Robust Dynamic Assortment Optimization in the Presence of Outlier Customers *

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Abstract

We consider the dynamic assortment optimization problem under the multinomial logit model (MNL) with unknown utility parameters. The main question investigated in this paper is model mis-specification under the ε -contamination model, which is a fundamental model in robust statistics and machine learning. In particular, throughout a selling horizon of length T, we assume that customers make purchases according to a well specified underlying multinomial logit choice model in a $(1-\varepsilon)$ -fraction of the time periods, and make arbitrary purchasing decisions instead in the remaining ε -fraction of the time periods. In this model, we develop a new robust online assortment optimization policy via an active elimination strategy. We establish both upper and lower bounds on the regret, and show that our policy is optimal up to logarithmic factor in T when the assortment capacity is constant. Furthermore, we develop a fully adaptive policy that does not require any prior knowledge of the contamination parameter ε . Our simulation study shows that our policy outperforms the existing policies based on upper confidence bounds (UCB) and Thompson sampling.

Keywords:Dynamic assortment optimization, regret analysis, robustness, ε -contamination model, active elimination.

1 Introduction

A wide range of operations problems, ranging from assortment optimization to supply chain management, are built on an underlying probabilistic model. When real world outcomes follow this model, existing optimization techniques are able to provide accurate solutions. However, these model assumptions are only abstractions of reality and do not perfectly capture the sophisticated natural environment. In other words, these models are inherently mis-specified to a certain degree. Accordingly, model mis-specification and robust estimation have been an important topic

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in the statistics literature (Huber & Ronchetti, 2011). However, this literature primarily focuses on estimation or prediction from a given dataset, which is insufficient for modern operations settings where decision making plays a vital role. Unfortunately, most decision-making policies are derived from optimization problems that explicitly rely on the probabilistic model, so they are inherently not robust to model mis-specification. Can we design robust policies for these operations problems?

This paper studies model mis-specification for an important problem in revenue management — dynamic assortment optimization, under a popular ε -contamination model (which will be introduced in the next paragraph). Assortment optimization has a wide range of applications in retailing and online advertising. Given a large number of substitutable products, the assortment optimization problem involves selecting a subset of products (a.k.a., an assortment) to offer a customer such that the expected revenue is maximized. To model customers' choice behavior when facing a set of offered products, discrete choice models have been widely used, and one of the most popular such models is the multinomial logit model (MNL) (McFadden, 1974). In dynamic assortment optimization, the customers' choice behavior (e.g., mean utilities of products in an MNL) is not known a priori and must be learned online, which is often the case in practice, as historical data is often insufficient (e.g., fast fashion sale or online advertising). More specifically, the seller offers an assortment to each arriving customer for a finite time horizon T, observes the purchase behavior of the customer and then updates the utility estimate. The goal of the seller is to maximize the cumulative expected revenue over T periods. Due to its practical relevance, dynamic assortment optimization has received much attention in literature. (Caro & Gallien, 2007; Rusmevichientong et al., 2010; Saure & Zeevi, 2013; Agrawal et al., 2019, 2017).

All of these existing works assume that each arriving customer makes her purchase according to an underlying choice model. Yet, in practice, a small fraction of customers could make "outlier" purchases. To model such outlier purchases, we adopt a natural robust model in the statistical literature — the ε -contamination model (Huber, 1964), which dates back to the 1960s and is perhaps the most widely used model in robust statistics. In the general setup of the ε -contamination model, we are given n i.i.d. samples drawn from a distribution $(1 - \varepsilon)P_{\theta} + \varepsilon Q$, where P_{θ} denotes the distribution of interest P, parameterized by θ (e.g., a Gaussian distribution with mean θ), and Q is an arbitrary contamination distribution. The parameter $\varepsilon > 0$, which is usually very small, reflects the level at which contamination occurs, so a larger ε value means more observations are contaminated. The standard objective is to identify or estimate the parameter θ of the distribution of interest, in the presence of corrupted observations from Q. For the purpose of dynamic assortment optimization in the presence of outlier customers, the P_{θ} distribution represents the choice model for the majority of customers, which are "typical," (with θ being the parameter of an underlying MNL choice model of interest), while the Q distribution corresponds to choice models of "outlier" customers and ε reflects the proportion of outlier customers. For dynamic assortment optimization, we also deviate from the standard parameter estimation objective and focus on designing online decision-making policies.

In the classical ε -contamination model, the "outlier distribution" Q stays stationary for all samples, To make the contamination model more practical in the online assortment optimization setting, we strengthen the model from two aspects:

1. Instead of assuming a fixed corruption distribution Q for all outlier customers, we allow Q to change over different time periods (i.e., Q_t is the outlier distribution for customers at time

period t);

2. Instead of assuming that each time t is corrupted "uniformly at random", we assume that outlier customers appear in at most εT time periods. The purchase pattern and arrivals of outlier customers can, however, be arbitrary and even *adaptive* to the assortment decisions or customer purchase activities prior to time period t. The corrupted time periods and associated Q_t 's are unknown to the seller.

This setting is much richer than the "random arrival setting" and more realistic in practice. Indeed, in a holiday season, consecutive time periods might contain anomalous or outlier purchasing behavior, which cannot be capture by "random corruption" in the original ε -contamination model. The details of our outlier customer model will be rigorously specified in Section 3.

The main goal of the paper is to develop a robust dynamic assortment policy under this ε -contaminated MNL. Our first observation is that popular policies in the literature including Upper-Confidence-Bounds (UCB) (Agrawal et al., 2019) and Thompson sampling (Agrawal et al., 2017) no longer work in this model. The reason is that these policies cannot use typical customers that arrive later in the selling period to correct for misleading customers that arrive early on, and hence even a small number of outlier customers can lead to poor performance. Further, while it is well known that *randomization* is crucial in any adversarial setting (see, e.g., (Auer et al., 2002; Bubeck & Cesa-Bianchi, 2012)) to hedge against outliers, UCB is a deterministic policy, while Thompson sampling provides very little randomization via posterior sampling. We explain these failures in more detail in Secs. 3 and 6 later in this paper.

To address the contaminated setting, we develop a novel active elimination algorithm for robust dynamic planning, which gradually eliminates those items that are not in the optimal assortment with high probability (see Algorithm 1). Compared to the existing methods mentioned above (Agrawal et al., 2019, 2017), our active elimination method has several important technical novelties. First, our active elimination policy implements the randomization in a much more explicit way by sampling from a carefully constructed small set of "active" products. Second, the existing UCB and Thompson sampling algorithms for MNL rely on an epoch-based strategy (i.e., repeatedly offering the same assortment until no purchase) to enable an unbiased estimation of utility parameters. This procedure is inherently fragile since the stopping time of an epoch relies on a single no-purchase activity, which can be easily manipulated by outlier customers; a few outliers can greatly affect the stopping times. The failure of such an epoch-based strategy implies that unbiased estimation of utility parameters is no longer possible. To overcome this challenge, we propose a new utility estimation strategy based on geometrically increasing offering time periods. We conduct a careful perturbation analysis to control the bias of these estimates, which leads to new confidence bounds for our active elimination algorithm (see Sec. 4 for more details).

We provide theoretical guarantees for our proposed robust policy via regret analysis and information-theoretic lower bounds. In particular, let T be the selling horizon, N the total number of products, and K the cardinality constraint of an assortment (see Sec. 3). For the reasonable setting where ε is not too large, our active elimination algorithm (Algorithm 1) achieves $\widetilde{O}(\varepsilon K^2T + \sqrt{KNT})$ regret when ε (or a reasonable upper bound of ε) is known (see Theorem 1), where $\widetilde{O}(\cdot)$ only suppresses $\log(T)$ factors. Compared to the $\Omega(\varepsilon T + \sqrt{NT})$ lower bound (see Theorem 2), our upper bound is tight up to polynomial factors involving K and other logarithmic factors. We also remark that the special case of $\varepsilon=0$ reduces to the existing setting studied in (Agrawal et al., 2019, 2017;

Chen & Wang, 2018) in which no outlier customers are present. Compared to existing results, our regret bound is tight except for an additional $O(\sqrt{K})$ factor, which represents the cost of being adaptive to outlier customers (see Sec. 4.2 for more discussions). We emphasize that in a typical assortment optimization problem, the capacity of an assortment K is usually a small constant, especially relative to T and N.

The above result assumes that an upper bound on the outlier proportion ε is given as prior knowledge. While in some cases we may be able to estimate ε from historical data, this is not always possible, which motivates the design of fully adaptive policies that do not require ε as an input. Inspired by the "multi-layer active arm race" from the multi-armed bandits literature (Lykouris et al., 2018), we propose an adaptive robust dynamic assortment optimization policy in Algorithm 3. Our policy runs multiple "threads" of known- ε algorithms on a geometric grid of ε values in parallel, and, as we show, achieves $\widetilde{O}(\varepsilon T + \sqrt{NT})$ regret, where \widetilde{O} suppresses $\log(T)$ and K factors (see Theorem 3). Algorithm 3 and its analysis in Sec. 5 provide more details.

The rest of the paper is organized as follows. Sec. 2 introduces the related work. Sec. 3 describes the problem formulation. The first active elimination policy and the regret bounds are presented in Sec. 4, while the adaptive algorithm is presented in Sec. 5. Numerical illustration are provided in Sec. 6 with the conclusion in Sec. 7. The proof the lower bound result is provided in the appendix. Proofs of some technical lemmas are relegated to the supplementary material.

2 Related works

Static assortment optimization with known choice behavior has been an active research area since the seminal works by van Ryzin & Mahajan (1999) and Mahajan & van Ryzin (2001). Motivated by fast-fashion retailing, dynamic assortment optimization, which adaptively learns unknown customers' choice behavior, has received increasing attention in the context of data-driven revenue management. The work by Caro & Gallien (2007) first studied dynamic assortment optimization problem under the assumption that demands for different products are independent. Recent works by Rusmevichientong et al. (2010); Saure & Zeevi (2013); Agrawal et al. (2019, 2017); Chen & Wang (2018); Wang et al. (2018) incorporated MNL models into dynamic assortment optimization and formulated the problem as an online regret minimization problem. In particular, for the standard MNL model, Agrawal et al. (2019) and Agrawal et al. (2017) developed UCB and Thompson sampling based approaches for online assortment optimization. Moreover, some recent work (Cheung & Simchi-Levi, 2017; Chen et al., 2018; Oh & Iyengar, 2019) study dynamic assortment optimization based on contextual MNL models, where the utility takes the form of an inner product between a feature vector and the coefficients. The present work focuses on the standard non-contextual MNL model, but a natural direction for future work is to extend our results to the contextual setting.

All works outlined above assume an underlying MNL choice model is correctly specified. However, model mis-specification is common in practice, and robust statistics, one of the most important branches in statistics, is a natural tool to address such mis-specification. The ε -contamination model, which was proposed by P. J. Huber (Huber, 1964), is perhaps the most widely used robust model and has recently attracted much attention from the machine learning community (see, e.g., Chen et al. (2016); Diakonikolas et al. (2017, 2018) and reference therein). Despite this attention, online learning in the ε -contamination model or its generalizations is relatively unexplored. In the

online setting, Esfandiari et al. (32018) studied online allocation under a mixing adversarial and stochastic model but the setting does not require any learning component. For online learning, the recent work of Lykouris et al. (2018) studies the contaminated stochastic multi-armed bandit (MAB), but, due to the complex structure of discrete choice models, these results do not directly apply to our setting. Indeed, a straightforward analogy between assortment optimization and MAB is to treat each feasible assortment as an arm, but directly using this mapping will result in a large regret due to the exponentially many possible assortments.

In learning and decision-making settings, a few recent work investigate the impact of model mis-specification in revenue management, e.g., Cooper et al. (2006) for capacity booking problems and Besbes & Zeevi (2015) for dynamic pricing. In particular, Besbes & Zeevi (2015) show that a class of pricing policies based on linear demand functions perform well even when the underlying demand is not linear. Cooper et al. (2006) also identified some cases where simple decisions are optimal under mis-specification. However, our setting is quite different, as the widely used UCB and Thompson sampling policies are not robust under our model. On the other hand, our new active-elimination policy is robust to model mis-specification and additional achieves near-optimal regret when the model is well-specified.

