Properties of the most diversified portfolio

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This paper expands upon "Toward Maximum Diversification", a 2008 paper by Choueifaty and Coignard (Journal of Portfolio Management 35(1), 40–51). We present new mathematical properties of the diversification ratio and most diversified portfolio (MDP), and investigate the optimality of the MDP in a mean-variance framework. We also introduce a set of "portfolio invariance properties", providing the basic rules an unbiased portfolio construction process should respect. In the light of these rules, the MDP is then compared with popular methodologies (equal weights, equal risk contribution, minimum variance), and their performance is investigated over the past decade, using the MSCI World as the reference universe. We believe that the results obtained in this paper show that the MDP is a strong candidate for being the undiversifiable portfolio, and as such delivers investors the full benefit of the equity premium.

1 INTRODUCTION

Ever since its introduction in the 1960s, the capital asset pricing model (CAPM) has come under intense scrutiny. In particular, the efficiency of the market capitalization weighted index has been questioned, with academics and practitioners offering numerous investment alternatives. In a seminal paper, Haugen and Baker (1991) concisely proclaimed that "matching the market is an inefficient investment strategy". They

argued that theory can still predict cap-weighted portfolios to be inefficient invest-
ments, even assuming that investors rationally optimize the relationship between risk
and expected return in equilibrium, in an "informationally efficient" capital market.
Putting theory into practice, Haugen and Baker presented one of the first empirical
studies of the minimum-variance portfolio. Over the period 1972–89, this portfolio
delivered returns equal to or greater than those of a broad market cap-weighted index
of US stocks, while achieving consistently lower volatility, thus demonstrating the
ex post inefficiency of the market cap-weighted index. Nearly fifteen years later,
Arnott *et al* (2005) created indexes with alternative measures of company size based
on fundamental metrics. They showed that such indexes were "more mean-variance
efficient" than market cap-weighted indexes, further challenging the CAPM. Subse-
quently, Choueifaty (2006) introduced the concept of maximum diversification, via a
formal definition of portfolio diversification: the diversification ratio (DR). Choueifaty
further went on to describe the portfolio which maximizes the DR – the most diver-
sified portfolio (MDP) – as an efficient alternative to the market cap-weighted index.

This paper expands upon Choueifaty and Coignard (2008), which introduced the
concepts of DR and MDP to a broad audience. First, we explore the mathematical
properties of the DR. We also establish a new equivalent definition of the long-only
MDP, and generalize the condition for the optimality of the MDP in a mean-variance
framework. Next, we compare the MDP with three well-known long-only quantitative
portfolio construction approaches: the equal-weighted (EW), equal-risk-contribution
(ERC) (Maillard *et al* 2010) and minimum-variance (MV) portfolios (Haugen and
Baker 1991; Clarke *et al* 2006). We introduce a set of basic invariance properties that
an unbiased portfolio construction process should respect, and then examine each
approach in light of these properties, using synthetic examples. Finally, using one of
the broadest equity universes available – the MSCI World – we study the empirical
performance of the four portfolios over the past decade.

2 PROPERTIES OF THE DIVERSIFICATION RATIO

2.1 Definition

Numerous definitions of diversifications have been proposed since the seminal work
of Markowitz (1952). Choueifaty (2006) proposed an actual measure of portfolio
diversification, called the DR, which he defined as the ratio of the portfolio's weighted
average volatility to its overall volatility. This measure embodies the very nature of
diversification, whereby the volatility of a long-only portfolio of assets is less than or
equal to the weighted sum of the assets' volatilities. As such, the DR of a long-only
portfolio is greater than or equal to 1, and equals unity for a single-asset portfolio.
Consider, for example, an equal-weighted portfolio of two independent assets with

the same volatility: its DR is equal to $\sqrt{2}$, and to $\sqrt{N}$ for N independent assets.[1] In essence, the DR of a portfolio measures the diversification gained from holding assets that are not perfectly correlated. We formalize this intuition by introducing a formal definition as well as establishing several properties of the DR. Note that all portfolios in this paper are constrained to be long-only, unless otherwise stated.[2]

We consider a universe of N risky single assets $\{S_1, \ldots, S_N\}$, with volatility $\boldsymbol{\sigma} = (\sigma_i)$, correlation matrix $C = (\rho_{i,j})$ and covariance matrix $\Sigma = (\rho_{i,j}\sigma_i\sigma_j)$, with $1 \leqslant i, j \leqslant N$. Let $\boldsymbol{w} = (w_i)$ be the weights of a long-only portfolio, $\boldsymbol{\sigma}(\boldsymbol{w})$ its volatility, and $\langle \boldsymbol{w} \mid \boldsymbol{\sigma} \rangle = \sum_i w_i \sigma_i$ its average volatility. The diversification ratio $\mathrm{DR}(\boldsymbol{w})$ of a portfolio is defined as the ratio of its weighted average volatility and its volatility,

$$\mathrm{DR}(\boldsymbol{w}) = \frac{\langle \boldsymbol{w} \mid \boldsymbol{\sigma} \rangle}{\sigma(\boldsymbol{w})}. \tag{2.1}$$

2.2 DR decomposition

It is intuitive that portfolios with "concentrated" weights and/or highly correlated holdings would be poorly diversified, and hence be characterized by relatively low DRs. Here we formalize this intuition by decomposing the DR of a portfolio into its weighted-correlation and weighted-concentration measures. As shown in Appendix A, the DR decomposition is

$$\mathrm{DR}(\boldsymbol{w}) = [\rho(\boldsymbol{w})(1 - \mathrm{CR}(\boldsymbol{w})) + \mathrm{CR}(\boldsymbol{w})]^{-1/2}, \tag{2.2}$$

where $\rho(\boldsymbol{w})$ is the volatility-weighted average correlation of the assets in the portfolio,

$$\rho(\boldsymbol{w}) = \frac{\sum_{i \neq j}(w_i\sigma_i w_j\sigma_j)\rho_{i,j}}{\sum_{i \neq j}(w_i\sigma_i w_j\sigma_j)}, \tag{2.3}$$

and $\mathrm{CR}(\boldsymbol{w})$ is the volatility-weighted concentration ratio (CR) of the portfolio, with

$$\mathrm{CR}(\boldsymbol{w}) = \frac{\sum_i (w_i\sigma_i)^2}{(\sum_i w_i\sigma_i)^2}. \tag{2.4}$$

A fully concentrated long-only portfolio has unit CR (a one asset portfolio), while an equal volatility weighted portfolio has the lowest CR, equal to the inverse of the

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[1] In effect, the average volatility of the assets is equal to their common volatility, and the volatility of the portfolio equals their common volatility divided by the square root of the number of assets. We refer the reader to the "Definition of the diversification ratio" section of Choueifaty and Coignard (2008), for more examples.

