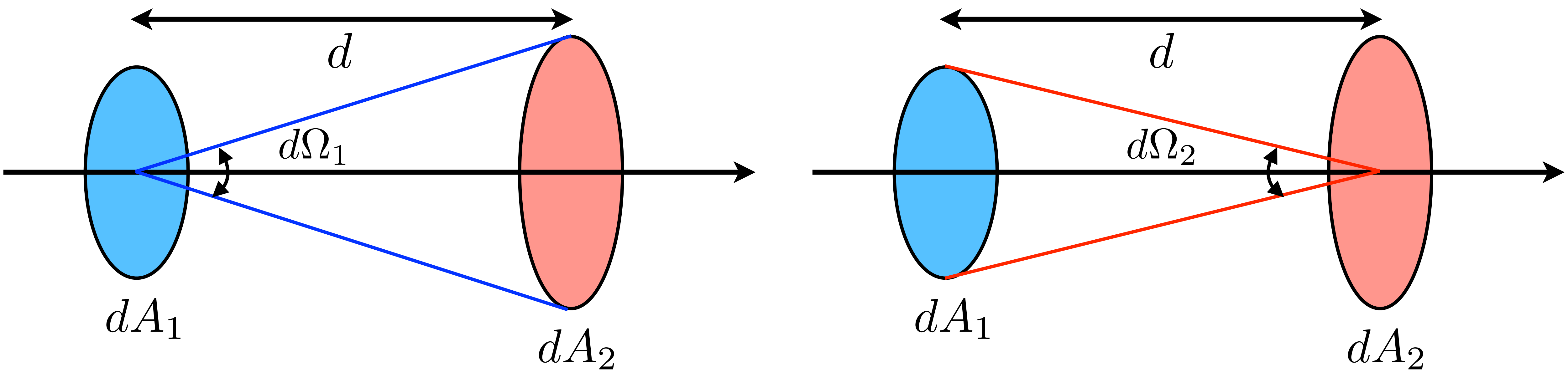
- How does intensity changes along a ray in free space
  - Suppose a bundle of rays and any two points along the rays and construct two "infinitesimal" areas dA_1 and dA_2 normal to the rays at these points.
  - What are the energies carried by the rays passing through both areas?

energy passing through 1

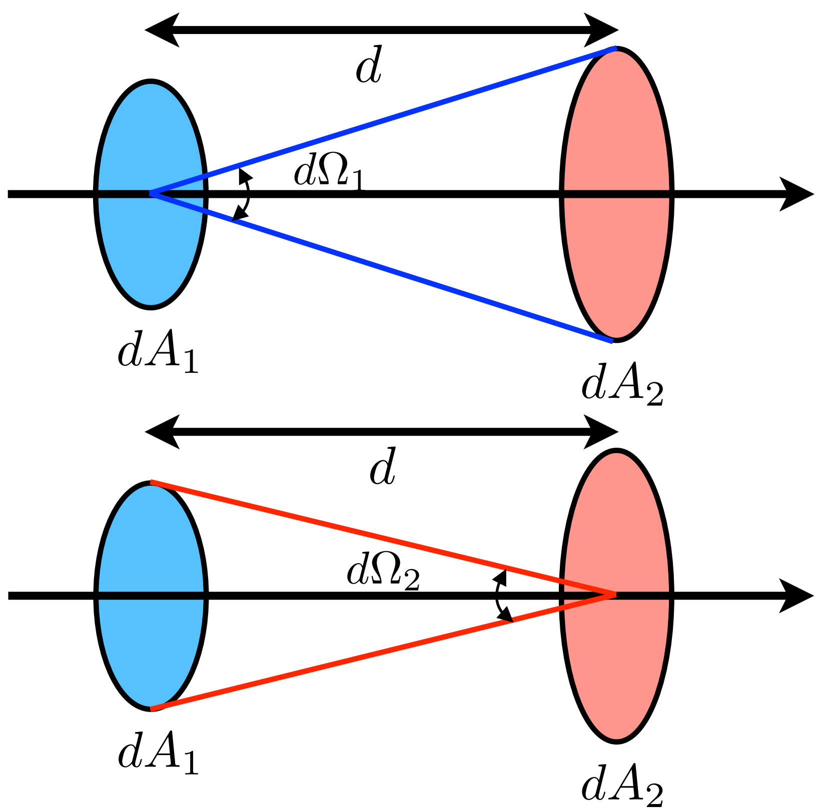
$$dE_1 = I_1 dA_1 d\Omega_1 d\nu dt$$

energy passing through 2

$$dE_2 = I_2 dA_2 d\Omega_2 d\nu dt$$



- Here, $d\Omega_1$ is the solid angle subtended by dA_2 as seen from the location 1 and $d\Omega_2$ is the solid angle subtended by dA_1 as seen from the location 2.



$$d\Omega_1 = \frac{dA_2}{d^2}$$

$$d\Omega_2 = \frac{dA_1}{d^2}$$

Conservation of energy:
Because energy is conserved,

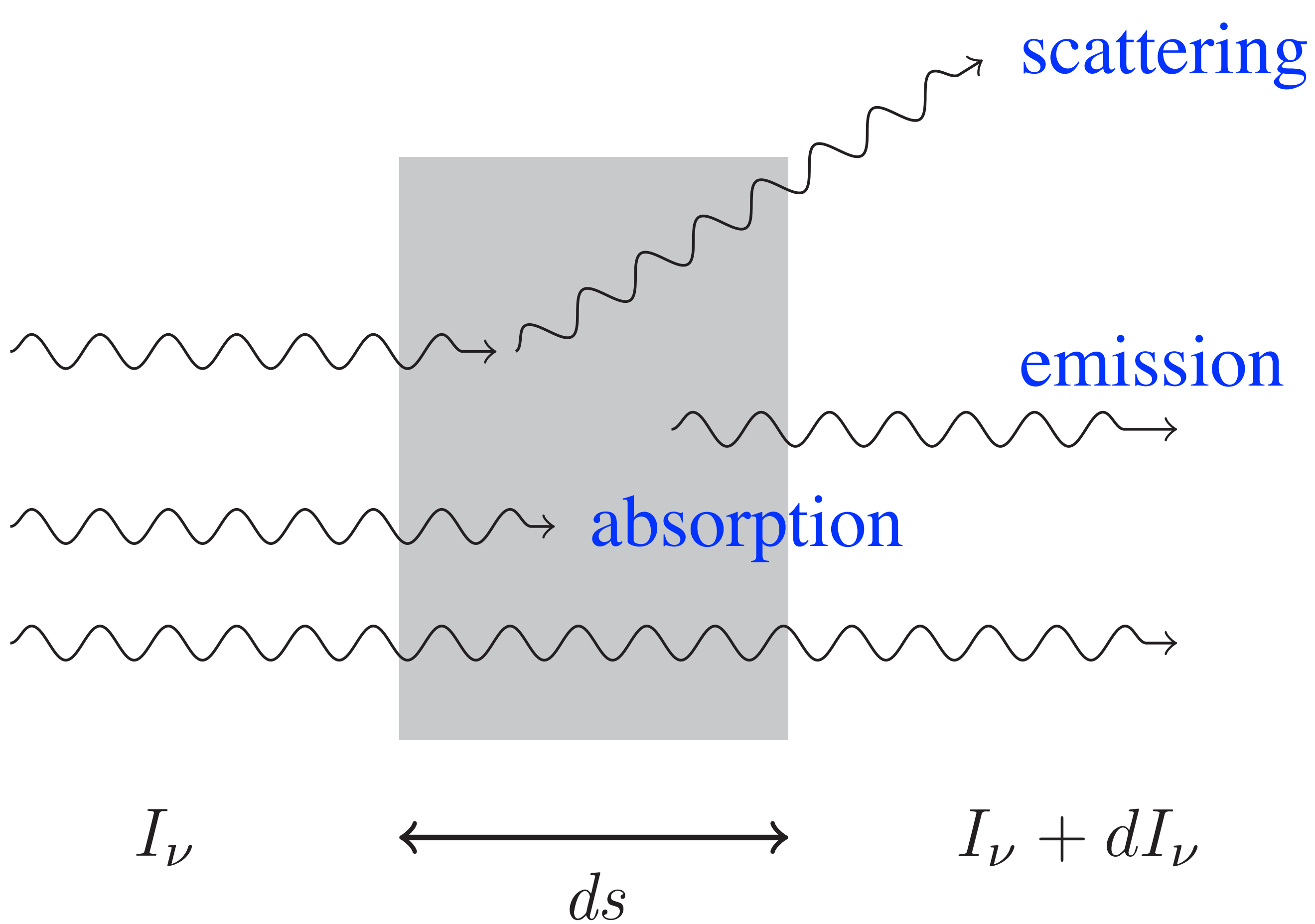
$$dE_1 = dE_2 \quad \rightarrow \quad I_1 = I_2$$

- Conclusion (***the constancy of intensity***): $I_1 = I_2 \longrightarrow \boxed{\frac{dI}{ds} = 0}$
  - the specific intensity remains the same as radiation propagates through free space.

- We receive the same specific intensity at the telescope as is emitted at the source.
  - Imagine looking at a uniformly lit wall and walking toward it. As you get closer, a field-of-view with fixed angular size will see a progressively smaller region of the wall, but this is exactly balanced by the inverse square law describing the spreading of the light rays from the wall.

< Radiative Transfer Equation > — Emission & Absorption

- In reality, as a ray passes through matter, energy may be added, subtracted, or scattered from it by emission, absorption, or scattering.
  - The intensity will not in general remain constant.
  - These interactions are described by the ***radiative transfer equation***.



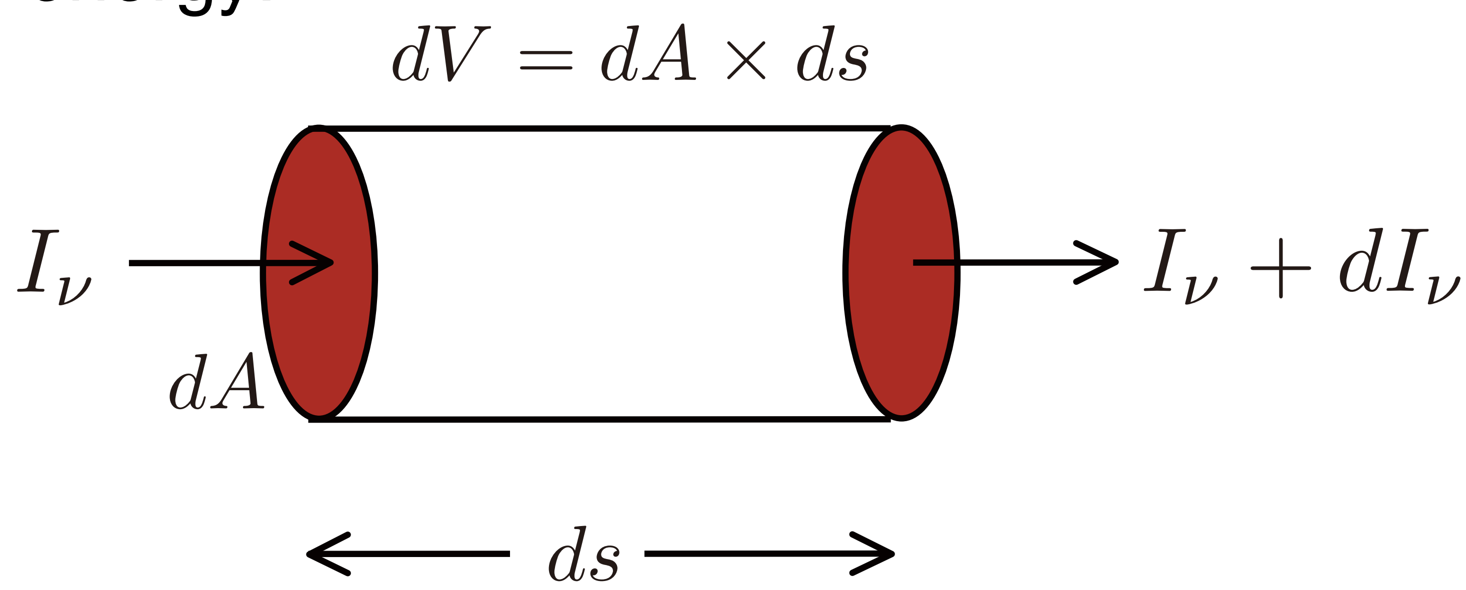
loss: scattering & absorption

gain: emission

[J. P. Williams]
Introduction to the Interstellar Medium

Emission

- If the radiation travels through a medium which is itself emitting radiation, that will add to the energy:



