Linear Equations

linear equation - variables $x_1, \dots, x_n$ is an equation

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

b ← real/complex numbers

a_1 ← real/complex numbers

n ← any positive integer (normally 2→5) dimension

$4x_1 - 5x_2 + 2 = x_1 \longrightarrow 3x_1 - 5x_2 = -2$ linear equation

$4x_1 - 5x_2 = x_1x_2 \longrightarrow$ X nonlinear form

System of linear equations - combination of 2 or more linear equations

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \quad\; - 4x_3 &= -7 \end{aligned} \longrightarrow (5, 6.5, 3)$$

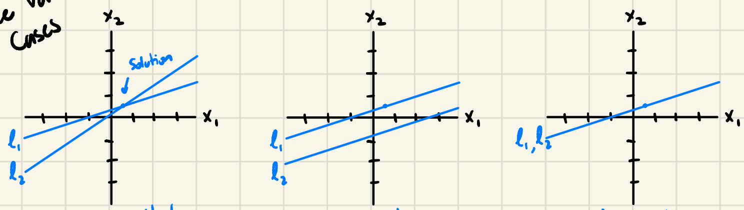
solution to system

Solution - a list $(s_1, s_2 \dots s_n)$ of numbers that makes each equation true when $s_n \rightarrow x_n$ (substituted)

Solution Set - Set of all possible solutions

Equivalent - IF 2 systems have = solution sets, the systems are equivalent

Three Variable Cases (every case in linear equations)



non-parallel parallel identical

Row Operations

- Replacement/Addition (Add a multiple from 1 row to another)
- Interchange (Interchange 2 rows)
- Scaling (Multiply a row with a non-zero scalar)

} DOES NOT CHANGE EQUATION

$R_1 \quad x_1 - 2x_2 + x_3 = 0$

$R_2 \quad 2x_2 - 8x_3 = 8$

$R_3 \quad 5x_1 \quad - 5x_3 = 10$

$R_1 + R_2 \rightarrow R_1 \quad x_1 + 0x_2 - 7x_3 = 8$

$R_2 / 2 \rightarrow R_2 \quad x_2 - 4x_3 = 4$

$R_3 - 5R_1 \rightarrow R_3 \quad 10x_2 - 10x_3 = 10$

$R_3 - 10R_2 \rightarrow R_3 \quad x_1 - 7x_3 = 8$

$x_2 - 4x_3 = 4$

$30x_3 = -30 \rightarrow x_3 = -1$

substitute into

$R_{1new} \rightarrow x_3 = -1, x_1 = 1$

$R_{2new} \rightarrow x_3 = -1, x_2 = 0$

Solution: $(1, 0, -1)$

Consistent Systems

Augmented Matrix - linear equation → matrix

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 7 \end{aligned} \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 7 \end{array}\right)$$

Consistent - Has <u>at least</u> 1 solution (linear equation)

Row Equivalent - If a sequence of row operations transforms 1 matrix → into the other

- If the Augmented Matrices of 2 linear systems are row equivalent, then they have the same solution set

- Ex. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \underset{\text{equivalent}}{\overset{\text{row}}{\sim}} B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$R_1 - R_2 \rightarrow R_1$

A augmented $\vec{b}$ = $(A \mid \vec{b})$

- Ex. $\vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $(A \mid \vec{b}) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right)$

Echelon Form and RREF

Echelon Form ① All zero rows (if any present) are at the bottom

② First non-zero entry (leading entry) of a row is to the <u>right</u> of any leading entries in the row above

③ All entries below a leading entry are zero

- Ex. $A = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ echelon form

$B = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \end{pmatrix}$ ② ③ NOT echelon form

Row Reduced Echelon Form ① All leading entries, <u>if any</u>, = 1

② Leading entries are the only <u>nonzero</u> entry in their column

- Ex. $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ row reduced echelon form

$B = \begin{pmatrix} 1 & 0 & 6 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ echelon form

Row Reduction Algorithm

one of many!

pivot position - a pivot position in Matrix "A" is a location that corresponds to a leading row in the RREF form of "A"

pivot column - a column of A that contains a pivot position

- Ex. $\begin{pmatrix} 0 & -3 & -6 & 9 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \end{pmatrix} \xrightarrow[R_3 - 2R_2]{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & -2 & -1 & 3 \\ 0 & -3 & -6 & 9 \\ 0 & 1 & 2 & -3 \end{pmatrix} \xrightarrow[R_2(-\frac{1}{3})]{R_1(-1)} \begin{pmatrix} 1 & 2 & 1 & -3 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & 2 & -3 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 2 & 1 & -3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

echelon, not RREF — RREF

1st and 2nd columns are pivotal b/c they contain the leading entry in the echelon form