[2] Definitions are provided accordingly.

number of assets it contains.[3] The CR introduces a generalization of the Herfindahl–Hischman index (Rhoades 1993; Bouchaud and Potters 2009, p. 205) used, for example, by US authorities as a sector concentration measure. In effect, the CR measures not only the concentration of weights, but also the concentration of risks; assets are weighted proportionally to their volatilities.

The above DR decomposition explicitly shows that the DR increases when the average correlation and/or the CR decreases. In the extreme, if correlations increase to unity, the DR is equal to 1, regardless of the value of the CR, as portfolios of assets are no more diversified than a single asset. We note that when pairwise asset correlations are equal, the DR varies only via the CR, and maximizing the DR is equivalent to minimizing the CR.

2.3 The DR as a measure of degrees of freedom

We provide an intuitive interpretation of the DR, by first considering a universe of F independent risk factors, and a portfolio whose exposure to each risk factor is inversely proportional to the factor's volatility. Such a portfolio allocates its risk budget equally across all risk factors, and will have a DR squared (DR^2) equal to F.[4] As such, its DR^2 is equal to the number of independent risk factors, or degrees of freedom, represented in the portfolio. Therefore, the DR^2 of any portfolio of assets can be interpreted as the number F of independent risk factors, necessary for a portfolio that allocates equal risk to independent risk factors, to achieve the same DR. As such, F can be interpreted as the effective number of independent risk factors, or degrees of freedom, represented in the portfolio.

For example, the DR of an index, such as the MSCI World, was 1.7 as of the end of 2010, implying that a passive MSCI World investor would have been effectively exposed to $1.7^2 \approx 3$ independent risk factors, in our interpretation. Taking this a step further, if we seek to maximize the DR, the resulting DR would equal the square root of the effective number of independent risk factors available in the entire market. At the end of 2010, this resulted in a DR of 2.6, or $2.6^2 \approx 7$ effective degrees of freedom. An interpretation of this result is that the market cap-weighted index misses out on the opportunity to effectively diversify across about four additional independent risk factors.

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[3] The Herfindal index attains its minimum value for an equal-weighted portfolio. In our case, it suffices to rescale the portfolio weights by their associated volatilities to obtain this result.

[4] In effect, noting c the proportionality constant between the weights of the portfolio and the inverse of the volatilities, the numerator of the DR equals c times F, while its denominator equals c times the square root of F.

3 THE MOST DIVERSIFIED PORTFOLIO

3.1 Definition

The attractive properties of the DR naturally lead to the construction of maximally diversified long-only portfolios, defined as

$$\boldsymbol{w}^{\mathrm{MDP}} = \arg \max_{\boldsymbol{w} \in \boldsymbol{\Pi}^+} \mathrm{DR}(\boldsymbol{w}), \tag{3.1}$$

where $\boldsymbol{\Pi}^+$ is the set of long-only portfolios with weights summing to 1.[5] As shown in Appendix B, the long-only MDP always exists and is unique when the covariance matrix Σ is definite. Furthermore, the portfolio's weights satisfy the first-order condition

$$\Sigma \boldsymbol{w}^{\mathrm{MDP}} = \frac{\sigma(\boldsymbol{w}^{\mathrm{MDP}})}{\mathrm{DR}(\boldsymbol{w}^{\mathrm{MDP}})} \boldsymbol{\sigma} + \boldsymbol{\lambda}, \tag{3.2}$$

where the nonnegative (dual) variables $\boldsymbol{\lambda}$ are such that $\min(\boldsymbol{\lambda}, \boldsymbol{w}^{\mathrm{MDP}}) = \boldsymbol{0}$.

3.2 The core properties of the MDP

An equivalent definition[6] of the MDP, which we call the core property of the MDP (1), provides a very intuitive understanding of its nature.

(1) Any stock not held by the MDP is more correlated to the MDP than any of the stocks that belong to it. Furthermore, all stocks belonging to the MDP have the same correlation to it.

This property illustrates that all assets in the universe considered are effectively represented in the MDP, even if the portfolio does not physically hold them. For example, an MDP portfolio constructed using S&P 500 stocks may hold approximately 50 stocks. That does not mean, however, that this portfolio is not diversified, as the 450 stocks it does not hold are more correlated to the MDP compared with the 50 stocks it actually holds. This is consistent with the notion that the MDP is the undiversifiable portfolio.

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[5] Note that this section treats the long-only constrained MDP. We refer the reader to the "Properties" section of Choueifaty and Coignard (2008) for results addressing the unconstrained (long–short) case. Also, the objective is finite providing there does not exist a zero volatility portfolio in $\boldsymbol{\Pi}^+$. In an equity context, that amounts to assume there does not exist a long-only, zero risk portfolio bearing a positive risk premium. This is of course the case when Σ is definite.

[6] We show in Appendix B that, when the covariance matrix is definite, the MDP is the only portfolio respecting this property, which uniquely defines the MDP. When this is not the case, all portfolios respecting this property have maximal diversification, and are fully correlated.

The core property of the MDP (1) is established in Appendix B with the help of the above first-order condition. It is also equivalent to the following alternative definition, which is more involved, and forms the basis of its proof. For this reason, we call it the core property of the MDP (2).

(2) The long-only MDP is the long-only portfolio such that the correlation between any other long-only portfolio and itself is greater than or equal to the ratio of their DRs.[7]

Equivalently, for any long-only portfolio with weights $\boldsymbol{w}$:

$$\rho_{\boldsymbol{w},\boldsymbol{w}^{\text{MDP}}} \geqslant \frac{\text{DR}(\boldsymbol{w})}{\text{DR}(\boldsymbol{w}^{\text{MDP}})}. \tag{3.3}$$

Accordingly, the more diversified a long-only portfolio is, the greater its correlation with the MDP. Note that when the covariance matrix Σ is not definite, all portfolios satisfying the core property (1) or (2), equivalently maximize the DR. As such, (3.3) also shows that all solutions are equivalent, as they have a correlation of one between themselves.