- Spontaneous "***emission coefficient***" or "***emissivity***" j_ν is the amount of energy emitted per unit volume, per unit solid angle, per unit time, and per unit frequency:

$$dE = j_\nu dV d\Omega dt d\nu \quad (j_\nu : \mathrm{erg\ cm^{-3}\ s^{-1}\ sr^{-1}\ Hz^{-1}})$$

- In going a distance ds, a beam of cross section dA travels through a volume $dV = dAds$. Thus the intensity added to the beam is by ds is

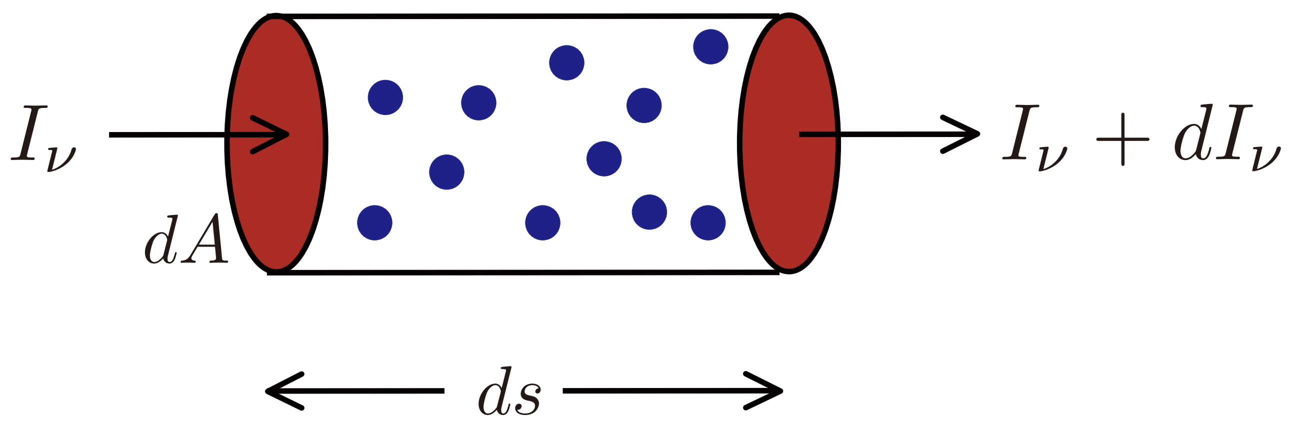
$$dE = (dI_\nu) dA d\Omega dt d\nu \longrightarrow \boxed{dI_\nu = j_\nu ds}$$

- Therefore, the equation of radiative transfer for pure emission becomes:

$$\frac{dI_\nu}{ds} = j_\nu$$

Absorption

- If the radiation travels through a medium which absorbs radiation, the energy in the beam will be reduced:



- Let n denote the number density of absorbers (particles per unit volume).
- Assume that each absorber has a cross-sectional area of σ_ν (in units of cm^2).

‘geometric’ cross section: $\sigma = \pi r^2$ for a spherical particle with a

- If a beam travels through ds, total area of absorbers is

$$\text{number of absorbers } \times \text{cross section} = \left(n \times dA \times ds\right) \times \sigma_\nu$$

Fraction of radiation absorbed = Fraction of area blocked

$$\frac{dI_\nu}{I_\nu} = -\frac{ndAds\sigma_\nu}{dA} = -n\sigma_\nu ds$$

$$dI_\nu = -n\sigma_\nu I_\nu ds \equiv -\alpha_\nu I_\nu ds$$

$$\longrightarrow \quad \boxed{\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu}$$

- ***Absorption coefficient*** is defined as $\alpha_\nu \equiv n\sigma_\nu$ (units: cm^{-1}), meaning the ***total cross-sectional area per unit volume***.

$$\boxed{\begin{aligned} \alpha_\nu &= n\sigma_\nu \quad [\mathrm{cm}^{-1}] \\ &= \rho\kappa_\nu \end{aligned}}$$

where ρ is the mass density and κ_ν is called the **mass absorption coefficient** or the **opacity coefficient**.

Emission + Absorption

- ***Radiative transfer equation*** with both absorption and emission is

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

absorption emission

- We can rewrite the radiative transfer equation using the optical depth as a measure of ‘distance’ rather than s:

$$\frac{dI_\nu}{\alpha_\nu ds} = -I_\nu + \frac{j_\nu}{\alpha_\nu}$$

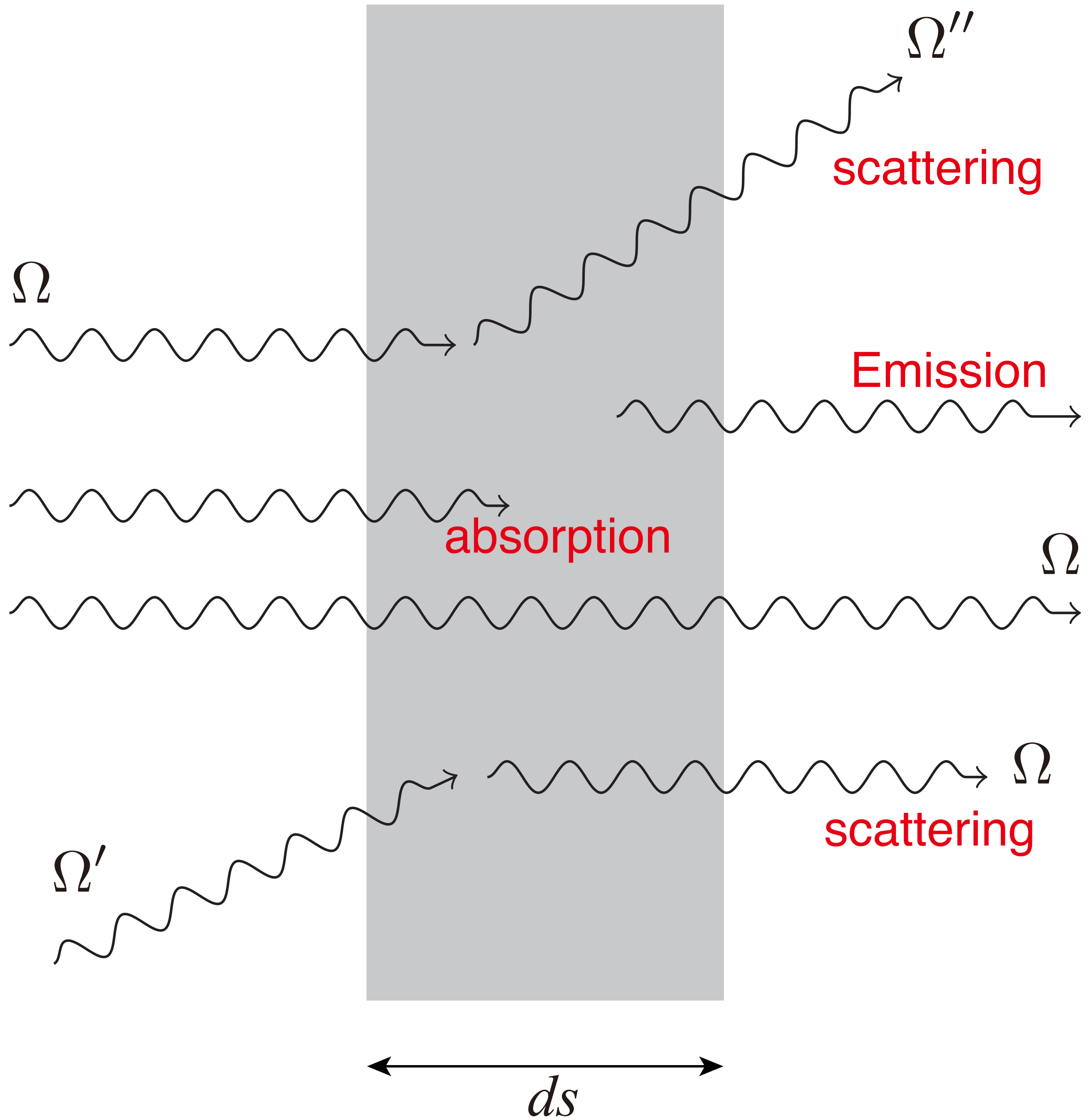
$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

- where $S_\nu \equiv j_\nu / \alpha_\nu$ ***is called the source function***. This is an alternative and sometimes more convenient way to write the equation.

Emission + Absorption + Scattering

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\mathrm{ext}} I_\nu + j_\nu + \underbrace{\alpha_\nu^{\mathrm{scatt}} \int \Phi_\nu(\Omega' \to \Omega) I_\nu(\Omega') d\Omega'}$$

$$\Rightarrow \quad j_\nu^{\mathrm{scatt}}$$



- extinction cross section

$$\sigma_\nu^{\mathrm{ext}} = \sigma_\nu^{\mathrm{abs}} + \sigma_\nu^{\mathrm{scatt}}$$

- extinction coefficient

$$\alpha_\nu^{\mathrm{ext}} = \alpha_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{scatt}}$$
$$= n\sigma_\nu^{\mathrm{abs}} + n\sigma_\nu^{\mathrm{scatt}}$$

- scattering phase function

$$\Phi_\nu(\Omega' \to \Omega)$$

$$\int \Phi_\nu(\Omega' \to \Omega) d\Omega = 1$$

$$\int \Phi_\nu(\Omega' \to \Omega) d\Omega' = 1$$

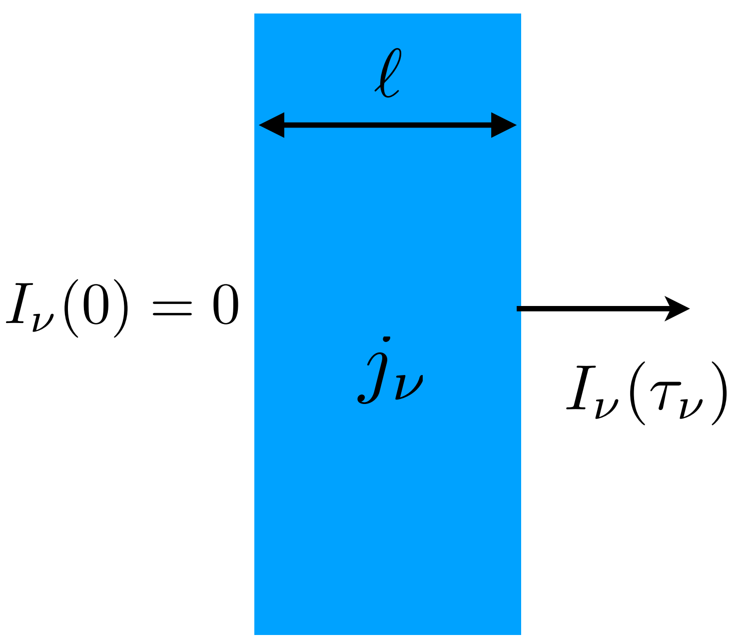
Solution: Emission Only

- For pure emission, $\alpha_\nu = 0$

$$\frac{dI_\nu}{ds} = j_\nu$$

$$I_\nu(s) = I_\nu(0) + \int_0^s j_\nu(s')ds'$$

- The brightness increase is equal to the emission coefficient integrated along the line of sight.