Algorithm ① Swap the 1st row with lower one so leftmost nonzero entry is in 1st row

② Scale 1st row so its leading entry = 1

③ Use row replacement so all entries above/below = 0

} Repeat for each row

Existence and Uniqueness

basic variables — Variables that correspond to a pivot (# of columns, ex. x_1, x_3, x_5)

free variables — Variables that are not basic (any value)

- Any choice of the free variables → solution of system
- Matrices do not have variables (it does have pivots)
- Augmented matrices are associated with coefficients and have variables

Consistent — A linear system is consistent only if the last column does not have a pivot

① It has a unique solution only if there are no free variables

② It has infinite solutions that are parameterized by free variables

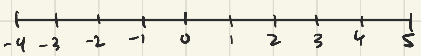
Vector Equations

Vectors – gives deeper insight into properties of systems / solutions ($\mathbb{R}^n$)

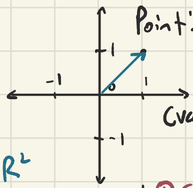
$\mathbb{R}$ = all real numbers

$\mathbb{R}^n$ = all ordered n-tuples of real numbers $(x_1, x_2 \dots x_n)$

when $n=1$, get $\mathbb{R}$ back: $\mathbb{R}^1 = \mathbb{R}$



↙ $\mathbb{R}^1$



Point: $(1,1)$

Vector: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(vectors are written as column vectors)

$\mathbb{R}^2$

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

① Scalar Multiples $c\vec{u} = c\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix}$

② Vector Addition $\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$

Linear Combinations

Definition: if vectors are $\vec{v}_1, \vec{v}_2 \dots \vec{v}_p$, and scalars $c_1, c_2 \dots c_p$, vector $\vec{y}$ where is

$$y = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_p\vec{v}_p$$

linear combination sum

↑ "weights"

Example: Can $\vec{y}$ be represented by $\vec{v}_1$ and $\vec{v}_2$?

$\vec{y} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ⟶

Solution – find c_1 and c_2 so that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{y}$

$c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} c_1 - c_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

⟶ $c_1 = 2$, $c_2 = 1$, ⟵

so this system has a solution

Any 2 vectors in $\mathbb{R}^2$ that are not scalar multiples will span $\mathbb{R}^2$ – any vector can be represented as a linear combination of 2 vectors

Span

Definition: Given vectors $\vec{v}_1, \vec{v}_2 \dots$ and scalars $c_1, c_2 \dots$ the set of <u>all</u> linear combinations of $\vec{v}_1, \vec{v}_2 \dots$ is the span

Example: IS $\vec{y}$ in the <u>span</u> of vectors $\vec{v}_1$ and $\vec{v}_2$?

could be written as a linear combination of $v_1 + v_2$

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} \text{ and } \vec{y} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix}$$

①
$$c_1 \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -3 & 6 & 15 \end{array}\right] \xrightarrow[R_3+3R_1]{R_2+2R_1} \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 12 & 36 \end{array}\right] \xrightarrow{R_3/12 \quad R_2/9} \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right] \Big\} \text{ Impossible, so NOT in span}$$

inconsistent

In General: Any 2 non-parallel vectors in $\mathbb{R}^3$ span a plane that passes through the origin. Any vector in that plane is <u>also</u> in the span of the 2 vectors

Matrix-Vector Product

Symbol	Meaning
$\in$	Belongs To
R^n	Set of vectors with n real-valued elements
$R^{m \times n}$	Set of matrices with m rows / n columns

Example:

$\vec{x} \in R^5 = \vec{x}$ is a vector with 5 real-valued components

IF $A \in R^{m \times n}$ has columns $\vec{a}_1, \dots \vec{a}_n$ and $\vec{x} \in R^n$, then the matrix vector product $A\vec{x}$ is a linear comb.

$$A\vec{x} = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{a}_1 & \dots & \vec{a}_n \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{x_1 \vec{a}_1 + \dots x_n \vec{a}_n}_{\text{Linear Combination}}$$

* $A\vec{x}$ is in the <u>span</u> of A

Example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} 2 & 0 \\ 0 & -9 \end{pmatrix} = 2a_1 - 9a_2$$

Existence of Solutions

$A\vec{x} = \vec{b}$ <u>has</u> a solution <u>if and only if</u> $\vec{b}$ is a linear combination of columns of A

Example:

what is $\vec{b}$?

