Astrophysics

Lecture 05

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Radiation from Moving Charges 2

Dipole Approximation (the radiation from many particles)

- Consider many particles with positions $\mathbf{r}_i$, velocities $\mathbf{u}_i$, and charges q_i $(i = 1,2,3,\cdots, N)$. The radiation field at large distances can be found by adding together the $\mathbf{E}_{\rm rad}$ from each particle.
- However, the radiation field equations refer to conditions at retarded time, and the retarded times will differ for each particle. Therefore, we must keep track of the phase relations between the particles.

  Dipole approximation: there are situations in which it is possible to ignore this difficulty. **The condition for dipole approximation is simply equivalent to the nonrelativistic condition.**
- Let R_0 be the distance from some point in the system to the field point. Then, $R_i = R_0 + \ell_i \approx R_0$ as $R_0 \gg \ell_i$. Finally, we have

$$\mathbf{E}_{\rm rad} = \sum_i \frac{q_i}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}_i)}{R_i} \longrightarrow \mathbf{E}_{\rm rad} \approx \frac{1}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \sum_i q_i \dot{\mathbf{u}}_i)}{R_0}$$

$$\boxed{\mathbf{E}_{\rm rad} = \frac{\mathbf{n} \times \left(\mathbf{n} \times \ddot{\mathbf{d}}\right)}{c^2 R_0}}$$

where the electric dipole moment is defined as

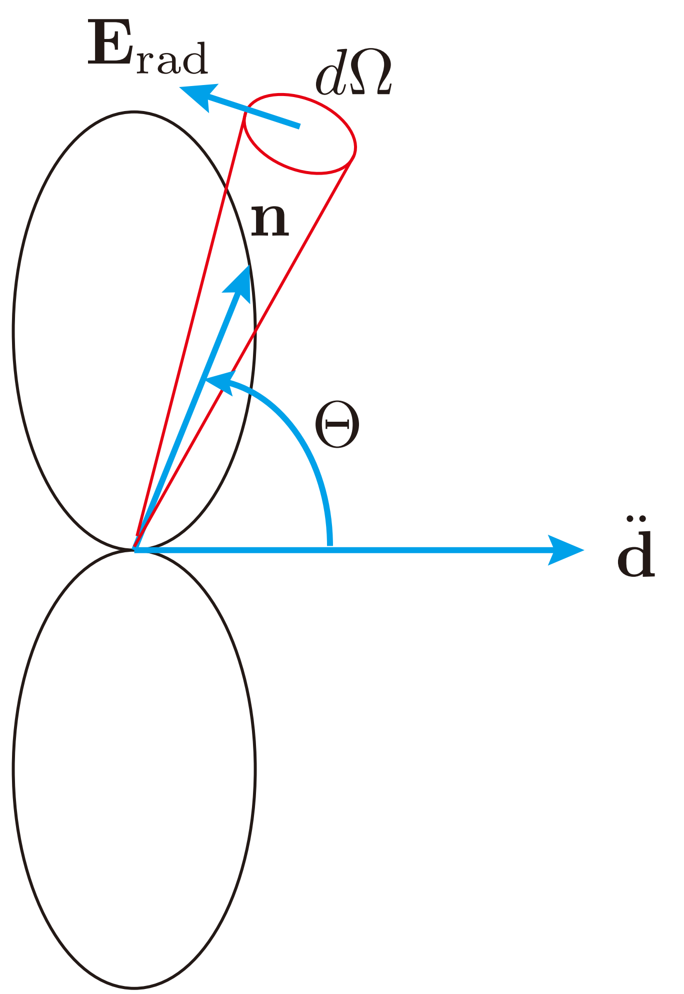
$$\boxed{\mathbf{d} = \sum_i q_i \mathbf{r}_i}$$

Note that <u>the right-hand side of the above equations must still be evaluated at a retarded time</u>, but using any point within the region, say, the position used to define R_0.

- As before, for a single particle, we find the generalized formulas for the radiation pattern and the total power, which are called the dipole approximation:

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\sin^2\Theta, \qquad P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}$$

Note that the instantaneous polarization of **E** lines in the plane of $\ddot{\mathbf{d}}$ and **n**.



- **Spectrum of radiation in the dipole approximation**:

For simplicity we assume that **d** always lies in a single direction. Then, the magnitude of the electric field is given by

$$\boxed{E(t) = \ddot{d}(t)\frac{\sin\Theta}{c^2 R_0}}$$

where $d(t)$ is the magnitude of the dipole moment.

Fourier transform of $d(t)$ is defined as $d(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\bar{d}(\omega)d\omega$

Then, $\ddot{d}(t) = -\int_{-\infty}^{\infty} \omega^2 e^{-i\omega t}\bar{d}(\omega)d\omega$

$$\boxed{\bar{E}(\omega) = -\frac{1}{c^2 R_0}\omega^2\bar{d}(\omega)\sin\Theta}$$

- The energy per unit solid angle per frequency range in the dipole approximation is given by

$$\frac{dW}{d\omega d\Omega} = R_0^2 \frac{dW}{d\omega dA}$$
$$\frac{dW}{d\omega dA} = c\left|\bar{E}(\omega)\right|^2 \longrightarrow \frac{dW}{d\omega d\Omega} = \frac{\omega^4}{c^3}\left|\bar{d}(\omega)\right|^2 \sin^2\Theta$$

The total energy per frequency range is

$$\boxed{\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\left|\bar{d}(\omega)\right|^2} \longleftarrow \frac{dW}{d\omega} = \int \frac{dW}{d\omega d\Omega} d\Omega$$

- The above formulas describe an interesting property of dipole radiation, namely, that the spectrum of the emitted radiation is related directly to the frequencies of oscillation of the dipole moment. However, this property is not true for particles with relativistic velocities.

- It is also worthwhile to note the dependence of $\omega^4 \propto \lambda^{-4}$ in the power spectrum.

A general Multipole Expansion*

- The above treatment was obtained only qualitatively. We would like to be more explicit.
- Recall that the vector potential is

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}' \int dt' \, \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

- Consider a Fourier analysis of the sources and fields:

$$\mathbf{j}_\omega(\mathbf{r}) = \int \mathbf{j}(\mathbf{r},t)e^{i\omega t}dt$$

$$\mathbf{A}_\omega(\mathbf{r}) = \int \mathbf{A}(\mathbf{r},t)t^{i\omega t}dt$$

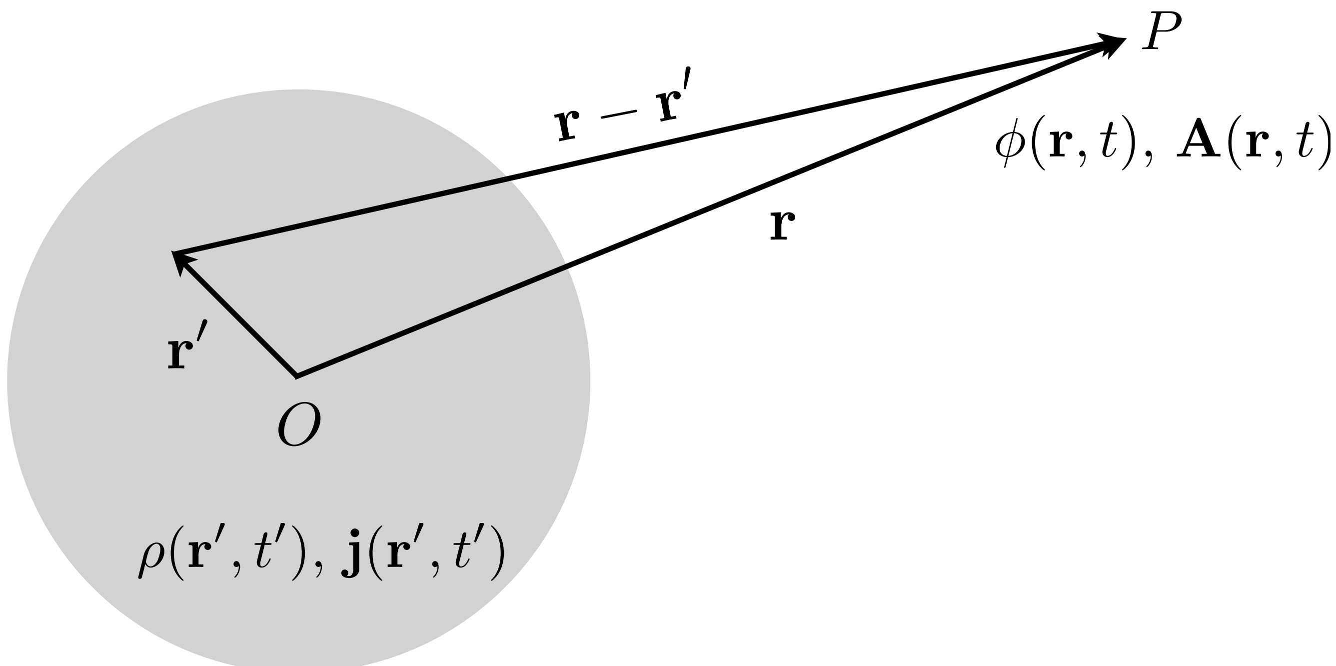
Then, the vector potential becomes

$$\begin{aligned}
\mathbf{A}_\omega(\mathbf{r}) &= \frac{1}{c}\int d^3\mathbf{r}' \int dt' \int dt \, \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}e^{i\omega t}\delta\left(t'-t+|\mathbf{r}-\mathbf{r}'|/c\right) \\
&= \frac{1}{c}\int d^3\mathbf{r}' \int dt' \, \frac{\mathbf{j}(\mathbf{r},t')}{|\mathbf{r}-\mathbf{r}'|}e^{i\omega t'}e^{i\omega|\mathbf{r}-\mathbf{r}'|/c} \\
&= \frac{1}{c}\int d^3\mathbf{r}' \, \frac{\mathbf{j}_\omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|} \qquad \leftarrow \quad k \equiv \omega/c
\end{aligned}$$

Note that this exponential term is caused by the retardation.