$$I_\nu = j_\nu \ell$$
$$\text{if } I_\nu(0) = 0 \text{ and } j_\nu = \text{constant}$$

Solution: Absorption Only

- Pure absorption: $j_\nu = 0$

  Rearranging the previous equation, we obtain the equation of radiative transfer for pure absorption:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

  - The amount of reduced energy depends on how much radiation we already have.
  - Integrate to find how radiation changes along path:

$$\int_0^s \frac{dI_\nu}{I_\nu} = -\int_0^s \alpha_\nu(s')ds'$$

$$\left[\ln I_\nu\right]_0^s = -\int_0^s \alpha_\nu(s')ds'$$

$$I_\nu(s) = I_\nu(0)\exp\left[-\int_0^s \alpha_\nu(s')ds'\right]$$

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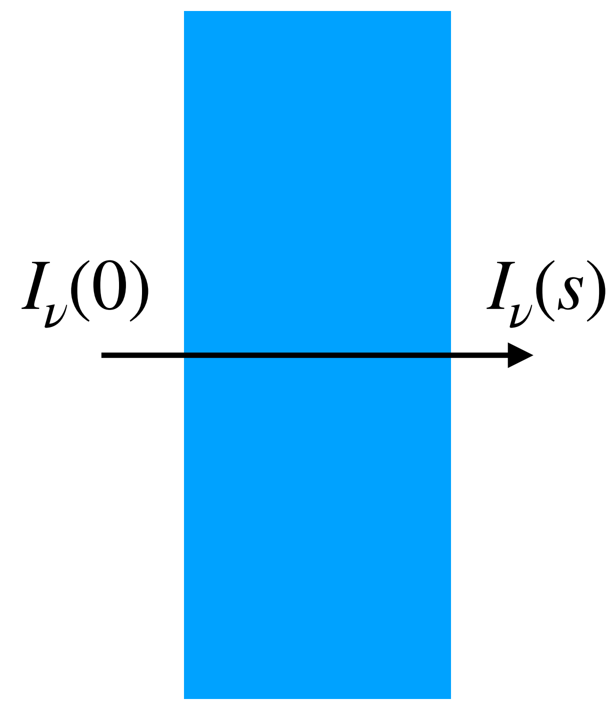


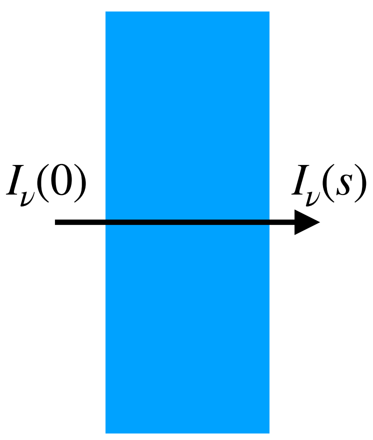
- The brightness decreases along the ray by the exponential of the absorption coefficient integrated along the line of sight.

- ***Optical depth***:

  Imagine radiation traveling into a cloud of absorbing gas, the exponential defines a scale over which radiation is attenuated.

  We define the optical depth τ_ν as:

$$\tau_\nu(s) = \int_0^s \alpha_\nu(s')ds' \quad \text{or} \quad d\tau_\nu = \alpha_\nu ds$$



Transmitted Light

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu}$$

Absorbed Light

$$I_\nu^{\text{abs}}(\tau_\nu) = I_\nu(0)(1 - e^{-\tau_\nu})$$

- A medium is said to be ***optically thick*** at a frequency ν if the optical depth for a typical path through the medium satisfies:

$$\tau_\nu(s) > 1 \qquad I_\nu(\tau_\nu) \to 0 \qquad I_\nu^{\text{abs}}(\tau_\nu) \to I_\nu(0)$$

- The medium is ***optically thin*** if, instead:

$$\tau_\nu(s) < 1 \qquad I_\nu(\tau_\nu) \to I_\nu(0) \qquad I_\nu^{\text{abs}}(\tau_\nu) \to 0$$

An optically thin medium is one which a typical photon of frequency ν can pass through without being (significantly) absorbed.

- If the absorption coefficient is a constant (example: a uniform density gas of ionized hydrogen), then we obtain

$$I_\nu(s) = I_\nu(0)e^{-\alpha_\nu s}$$

specific intensity after distance s

initial intensity at $s = 0$.

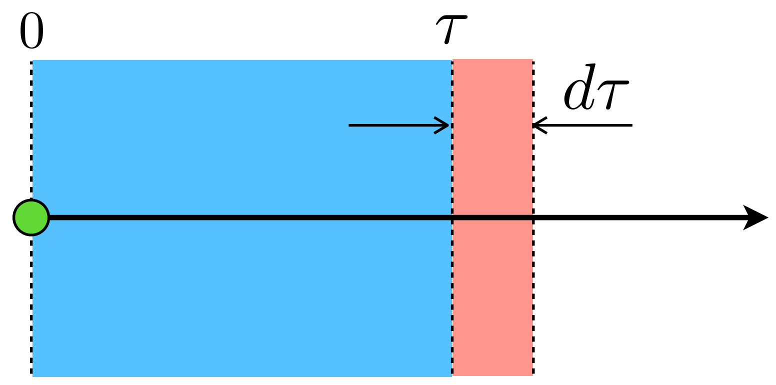
radiation exponentially absorbed with distance

- ***Attenuation***
  - Imagine radiation traveling into a cloud of absorbing gas, exponential defines a scale over which radiation is attenuated.
  - When $\int_0^s \alpha_\nu(s')ds' = 1$, the intensity will be reduced to $1/e$ of its original value.

Mean Free Path

- From the exponential absorption law,

  the **probability that a photon survives from optical depth 0 to τ_ν and is subsequently absorbed between τ_ν and $\tau_\nu + d\tau_\nu$** is



$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu}$

probability

$$\frac{|dI_\nu|}{I_\nu(0)} = \frac{1}{I_\nu(0)}\left|\frac{dI_\nu}{d\tau_\nu}\right| d\tau_\nu = e^{-\tau_\nu} d\tau_\nu \quad \rightarrow \quad P(\tau_\nu) = e^{-\tau_\nu}$$

= **probability density function** for the absorption at an optical depth τ_ν

- The ***mean optical depth*** traveled is thus equal to unity:

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu P(\tau_\nu) d\tau_\nu = \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1$$

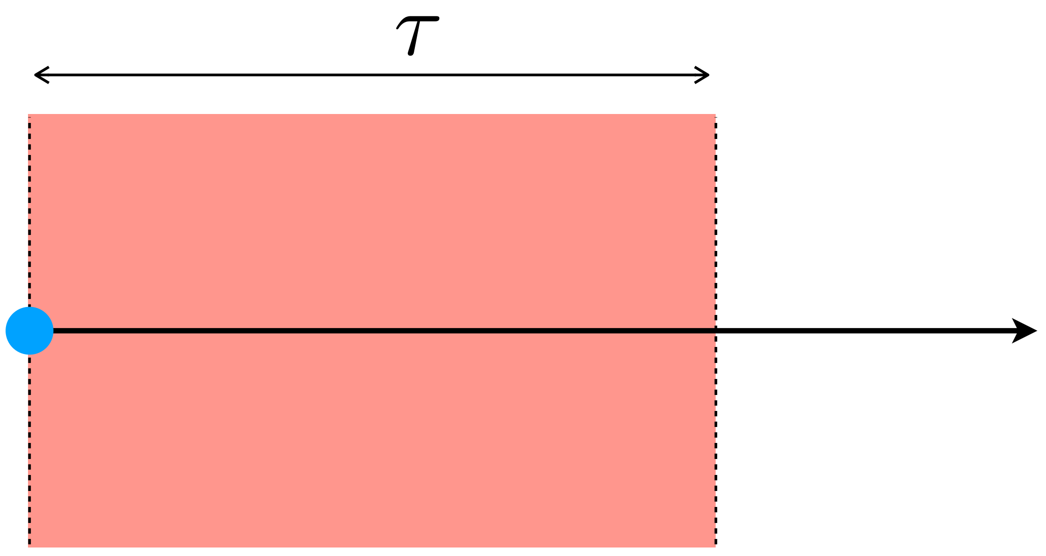
- ***The mean free path is defined as the average distance a photon can travel through an absorbing material until it is absorbed.*** In a homogeneous medium, the mean free path is determined by

$$\langle \tau_\nu \rangle = \alpha_\nu \ell_{\rm mfp} = 1 \quad \rightarrow \quad \boxed{\ell_{\rm mfp} = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}}$$

- A local mean path at a point in an inhomogeneous material can be also defined.