3 Problem formulation and motivation

There are N items, each associated with a known revenue parameter $r_i \in [0,1]$ and an unknown utility parameter $v_i \in [0,1]$. At each time t a customer arrives, for a total of T time periods. The retailer then provides an assortment $S_t \subseteq [N]$ to the customer, subject to a capacity constraint $|S_t| \leq K$. The customer then chooses at most one item $i_t \in S_t$ to purchase, upon which the retailer collects a revenue of r_{i_t} . If the customer chooses to purchase nothing (denoted by $i_t = 0$), then the retailer collects no revenue.

At each time t, the arriving customer is assumed to be one of the following two types:

1. A **typical** customer makes purchases $i_t \in S_t \cup \{0\}$ according to a multinomial-logit (MNL) choice model

$$\Pr[i_t = i|S_t] = \frac{v_i}{v_0 + \sum_{j \in S_t} v_j}, \quad v_0 = 1.$$
(1)

We assume that $v_i \in [0, 1]$;

2. An **outlier** customer makes purchases $i_t \in S_t \cup \{0\}$ according to an arbitrary unknown distribution Q_t (marginalized on $S_t \cup \{0\}$). Q_t can potentially change with t.

We note that the MNL model in Eq. (1) together with the constraint that $v_i \in [0, 1]$ implies that "no purchase" is the most probable outcome for a *typical* customer, which is a common assumption in the literature. We do not impose such an assumption on outlier customers as the distributions Q_t can be completely arbitrary.

We consider the following ε -contamination model:

(A1) (Bounded adversaries). The number of outlier customers throughout T time periods does not exceed εT , where $\varepsilon \in [0,1)$ is a problem parameter;

(A2) (Adaptive adversaries). The choice model Q_t for an outlier customer at time t can be adversarially and adaptively chosen, based on the previous customers, offered assortments, and past purchasing activity.

A rigorous mathematical formulation is as follows: For any time period $t=1,2,\cdots,T$, let $\phi_t \in \{0,1\}$ be the indicator variable of whether customer at time t is an outlier ($\phi_t=1$ if customer t is an outlier and 0 otherwise), $S_t \subseteq [N]$ be the assortment provided at time $t, i_t \in S_t \cup \{0\}$ be the purchasing activity of the customer. The protocol is formally defined as follows:

Definition 1 (Definition of protocol). We define the following:

- 1. An adaptive adversary consists of T arbitrary measurable functions $\mathfrak{A}_1, \cdots, \mathfrak{A}_T$, where \mathfrak{A}_t : $\{\phi_{\tau}, Q_{\tau}, S_{\tau}, i_{\tau}\}_{\tau \leq t-1} \mapsto (\phi_t, Q_t)$ produces the type of the customer (typical or outlier) ϕ_t and the outlier distribution Q_t at time period t, from the filtration $\mathcal{F}_{t-1} = \{\phi_{\tau}, Q_{\tau}, S_{\tau}, i_{\tau}\}_{\tau \leq t-1}$;
- 2. An admissible policy consists of T random functions $\mathfrak{P}_1, \dots, \mathfrak{P}_T$, where $\mathfrak{P}_t : \{S_{\tau}, i_{\tau}\}_{\tau \leq t-1} \mapsto S_t$ produces a randomized assortment $S_t \subseteq [N]$, $|S_t| \leq K$ at time period t, from the filtration $\mathcal{G}_{t-1} = \{S_{\tau}, i_{\tau}\}_{\tau \leq t-1}$;
- 3. If $\phi_t = 0$ then i_t is realized according to model (1) conditioned on S_t ; otherwise if $\phi_t = 1$ then i_t is realized according to model Q_t .

The objective of the retailer is to develop an admissible dynamic assortment optimization strategy that is competitive with S^* , the optimal assortment for typical customers. In particular, the performance of the retailer is measured by the cumulative *regret*, which is defined as,

Regret :=
$$\mathbb{E} \sum_{t=1}^{T} R(S^*) - R(S_t)$$
, where $S^* = \arg \max_{|S| \le K} R(S)$, $R(S) = \frac{\sum_{i \in S} r_i v_i}{1 + \sum_{i \in S} v_i}$, (2)

where R(S) is the expected revenue when offering the assortment S.

To motivate our policy, we first briefly explain why the popular Upper-confidence-bounds (UCB) and Thompson sampling fail in the presence of outlier customers. These algorithms are designed for the uncontaminated setting where $\varepsilon=0$, so the confidence bounds (in UCB policies) and posterior updates (in Thompson sampling policies) are designed under the assumption that *all* customers follow the same MNL model. Unfortunately, in the presence of outlier customers the confidence intervals are too narrow and the posterior updates are too aggressive. With these update strategies, a small number of outlier customers preferring items unpopular to typical customers could "swing" the algorithms' parameter estimates, which can lead to the belief that these unpopular items are actually popular. This subsequently leads to poor exploration of the popular items, which eventually hurts performance. As a numerical demonstration, we construct a concrete setting in Sec. 6 where the performance of UCB and Thompson sampling policies degrades considerably in the presence of outlier customers.

4 An active-elimination policy

We propose an *active-elimination* policy for dynamic assortment optimization in the presence of outlier customers. A pseudo-code description of the proposed algorithm is given in Algorithm 1.

Algorithm 1 An active-elimination algorithm for robust dynamic assortment optimization.

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1: Input: time horizon T, outlier proportion \overline{\varepsilon}, revenue parameters \{r_i\}, capacity constraint K.
 2: Output: a sequence of assortments \{S_t\}_{t=1}^T attaining good regret.
 3: Set \widehat{v}^{(0)} \equiv 1, \widehat{\Delta}_{\overline{\varepsilon}}(0) = 1, \mathcal{A}^{(0)} = [N], T_0 = 128(K+1)^2 N \ln T;
 4: for \tau = 0, 1, 2, \cdots do
              *Compute \gamma^{(\tau)} = \max_{|S| < K, S \subset \mathcal{A}^{(\tau)}} R(S; \widehat{v}^{(\tau)});
 5:
             ^{\dagger} \text{ Compute } S_{\tau}^{(i)} = \arg\max_{S \subseteq \mathcal{A}^{(\tau)}, |S| \leq K, i \in S} R(S; \widehat{v}^{(\tau)}) \text{ for every } i \in \mathcal{A}^{(\tau)};
 6:
              Update \mathcal{A}^{(\tau+1)} = \{ i \in \mathcal{A}^{(\tau)} : R(S_{\tau}^{(i)}; \widehat{v}^{(\tau)}) + 2\widehat{\Delta}_{\overline{\varepsilon}}(\tau) \ge \gamma^{(\tau)} \};
 7:
              Set n_i = 0 and n_0(i) = 0 for all i \in \mathcal{A}^{(\tau+1)}; set T_{\tau} = 2^{\tau} T_0;
 8:
              for the next T_{\tau} time periods do
 9:
                     Sample i \in \mathcal{A}^{(\tau+1)} uniformly at random;
10:
                    Provide the assortment S_{\tau}^{(i)} to the incoming customer and observe purchase i_t;
11:
                     Update n_i \leftarrow n_i + 1\{i_t = i\} and n_0(i) \leftarrow n_0(i) + 1\{i_t = 0\};
12:
              end for
13:
              Update estimates \widehat{v}_i^{(\tau+1)} = \max\{1, n_i/n_0(i)\} for every i \in \mathcal{A}^{(\tau+1)};
14:
             Define \overline{\varepsilon}_{\tau} = \min\{1, \overline{\varepsilon}T/T_{\tau}\}, N_{\tau} = |\mathcal{A}^{(\tau+1)}| and compute error upper bound as
15:
       \widehat{\Delta}_{\overline{\varepsilon}}(\tau+1) = \begin{cases} 1, & T_{\tau} < \frac{\overline{\varepsilon}T}{4(K+1)}; \\ 16K(K+1)\left(\frac{\overline{\varepsilon}_{\tau}}{2} + \sqrt{\frac{\overline{\varepsilon}_{\tau}N_{\tau}\ln T}{T_{\tau}}} + \frac{2N_{\tau}\ln T}{3T_{\tau}}\right) + 16\sqrt{\frac{KN_{\tau}\ln T}{T_{\tau}}}, & \text{otherwise}; \end{cases}
16: end for
17: Remarks:
18: * For any set of \{\widehat{v}\}, R(S; \widehat{v}) = (\sum_{i \in S} r_i \widehat{v}_i)/(1 + \sum_{i \in S} \widehat{v}_i);
19: *, † Both optimization can be computed efficiently. See Sec. 4.1 for details.
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While Algorithm 1 requires the knowledge of ε (or an upper bound $\overline{\varepsilon}$, see Theorem 1) as input, we emphasize that such requirement can be completely removed by designing more complex policies, as we will show in Sec. 5. To highlight our main idea, we state Algorithm 1 upfront as the prior knowledge of ε simplifies both the algorithm and its analysis.

At a high level, Algorithm 1 operates in *epochs* $\tau=0,1,\cdots$ with geometrically increasing lengths, and only performs item estimation or assortment updates between epochs. At any time t, the algorithm maintains an active set of items $\mathcal{A}\subseteq [N]$ consisting of all items that could potentially form a "good" assortment, and estimates of parameters $\{\widehat{v}_i\}$ for all active items i in \mathcal{A} . For each time period t in a single epoch t, a *random* item t is sampled from the current active item set and a "near-optimal" assortment is built, which must contain the target item t. Once an epoch t ends, parameter estimates of t are updated and the active set t is shrunk based on the updated estimates to exclude sub-optimal items. We will ensure that with high probability, the optimal assortment t is always a subset of active sets for all epochs (see Lemma 3).

We now detail all notation used in Algorithm 1:

- $\tau_0 \in \mathbb{N}$: the indices of *epochs* whose lengths increase geometrically $(T_{\tau} = 2^{\tau}T_0)$;
- $\widehat{v}^{(\tau)} \in [0,1]^N$: the estimates of preference parameters (of typical customers) at epoch τ ;

Algorithm 2 Assortment optimization with additional constraints

- 1: **Input**: revenue parameters $\{r_i\}_{i=1}^n$, estimated preference parameters $\{\widehat{v}_i\}_{i=1}^n$, must-have item i, capacity constraint K, stopping accuracy δ ;
- 2: **Output**: assortment \widehat{S} , $|\widehat{S}| \leq K$, $i \in \widehat{S}$ that maximizes $R(\widehat{S}; \widehat{v})$.
- 3: Initialization: $\alpha_{\ell} = 0$ and $\alpha_{u} = 1$; $\widehat{S} = \emptyset$;
- 4: while $\alpha_u \alpha_\ell > \delta$ do
- 5: $\alpha_{\text{mid}} \leftarrow (\alpha_{\ell} + \alpha_{u})/2;$
- 6: For each $j \neq i$, sort $\psi_j := (r_j \alpha_{\text{mid}}) \widehat{v}_j$ in descending order, and let $\Psi := \{j \neq i : \psi_j \geq 0\}$ be the subset consisting of all items other than i with non-negative ψ_j ;
- 7: Compute $t := \psi_i + \text{ the } (K-1) \psi_j \text{ in } \Psi \text{ with the largest values;}$
- 8: If $t \geq \alpha_{\text{mid}}$ then set $\widehat{S} = \{i\} \cup \{\text{the } (K-1) \text{ items in } \Psi \text{ with the largest } \psi_j \text{ values} \}$ and $\alpha_\ell \leftarrow \alpha_{\text{mid}}$; else set $\alpha_u \leftarrow \alpha_{\text{mid}}$.
- 9: end while
 - $\mathcal{A}^{(\tau+1)}\subseteq [N]$: the subset of active items, which are to be explored uniformly at random in epoch τ ;
 - $\gamma^{(\tau)} \in [0,1]$ (see step 5): the estimated expected revenue of the optimal assortment calculated based on the active item subset $\mathcal{A}^{(\tau+1)}$ and current preference estimates $\widehat{v}^{(\tau)}$;
 - $S_{\tau}^{(i)} \subseteq [N]$ (see step 6): an optimal assortment computed based on $\mathcal{A}^{(\tau+1)}$ and $\widehat{v}^{(\tau)}$, which must include the specific item i; this assortment is used to explore and estimate the the utility parameter v_i of item i;
 - $n_i, n_0(i) \in \mathbb{N}$ (see step 12): counters used in the estimate of v_i ; note that for any supplied assortment $S_{\tau}^{(i)}$, we only record the number of times a customer purchases item i (accumulated by n_i), and the number of times a customer makes no purchases (accumulated by $n_0(i)$); other purchasing activities (e.g., purchases of an item $\ell \in S_{\tau}^{(i)}$ other than i) will not be recorded:
 - $\widehat{\Delta}_{\overline{\varepsilon}}(\tau+1) \in [0,1]$: length of confidence intervals used to eliminate items from $\mathcal{A}^{(\tau+1)}$; its length depends on both the epoch index τ and the prior knowledge of the outlier proportion $\overline{\varepsilon}$:

In the rest of the section, we first give a brief description of how to compute $\widehat{S}_{\tau}^{(i)}$ in Line 6 efficiently. Then we detail the regret upper bound of Algorithm 1 and provide the the proof.

