Astrophysics

Lecture 05
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Radiation from Moving Charges 2

Dipole Approximation (the radiation from many particles)

- Consider many particles with positions $\mathbf{r}_i$, velocities $\mathbf{u}_i$, and charges q_i $(i = 1,2,3,\cdots,N)$. The radiation field at large distances can be found by adding together the $\mathbf{E}_{\rm rad}$ from each particle.
- However, the radiation field equations refer to conditions at retarded time, and the retarded times will differ for each particle. Therefore, we must keep track of the phase relations between the particles.

  There are situations in which it is possible to ignore this difficulty:

  Let L = typical size of the system
  τ = typical time scale for variations within the system

  If $\tau \gg L/c$ (light-travel-time), the differences in retarded time across the source are negligible.

  Note that τ can represent the time scale over which significant changes in the radiation field, and this in turn determines typical characteristic frequency of the emitted radiation. ***This condition is equivalent to the condition for the characteristic frequency (or characteristic wavelength) :***

$$\nu \approx \frac{1}{\tau} \ll \frac{c}{L} \quad \text{or} \quad \lambda = \frac{c}{\nu} \gg L$$

  In other words, ***the differences in retarded times can be ignored when the system size is much smaller than the characteristic wavelength.***

- We may also characterize τ as the time a particle takes to change its motion substantially.

  Let ℓ be a characteristic scale of the particle's orbit and u be a typical velocity, then $\tau \sim \ell/u$. The above condition $\tau \gg L/c$ then imply $u/c = \ell/(\tau c) \ll \ell/L$

  But since $\ell < L$, **the condition for dipole approximation is simply equivalent to the nonrelativistic condition:**

$$\boxed{u \ll c}$$

  With the above conditions met we can use the nonrelativistic form of the radiation fields:

$$\boxed{\mathbf{E}_{\text{rad}} = \sum_i \frac{q_i}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}_i)}{R_i}}$$

- Let R_0 be the distance from some point in the system to the field point. Then, $R_i = R_0 + \ell_i \approx R_0$ as $R_0 \gg \ell_i$. Finally, we have

$$\mathbf{E}_{\text{rad}} \approx \frac{1}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \sum_i q_i \dot{\mathbf{u}}_i)}{R_0} \quad \rightarrow \quad \boxed{\mathbf{E}_{\text{rad}} = \frac{\mathbf{n} \times \left(\mathbf{n} \times \ddot{\mathbf{d}}\right)}{c^2 R_0}}$$

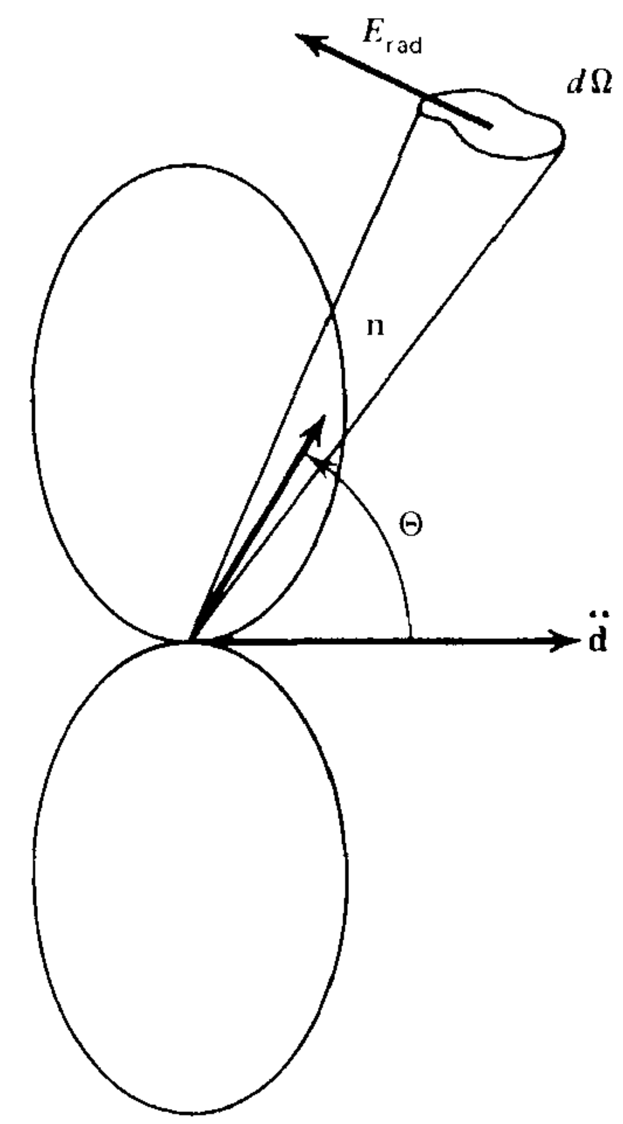
  where the electric dipole moment is defined as $\boxed{\mathbf{d} = \sum_i q_i \mathbf{r}_i}$

  Note that <u>the right-hand side of the above equations must still be evaluated at a retarded time</u>, but using any point within the region, say, the position used to define R_0.

- As before, for a single particle, we find the generalized formulas for the radiation patter and the total power, which are called the dipole approximation:

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\sin^2\Theta, \quad P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}$$

Note that the instantaneous polarization of **E** lines in the plane of $\ddot{\mathbf{d}}$ and **n**.



- **Spectrum of radiation in the dipole approximation**:

For simplicity we assume that **d** always lies in a single direction. Then, the magnitude of the electric field is given by

$$E(t) = \ddot{d}(t)\frac{\sin\Theta}{c^2 R_0}$$

where $d(t)$ is the magnitude of the dipole moment.

Fourier transform of $d(t)$ is defined as $d(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\bar{d}(\omega)d\omega$

Then, $\ddot{d}(t) = -\int_{-\infty}^{\infty} \omega^2 e^{-i\omega t}\bar{d}(\omega)d\omega$

$$\bar{E}(\omega) = -\frac{1}{c^2 R_0}\omega^2\bar{d}(\omega)\sin\Theta$$

- The energy per unit solid angle per frequency range in the dipole approximation is given by

$$\begin{aligned} \frac{dW}{d\omega d\Omega} &= R_0^2 \frac{dW}{d\omega dA} \\ \frac{dW}{d\omega dA} &= c \left| \bar{E}(\omega) \right|^2 \end{aligned} \longrightarrow \frac{dW}{d\omega d\Omega} = \frac{\omega^4}{c^3} \left| \bar{d}(\omega) \right|^2 \sin^2 \Theta$$

The total energy per frequency range is

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} \left| \bar{d}(\omega) \right|^2$$

- The above formulas describe an interesting property of dipole radiation, namely, that the spectrum of the emitted radiation is related directly to the frequencies of oscillation of the dipole moment. However, this property is not true for particles with relativistic velocities.

- It is also worthwhile to note the dependence of $\omega^4 \propto \lambda^{-4}$ in the power spectrum.

A general Multipole Expansion*

- The above treatment was obtained only qualitatively. We would like to be more explicit.
- Recall that the vector potential is

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{c}\int d^3\mathbf{r}'\int dt'\,\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

- Consider a Fourier analysis of the sources and fields:

$$\mathbf{j}_\omega(\mathbf{r})=\int \mathbf{j}(\mathbf{r},t)e^{i\omega t}dt$$
$$\mathbf{A}_\omega(\mathbf{r})=\int \mathbf{A}(\mathbf{r},t)t^{i\omega t}dt$$

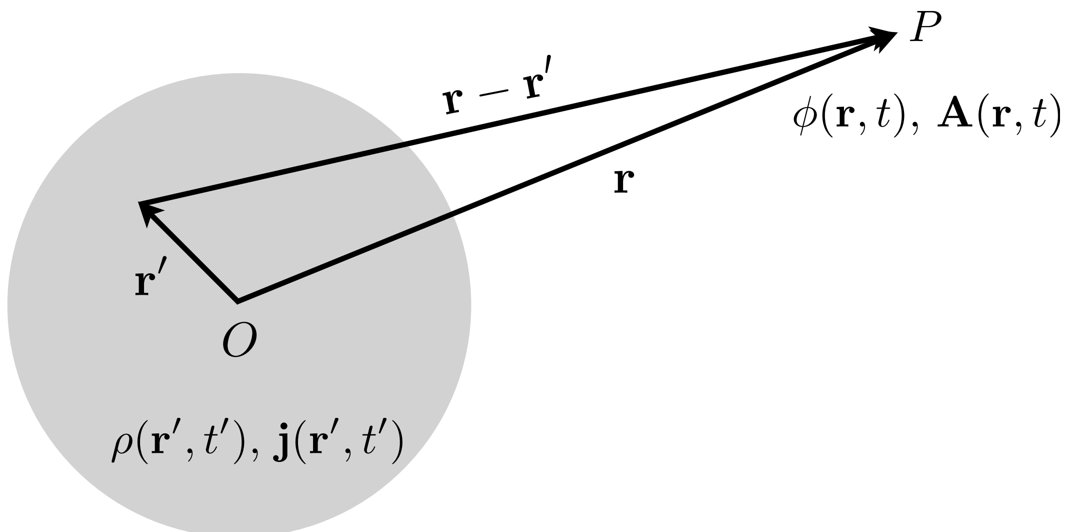
Then, the vector potential becomes

$$\begin{aligned}\mathbf{A}_\omega(\mathbf{r})&=\frac{1}{c}\int d^3\mathbf{r}'\int dt'\int dt\,\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}e^{i\omega t}\delta\left(t'-t+|\mathbf{r}-\mathbf{r}'|/c\right)\\&=\frac{1}{c}\int d^3\mathbf{r}'\int dt'\,\frac{\mathbf{j}(\mathbf{r},t')}{|\mathbf{r}-\mathbf{r}'|}e^{i\omega t'}e^{i\omega|\mathbf{r}-\mathbf{r}'|/c}\\&=\frac{1}{c}\int d^3\mathbf{r}'\,\frac{\mathbf{j}_\omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}\qquad\leftarrow\quad k\equiv\omega/c\end{aligned}$$

Note that this exponential term is caused by the retardation.

This equation now relate single Fourier components of source **j** and potential **A**.