- The probability of a photon being absorbed within an optical depth τ_ν is

$$\int_0^{\tau_\nu} P(\tau'_\nu) d\tau'_\nu = 1 - e^{-\tau_\nu}$$



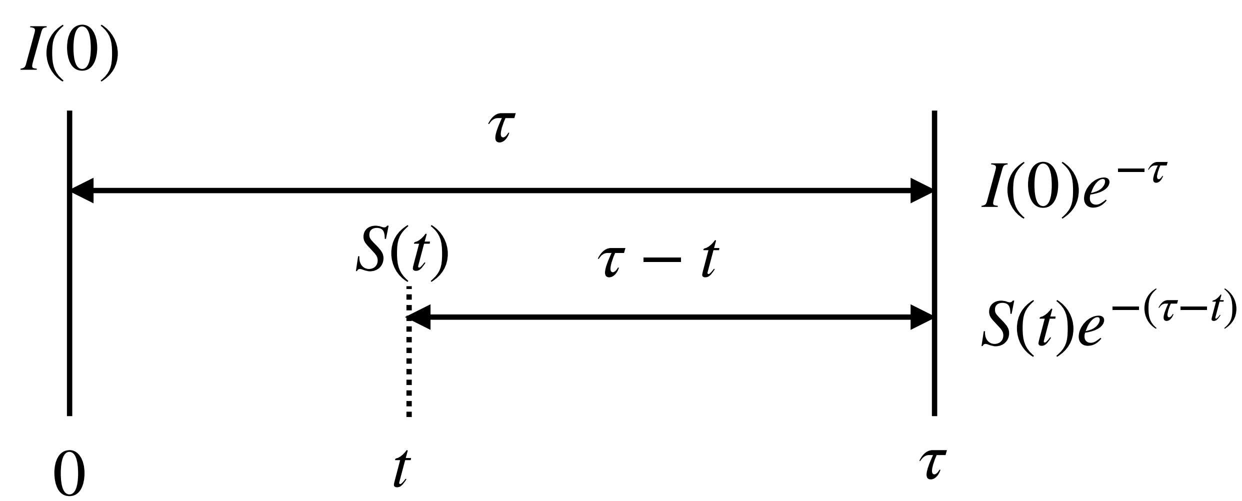
Formal Solution of the RT equation

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

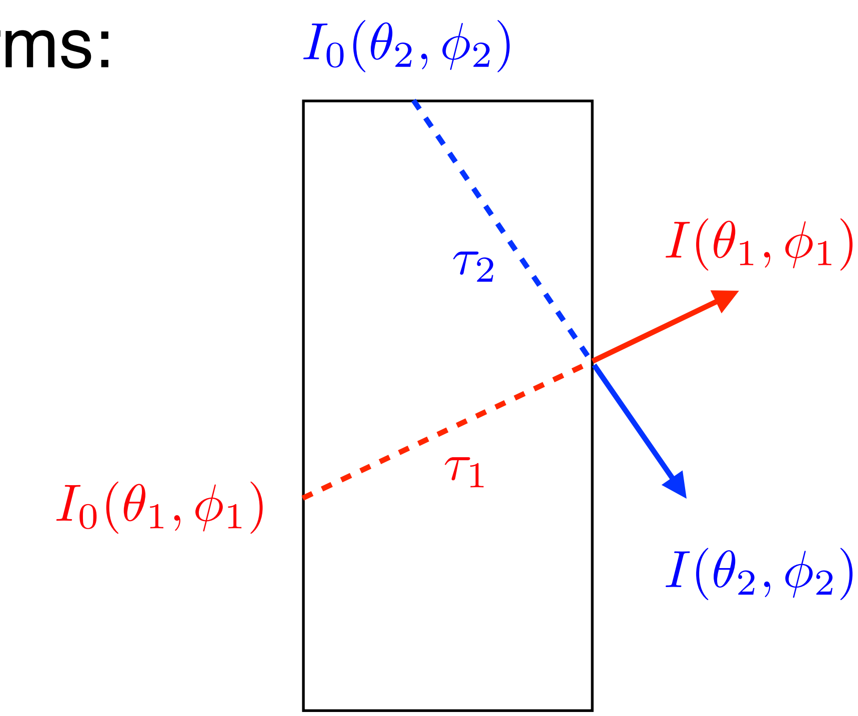
$$\frac{d}{d\tau_\nu}\left(e^{-\tau_\nu} I_\nu\right) = e^{-\tau_\nu} S_\nu$$

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(t_\nu)e^{-(\tau_\nu - t_\nu)}dt_\nu$$



- The solution is easily interpreted as the sum of two terms:
  - the initial intensity diminished by absorption
  - the integrated source diminished by absorption.
- For a constant source function, the solution becomes

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right) \end{aligned}$$



Relaxation

- "Relaxation"

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$I_\nu > S_\nu \quad \rightarrow \quad \frac{dI_\nu}{d\tau_\nu} < 0, \quad \text{then } I_\nu \text{ tends to decrease along the ray}$$

$$I_\nu < S_\nu \quad \rightarrow \quad \frac{dI_\nu}{d\tau_\nu} > 0, \quad \text{then } I_\nu \text{ tends to increase along the ray}$$

- ***The source function is the quantity that the specific intensity tries to approach***, and does approach if given sufficient optical depth.

$$\text{As} \quad \tau_\nu \rightarrow \infty, \quad I_\nu \rightarrow S_\nu$$

Thermal equilibrium

- In general, equilibrium means a state of balance.
  - ***Thermal equilibrium* refers to steady states of temperature, which defines the average energy of material or photons.**
  - In a state of (complete) ***thermodynamic equilibrium (TE)***, no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system. ***In TE, matter and radiation are in equilibrium at the same temperature T.***
  - If the material is (locally) in thermodynamic equilibrium at a well-defined temperature *T*, ***it is said to be in local thermodynamic equilibrium (LTE) even if it is not in equilibrium with the radiation field.***
- ***Note that thermal equilibrium differ from thermodynamic equilibrium.***

The state of LTE

- Macroscopically, LTE is characterized by the following three equilibrium distributions:
  - **Maxwellian velocity distribution** of particles, written here in terms of distribution for the absolute values of velocity,

$$f(v)dv = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) 4\pi v^2 dv$$

    where m is the particle mass and k the Boltzmann constant.

  - **Boltzmann excitation equation**,

$$\frac{n_i}{N_I} = \frac{g_i}{U_I} e^{-E_i/kT} \qquad \text{where } U_I = \sum_i g_i e^{-E_i/kT}$$

    where n_i is the population of level i, g_i is its statistical weight, and E_i is the level energy, measured from the ground state; N_I and U_I are the total number density and the partition function of the ionization state I to which level i belongs, respectively.

  - **Saha ionization equation**,

$$\frac{N_I}{N_{I+1}} = n_e \frac{U_I}{U_{I+1}} \left(\frac{h^2}{2\pi m_e kT}\right)^{3/2} e^{\chi_I/kT}$$

    where χ_I is the ionization potential of ion I.

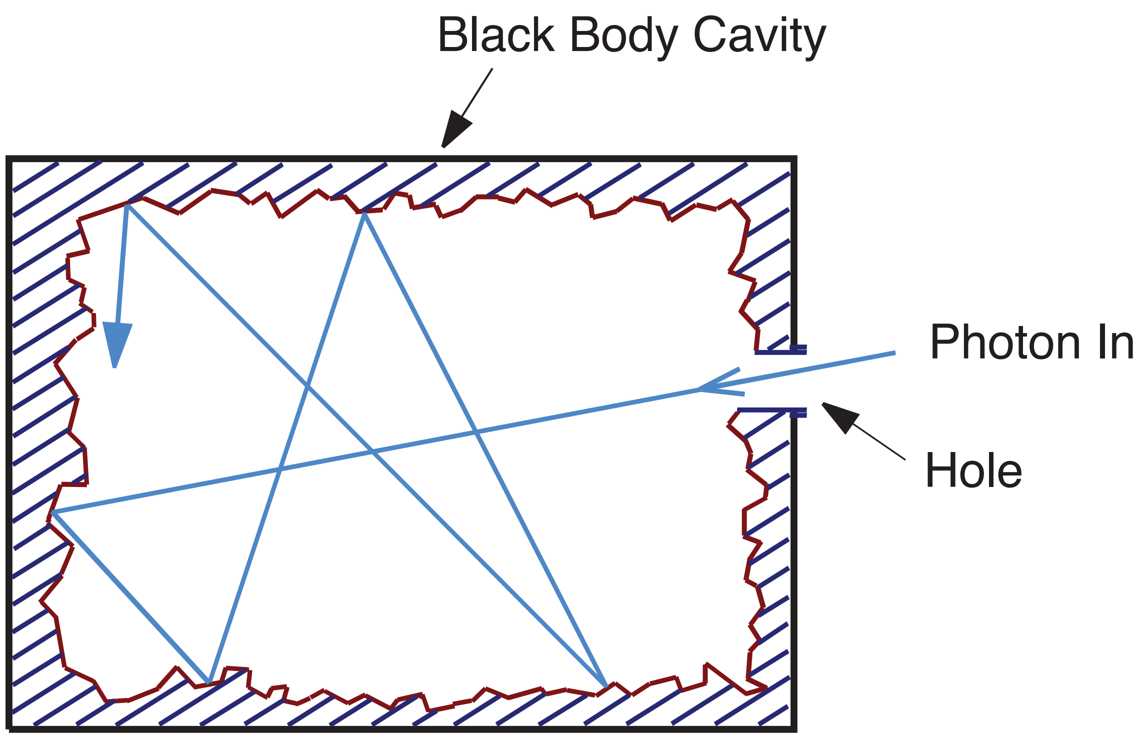
- Microscopically, LTE holds if all atomic processes are in detailed balance, i.e., if the number of processes $A \rightarrow B$ is exactly balanced by the number of inverse processes $B \rightarrow A$.

Blackbody Radiation

- In everyday life, when observing an object, what is detected is the light that it is reflecting.
  - For instance, if when looking at a red object, the reason why it is red is that the object in question is absorbing most colors except red, which is being reflected.
  - In sunlight or light emitted by most household bulbs, there exist all of the colors of visible part of the electromagnetic spectrum.
  - Meanwhile, black objects absorb most of the visible light they receive.
- Greenhouse Effect
  - The average temperature on the Earth’s surface is regulated by the amount of energy it receives from the Sun and the amount irradiated to space.
  - The Earth’s atmosphere is transparent to the visible part of the electromagnetic spectrum. Since the temperature at the Sun’s surface is approximately 5800 K, its spectrum maximum is in the visible region and thus a lot of energy crosses the atmosphere and reaches the Earth ’ s surface.
  - The Earth’s surface has an approximate temperature of 290 K and emits mostly infrared radiation.
  - However, molecules such as H_2O and CO_2 can absorb infrared radiation and thus keep some heat in the terrestrial system. If it wasn’t for the atmosphere, the temperature at our planet ’ s surface would be more than 30 degrees cooler than it is now.
  - Human activity, such as the burning of fossil fuels, has increased the amount of pollutants (mostly CO_2) in our atmosphere.
  - The increase of the abundances of these gases, called greenhouse gases, amplifies the opacity of the atmosphere to infrared radiation, which decreases the amount of energy lost to space.
  - This process leads to a slight increase of the Earth’s temperature and is called the greenhouse effect.

Blackbody

- Think of a container bounded by opaque walls with a very small hole in one wall.
  - A photon entering the hole has only a very small probability of escaping; instead, it is repeatedly reflected from the interior surface until it is ultimately absorbed within the container.
  - Under such conditions, the material and photons continually share their kinetic energies.
  - In perfect thermal equilibrium, ***the average particle kinetic energy will be equal to the average photon energy, and a unique temperature T can be defined.***

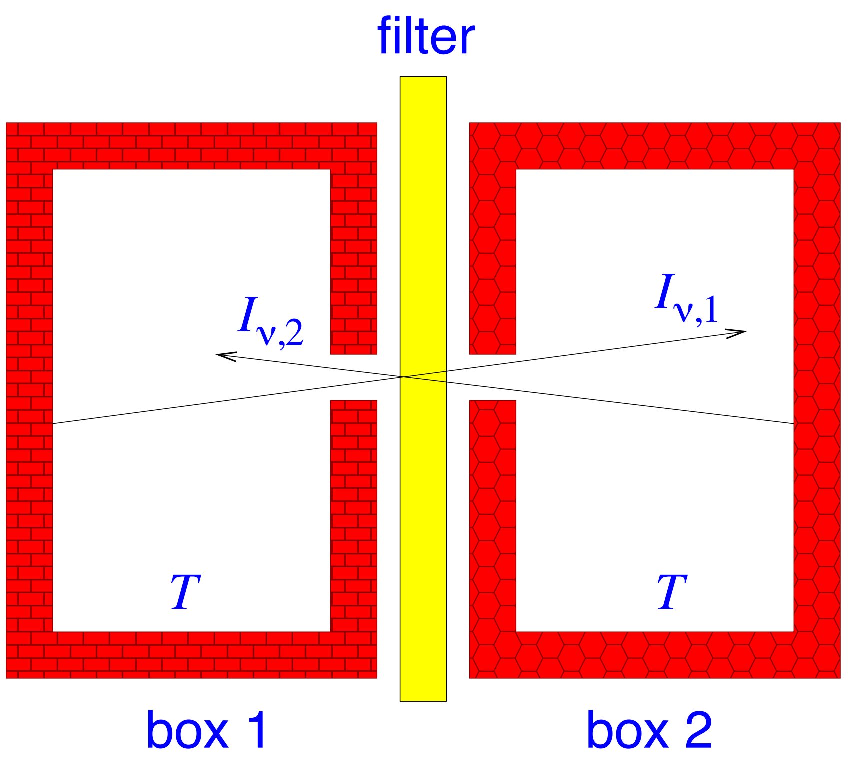


(c) D. F. Gra

- A **blackbody** is an idealized physical body that absorbs all incident radiation regardless of frequency or angle of incidence (i.e., perfect absorber). The above cavity can be regarded to be a blackbody.
- ***A small fraction of the internal radiation leaks out of the hole, and we can measure its spectrum;*** *the leakage is so small that the thermodynamics of the container is essentially maintained.*
- Radiation from a blackbody in thermal equilibrium is called the **blackbody radiation**.