3.3 Optimality properties of the MDP in a mean-variance framework

We show in this section that, provided "risk is rewarded" – a central tenet of finance – maximizing the DR is equivalent to maximizing investors' ubiquitous mean-variance utility. In that case, the MDP is the optimal, equilibrium portfolio.[8]

Consider a homogeneous investment universe of single assets where we have no reason to believe, *ex ante*, that any single asset will reward risk more than another. In this universe, the *ex ante* Sharpe ratios of single assets are identical, and thus each asset's expected excess return (EER) is proportional to its volatility: risk is homogeneously rewarded. Assume that a risk-free asset is available, with rate r_f. Denoting by $r_1, \ldots, r_N$ the single assets' returns and by k a positive constant, single assets' EERs satisfy

$$E(r_i) - r_f = k\sigma_i. \tag{3.4}$$

As such, for any portfolio of single assets with weights $\boldsymbol{w}$ and return $r_{\boldsymbol{w}}$,

$$E(r_{\boldsymbol{w}}) - r_f = \sum_{i=1}^{N} w_i (E(r_i) - r_f) = k\langle \boldsymbol{w} \mid \boldsymbol{\sigma} \rangle.$$

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[7] A similar property was derived in Choueifaty and Coignard (2008) in a simpler unconstrained (long/short) setting.

[8] This ideal setting is of course far from reality. Note, however, that the assumptions entertained here are not prerequisites for the MDP's outperformance in other contexts, in particular in the real world.

Using the definition of the diversification ratio, we finally obtain

$$E(r_{\boldsymbol{w}}) - r_f = k\sigma(\boldsymbol{w})\text{DR}(\boldsymbol{w}). \qquad (3.5)$$

Equation (3.5) shows that portfolios' EERs are proportional to their volatilities multiplied by their DRs.[9] Dividing both sides of this equation by $\sigma(\boldsymbol{w})$ demonstrates that in this homogenous universe, maximizing the DR is equivalent to maximizing the Sharpe ratio.

Going a step further, assume that all CAPM assumptions hold as in Sharpe (1991), whose Nobel lecture includes a very clear, self-contained, exposé of the CAPM. One central assumption is that "all investors are in agreement concerning expected returns and (asset) covariances". When equilibrium prices are attained, both expected returns and covariances are determined in such a way that markets clear. Let us explore further the case where all investors also agree that single assets' EERs are proportional to their volatilities. In this setting, assets' EERs depend on volatilities and on the proportionality constant k (constant across assets).[10] As such, assuming that equilibrium prices are attained, both asset covariances and the constant k are determined in equilibrium. Provided that markets have cleared, we still obtain the security market line relationship.[11] Also, as a risk-free asset is available, the portfolio of risky assets held by all investors maximizes the Sharpe ratio,[12] which in this particular situation also maximizes the DR, as EERs are proportional to volatilities. As a result, this portfolio

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[9] Assuming that single assets' Sharpe ratios are constant clearly does not mean that all portfolios also have constant Sharpe ratios, as their Sharpe ratios are proportional to their DR, the value of which varies across portfolios. As such, there is no internal inconsistency as noted in Chow *et al* (2011), when assuming that single assets' EERs are proportional to their volatilities and not those of portfolios.

[10] Assuming that single assets' EERs are proportional to their volatilities does not mean that assets' EERs are fixed prior to equilibrium, as they depend on the value of k which will be determined in equilibrium. In effect, (3.7) shows that in equilibrium, k is equal to the Sharpe ratio of the MDP, multiplied by the constant correlation of all assets to the MDP.

[11] Using Sharpe's notation, adding the assumption that EERs are proportional to volatility imposes the additional requirement that investors' expectations are such that $E_i = k\sigma_i$. However, investors' first-order condition for portfolio optimality (2) in Sharpe's lecture is unchanged, as is its aggregation over all investors (3), which form the basis for the CAPM's pricing (5) in Sharpe's lecture. Further assuming that a risk-free asset is available leads to (3.4), which is the Security Market Line relationship we refer to in this paper. The requirement that $E_i = k\sigma_i$ naturally carries over to this last equation. It remains to be seen, however, whether equilibrium can be reached with such additional requirement. See also footnote 12.

[12] When a risk-free asset is available with unlimited lending/borrowing, maximizing the mean variance utility function gives the same portfolio of risky assets, compared with directly maximizing the Sharpe ratio. The risk tolerance of the investor then determines the proportion of cash held.

is the MDP, and the security market line relationship reads

$$E(r_i) - r_f = \rho_{i\,\mathrm{MDP}} \frac{\sigma_i}{\sigma_{\mathrm{MDP}}} (E(r_{\mathrm{MDP}}) - r_f). \qquad (3.6)$$

It is demonstrated in Appendix B that the correlation of any asset to the unconstrained MDP is the same. Noting ρ_{MDP} this correlation, we finally obtain the pricing equation:

$$E(r_i) - r_f = \rho_{\mathrm{MDP}} \frac{\sigma_i}{\sigma_{\mathrm{MDP}}} (E(r_{\mathrm{MDP}}) - r_f). \qquad (3.7)$$

Naturally, this last result is consistent with the initial hypothesis that assets' EERs are proportional to volatility. It also shows that in equilibrium,[13] the identical Sharpe ratio of single assets is equal to the Sharpe ratio of the equilibrium portfolio, the MDP, multiplied by the constant correlation of all assets to this portfolio. Importantly, it also demonstrates that we still have the original CAPM result that assets are rewarded in proportion to their systematic risk exposure, which in this setting corresponds to their exposure to the MDP.

4 COMPARISON OF QUANTITATIVE PORTFOLIOS

4.1 Portfolio invariance properties

We propose in this section a set of basic properties that an unbiased, agnostic portfolio construction process should respect, based on common sense and reasonable economic grounds. A starting point is the fact that portfolios resulting from these processes are highly dependent upon the structure of the universe of assets considered. As such, it may be reasonable to require an unbiased process to produce exactly the same portfolio when considering a universe equivalent to the original one. We formalize this idea in the following three portfolio invariance properties.

(1) *Duplication invariance:* consider a universe where an asset is duplicated (for example, due to multiple listings of the same asset). An unbiased portfolio construction process should produce the same portfolio, regardless of whether the asset was duplicated.

(2) *Leverage invariance:* imagine that a company chooses to deleverage (leverage). All else being equal, the weights allocated by the portfolio to the company's underlying business(es) should not change, as its cash exposure is dealt with separately.