- Let's choose an origin of coordinates inside the source of size L. Then, at field points such that $r \gg L$, we will expand the potential in a power series of kr'.



$$
\begin{aligned}
|\mathbf{r}-\mathbf{r}'| &= \left[(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')\right]^{1/2} = \left[r^2 - 2\,(\mathbf{r}\cdot\mathbf{r}') + r'^2\right]^{1/2} \\
&= r\left[1 - \frac{2\,(\mathbf{r}\cdot\mathbf{r}')}{r^2} + \frac{r'^2}{r^2}\right]^{1/2} \\
&\approx r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \cdots\right) && \leftarrow\ (1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots \\
&= r - \mathbf{n}\cdot\mathbf{r}' + \cdots && \leftarrow\ \text{Here,}\ \ \mathbf{n} \equiv \frac{\mathbf{r}}{r} \quad (\mathbf{n}\ \text{points toward the field point}\ \mathbf{r})
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r}\left[1 - \frac{2\,(\mathbf{r}\cdot\mathbf{r}')}{r^2} + \frac{r'^2}{r^2}\right]^{-1/2} \approx \frac{1}{r}\left(1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \cdots\right) \\
&= \frac{1}{r} + \frac{\mathbf{n}\cdot\mathbf{r}'}{r^2} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}_\omega(\mathbf{r}) &= \frac{1}{c}\int \frac{\mathbf{j}_\omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{ik|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \\
&\approx \frac{1}{c}\int \mathbf{j}_\omega(\mathbf{r}')\frac{1}{r}\left(1+\frac{\mathbf{n}\cdot\mathbf{r}'}{r}\right) e^{ikr} e^{-ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}' \\
&= \frac{e^{ikr}}{cr}\left[\int \mathbf{j}_\omega(\mathbf{r}') e^{-ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}' + \frac{1}{r}\int (\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}_\omega(\mathbf{r}') e^{-ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}'\right]
\end{aligned}
$$

(1) The factor exp(ikr) outside the integral expresses **the effect of retardation from the source as a whole**.

(2) The factor exp($-ik\mathbf{n}\cdot\mathbf{r}'$) inside the integral expresses **the relative retardation of each element** of the source.

In our slow motion approximation, $kL = 2\pi L/\lambda \ll 1$, the first and second integrals can be approximated, respectively, to be

$$
\begin{aligned}
&\int \mathbf{j}_\omega(\mathbf{r}') e^{ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}' \approx \int \mathbf{j}_\omega(\mathbf{r}')\left[1 - ik\mathbf{n}\cdot\mathbf{r}' + \cdots\right] d^3\mathbf{r}' \\
&\int (\mathbf{n}\cdot\mathbf{r}')\mathbf{j}_\omega(\mathbf{r}') e^{ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}' \approx \int (\mathbf{n}\cdot\mathbf{r}')\mathbf{j}_\omega(\mathbf{r}')\left[1 + \cdots\right] d^3\mathbf{r}'
\end{aligned}
\quad \leftarrow \quad e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}
$$

Then, the vector potential becomes

$$
\mathbf{A}_\omega(\mathbf{r}) \approx \frac{e^{ikr}}{cr}\left[\int \mathbf{j}_\omega(\mathbf{r}') d^3\mathbf{r}' + \left(\frac{1}{r} - ik\right)\int (\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}_\omega(\mathbf{r}') d^3\mathbf{r}' + \mathcal{O}\left((\mathbf{n}\cdot\mathbf{r}')^2\right)\right]
$$

- The "electric" dipole approximation results from taking just the first term in the above equation:

$$\mathbf{A}_\omega(\mathbf{r})|_{\text{dipole}} \approx \frac{e^{ikr}}{cr} \int \mathbf{j}_\omega(\mathbf{r}') d^3\mathbf{r}'$$

- The second term give the "electric" quadrupole and "magnetic" dipole terms.

$$\mathbf{A}_\omega(\mathbf{r}) \approx \frac{e^{ikr}}{cr} \left(\frac{1}{r} - ik\right) \int (\mathbf{n}\cdot\mathbf{r}')\, \mathbf{j}_\omega(\mathbf{r}') d^3\mathbf{r}'$$

The term inside the integral can be expressed in terms of a symmetric and asymmetric terms for $\mathbf{r}'$ and $\mathbf{j}$.

$$\begin{aligned}(\mathbf{n}\cdot\mathbf{r}')\mathbf{j} &= \frac{1}{2}\left[(\mathbf{n}\cdot\mathbf{r})'\mathbf{j} + (\mathbf{n}\cdot\mathbf{j})\mathbf{r}'\right] + \frac{1}{2}\left[(\mathbf{n}\cdot\mathbf{r})'\mathbf{j} - (\mathbf{n}\cdot\mathbf{j})\mathbf{r}'\right] \\ &= \frac{1}{2}\left[(\mathbf{n}\cdot\mathbf{r})'\mathbf{j} + (\mathbf{n}\cdot\mathbf{j})\mathbf{r}'\right] + \frac{1}{2}(\mathbf{r}'\times\mathbf{j})\times\mathbf{n}\end{aligned}$$

The first and second terms correspond to the electric quadrupole and magnetic dipole terms, respectively.

- Lamor's formula is obtained by assuming $kr \gg 1$, or in other words, by taking the far zone approximation in addition to the dipole approximation.

Thomson Scattering ((free) Electron Scattering)

- Recall the dipole formula $$\frac{dP}{d\Omega} = \frac{dW}{dt d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\sin^2\Theta, \quad P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}$$
- Let us consider the **process in which a free charged particle (electron) radiates in response to an incident electromagnetic wave**.
- In non-relativistic case, we may neglect magnetic force.

  magnetic/electric force ratio in Lorentz force: $F_B/F_E \sim (v/c)B/E = v/c \ll 1$
- Consider a monochromatic wave with frequency ω_0 and linearly polarized in direction $\hat{\boldsymbol{\epsilon}}$:

$$\mathbf{E} = \hat{\boldsymbol{\epsilon}} E_0 \sin\omega_0 t$$

Thus the force on a particle with the charge *e* is

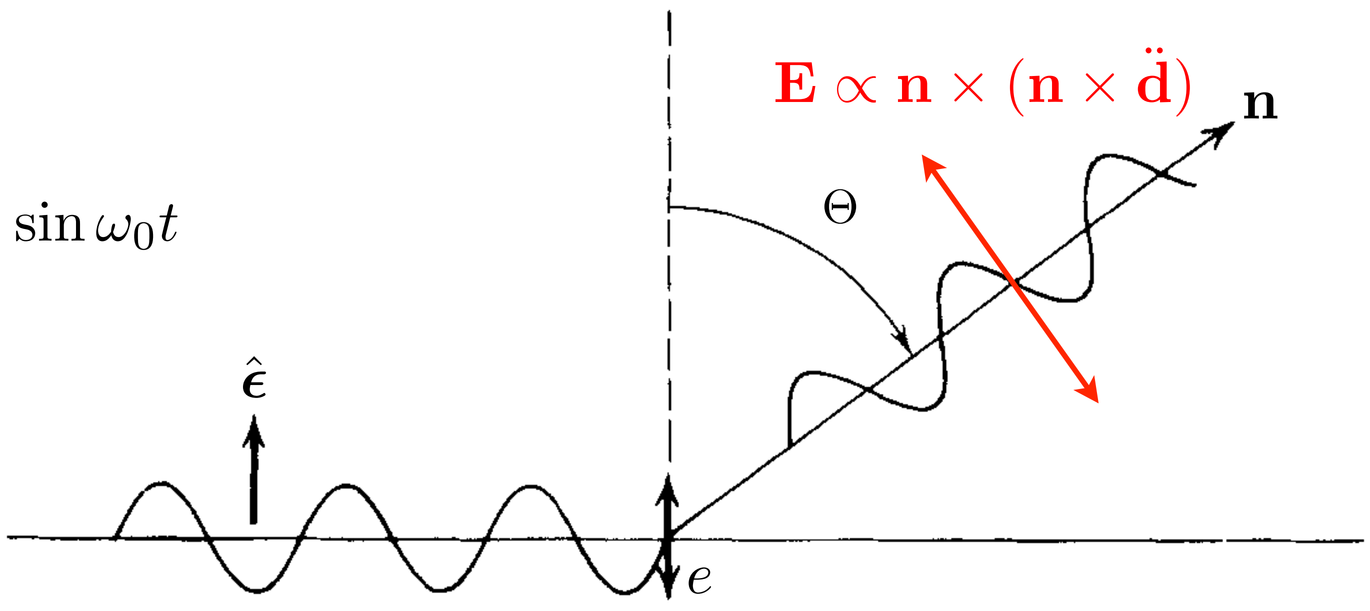
$$\mathbf{F} = e\mathbf{E} = \hat{\boldsymbol{\epsilon}} e E_0 \sin\omega_0 t$$

the acceleration of the electron is

$$\ddot{\mathbf{r}} = \hat{\boldsymbol{\epsilon}}\frac{eE_0}{m}\sin\omega_0 t, \quad \ddot{\mathbf{d}} = e\ddot{\mathbf{r}} = \hat{\boldsymbol{\epsilon}}\frac{e^2E_0}{m}\sin\omega_0 t$$

the dipole moment is

$$\mathbf{d} = -\hat{\boldsymbol{\epsilon}}\left(\frac{e^2E_0}{m\omega_0^2}\right)\sin\omega_0 t$$



- We obtain the time-averaged power per solid angle $\left(\langle \sin^2 \omega_0 t \rangle = 1/2\right)$:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{< \ddot{\mathbf{d}}^2 >}{4\pi c^3} \sin^2 \Theta = \frac{e^4 E_0^2}{8\pi m^2 c^3} \sin^2 \Theta, \quad \langle P \rangle = \frac{e^4 E_0^2}{3m^2 c^3}$$

Note that the time-averaged incident flux is

$$\langle S \rangle = \frac{c}{8\pi} E_0^2$$

The **differential cross section**, $\dfrac{d\sigma}{d\Omega}$, for linearly polarized radiation is obtained by

$$\frac{d\sigma}{d\Omega} \equiv \left\langle \frac{dP}{d\Omega} \right\rangle / \langle S \rangle , \quad \boxed{\therefore \frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \sin^2 \Theta = r_0^2 \sin^2 \Theta, \quad r_0 \equiv \frac{e^2}{mc^2}}$$

where the quantity r_0 gives a measure of the "size" of the point charge. (Note electrostatic potential energy $e\phi = e^2/r_0$).

For an electron, the classical electron radius has a value $r_0 = 2.82 \times 10^{-13}$ cm.

The total cross section is found by integrating over solid angle.

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi r_0^2 \int_{-1}^{1} (1 - \mu^2) d\mu = \frac{8\pi}{3} r_0^2$$

For an electron, the scattering process is then called Thomson scattering or electron scattering, and the **Thomson cross section** is

$$\boxed{\sigma_{\mathrm{T}} = \frac{8\pi}{3} r_0^2 = 6.652 \times 10^{-25} \ \mathrm{cm}^2}$$

- Note:

  The total and differential cross sections are frequency independent.

  The scattered radiation is linearly polarized in the plane of the incident polarization vector $\hat{\epsilon}$ and the direction of scattering $\mathbf{n}$.

  $\sigma \propto 1/m^2$: electron scattering is larger than ions by a factor of $(m_{\rm p}/m_e)^2 = (1836)^2 \approx 3.4 \times 10^6$.

  We have implicitly assumed that electron recoil is negligible. This is only valid for nonrelativistic energies. For higher energies, the (quantum-mechanical) Klein-Nishina cross section has to be used.

- What is **the cross section for scattering of unpolarized radiation**?

  An unpolarized beam can be regarded as the independent superposition of two linear-polarized beams with perpendicular axes.