This equation now relate single Fourier components of source $\mathbf{j}$ and potential $\mathbf{A}$.

- Let's choose an origin of coordinates inside the source of size L. Then, at field points such that $r \gg L$, we will expand the potential in a power series of kr'.



$$
\begin{aligned}
|\mathbf{r}-\mathbf{r}'| &= \left[(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')\right]^{1/2} = \left[r^2 - 2\,(\mathbf{r}\cdot\mathbf{r}') + r'^2\right]^{1/2} = r\left[1 - \frac{2\,(\mathbf{r}\cdot\mathbf{r}')}{r^2} + \frac{r'^2}{r^2}\right]^{1/2} \\
&\approx r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \cdots\right) \qquad \leftarrow\ (1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots \\
&= r - \mathbf{n}\cdot\mathbf{r}' + \cdots \qquad \leftarrow\ \text{Here,}\ \ \mathbf{n} \equiv \frac{\mathbf{r}}{r}\quad \text{(}\mathbf{n}\text{ points toward the field point }\mathbf{r}\text{)}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r}\left[1 - \frac{2\,(\mathbf{r}\cdot\mathbf{r}')}{r^2} + \frac{r'^2}{r^2}\right]^{-1/2} \approx \frac{1}{r}\left(1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \cdots\right) \\
&= \frac{1}{r} + \frac{\mathbf{n}\cdot\mathbf{r}'}{r^2} + \cdots
\end{aligned}
$$

$$\begin{aligned}\mathbf{A}_\omega(\mathbf{r}) &= \frac{1}{c}\int \frac{\mathbf{j}_\omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{ik|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \\ &\approx \frac{1}{c}\int \mathbf{j}_\omega(\mathbf{r}')\frac{1}{r}\left(1+\frac{\mathbf{n}\cdot\mathbf{r}'}{r}\right) e^{ikr} e^{-ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}' \\ &= \frac{e^{ikr}}{cr}\left[\int \mathbf{j}_\omega(\mathbf{r}') e^{-ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}' + \frac{1}{r}\int (\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}_\omega(\mathbf{r}') e^{-ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}'\right]\end{aligned}$$

(1) The factor $\exp(ikr)$ outside the integral expresses **the effect of retardation from the source as a whole**.

(2) The factor $\exp(-ik\mathbf{n}\cdot\mathbf{r}')$ inside the integral expresses **the relative retardation of each element** of the source.

In our slow motion approximation, $kL = 2\pi L/\lambda \ll 1$, the first and second integrals can be approximated, respectively, to be

$$\int \mathbf{j}_\omega(\mathbf{r}') e^{ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}' \approx \int \mathbf{j}_\omega(\mathbf{r}')\left[1 - ik\mathbf{n}\cdot\mathbf{r}' + \cdots\right] d^3\mathbf{r}'$$

$$\int (\mathbf{n}\cdot\mathbf{r}')\mathbf{j}_\omega(\mathbf{r}') e^{ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}' \approx \int (\mathbf{n}\cdot\mathbf{r}')\mathbf{j}_\omega(\mathbf{r}')\left[1 + \cdots\right] d^3\mathbf{r}'$$

$$\leftarrow \quad e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Then, the vector potential becomes

$$\mathbf{A}_\omega(\mathbf{r}) \approx \frac{e^{ikr}}{cr}\left[\int \mathbf{j}_\omega(\mathbf{r}') d^3\mathbf{r}' + \left(\frac{1}{r} - ik\right)\int (\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}_\omega(\mathbf{r}') d^3\mathbf{r}' + \mathcal{O}\left((\mathbf{n}\cdot\mathbf{r}')^2\right)\right]$$

- The "electric" dipole approximation results from taking just the first term in the above equation:

$$\mathbf{A}_\omega(\mathbf{r})|_{\text{dipole}} \approx \frac{e^{ikr}}{cr} \int \mathbf{j}_\omega(\mathbf{r}') d^3\mathbf{r}'$$

- The second term give the "electric" quadrupole and "magnetic" dipole terms.

$$\mathbf{A}_\omega(\mathbf{r}) \approx \frac{e^{ikr}}{cr} \left(\frac{1}{r} - ik \right) \int (\mathbf{n} \cdot \mathbf{r}')\, \mathbf{j}_\omega(\mathbf{r}') d^3\mathbf{r}'$$

The term inside the integral can be expressed in terms of a symmetric and asymmetric terms for $\mathbf{r}'$ and $\mathbf{j}$.

$$\begin{aligned}(\mathbf{n} \cdot \mathbf{r}')\mathbf{j} &= \frac{1}{2}\left[(\mathbf{n} \cdot \mathbf{r}')\mathbf{j} + (\mathbf{n} \cdot \mathbf{j})\mathbf{r}'\right] + \frac{1}{2}\left[(\mathbf{n} \cdot \mathbf{r}')\mathbf{j} - (\mathbf{n} \cdot \mathbf{j})\mathbf{r}'\right] \\ &= \frac{1}{2}\left[(\mathbf{n} \cdot \mathbf{r}')\mathbf{j} + (\mathbf{n} \cdot \mathbf{j})\mathbf{r}'\right] + \frac{1}{2}(\mathbf{r}' \times \mathbf{j}) \times \mathbf{n}\end{aligned}$$

The first and second terms correspond to the electric quadrupole and magnetic dipole terms, respectively.

- Lamor's formula is obtained by assuming $kr \gg 1$, or in other words, by taking the far zone approximation in addition to the dipole approximation.

Free electron: Thomson Scattering (Electron Scattering)

- Recall the dipole formula

$$\boxed{\frac{dP}{d\Omega} = \frac{dW}{dt d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\sin^2\Theta, \quad P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}}$$

- Let us consider the **process in which a free charged particle (electron) radiates in response to an incident electromagnetic wave**, as an important application of the dipole formula.

In non-relativistic case, we may neglect magnetic force. (recall $E = B$)

magnetic/electric force ratio in Lorentz force: $F_B/F_E \sim (v/c)B/E = v/c \ll 1$

Consider a monochromatic wave with frequency ω_0 and linearly polarized in direction $\hat{\boldsymbol{\epsilon}}$:

$$\mathbf{E} = \hat{\boldsymbol{\epsilon}} E_0 \sin\omega_0 t$$

Thus the force on a particle with the charge e is

$$\mathbf{F} = e\mathbf{E} = \hat{\boldsymbol{\epsilon}} e E_0 \sin\omega_0 t$$

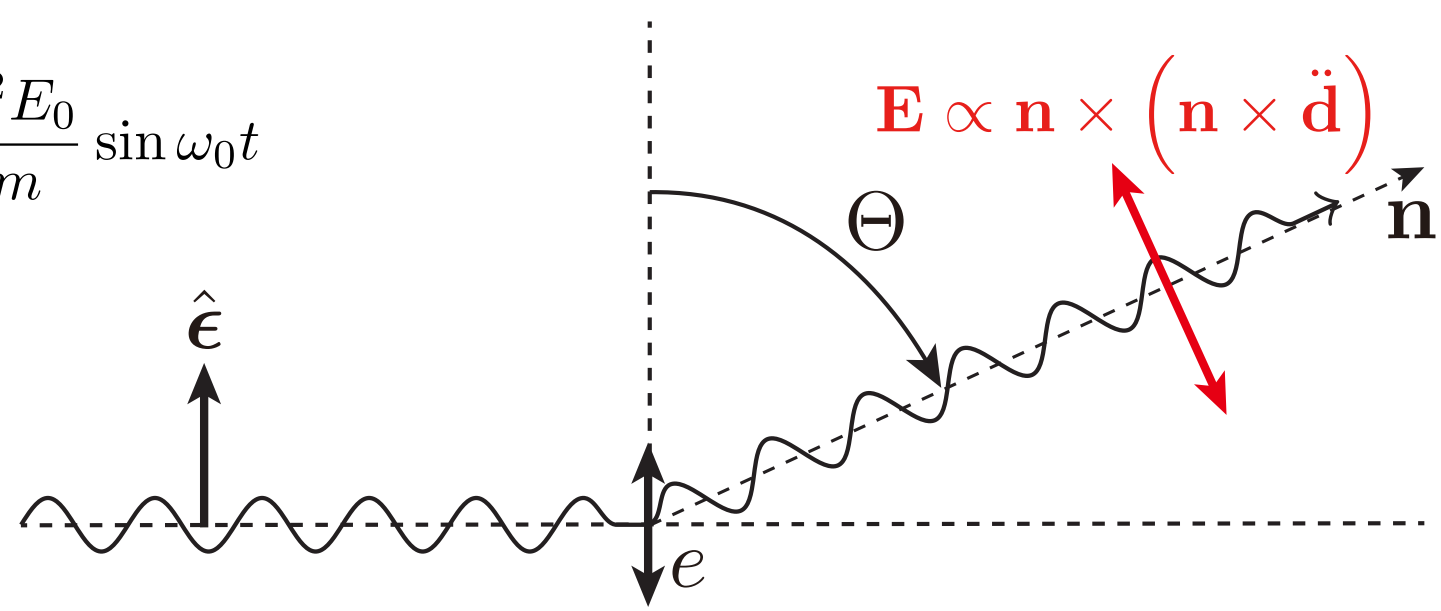
the acceleration of the electron is

$$\ddot{\mathbf{r}} = \hat{\boldsymbol{\epsilon}}\frac{eE_0}{m}\sin\omega_0 t, \quad \ddot{\mathbf{d}} = e\ddot{\mathbf{r}} = \hat{\boldsymbol{\epsilon}}\frac{e^2E_0}{m}\sin\omega_0 t$$

the dipole moment is

$$\boxed{\mathbf{d} = -\hat{\boldsymbol{\epsilon}}\left(\frac{e^2E_0}{m\omega_0^2}\right)\sin\omega_0 t}$$

This describes an oscillating dipole.