Blackbody radiation is the universal function.

- ***Now, consider another cavity (box 2), also at the same temperature***, but made of different material or shape and connect two cavities with a filter transparent only in the narrow frequency range ν and $\nu + d\nu$.
  - ***In equilibrium at T, radiation should transfer no net energy from one cavity to the other.*** Otherwise, one cavity will cool down and the other heats up; this violates the second law of thermodynamics.
  - Therefore, the intensity or spectrum that passes through the holes should be a universal function of *T* and should be isotropic (and unpolarized).
  - The intensity and spectrum of the radiation emerging from the hole should be independent of the wall material (e.g., wood, copper, or concrete, etc) and any absorbing material that may be inside the cavity.
  - The universal function is called the Planck function $B_\nu(T)$.
  - This is the blackbody radiation.



Kirchhoff's Law in TE

- In (full) thermodynamic equilibrium at temperature T, by definition, we know that

$$\frac{dI_\nu}{ds} = 0 \quad \text{and} \quad I_\nu = B_\nu(T)$$

  - We also note that

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

  - Then, we can obtain ***the Kirchhoff's law for a system in TE***:

$$\boxed{\frac{j_\nu(T)}{\alpha_\nu(T)} = B_\nu(T)}$$

  - This is remarkable because it connects the properties $j_\nu(T)$ and $\alpha_\nu(T)$ of any kind of matter to the single universal spectrum $B_\nu(T)$.

Kirchhoff's Law in LTE

- Recall that Kirchhoff's law was derived for a system in thermodynamic equilibrium.

- ***Kirchhoff's law applies not only in TE but also in LTE***:
  - Recall that $B_\nu(T)$ is independent of the properties of the radiating /absorbing material.
  - In contrast, both $j_\nu(T)$ and $\alpha_\nu(T)$ depend only on the materials in the cavity and on the temperature of that material; they do not depend on the ambient radiation field or its spectrum.
  - Therefore, the Kirchhoff's law should be true even for the case of LTE.
  - ***In LTE, the ratio of the emission coefficient to the absorption coefficient is a function of temperature and frequency only.***
  - This generalized version of Kirchhoff's law is an exceptionally valuable tool for calculating the emission coefficient from the absorption coefficient or vice versa.

Implications of Kirchhoff's Law

- A good absorber is a good emitter, and a poor absorber is a poor emitter. (In other words, a good reflector must be a poor absorber, and thus a poor emitter.)

$$j_\nu = \alpha_\nu B_\nu(T) \quad \rightarrow \quad j_\nu \text{ increases as } \alpha_\nu \text{ increases}$$

  - It is not possible to thermally radiate more energy than a blackbody, at equilibrium.

$$j_\nu < B_\nu(T) \quad \text{because} \quad \alpha_\nu < 1$$

  - The radiative transfer equation in LTE can be rewritten:

$$\boxed{\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T)}$$

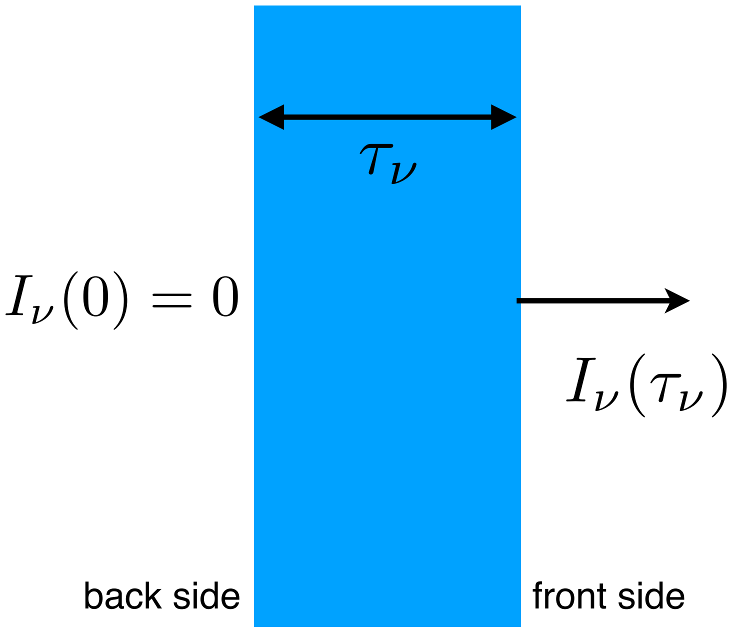
- ***Blackbody radiation vs. Thermal radiation***
  - ***Blackbody radiation*** means $I_\nu = B_\nu(T)$. An object for which the intensity is the Planck function is emitting blackbody radiation.
  - ***Thermal radiation is defined to be radiation emitted by "matter" in LTE***. Thermal radiation means $S_\nu = B_\nu(T)$. An object for which the source function is the Planck function is emitting thermal radiation.
  - ***Thermal radiation becomes blackbody radiation only for optically thick media.***

Blackbody Radiation vs. Thermal Radiation

- To see the difference between thermal and blackbody radiation,
  - Consider a slab of material with optical depth τ_ν that is producing thermal radiation.
  - If no light is falling on the back side of the slab, the intensity that is measured on the front side of the slab is

$$I_\nu(0) = 0,\quad S_\nu = B_\nu \longrightarrow$$

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)$$
$$= B_\nu\left(1 - e^{-\tau_\nu}\right)$$



  - If the slab is optical thick at frequency ν $(\tau_\nu \gg 1)$, then

$$\boxed{I_\nu = B_\nu \quad \text{as } \tau_\nu \to \infty}$$

  - If the slab is optically thin $(\tau_\nu \ll 1)$, then

$$\boxed{I_\nu \approx \tau_\nu B_\nu \ll B_\nu \quad \text{as } \tau_\nu \ll 1}$$

This indicates that the radiation, although it is thermal, will not be blackbody radiation.

Thermal radiation becomes blackbody radiation only for optical thick media.

< Planck Spectrum (Spectrum of blackbody radiation) >

- There is no perfect blackbody.
  - However, the cosmic microwave background comes quite close; stars can sometimes be usefully approximated as blackbodies.
  - By the end of the 19th century, the blackbody spectrum was fairly well known empirically, from laboratory studies. In 1900, Max Planck, using his idea of quantized energies, derived the blackbody spectrum.

See Radiative Transfer (Rybicki & Lightman) for the derivation of the Planck function "Fundamentals of Statistical and Thermal Physics" (Federick Reif) or "Astrophysical Concepts" (Harwit) for more details.

Spectrum of blackbody radiation

- The frequency dependence of blackbody radiation is given by the ***Planck function***:

$$B_\nu d\nu = B_\lambda d\lambda, \quad \nu = c/\lambda$$

$$\boxed{B_\nu(T) = \frac{2h\nu^3/c^2}{\exp\left(h\nu/k_{\mathrm{B}}T\right) - 1} \quad \text{or} \quad B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp\left(hc/\lambda k_{\mathrm{B}}T\right) - 1}}$$

$$h = 6.63 \times 10^{-27} \text{ erg s (Planck's constant)}$$

$$k_{\mathrm{B}} = 1.38 \times 10^{-16} \text{ erg K}^{-1} \text{ (Boltzmann's constant)}$$

- ***Energy density:***

$$u_\nu(T) = \frac{4\pi}{c} B_\nu(T) = \frac{8\pi h\nu^3/c^3}{\exp\left(h\nu/k_{\mathrm{B}}T\right) - 1}$$

Note that the textbook Ryden's "Interstellar and Intergalactic Medium" use the symbol $\varepsilon_\nu(T)$ to denote the energy density.

- Photon occupation number:
  - The photon occupation number is dimensionless, and is simply **the average number of photons per mode per polarization**.

$$n_\gamma(\nu) = \frac{1}{4\pi\rho_s}\frac{u_\nu}{h\nu} = \frac{\langle E\rangle}{h\nu}$$

$$\rho_s = \frac{2\nu^2}{c^3}$$

$$\boxed{n_\gamma(\nu) = \frac{c^2}{2h\nu^3} I_\nu}$$

  - If the radiation field is a blackbody, the photon occupation number is given by

$$\boxed{n_\gamma(\nu; T) = \frac{1}{\exp\left(h\nu/k_\mathrm{B}T\right) - 1}}$$

Bose-Einstein statistics

Stefan-Boltzmann Law

- Emergent flux is proportional to T^4.

$$F = \pi \int_0^\infty B_\nu(T) d\nu = \pi B(T) \longleftarrow B(T) = \int_0^\infty B_\nu(T) d\nu = \frac{ac}{4\pi} T^4 = \frac{\sigma_{\rm SB}}{\pi} T_4$$

$$F = \sigma_{\rm SB} T^4$$

$$\text{Stephan} - \text{Boltzmann constant}: \sigma_{\rm SB} = \frac{2\pi^5 k_{\rm B}^4}{15 c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^2 \text{ s}^{-1} \text{ K}^{-4} \text{ sr}^{-1}$$

- Total energy density ***(another form of the Stefan-Boltzmann law)***

$$u = \frac{4\pi}{c} \int_0^\infty B_\nu(T) d\nu = \frac{4\pi}{c} B(T) \qquad u(T) = \left(\frac{T}{3400 \text{ K}} \right)^4 \text{ erg cm}^{-3}$$

$$u = aT^4$$

$$\text{radiation constant}: a \equiv \frac{4\sigma_{\rm SB}}{c} = 7.57 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$$

- Total photon number density

$$N_{\rm photon} = \int_0^\infty \frac{u_\nu}{h\nu} d\nu \approx 20 T^3 \text{ cm}^{-3}$$

Rayleigh-Jeans Law & Wien Law

Rayleigh-Jeans Law (low-energy limit)