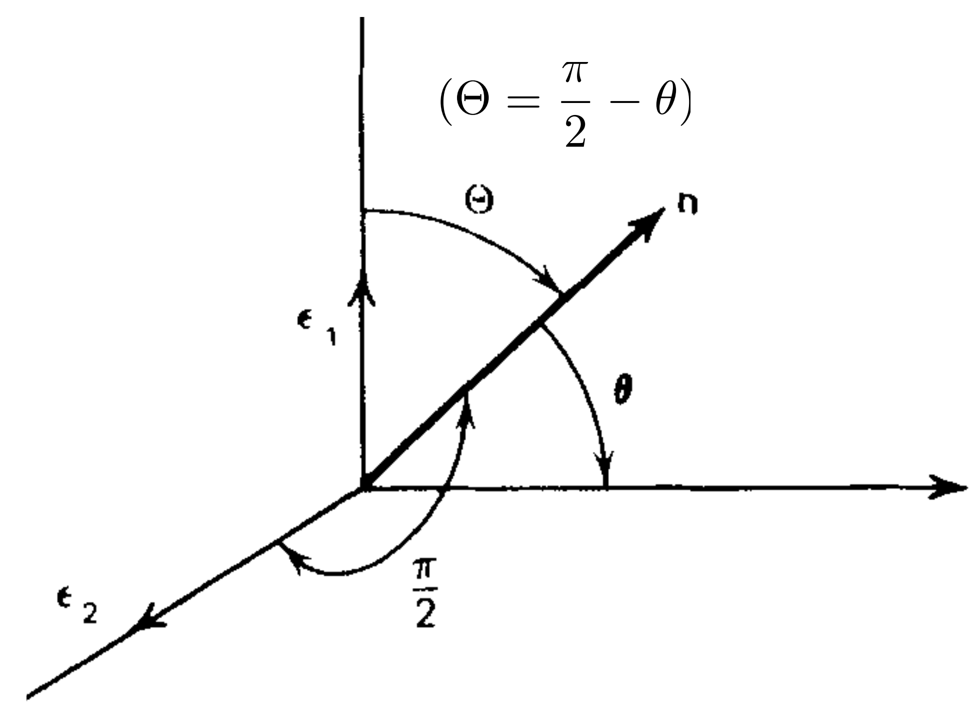
  Let us assume that $\mathbf{n}$ = direction of scattered radiation

  $\mathbf{k}$ = direction of incident radiation

  Choose

  the first electric field along $\hat{\epsilon}_1$, which is in the $\mathbf{n} - \mathbf{k}$ plane

  the second one along $\hat{\epsilon}_2$ orthogonal to this plane and to $\mathbf{n}$



- Let Θ = angle between $\boldsymbol{\epsilon}_1$ and $\mathbf{n}$, and note that angle between $\boldsymbol{\epsilon}_2$ and $\mathbf{n}$ = $\pi/2$.

  $\theta = \pi/2 - \Theta$ = angle between the scattered wave and incident wave

  Then, the differential cross section for unpolarized radiation

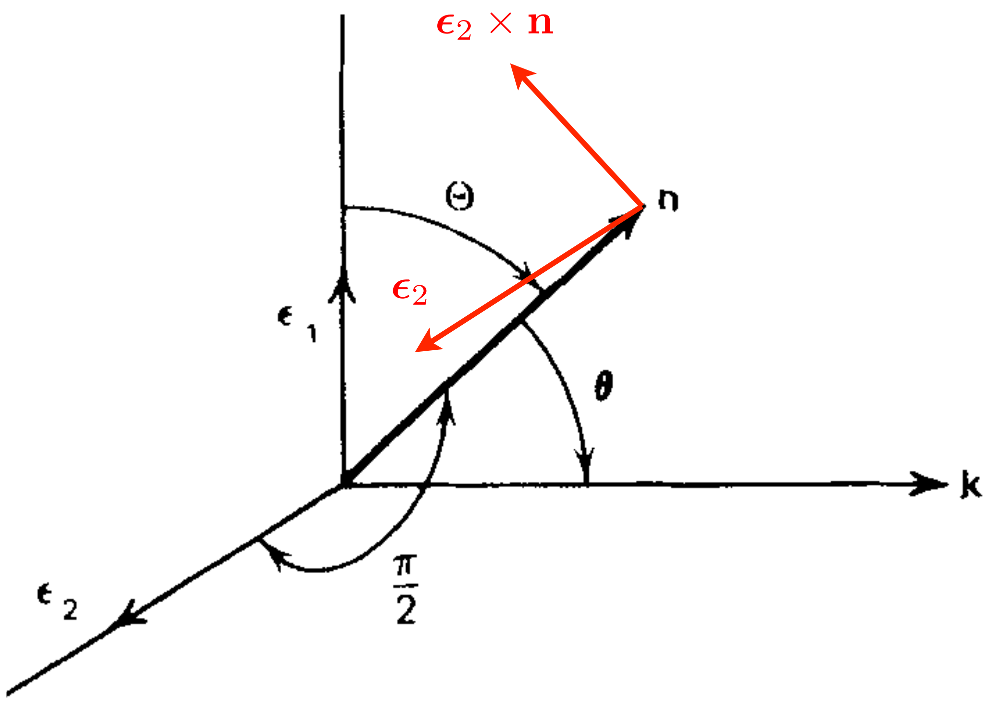
  is the average of the cross sections for scattering of two electric fields.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2}\left[\left(\frac{d\sigma}{d\Omega}\right)_{\epsilon_2} + \left(\frac{d\sigma}{d\Omega}\right)_{\epsilon_1}\right]$$

$$= \frac{1}{2}\left[\left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\text{pol}} + \left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\text{pol}}\right]$$

$$= \frac{1}{2} r_0^2 \left(1 + \sin^2\Theta\right)$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2} r_0^2 \left(1 + \cos^2\theta\right)}$$



This depends only on the angle between the incident and scattered directions, as it should for unpolarized radiation.

Total cross section: $$\sigma_{\text{unpol}} = \int \left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} d\Omega = \pi r_0^2 \int_{-1}^{1} (1 + \mu^2) d\mu$$

$$= \frac{8\pi}{3} r_0^2$$

$$= \sigma_{\text{pol}}$$

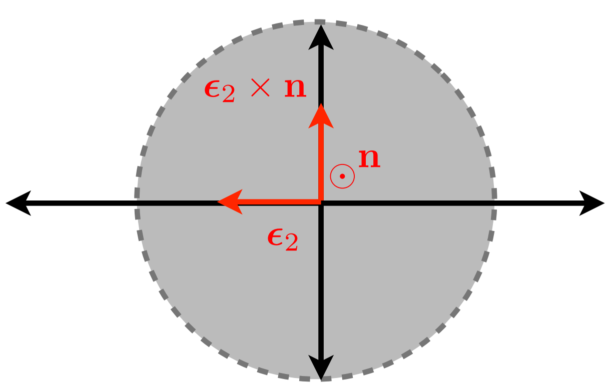
Properties of Thomson Scattering

- Forward-backward symmetry: differential cross section is symmetric under $\theta \to -\theta$.
- Total cross section of unpolarized incident radiation = total cross section for polarized incident radiation. This is because the electron at rest has no preferred direction defined.
- **Scattering creates polarization**

  The scattered intensity is proportional to $1+\cos^2\theta$, of which 1 arises from the incident electric field along $\boldsymbol{\epsilon}_2$ and $\cos^2\theta$ from the incident electric field along $\boldsymbol{\epsilon}_1$.

  "$\cos^2\theta$" of the polarization along $\boldsymbol{\epsilon}_2$ will be cancelled out by the independent polarization along $\boldsymbol{\epsilon}_2 \times \mathbf{n}$.

  Therefore, the degree of polarization of the scattered wave:



$$\Pi = \frac{1-\cos^2\theta}{1+\cos^2\theta}$$

  Electron scattering of a completely unpolarized incident wave produces a scattered wave with some degree of polarization.

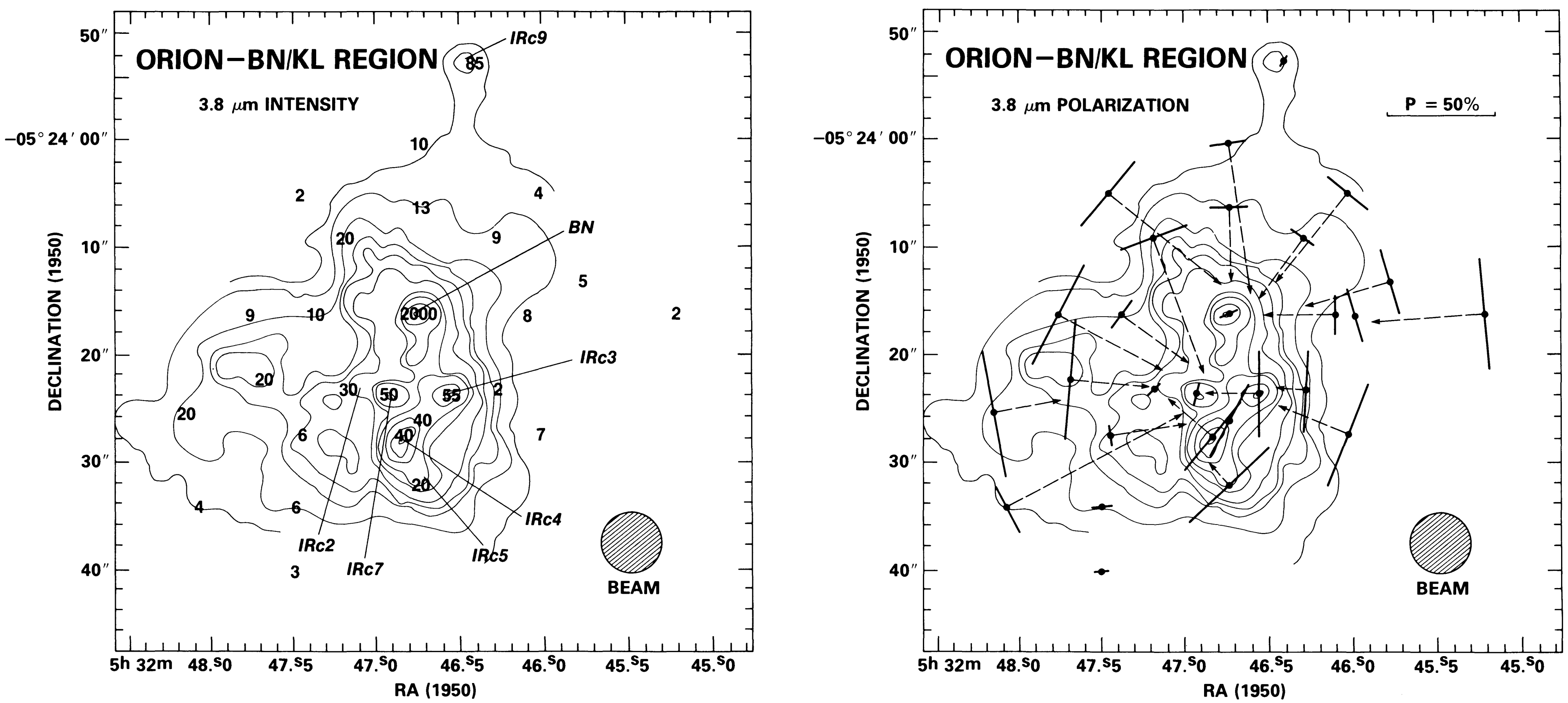
  <u>No net polarization along the incident direction</u> $(\theta = 0)$, since, by symmetry, all directions are equivalent.