- We obtain the time-averaged power per solid angle $\left(\langle \sin^2 \omega_0 t \rangle = 1/2\right)$:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{< \ddot{\mathbf{d}}^2 >}{4\pi c^3} \sin^2 \Theta = \frac{e^4 E_0^2}{8\pi m^2 c^3} \sin^2 \Theta, \quad \langle P \rangle = \frac{e^4 E_0^2}{3m^2 c^3}$$

Recall that the time-averaged incident flux is $\langle S \rangle = \dfrac{c}{8\pi} E_0^2$

The **differential cross section**, $\dfrac{d\sigma}{d\Omega}$, for linearly polarized radiation is obtained by

$$\frac{d\sigma}{d\Omega} \equiv \left\langle \frac{dP}{d\Omega} \right\rangle \Big/ \langle S \rangle \qquad \boxed{\frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \sin^2 \Theta = r_0^2 \sin^2 \Theta, \quad r_0 \equiv \frac{e^2}{mc^2}}$$

where the quantity r_0 gives a measure of the "size" of the point charge. (Note electrostatic potential energy $e\phi = e^2/r_0 = mc^2$). For an electron, the classical electron radius has a value $r_0 = 2.82 \times 10^{-13}$ cm.

The total cross section is found by integrating over solid angle ($\mu \equiv \cos \Theta$).

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi r_0^2 \int_{-1}^{1} (1 - \mu^2) d\mu = \frac{8\pi}{3} r_0^2$$

For an electron, this scattering process is called Thomson scattering or electron scattering, and the **Thomson cross section** is

$$\boxed{\sigma_{\mathrm{T}} = \frac{8\pi}{3} r_0^2 = 6.652 \times 10^{-25} \ \mathrm{cm}^2}$$

- Properties:

  The total and differential cross sections are frequency independent.

  The scattered radiation is linearly polarized in the plane of the incident polarization vector $\hat{\boldsymbol{\epsilon}}$ and the direction of scattering $\mathbf{n}$.

  $\sigma \propto 1/m^2$: electron scattering is larger than ions by a factor of $(m_p/m_e)^2 = (1836)^2 \approx 3.4 \times 10^6$.

  We have implicitly assumed that electron recoil is negligible. This assumption is only valid for nonrelativistic energies. For higher energies, the (quantum-mechanical) Klein-Nishina cross section has to be used.

- The **cross section for Thomson scattering of unpolarized radiation**

  An unpolarized beam can be regarded as the independent superposition of two linear-polarized beams (with the same strengths) with perpendicular axes.

  Let us assume that $\mathbf{n}$ = direction of scattered radiation

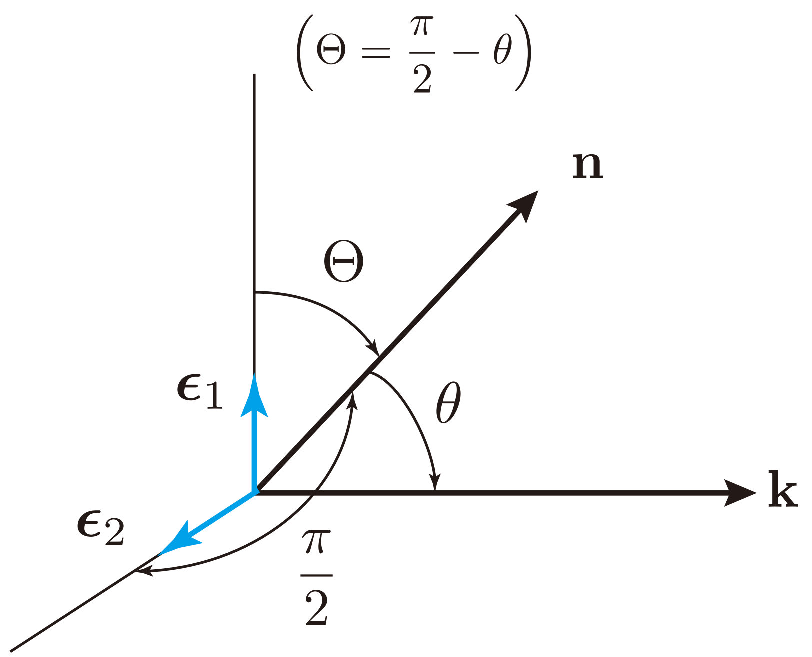
  $\mathbf{k}$ = direction of incident radiation

  scattering plane = a plane which contains $\mathbf{n}$ *and* $\mathbf{k}$.

  Let's choose

  the first electric field along $\hat{\boldsymbol{\epsilon}}_1$ to be is in the $\mathbf{n} - \mathbf{k}$ plane and

  the second one along $\hat{\boldsymbol{\epsilon}}_2$ orthogonal to this plane and to $\mathbf{n}$.



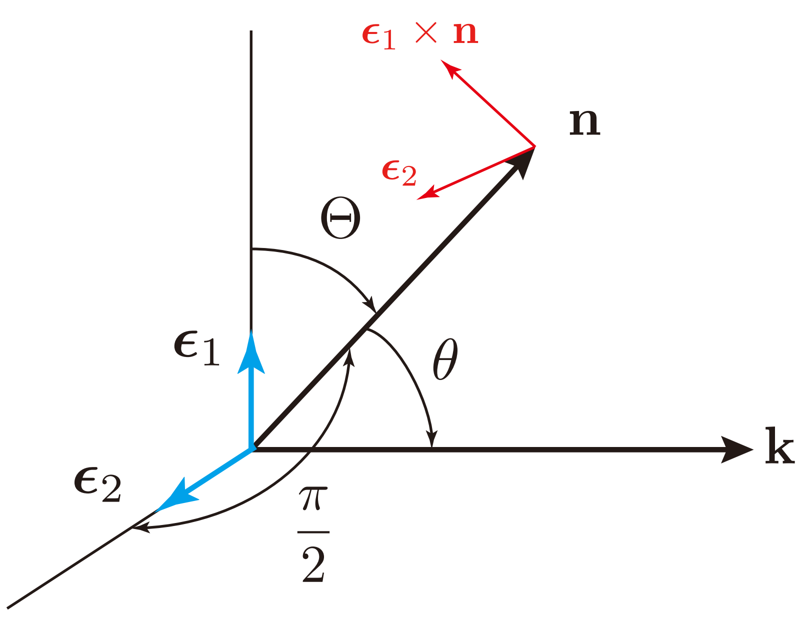
- Let Θ = angle between $\hat{\boldsymbol{\epsilon}}_1$ and $\mathbf{n}$, and note that angle between $\hat{\boldsymbol{\epsilon}}_2$ and $\mathbf{n}$ is $\pi/2$.

  $\theta = \pi/2 - \Theta$ = angle between the scattered wave and incident wave

Then, the differential cross section for unpolarized radiation is given by the average of the cross sections for scattering of two electric fields.

$$
\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} &= \frac{1}{2}\left[\left(\frac{d\sigma}{d\Omega}\right)_{\epsilon_2} + \left(\frac{d\sigma}{d\Omega}\right)_{\epsilon_1}\right] \\
&= \frac{1}{2}\left[\left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\text{pol}} + \left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\text{pol}}\right] \\
&= \frac{1}{2} r_0^2 \left(1 + \sin^2\Theta\right)
\end{aligned}
$$

$$
\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2} r_0^2 \left(1 + \cos^2\theta\right)}
$$



This the scattering phase function for Thomson scattering or Rayleigh scattering.

This depends only on the angle between the incident and scattered directions, as it should for unpolarized radiation.

Total cross section:

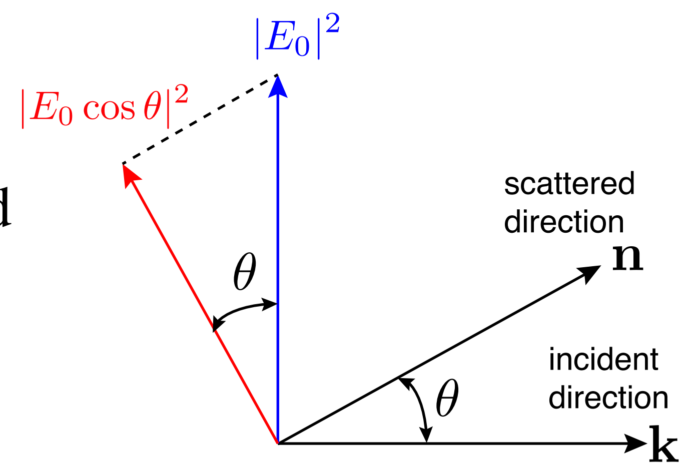
$$
\sigma_{\text{unpol}} = \int \left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} d\Omega = \pi r_0^2 \int_{-1}^{1} (1+\mu^2)\,d\mu = \frac{8\pi}{3} r_0^2
$$

$$
\boxed{\sigma_{\text{unpol}} = \sigma_{\text{pol}} = \frac{8\pi}{3} r_0^2}
$$

Properties of Thomson Scattering

- Forward-backward symmetry: differential cross section is symmetric under $\theta \to -\theta$.
- Total cross section of unpolarized incident radiation = total cross section for polarized incident radiation. This is because the electron at rest has no preferred direction defined.
- **Scattering creates polarization**

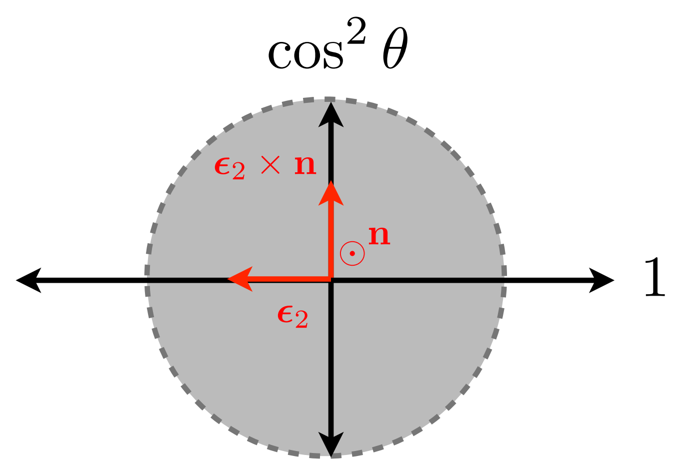
The scattered intensity is proportional to $1 + \cos^2\theta$, of which 1 arises from the incident electric field along $\hat{\epsilon}_2$ and $\cos^2\theta$ from the incident electric field along $\hat{\epsilon}_1$.