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 8 & 4 \\ 0 & 1 & -2 \end{pmatrix} \vec{x} = \vec{b} \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 2 & 8 & 4 & b_2 \\ 0 & 1 & -2 & b_3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 2 & -4 & b_2 - 2b_1 \\ 0 & 1 & -2 & b_3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 2 & -4 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - \frac{1}{2}b_2 - b_1 \end{array}\right)$$

has to = 0

$$b_3 - \tfrac{1}{2}b_2 - b_1 = 0$$

$$b_1 = b_3 - \tfrac{1}{2}b_2$$

$$\boxed{\vec{b} = \begin{pmatrix} b_3 - \frac{1}{2}b_2 \\ b_2 \\ b_3 \end{pmatrix}}$$

Solution

Representation:

① List of Equations $2x_1 + 3x_2 = 7, \; x_1 - x_2 = 5$

② Augmented Matrix $\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$

③ Vector Equation $x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

④ Matrix Equation $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

Homogeneous Systems

Linear Systems of the form $A\vec{x} = \vec{0}$ are homogeneous $\longrightarrow$ always has trivial solution, $\vec{x} = \vec{0}$

Linear Systems of the form $A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$ are inhomogeneous

↳ Could have non-trivial solutions

① Has a free variable

② A has a column with <u>no</u> pivot

Example:

homogeneous

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & -1 & -5 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & -1 & -5 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

pivot columns

free variable

Solution Set:

$$\vec{x} = \begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix}$$

1st Row $x_1 - 2x_3 = 0$

$x_1 = 2x_3$

2nd Row $x_2 + x_3 = 0$

$x_2 = -x_3$

Parametric Vector Form

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 2 & -1 & -5 & 1 \\ 1 & 0 & -2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & -3 & -3 & -3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \vec{x} = \begin{pmatrix} 1 + 2x_3 \\ 1 - x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

In General:

IF free variables for $A\vec{x} = \vec{0}$ are $x_k \dots x_n$, then all solutions <u>can</u> be written

↳ $\vec{x} = x_k\vec{v}_k + x_{k+1}\vec{v}_{k+1} + \dots + x_n\vec{v}_n$

Steps:

① Row Reduction to RREF

② Solutions expressed as a vector with how variables are related

Linear Independence

Linear Independent: A set of vectors are linearly independent if $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$ has only the trivial solution

↳ otherwise, it is linearly dependent

$\{\vec{v}_1 \dots \vec{v}_k\}$ are linearly dependent if there are real values $c_1, c_2, \dots c_k$ <u>not all zero</u> so that:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{c}_k = \vec{0}$$

① Set the <u>linear combination</u> to the <u>zero vector</u>

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_k\vec{v}_k = \underbrace{[\vec{v}_1\ \vec{v}_2 \dots \vec{v}_k]}_{V} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_3 \end{bmatrix}}_{\vec{c}} = V\vec{c} = \vec{0}\ ?$$

Linear Independence: There is <u>no</u> non-zero solution $\vec{c}$

Linear Dependence: There is a non-zero solution $\vec{c}$

Example: For what value h is the set linearly independent?

$$\begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}, \begin{bmatrix} 1 \\ h \\ 1 \end{bmatrix}, \begin{bmatrix} h \\ 1 \\ 1 \end{bmatrix}$$

$\rightarrow$ IF independent, $c_1\begin{pmatrix} 1 \\ 1 \\ h \end{pmatrix} + c_2\begin{pmatrix} 1 \\ h \\ 1 \end{pmatrix} + c_3\begin{pmatrix} h \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

FOR ONLY $c_1 = c_2 = c_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & h & 0 \\ 1 & h & 1 & 0 \\ h & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & h & 0 \\ 0 & h-1 & 1-h & 0 \\ 0 & 1-h & 1-h^2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & h & 0 \\ 0 & h-1 & 1-h & 0 \\ 0 & 0 & 2-h-h^2 & 0 \end{array}\right]$$

IF $2-h-h^2=0$, vectors are dependent, because c_3 is free ($\mathbb{R}$)

Linear Independence Theorems

Smallest number of vectors needed for parametric solution

IF $\vec{v}_1, \vec{v}_2 \in R^n$, **WHEN** is the set $\{\vec{v}_1, \vec{v}_2\}$ linearly dependent?