  <u>100% polarization perpendicular to the incident direction</u> $(\theta = \pi/2)$, since the electron's motion is confined to a plane normal to the incident direction.

Astrophysical Applications of Polarization by Scattering

- Detection of a concentric pattern of polarization vectors in an extended region indicates that the light comes via scattering from a central point source.

Werner et al. (1983, ApJL, 265, L13)



- Left map shows the IR intensity map at 3.8 um of the Becklin-Neugebauer/Kleinmann-Low region of Orion. It is not easy to identify which bright spots correspond to locations of possible protostars.
- However, the polarization map singles out only two positions of intrinsic luminosity: IRc2 (now known to be an intense protostellar wind) and BN (suspected to be a relatively high-mass star)
- All the other bright spots (IRc3 through 7) correspond to IR reflection nebulae.

Harmonic Oscillator: classical model to the motion of an electron in an atom

- **<u>Lorentz Oscillator Model</u> to describe the interaction between atoms and electric fields**: The electron (with small mass) is bound to the nucleus of the atom (with a much larger mass) by a force that behaves according to Hooke's Law (a spring-like force). An applied electric field would then interact with the charge of the electron, causing "stretching" or "compression" of the spring.
- **The electron's equation of motion**:

$$\boxed{m\ddot{\mathbf{x}} = -k\mathbf{x} + \mathbf{F}_{\mathrm{ext}} + \mathbf{F}_{\mathrm{rad}}}$$

$$k = m\omega_0^2 \ , \ \text{where } k = \text{spring constant}$$

$$\omega_0 = \text{natural (fundamental or resonant) freqency}$$

$\mathbf{F}_{\mathrm{ext}} =$ external force, driving force, or external electric field

$\mathbf{F}_{\mathrm{rad}} =$ radiation reaction force (radiation damping)
the damping of a charge's motion which arises because
of the emission of radiation)

Free Oscillator: radiation damping

- **Undriven Harmonically Bound Particles** (free oscillator)

  Since an oscillating electron represents a continuously accelerating charge, the electron will radiate energy. The energy radiated away must come from the particle's own energy (energy conservation). In other words, **there must be a force acting on a particle by virtue of the radiation it produces. This is called the radiation reaction force.**

  Let's derive the formula for the radiation reaction force from the fact that the energy radiated must be compensated for by the work done against the radiation reaction force.

  (1) On one hand, the radiative loss rate of energy, averaged over one cycle of the oscillating dipole, can be represented by the radiative reaction force:

$$\frac{dW}{dt} = \langle \mathbf{F}_{\rm rad} \cdot \dot{\mathbf{x}} \rangle$$

  (2) On the other hand, from the Larmor's formula for a dipole, the radiative loss will be:

$$\frac{dW}{dt} = -\frac{2e^2 \left\langle |\ddot{\mathbf{x}}|^2 \right\rangle}{3c^3}$$

Free Oscillator: Abraham-Lorentz formula

$$\therefore \quad \langle \mathbf{F}_{\text{rad}} \cdot \dot{\mathbf{x}} \rangle = -\frac{2e^2 \left\langle |\ddot{\mathbf{x}}|^2 \right\rangle}{3c^3}$$

Here, $\displaystyle \left\langle |\ddot{\mathbf{x}}|^2 \right\rangle \equiv \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}} dt$

$$= \frac{1}{\tau} \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \Big|_{-\tau/2}^{\tau/2} - \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt$$

We assume that the initial and final states are the same: $\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}(-\tau/2) = \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}(\tau/2)$

Then,

$$\left\langle |\ddot{\mathbf{x}}|^2 \right\rangle = -\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt = -\left\langle \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \right\rangle \quad \rightarrow \quad \langle \mathbf{F}_{\text{rad}} \cdot \dot{\mathbf{x}} \rangle = \frac{2e^2 \left\langle \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \right\rangle}{3c^3}$$

Therefore, we can obtain $\boxed{\mathbf{F}_{\text{rad}} = \dfrac{2e^2 \dddot{\mathbf{x}}}{3c^3} \quad : \text{ Abraham-Lorentz formula}}$

- **Abraham-Lorentz formula**:

$$\boxed{\mathbf{F}_{\mathrm{rad}} = \frac{2e^2 \dddot{\mathbf{x}}}{3c^3}}$$

This formula depends on the derivative of acceleration. This increases the degree of the equation of motion of a particle and can lead to some nonphysical behavior if not used properly and consistently.

For a simple harmonic oscillator with a frequency ω_0, we can avoid the difficulty by using

$$\dddot{\mathbf{x}} = -\omega_0^2 \dot{\mathbf{x}}$$

This is a good assumption as long as the energy is to be radiated on a time scale that is long compared to the period of oscillation. In this regime, radiation reaction may be considered as a *perturbation* on the particle's motion. We then rewrite the radiation reaction force as

$$\mathbf{F}_{\mathrm{rad}} = -\frac{2e^2\omega_0^2}{3c^3}\dot{\mathbf{x}} = -m\gamma\dot{\mathbf{x}}, \qquad \gamma \equiv \frac{2e^2\omega_0^2}{3mc^3} \quad : \quad \text{damping constant}$$

Note $\gamma = A_{21}$ in Quantum Mechanics

Condition for this approximation:

T = the time interval over which the kinetic energy of the particle is changed substantially by the emission of radiation:

$$T \sim \frac{mv^2}{dW/dt} \sim \frac{3mc^3}{2e^2}\left(\frac{v}{a}\right)^2$$

t_p = the typical orbital time scale for the particle: $t_p \sim \frac{v}{a}$ or $t_p = \frac{2\pi}{\omega_0}$

Then, the condition is

$$\left(\text{electron radius, } r_e = \frac{e^2}{mc^2}\right)$$

$$\frac{T}{t_p} \gg 1 \quad \rightarrow \quad \frac{3mc^3}{2e^2}t_p = \frac{t_p}{\tau_c} \gg 1 \quad \rightarrow \quad t_p \gg \tau_c \equiv \frac{2}{3}\frac{e^2}{mc^3} = \frac{2}{3}\frac{r_e}{c}(\sim 10^{-23}\ \mathrm{s})$$

where τ_c is the time for radiation to cross a distance comparable to the classical electron radius.

In terms of frequency of the oscillator, this condition is equivalent to:

$$\frac{2\pi}{\tau_c} = 3\pi\frac{c}{r_e} \equiv \omega_c \gg \omega_0$$

In terms of wavelength of the oscillator,

$$\lambda_0 = \frac{2\pi c}{\omega_0} \gg \lambda_c \equiv \frac{2\pi c}{\omega_c} = \frac{2}{3}r_e \ (\sim\ 2 \times 10^{-13}\ \mathrm{cm} = 2 \times 10^{-5}\text{Å})$$

Therefore, **in most cases, the approximation is valid.**

At this limit:

$$
\begin{aligned}
\frac{\gamma}{\omega_0} &= \frac{2e^2}{3mc^2}\frac{\omega_0}{c} \\
&= \frac{2}{3}\frac{r_e}{\lambda_0}2\pi \\
\therefore \frac{\gamma}{\omega_0} &\ll 1 \text{ for } \lambda_0 \gg r_e = 2.82 \times 10^{-13} \text{ cm}
\end{aligned}
$$

- Equation of motion of the electron in a Lorentz atom:

$$\ddot{\mathbf{x}} + \gamma \dot{\mathbf{x}} + \omega_0^2 \mathbf{x} = 0$$

This equation may be solved by assuming that $x(t) \propto e^{\alpha t}$.

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \;\rightarrow\; \alpha = -(\gamma/2) \pm \sqrt{(\gamma/2)^2 - \omega_0^2}$$
$$= -\gamma/2 \pm i\omega_0 + \mathcal{O}(\gamma^2/\omega_0^2)$$

Assuming initial conditions

$$x(0) = x_0,\; \dot{x}(0) = 0 \text{ at } t = 0$$

we have

$$x(t) = \frac{1}{2} x_0 \left[e^{-(\gamma/2 - i\omega_0)t} + e^{-(\gamma/2 + i\omega_0)t} \right] = x_0 e^{-\gamma/2} \cos\omega_0 t \longrightarrow \text{Damping oscillator}$$

- Power spectrum:

$$\bar{x}(\omega) = \frac{1}{2\pi} \int_0^\infty x(t) e^{i\omega t} dt = \frac{x_0}{4\pi} \left[\frac{1}{\gamma/2 - i(\omega + \omega_0)} + \frac{1}{\gamma/2 - i(\omega - \omega_0)} \right]$$

This becomes large in the vicinity of $\omega = \omega_0$ and $\omega = -\omega_0$.

We are ultimately interested only in positive frequencies, and only in regions in which the values become large. Therefore, we obtain

$$\bar{x}(\omega) \approx \frac{x_0}{4\pi} \frac{1}{\gamma/2 - i(\omega - \omega_0)}, \quad |\bar{x}(\omega)|^2 = \left(\frac{x_0}{4\pi} \right)^2 \frac{1}{(\omega - \omega_0)^2 + (\gamma/2)^2}$$

Emission Line profile

Recall
$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} e^2 |\bar{x}(\omega)|^2$$

Energy radiated per unit frequency:

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} \frac{e^2 x_0^2}{(4\pi)^2} \frac{1}{(\omega-\omega_0)^2 + (\gamma/2)^2} = \frac{1}{2} m \left(\frac{\omega^4}{\omega_0^2}\right) x_0^2 \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$
$$\approx \frac{1}{2} m \omega_0^2 x_0^2 \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

For a harmonic oscillator, note that the equation of motion is $\mathbf{F} = -k\mathbf{x} = -m\omega_0^2 \mathbf{x}$, spring constant is $k = m\omega_0^2$, and the potential energy (energy stored in spring) is $(1/2)kx_0^2$.