" $\cos^2\theta$ " of the polarization along $\hat{\epsilon}_2$ will be cancelled out by the independent polarization along $\hat{\epsilon}_2 \times \mathbf{n}$.

Therefore, the degree of polarization of the scattered wave:

$$\Pi = \frac{1 - \cos^2\theta}{1 + \cos^2\theta}$$



Electron scattering of a completely unpolarized incident wave produces a scattered wave with some degree of polarization.

<u>No net polarization along the incident direction</u> ($\theta = 0$), since, by symmetry, all directions are equivalent.

<u>100% polarization perpendicular to the incident direction</u> ($\theta = \pi/2$), since the electron's motion is confined to a plane normal to the incident direction.

Classical model to the motion of an electron in an atom

- **<u>Lorentz Oscillator Model</u> to describe the interaction between atoms and electric fields**: The electron (with a small mass) is bound to the nucleus of the atom (with a much larger mass) by a force that behaves according to Hooke's Law (a spring-like force). An applied electric field would then interact with the charge of the electron, causing "stretching" or "compression" of the spring.
- Hooke's law: the force needed to extend or compress a spring by some distance (x) scales linearly with respect to that distance — that is, $F = kx$, where k is a constant factor characteristic of the spring (spring constant).
- **The electron's equation of motion**:

$$\boxed{m\ddot{\mathbf{x}} = -k\mathbf{x} + \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{rad}}}$$

$$k = m\omega_0^2 \ , \ \text{where } k = \text{spring constant}$$

$$\omega_0 = \text{natural (fundamental or resonant) freqency}$$

$\mathbf{F}_{\text{ext}} =$ external force, driving force, or external electric field

$\mathbf{F}_{\text{rad}} =$ radiation reaction force (radiation damping)
the damping of a charge's motion which arises because
of the emission of radiation)

[1] Spontaneous Emission: Radiation Damping, Free Oscillator

- **Undriven Harmonically Bound Particles** (free oscillator)

  Since an oscillating electron represents a continuously accelerating charge, the electron will radiate energy. The energy radiated away must come from the particle's own energy (energy conservation). In other words, **there must be a force acting on a particle by virtue of the radiation it produces. This is called the radiation reaction force.**

  Let's derive the formula for the radiation reaction force from the fact that the energy radiated must be compensated for by the work done against the radiation reaction force.

  (1) On one hand, the radiative loss rate of energy, **averaged over one cycle** of the oscillating dipole, can be represented by the radiative reaction force:

$$\frac{dW}{dt} = \langle \mathbf{F}_{\mathrm{rad}} \cdot \dot{\mathbf{x}} \rangle$$

  (2) On the other hand, from the Larmor's formula for a dipole, the radiative loss will be:

$$\frac{dW}{dt} = -\frac{2e^2 \left\langle |\ddot{\mathbf{x}}|^2 \right\rangle}{3c^3}$$

Abraham-Lorentz formula

$$\therefore \quad \langle \mathbf{F}_{\rm rad} \cdot \dot{\mathbf{x}} \rangle = -\frac{2e^2 \left\langle |\ddot{\mathbf{x}}|^2 \right\rangle}{3c^3}$$

Here, $\left\langle |\ddot{\mathbf{x}}|^2 \right\rangle \equiv \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}} dt$

$$= \frac{1}{\tau} \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \Big|_{-\tau/2}^{\tau/2} - \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt \longleftarrow \boxed{\int (f'g)dx = fg - \int (fg')dx}$$

Here, τ is the oscillation period.

We assume that the initial and final states are the same: $\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}(-\tau/2) = \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}(\tau/2)$

Then,

$$\left\langle |\ddot{\mathbf{x}}|^2 \right\rangle = -\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt = -\langle \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \rangle \quad \rightarrow \quad \langle \mathbf{F}_{\rm rad} \cdot \dot{\mathbf{x}} \rangle = \frac{2e^2 \langle \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \rangle}{3c^3}$$

Therefore, we can obtain $\boxed{\mathbf{F}_{\rm rad} = \frac{2e^2 \dddot{\mathbf{x}}}{3c^3} \quad : \text{ Abraham-Lorentz formula}}$

- **<u>Abraham-Lorentz formula</u>**:

$$\mathbf{F}_{\mathrm{rad}} = \frac{2e^2 \dddot{\mathbf{x}}}{3c^3}$$

This formula depends on the derivative of acceleration. This increases the degree of the equation of motion of a particle and can lead to some nonphysical behavior if not used properly and consistently.

For a simple harmonic oscillator with a frequency ω_0, we can avoid the difficulty by using

$$\dddot{\mathbf{x}} = -\omega_0^2 \dot{\mathbf{x}} \longleftarrow$$

$$\begin{aligned} x &= \sin \omega t \\ \dot{x} &= \omega_0 \cos \omega_0 t \\ \ddot{x} &= -\omega_0^2 \sin \omega_0 t \end{aligned} \qquad \begin{aligned} \dddot{x} &= -\omega_0^3 \cos \omega_0 t \\ &= -\omega_0^2 \dot{x} \end{aligned}$$

- ***This is a good assumption as long as the energy is to be radiated on a time scale that is long compared to the period of oscillation.*** In this regime, radiation reaction may be considered as a *perturbation* on the particle's motion. We then rewrite the radiation reaction force as

$$\mathbf{F}_{\mathrm{rad}} = -\frac{2e^2\omega_0^2}{3c^3}\dot{\mathbf{x}} = -m\gamma\dot{\mathbf{x}},$$

$$\gamma \equiv \frac{2e^2\omega_0^2}{3mc^3}$$ damping constant (Einstein A coefficient)

Note $\gamma = A_{21}$ in Quantum Mechanics

$$m\ddot{\mathbf{x}} + k\mathbf{x} + m\gamma\dot{\mathbf{x}} = 0$$

This is the equation for a string-mass system subject to friction damping.

Condition for this approximation:

T = the time interval over which the kinetic energy of the particle is changed substantially by the emission of radiation:

$$T \sim \frac{mv^2}{dW/dt} \sim \frac{3mc^3}{2e^2}\left(\frac{v}{a}\right)^2$$

Here, a = acceleration.

t_p = the typical orbital time scale for the particle: $t_p \sim \dfrac{v}{a}$ or $t_p = \dfrac{2\pi}{\omega_0}$

Then, the condition is

$$\left(\text{electron radius, } r_e = \frac{e^2}{mc^2}\right)$$

condition: $\boxed{\dfrac{T}{t_p} \gg 1} \rightarrow \dfrac{3mc^3}{2e^2}t_p = \dfrac{t_p}{\tau_c} \gg 1 \rightarrow t_p \gg \tau_c \equiv \dfrac{2}{3}\dfrac{e^2}{mc^3} = \dfrac{2}{3}\dfrac{r_e}{c}(\sim 10^{-23}\ \mathrm{s})$

where τ_c is the time for radiation to cross a distance comparable to the classical electron radius.

In terms of frequency of the oscillator, this condition is equivalent to:

$$\frac{2\pi}{\tau_c} = 3\pi\frac{c}{r_e} \equiv \omega_c \gg \omega_0 = \frac{2\pi}{\tau_p}$$

In terms of wavelength of the oscillator,

$$\lambda_0 = \frac{2\pi c}{\omega_0} \gg \lambda_c \equiv \frac{2\pi c}{\omega_c} = \frac{2}{3}r_e\ (\sim\ 2\times 10^{-13}\ \mathrm{cm} = 2\times 10^{-5}\text{Å})$$

Therefore, **in most cases, the approximation is valid.**

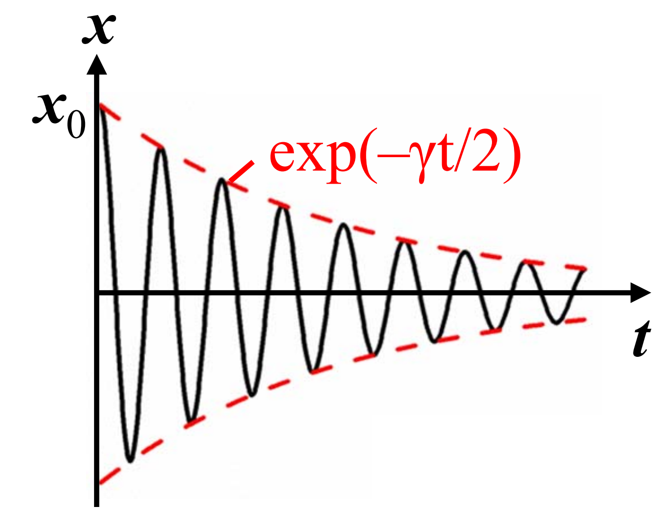
<u>At this limit:</u>

$$\begin{aligned}\frac{\gamma}{\omega_0} &= \frac{2e^2}{3mc^2}\frac{\omega_0}{c}\\ &= \frac{2}{3}\frac{r_e}{\lambda_0}2\pi\\ \therefore \frac{\gamma}{\omega_0} &\ll 1 \text{ for } \lambda_0 \gg r_e = 2.82 \times 10^{-13} \text{ cm}\end{aligned}$$

- <u>Equation of motion of the electron in a Lorentz atom:</u>

$$\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x} = 0$$

This equation may be solved by assuming that $x(t) \propto e^{\alpha t}$.