↳ there exists $c_1 + c_2$ (<u>NOT</u> zero) where $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$

↓

2 cases

1) If $\vec{v}_1$ or $\vec{v}_2$ is the <u>zero</u> vector, c_1/c_2 can be anything
2) If $\vec{v}_1 \neq \vec{0}$ and $\vec{v}_2 \neq \vec{0}$, then $\vec{v}_2 = -\frac{c_1}{c_2}\vec{v}_1$, so $\vec{v}_1$ and $\vec{v}_2$ ARE MULTIPLES
   ↳ vectors are parallel (one vector is in span of another)

Theorems:

1) More Vectors than Elements – if $\vec{v}_1 \ldots v_k$ are in R^n and $k > n$, $\{\vec{v}_1 \ldots \vec{v}_k\}$ is <u>linearly dependent</u>
2) Set contains Zero Vector – if any 1 or more of $\vec{v}_1 \ldots \vec{v}_k$ is a <u>zero vector</u>, then ↑
3) Last column is the sum of the first two = dependent

Domain, Codomain, Range

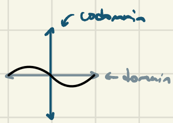
A as any $m \times n$ matrix ↓

$$T: R^n \to R^m,\ T(\vec{x}) = A\vec{x}$$

Matrix Transformation

domain of T is R^n → horizontal axis (input)

codomain of T is R^m → vertical axis (output)



Range: How high/low the line reaches

↳ Ex. Range of $\sin(x)$ is $[-1, 1]$

Vector $T(\vec{x})$ is image of $\vec{x}$ under T

range – set of all possible images of $T(\vec{x})$

5 ways to represent $A\vec{x} = \vec{b}$

- set of linear equations
- augmented matrix
- matrix equation
- vector equation
- linear transformation equation

Ex. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $T(\vec{x}) = A\vec{x}$

For $A\vec{x}$ to be defined, $\vec{x} \in R^2$ → domain is

T domain: R^2

T codomain: R^3

$\vec{u}$ under T image: $T(\vec{u}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}3 + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}4 = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$

T range: $T = A(\vec{x}) = x_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ = range is span $\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$

Calculate $\vec{v} \in R^2$ so that $T(\vec{v}) = \vec{b} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$ ← If this is possible, $T = A\vec{v} = \vec{b}$

$\left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right) \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix} \to \vec{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

Give a $\vec{c} \in R^3$ so there is no $\vec{v}$ with $T(\vec{v}) = \vec{c}$

Try: $\vec{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ → is $\vec{c}$ in the span of A? → $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$ → NOT CONSISTENT $\begin{array}{l} 11 \to 1 \\ 11 \to 3 \end{array}$

$\vec{c}$ is NOT in range of T

$\vec{c}$ is NOT in span of T

Geometric Interpretations of Linear Transformations

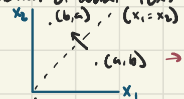
linear transformation: $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if $\left\{\begin{array}{l} T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v}) \text{ for } \underline{\text{all}}\ \vec{u},\vec{v} \text{ in } \mathbb{R}^n \\ T(c\vec{v}) = cT(\vec{v}) \text{ for } \underline{\text{all}}\ \vec{v} \in \mathbb{R}^n, \text{ and } c \text{ in } \mathbb{R} \end{array}\right\}$

principle of superposition: $T(c_1\vec{v}_1 + \ldots + c_k\vec{v}_k) = c_1T(\vec{v}_1) + \ldots c_kT(\vec{v}_k)$

↳ <u>EVERY</u> matrix transformation T_A is linear

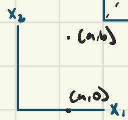
Ex. T is the linear transformation $T(\vec{x}) = A\vec{x}$, give a geometric interpretation of what $T(\vec{x})$ does to vectors in $\mathbb{R}^2$

1) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, let $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} \to T(\vec{x}) = A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} a \\ b \end{pmatrix}, \to \begin{pmatrix} b \\ a \end{pmatrix}$



→ reflection through $x_1 = x_2$

2) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, let $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} \to T(\vec{x}) = A\vec{x} = \begin{pmatrix} a \\ 0 \end{pmatrix}$



→ projection onto x_1-axis

3) $A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, let $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} \to T(\vec{x}) = A\vec{x} = \begin{pmatrix} ak \\ bk \end{pmatrix}$ → scaling by k

Ex. T_A in $\mathbb{R}^3$

1) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, let $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \to T(\vec{x}) = A(\vec{x}) = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ → projecting onto x_1x_2-plane (0 for x_3 value)

2) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, let $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \to T(\vec{x}) = A(\vec{x}) = \begin{pmatrix} a \\ -b \\ c \end{pmatrix}$ → reflection through x_1x_3-plane

Ex. Constructing Matrix of Transformation

$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix}$, $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}$

↳ what is matrix A so $T = Ax$?