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[13] In this particular setting, any given market portfolio can be attained as the result of an equilibrium. It suffices, for example, that investors agree on zero expected correlations between assets, with expected volatilities being inversely proportional to the market portfolio's weights. In such a case, the market portfolio maximizes the Sharpe ratio, as well as the DR.

TABLE 1 Portfolio weights and risk contributions (%).

Portfolio	Weights		Risk contributions	
	A	*B*	*A*	*B*
EW	50	50	71	29
ERC	33	67	50	50
MV	—	100	—	100
MDP	33	67	50	50

(3) *Polico invariance:* the addition of a positive linear combination of assets already belonging to the universe (for example, the creation of a long-only leveraged exchange-traded fund on a subset of the universe) should not affect the portfolio's weights to the original assets, as they were already available in the original universe. We abbreviate "positive linear combination" to "polico".

4.2 Comparison of well-known quantitative approaches

Among the alternatives to cap-weighted indexes that have been proposed, we compare the EW portfolio, the MV portfolio, the ERC portfolio and the MDP.[14] These portfolios are related to cap-weighted indexes, insofar as they are all fully invested, unlevered, long-only, and provide comparable access to the equity risk premium. The MV portfolio, for example, minimizes volatility across all long-only portfolios, with weights summing to 1. We examine each of these portfolios in the context of the aforementioned portfolio invariance properties.

We consider a simple universe $\{A, B\}$ of two assets A and B, with volatilities $\sigma_A = 20\%$; $\sigma_B = 10\%$, respectively, and pairwise correlation $\rho_{AB} = 50\%$. The portfolio weights and risk contributions[15] for each of the above four approaches are given in Table 1.

By construction, the EW portfolio sees its largest risk contributions coming from the most volatile asset, whereas the MV invests 100% of its holdings in the low-risk asset.[16] Only the MDP and ERC portfolios provide a truly diversified risk allocation

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[14] We have omitted from this list the equal volatility weighted portfolio for the sake of brevity. Note that this portfolio is leverage invariant but neither duplication nor polico invariant, and is similar to the ERC in that respect.

[15] The risk contribution of an asset is defined here as the product of its portfolio weight and its marginal contribution to volatility, divided by the portfolio's overall volatility.

[16] The fact that the MV portfolio assigns a zero weight to asset A may come as a surprise, but there is no mistake here.

TABLE 2 Portfolio weights (%).

Portfolio	New weights			New weights to the original assets		Original weights		Compliant
	A	*A*	*B*	*A*	*B*	*A*	*B*	
EW	33	33	33	67	33	50	50	No
ERC	23	23	54	46	54	33	67	No
MV	—	—	100	—	100	—	100	Yes
MDP	17	17	67	33	67	33	67	Yes

in this case, as seen from their risk contributions. In the next three subsections, we examine whether these portfolio construction methodologies respect the portfolio invariance properties.

4.3 Duplication invariance

Consider a new universe derived from the first one, where asset A is duplicated: $\{A, A, B\}$. Each of the four portfolios assigns weights as shown in Table 2.

Both the MV and MDP are duplication invariant, as their weights in the original assets are unchanged. The duplication invariance of the MV and MDP is true in general.[17] However, both the EW and ERC are not invariant, which shows that they are biased toward assets with multiple representations.

4.4 Leverage invariance

Consider the new universe $\{LA, B\}$ following the replacement of A with LA, a combination of one-quarter A and three-quarters cash. This leads to the following figures: $\sigma_{LA} = 5\%, \sigma_B = 10\%$ and $\rho_{LA,B} = 50\%$, and to the corresponding portfolio weights given in Table 3 on the facing page

The MDP and ERC are leverage invariant. This is true in general, and is shown for the MDP in Appendix C. On the contrary, the EW and MV portfolio are not leverage invariant, as the former invests a smaller weight in asset A and the latter is now fully concentrated in asset A, and not B. This shows that both the MV and EW are biased with respect to assets' leverage.

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[17] Since the introduction of a redundant asset leads to a redundant equation in the first-order conditions associated to the MV and MDP programs.

TABLE 3 Leverage invariance portfolio weights (%).

	New weights		New weights to the original assets		Original weights		
Portfolio	LA	B	A	B	A	B	Compliant
EW	50	50	20	80	50	50	No
ERC	67	33	33	67	33	67	Yes
MV	100	—	100	—	—	100	No
MDP	67	33	33	67	33	67	Yes

TABLE 4 Polico portfolio weights (%).

	New weights			New weights to the original assets		Original weights		
Portfolio	A	B	Polico	A	B	A	B	Compliant
EW	33	33	33	55	45	50	50	No
ERC	21	46	34	41	59	33	67	No
MV	—	69	31	17	83	—	100	No
MDP	33	67	—	33	67	33	67	Yes

4.5 Polico invariance

To illustrate polico invariance, a polico[18] containing half A, one-quarter B and one-quarter cash is added to the new universe $\{A, B, \text{polico}\}$, leading to $\sigma_{\text{polico}} = 11.46\%$, $\rho_{A,\text{polico}} = 98.2\%$, $\rho_{B,\text{polico}} = 65.5\%$ and to the following portfolio weights given in Table 4.

The MDP is polico invariant, as it does not select the polico, and has unchanged overall weights. This general fact, demonstrated in Appendix C, shows that the MDP is robust to the misspecification of the nature of policos. In effect, the polico was treated in this example as a single asset, and not as a portfolio (its DR was assumed to equal 1). On the contrary, the EW, ERC and MV portfolios are not polico invariant, as the EW and ERC are biased toward assets with multiple representations, and the EW and MV are biased with respect to leverage. Note that in this situation the MV has positive weights on both A and B, due to the selection of the polico.

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[18] As defined earlier, a polico is a linear combination of assets (a leveraged long-only portfolio).

TABLE 5 Invariance properties.

Portfolio	Duplication	Leverage	Polico
EW	No	No	No
ERC	No	Yes	No
MV	Yes	No	No
MDP	Yes	Yes	Yes

4.6 Summary of results

To summarize, Table 5 describes the invariance properties respected by each portfolio.

The MDP's goal is to maximize diversification and, as such, to be unbiased. The fact that the MDP satisfies all three portfolio invariance properties is consistent with this goal. The other portfolios studied here make implicit bets. The EW and ERC portfolios reflect the belief that representativeness can only be achieved by investing in all stocks present in the universe. The EW and MV portfolios make implicit bets on companies' leverage.