4.1 Solving the optimization problem

The implementation of most steps of Algorithm 1 is straightforward, except for the computation of the assortments $S_{\tau}^{(i)}$, which require futher algorithmic development. This computation can be formulated as the following combinatorial optimization problem:

$$\max_{|S| \le K, i \in S} R(S; \widehat{v}) = \max_{|S| \le K, i \in S} \frac{\sum_{j \in S} r_j \widehat{v}_j}{1 + \sum_{j \in S} \widehat{v}_j},\tag{3}$$

for a specific $i \in [N]$. This optimization problem is similar to the classical capacity-constrained assortment optimization (see, e.g., Rusmevichientong et al. (2010)), but the additional constraint $i \in S$ in (3) yields a subtle difference. For the purpose of completeness, we provide an efficient optimization method with binary search for solving Eq. (3). Pseudo-code is provided in Algorithm 2.

For any $\alpha \in (0,1]$, we want to check whether there exists $S \subseteq [N]$, $|S| \le K$, $i \in S$ such that $R(S; \widehat{v}) \ge \alpha$, or equivalently $\sum_{j \in S} r_j \widehat{v}_j \ge \alpha + \alpha \sum_{j \in S} \widehat{v}_j$. Re-organizing the terms, we only need to check whether there exists $|S| \le K$, $i \in S$ such that $\sum_{j \in S} (r_j - \alpha) \widehat{v}_j \ge \alpha$. Because $i \in S$ must hold, we only need to check whether there exists $S' \subseteq [N] \setminus \{i\}$, $|S'| \le K - 1$ such that

$$(r_i - \alpha)\widehat{v}_i + \sum_{j \in S'} (r_j - \alpha)\widehat{v}_j \ge \alpha. \tag{4}$$

This can be accomplished by including all $j \in [N] \setminus \{i\}$ with the largest (K-1) positive values of $(r_j - \alpha) \widehat{v}_j$ into the set of S' and check whether Eq. (4). If Eq. (4) holds, the current revenue value of α can be obtained and otherwise the current value of α cannot be obtained. We then solve the optimization problem by a standard binary search on α . We also note that $\gamma^{(\tau)}$ in Line 5 is a standard static capacitated assortment optimization, which can be solved efficiently (see Rusmevichientong et al. (2010)).

4.2 Regret analysis

The following theorem is our main regret upper bound result for Algorithm 1.

Theorem 1. Suppose $\overline{\varepsilon} \ge \varepsilon$ and $N \le T$. Then there exists a universal constant $C_0 < \infty$ such that, for sufficiently large T, the regret of Algorithm 1 is upper bounded by

$$C_0 \times \left(\overline{\varepsilon}K^2T\log T + (K^2\sqrt{\overline{\varepsilon}} + \sqrt{K})\sqrt{NT\log^3 T} + K^2N\log^2 T.\right).$$

Furthermore, if $\overline{\varepsilon} \lesssim 1/K^3$ holds then the regret upper bound can be simplified to

$$O\left(\overline{\varepsilon}K^2T\log T + \sqrt{KNT\log^3 T}\right).$$

We complement Theorem 1 with the following *lower bound* result, with the proof in the appendix.

Theorem 2. There exists a universal constant $c_0 > 0$ such that, for any policy π , its worst-case regret for problem instances with T customers, N items, K < N/4 assortment capacity constraint and $\lfloor \varepsilon T \rfloor$ outlier customers ($0 \le \varepsilon < 1$) is lower bounded by $c_0 \times (\varepsilon T + \sqrt{NT})$.

Remark 1. The regret upper bound matches the $\Omega(\varepsilon T + \sqrt{NT})$ lower bound established in Theorem 2, up to low-degree polynomial terms depending only on K and other logarithmic terms.

An important special case of Theorem 1 is $\varepsilon = \overline{\varepsilon} = 0$, which reduces to the well-studied dynamic assortment optimization problem without outlier customers. For such settings, Agrawal et al. (2017, 2019) give algorithms with a regret upper bound of $\widetilde{O}(\sqrt{NT})$, which matches the lower

bound of $\Omega(\sqrt{NT})$ given in (Chen & Wang, 2018) up to poly-logarithmic terms. Comparing their results to Theorem 1, we observe that our result at $\varepsilon = \overline{\varepsilon} = 0$ matches the $\widetilde{O}(\sqrt{NT})$ regret bound except for an additional term of $O(\sqrt{K})$. This $O(\sqrt{K})$ factor stems from our active elimination protocol, which is essential for handling outlier customers when $\varepsilon > 0$. We believe removing this factor is technically quite challenging, and leave it as an interesting open question. We also note that the capacity constraint K is typically a very small constant in practice, and hence an additional $O(\sqrt{K})$ term is likely negligible.

Our regret upper bound in Theorem 1 also yields meaningful guarantees when ε is not zero. For example, with $\varepsilon = O(T^{-1/4})$, meaning that $O(T^{3/4})$ out of T customers are outliers, Theorem 1 provides an $O(K^2T^{3/4}\log T)$ regret upper bound. This guarantee is non-trivial because it is sublinear in T, although it is larger than the standard $\widetilde{O}(\sqrt{NT})$ bound for the uncontaminated setting. Thus, Theorem 1 reveals the trade-off and impact of a small proportion of outlier customers on the performance of dynamic assortment optimization algorithms/systems.

4.3 Proof sketch of Theorem 1

In this section we sketch the proof of Theorem 1. Key lemmas and their implications are given, while the complete proofs of the presented lemmas are deferred to the supplementary material accompanying this paper.

We first state a lemma that upper bounds the estimation error $|\widehat{v}_i^{(\tau+1)} - v_i|$:

Lemma 1. Suppose $T_0 \geq 128(K+1)^2 N_\tau \ln T$ and $\min\{1, \varepsilon T/T_\tau\} \leq 1/4(K+2)$. With probability $1 - O(\tau_0 N/T^2)$ it holds for all τ satisfying $T_\tau \geq \max\{\overline{\varepsilon}, \varepsilon\}T/4(K+1)$ and $i \in \mathcal{A}^{(\tau+1)}$ that $|\widehat{v}_i^{(\tau+1)} - v_i| \leq \Delta_\varepsilon^*(i, \tau+1)$, where

$$\Delta_{\varepsilon}^{*}(i,\tau+1) = 8(K+1) \left(\frac{\varepsilon_{\tau}}{2} + \sqrt{\frac{\varepsilon_{\tau} N_{\tau} \ln T}{T_{\tau}}} + \frac{2N_{\tau} \ln T}{3T_{\tau}} \right) + 8\sqrt{\frac{(1+V_{S})v_{i}N_{\tau} \ln T}{T_{\tau}}}, \quad (5)$$

where ε_{τ} is defined as $\varepsilon_{\tau} = \min\{1, \varepsilon T/T_{\tau}\}$, $N_{\tau} = |\mathcal{A}^{(\tau+1)}|$ and $V_S = \sum_{j \in S_{\tau}^{(i)}} v_j$.

Lemma 1 shows that, with high probability, the estimation error between $\widehat{v}_i^{(\tau+1)}$ and v_i , the true preference parameter of item i for typical customers, can be upper bounded by $\Delta_{\varepsilon}^*(i,\tau+1)$ which is a function of K, τ , T, ε and $N_{\tau}=|\mathcal{A}^{(\tau+1)}|$. It should be noted that the definition of $\Delta_{\varepsilon}^*(i,\tau+1)$ involves unknown quantities (mostly $V_S=\sum_{j\in S_{\tau}^{(i)}}v_j$) and hence cannot be directly used in an algorithm. The definition of $\widehat{\Delta}_{\overline{\varepsilon}}(\tau+1)$ in Algorithm 1, on the other hand, involves only known quantities and estimates. In Corollary 1, we will establish the connection between $\Delta_{\varepsilon}^*(i,\tau+1)$ and $\widehat{\Delta}_{\overline{\varepsilon}}(\tau+1)$.

Our next lemma derives how the estimated expected revenue $R(S; \widehat{v})$ deviates from the true value R(S; v) by using upper bounds on the estimation errors between \widehat{v} and v:

Lemma 2. For any $S \subseteq [N]$, $|S| \leq K$ and $\{\widehat{v}_i\}$, it holds that

$$|R(S; \widehat{v}) - R(S; v)| \le \frac{2\sum_{i \in S} |\widehat{v}_i - v_i|}{1 + \sum_{i \in S} v_i}.$$

The proof uses only elementary algebra.

Combining Lemmas 1 and 2, we show that the $\widehat{\Delta}_{\overline{\varepsilon}}(\tau)$ quantities defined in our algorithm serve as valid upper bounds on the estimation error between $R(S; \widehat{v}^{(\tau)})$ and R(S; v):

Corollary 1. For every τ and $|S| \leq K$, $S \subseteq \mathcal{A}^{(\tau)}$, conditioned on the success events on epochs up to τ , it holds that $|R(S; \widehat{v}^{(\tau)}) - R(S; v)| \leq \widehat{\Delta}_{\varepsilon}(\tau) \leq \widehat{\Delta}_{\max\{\varepsilon,\overline{\varepsilon}\}}(\tau)$, where $\widehat{\Delta}$ is defined in Algorithm 1.

Our next lemma is an important structural lemma which states that, with high probability, any item in the optimal assortment S^* is never excluded from active item sets $\mathcal{A}^{(\tau+1)}$ for all epochs τ .

Lemma 3. If $\overline{\varepsilon} \geq \varepsilon$ then with probability $1 - O(\tau_0 N/T^2)$ it holds that $S^* \subseteq \mathcal{A}^{(\tau)}$ for all τ .

This structural lemma yields two important consequences: first, since "good" items remain within the active item subsets $\mathcal{A}^{(\tau+1)}$, each of the assortments $S_{\tau}^{(i)}$ computed at step 6 of Algorithm 1 will have relatively high expected revenue. Second, the fact that $S^* \subseteq \mathcal{A}^{(\tau+1)}$ implies that the optimistic estimates $\gamma^{(\tau)}$ will always be based on the expected revenue of the actual optimal assortment $R(S^*; v)$. This justifies the elimination step 7 in which we discard all items whose best assortment has significantly lower revenue than $\gamma^{(\tau)}$.

The proof of Lemma 3 is based on an inductive argument, which shows that if S^* belongs to $\mathcal{A}^{(\tau)}$ at the beginning of every epoch τ , then any item in S^* will not be removed (with high probability) by step 7. The intuition for this is that the optimal assortment containing any $i \in S^*$ is S^* itself, whose revenue cannot be to far away from $\gamma^{(\tau)}$ due to Lemmas 1 and 2. The complete proof of Lemma 3 is provided in the supplementary material.

Finally, our last technical lemma upper bounds the per-period regret incurred by Algorithm 1.