From
$$\int_{-\infty}^{\infty} \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2} d\omega = \frac{1}{\pi} \tan^{-1}\{2(\omega-\omega_0)/\gamma\}|_{-\infty}^{\infty} = 1$$

Total emitted energy = initial potential energy of the oscillator:

$$W = \int_0^{\infty} \frac{dW}{d\omega} d\omega = \frac{1}{2} k\omega_0^2$$

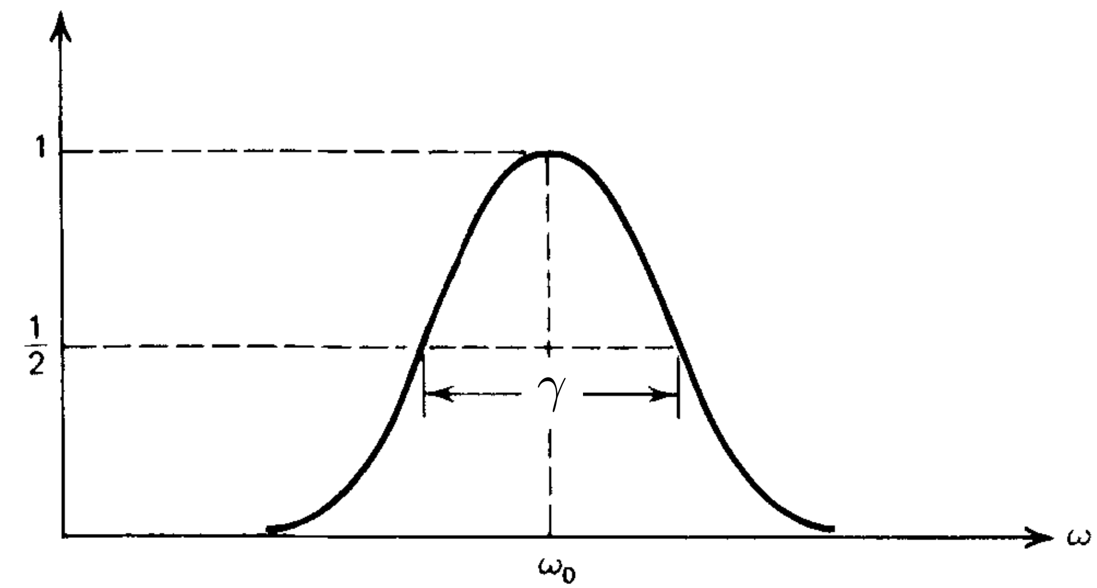
Profile of the emitted spectrum:
$$\boxed{\phi(\omega) = \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}}$$
Lorentz (natural) profile

Damping constant is the full width at half maximum (FWHM).

$$\phi(\omega) = \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

$$\phi(\nu) = \frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2}$$

Note $\phi(\omega)d\omega = \phi(\nu)d\nu$



The line width $\Delta\omega = \gamma$ is a universal constant when expressed in terms of wavelength:

$$\begin{aligned} \lambda &= \frac{2\pi c}{\omega} \\ \Delta\lambda &= 2\pi c \frac{\Delta\omega}{\omega^2} = 2\pi c \frac{2}{3}\frac{r_e}{c} \longleftarrow \quad \left(\Delta\omega = \gamma = \frac{2}{3} r_e \frac{\omega_0^2}{c}\right) \\ &= \frac{4}{3}\pi r_e \\ &= 1.2\times 10^{-4}\text{Å} \end{aligned}$$

In Quantum Mechanics, the line width is not a universal constant.

Driven Oscillator (scattering)

- **Driven Harmonically Bound Particles** (forced oscillators)

Electron's equation of motion (electric charge = -e):

$$\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x} = -\frac{e\mathbf{E}_0}{m}e^{i\omega t}$$

Rybicki & Lightman use the following equation.

$$\ddot{\mathbf{x}} - (\gamma/\omega_0^2)\dddot{\mathbf{x}} + \omega_0^2\mathbf{x} = -\frac{e\mathbf{E}_0}{m}e^{i\omega t}$$

Steady-state solution of this equation:

$$\mathbf{x} = \mathbf{x}_0 e^{i\omega t} \equiv |\mathbf{x}_0|e^{i(\omega t+\delta)} \quad \rightarrow \quad (-\omega^2 + i\omega\gamma + \omega_0^2)\mathbf{x}_0 e^{i\omega t} = -\frac{e\mathbf{E}_0}{m}e^{i\omega t}$$

$$\mathbf{x}_0 = \frac{(e/m)\mathbf{E}_0}{(\omega^2 - \omega_0^2) - i\omega\gamma}$$

$$\mathbf{x}_0 = |\mathbf{x}_0|e^{i\delta} \propto (\omega^2 - \omega_0^2) + i\omega\gamma \quad \rightarrow \quad \delta = \tan^{-1}\left(\frac{\omega\gamma}{\omega^2 - \omega_0^2}\right)$$

The response is slightly out of phase with respect to the imposed field.

For $\omega > \omega_0$, the particle "leads" the driving force and for $\omega < \omega_0$ it "lags."

Time-averaged total power radiated:

$$P = \left\langle \frac{dW}{dt} \right\rangle = \frac{2e^2\left\langle |\ddot{\mathbf{x}}|^2 \right\rangle}{3c^3} = \frac{e^2\omega^4|\mathbf{x}_0|^2}{3c^3}$$

$$= \frac{e^4E_0^2}{3m^2c^3}\frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}$$

- Scattering cross section:

$$\sigma_{\rm sca} \equiv \frac{\langle P \rangle}{\langle S \rangle}, \quad \langle S \rangle = \frac{c}{8\pi} E_0^2 \quad \longrightarrow \quad \sigma_{\rm sca}(\omega) = \frac{8\pi e^4}{3m^2c^4} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}$$

$$= \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}$$

- Limiting Cases of Interest

(a) $\omega \gg \omega_0$ (Thomson scattering by free electron)

$$\boxed{\sigma_{\rm sca} = \sigma_T = \frac{8\pi}{3} r_e^2}$$

At high incident energies, the binding becomes negligible.

(b) $\omega \ll \omega_0$ (Rayleigh scattering by bound electron)

$$\boxed{\sigma_{\rm sca} = \sigma_T \left(\frac{\omega}{\omega_0} \right)^4}$$

The electric filed appears nearly static and produces a nearly static force.

Blue color of the sky at sunrise:

Red color of the sun at sunset: when the path through the atmosphere is longer, the blue and green components are removed almost completely leaving the longer wavelength orange and red.

(c) $\omega \approx \omega_0$ (Resonance scattering of line radiation)

$$\sigma_{\rm sca}(\omega) \approx \sigma_T \frac{\omega_0^4}{(\omega-\omega_0)^2(2\omega_0)^2 + (\omega_0\gamma)^2}$$

$$= \sigma_T \frac{\omega_0^2/4}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

Note $\sigma_{\rm scat}(\omega) = \sigma_\nu(\nu)$

$$\sigma_T \frac{\omega_0^2}{4} = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \times \frac{1}{4} \times \left(\gamma \frac{3}{2}\frac{mc^3}{e^2\omega_0^2}\right) = 2\pi^2 \frac{e^2}{mc}(\gamma/2\pi) \longrightarrow$$

$$\boxed{\begin{aligned} \sigma_{\rm sca}(\omega) &= \frac{2\pi^2 e^2}{mc}\frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2} \\ \sigma_{\rm sca}(\nu) &= \frac{\pi e^2}{mc}\frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2} \end{aligned}}$$

In the neighborhood of the resonance, the shape of the scattering cross section is the same as the emission line profile from the free oscillator.

Total scattering cross section: $\displaystyle \int_0^\infty \sigma(\omega)d\omega = \frac{2\pi^2 e^2}{mc}, \quad \int_0^\infty \sigma(\nu)d\nu = \frac{\pi e^2}{mc}$

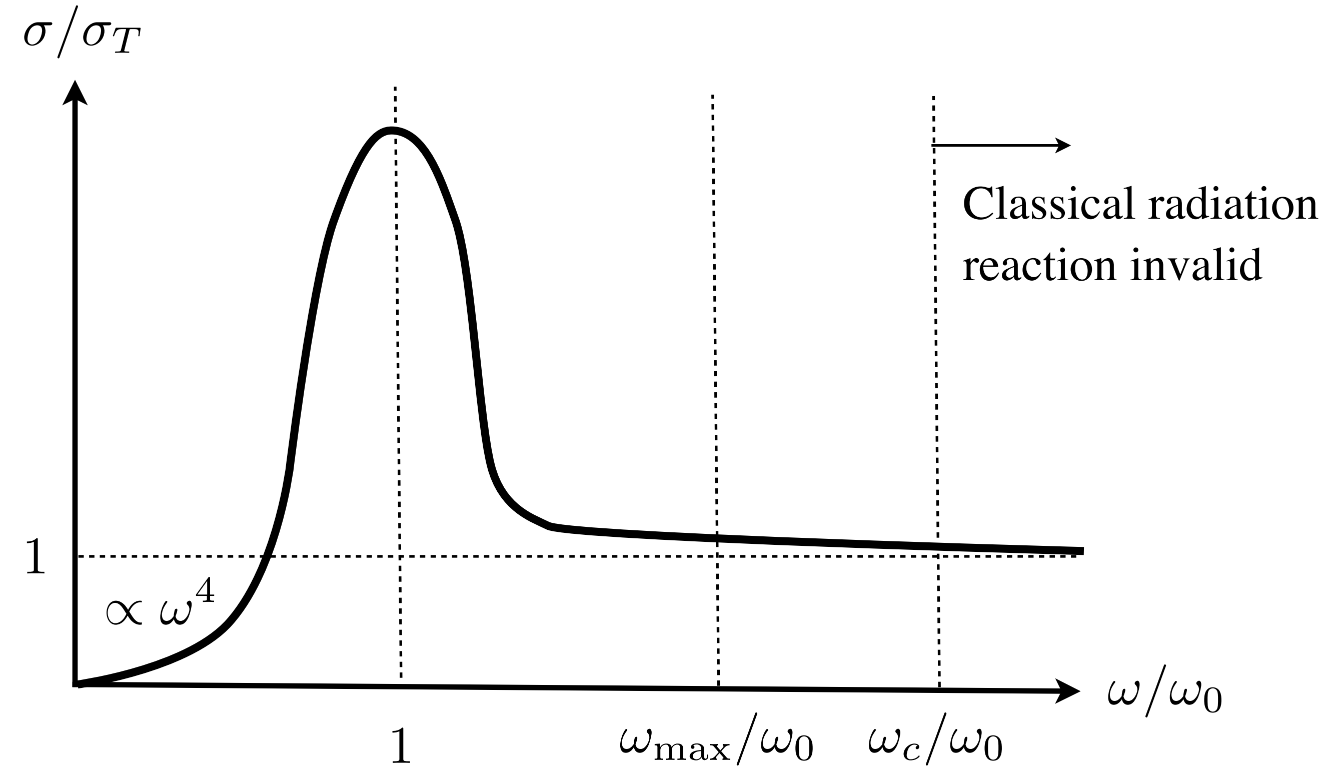
In evaluating this integral, we have apparently neglected a divergence, since the cross section approaches σ_T for large ω .

However, note that the approximate formula for radiation reaction is only valid for $\omega_0 \ll \omega_c$. Therefore, we must cut off the integral at a $\omega_{\rm max}$ such that $\omega_0 \ll \omega_{\rm max} \ll \omega_c$.

We also note that the contribution to the integral from the constant Thomson limit is less than

$$\int_0^{\omega_{\max}} \sigma_T d\omega = \sigma_T \omega_{\max} \ll \sigma_T \omega_c = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \times 3\pi\left(\frac{mc^3}{e^2}\right) = \frac{8\pi^2 e^2}{mc} \approx \int_0^{\infty} \sigma_{\rm sca}(\omega) d\omega$$

The contribution is therefore negligible.



<u>In the quantum theory of spectral lines,</u>

we obtain similar formulas, which are conveniently stated in terms of the classical results as

$$\boxed{\int_0^{\infty} \sigma(\nu) d\nu = \frac{\pi e^2}{mc} f_{nn'}}$$

where $f_{nn'}$ is called the ***<u>oscillator strength</u>*** or ***<u>f-value</u>*** for the transition between states *n* and *n'*.