5 EMPIRICAL STUDY

5.1 Setup

In this section, we compare the performance of five long-only portfolios: the market capitalization-weighted index (MKT), the ERC portfolio, the EW portfolio, the MDP and the MV portfolio.

Our investment universe for backtesting is the MSCI World,[19] which contains approximately 1500 stocks, spread across developed markets globally. The MSCI creates the well-known MSCI World minimum volatility index (MsMV). The MsMV would seem a natural candidate for our MV portfolio since it is also constructed using the MSCI World universe. Its construction methodology is well documented by the index provider,[20] with data available since the creation of the euro, ie, December 31, 1998. However, a careful reading of MSCI's minimum volatility methodology reveals a complex set of minimum and maximum weight, country, sector and turnover constraints, and also minimum and maximum exposure to various risk factors. As a result, the MsMV may not be representative of an MV portfolio; for this reason we have implemented a simpler version of MV, in addition to the MsMV.

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[19] See Choueifaty and Coignard (2008) for empirical results on the smaller, single-currency US and European universes.
[20] See www.mscibarra.com/.

The ERC, EW, MV and MDP portfolios were rebalanced semi-annually,[21] and stocks belonging to the MSCI index were selected at each rebalancing date. In order to avoid liquidity and price availability issues in such a broad universe, we only considered, at each rebalancing date, the top half of stocks by market capitalization[22] (793 stocks on average, representing 90% of the index capitalization). To allow for a fair comparison between our portfolios and MKT, we also built a synthetic market cap-weighted index labeled MKT/2, comprised of the top half of stocks ranked by market capitalization. For an appropriate comparison with the MsMV portfolio, we simply added a maximum weight, a regional constraint, as well as a turnover penalty to the MV and MDP construction.[23]

In order to use data reflecting as much recent information as possible, we estimated the covariance matrices for the ERC, MV and MDP using a one-year window of past daily returns,[24] at each rebalancing date. To account for the impact of time zone differences, we developed a "plesiochronous correlation estimator",[25] which allows for the joint estimation of asset correlations, while taking into account the time delay between observations.[26] As having fewer observations than the number of assets results in a nondefinite covariance matrix, we have also considered using a basic, yet robust, method consisting of shrinking half of the correlation matrix towards the

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[21] Portfolios are rebalanced at the end of May and November, as is the MsMV.

[22] At each rebalancing date, we eliminated all stocks with less than six months' price history, and selected the top half of the remaining stocks by market capitalization. Local currency total returns were converted to USD, according to the MSCI methodology, and MSCI's official forex data was used. Total returns and market capitalization were obtained through Bloomberg.

[23] Having in mind MSCI's methodology, we added a 1.5% maximum weight constraint, and a maximum weight by time zone (America, Europe, Asia), to ensure allocation to the zones do not exceed those of the MSCI World (MKT) by more than 5%. We also added a turnover reduction penalty to the MDP and MV objective functions, such that the annualized tracking error to the unpenalized problem was no greater than 1.5%. We did not add those constraints to the ERC portfolio, as they were generally satisfied in its unconstrained version.

[24] Having fewer observations than the number of assets results in a nondefinite covariance matrix. This was not an issue for the MDP and MV in the back tests presented here, as all portfolios contained fewer assets than observations (159 on average for the MV and 137 for the MDP), and were shown to be the unique solution of their optimization programs.

[25] Plesio means "near" in Greek, thus plesiochronous can be understood as "almost synchronous". We chose this term to represent the fact that even if the Japanese and US stock markets, for example, never trade simultaneously, their time delay is mostly constant.

[26] This estimator was developed in spirit of the work done by Hayashi and Yoshida (2005). See also Hoffmann *et al* (2009) for further references.

identity matrix.[27] Portfolios built using these positive symmetric definite matrixes are labeled ERC$_{PSD}$, MDP$_{PSD}$ and MV$_{PSD}$.

Finally, while it is straightforward to verify whether a portfolio has the ERC property, a direct implementation of the numerical optimization algorithms, as proposed in Maillard *et al* (2010), can require prohibitive computation time. For our purposes, we chose to implement the optimization problem (7) of their paper, which provides a unique, well-defined, long-only portfolio that respects the ERC property.[28]

5.2 Performance

The portfolios' empirical performance is shown in Figure 1 on the facing page and summarized in Table 6 on the facing page. All portfolios outperform MKT, which is consistent with the documented inefficiency of market cap-weighted indexes, even when assuming unrealistically high all-in trading costs of the order of 2%[29] to account for their higher turnover. The ERC, MV and MDP deliver significantly higher returns and lower volatility, whereas the EW outperforms the market cap-weighted index with comparable volatility. The ERC, in turn, functions as a risk-weighted version of the EW, with marginally higher returns and significantly lower risk. Among the portfolios with the lowest risk, the MsMV registers a modest performance advantage, with significantly less volatility than the cap-weighted index. Its MV counterpart, which has fewer constraints, has the lowest realized volatility, with returns similar in magnitude to the ERC portfolio.

Table 7 on page 16 provides performance for the ERC$_{PSD}$, MDP$_{PSD}$ and MV$_{PSD}$ portfolios. Overall returns and volatilities are mostly unchanged[30] compared with original versions of these portfolios. However, using the shrinkage method lessens turnover by 5 to 10%, with the MV and MDP portfolios holding 41 and 24 more

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[27] This method produces positive-definite matrices with eigenvalues greater than 0.5, and associated covariance matrices that are also positive-definite. We choose a high, constant, shrinkage intensity to ensure robustness. For references, see Ledoit and Wolf (2004).

[28] The solution is unique, provided of course that the covariance matrix is definite. We found that with a standard PC (Intel Xeon at 2.66 GHz with 8 Gb of RAM), those optimizations required less than a couple of seconds to converge, even when considering one thousand assets.

[29] Unrealistic all-in trading costs of 3.4% (respectively, 2.4%) would be needed for the MDP's higher turnover to be such that its outperformance relative to the market cap benchmark reduces to zero (respectively, for both the EW and ERC).

[30] This may come as a surprise to practitioners used to long–short portfolio optimizations and to observing drastically different (and improved) results. However, the MV and MDP portfolios considered in this paper are long-only, and contain fewer assets than observations. As such, they are much less sensitive to the estimation errors of the covariance matrix. Also, the long-only constraint has already an effect similar to using a robust estimation technique (see Jagannathan and Ma 2003).