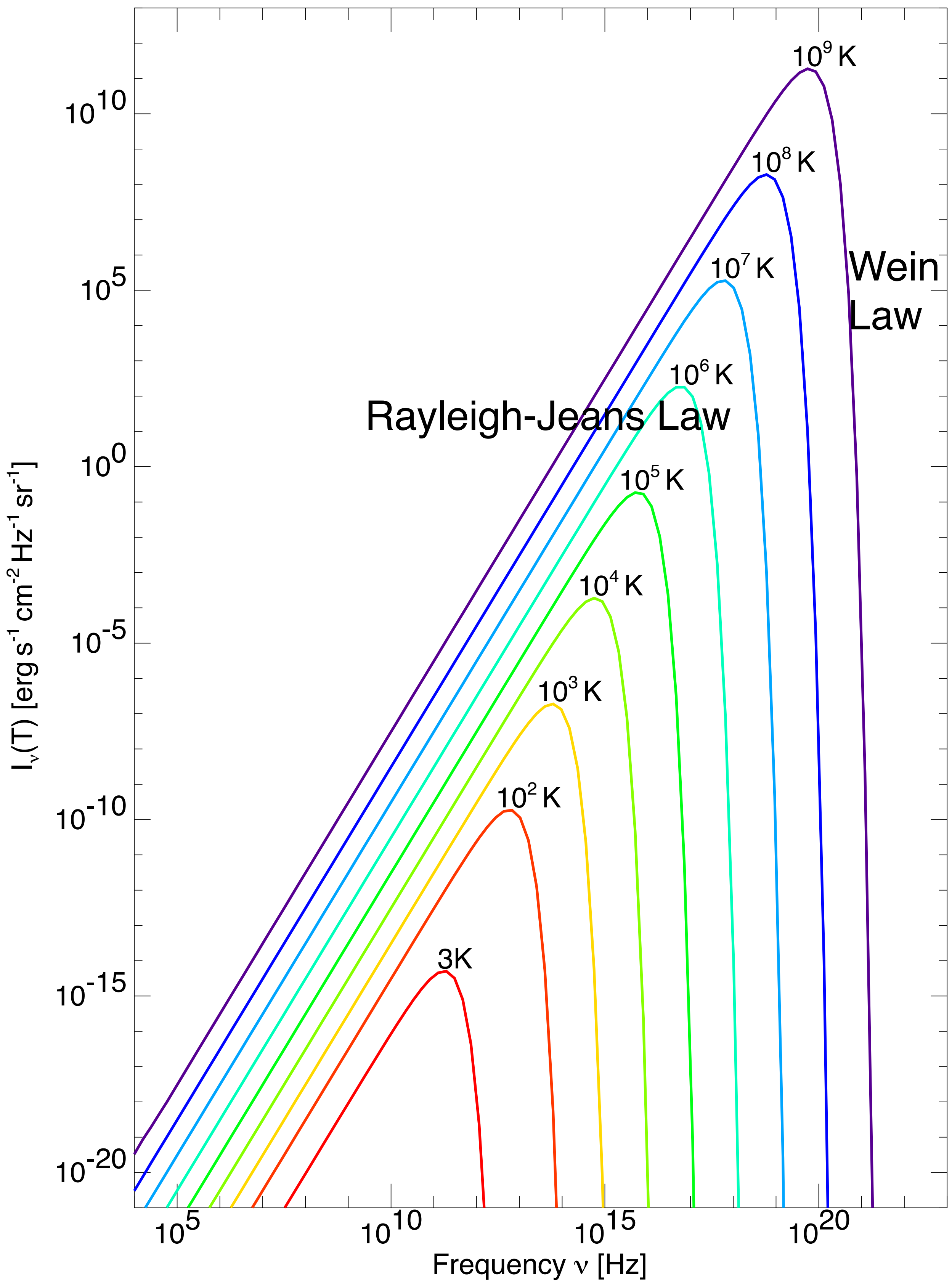
$$h\nu \ll k_{\rm B}T \quad \left(\nu \ll 2\times 10^{10}\,(T/1\,{\rm K})\,{\rm Hz}\right)$$

$$\boxed{I_\nu^{RJ}(T) = \frac{2\nu^2}{c^2}k_{\rm B}T}$$

Wien Law (high-energy limit)

$$h\nu \gg k_{\rm B}T$$

$$\boxed{I_\nu^{W}(T) = \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{k_{\rm B}T}\right)}$$



Characteristic Temperatures

- ***Brightness Temperature:***
  - The brightness temperature is defined to be the temperature such that a blackbody at that temperature would have specific intensity:

$$I_\nu = B_\nu(T_b) \quad \rightarrow \quad T_b(\nu) = \frac{h\nu/k_\mathrm{B}}{\ln\left[1 + 2h\nu^3/(c^2 I_\nu)\right]}$$

- ***Antenna Temperature:***
  - Radio astronomers are used to working at very low frequencies. They define the antenna temperature as being the brightness temperature in the Rayleigh-Jeans tail.

$$I_\nu = \frac{2\nu^2}{c^2} k_\mathrm{B} T_b \quad \rightarrow \quad T_A \equiv \frac{c^2}{2k_\mathrm{B}\nu^2} I_\nu$$

  - Radiative transfer equation in the RJ limit:
    - If the matter has its energy levels populated according to an excitation temperature $T_\mathrm{exc} \gg h\nu/k_\mathrm{B}$, then the source function is given by $S_\nu(T_\mathrm{exc}) = \left(2\nu^2/c^2\right) k_\mathrm{B} T_\mathrm{exc}$ from the generalized Kirchhoff's law.
    - Then, RT equation becomes $\boxed{\dfrac{dT_A}{d\tau_\nu} = -T_A + T_\mathrm{exc} \quad \text{if } h\nu \ll k_\mathrm{B} T_\mathrm{exc}}$ $\qquad \dfrac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$

$$T_A = T_A(0)e^{-\tau_\nu} + T_\mathrm{exc}(1 - e^{-\tau_\nu}) \quad \text{if } T_\mathrm{exc} \text{ is constant.}$$

- ***Color Temperature:***
  - By fitting the spectrum to a blackbody curve without regarding to vertical scale (absolute intensity scale), a color temperature T_c is obtained.
  - The color temperature correctly gives the temperature of a blackbody source of unknown absolute scale.
- ***Effective Temperature:***
  - The effective temperature of a source is obtained by equating the actual flux F to the flux of a blackbody at temperature $T_{\rm eff}$.

$$F = \int\int I_\nu \cos\theta d\nu d\Omega = \sigma T_{\rm eff}^4$$

Stefan-Boltzmann law

- ***Excitation Temperature:***
  - The excitation temperature of level u relative to level ℓ is defined by

Boltzmann distribution

$$\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} \exp\left(-\frac{E_{u\ell}}{k_{\rm B} T_{\rm exc}}\right) \quad \rightarrow \quad T_{\rm exc} \equiv \frac{E_{u\ell}/k_{\rm B}}{\ln\left(\frac{n_\ell/g_\ell}{n_u/g_u}\right)} \quad (E_{u\ell} \equiv E_u - E_\ell)$$

  - Radio astronomers studying the 21 cm line sometimes use the term “spin temperature” $T_{\rm spin}$ for excitation temperature.

Source Function for Multiple Processes

- When multiple processes contribute to local emission and extinction, the RT equation and the total source function are given by

$$\frac{dI_\nu}{ds} = -\left(\sum \alpha_\nu\right) I_\nu + \sum j_\nu \quad \Rightarrow \quad d\tau_\nu = \sum \alpha_\nu ds \ \text{ and } \ S_\nu^{\text{tot}} = \frac{\sum j_\nu}{\sum \alpha_\nu}$$

- For example, the source function at a frequency ν within a spectral line is

$$S_\nu^{\text{tot}} = \frac{j_\nu^c + j_\nu^l}{\alpha_\nu^c + \alpha_\nu^l} = \frac{S_\nu^c + \eta_\nu S_\nu^l}{1 + \eta_\nu}$$

Here, $\eta_\nu \equiv \dfrac{\alpha_\nu^l}{\alpha_\nu^c}$ = line-to-continuum extinction ratio

$S_\nu^c \equiv \dfrac{j_\nu^c}{\alpha_\nu^c}$ = continuum source function

$S_\nu^l \equiv \dfrac{j_\nu^l}{\alpha_\nu^l}$ = line source function

[Note]

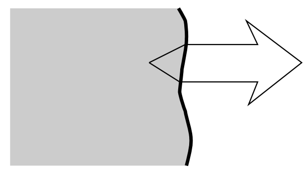
$I_\nu,\ F_\nu,\ j_\nu,\ S_\nu$ $\Rightarrow$ The subscript ν implies measurement per frequency interval.

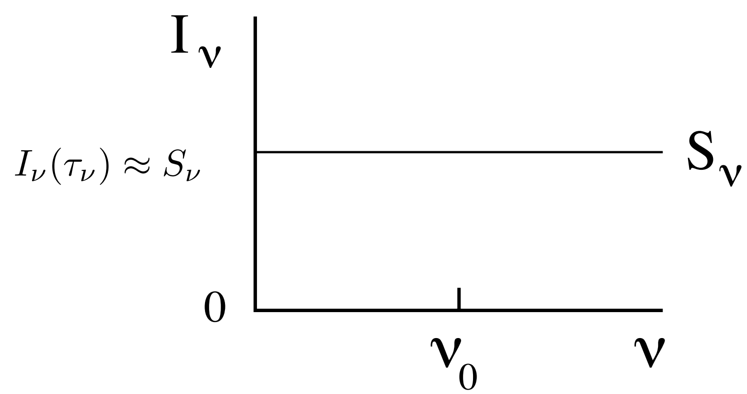
$\alpha_\nu,\ \sigma_\nu,\ \kappa_\nu$ $\Rightarrow$ The subscript ν simply expresses frequency dependence.

Spectral Lines from a Homogeneous Object with a constant Source Function

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)$$

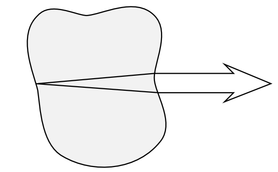
$\tau_\nu(D) >> 1$

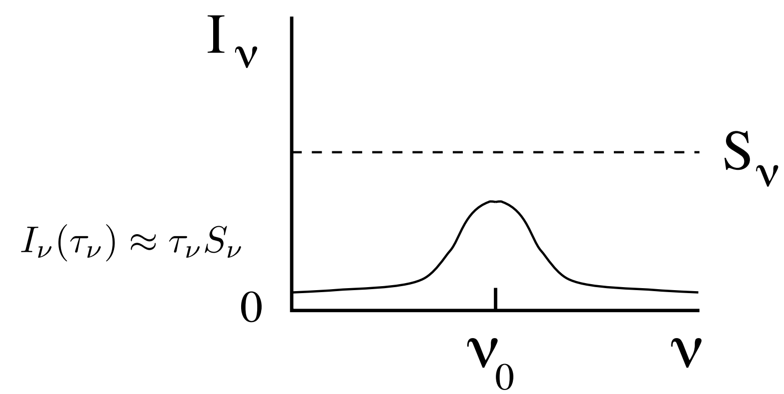




No lines emerge when the object is optically thick.

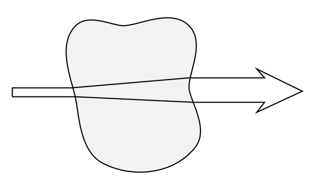
$\tau_\nu(D) < 1$
$I_\nu(0) = 0$

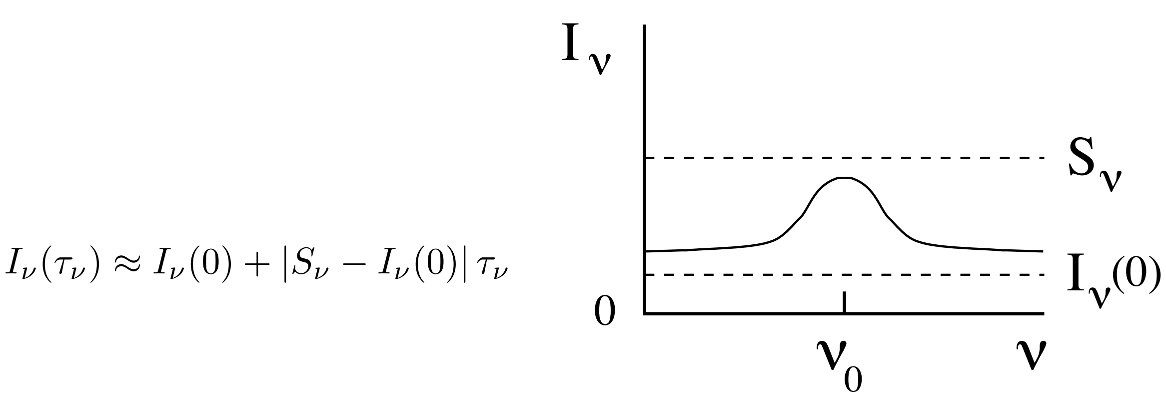




When the object is optically thin and not back-lit ($I_\nu(0) = 0$), emission lines emerge.