Lemma 4. Suppose $S^* \subseteq \mathcal{A}^{(\tau)}$ holds for all τ . Then with probability $1 - O(\tau_0 N/T^2)$, for every $\tau < \tau_0$ and $i \in \mathcal{A}^{(\tau+1)}$, it holds that $R(S^*; v) - R(S^{(i)}_{\tau}; v) < 4\widehat{\Delta}_{\overline{\tau}}(\tau)$.

Given the established technical lemmas, we are now ready to give the proof of Theorem 1.

Proof. Let τ^* be the smallest integer such that $T_{\tau^*} \geq \overline{\varepsilon}T/4(K+1)$. For all epochs $\tau < \tau^*$, the induced cumulative regret can be upper bounded by

$$\sum_{\tau < \tau^*} T_{\tau} \le T_{\tau^*} \le \overline{\varepsilon} T. \tag{6}$$

In the rest of this proof we upper bound the regret incurred from epochs $\tau \geq \tau^*$. By Lemma 4, the regret incurred by a single time period in epoch τ is upper bounded by $4\widehat{\Delta}_{\overline{\varepsilon}}(\tau)$. The total regret accumulated in epoch τ is then upper bounded by $4\widehat{\Delta}_{\overline{\varepsilon}}(\tau) \times T_{\tau}$. Hence, the regret accumulated on

the entire T time periods is upper bounded by

$$\sum_{\tau=0}^{\tau_0} 4\widehat{\Delta}_{\overline{\varepsilon}}(\tau)T_{\tau}$$

$$\lesssim \sum_{\tau=0}^{\tau_0} \left(K^2 \overline{\varepsilon}_{\tau} + K^2 \sqrt{\frac{\overline{\varepsilon}_{\tau} |\mathcal{A}^{(\tau+1)}| \log T}{T_{\tau}}} + \frac{K^2 |\mathcal{A}^{(\tau+1)}| \log T}{T_{\tau}} + \sqrt{\frac{K |\mathcal{A}^{(\tau+1)}| \log T}{T_{\tau}}} \right) \times T_{\tau}$$

$$\leq \sum_{\tau=0}^{\tau_0} \left(\frac{K^2 \overline{\varepsilon}T}{T_{\tau}} + K^2 \sqrt{\frac{\overline{\varepsilon} |\mathcal{A}^{(\tau+1)}| T \log T}{T_{\tau}^2}} + \frac{K^2 |\mathcal{A}^{(\tau+1)}| \log T}{T_{\tau}} + \sqrt{\frac{K |\mathcal{A}^{(\tau+1)}| \log T}{T_{\tau}}} \right) \times T_{\tau}$$

$$\leq \tau_0 K^2 \overline{\varepsilon}T + K^2 \sqrt{\overline{\varepsilon}T \log T} \left(\sum_{\tau \leq \tau_0} \sqrt{|\mathcal{A}^{(\tau+1)}|} \right) + K^2 \log T \left(\sum_{\tau \leq \tau_0} |\mathcal{A}^{(\tau+1)}| \right)$$

$$+ \sqrt{K \log T} \left(\sum_{\tau \leq \tau_0} \sqrt{T_{\tau} |\mathcal{A}^{(\tau+1)}|} \right) + K^2 \log T \left(\sum_{\tau \leq \tau_0} |\mathcal{A}^{(\tau+1)}| \right)$$

$$\leq \tau_0 K^2 \overline{\varepsilon}T + \tau_0 K^2 \sqrt{\overline{\varepsilon}NT \log T} + \tau_0 K^2 N \log T + \sqrt{K \log T} \times \sqrt{\sum_{\tau \leq \tau_0} |\mathcal{A}^{(\tau+1)}|} \times \sqrt{\sum_{\tau \leq \tau_0} T_{\tau}}$$

$$\leq K^2 \overline{\varepsilon}T \log T + K^2 \sqrt{\overline{\varepsilon}NT \log^3 T} + \sqrt{K \log T} \times \sqrt{\tau_0 N} \times \sqrt{T} + K^2 N \log^2 T$$

$$\lesssim \overline{\varepsilon}K^2 T \log T + (K^2 \sqrt{\overline{\varepsilon}} + \sqrt{K}) \sqrt{NT \log^3 T} + K^2 N \log^2 T.$$
(9)

Here in Eq. (8), we apply Cauchy-Schwartz inequality. The final inequality holds because $\tau_0 = O(\log T)$.

5 Adaptation to unknown outlier proportion ε

In this section we describe a more complex algorithm for robust dynamic assortment optimization where the outlier proportion ε is *unknown* a priori to the retailer. Inspired by the "multi-layer active arm race" for multi-armed bandits in Lykouris et al. (2018), Algorithm 3 runs multiple "threads" of known- ε algorithms on a geometric grid of ε values in parallel with a careful coordination among the threads.

The pseudo-code of the proposed adaptive algorithm is given in Algorithm 3. We note that for two threads j' < j, we have $\widehat{\varepsilon}_{j'} > \widehat{\varepsilon}_{j}$, which implies that the confidence interval length $\widehat{\Delta}_{\widehat{\varepsilon}_{j'}}(\tau+1)$ is typically longer than $\widehat{\Delta}_{\widehat{\varepsilon}_{j'}}(\tau+1)$. Therefore, the thread j' is less aggressive than the thread j in terms of eliminating items, i.e., an item eliminated by thread j can possibly be still kept in active set in thread j'. More detailed explanations of key steps in Algorithm 3 are summarized below:

1. Independence of threads: different threads j < J, which correspond to different hypothetical values of ε (denoted as $\widehat{\varepsilon}_j$), are largely independent from each other, maintaining their own parameter estimates $\widehat{v}^{(\tau),j}$, active item set $\mathcal{A}_j^{(\tau+1)}$ and confidence intervals $\widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau+1)$.

Algorithm 3 Dynamic assortment optimization robust to unknown outlier proportion ε .

```
1: Input: lower bound on outlier proportion \underline{\varepsilon} = 2^{-J}, J = \lfloor \log_2 \sqrt{N/T} \rfloor + 1;
  2: Output: a sequence of assortments \{S_t\}_t attaining good regret for any \varepsilon;
  3: Construct a grid of outlier proportion values \{\widehat{\varepsilon}_j\}_{j=0}^{J-1} where \widehat{\varepsilon}_j = 2^{-j};
  4: Construct J threads j < J, each with \widehat{\varepsilon}_j outlier proportion;
  5: For each i \in [N] and j < J, set \widehat{v}^{(0),j} \equiv 1, \widehat{\Delta}_{\widehat{\varepsilon}_j}(0) = 1, \mathcal{A}_j^{(0)} = [N], T_0 = 64(K+1)^2 \ln T;
  6: for \tau = 0, 1, 2 \cdots do
              \begin{aligned} & \textbf{for } j = 0, 1, \cdots, J-1 \ \textbf{do} \\ & \text{If } j > 0 \text{ then update } \mathcal{A}_j^{(\tau)} = \mathcal{A}_j^{(\tau)} \cap \mathcal{A}_{j-1}^{(\tau+1)}; \\ & ^*\text{Compute } \gamma_j^{(\tau)} \text{ and } S_{\tau,j}^{(i)} \text{ for each } i \in \mathcal{A}_j^{(\tau)} \text{ and update } \mathcal{A}_j^{(\tau+1)}; \end{aligned}
  7:
  8:
  9:
10:
              end for
               for the next T_{\tau} = 2^{\tau} T_0 time periods do
11:
                      Sample thread j < J with probability \wp_j := 2^{-(J-j)}/(1-2^{-J});
12:
                     Sample item i \in \mathcal{A}_i^{(\tau+1)} uniformly at random;
13:
                     \begin{array}{l} \text{if} \ ^\dagger \text{there exists} \ \widehat{\varepsilon}_k > \widehat{\varepsilon}_j \ \text{such that} \ R(\widehat{S}^{(i)}_{\tau,j};\widehat{v}^{(\tau),k}) < \gamma_k^{(\tau)} - 7\widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau) \ \text{then} \\ \text{Re-start Algorithm 3 with} \ J \leftarrow J - 1; \end{array}
14:
15:
                     end if
16:
                     Provide assortment S_{\tau,j}^{(i)} to the incoming customer and observes purchase i_t;
17:
                     Update n_i^j \leftarrow n_i^j + \mathbf{1}\{i_t = i\} and n_0^j(i) \leftarrow n_0^j(i) + \mathbf{1}\{i_t = 0\};
18:
19:
              Update estimates \widehat{v}_i^{(\tau+1),j} = \max\{1, n_i^j/n_0^j(i)\} for all j \leq J and i \in \mathcal{A}_i^{(\tau+1)};
20:
              For every j \leq J, compute \widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau+1) with T, T_{\tau} replaced by T_j := \wp_j T and T_{\tau,j} := \wp_j T_{\tau};
21:
22: end for
23: * Using the procedure outlined in Algorithm 2.
24: \dagger \ \widehat{v}^{(\tau),k} and \gamma_k^{(\tau)} are estimates of v and computed \gamma^{(\tau)} values maintained in thread k.
```

Coordination among threads only appear in two steps in Algorithm 3: Step 8, which maintains a hierarchical "nested" structure of the active item sets $\mathcal{A}_j^{(\tau+1)}$ among the threads, and Step 15, which provides update rules for $J \leftarrow J-1$ by comparing the obtained optimistic assortment among different threads. Further details are given in subsequent bullets.

2. Heterogeneous sampling of different threads: at each time period t when a potential customer arrives, a random thread j < J is selected to provide assortments. The random thread, however, is not selected uniformly at random but according to a specifically designed distribution, with the probability of selecting thread j equals $\wp_j = 2^{-(J-j)}/(1-2^{-J})$. Intuitively, such a sampling distribution "favors" threads with smaller hypothetical $\widehat{\varepsilon}_j$ values.

Such a sampling distribution is motivated by the fact that threads with larger $\widehat{\varepsilon}_j$ values typically incur large regret, because their elimination rules are too conservative and therefore too many sub-optimal items i would remain in their active item subset $\mathcal{A}_j^{(\tau+1)}$. Therefore, the probability of choosing threads with larger $\widehat{\varepsilon}_j$ values should be small in order to remedy their potential large regret incurred per time period.

On the other hand, while threads corresponding to smaller $\hat{\varepsilon}_i$ values might also incur large

regret due to their overly aggressive elimination rules (which could potentially eliminate the actual optimal assortment S^* from their active set $\mathcal{A}_j^{(\tau+1)}$), such events can be reliably detected using the checking rule described in Step 15, as we describe in more details in the next bullet.

3. Coordination and interaction among threads: as we mentioned in the first bullet, the coordination and interaction among different threads only happen in Steps 8 and 15 in Algorithm3. In this bullet we discuss these two steps in more details.

Step 8 aims at maintaining a "nested" structure among the active subsets $\mathcal{A}_j^{(\tau+1)}$, such that $\mathcal{A}_j^{(\tau+1)}\subseteq\mathcal{A}_{j'}^{(\tau+1)}$ for any $j'\leq j$ at any epoch τ . We remark that such a nested structure should be expected even without this step, because thread $j'\leq j$ is less aggressive than thread j, in the sense that confidence intervals $\widehat{\Delta}_{\widehat{\varepsilon}_{j'}}(\tau+1)$ is typically longer than $\widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau+1)$. Hence, one should expect larger active set for thread j'. Nevertheless, due to stochastic fluctuations such nested structures might be violated. Therefore, to explicitly enforce such a nested structure, we implement the intersection of active sets in Step 8 at the start of every epoch τ .

Step 15 is an important step that tries to detect the event of $\widehat{\varepsilon}_{J-1}$ being too small compared to the actual (unknown) outlier level ε . As we have mentioned in the previous bullet, if $\widehat{\varepsilon}_j$ is much smaller than true ε , the elimination rule of thread j might be too aggressive, leading to the optimal assortment S^* being eliminated from the active subset $\mathcal{A}_j^{(\tau+1)}$, and subsequently resulting in a large regret. Step 15 detects such events by comparing the optimistic assortment $S_{\tau,j}^{(\cdot)}$ with threads j' < j, which are associated with less aggressive elimination rules and hence would retain the optimal assortment S^* with high probability. The condition in Step 15 indicates the detection of a contradiction, suggesting the optimal assortment S^* is no longer retained in $\mathcal{A}_j^{(\tau+1)}$, which subsequently lead to the conclusion that $\widehat{\varepsilon}_j$ is too small. Then we terminate the current thread and restart the algorithm with $J \leftarrow J-1$.