FIGURE 1 Comparison of quantitative portfolio performances.
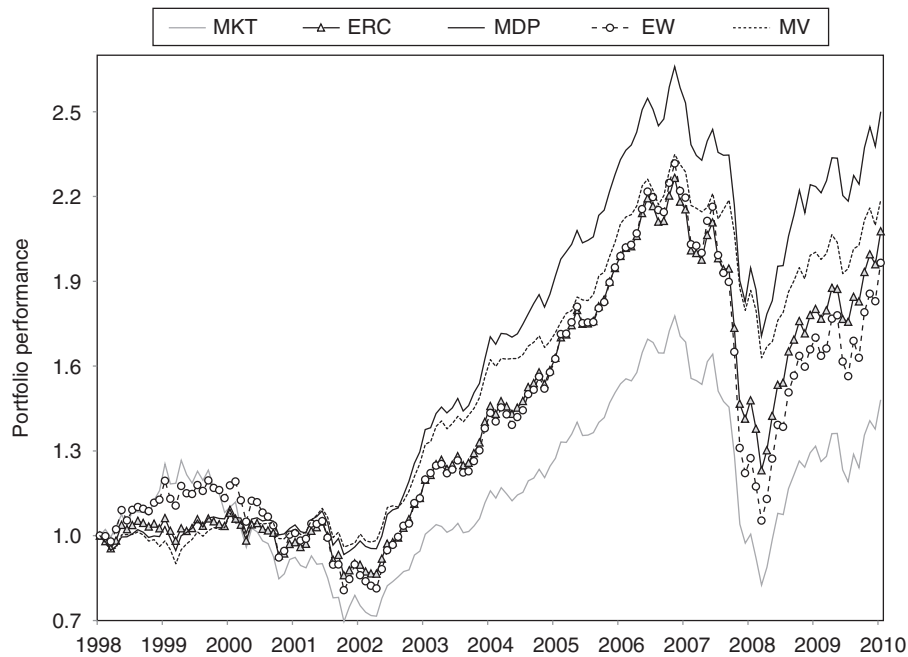
TABLE 6 Comparison of quantitative portfolio performances, 1999–2010.

Statistic	MKT	MKT/2	MMV	EW	ERC	MV	MDP
Return	3.1%	2.9%	4.2%	5.8%	6.3%	6.7%	7.9%
Volatility (monthly)	16.6%	16.3%	11.7%	16.7%	13.1%	10.0%	11.4%
Volatility (daily)	17.2%	17.2%	12.3%	16.4%	12.9%	10.0%	11.2%
Turnover (one way)	14%	11%	23%	29%	50%	76%	82%
Tracking Error (daily)	0.0%	0.8%	7.6%	3.6%	6.7%	10.4%	9.2%
DR (daily)	2.3	2.2	2.8	2.5	3.0	3.4	3.7
nbStocks (avg)	1586	793	250	793	793	159	137
Sharpe (monthly)	−0.00	−0.01	0.05	0.16	0.24	0.36	0.42
Sharpe (daily)	−0.00	−0.01	0.06	0.16	0.24	0.36	0.43

stocks, respectively. This can be expected, as shrunken correlation matrices are by design more stable over time, with the MV and MDP implicitly shrunk toward the equal-variance-weighted and equal-volatility-weighted portfolios.

TABLE 7 Performances using a robust correlation matrix estimation method, 1999–2010.

Statistic	ERC$_{PSD}$	MV$_{PSD}$	MDP$_{PSD}$
Return	6.2%	6.7%	7.8%
Volatility (monthly)	13.4%	10.2%	11.5%
Volatility (daily)	13.2%	10.2%	11.3%
Turnover (one way)	45%	66%	76%
Tracking error (daily)	6.2%	10.1%	9.1%
DR (daily)	2.9	3.2	3.6
nbStocks (avg)	793	200	161
Sharpe (monthly)	0.23	0.35	0.41
Sharpe (daily)	0.24	0.35	0.41

Unsurprisingly, the market cap-weighted portfolio has the lowest DR, given its high concentration in large cap stocks and risk factors.[31] The EW portfolio's diversified holdings result in slightly higher diversification, albeit less than the other portfolios, which use asset covariance information. The MDP both presents the highest DR – its primary objective – and also the highest Sharpe ratio. As such, it is the closest candidate to being the tangency portfolio. Overall, both the MV portfolio and the MDP come close to delivering on their respective claims: to minimize *ex post* volatility for the former, and to maximize *ex post* Sharpe ratio for the latter.

5.3 Fama–French regression

Table 8 on the facing page shows the results of a series of Fama and French (1993) three-factor regressions for each portfolio construction methodology. The factors are labeled MKT for the MSCI World Gross USD index in excess of the one-month LIBOR, HML for the performance difference between the MSCI World value and growth indexes, and SMB for the performance difference between the smallest 30% and the largest 30% of stocks by market capitalization. Month-end data was used, with excess returns computed using US one-month LIBOR. Alphas are reported using annualized compounded returns.

All nonmarket capitalization strategies[32] have positive SMB factor exposure and are thus less biased toward large capitalization stocks than the market cap-weighted index. Unsurprisingly, the EW has the largest exposure to SMB, both in terms of

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[31] For example, the financial sector weighted 24.4% of the MSCI EMU, representing 34.6% of its total risk, on average, over the year 2010.

[32] The results obtained for the ERC$_{PSD}$, MDP$_{PSD}$ and MV$_{PSD}$ are not reported, as they are extremely close to their original counterparts. The only noticeable change concerns the R^2 values of the regressions, which increase by a few percent.

TABLE 8 Fama–French monthly regression coefficients, 1999–2010.