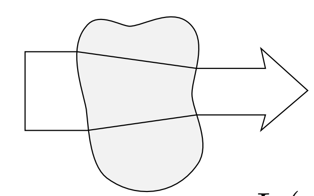
$\tau_\nu(D) < 1$
$I_\nu(0) < S_\nu$

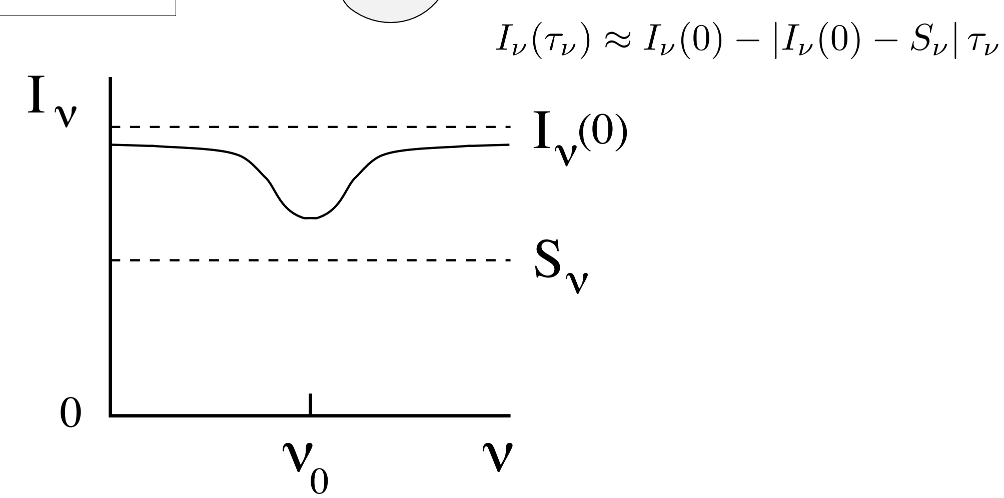




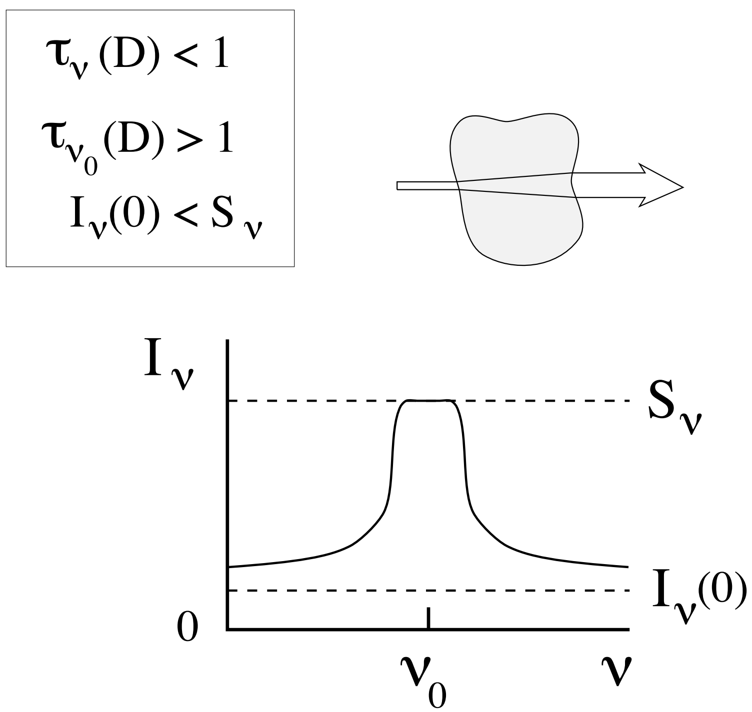
When the object is optically thin and illuminated with $I_\nu(0) < S_\nu$, emission lines emerge.

$\tau_\nu(D) < 1$
$I_\nu(0) > S_\nu$



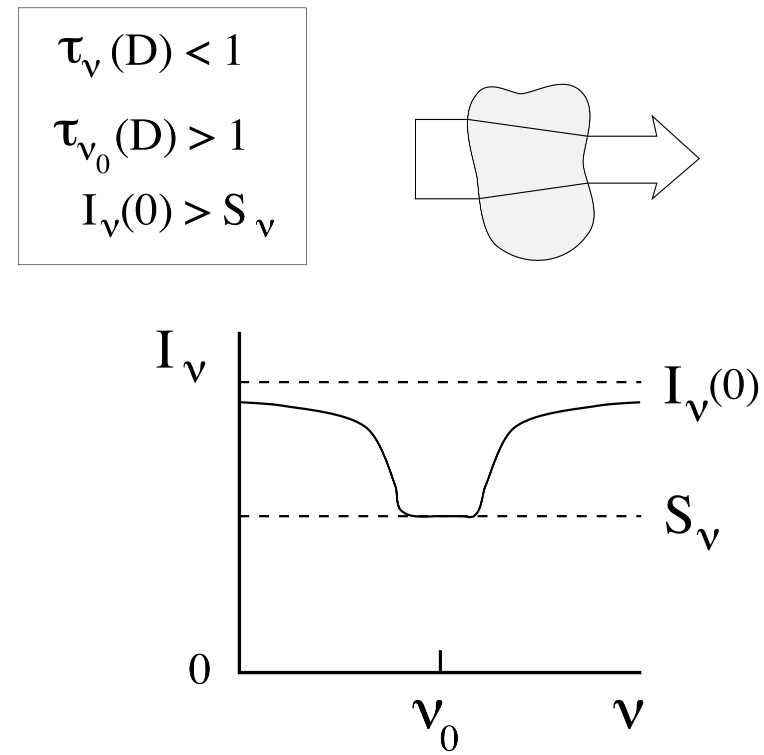


Absorption lines emerge only when the object is optically thin and $I_\nu(0) > S_\nu$.



The emergent lines saturate to $I_\nu(0) \approx S_\nu$ when the object is optically thick at line center (ν_0), but optically thin at $\nu \neq \nu_0$.

$$\begin{aligned} I_\nu(\tau_\nu) &\approx S_\nu && \text{at } \nu \approx \nu_0 \\ &\approx I_\nu(0) + |S_\nu - I_\nu(0)|\,\tau_\nu && \text{at } \nu \neq \nu_0 \end{aligned}$$

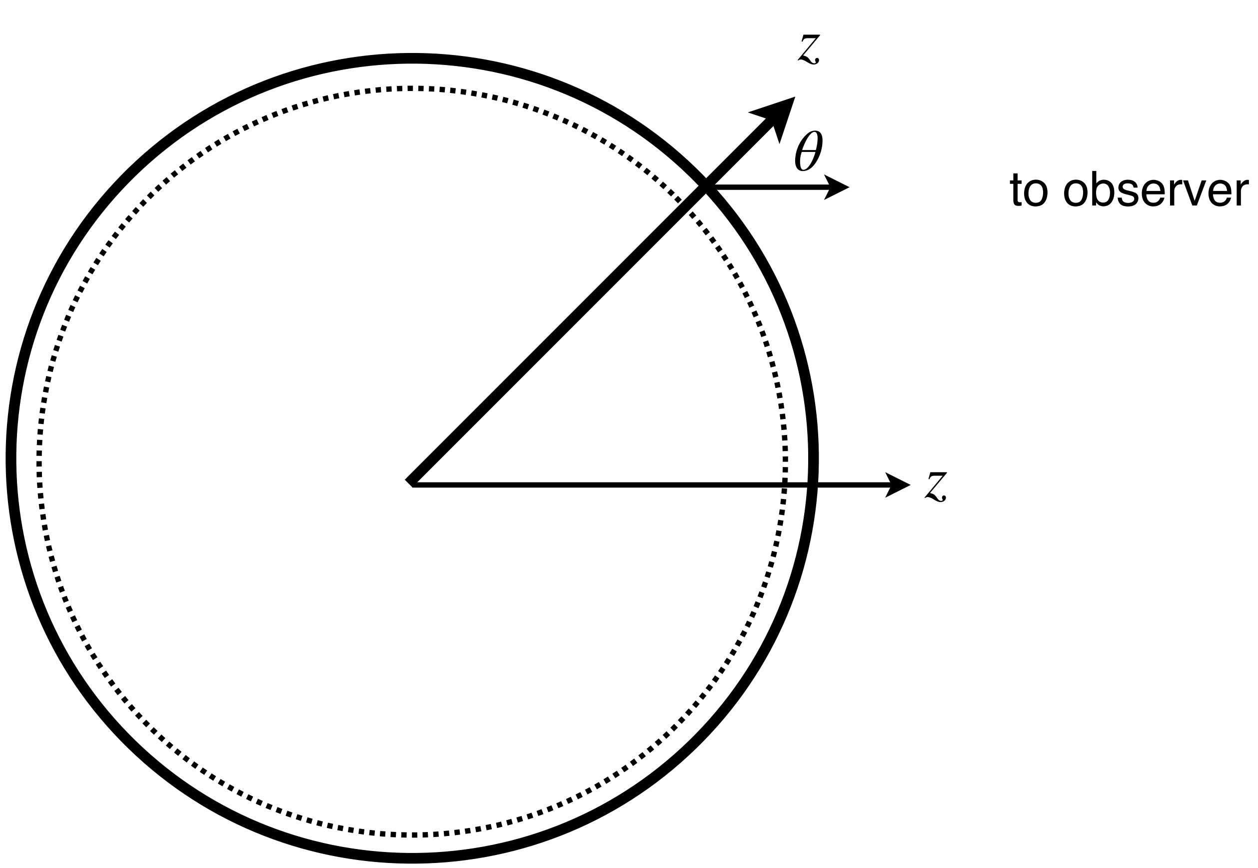


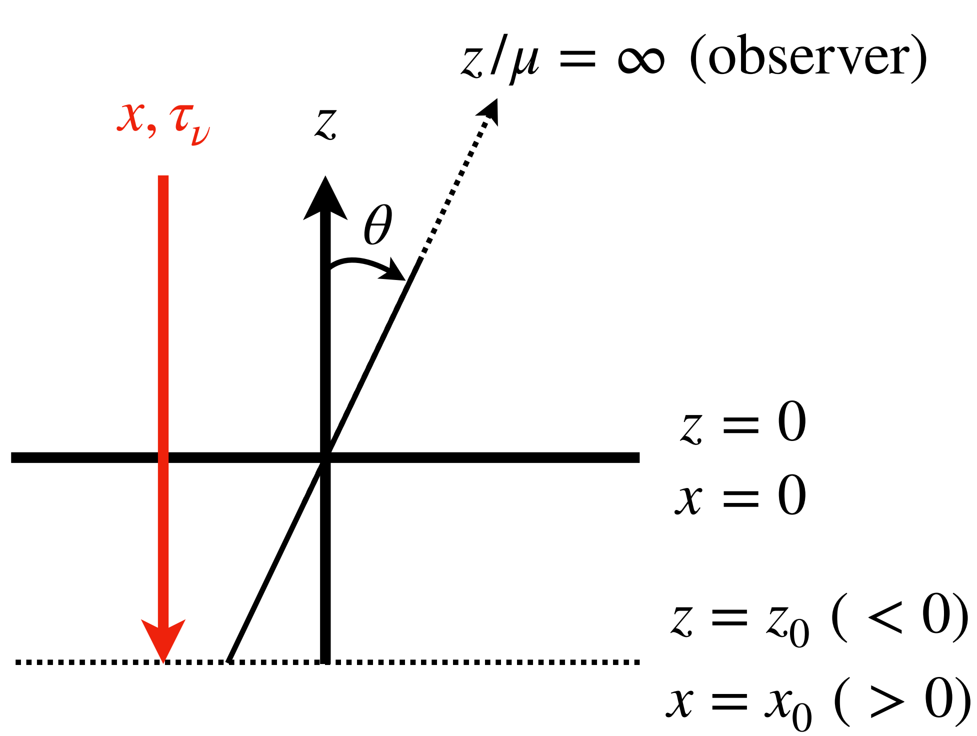
The emergent lines saturate to $I_\nu(0) \approx S_\nu$ when the object is optically thick at line center (ν_0), but optically thin at $\nu \neq \nu_0$.