In the rest of this section we state our regret upper bound result for the adaptive algorithm 3, as well as a sketch of its proof.

5.1 Regret analysis and proof sketch

We establish the following regret upper bound for Algorithm 3.

Theorem 3. Suppose Algorithm 3 is run with an initial value of $J = \lfloor \log_2(\sqrt{N/T}) \rfloor + 1$. Then there exists a constant $C_1 = \text{poly}(K, \log(NT))$ such that, for any $\varepsilon \in [0, 1/2]$ and sufficiently large T, the regret of Algorithm 3 is upper bounded by

$$C_1 \times (\varepsilon T + \sqrt{NT}).$$

Remark 2. In the statement of Theorem 3, $C_1 = \text{poly}(K, \log(NT))$ means $C_1 = (K \log(NT))^c$ for some universal constant $c < \infty$. For notational simplicity we did not work out the exact constant c in the expression of C_1 .

The complete proof of Theorem 3 as well as the proofs of technical lemmas are relegated to the supplementary material. Here we sketch key steps in the proof.

The first step of our proof is the following lemma, which shows that for threads with larger $\hat{\varepsilon}_j$ values, the optimal assortment S^* is never removed from their active item sets with high probability.

Lemma 5. With probability
$$1 - O(\tau_0 NJ/T^2)$$
 it holds for all τ and $\widehat{\varepsilon}_j \geq \varepsilon$ that $S^* \subseteq \mathcal{A}_j^{(\tau)}$.

Lemma 5 is similar in spirit to the structural results established in Lemma 3 for Algorithm 1, with the difference of being only applicable to threads j with $\widehat{\varepsilon}_j \geq \varepsilon$. For the other threads with $\widehat{\varepsilon}_j < \varepsilon$, because their corresponding elimination rule being too aggressive due to under-estimation of the true parameter ε , it is not guaranteed that $S^* \subseteq \mathcal{A}_j^{(\tau+1)}$ holds, and their incurred regret will be upper bounded in other ways detailed later in this section.

Our next lemma analyzes the important step 15 of Algorithm 3:

Lemma 6. If $\hat{\varepsilon}_J \geq \varepsilon$ then with probability $1 - O(\tau_0 NJ/T)$, Algorithm 3 will not be re-started.

At a higher level, Lemma 6 states that with high probability, whenever the step 15 is triggered (which causes $J \leftarrow J - 1$ and the re-start of the entire algorithm), the smallest hypothetical value $\widehat{\varepsilon}_J$ remains below the actual value of ε . This ensures that the actual ε always falls between $\widehat{\varepsilon}_0$ and $\widehat{\varepsilon}_J$ throughout the entire T time periods.

The proof of Lemma 6 is highly non-trivial is based on Lemma 5. In particular, the condition in step 15 of Algorithm 3 compares the obtained optimistic assortments $S_{\tau,j}^{(i)}$ in thread j with estimates in threads j' < j, which have larger $\widehat{\varepsilon}_j$ values. If, hypothetically, $\widehat{\varepsilon}_j$ is larger than or equal to ε , then by Lemma 6 the actual optimal assortment S^* should be retained in $\mathcal{A}_{j'}^{(\tau+1)}$ for all $j' \leq j$, and therefore the estimated optimality of $S_{\tau,j}^{(i)}$ should be consistent in all threads $j' \leq j$. Hence, any inconsistency as detected by step 15 would suggest $\widehat{\varepsilon}_j < \varepsilon$, in which case decreasing J is justified.

We proceed to describe two lemmas upper bounding regret accumulated by different threads of Algorithm 3. We first define some notations. For $0 \le j < J$, denote $\mathsf{R}(\widehat{\varepsilon}_j)$ as the cumulative regret of all time periods during which thread j is run. Clearly, the total regret incurred is upper bounded by $\sum_{j < J} \mathsf{R}(\widehat{\varepsilon}_j)$. Using linearity of the expectation, it then suffices to upper bound $\mathbb{E}[\mathsf{R}(\widehat{\varepsilon}_j)]$ for every j < J. The next two lemmas then upper bound $\mathbb{E}[\mathsf{R}(\widehat{\varepsilon}_j)]$ for two different scenarios. For notational simplicity we use \le to hide $\mathsf{poly}(K, \log(NT))$ factors.

Lemma 7. For all
$$j < J$$
 satisfying $\widehat{\varepsilon}_j \geq \varepsilon$, $\mathbb{E}[\mathsf{R}(\widehat{\varepsilon}_j)] \lesssim \sum_{\tau \leq \tau_0} \mathbb{E}[\widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau) \times \wp_j T_\tau]$.

Lemma 8. For all j < J satisfying $\widehat{\varepsilon}_j < \varepsilon$ and any $\widehat{\varepsilon}_k > \max\{\widehat{\varepsilon}_j, \varepsilon\}$, it holds that $\mathbb{E}[\mathsf{R}(\widehat{\varepsilon}_j)] \lesssim \sum_{\tau \leq \tau_0} \mathbb{E}[\widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau) \times \wp_j T_{\tau}]$.

The lemmas 7 and 8 upper bound the total accumulated regret of threads $0 \le j < J$, separately for the case of $\widehat{\varepsilon}_j \ge \varepsilon$ and $\widehat{\varepsilon}_j < \varepsilon$. The case of $\widehat{\varepsilon}_j \ge \varepsilon$ is relatively easier to prove, since $S^* \subseteq \mathcal{A}_j^{(\tau+1)}$ as shown in Lemma 5 and similar proofs to Theorem 1 for non-adaptive algorithms could be used. The case of $\widehat{\varepsilon}_j < \varepsilon$ is, on the other hand, much more difficult because S^* might be eliminated by these threads, and the stopping rule in step 15 of Algorithm 3 is carefully analyzed to upper bound the regret incurred by these threads. The complete proofs of both lemmas, as well as the complete proof of Theorem 3, are deferred to the supplementary material.

6 Numerical illustration

The main purpose of the numerical illustration is to demonstrate the robustness of our proposed policy and its benefit over existing non-robust policies in dynamic assortment optimization literature, including Thompson sampling (TS) (Agrawal et al., 2017) and Upper Confidence Bounds (UCB) (Agrawal et al., 2019). To this end, we construct the following data instance:

- 1. K out of N items have revenue parameters $r_i \equiv 1$ and preference parameters $v_i \equiv 0$;
- 2. For the other (N K) items, both their revenue and preference parameters (r_i, v_i) are uniformly distribution on [0.1, 0.2];
- 3. For the first $\lfloor \varepsilon T \rfloor$ time periods, the arriving customers are outliers with choice models $Q_t \equiv Q$, where Q is an MNL-parameterized choice model with preference parameters set as $v_i' = 1$ if $v_i = 0$ and $v_i' = v_i$ otherwise.

This constructed instance aims to reflect two important properties of outlier customers in practice. First, outlier customers would have quite different preferences from typical customers. Second, outlier customers frequently arrive in consecutive time chunks, for example during a particular period of holiday season. It is important that the retailers' assortment recommendation will not be swung by such chunks of outlier customers after they arrived and left.

In particular, the created problem instance consists of K items with very high revenue, but very low preference parameters such that few customers will buy them. Under normal circumstances, a dynamic assortment optimization algorithm could identify the popularity of these K items very fast and stop providing them. However, with outlier customers, the dynamic assortment planning problem becomes much challenging as outlier customers might prefer these K items over the other ones, essentially generating high "fake" revenue as these items are both popular (among outlier customers) and profitable. A robust algorithm shall not be severely impacted by such outlier customers.

For the baseline methods, the TS method is tuning-free with a non-informative $\operatorname{Beta}(1,1)$ prior on each item. For the UCB algorithm, we find the value in the multiplier (C_1) when constructing upper confidence bands that give the best performance (in the original paper of Agrawal et al. (2019) C_1 is set to $C_1 = 48$ for theoretical purposes). Each method is run for 100 independent trials and the mean average regret (i.e., the cumulative regret over T) is reported. The standard deviation of all the methods are sufficiently small and thus omitted for better visualization.

In Figure 1, we report the results for all methods under various settings of T, N, K and ε . The experimental settings are chosen as $N \in \{100, 300\}$, $K \in \{10, 20\}$, $\varepsilon \in \{0.05, 0.1\}$ and T ranging from T=1,000 to T=20,000. From Figure 1, we can see that our proposed algorithms will stabilize at a mean regret level (0.02 to 0.06) much lower than the non-robust TS and UCB methods. Before reaching the minimum regret level, with longer time horizon the average regret will decrease for our methods, while for TS/UCB the regret will *not* decrease with larger T, especially when ε is large. Our results thus confirm the effectiveness of our proposed algorithms for robust dynamic assortment optimization.

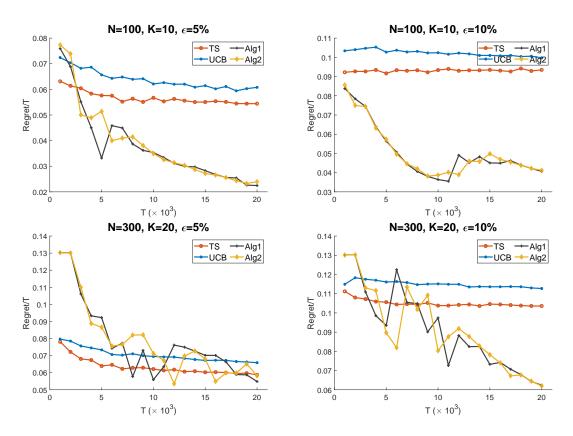


Figure 1: Comparison of average regret (i.e., regret divided by T) between our proposed algorithms and baselines. The time horizon T ranges from 1,000 to 20,000.

7 Conclusions and Future Work

In this paper, we extend the ε -contamination model from statistics to the setting of online learning and study the dynamic assortment optimization problem under the model. We propose a new active elimination policy robust to the adversarial corruptions and establish the regret bound. We further develop an adaptive policy that does not require any prior knowledge of the corruption proportion ε .

Although we have mentioned that the cardinality K is usually a small constant in practice, it is a technically interesting open problem to further sharpen K in our regret bound. However, from our experience, making K tight can be technically very challenging. Beyond this technical question, we hope that this work can attract more attentions on model mis-specification in revenue management, which is indeed an important issue in practice. Most statistical estimation procedure or learning algorithms are built on certain models. However, model mis-specification widely occurs in real applications and the contamination model has been widely adopted for modeling mis-specification since it was proposed in 1960's. The idea developed in this work can be useful for solving other operations management problems under contamination models (e.g., dynamic pricing).

Appendix: proof of lower bound (Theorem 2)

It suffices to prove that the regret of any π is lower bounded by $\Omega(\max\{\varepsilon T, \sqrt{NT}\})$, because $\varepsilon T + \sqrt{NT} \le 2\max\{\varepsilon T, \sqrt{NT}\}$. An $\Omega(\sqrt{NT})$ lower bound has already been established in (Chen & Wang, 2018) without the introduction of outlier customers. Hence to prove Theorem 2 we only need to establish an $\Omega(\varepsilon T)$ regret lower bound.