$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \;\rightarrow\; \alpha = -(\gamma/2) \pm \sqrt{(\gamma/2)^2 - \omega_0^2}$$

Here, we assumed $\gamma \ll \omega_0$. $\quad = -\gamma/2 \pm i\omega_0 + \mathcal{O}(\gamma^2/\omega_0^2)$

Assuming initial conditions

$$x(0) = x_0,\; \dot{x}(0) = 0 \text{ at } t = 0$$

we have

$$\boxed{x(t) = \frac{1}{2}x_0\left[e^{-(\gamma/2 - i\omega_0)t} + e^{-(\gamma/2 + i\omega_0)t}\right] = x_0 e^{-\gamma t/2}\cos\omega_0 t} \longrightarrow \text{Damping oscillator}$$

- <u>Power spectrum</u>:

Here, $x(t) = 0$ for $t < 0$.

$$\bar{x}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{i\omega t}dt = \frac{x_0}{4\pi}\left[\frac{1}{\gamma/2 - i(\omega + \omega_0)} + \frac{1}{\gamma/2 - i(\omega - \omega_0)}\right]$$

This becomes large in the vicinity of $\omega = \omega_0$ and $\omega = -\omega_0$.

We are ultimately interested only in positive frequencies, and only in regions in which the values become large. Therefore, we obtain

$$\bar{x}(\omega) \approx \frac{x_0}{4\pi}\frac{1}{\gamma/2 - i(\omega - \omega_0)}, \quad |\bar{x}(\omega)|^2 = \left(\frac{x_0}{4\pi}\right)^2 \frac{1}{(\omega - \omega_0)^2 + (\gamma/2)^2}$$

Emission Line profile

Recall the Lamor's formula

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} e^2 |\bar{x}(\omega)|^2$$

<u>Energy radiated per unit frequency</u>:

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} \frac{e^2 x_0^2}{(4\pi)^2} \frac{1}{(\omega-\omega_0)^2 + (\gamma/2)^2} = \frac{1}{2} m \left(\frac{\omega^4}{\omega_0^2}\right) x_0^2 \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$
$$\approx \frac{1}{2} m\omega_0^2 x_0^2 \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

For a harmonic oscillator, note that the equation of motion is $\mathbf{F} = -k\mathbf{x} = -m\omega_0^2\mathbf{x}$, the spring constant is $k = m\omega_0^2$, and the potential energy (energy stored in spring) is $(1/2)kx_0^2$.

From

$$\int_{-\infty}^{\infty} \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2} d\omega = \frac{1}{\pi} \tan^{-1}\{2(\omega-\omega_0)/\gamma\}|_{-\infty}^{\infty} = 1$$

Total emitted energy = initial potential energy of the oscillator:

$$W = \int_0^{\infty} \frac{dW}{d\omega} d\omega = \frac{1}{2} k x_0^2 = \frac{1}{2} m\omega_0^2 x_0^2$$

Profile of the emitted spectrum:

$$\boxed{\phi(\omega) = \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}}$$

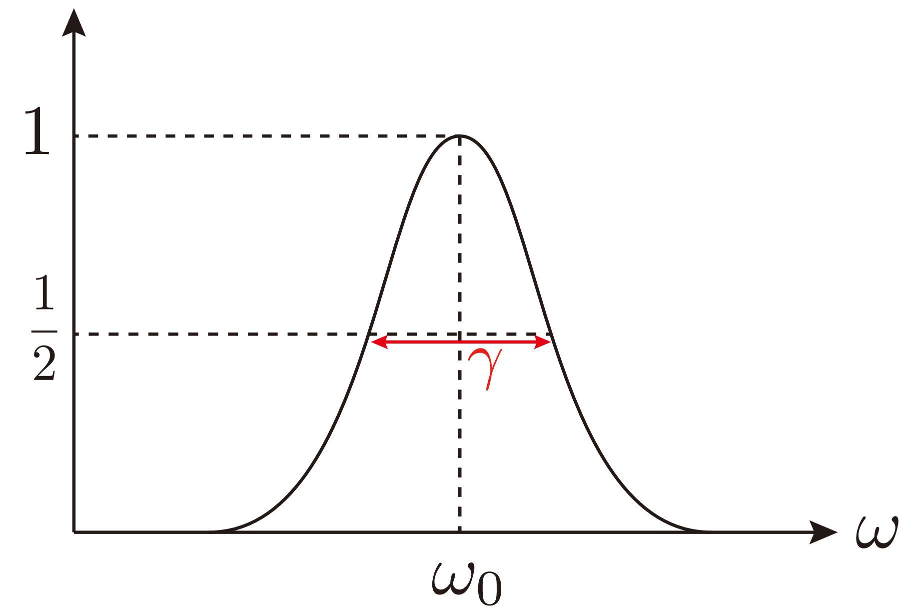
This is the Lorentz (natural) line profile for the spontaneous emission.

Damping constant is the full width at half maximum (FWHM).

$$\phi(\omega) = \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

$$\boxed{\phi(\nu) = \frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2}}$$

Note: $\phi(\omega)d\omega = \phi(\nu)d\nu$



The line width $\Delta\omega = \gamma$ is a universal constant when expressed in terms of wavelength:

$$\lambda = \frac{2\pi c}{\omega}$$

$$\Delta\lambda = 2\pi c\frac{\Delta\omega}{\omega^2} = 2\pi c\frac{2}{3}\frac{r_e}{c} \longleftarrow \left(\Delta\omega = \gamma = \frac{2}{3}r_e\frac{\omega_0^2}{c}\right)$$

$$= \frac{4}{3}\pi r_e$$

$$= 1.2\times 10^{-4}\text{Å}$$

However, in Quantum Mechanics, the line width is not a universal constant.

[2] Scattering: Driven Oscillator

- **Driven Harmonically Bound Particles** (forced oscillators)

Electron's equation of motion (electric charge = -e): $\mathbf{F}_{\rm ext} = -e\mathbf{E}_0 e^{i\omega t}$

$$\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x} = -\frac{e\mathbf{E}_0}{m}e^{i\omega t}$$

Steady-state solution of this equation:

Rybicki & Lightman use the following equation.

$$\ddot{\mathbf{x}} - (\gamma/\omega_0^2)\dddot{\mathbf{x}} + \omega_0^2\mathbf{x} = -\frac{e\mathbf{E}_0}{m}e^{i\omega t}$$

$$\mathbf{x} = \mathbf{x}_0 e^{i\omega t} \equiv |\mathbf{x}_0| e^{i(\omega t + \delta)} \quad \rightarrow \quad (-\omega^2 + i\omega\gamma + \omega_0^2)\mathbf{x}_0 e^{i\omega t} = -\frac{e\mathbf{E}_0}{m}e^{i\omega t}$$

$$\mathbf{x}_0 = \frac{(e/m)\mathbf{E}_0}{(\omega^2 - \omega_0^2) - i\omega\gamma}$$

$$\mathbf{x}_0 = |\mathbf{x}_0| e^{i\delta} \propto (\omega^2 - \omega_0^2) + i\omega\gamma \quad \rightarrow \quad \delta = \tan^{-1}\left(\frac{\omega\gamma}{\omega^2 - \omega_0^2}\right)$$

The response is slightly out of phase with respect to the imposed field.

For $\omega > \omega_0$, the particle "leads" the driving force and for $\omega < \omega_0$, it "lags."

Time-averaged total power radiated:

$$P = \left\langle \frac{dW}{dt} \right\rangle = \frac{2e^2 \left\langle |\ddot{\mathbf{x}}|^2 \right\rangle}{3c^3} = \frac{e^2\omega^4|\mathbf{x}_0|^2}{3c^3}$$

$$= \frac{e^4 E_0^2}{3m^2c^3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}$$

- **Scattering cross section:**

$$\sigma_{\rm sca} \equiv \frac{\langle P \rangle}{\langle S \rangle}, \quad \langle S \rangle = \frac{c}{8\pi} E_0^2 \quad \longrightarrow \quad \sigma_{\rm sca}(\omega) = \frac{8\pi e^4}{3m^2c^4} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}$$

$$= \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}$$

- **Limiting Cases of Interest**

(a) $\omega \gg \omega_0$ (**Thomson scattering** by free electron)

$$\boxed{\sigma_{\rm sca} = \sigma_T = \frac{8\pi}{3} r_e^2}$$

At high incident energies, the binding becomes negligible. Therefore, this corresponds to the case of a free electron.

(b) $\omega \ll \omega_0$ (**Rayleigh scattering** by bound electron)

This is the case where the electric filed appears nearly static and produces a nearly static force.

$$\boxed{\sigma_{\rm sca} = \sigma_T \left(\frac{\omega}{\omega_0} \right)^4 = \sigma_T \left(\frac{\lambda_0}{\lambda} \right)^4}$$

- Rayleigh scattering refers to the scattering of light by particles smaller than the wavelength of the light.
- (blue color of the sky at sunrise) The dependence results in the indirect blue light coming from all regions of the sky.
- (red color of the sun at sunset) Conversely, glancing toward the Sun, the colors that were not scattered away - the longer wavelengths such as red and yellow light - are directly visible, giving the Sun itself a slightly yellowish color.
- However, view from space, the sky is black and the Sun is white.