$$\begin{aligned} I_\nu(\tau_\nu) &\approx S_\nu && \text{at } \nu \approx \nu_0 \\ &\approx I_\nu(0) - |I_\nu(0) - S_\nu|\,\tau_\nu && \text{at } \nu \neq \nu_0 \end{aligned}$$

Radiative Transfer in Stellar Atmospheres

- In the context of stellar atmospheres, one often adopts axial symmetry with the z-axis radially outward along the axis of symmetry (perpendicular to the surface of a spherical star consisting of horizontally homogeneous shells). *The optical depth is measured along the viewing direction.*
- The geometrical thickness of the photosphere in most stars is very small compared to the stellar radius.
  - The solar photosphere is ~700 km thick, or ~0.1% of the solar radius.
  - Under these conditions, ***a plane-parallel approximation can be made.***





- The viewing angle μ is defined by $\mu = \cos\theta$ where θ is the angle between the line of sight and the z-axis.
- In stellar atmospheres, the total optical thickness, measured from the stellar center, along the line of sight is far too large to be any interest. Therefore, we define the optical depth τ_ν measured along the radial line of sight from the observer's location at $x = -\infty$ $(z = \infty)$. Then, the optical depth for a geometrical location with $x = x_0$ $(z = z_0)$ is given by

$$\tau_\nu(x_0) = \int_{-\infty}^{x_0} \alpha_\nu dx \quad \Rightarrow \quad \int_0^{x_0} \alpha_\nu dx \;\; \text{as } \alpha_\nu = 0 \text{ for } x < 0$$

- For a frequency within a spectral line, the total optical depth is given by

$$d\tau_\nu^{\text{total}} = \left(\alpha_\nu^c + \alpha_\nu^l\right) dx = (1 + \eta_\nu)\, d\tau_\nu^c$$

- The radiative transfer equation in plane-parallel geometry is then given by

$$ds = -\frac{dx}{\mu} \quad \Rightarrow \quad \boxed{\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu}$$

- Formal solution:

$$\frac{dI}{d(\tau/\mu)} = I - S$$

$$\frac{d}{d(\tau/\mu)}\left(e^{-\tau/\mu} I\right) = -e^{-\tau/\mu} S$$

$$I(\tau) = I(\tau^b) e^{-(\tau^b - \tau)/\mu} - \int_{\tau^b}^{\tau} S(t) e^{-(t-\tau)} dt/\mu$$

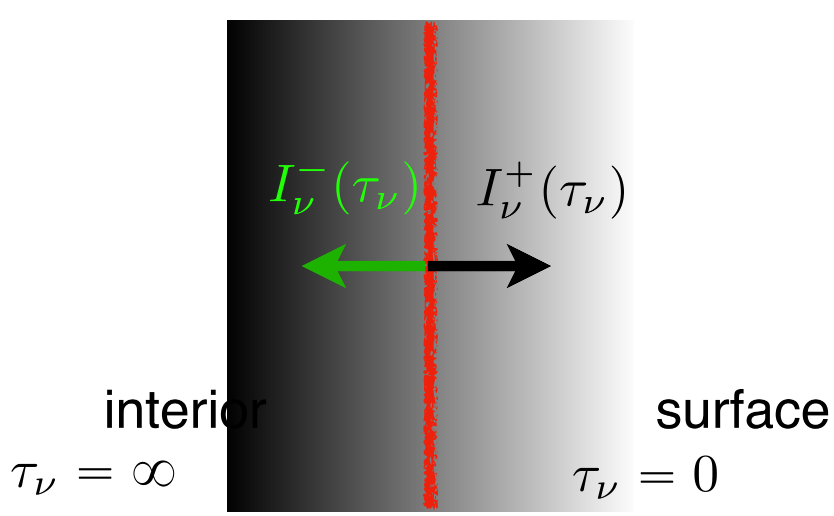
  - The inward directed intensity ($\mu < 0$) and outward directed intensity ($\mu > 0$) are respectively given by

$$\boxed{\begin{aligned} I_\nu^-(\tau_\nu, \mu) &= -\int_0^{\tau_\nu} S_\nu(t_\nu) e^{-(t_\nu - \tau_\nu)/\mu} dt_\nu/\mu \quad \text{for } \mu < 0 \\ I_\nu^+(\tau_\nu, \mu) &= +\int_{\tau_\nu}^{\infty} S_\nu(t_\nu) e^{-(t_\nu - \tau_\nu)/\mu} dt_\nu/\mu \quad \text{for } \mu > 0 \end{aligned}}$$

boundary conditions:

$$I^-(\tau^b, \mu) = 0 \quad \text{at } \tau^b = 0$$

$$e^{-\tau^b} = 0 \quad \text{at } \tau^b = \infty$$



Eddington-Barbier Approximation

- Suppose that the source function can be approximated as a polynomial in optical depth:

$$S_\nu(\tau_\nu) = \sum_{n=0}^{\infty} a_n \tau_\nu^n = a_0 + a_1\tau_\nu + a_1\tau_\nu^2 + \cdots$$

  - Then, the emergent intensity at the stellar surface ($\tau_\nu = 0,\ \mu > 0$) is given by

$$\begin{aligned} I_\nu^+(0,\mu) &= \int_0^\infty S_\nu(t_\nu) e^{-t_\nu/\mu} dt_\nu/\mu \\ &= \sum_{n=0}^{\infty} n! a_n \mu^n = a_0 + a_1\mu + 2a_2\mu^2 + \cdots \end{aligned}$$

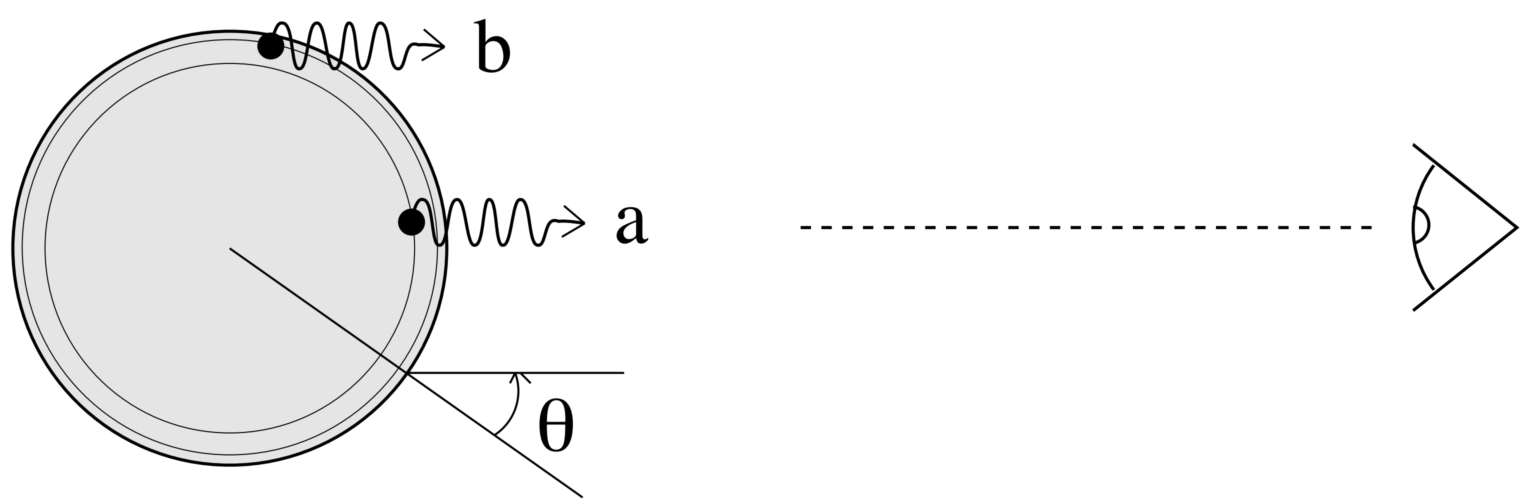
  - Truncation of both expansions after the first two terms produces the Eddington-Barbier approximation:

$$\boxed{I_\nu^+(0,\mu) \approx S_\nu(\tau_\nu = \mu)}$$

  - Likewise for the emergent flux:

$$\begin{aligned} \mathcal{F}_\nu^+(0) &= 2\pi \int_0^1 \mu I_\nu^+(0,\mu) d\mu = \pi \sum_{n=0}^{\infty} \frac{2n!}{n+2} a_n \\ &= \pi \left(a_0 + \frac{2}{3} a_1 + a_2 + \cdots \right) \end{aligned}$$

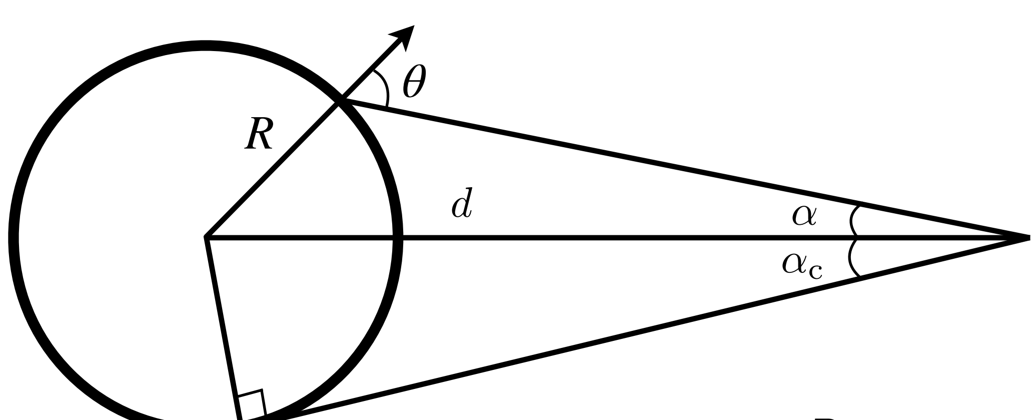
$$\longrightarrow \quad \boxed{\mathcal{F}_\nu^+(0) \approx \pi S_\nu(\tau_\nu = 2/3)}$$



- Flux as measured by an observer:

① From the conservation of energy: $\boxed{F_\nu = \frac{4\pi R^2}{4\pi d^2}\mathcal{F}_\nu^+ \approx \frac{R^2}{d^2}\pi S_\nu(\tau_\nu = 2/3)}$ Here, R = stellar radius, and d = distance to the star.

② From the definition of flux: $F_\nu = 2\pi \int_0^{\alpha_c} I_\nu(0,\mu)\cos\alpha\sin\alpha d\alpha$



$$\sin\alpha_c = \frac{R}{d}$$

sine law: $\frac{\sin\alpha}{R} = \frac{\sin(\pi-\theta)}{d} = \frac{\sin\theta}{d}$

$$\Rightarrow \cos^2\alpha = \left(1 - \frac{R^2}{d^2}\right) + \frac{R^2}{d^2}\cos^2\theta$$

$$\cos\alpha\sin\alpha\, d\alpha = \frac{R^2}{d^2}\cos\theta\sin\theta\, d\theta$$

$$\therefore F_\nu = 2\pi\frac{R^2}{d^2}\int_0^{\pi/2} I_\nu(0,\mu)\cos\theta\sin\theta d\theta = \frac{R^2}{d^2}\mathcal{F}_\nu^+$$