Consider two problem instances $P_1 = \{r_i, v_i\}_{i=1}^n$ and $P_2 = \{r_i, v_i'\}_{i=1}^n$ with shared revenue parameters $\{r_i\}_{i=1}^n$ and different preference parameters (for typical customers) $\{v_i\}_{i=1}^n$, $\{v_i'\}_{i=1}^n$, such that for any assortment $S \subseteq [N]$, $|S| \le K$, $\max\{R(S_1^*|P_1) - R(S|P_1), R(S_2^*|P_2) - R(S|P_2)\} = \Omega(1)$, where S_1^* and S_2^* are the optimal assortments under P_1 and P_2 , respectively. The existence and explicit construction of such problem instances can be found in (Chen & Wang, 2018). Now consider the case in which all of the first $\lfloor \varepsilon T \rfloor$ customers are outliers, associated with the *same* outlier choice model Q under both P_1 and P_2 . Because the choice model of the outlier customers are the same, no algorithm can distinguish P_1 from P_2 during the first $\lfloor \varepsilon T \rfloor$ time periods with success probability larger than 1/2. Therefore, the worst-case regret (under P_1 and P_2) of any algorithm is at least $\Omega(\varepsilon T)$, which is to be demonstrated.

Appendix: additional proofs

A Proofs of technical lemmas for Theorem 1

A.1 Proof of Lemma 1

Lemma 1 (restated). Suppose $T_0 \geq 128(K+1)^2 N_\tau \ln T$ and $\min\{1, \varepsilon T/T_\tau\} \leq 1/4(K+2)$. With probability $1 - O(\tau_0 N/T^2)$ it holds for all τ satisfying $T_\tau \geq \max\{\overline{\varepsilon}, \varepsilon\}T/4(K+1)$ and $i \in \mathcal{A}^{(\tau+1)}$ that $|\widehat{v}_i^{(\tau+1)} - v_i| \leq \Delta_\varepsilon^*(i, \tau+1)$, where

$$\Delta_{\varepsilon}^{*}(i,\tau+1) = 8(K+1) \left(\frac{\varepsilon_{\tau}}{2} + \sqrt{\frac{\varepsilon_{\tau} N_{\tau} \ln T}{T_{\tau}}} + \frac{2N_{\tau} \ln T}{3T_{\tau}} \right) + 8\sqrt{\frac{(1+V_{S})v_{i}N_{\tau} \ln T}{T_{\tau}}}, \quad (10)$$

where ε_{τ} is defined as $\varepsilon_{\tau} = \min\{1, \varepsilon T/T_{\tau}\}$, $N_{\tau} = |\mathcal{A}^{(\tau+1)}|$ and $V_S = \sum_{j \in S_{\tau}^{(i)}} v_j$.

Proof. Denote $N_{\tau}:=|\mathcal{A}^{(\tau+1)}|$ and let \mathcal{T}_{τ} be the T_{τ} consecutive time periods during which assortments $S_{\tau}^{(i)}$, $i\in\mathcal{A}^{(\tau+1)}$ are offered uniformly at random. Let also \mathcal{T}_{τ} be the set of all time periods at epoch τ . For each $i\in\mathcal{A}^{(\tau+1)}$ and $t\in\mathcal{T}_{\tau}$, define indicator variable $I_{ti}(0)=1$ if $S_{\tau}^{(i)}$ is offered at time t and the no-purchase action $i_t=0$ is taken from the incoming customer, and $I_{ti}(0)=0$ otherwise. Similarly, define $I_{ti}(i)=1$ if $S_{\tau}^{(i)}$ is offered at time t and the purchase of item $i,i_t=i$, is observed from the incoming customer. We then have, by definition, that,

$$\frac{n_0(i)}{T_\tau} = \frac{1}{T_\tau} \sum_{t \in \mathcal{T}_\tau} I_{ti}(0), \qquad \frac{n_i}{T_\tau} = \frac{1}{T_\tau} \sum_{t \in \mathcal{T}_\tau} I_{ti}(i). \tag{11}$$

Recall the definition that $V_S := \sum_{j \in S_{\tau}^{(j)}} v_j$. Let \mathcal{T}_{τ}^* and \mathcal{T}_{τ}^o be the time periods corresponding to typical and outlier customers, respectively. The expectation of $n_0(i)/T_{\tau}$ can subsequently be

calculated as

$$\mathbb{E}\left[\frac{n_0(i)}{T_{\tau}}\right] = \frac{1}{T_{\tau}} \left(\sum_{t \in \mathcal{T}_{\tau}^*} \frac{1}{N_{\tau}(1 + V_S)} + \sum_{t \in \mathcal{T}_{\tau}^o} \frac{\Pr[i_t = 0 | Q_t]}{N_{\tau}} \right)$$
$$= \frac{1}{N_{\tau}(1 + V_S)} + \frac{1}{N_{\tau}T_{\tau}} \sum_{t \in \mathcal{T}_{\tau}^o} \left(\Pr[i_t = 0 | Q_t] - \frac{1}{1 + V_S} \right).$$

Note that since assortments are selected at random, the events " $S_{\tau}^{(i)}$ is offered at time t" and "customer arriving at time t is an outlier" are independent, which is crucial in the above derivation. Define $\alpha_0 := \frac{1}{|T_{\tau}^o|} \sum_{t \in \mathcal{T}_{\tau}^o} (\Pr[i_t = 0|Q_t] - 1/(1+V_S))$. It is clear by definition that $|\alpha_0| \leq 1$. Furthermore, because at most εT customers are outliers throughout the entire T time periods, we know that $|\mathcal{T}_{\tau}^o| \leq \min\{T_{\tau}, \varepsilon T\}$ and hence $\widetilde{\varepsilon}_{\tau} := |\mathcal{T}_{\tau}^o|/T_{\tau} \leq \min\{1, \varepsilon T/T_{\tau}\} = \varepsilon_{\tau}$. Subsequently, we have

$$\mathbb{E}\left[\frac{n_0}{T_\tau}\right] = \frac{1}{N_\tau(1+V_S)} + \frac{\widetilde{\varepsilon}_\tau \alpha_0}{N_\tau}.\tag{12}$$

Similarly, for n_i we have

$$\mathbb{E}\left[\frac{n_i}{T_\tau}\right] = \frac{v_i}{N_\tau(1+V_S)} + \frac{\widetilde{\varepsilon}_\tau \alpha_i}{N_\tau},\tag{13}$$

where $\alpha_i = \frac{1}{|\mathcal{T}_{\tau}^o|} \sum_{t \in \mathcal{T}_{\tau}^o} (\Pr[i_t = i|Q_t] - v_i/(1 + V_S))$ which also satisfies $|\alpha_i| \leq 1$.

It is easy to verify that the partial sums $\sum_{t \in \mathcal{T}_{\tau}, t < s} I_{ti}(j) - \mathbb{E}[I_{ti}(j)|\mathcal{F}_{s-1}]$ form martingales for both $j \in \{0, i\}$, because the decision of whether customer t is an outlier is independent from the event $\mathbf{1}\{S_t = S_{\tau}^{(i)}\}$. The variances of n_i and n_i can also be upper bounded as $\mathbb{V}[n_0] \leq T_{\tau} \varepsilon_{\tau}/N_{\tau} + T_{\tau}/N_{\tau}(1+V_S)$ and $\mathbb{V}[n_i] \leq T_{\tau} \varepsilon_{\tau}/N_{\tau} + T_{\tau} v_i/N_{\tau}(1+V_S)$. Subsequently, nvoking Bernstein's inequality (Lemma 8), we have that

$$\frac{n_0}{T_{\tau}} = \frac{\widetilde{\varepsilon}_{\tau} \alpha_0}{N_{\tau}} + \frac{1}{N_{\tau} (1 + V_S)} + \eta_0, \quad \frac{n_i}{T_{\tau}} = \frac{\widetilde{\varepsilon}_{\tau} \alpha_i}{N_{\tau}} + \frac{v_i}{N_{\tau} (1 + V_S)} + \eta_i,$$

where

$$\Pr\left[|\eta_0| > \frac{4\ln T}{3T_\tau} + 2\sqrt{\frac{\varepsilon_\tau \ln T}{N_\tau T_\tau}} + 2\sqrt{\frac{\ln T}{(1+V_S)N_\tau T_\tau}}\right] \le \frac{2}{T^2};$$

$$\Pr\left[|\eta_i| > \frac{4\ln T}{3T_\tau} + 2\sqrt{\frac{\varepsilon_\tau \ln T}{N_\tau T_\tau}} + 2\sqrt{\frac{v_i \ln T}{(1+V_S)N_\tau T_\tau}}\right] \le \frac{2}{T^2}.$$

The estimate $\hat{v}_i = n_i/n_0$ then admits the form of

$$\widehat{v}_i = \frac{v_i + (1 + V_S)(\widetilde{\varepsilon}_\tau \alpha_i + \eta_i N_\tau)}{1 + (1 + V_S)(\widetilde{\varepsilon}_\tau \alpha_0 + \eta_0 N_\tau)} = v_i + \frac{(1 + V_S)(\widetilde{\varepsilon}_\tau \alpha_i + \eta_i N_\tau - v_i(\widetilde{\varepsilon}_\tau \alpha_0 + \eta_0 N_\tau))}{1 + (1 + V_S)(\widetilde{\varepsilon}_\tau \alpha_0 + \eta_0 N_\tau)}.$$

Additionally, by Hoeffding's inequality, it also holds that

$$\Pr\left[\max\{|\eta_0|, |\eta_i|\} > \sqrt{\frac{8\ln T}{N_\tau T_\tau}}\right] \le \frac{4}{T^2}.$$
(14)

Because $V_S \leq K$, $\alpha_0, \alpha_1 \in [0, 1]$ and $|\eta_0|, |\eta_i| \leq \sqrt{8 \ln T/N_\tau T_\tau}$ with probability $1 - 4/T^2$, we have that (with probability $1 - O(T^{-2})$)

$$\left| \widehat{v}_i - v_i \right| \le \frac{(1 + V_S)(2\varepsilon_\tau + 8N_\tau \ln T / (3T_\tau) + 4\sqrt{\varepsilon_\tau N_\tau \ln T / T_\tau} + 4\sqrt{v_i N_\tau \ln T / (1 + V_S) T_\tau})}{1 - (K+1)\varepsilon_\tau - (K+1)\sqrt{8N_\tau \ln T / T_\tau}}.$$

Provided that $\varepsilon_{\tau} \leq 1/4(K+2)$

$$\left|\widehat{v}_i - v_i\right| \le 4(K+1)\varepsilon_\tau + 8(K+1)\sqrt{\frac{\varepsilon_\tau N_\tau \ln T}{T_\tau}} + \frac{16(K+1)N_\tau \ln T}{3T_\tau} + 8\sqrt{\frac{(1+V_S)v_i N_\tau \ln T}{T_\tau}}$$
$$= \Delta_\varepsilon^*(i, \tau+1),$$

which completes the proof of Lemma 1.

A.2 Proof of Lemma 2

Lemma 2 (restated). For any $S \subseteq [N]$, $|S| \leq K$ and $\{\hat{v}_i\}$, it holds that

$$|R(S; \widehat{v}) - R(S; v)| \le \frac{2\sum_{i \in S} |\widehat{v}_i - v_i|}{1 + \sum_{i \in S} v_i}.$$

Proof. Expanding the definitions of $R(S; \hat{v})$ and R(S; v), we have

$$\begin{split} \left| R(S; \widehat{v}) - R(S; v) \right| &= \left| \frac{\sum_{i \in S} r_i \widehat{v}_i}{1 + \sum_{i \in S} \widehat{v}_i} - \frac{\sum_{i \in S} r_i v_i}{1 + \sum_{i \in S} v_i} \right| \\ &= \left| \frac{\left(\sum_{i \in S} r_i \widehat{v}_i \right) (1 + \sum_{i \in S} v_i) - \left(\sum_{i \in S} r_i v_i \right) (1 + \sum_{i \in S} \widehat{v}_i)}{(1 + \sum_{i \in S} \widehat{v}_i) (1 + \sum_{i \in S} v_i)} \right| \\ &\leq \frac{\left(\sum_{i \in S} r_i \widehat{v}_i \right) \left(\sum_{i \in S} \left| v_i - \widehat{v}_i \right| \right) + \left(1 + \sum_{i \in S} \widehat{v}_i \right) \left(\sum_{i \in S} r_i \left| \widehat{v}_i - v_i \right| \right)}{(1 + \sum_{i \in S} \widehat{v}_i) (1 + \sum_{i \in S} v_i)} \\ &\leq \frac{2 \sum_{i \in S} \left| \widehat{v}_i - v_i \right|}{1 + \sum_{i \in S} v_i}. \end{split}$$

A.3 Proof of Corollary 1

Corollary 1 (restated). For every τ and $|S| \leq K$, $S \subseteq \mathcal{A}^{(\tau)}$, conditioned on the success events on epochs up to τ , it holds that $|R(S; \widehat{v}^{(\tau)}) - R(S; v)| \leq \widehat{\Delta}_{\varepsilon}(\tau) \leq \widehat{\Delta}_{\max\{\varepsilon,\overline{\varepsilon}\}}(\tau)$, where $\widehat{\Delta}$ is defined in Algorithm 1.