Resonance Lines

Draine, Physics of the interstellar and intergalactic medium

Table 9.4 Selected Resonance Lines[a] with $\lambda < 3000$ Å

	Configurations	ℓ	u	$E_\ell/hc(\mathrm{cm}^{-1})$	$\lambda_{\mathrm{vac}}(\mathrm{\AA})$	$f_{\ell u}$
C IV	$1s^22s - 1s^22p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1550.772	0.0962
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1548.202	0.190
N V	$1s^22s - 1s^22p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1242.804	0.0780
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1242.821	0.156
O VI	$1s^22s - 1s^22p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1037.613	0.066
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1037.921	0.133
C III	$2s^2 - 2s2p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	977.02	0.7586
C II	$2s^22p - 2s2p^2$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{D}^o_{3/2}$	0	1334.532	0.127
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{D}^o_{5/2}$	63.42	1335.708	0.114
N III	$2s^22p - 2s2p^2$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{D}^o_{3/2}$	0	989.790	0.123
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{D}^o_{5/2}$	174.4	991.577	0.110
C I	$2s^22p^2 - 2s^22p3s$	$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	0	1656.928	0.140
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	16.40	1656.267	0.0588
		$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	43.40	1657.008	0.104
N II	$2s^22p^2 - 2s2p^3$	$^3\mathrm{P}_0$	$^3\mathrm{D}^o_1$	0	1083.990	0.115
		$^3\mathrm{P}_1$	$^3\mathrm{D}^o_2$	48.7	1084.580	0.0861
		$^3\mathrm{P}_2$	$^3\mathrm{D}^o_3$	130.8	1085.701	0.0957
N I	$2s^22p^3 - 2s^22p^23s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1199.550	0.130
		$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{3/2}$	0	1200.223	0.0862
O I	$2s^22p^4 - 2s^22p^33s$	$^3\mathrm{P}_2$	$^3\mathrm{S}^o_1$	0	1302.168	0.0520
		$^3\mathrm{P}_1$	$^3\mathrm{S}^o_1$	158.265	1304.858	0.0518
		$^3\mathrm{P}_0$	$^3\mathrm{S}^o_1$	226.977	1306.029	0.0519
Mg II	$2p^63s - 2p^63p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	2803.531	0.303
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	2796.352	0.608
Al III	$2p^63s - 2p^63p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1862.790	0.277
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1854.716	0.557

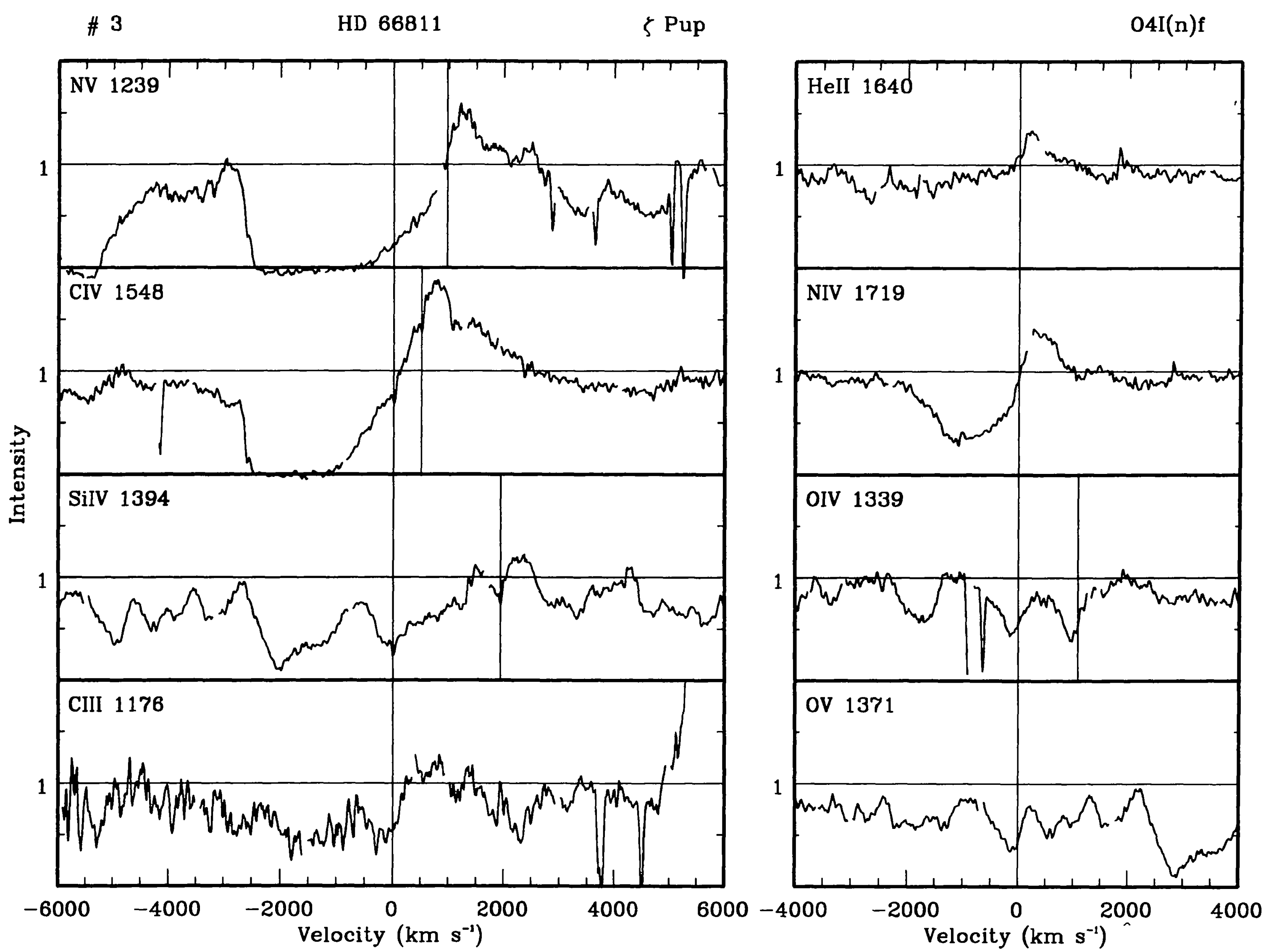
Table 9.4 contd.

	Configurations	ℓ	u	$E_\ell/hc\,(\mathrm{cm}^{-1})$	$\lambda_{\mathrm{vac}}(\mathrm{\AA})$	$f_{\ell u}$
Mg I	$2p^63s^2 - 2p^63s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	2852.964	1.80
Al II	$2p^63s^2 - 2p^63s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	1670.787	1.83
Si III	$2p^63s^2 - 2p^63s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	1206.51	1.67
P IV	$2p^63s^2 - 2p^63s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	950.655	1.60
Si II	$3s^23p - 3s^24s$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{S}_{1/2}$	0	1526.72	0.133
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{S}_{1/2}$	287.24	1533.45	0.133
P III	$3s^23p - 3s3p^2$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{D}_{3/2}$	0	1334.808	0.029
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{D}_{5/2}$	559.14	1344.327	0.026
Si I	$3s^23p^2 - 3s^23p4s$	$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	0	2515.08	0.17
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	77.115	2507.652	0.0732
		$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	223.157	2516.870	0.115
P II	$3s^23p^2 - 3s3p^3$	$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	0	1301.87	0.038
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	164.9	1305.48	0.016
		$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	469.12	1310.70	0.115
S III	$3s^23p^2 - 3s3p^3$	$^3\mathrm{P}_0$	$^3\mathrm{D}^o_1$	0	1190.206	0.61
		$^3\mathrm{P}_1$	$^3\mathrm{D}^o_2$	298.69	1194.061	0.46
		$^3\mathrm{P}_2$	$^3\mathrm{D}^o_3$	833.08	1200.07	0.51
Cl IV	$3s^23p^2 - 3s3p^3$	$^3\mathrm{P}_0$	$^3\mathrm{D}^o_1$	0	973.21	0.55
		$^3\mathrm{P}_1$	$^3\mathrm{D}^o_2$	492.0	977.56	0.41
		$^3\mathrm{P}_2$	$^3\mathrm{D}^o_3$	1341.9	984.95	0.47
P I	$3s^23p^3 - 3s^23p^24s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1774.951	0.154
S II	$3s^23p^3 - 3s^23p^24s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1259.518	0.12
Cl III	$3s^23p^3 - 3s^23p^24s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1015.019	0.58
S I	$3s^23p^4 - 3s^23p^34s$	$^3\mathrm{P}_2$	$^3\mathrm{S}^o_1$	0	1807.311	0.11
		$^3\mathrm{P}_1$	$^3\mathrm{S}^o_1$	396.055	1820.343	0.11
		$^3\mathrm{P}_0$	$^3\mathrm{S}^o_1$	573.640	1826.245	0.11
Cl II	$3s^23p^4 - 3s3p^5$	$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	0	1071.036	0.014
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	696.00	1079.080	0.00793
		$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	996.47	1075.230	0.019
Cl I	$3s^23p^5 - 3s^23p^44s$	$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{P}_{3/2}$	0	1347.240	0.114
		$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{P}_{3/2}$	882.352	1351.657	0.0885
Ar II	$3s^23p^5 - 3s3p^6$	$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{S}_{1/2}$	0	919.781	0.0089
		$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{S}_{1/2}$	1431.583	932.054	0.0087
Ar I	$3p^6 - 3p^54s$	$^1\mathrm{S}_0$	$^2[1/2]^o$	0	1048.220	0.25