(c) $\omega \approx \omega_0$ (**Resonance scattering** of line radiation)

$$\sigma_{\rm sca}(\omega) \approx \sigma_T \frac{\omega_0^4}{(\omega-\omega_0)^2(2\omega_0)^2 + (\omega_0\gamma)^2}$$
$$= \sigma_T \frac{\omega_0^2/4}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

$$\sigma_T \frac{\omega_0^2}{4} = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \times \frac{1}{4} \times \left(\gamma \frac{3}{2}\frac{mc^3}{e^2\omega_0^2}\right) = 2\pi^2 \frac{e^2}{mc}(\gamma/2\pi) \longrightarrow$$

Note that $\nu = 2\pi\omega$ and $\sigma_{\rm sca}(\nu) = \sigma_{\rm scat}(\omega)/2\pi$

$$\boxed{\begin{aligned} \sigma_{\rm sca}(\omega) &= \frac{2\pi^2 e^2}{mc}\frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2} \\ \sigma_{\rm sca}(\nu) &= \frac{\pi e^2}{mc}\frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2} \end{aligned}}$$

In the neighborhood of the resonance, ***the shape of the scattering cross section is the same as the emission line profile from the free oscillator.***

Total scattering cross section:
$$\int_0^\infty \sigma(\omega)d\omega = \frac{2\pi^2 e^2}{mc}, \quad \boxed{\int_0^\infty \sigma(\nu)d\nu = \frac{\pi e^2}{mc}}$$

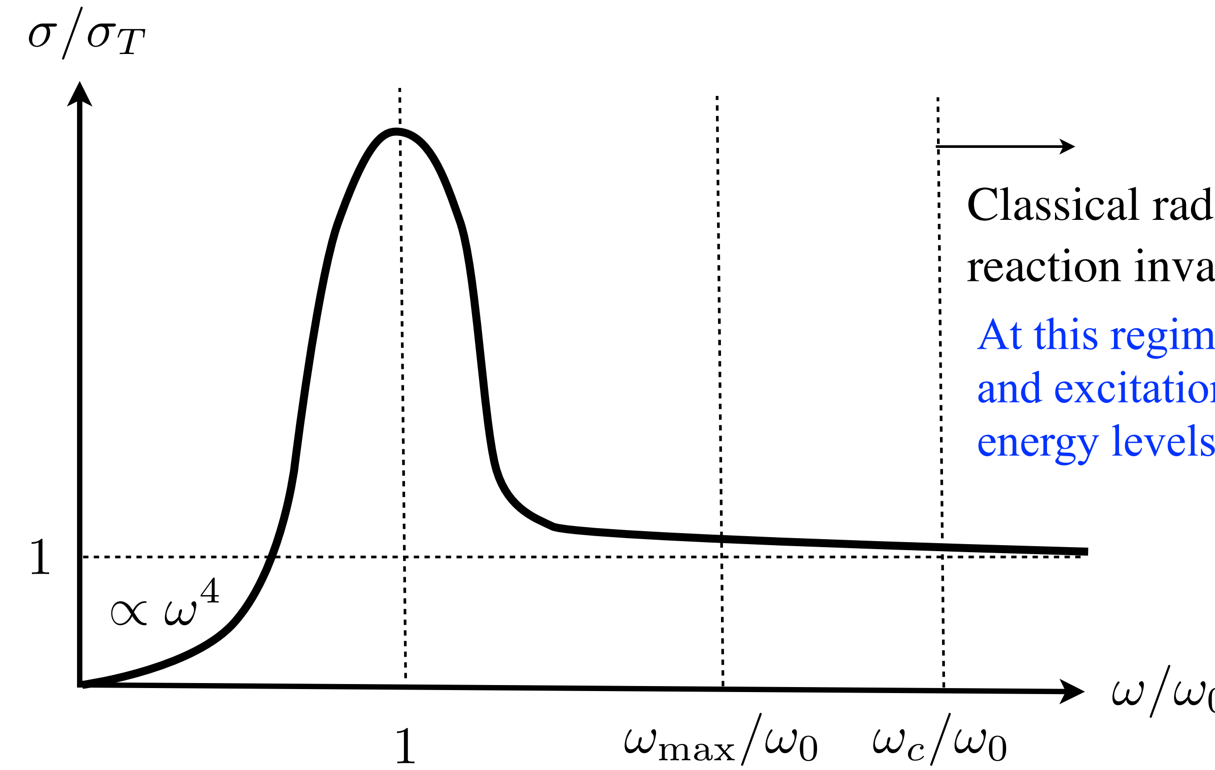
In evaluating this integral, we have apparently neglected a divergence, since the cross section approaches σ_T for large ω.

However, note that the approximate formula for radiation reaction is only valid for $\omega_0 \ll \omega_c$. Therefore, we must cut off the integral at a $\omega_{\rm max}$ such that $\omega_0 \ll \omega_{\rm max} \ll \omega_c$.

We also note that the contribution to the integral from the constant Thomson limit is less than

$$\int_0^{\omega_{\max}} \sigma_T d\omega = \sigma_T \omega_{\max} \ll \sigma_T \omega_c = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \times 3\pi\left(\frac{mc^3}{e^2}\right) = \frac{8\pi^2 e^2}{mc} \approx \int_0^{\infty} \sigma_{\text{sca}}(\omega) d\omega$$

The contribution is therefore negligible.



<u>In the quantum theory of spectral lines,</u>

we obtain similar formulas, which are

conveniently stated in terms of the classical results as

$$\boxed{\int_0^{\infty} \sigma(\nu) d\nu = \frac{\pi e^2}{mc} f_{nn'}}$$

where $f_{nn'}$ is called the ***<u>oscillator strength</u>*** or ***<u>f-value</u>*** for the transition between states *n* and *n'*.

Resonance Line

- Resonance Line

  A spectral line caused by an electron jumping **between the ground state and the first energy level** in an atom or ion. It is the longest wavelength line produced by a jump to or from the ground state.

  Because ***the majority of electrons are in the ground state in many astrophysical environments***, and because the energy required to reach the first level is the least needed for any transition, resonance lines are usually the strongest lines in the spectrum for any given atom or ion.

Resonance Lines

Draine, Physics of the interstellar and intergalactic medium

Table 9.4 Selected Resonance Lines[a] with $\lambda < 3000$ Å

	Configurations	ℓ	u	$E_\ell/hc(\,\mathrm{cm}^{-1})$	$\lambda_{\mathrm{vac}}(\,\mathrm{Å})$	$f_{\ell u}$
C IV	$1s^22s-1s^22p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1550.772	0.0962
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1548.202	0.190
N V	$1s^22s-1s^22p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1242.804	0.0780
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1242.821	0.156
O VI	$1s^22s-1s^22p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1037.613	0.066
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1037.921	0.133
C III	$2s^2-2s2p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	977.02	0.7586
C II	$2s^22p-2s2p^2$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{D}^o_{3/2}$	0	1334.532	0.127
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{D}^o_{5/2}$	63.42	1335.708	0.114
N III	$2s^22p-2s2p^2$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{D}^o_{3/2}$	0	989.790	0.123
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{D}^o_{5/2}$	174.4	991.577	0.110
C I	$2s^22p^2-2s^22p3s$	$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	0	1656.928	0.140
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	16.40	1656.267	0.0588
		$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	43.40	1657.008	0.104
N II	$2s^22p^2-2s2p^3$	$^3\mathrm{P}_0$	$^3\mathrm{D}^o_1$	0	1083.990	0.115
		$^3\mathrm{P}_1$	$^3\mathrm{D}^o_2$	48.7	1084.580	0.0861
		$^3\mathrm{P}_2$	$^3\mathrm{D}^o_3$	130.8	1085.701	0.0957
N I	$2s^22p^3-2s^22p^23s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1199.550	0.130
		$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{3/2}$	0	1200.223	0.0862
O I	$2s^22p^4-2s^22p^33s$	$^3\mathrm{P}_2$	$^3\mathrm{S}^o_1$	0	1302.168	0.0520
		$^3\mathrm{P}_1$	$^3\mathrm{S}^o_1$	158.265	1304.858	0.0518
		$^3\mathrm{P}_0$	$^3\mathrm{S}^o_1$	226.977	1306.029	0.0519
Mg II	$2p^63s-2p^63p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	2803.531	0.303
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	2796.352	0.608
Al III	$2p^63s-2p^63p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{1/2}$	0	1862.790	0.277
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^o_{3/2}$	0	1854.716	0.557

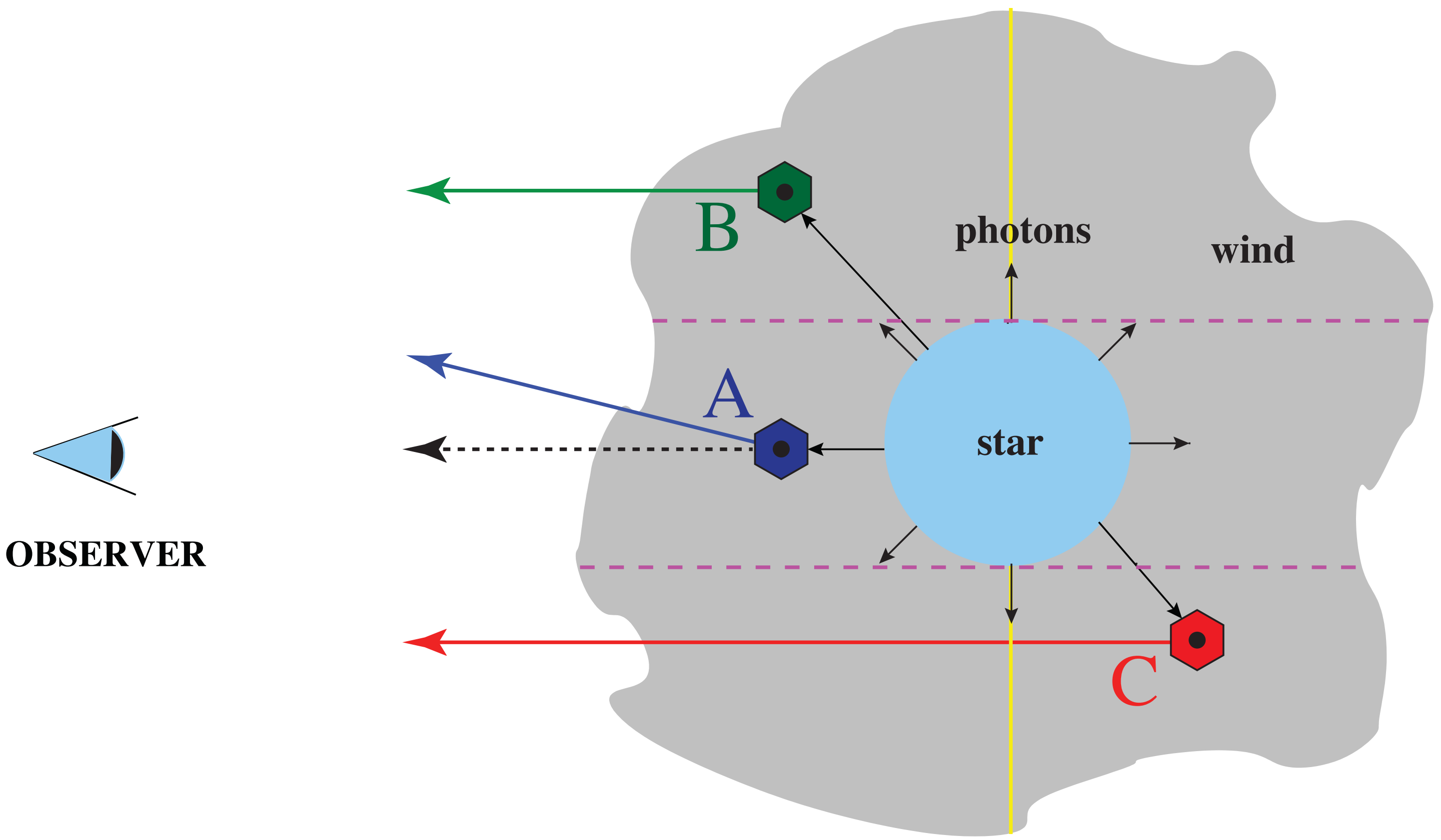
Table 9.4 contd.