Proof. Note that it suffices to prove the first inequality, because $\widehat{\Delta}$ is a monotonically increasing function in ε . Also, we only need to consider the case of $\varepsilon_{\tau-1} \leq 1/4(K+1)$, as $\widehat{\Delta}_{\varepsilon}(\tau) = 1$ otherwise which trivially upper bounds $|R(S; \widehat{v}^{(\tau)}) - R(S; v)|$. Invoking Lemma 2 and the upper bound $|\widehat{v}_i^{(\tau)} - v_i| \leq \Delta_{\varepsilon}^*(i, \tau)$ in Lemma 1, we have (recall the definition that $V_S = \sum_{i \in S} v_i$)

$$\begin{aligned} &|R(S; \widehat{v}^{(\tau)}) - R(S; v)| \leq \frac{2\sum_{i \in S} \Delta_{\varepsilon}^{*}(i, \tau)}{1 + V_{S}} \\ &\leq 2K \times 8(K+1) \left(\frac{\varepsilon_{\tau-1}}{2} + \sqrt{\frac{\varepsilon_{\tau-1} N_{\tau-1} \ln T}{T_{\tau-1}}} + \frac{2N_{\tau-1} \ln T}{3T_{\tau-1}} \right) \\ &+ 16\sqrt{\frac{(1+V_{S})N_{\tau-1} \ln T}{T_{\tau-1}}} \times \frac{\sum_{i \in S} \sqrt{v_{i}}}{1 + V_{S}} \\ &\leq 16K(K+1) \left(\frac{\varepsilon_{\tau-1}}{2} + \sqrt{\frac{\varepsilon_{\tau-1} N_{\tau-1} \ln T}{T_{\tau-1}}} + \frac{2N_{\tau-1} \ln T}{3T_{\tau-1}} \right) + 16\sqrt{\frac{(1+V_{S})N_{\tau-1} \ln T}{T_{\tau-1}}} \frac{\sqrt{KV_{S}}}{1 + V_{S}} \\ &\leq 16K(K+1) \left(\frac{\varepsilon_{\tau-1}}{2} + \sqrt{\frac{\varepsilon_{\tau-1} N_{\tau-1} \ln T}{T_{\tau-1}}} + \frac{2N_{\tau-1} \ln T}{3T_{\tau-1}} \right) + 16\sqrt{\frac{KN_{\tau-1} \ln T}{T_{\tau-1}}} \\ &= \widehat{\Delta}_{\varepsilon}(\tau). \end{aligned}$$

Here in the third inequality we use Cauchy-Schwarz inequality on $\sum_{i \in S} \sqrt{v_i}$; more specifically, $\sum_{i \in S} \sqrt{v_i} = \sum_{i \in S} \sqrt{v_i} \times 1 \le \sqrt{\sum_{i \in S} 1} \times \sqrt{\sum_{i \in S} v_i} \le \sqrt{K} \times \sqrt{V_S}$, as $|S| \le K$ and $V_S = \sum_{i \in S} v_i$, so that $\sqrt{V_S} \le \max\{1, V_S\}$.

A.4 Proof of Lemma 3

Lemma 3 (restated). If $\overline{\varepsilon} \geq \varepsilon$ then with probability $1 - O(\tau_0 N/T^2)$ it holds that $S^* \subseteq \mathcal{A}^{(\tau)}$ for all τ .

Proof. We use induction to prove this lemma. Let τ^* be the smallest integer such that $T_{\tau^*} \geq \overline{\varepsilon}T/4(K+1)$. Because $\mathcal{A}^{(0)} = \cdots = \mathcal{A}^{(\tau^*)} = [N]$, the lemma clearly holds for τ^* . Next, conditioned on $S^* \subseteq \mathcal{A}^{(\tau)}$, we will prove that $S^* \subseteq \mathcal{A}^{(\tau+1)}$.

Let \widehat{S} be the solution of step (*) in Algorithm 1. If $\tau = \tau^*$, then $R(\widehat{S}; v) \leq R(S^*; v) + \widehat{\Delta}_{\overline{\varepsilon}}(\tau)$ because $\widehat{\Delta}_{\overline{\varepsilon}}(\tau) = 1$. If $\tau > \tau^*$, we have

$$\gamma^{(\tau)} = R(\widehat{S}; \widehat{v}^{(\tau)}) \le R(\widehat{S}; v) + \widehat{\Delta}_{\varepsilon}(\tau) \le R(S^*; v) + \widehat{\Delta}_{\varepsilon}(\tau) \le R(S^*; v) + \widehat{\Delta}_{\overline{\varepsilon}}(\tau),$$

where the first inequality holds by Corollary 1, and the last inequality holds by monotonicity of $\widehat{\Delta}_{\varepsilon}(\tau)$. In both cases, it holds that

$$\gamma^{(\tau)} \le R(S^*; v) + \widehat{\Delta}_{\overline{\varepsilon}}(\tau). \tag{15}$$

For any $i \in S^*$, let $S_{\tau}^{(i)}$ be the solution of (\dagger) of Algorithm 1. Because $S^* \subseteq \mathcal{A}^{(\tau)}$ by the induction hypothesis, we know that S^* is a feasible solution to the optimization question of (\dagger) and

therefore

$$R(S_{\tau}^{(i)}; \widehat{v}^{(\tau)}) \ge R(S^*; \widehat{v}^{(\tau)}) \ge R(S^*; v) - \widehat{\Delta}_{\overline{\varepsilon}}(\tau). \tag{16}$$

Combining Eqs. (15,16) we have (with high probability) that $R(S_{\tau}^{(i)}; \widehat{v}) \geq \gamma^{(\tau)} - 2\widehat{\Delta}_{\overline{\varepsilon}}(\tau)$, and hence $i \in \mathcal{A}^{(\tau+1)}$. Repeat the argument for all $i \in S^*$ we have proved that $S^* \subseteq \mathcal{A}^{(\tau+1)}$ with high probability.

A.5 Proof of Lemma 4

Lemma 4 (restated). Suppose $S^* \subseteq \mathcal{A}^{(\tau)}$ holds for all τ . Then with probability $1 - O(\tau_0 N/T^2)$, for every $\tau \leq \tau_0$ and $i \in \mathcal{A}^{(\tau+1)}$, it holds that $R(S^*; v) - R(S^{(i)}_{\tau}; v) \leq 4\widehat{\Delta}_{\overline{\varepsilon}}(\tau)$.

Proof. Because $i \in \mathcal{A}^{(\tau+1)}$, we know that $R(S_{\tau}^{(i)}; \widehat{v}^{(\tau)}) \geq \gamma^{(\tau)} - 2\widehat{\Delta}_{\overline{\varepsilon}}(\tau)$. Additionally, because $S^* \subseteq \mathcal{A}^{(\tau)}$, we have that $\gamma^{(\tau)} = R(\widehat{S}_{\tau}; \widehat{v}^{(\tau)}) \geq R(S^*; \widehat{v}^{(\tau)}) \geq R(S^*; v) - \widehat{\Delta}_{\overline{\varepsilon}}(\tau)$, with high probability by invoking Corollary 1. Subsequently,

$$R(S_{\tau}^{(i)}; v) \ge R(S_{\tau}^{(i)}; \widehat{v}^{(\tau)}) - \widehat{\Delta}_{\overline{\varepsilon}}(\tau) \ge \gamma^{(\tau)} - 3\widehat{\Delta}_{\overline{\varepsilon}}(\tau) \ge R(S^*; v) - 4\widehat{\Delta}_{\overline{\varepsilon}}(\tau),$$

which is to be demonstrated.

B Proofs of technical lemmas of Theorem 3

First, note that if $\varepsilon \lesssim \sqrt{N/T}$, the εT term in Theorem 3 will be dominated by the \sqrt{NT} term and is therefore not important. Hence, throughout the rest of this section we shall assume without loss of generality that $\varepsilon \geq \sqrt{N/T}$, which also means that $\widehat{\varepsilon}_J \leq \varepsilon$ in the beginning.

B.1 Proof of Lemma 5

Lemma 5 (restated). With probability $1 - O(\tau_0 N J/T^2)$ it holds for all τ and $\widehat{\varepsilon}_j \geq \varepsilon$ that $S^* \subseteq \mathcal{A}_j^{(\tau)}$.

Proof. Because each thread j < J is sampled at random with probability \wp_j , the expected total number of outlier customers thread j encounters is upper bounded by $\varepsilon \times \wp_j T = \mathbb{E}[\varepsilon T_j]$. Hence, by Bernstein's inequality and the union bound, for $\varepsilon \gtrsim \sqrt{N/T}$, with probability at least $1 - O(T^2)$ the total number of outlier customers thread j encounters is upper bounded by $O(\varepsilon T_j \log T)$. In the rest of this proof, we will consider $\varepsilon \to \varepsilon \log T$ instead of merely ε , which only adds multiplicative $\log T$ factors to the regret bound in Theorem 3. With such considerations, for all j < J satisfying $\widehat{\varepsilon}_j \geq \varepsilon$, Lemma 1 and Corollary 1 in the previous proof of Theorem 1 would remain valid.

The rest of the proof is quite similar to the proof of Lemma 3, except we have to take into consideration the effect of Step 8 of Algorithm 3. The proof is again done via induction: at the first epoch $\tau=0$ we have $\mathcal{A}_j^{(\tau)}=[N]$ and the lemma clearly holds. Now assume the lemma holds for some τ , we want to prove $S^*\subseteq\mathcal{A}_j^{(\tau+1)}$ for all $\widehat{\varepsilon}_j\geq\varepsilon$.

Fix arbitrary $i \in S^* \subseteq \mathcal{A}_j^{(\tau)}$ and assume by way of contradiction that $i \notin \mathcal{A}_j^{(\tau+1)}$. Then there exists $k \leq j$ such that $R(S_{\tau,k}^{(i)}; \widehat{v}^{(\tau),k}) < \gamma_k^{(\tau)} - 2\widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau)$. Additionally, because $S_{\tau,k}^{(i)}$ is the maximizer of $R(S; \widehat{v}^{(\tau),k})$ for all $|S| \leq k$, $i \in S$, it holds that $R(S_{\tau,k}^{(i)}; \widehat{v}^{(\tau),k}) \geq R(S^*; \widehat{v}^{(\tau),k})$. Let also \widehat{S}_k be

the assortment attaining $\gamma_k^{(\tau)}$ (i.e., $R(\widehat{S}_k; \widehat{v}^{(\tau),k}) = \gamma_k^{(\tau)}$). Then, invoking Corollary 1, we have with probability $1 - O(NJ/T^2)$ that

$$R(S^*; v) \leq R(S^*; \widehat{v}^{(\tau),k}) + \widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau) \leq R(S_{\tau,k}^{(i)}; \widehat{v}^{(\tau),k}) + \widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau)$$
$$< \gamma_k^{(\tau)} - \widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau) = R(\widehat{S}_k; \widehat{v}^{(\tau),k}) - \widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau) \leq R(\widehat{S}_k; v) \leq R(S^*; v),$$

leading to the desired contradiction.

B.2 Proof of Lemma 6

Lemma 6 (restated). If $\widehat{\varepsilon}_J \geq \varepsilon$ then with probability $1 - O(\tau_0 NJ/T)$, Algorithm 3 will not be re-started.