Portfolio	MKT	SMB	HML	Alpha	R^2
EW	0.96	0.41	0.06	0.04%	99%
t-statistic	132.28	21.63	4.33	0.09	
ERC	0.71	0.41	0.14	0.14%	93%
t-statistic	40.60	8.99	3.99	0.20	
MMV	0.65	0.15	0.19	−0.56%	87%
t-statistic	29.72	2.66	4.29	0.83	
MV	0.46	0.23	0.23	1.35%	70%
t-statistic	16.32	3.15	4.02	0.83	
MDP	0.57	0.31	0.16	2.26%	80%
t-statistic	21.29	4.35	2.85	1.46	

slope coefficient and statistical significance, as well as the largest market exposure of the strategies. For portfolios using a risk matrix (ERC, MMV, MV and MDP), market exposures are substantially less than 1, with the lowest being MV, followed by the MDP. All of the strategies load positively on HML, with the MV showing the largest exposure, as measured by both factor loading and statistical significance, consistent with its bias toward low volatility (value) stocks. The MV and MDP exhibit the lowest R^2, revealing that the market cap-weighted index and the other two factors fail to explain a relatively large part of the performance of these two portfolios. Interestingly, the MsMV shows a negative Fama–French alpha, indicating that the numerous constraints placed on its construction may in fact be destructive of value. Finally, the MDP delivers the highest alpha of the five strategies tested, indicating that the performance of the MDP is significantly higher than what its Fama–French factor exposures would predict. This is consistent with the MDP's goal of delivering maximum diversification, and thus a balanced exposure to the effective risk factors available in the universe.

6 CONCLUSION

In this paper, we have introduced additional properties of the DR and of the MDP, and proposed a basic set of rules that an unbiased, agnostic portfolio construction process should respect: the portfolio invariance properties. We find that the MDP adheres to these rules. Furthermore, using the MSCI World index as a reference universe to compare the performance of the MDP with other approaches, we find that the MDP stands out, both in terms of relative performance and exposure to Fama–French factors.

Classical financial theory defines the equity risk premium as the return of the undiversifiable portfolio. In developing the MDP, our goal was to articulate a theory and a consistent construction methodology that deliver the full benefit of the equity risk premium to investors and their trustees, and we believe that our work shows that the MDP is a strong candidate for being the undiversifiable portfolio.

APPENDIX A

A.1 Notation

In the following, we denote by $\odot$ the element-wise product of two vectors and by $\oslash$ the element-wise division of two vectors.

A.2 Diversification ratio decomposition

Noting $\tilde{\boldsymbol{w}} = \boldsymbol{w} \odot \boldsymbol{\sigma}$, the variance of the portfolio with weights $\boldsymbol{w}$ can be written as

$$\sigma^2(\boldsymbol{w}) = \sum_i \tilde{w}_i^2 + 2\sum_{i<j} \tilde{w}_i \tilde{w}_j \rho_{i,j} = \sum_i \tilde{w}_i^2 + \rho(\boldsymbol{w}) \sum_{i \neq j} \tilde{w}_i \tilde{w}_j.$$

Noting that

$$\sum_{i \neq j} \tilde{w}_i \tilde{w}_j = \left(\sum_i \tilde{w}_i \right)^2 - \sum_i \tilde{w}_i^2$$

leads to

$$\sigma^2(\boldsymbol{w}) = (1 - \rho(\tilde{\boldsymbol{w}})) \sum_i \tilde{w}_i^2 + \rho(\boldsymbol{w}) \left(\sum_i \tilde{w}_i \right)^2.$$

Then, dividing this equality by $(\sum_i \tilde{w}_i)^2$ gives the decomposition

$$\frac{1}{\mathrm{DR}(\boldsymbol{w})^2} = (1 - \rho(\boldsymbol{w})) \, \mathrm{CR}(\boldsymbol{w}) + \rho(\boldsymbol{w}).$$

APPENDIX B

B.1 Most diversified portfolios' existence and uniqueness

The MDP optimization is a quadratic programming problem on a convex set: thanks to the fact that the DR is invariant by scalar multiplication, this is equivalent to $\min_{\boldsymbol{w}} \frac{1}{2} \boldsymbol{w}' \Sigma \boldsymbol{w}$, constrained by $w_i \geqslant 0$ and $\sum_i w_i \sigma_i = 1$, with weights rescaled to sum to 1 afterwards. The existence follows; uniqueness also follows if the covariance matrix is definite (see Berkovitz 2001, pp. 210–215).

B.2 Most diversified portfolios' first-order conditions

We first apply the Karush–Kuhn–Tucker (KKT) theorem: all admissible points qualify, according to the linear independence constraint qualification (equality and inequality are independent unless all the inequality constraints are active, which would mean that $w = 0$). The log of our positive objective function is

$$f(w) = \ln(\text{DR}(w)) = \ln\langle \sigma \mid w \rangle - \tfrac{1}{2} \ln\langle \Sigma w \mid w \rangle,$$

with

$$\nabla f_w = \frac{1}{\langle \sigma \mid w \rangle}\sigma - \frac{1}{2}\frac{1}{\langle \Sigma w \mid w \rangle} 2\Sigma w.$$

The KKT theorem states that at optimal points w^*, there exist a vector $v \in \mathbb{R}^N$ and a scalar μ such that

$$\frac{1}{\langle \sigma \mid w^* \rangle}\sigma - \frac{1}{\sigma(w^*)^2}\Sigma w^* + \mu\mathbf{1} + v = 0,$$

$$\min(v, w^*) = \mathbf{0},$$

$$\langle w^* \mid \mathbf{1} \rangle = 1.$$

Multiplying the first condition on the left by the transpose of w^*, shows that μ must be 0, and that the first condition is independent of the constraint that weights sum to 1. This does not come as a surprise, as the DR is invariant by scalar multiplication. Define $\lambda = \sigma^2(w^*)v$; an optimal point w^* is necessarily associated to a vector $\lambda \in \mathbb{R}^N$ satisfying

$$\Sigma w^* = \frac{\sigma(w^*)^2}{\langle \sigma \mid w^* \rangle}\sigma + \lambda,$$

$$\min(\lambda, w^*) = \mathbf{0},$$

$$\langle w^* \mid \mathbf{1} \rangle = 1.$$

B.3 The core property of the most diversified portfolio (2)

We first show that the MDP respects the core property (2). By definition, the correlation of the MDP to any other portfolio reads

$$\rho_{w,w^{\text{MDP}}} = \frac{\langle \Sigma w^{\text{MDP}} \mid w \rangle}{\sigma(w^{\text{MDP}})\sigma(w)}.$$

Given that

$$\Sigma w^{\mathrm{MDP}} = \delta \sigma + \lambda, \quad \text{with } \delta = \frac{\sigma(w^{\mathrm{MDP}})}{\mathrm{DR}(w^{\mathrm{MDP}})},$$

we have

$$\rho_{w,w^{\mathrm{MDP}}} = \frac{1}{\mathrm{DR}(w^{\mathrm{MDP}})\sigma(w)} \left\langle \sigma + \frac{1}{\delta}\lambda \,\middle|\, w \right\rangle.$$