	Configurations	ℓ	u	$E_\ell/hc\,(\,\mathrm{cm}^{-1})$	$\lambda_{\mathrm{vac}}(\,\mathrm{Å})$	$f_{\ell u}$
Mg I	$2p^63s^2-2p^63s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	2852.964	1.80
Al II	$2p^63s^2-2p^63s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	1670.787	1.83
Si III	$2p^63s^2-2p^63s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	1206.51	1.67
P IV	$2p^63s^2-2p^63s3p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^o_1$	0	950.655	1.60
Si II	$3s^23p-3s^24s$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{S}_{1/2}$	0	1526.72	0.133
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{S}_{1/2}$	287.24	1533.45	0.133
P III	$3s^23p-3s3p^2$	$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{D}_{3/2}$	0	1334.808	0.029
		$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{D}_{5/2}$	559.14	1344.327	0.026
Si I	$3s^23p^2-3s^23p4s$	$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	0	2515.08	0.17
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	77.115	2507.652	0.0732
		$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	223.157	2516.870	0.115
P II	$3s^23p^2-3s3p^3$	$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	0	1301.87	0.038
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	164.9	1305.48	0.016
		$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	469.12	1310.70	0.115
S III	$3s^23p^2-3s3p^3$	$^3\mathrm{P}_0$	$^3\mathrm{D}^o_1$	0	1190.206	0.61
		$^3\mathrm{P}_1$	$^3\mathrm{D}^o_2$	298.69	1194.061	0.46
		$^3\mathrm{P}_2$	$^3\mathrm{D}^o_3$	833.08	1200.07	0.51
Cl IV	$3s^23p^2-3s3p^3$	$^3\mathrm{P}_0$	$^3\mathrm{D}^o_1$	0	973.21	0.55
		$^3\mathrm{P}_1$	$^3\mathrm{D}^o_2$	492.0	977.56	0.41
		$^3\mathrm{P}_2$	$^3\mathrm{D}^o_3$	1341.9	984.95	0.47
P I	$3s^23p^3-3s^23p^24s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1774.951	0.154
S II	$3s^23p^3-3s^23p^24s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1259.518	0.12
Cl III	$3s^23p^3-3s^23p^24s$	$^4\mathrm{S}^o_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1015.019	0.58
S I	$3s^23p^4-3s^23p^34s$	$^3\mathrm{P}_2$	$^3\mathrm{S}^o_1$	0	1807.311	0.11
		$^3\mathrm{P}_1$	$^3\mathrm{S}^o_1$	396.055	1820.343	0.11
		$^3\mathrm{P}_0$	$^3\mathrm{S}^o_1$	573.640	1826.245	0.11
Cl II	$3s^23p^4-3s3p^5$	$^3\mathrm{P}_2$	$^3\mathrm{P}^o_2$	0	1071.036	0.014
		$^3\mathrm{P}_1$	$^3\mathrm{P}^o_2$	696.00	1079.080	0.00793
		$^3\mathrm{P}_0$	$^3\mathrm{P}^o_1$	996.47	1075.230	0.019
Cl I	$3s^23p^5-3s^23p^44s$	$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{P}_{3/2}$	0	1347.240	0.114
		$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{P}_{3/2}$	882.352	1351.657	0.0885
Ar II	$3s^23p^5-3s3p^6$	$^2\mathrm{P}^o_{3/2}$	$^2\mathrm{S}_{1/2}$	0	919.781	0.0089
		$^2\mathrm{P}^o_{1/2}$	$^2\mathrm{S}_{1/2}$	1431.583	932.054	0.0087
Ar I	$3p^6-3p^54s$	$^1\mathrm{S}_0$	$^2[1/2]^o$	0	1048.220	0.25

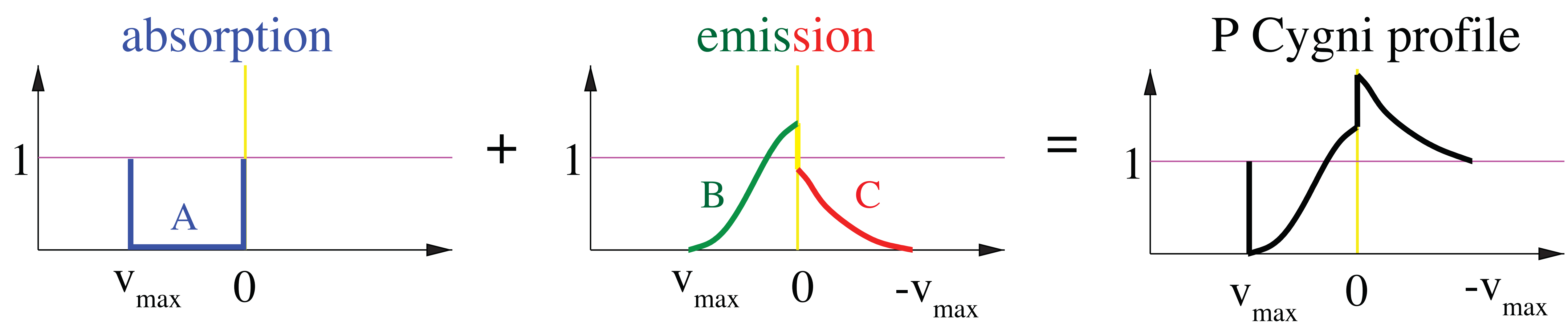
[a] Transition data from NIST Atomic Spectra Database v4.0.0 (Ralchenko et al. 2010)

P Cygni profile formation

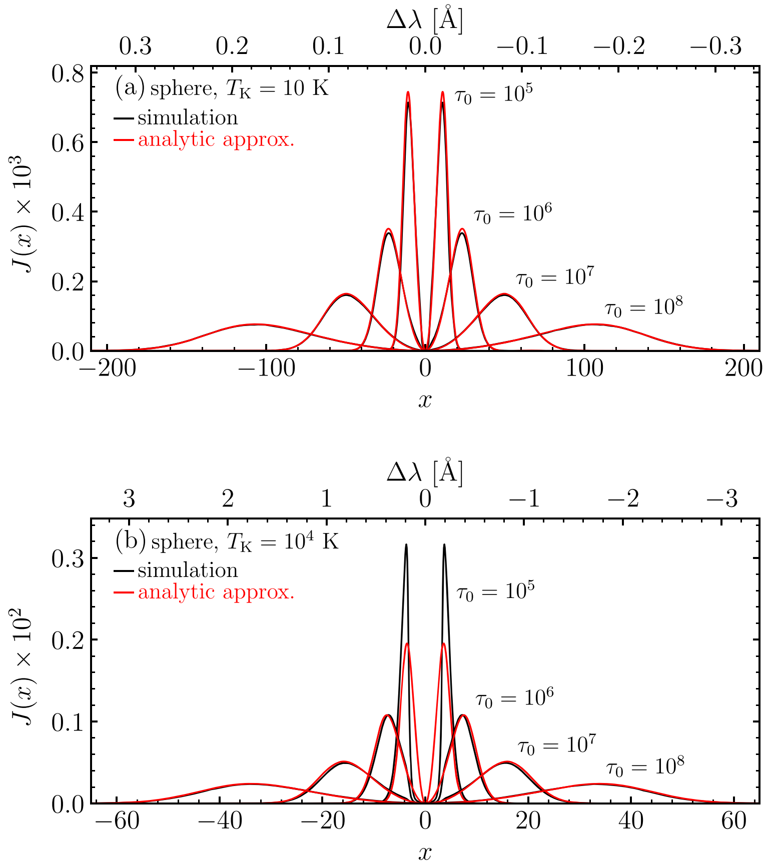
- The blueshifted absorption line is produced by material moving away from the star and toward us, whereas the emission come from other parts of the expanding shell.



