Math 3012: Combinatorics

↳ Lectures 10-20

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Author's Note

↳ This is solely covering the topics of the lecture videos, so I recommend you read the textbook in regards to working out homework

↳ Pretty much all info regarding computer science has been omitted (i.e. functions)

↳ Lecture numbers are on the left of titles

L.10 Girths and Forests

Forest – graphs with no cycles
↳ every component is a tree
↳ **girth** $= \infty$ ↳ $\chi(G) = 1$ or 2, always

Girth – size of the smallest cycle in G
↳ Constructions for triangle-free graphs have small **girths**
↳ Theorem: for every pair (g, t) of positive integers $g, t \geq 3$, there is a graph with **girth** G and $\chi = t$

Perfect Graphs & Complements

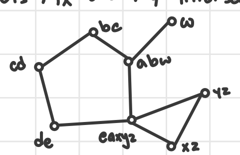
Perfect Graphs – if $\chi(G) = \omega(G)$ for <u>every</u> induced subgraph (subgraph that is a cycle)
↳ Any graph with an <u>odd cycle</u> as an induced subgraph is NOT **perfect**
↳ with odd cycles on 5 or more is NOT **perfect**

Complements (G^c) – the graphs have the same <u>vertex set</u> of G with edges not present in G
↳ pair xy of distinct vertices have an edge in G^c $\longleftrightarrow$ it does not have an edge in G
↳ If G^c has an odd cycle within an induced subgraph, G is NOT **perfect**

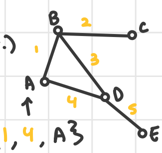
Berge's **Perfect Graph** Conjecture – A graph is **perfect** $\longleftrightarrow$ both G and G^c DO NOT have an odd cycle as an induced subgraph

Intersection Graphs

Let $F = \{A_x : x \in X\}$ be a family of sets. We associate with F an **intersection graph** G where the <u>vertices</u> of G are the <u>elements</u> of X and xy is an edge in G when the sets A_x and A_y intersect
↳ 2 vertices are adjacent $\longleftrightarrow$ their sets intersect $\longrightarrow$

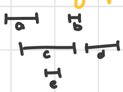


(ex.)



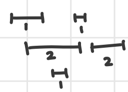
$\uparrow$
$\{1, 4, A\}$
edges
differentiate

↳ Applications of **intersection graphs** – **interval graphs**
↳ **Interval graphs**: **intersection graphs** of a family of closed intervals



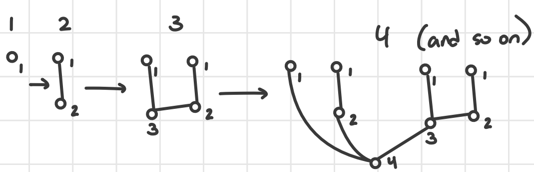
} c and a intersect, but b and e don't

↳ To color: Apply Greedy Algorithm and color in order of left endpoints →



Coloring as a Two-Player Game

Let's say you have a builder and a colorer
↳ an **online coloring**
↳ the builder can make the colorer color with <u>any</u> $\chi(G)$ without <u>ever</u> making a cycle:

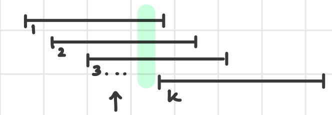


These form forests, which have $\chi(G) = 2$ or 1, but because the builder can keep adding vertices, $\chi(G) = \infty$

↳ relates to an **adversarial algorithm**

Interval Graphs and First-Fit

Idea: if you use color k, there will be a clique of size k in the graph
↳ applies for k=1

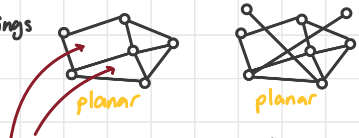


all intervals from 1,2,...,k-1 have been used, AND overlap with k
↳ so, $\chi = \omega$

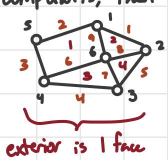
Theorem by Kierstead and WTT : $\chi = 3\omega - 2$ for online interval graphs
↳ So there is a strategy for coloring interval graphs
↳ Used in dynamic storage when using variables in programs

6.11 Planar Graphs and Euler's Formula

Planar Graphs - if a graph can be drawn with no edge crossings
↳ If a graph is planar, you can draw it with straight lines
↳ Testing for planarity is in P



Euler's Formula - n, q, and f denote # of vertices, edges and faces in a plane drawing of a planar graph with t components, then : $n - q + f = 1 + t$



→ n = 6 → $6 - 9 + 5 = 1 + 1$
↳ q = 9
↳ f = 4+1 = 5
↳ t = 1

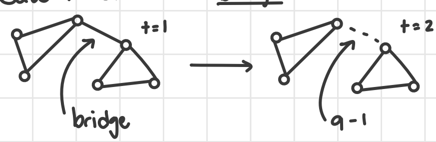
$2 = 2$ ✓

Bridge — an edge that, when removed, leads to a subgraph with t+1 components
↳ 2-connected — a connected graph with no bridges

