Chapter 6 : Regularization

6.1 Ridge

Seong yeon Park

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The Three Sisters of Newton
School of Mathematics, Statistics and Data Science
Sungshin Women's University

Outline

We formulate the least squares method for multiple regression with matrices.

$$L := \sum_{i=1}^{N} (y_i - \beta_0 - \beta_1 x_{i,1}, \cdots, -\beta_1 x_{i,p})^2,$$

$$L = \| y - X\beta \|^2 = (y - X\beta)^T (y - X\beta)$$
$$= y^T y - y^T X\beta - \beta^T X^T y - \beta^T X^T X\beta$$

- Partial differentiation with L

$$\nabla L := \frac{\partial L}{\partial \beta} = -X^T y - X^T y + 2X^T X\beta = -2X^T (y - X\beta)$$

- Set to zero to find the minimum value

$$-2X^T (y - X\beta) = 0$$

Multiple Regression Singular/Nonsingular Matrix

- When a matrix X^TX is nonsingular, we have

$$2X^TX\beta = 2X^Ty$$
$$\hat{\beta} = (X^TX)^{-1}X^Ty$$

- If matrix X^TX is singular, Determinant is too small, β becomes large and an inconvenient situation occurs.

Ridge Regression

- $\lambda \geq 0$ be a constant, we often use to minimize the square error plus by the squared norm of β multiplied by λ.
- Loss function of existing linear regression (sum of error squared)

$$L = \frac{1}{N} \parallel y - X\beta \parallel^2$$

- Loss function of ridge regression

$$L := \frac{1}{N} \parallel y - X\beta \parallel^2 + \lambda \parallel \beta \parallel_2^2$$

- $\lambda \parallel \beta \parallel_2^2$ is the regularization term of ridge regression.
- The larger the λ, the smaller the β size.

is the square of the L2 norm of β.

Differentiate Loss function of Ridge Regression

- Loss function of ridge regression

$$
\begin{aligned}
L &:= \frac{1}{N} \parallel y - X\beta \parallel^2 + \lambda \parallel \beta \parallel_2^2 \\
&= \frac{1}{N}(y - X\beta)^T(y - X\beta) + \lambda\beta^T\beta \\
&= \frac{1}{N}(y^Ty - y^TX\beta - \beta^TX^Ty + \beta^TX^TX\beta) + \lambda\beta^T\beta
\end{aligned}
$$

- Differentiate L by β,

$$
\begin{aligned}
\frac{\partial L}{\partial \beta} &= \frac{1}{N}(-2\beta^TX^Ty + 2X^TX\beta + 2\lambda\beta) \\
&= -\frac{2}{N}X^T(y - X\beta) + 2\lambda\beta = 0 \\
&= -\frac{1}{N}X^T(y - X\beta) + \lambda\beta = 0 \\
&= \frac{1}{N}X^T(y - X\beta) = \lambda\beta
\end{aligned}
$$

- This additional term serves to control the size of β.

Calculate the Weight $\hat{\beta}$

- If $X^T X + \lambda I$ is nonsingular,

$$
\begin{aligned}
\frac{1}{N} X^T (y - X\beta) &= \lambda\beta \\
X^T (y - X\beta) &= N\lambda\beta \\
X^T y - X^T X\beta &= N\lambda\beta \\
X^T y &= X^T X\beta + N\lambda\beta \\
X^T y &= (X^T X + N\lambda I)\beta
\end{aligned}
$$

- $\hat{\beta} = (X^T X + N\lambda I)^{-1} X^T y$

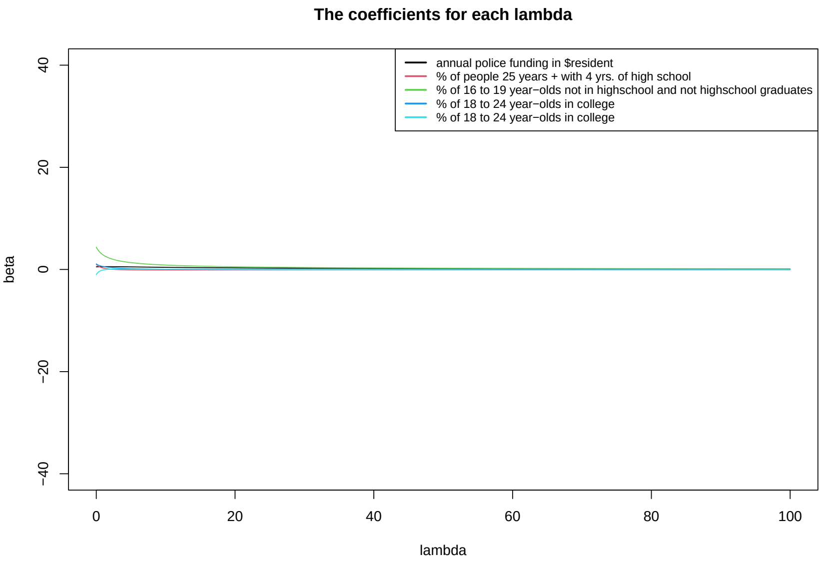
R Code for Ridge Regression

```r
ridge=function(X,y,lambda=0){
  X=as.matrix(X);p=ncol(X);n=length(y);X.bar=array(dim=p);s=array(dim=p)
  for (j in 1:p){X.bar[j]=mean(X[,j]);X[,j]=X[,j]-X.bar[j];};
  for (j in 1:p){s[j]=sd(X[,j]);X[,j]=X[,j]/s[j]};
  y.bar=mean(y);y=y-y.bar
  beta=drop(solve(t(x)%*%X+n*lambda*diag(p))%*%t(X)%*%y)
  for (j in 1:p)beta[j]=beta[j]/s[j]
  beta.0=y.bar-sum(X.bar*beta)
  return(list(beta=beta,beta.0=beta.0))
}
```

Example 48

```
df=read.table("crime.txt");x=df[,3:7];y=df[,1];p=ncol(x);
lambda.seq=seq(0,100,0.1);coef.seq=lambda.seq
plot(lambda.seq,coef.seq,xlim=c(0,100),ylim=c(-40,40),
     xlab="lambda",ylab="beta",main="The coefficients for each lambda",
     type="n",col="red")
for (j in 1:p){
  coef.seq=NULL;for(lambda in lambda.seq)coef.seq=c(coef.seq,
                                                        ridge(x,y,lambda)$beta[j])
  par(new=TRUE);lines(lambda.seq,coef.seq,col=j)
}
legend("topright",legend=
         c("annual police funding in $resident","% of people 25 years +
           with 4 yrs. of high school",
           "% of 16 to 19 year-olds not in highschool and not highschool
           graduates", "% of 18 to 24 year-olds in college",
           "% of 18 to 24 year-olds in college"),col=1:p,lwd=2,cex=.8)
```

Example 48


The coefficients for each lambda
annual police funding in $resident
% of people 25 years + with 4 yrs. of high school
% of 16 to 19 year−olds not in highschool and not highschool graduates
% of 18 to 24 year−olds in college
% of 18 to 24 year−olds in college
beta
lambda
40
20
0
−20
−40
0
20
40
60
80
100

Q & A

Thank You