credit: Joachim Puls
slightly modified

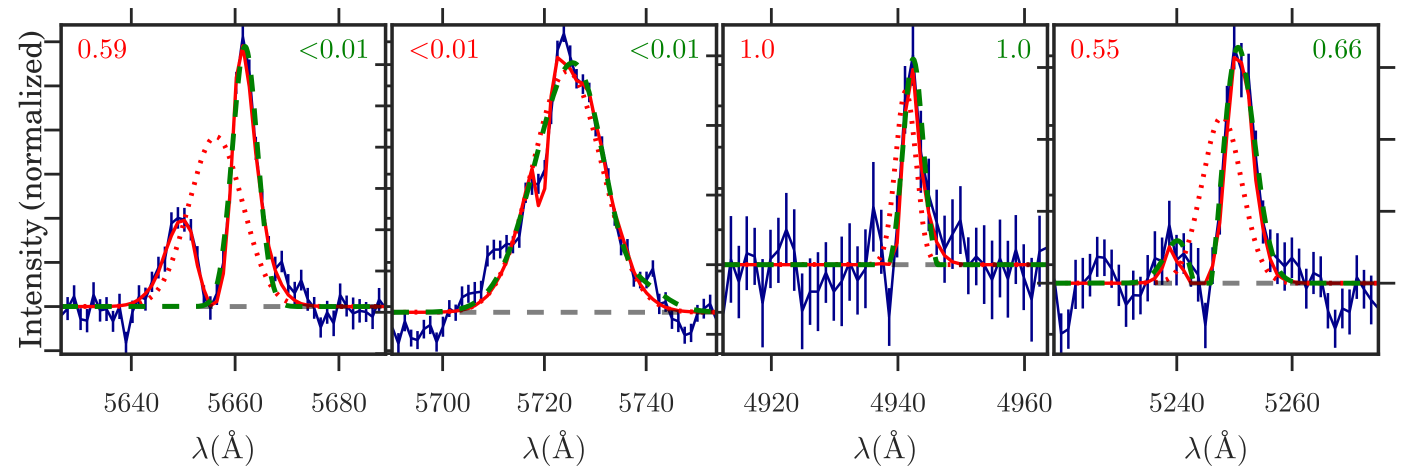


Lyα Resonance Scattering



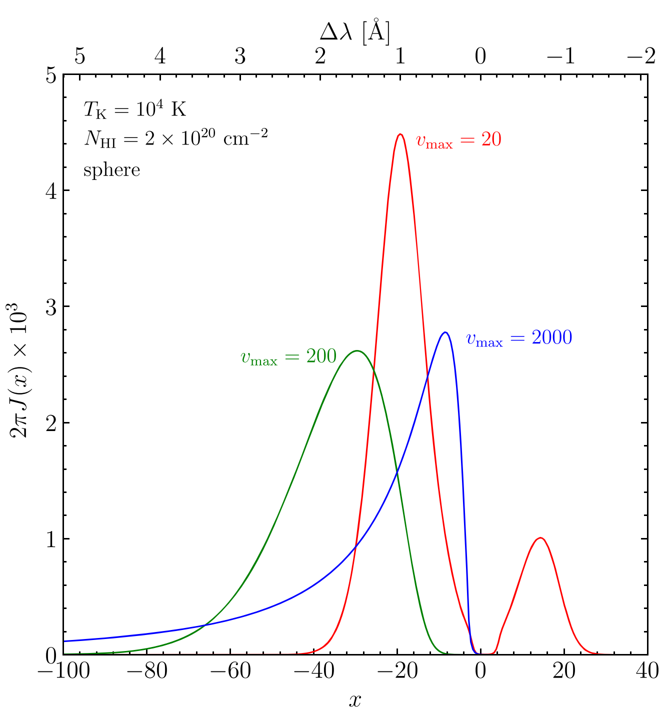
Seon & Kim (2020)

Emergent Lyα spectra from a static, homogeneous sphere at (a) $T = 10$ K and (b) 10^4 K, with different optical depths ($\tau_0 = 10^5 - 10^8$). The black curves are line profiles calculated with LaRT. The red curves denote an analytic series solution, which was derived from the series solution of Dijkstra et al. (2006).



Example spectral fits of the "shell" and "Gaussian-minus-Gaussian" models with all combinations of fitting failures/successes. The blue lines show the data, the red solid line the shell model fit (with the intrinsic spectrum as dotted red line), and the green dashed line shows the "Gaussian-minus-Gaussian" best fit. The numbers in the panel show the $p(\chi^2)$ values of the best fits in the corresponding colors.

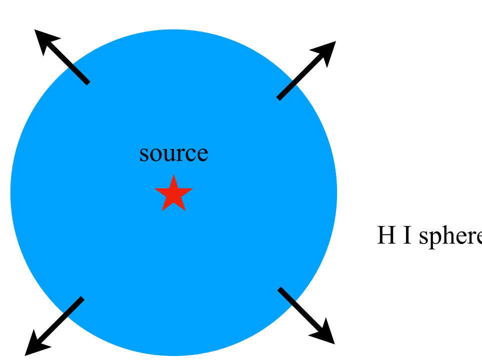
Gronke (2017)



Emergent Lyα for the dynamic motion test cases, in which the gas expands isotropically and has a temperature of $T = 10^4$ K and a column density of $N_{HI} = 2 \times 10^{20}$ cm^{-2}. The maximum velocity V_{max} of the Hubble- like outflow is denoted in units of km s^{-1}. The ordinate is the mean intensity integrated over the solid angle outgoing from the spherical surface.

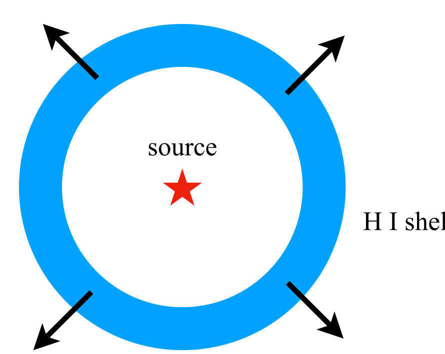
Seon & Kim (2020)

Hubble-like outflow



$$V = V_{max} \frac{r}{r_{max}}$$

spherically expanding shell model



$$V = V_{shell} \text{ (constant)}$$

Homework (due date: 10/23)

Solve the Problem 3.3 in the textbook.

3.3—Two oscillating dipole moments (radio antennas) $\mathbf{d}_1$ and $\mathbf{d}_2$ are oriented in the vertical direction and are a horizontal distance L apart. They oscillate in phase at the same frequency ω. Consider radiation at angle θ with respect to the vertical and in the vertical plane containing the two dipoles.

a. Show that

$$\frac{dP}{d\Omega} = \frac{\omega^4 \sin^2\theta}{8\pi c^3}\left(d_1^2 + 2d_1 d_2 \cos\delta + d_2^2\right),$$

where

$$\delta \equiv \frac{\omega L \sin\theta}{c}.$$

b. Thus show directly that when $L \ll \lambda$, the radiation is the same as from a single oscillating dipole of amplitude $d_1 + d_2$.

Hint (page 329 in the textbook)

3.3

a. Use Eq. (3.15a) with $q\dot{u} = -\omega^2 d\cos\omega t$ for each dipole, noting that the retarded times for each differ by $\Delta t = (L/c)\sin\theta$ (see Fig. S.3). Then

$$|\mathbf{E}_{\rm rad}| = -\frac{\omega^2}{rc^2}\left[d_1\cos\omega t + d_2\cos\omega(t-\Delta t)\right]\sin\theta$$

$$= -\frac{\omega^2}{rc^2}\left[(d_1 + d_2\cos\delta)\cos\omega t + d_2\sin\delta\sin\omega t\right]\sin\theta,$$

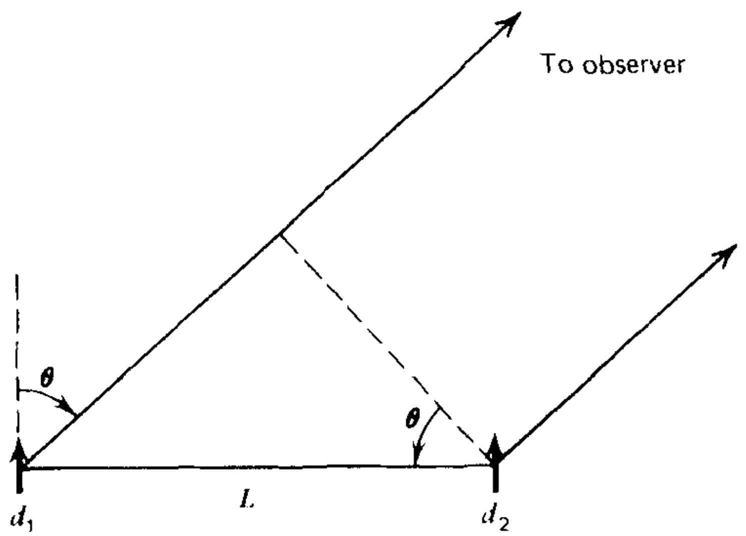


Figure S.3 Geometry for emission from two dipole radiators separated by distance L.

where $\delta = \omega\Delta t = \omega L\sin\theta/c$. Squaring and averaging over time, we find

$$\langle |E_{\rm rad}|^2\rangle = \frac{\omega^4\sin^2\theta}{2r^2c^4}\left[(d_1 + d_2\cos\delta)^2 + (d_2\sin\delta)^2\right]$$

$$= \frac{\omega^4\sin^2\theta}{2r^2c^4}\left(d_1^2 + 2d_1d_2\cos\delta + d_2^2\right).$$

We have finally,

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{cr^2}{4\pi}\langle |E_{\rm rad}|^2\rangle$$

$$= \frac{\omega^4\sin^2\theta}{8\pi c^3}\left(d_1^2 + 2d_1d_2\cos\delta + d_2^2\right).$$

b. When $L \ll \lambda$, we have $\delta \equiv 2\pi L\sin\theta/\lambda \ll 1$, and

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{\omega^4\sin^2\theta}{8\pi c^3}(d_1 + d_2)^2,$$

which is the radiation from an oscillating charge with dipole moment $d_1 + d_2$.