Keeping in mind that λ is nonnegative, for all long-only portfolios with nonnegative weights w,

$$\rho_{w,w^{\mathrm{MDP}}} \geqslant \frac{1}{\mathrm{DR}(w^{\mathrm{MDP}})\sigma(w)} \langle \sigma \mid w \rangle,$$

which leads us to the final result. We now prove that a portfolio w^* that respects the core property (2) necessarily maximizes the DR. As w^* respects the property (2), we have, for all long-only portfolios, $\rho_{w,w^*} \geqslant \mathrm{DR}(w)/\mathrm{DR}(w^*)$. Since correlations are not greater than unity, for all long-only portfolios we have $\mathrm{DR}(w^*) \geqslant \mathrm{DR}(w)$, which shows that w^* maximizes the DR across all long-only portfolios.

B.4 The core property of the most diversified portfolio (1)

Suppose that the MDP satisfies the core property (2). For any asset belonging to the MDP, the inequality given by the core property (2) becomes an equality as $\min(\lambda, w^{\mathrm{MDP}}) = 0$. Since the DR of a single asset equals 1, we have, for any given asset in the MDP,

$$\rho_{\mathrm{in},w^{\mathrm{MDP}}} = \frac{1}{\mathrm{DR}(w^{\mathrm{MDP}})}.$$

Now, using the core property (2) for any stock outside of the MDP, we finally obtain

$$\rho_{\mathrm{out},w^{\mathrm{MDP}}} \geqslant \frac{1}{\mathrm{DR}(w^{\mathrm{MDP}})} = \rho_{\mathrm{in},w^{\mathrm{MDP}}}.$$

Thus, the MDP satisfies the core property (1).

Conversely, suppose that a portfolio w^* satisfies the core property (1), for a given correlation ρ_{in,w^*}. Then, for any long-only portfolio w:

$$\rho_{w,w^*} = \frac{w^{\mathrm{T}}(\Sigma w^*)}{\sigma(w)\sigma(w^*)} = \frac{1}{\sigma(w)\sigma(w^*)} \sum_{i=1,\dots,N} w_i \sigma_i \sigma(w^*) \rho_{i,w^*}$$

$$\geqslant \rho_{\mathrm{in},w^*} \frac{\sum_{i=1,\dots,N} w_i \sigma_i}{\sigma(w)} = \rho_{\mathrm{in},w^*} \mathrm{DR}(w).$$

Applied with $w = w^*$, we have an equality. This shows that $\rho_{\mathrm{in},w^*}\mathrm{DR}(w^*) = 1$, and $\mathrm{DR}(w) \leqslant \rho_{w,w^*}\mathrm{DR}(w^*) \leqslant \mathrm{DR}(w^*)$. This demonstrates that w^* is the MDP, as it

has maximum diversification across all long-only portfolios. Overall, this shows that the core property of the MDP (1) is equivalent to the core property of the MDP (2).

B.5 Correlation of assets to the unconstrained most diversified portfolio

When removing the long-only constraint λ for all portfolios, possibly long–short,

$$\rho_{w,\boldsymbol{w}^{\text{MDP}}} = \frac{\text{DR}(\boldsymbol{w})}{\text{DR}(\boldsymbol{w}^{\text{MDP}})}.$$

In particular, the correlations of all assets to the MDP are constant, and equal to the inverse of the MDP's DR.

APPENDIX C

C.1 The most diversified portfolio is leverage invariant

The first-order condition for the MDP can be rewritten by splitting the covariance matrix into volatilities and correlations:

$$\boldsymbol{\sigma} \odot C(\boldsymbol{\sigma} \odot \boldsymbol{w}) = \delta\boldsymbol{\sigma} + \boldsymbol{\lambda}.$$

As volatilities are positive, this is equivalent to $C(\boldsymbol{\sigma}\odot\boldsymbol{w}) = \delta\mathbf{1}+\boldsymbol{\lambda}'$ with $\min(\boldsymbol{\lambda}', \boldsymbol{w}) = \mathbf{0}$ and $\boldsymbol{\lambda}' = \boldsymbol{\lambda} \oslash \boldsymbol{\sigma}$. Now, applying a positive leverage vector $\boldsymbol{L} = (L_i)_{i=1,\dots,A}$, the leveraged assets have the same correlation matrix C and volatilities $\boldsymbol{\sigma}^{\boldsymbol{L}} = (L_i\sigma_i)$. The portfolio $\boldsymbol{w}^{\boldsymbol{L}} = k\boldsymbol{w} \oslash \boldsymbol{L}$ is the MDP in the leveraged universe (with k a positive normalization constant, such that $\langle \boldsymbol{w} \mid \mathbf{1} \rangle = 1$), as it verifies the first-order condition $C(\boldsymbol{\sigma}^{\boldsymbol{L}} \odot \boldsymbol{w}^{\boldsymbol{L}}) = k\delta\mathbf{1} + k\boldsymbol{\lambda}'$ with $\min(k\boldsymbol{\lambda}', \boldsymbol{w}^{\boldsymbol{L}}) = \mathbf{0}$. This means that $\boldsymbol{\sigma}^{\boldsymbol{L}} \odot \boldsymbol{w}^{\boldsymbol{L}} = k\boldsymbol{\sigma} \odot \boldsymbol{w}$. Thus, the leverage invariance property is proved.

C.2 The most diversified portfolio is polico invariant

The core property of the MDP (2) shows that the MDP is such that any asset not selected by the MDP has a correlation greater than $1/\text{DR}(\text{MDP})$. This means the MDP is unchanged by adding to the universe any asset with a correlation strictly greater than $1/\text{DR}(\text{MDP})$. Furthermore, if we consider any polico Λ, we have

$$\rho(\Lambda, \text{MDP}) \geqslant \frac{\text{DR}(\Lambda)}{\text{DR}(\text{MDP})} > \frac{1}{\text{DR}(\text{MDP})},$$

as the DR of a polico is greater than 1. This means that when the polico is added to the universe, it is never selected, and the MDP remains unchanged (otherwise, we would have had $\rho(\Lambda, \text{MDP}) = 1/\text{DR}(\text{MDP})$ according to the core property (2)).

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