A is 3×2 And let $A = (\vec{a}_1\ \vec{a}_2)$ THEN $\left.\begin{array}{l} A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\vec{a}_1 + 0\vec{a}_2 = \vec{a}_1 \\ A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\vec{a}_1 + 1\vec{a}_2 = \vec{a}_2 \end{array}\right\} A = \begin{pmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{pmatrix}$

The Standard Vectors

standard vectors in $\mathbb{R}^n$ are vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3 \ldots \vec{e}_n$ where:

$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$ $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$ $\vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

all remaining entries

IF A is a $m \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n$ <u>then</u> $A\vec{e}_i = \vec{v}_i$, for $i = 1, 2, \ldots, n$

↳ MULTIPLYING a matrix with $\vec{e}_i$ gives column i of A

Ex.

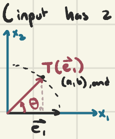
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \vec{e}_2 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to 0\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + 0\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$$

Ex.

linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ ("transform") $\to T(\vec{x}) = \vec{x}$ rotated counterclockwise by angle θ about $(0,0)$

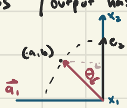
$T(x) = A\vec{x}$, we need $A \to A$ is 2×2 (input has 2 dimensions, so 2 rows | output has 2 dimensions, so 2 columns)

$\left.\begin{array}{l} A = (\vec{a}_1, \vec{a}_2) \\ Te_1 = A\vec{e}_1 = 1 \cdot \vec{a}_1 + 0\vec{a}_2 = \vec{a}_1 \end{array}\right\}$



(a,b), and $\left\{\begin{array}{l} \cos\theta = a/1 \\ \sin\theta = b/1 \end{array}\right.$

$T(e_1) = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \vec{a}_1$



$T(e_2) = \begin{pmatrix} -a \\ b \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

Counterclockwise {transformation}

$A = (\vec{a}_1, \vec{a}_2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta, & \cos\theta \end{pmatrix}$

Ex.

Define a transformation by $T(x_1,x_2) = (3x_1+x_2, 5x_1+7x_2, x_1+3x_2)$

↳ Is T one-to-one? Is T onto?

$T = A\vec{x}$, and $T(e_1) = \vec{a}_1$, $T(e_2) = \vec{a}_2$

$$\left.\begin{array}{l} T(e_1) = T(1,0) = (3+0, 5+0, 1+0) = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \\ T(e_2) = T(0,1) = (0+1, 0+7, 0+3) = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix} \end{array}\right\} T\vec{x} = A\vec{x},\ A = \begin{pmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{pmatrix}$$

Standard Matrices in R^2

Reflection over x_1-axis $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Reflection over y-axis $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	Reflection through $x_2 = x_1$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Reflection through $x_2 = -x_1$ $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	Projection onto x_1-axis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ Projection onto x_2-axis $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
horizontal contraction / expansion $\lvert k\rvert<1$ $k>1$ $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$	vertical contraction / expansion $\lvert k\rvert<1$ $k>1$ $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$	
horizontal shear left / right $k<1$ $k>1$ $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	vertical shear left / right $k<1$ $k>1$ $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$	

Ex.

Construct $A \in R^{2\times 2}$, T is linear transformation that rotates R^2 counterclockwise 90° and reflects through $x_1 = x_2$

counterclockwise $\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | reflects through $x_1 = x_2$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Onto and One-to-One

onto: $T: R^m \to R^n$ and for any $\vec{b} \in R^m$, $A\vec{x} = \vec{b}$ has a solution → always consistent

↑ existence property ↑

↳ T is onto if and only if its standard matrix has a pivot in every row

1) A spans R^m
2) Every row of A is pivotal

one-to-one: for all $\vec{b} \in R^m$ there is at most one (possibly no) $\vec{x} \in R^n$ so that $T(\vec{x}) = A\vec{x} = \vec{b}$

↳ does not assert existence for all $\vec{b}$

↑ uniqueness property ↑

↳ T is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{x} = \vec{0}$

↳ one-to-one only if every column is pivotal

1) Unique solution to $T(\vec{x}) = \vec{0}$ is trivial
2) A has linearly independent columns
3) Every column of A is pivotal

Ex.

a) A is a 2x3 standard matrix for one-to-one transform

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 1 \end{pmatrix}$ NOT possible, more columns than rows

$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ← one-to-one AND onto

Ex.

$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$

$T = Ax$, A is 3×2, $A = (T(\vec{e}_1), T(\vec{e}_2))$

$T(e_1) = T(1,0) = (3,5,1)$

$T(e_2) = T(0,1) = (1,7,3)$

$A = \begin{pmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{pmatrix}$

more rows than columns, CANNOT be onto

T is one-to-one (columns are linearly independent)