Proof. We only need to prove that, if $\widehat{\varepsilon}_J \geq \varepsilon$, then for any time period t, the condition at step 14 of Algorithm 3 is satisfied with probability at most $O(\tau_0 N J/T^2)$. Invoking Lemma 5, we know that with high probability $S^* \subseteq \mathcal{A}_j^{(\tau)}$ holds for all $\tau \leq \tau_0$ and $j \leq J$. Additionally, by algorithm design it is always guaranteed that $\mathcal{A}_j^{(\tau+1)} \subseteq \mathcal{A}_k^{(\tau)}$ for any $\widehat{\varepsilon}_k > \widehat{\varepsilon}_j$, and therefore $\widehat{S}_{\tau,j}^{(i)} \subseteq \mathcal{A}_j^{(\tau)}$ implies $\widehat{S}_{\tau,j}^{(i)} \subseteq \mathcal{A}_k^{(\tau)}$. Subsequently, invoking Corollary 1 we have with probability $O(\tau_0 N/T^2)$ that

$$R(\widehat{S}_{\tau,j}^{(i)}; \widehat{v}^{(\tau),k}) \ge R(\widehat{S}_{\tau,j}^{(i)}; v) - \widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau). \tag{17}$$

Since $i \in \mathcal{A}_j^{(\tau+1)}$, by the construction of $\mathcal{A}_j^{(\tau+1)}$ we know that $R(\widehat{S}_{\tau,j}^{(i)}; \widehat{v}^{(\tau),j}) \geq \gamma_j^{(\tau)} - 2\widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau)$. Let also \widehat{S}_j be the assortment attaining $\gamma_j^{(\tau)}$ (i.e., $R(\widehat{S}_j; \widehat{v}^{(\tau),j}) = \gamma_j^{(\tau)}$). Then invoking Corollary 1 again, we have with probability $1 - O(\tau_0 N/T^2)$ that

$$R(\widehat{S}_{\tau,j}^{(i)};v) \ge R(\widehat{S}_{\tau,j}^{(i)};\widehat{v}^{(\tau),j}) - \widehat{\Delta}_{\widehat{\varepsilon}_{j}}(\tau) \ge \gamma_{j}^{(\tau)} - 3\widehat{\Delta}_{\widehat{\varepsilon}_{j}}(\tau) = R(\widehat{S}_{j};\widehat{v}^{(\tau),j}) - 3\widehat{\Delta}_{\widehat{\varepsilon}_{j}}(\tau)$$

$$\stackrel{(*)}{\ge} R(S^{*};\widehat{v}^{(\tau),j}) - 3\widehat{\Delta}_{\widehat{\varepsilon}_{j}}(\tau) \ge R(S^{*};v) - 4\widehat{\Delta}_{\widehat{\varepsilon}_{j}}(\tau). \tag{18}$$

Here Eq. (*) holds because $S^* \subseteq \mathcal{A}_i^{(\tau)}$. Combining Eqs. (17) and (18), we have

$$R(\widehat{S}_{\tau,j}^{(i)}; \widehat{v}^{(\tau),k}) \ge R(S^*; v) - \widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau) - 4\widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau) \ge R(S^*; v) - 5\widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau), \tag{19}$$

where the last inequality holds because $\widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau) \leq \widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau)$ by definition. On the other hand, with \widehat{S}_k being the assortment attaining $\gamma_k^{(\tau)}$ (i.e., $R(\widehat{S}_k; \widehat{v}^{(\tau),k}) = \gamma_k^{(\tau)}$) and the fact that $S^* \subseteq \mathcal{A}_k^{(\tau)}$, invoking Corollary 1 we have with probability $1 - O(\tau_0 N/T^2)$ that

$$R(S^*; v) \ge R(\widehat{S}_k; v) \ge R(\widehat{S}_k; \widehat{v}^{(\tau),k}) - \widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau) = \gamma_k^{(\tau)} - \widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau). \tag{20}$$

Combining Eqs. (19) and (20) we have with probability $1 - O(\tau_0 N/T^2)$ that

$$R(\widehat{S}_{\tau,j}^{(i)}; \widehat{v}^{(\tau),k}) \ge \gamma_k^{(\tau)} - 6\widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau),$$

which is to be demonstrated.

B.3 Proof of Lemma 7

Lemma 7 (restated). For all $j \leq J$ satisfying $\widehat{\varepsilon}_j \geq \varepsilon$, $\mathbb{E}[\mathsf{R}(\widehat{\varepsilon}_j)] \lesssim \sum_{\tau \leq \tau_0} \mathbb{E}[\widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau) \times \wp_j T_\tau]$.

Proof. Because $\widehat{\varepsilon}_j \geq \varepsilon$, by Lemma 5 we know that $S^* \subseteq \mathcal{A}_j^{(\tau)}$ for all $\tau \leq \tau_0$ with high probability. Then, invoking Lemma 4, the regret incurred by thread j in a single time period is upper bounded by $O(\widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau))$. Because thread j is sampled with probability \wp_j , the expected number of time periods thread j is performed is $\wp_j T$. This completes the proof of Lemma 7.

B.4 Proof of Lemma 8

Lemma 8 (restated). For all j < J satisfying $\widehat{\varepsilon}_j < \varepsilon$ and any $\widehat{\varepsilon}_k > \max\{\widehat{\varepsilon}_j, \varepsilon\}$, it holds that $\mathbb{E}[\mathsf{R}(\widehat{\varepsilon}_j)] \lesssim \sum_{\tau \leq \tau_0} \mathbb{E}[\widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau) \times \wp_j T_{\tau}].$

Proof. Fix arbitrary $\tau \leq \tau_0$. According to step 14 of Algorithm 1, because the value of J does not decrease, we must have $R(\widehat{S}_{\tau,j}^{(i)}; \widehat{v}^{(\tau),k}) \geq \gamma_k^{(\tau)} - 7\widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau)$ for all assortments $\widehat{S}_{\tau,j}^{(i)}$ explored by thread j in epoch τ . Because $\widehat{\varepsilon}_k \geq \varepsilon$, we know that $S^* \subseteq \mathcal{A}_k^{(\tau)}$ with high probability, and using the same argument as in the proof of Lemma 4 we have with high probability that

$$R(S^*; v) - R(\widehat{S}_{\tau, j}^{(i)}; v) \lesssim \widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau),$$

which serves as an upper bound of the regret thread j incurs in a single time period it is performed. Therefore,

$$\mathbb{E}[\mathsf{R}(\widehat{\varepsilon}_j)] \lesssim \sum_{\tau \leq \tau_0} \mathbb{E}[\widehat{\Delta}_{\widehat{\varepsilon}_k}(\tau) \times \wp_j T_\tau],$$

which is to be demonstrated.

B.5 Proof of Theorem 3

Because we restart Algorithm 3 whenever J is reduced, and Lemma 6 shows that (with high probability) $\widehat{\varepsilon}_J \leq 2\varepsilon$ always holds. Note that it is possible for the value of $\widehat{\varepsilon}_J$ to be far smaller than the actual outlier proportion ε . In the rest of this section we shall assume without loss of generality that, throughout a consecutive of $T' \leq T$ time periods the value of J does not change, and furthermore $\widehat{\varepsilon}_J \leq 2\varepsilon$. The total regret over these T' time periods multiplying $J = O(\log T)$ would then be an upper bound on the total regret over the entire T time periods.

We first consider the regret incurred by thread j with $\widehat{\varepsilon}_j \geq \varepsilon$. By Lemma 7, the regret incurred by such a thread in epoch τ can be upper bounded by $O(\mathbb{E}[\widehat{\Delta}_{\widehat{\varepsilon}_j}T_{\tau,j}])$, because $T_{\tau,j} = \wp_j T_{\tau}$. Replacing T_{τ} by $T_{\tau,j}$ in the definition of $\widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau)$, we have that

$$\widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau) \lesssim \min\left\{1, \widehat{\varepsilon}_j T / T_{\tau,j}\right\} + \sqrt{\frac{|\mathcal{A}_j^{(\tau+1)}| \log T}{T_{\tau,j}}} + \frac{|\mathcal{A}_j^{(\tau+1)}| \log T}{T_{\tau,j}}.$$
(21)

On the other hand, because of the sampling protocol in Algorithm 3 we have that

$$\mathbb{E}\left[\sum_{\tau} T_{\tau,j}\right] = \wp_j T' \lesssim 2^{-(J-j)} T'. \tag{22}$$

Subsequently,

$$\mathbb{E}\sum_{\tau}\widehat{\Delta}_{\widehat{\varepsilon}_{j}}(\tau)T_{\tau,j} \lesssim \mathbb{E}\sum_{\tau}\widehat{\varepsilon}_{j}T_{\tau,j} + \sqrt{NT_{\tau,j}} + N.$$
(23)

Using Cauchy-Schwarz inequality and the concavity of $f(\cdot) = \sqrt{\cdot}$, we have

$$\mathbb{E}\sum_{\tau}\sqrt{T_{\tau,j}} \le \sqrt{\mathbb{E}\sum_{\tau}T_{\tau,j}} \lesssim \sqrt{2^{-(J-j)}T'}.$$
 (24)

Note that $\mathbb{E}\widehat{\varepsilon}_j T_j = 2^{-j} \times 2^{-(J-j)}T \le 2^{-J}T \lesssim \varepsilon T$, where the last inequality holds because $2^{-J} \lesssim \varepsilon$ thanks to Lemma 6. Combining this fact with Eqs. (23,24), we have that

$$\mathbb{E}\sum_{\tau} \widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau) \times \wp_j T_{\tau} \lesssim \varepsilon T + \sqrt{NT} + N \lesssim \varepsilon T + \sqrt{NT}.$$
 (25)

We next consider regret incurred by threads j < J with $\widehat{\varepsilon}_j < \varepsilon$. Let j^* be the largest integer such that $\widehat{\varepsilon}_{j^*} \geq \varepsilon$. Then by Lemma 8, the regret incurred by thread j in epoch τ is upper bounded by

$$\mathbb{E}[\widehat{\Delta}_{\widehat{\varepsilon}_{j^*}}(\tau) \times \wp_j T_\tau] \lesssim \mathbb{E}[\widehat{\Delta}_{\varepsilon}(\tau) \times \wp_j T_\tau],$$

where the inequality holds because $\widehat{\varepsilon}_{j^*} \leq 2\varepsilon$ by definition. Using the same analysis in Eqs. (21), (22), (23) and (24), we have that

$$\mathbb{E}\sum_{\tau}\widehat{\Delta}_{\widehat{\varepsilon}_j}(\tau)\times\wp_jT_{\tau}\lesssim \varepsilon T+\sqrt{2^{(J-j^*)}NT}.$$

Because $2^{-J}=\widehat{\varepsilon}_{J}\geq \sqrt{N/T}$ and $2^{-j^{*}}=\widehat{\varepsilon}_{j^{*}}\approx \varepsilon$, it is easy to verify that $\sqrt{2^{J-j^{*}}NT}\lesssim \sqrt{\varepsilon}N^{1/4}T^{3/4}$. Using the inequality $ab\leq (a^{2}+b^{2})/2$ we have that $\sqrt{\varepsilon}N^{1/4}T^{3/4}\leq \varepsilon T+\sqrt{NT}$.

C Tail inequalities

Lemma 8 (Bernstein's inequality for martingale process Freedman (1975)). Let X_1, \dots, X_n be centered random variables satisfying $|X_i| \leq M$ almost surely for all i, and that $\sum_{i \leq s} X_i$ for $s \leq n$ forms a martingale process. Then for any t > 0,

$$\Pr\left[\left|\sum_{i=1}^{n} X_i\right| > t\right] \le 2 \exp\left\{-\frac{t^2/2}{\sum_{i} \mathbb{E}[X_i^2] + Mt/3}\right\}.$$

As a corollary, for any $\delta > 0$,

$$\Pr\left[\left|\sum_{i=1}^{n} X_i\right| > \frac{2}{3} M \log(1/\delta) + \sqrt{2V^2 \log(1/\delta)}\right] \le 2\delta,$$

where $V^2 = \sum_i \mathbb{E}[X_i^2]$.

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