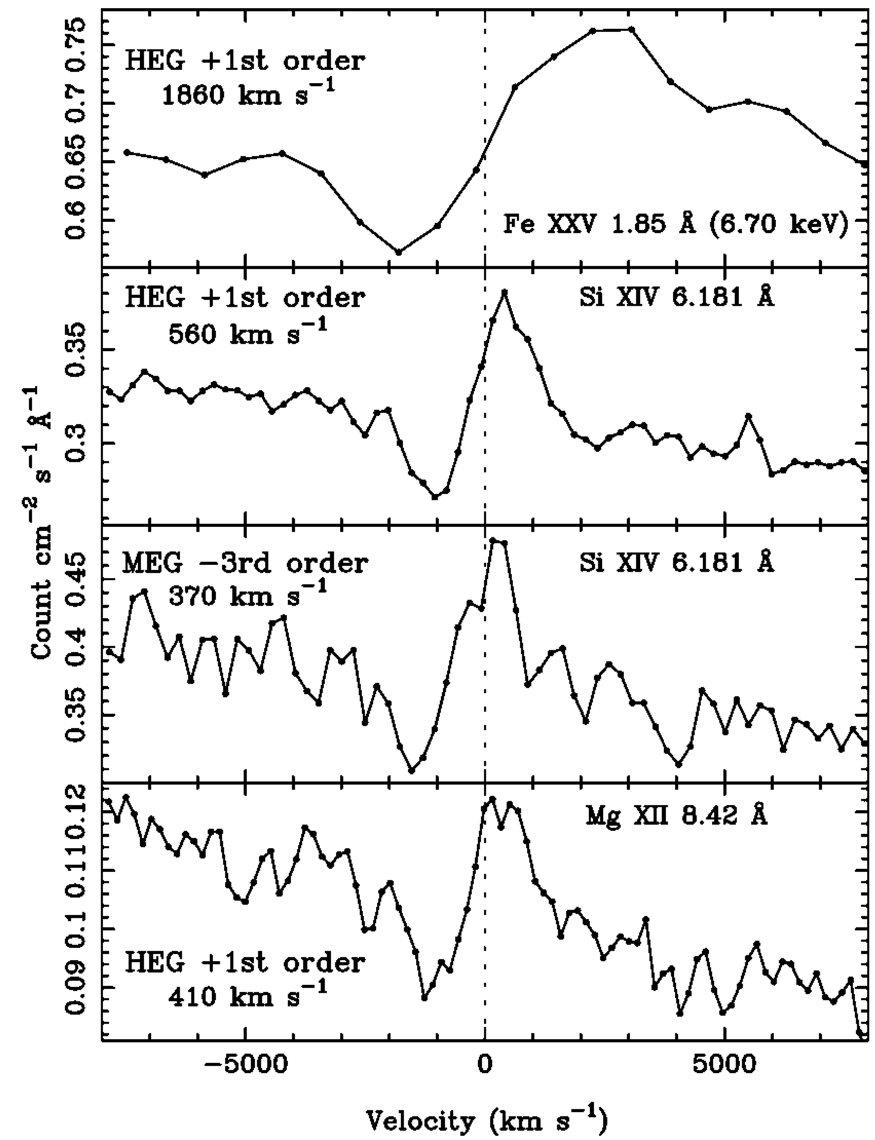
[a] Transition data from NIST Atomic Spectra Database v4.0.0 (Ralchenko et al. 2010)

P Cygni Profile

- The PCygni profile is characterized by strong emission lines with corresponding blueshifted absorption line.



zeta Puppis (Snow et al., 1994, ApJS, 95, 163)

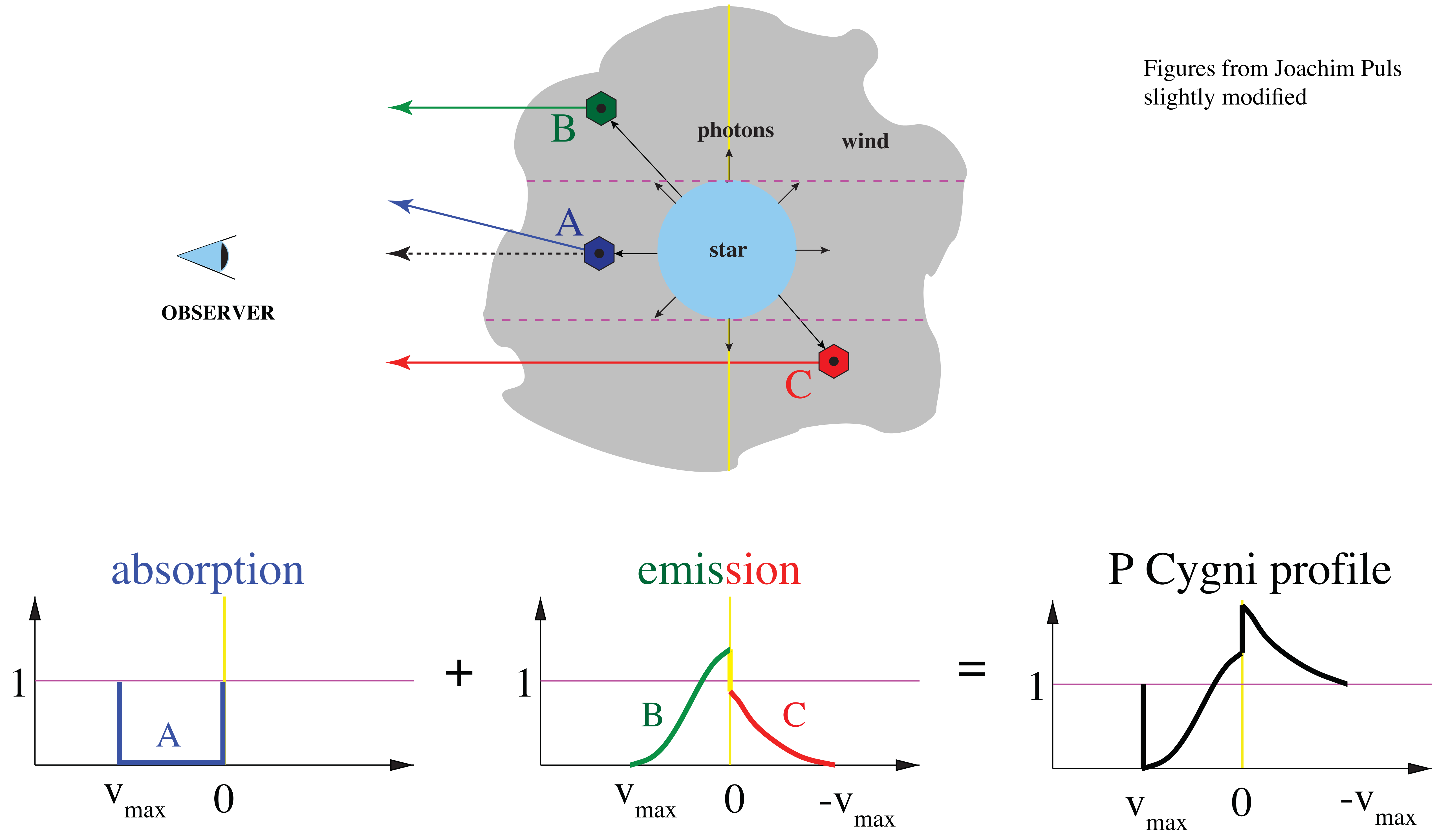


Circinus X-1
(Brandt & Schulz, 2000, ApJ, 544, L123)

P Cygni profile formation

- The blueshifted absorption line is produced by material moving away from the star and toward us, whereas the emission come from other parts of the expanding shell.

Figures from Joachim Puls slightly modified



Lyα Resonance Scattering

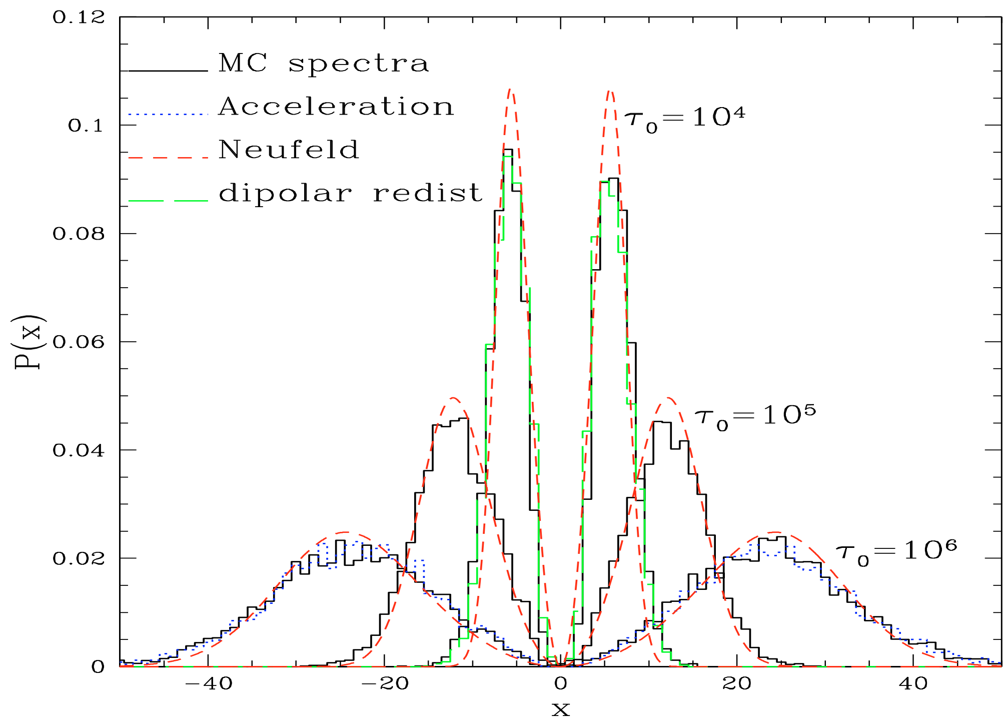
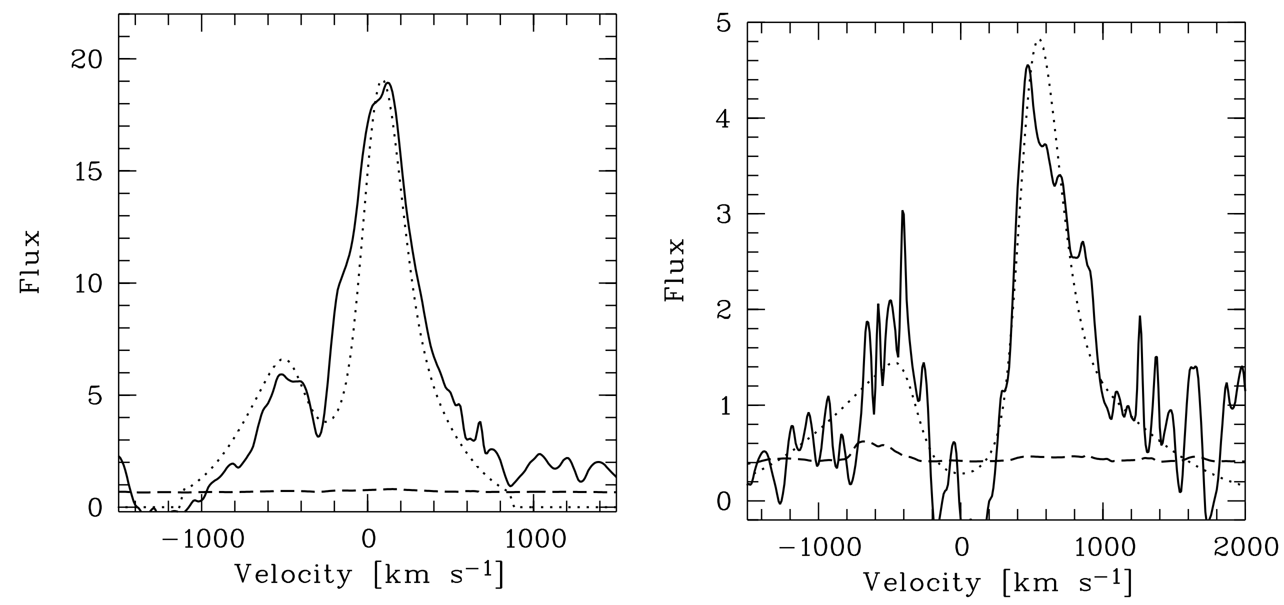


Fig. 1. Predicted emergent Lyα profiles for monochromatic line radiation emitted in a dust-free slab of different optical depths (solid lines) compared with analytic solutions from Neufeld (1990, dashed). The dotted blue curve shows the line profile obtained using a frequency redistribution function, which skips a large number of resonant core scatterings. The adopted conditions of the medium are: $T = 10$ K (i.e. $a = 1.5 \times 10^{-2}$) and $\tau_0 = 10^4, 10^5, 10^6$ *from top to bottom*. The green long-dashed curve, obtained with a dipolar angular redistribution, overlaps perfectly the black solid line obtained with the isotropic angular redistribution function, illustrating the fact that in static media, isotropy is a very good approximation.

Verhamme et al. (2006, A&A, 460, 397)



ID	v_{dis}(core) [km s^{-1}]	v_{dis}(shell) [km s^{-1}]	N_{HI} [cm^{-2}]	v_{outflow} [km s^{-1}]	z
4691	600	60	4×10^{17}	12	3.30
5215	500	125	$<2 \times 10^{16}$	125	3.15
7539	1140	190	2.5×10^{16}	190	3.29

Comparizon of the observed Lyα lines (solid lines) and the best-fit theoretical models. The dashed line indicates the noise level of the observed spectrum.

Tapken et al. (2007, A&A, 467, 63)

Raman Scattering*

- If the energy of the internal state (E_0) is less than that of the incoming photon $h\nu$, then a scattered photon of energy $h\nu - E_0$ can be produced.

  Raman scattering or the Raman effect is the inelastic scattering of a photon.

  When photons are scattered from an atom or molecule, **most photons are elastically scattered (i.e., Rayleigh scattering)**, such that the scattered photons have the same energy (frequency and wavelength) as the incident photons. However, a small fraction of the scattered photons (approximately 1 in 10 million) are scattered by an excitation, with the scattered photons having a frequency different from, and usually lower than, that of the incident photons.

- Example: **scattering of the O VI doublet (λλ1038, 1032) by neutral hydrogen.**

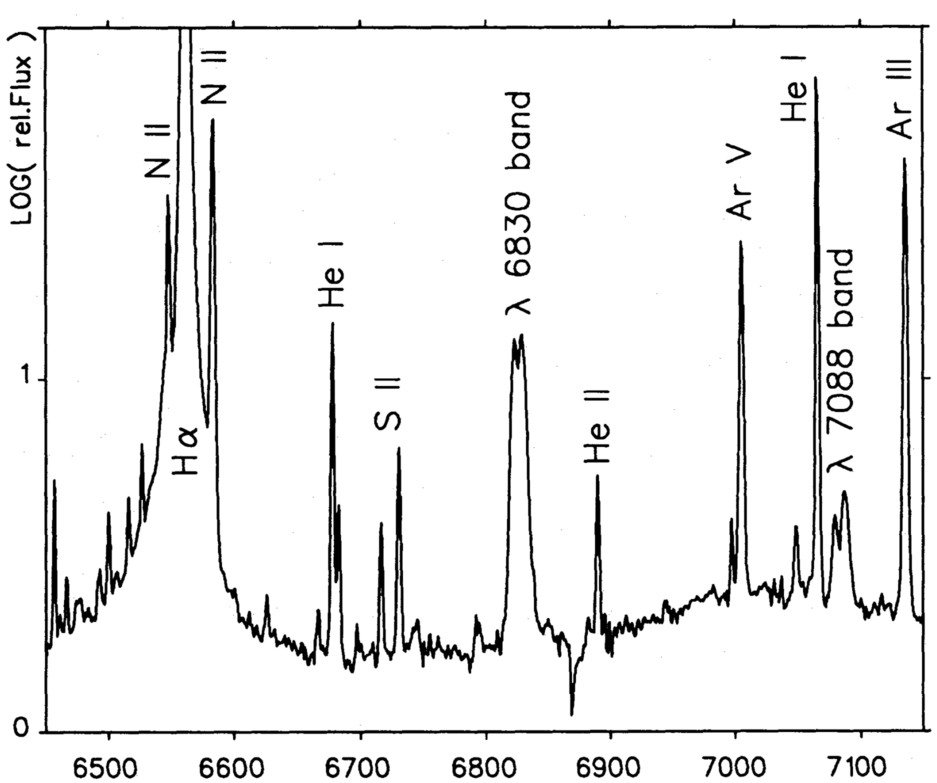
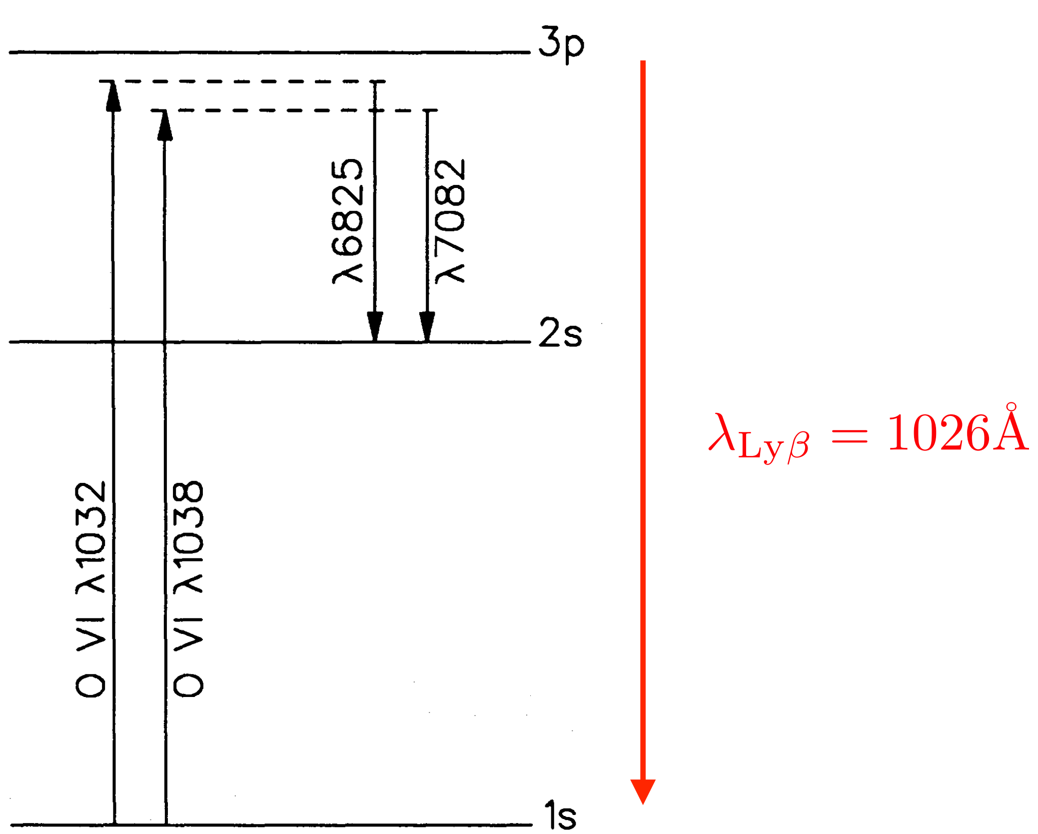


Fig.1. Raman scattered emission bands in the symbiotic star V1016 Cyg. The spectrum was obtained on the 1.93m telescope at the Observatoire de Haute Provence.



Schmid (1989, A&A, 211, L31)

Homework (due date: 10/11)

Solve the Problem 3.3 in the textbook.

3.3—Two oscillating dipole moments (radio antennas) $\mathbf{d}_1$ and $\mathbf{d}_2$ are oriented in the vertical direction and are a horizontal distance L apart. They oscillate in phase at the same frequency ω. Consider radiation at angle θ with respect to the vertical and in the vertical plane containing the two dipoles.

a. Show that

$$\frac{dP}{d\Omega}=\frac{\omega^4\sin^2\theta}{8\pi c^3}\left(d_1^2+2d_1d_2\cos\delta+d_2^2\right),$$

where

$$\delta\equiv\frac{\omega L\sin\theta}{c}.$$

b. Thus show directly that when $L\ll\lambda$, the radiation is the same as from a single oscillating dipole of amplitude d_1+d_2.

Hint (page 329 in the textbook)

3.3

a. Use Eq. (3.15a) with $q\dot{u} = -\omega^2 d\cos\omega t$ for each dipole, noting that the retarded times for each differ by $\Delta t = (L/c)\sin\theta$ (see Fig. S.3). Then

$$|\mathbf{E}_{\mathrm{rad}}| = -\frac{\omega^2}{rc^2}\left[d_1\cos\omega t + d_2\cos\omega(t-\Delta t)\right]\sin\theta$$
$$= -\frac{\omega^2}{rc^2}\left[(d_1 + d_2\cos\delta)\cos\omega t + d_2\sin\delta\sin\omega t\right]\sin\theta,$$

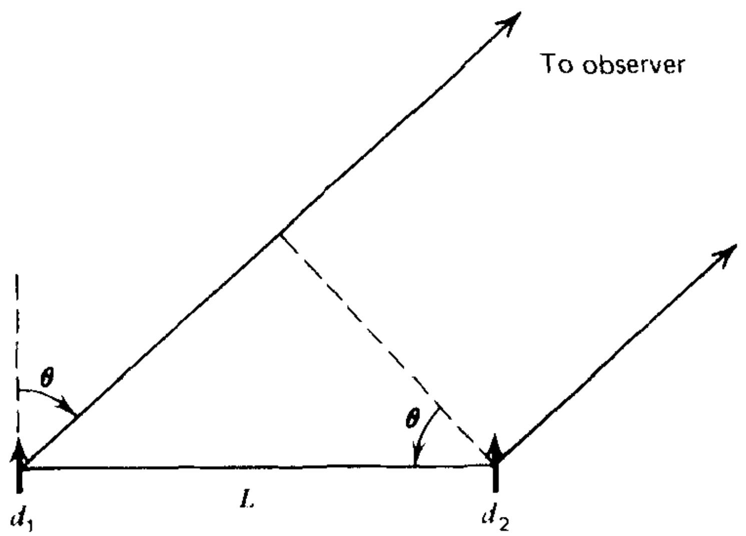


Figure S.3 Geometry for emission from two dipole radiators separated by distance L.

where $\delta = \omega\Delta t = \omega L\sin\theta/c$. Squaring and averaging over time, we find

$$\langle|E_{\mathrm{rad}}|^2\rangle = \frac{\omega^4\sin^2\theta}{2r^2c^4}\left[(d_1 + d_2\cos\delta)^2 + (d_2\sin\delta)^2\right]$$
$$= \frac{\omega^4\sin^2\theta}{2r^2c^4}(d_1^2 + 2d_1d_2\cos\delta + d_2^2).$$

We have finally,

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{cr^2}{4\pi}\langle|E_{\mathrm{rad}}|^2\rangle$$
$$= \frac{\omega^4\sin^2\theta}{8\pi c^3}(d_1^2 + 2d_1d_2\cos\delta + d_2^2).$$

b. When $L \ll \lambda$, we have $\delta \equiv 2\pi L\sin\theta/\lambda \ll 1$, and

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{\omega^4\sin^2\theta}{8\pi c^3}(d_1 + d_2)^2,$$

which is the radiation from an oscillating charge with dipole moment $d_1 + d_2$.