Proof of Euler's Formula

Fix value of n → 1) Base case: q=0 → f=1, t=1 ✓
2) Suppose q=k for some k ≥ 1, and = k+1 edges
↳ 2 Cases → G has edge e which is a bridge
↳ G has no bridges

Case 1: G has a bridge

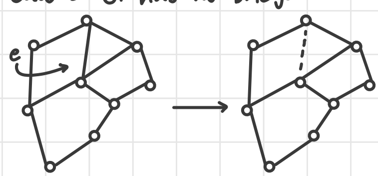


n, q, t, f

$n' = n, q' = q-1, t' = t+1, f' = f$

↳ Getting rid of a bridge by definition creates t+1 components and deletes an edge, so theorem holds

Case 2: G has no bridges



n, q, t, f

$n' = n, q' = q-1, t' = t, f' = f-1$

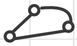
↳ $n - (q - \cancel{1}) + (f - \cancel{1}) = t + 1$, so Euler's theorem holds

The Max Number of Edges in a Planar Graph

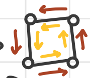
If G is a planar graph with $n \geq 3$ vertices and q edges: $q \leq 3n-6$

↳ Proof: Fix n and consider a planar drawing with the maximum number of edges. G is connected and has no bridges, so each edge belongs to exactly 2 faces.

↳ for each $m \geq 3$, let f_m be the number of faces whose boundary is a cycle of size m.

↳ each f has m sides  ← face with $m=3$ (f_3) → 

3(number of 3-side faces) = edges

↳ $3f_3 + 4f_4 + 5f_5 \ldots$ →  each edge counted twice, so $3f_3 + 4f_4 \ldots = 2q$

↳ $3f_3 + 4f_4 + 5f_5 \ldots = 2q$

↳ $3f_3 + 3f_4 + 3f_5 \ldots \leq$

$3(f_3 + f_4 + f_5 \ldots) \leq$ graph is connected, so $n - q + f = 2$ ← 1 component

$3(f) \leq 2q$ ← $f = 2 - n + q$

$3(2-n+q) \leq 2q$

$6 - 3n + 3q \leq 2q$
$\quad -3q \quad -3q$

$-1(6 - 3n \leq -q)$

→ $q \leq 3n - 6$

Four Color Theorem

Since the complete graph K_5 is non-planar, if G is a planar graph, then it has $\omega(G) \leq 4$

↳ Theorem: If G is planar, $\omega(G) \leq 4$, so it can at MOST be 4-colored

A graph H is a homeomorph of graph G if H is obtained by "inserting" 1 or more vertices on some of the edges of G

↳ if G is planar, all subgraphs of G are planar

↳ if H is a homeomorph of G, then H is planar ↔ G is planar

↳ G is non-planar if it contains a homeomorph of the complete graph K_5 as a subgraph

6.12 Two-Colorable Planar Graphs

If G is a two-colorable planar graph with $n \geq 3$ vertices and q edges, $q \leq 2n-4$

↳ Therefore, the complete bipartite graph $K_{3,3}$ is non-planar

↳ It's two-colorable but has $n=6$ vertices and 9 edges, and $9 \nleq 2(6)-4$

Kuratowski's Theorem: A graph is non-planar ↔ it contains a homeograph of the complete graph K_5 or of the complete bipartite graph $K_{3,3}$ as a subgraph

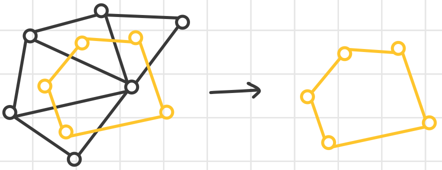
↳ If G is planar, $\omega(G) \leq 4$ (known as the four-color theorem)

↳ $\chi(G) \leq 4$

Planar Graphs and Planar Maps

Planar graphs have dual graphs that are also planar

↳ Dual graphs swap faces and vertices and add edges based on the boundaries of faces



Graph G   Dual Graph of G

Game Coloring for Graphs

Game Chromatic Number — least positive integer to color a graph with 2 players alternating coloring

↳ GCN of a planar graph $\leq 33 \to 17$ (Ongoing research)

List Chromatic Number — smallest integer t so that a proper coloring of the graph can <u>always</u> be found using colors from prescribed lists of size t, one list for each vertex

↳ ex. When $n = C(2t-1, t)$ (vertices), the complete bipartite graph $K_{n,n}$ has LCN $\geq t+1$

↳ LCN of a planar graph ≤ 5

L.13 Posets

Partially ordered Sets ("poe-set" Posets) — a set P with a binary relation $\leq$ that satisfies 3 conditions:

↳ $x \in P$, $x \leq x$ in P (reflexive property)

↳ $x, y \in P$, $x \leq y$ in P and $y \leq x$ in P, then $x = y$ (antisymmetric property)

↳ $x, y, z \in P$, $x \leq y$ in P and $y \leq z$ in P, then $x \leq z$ in P (transitive property)

Binary relation — a subset of $X \times X$ (cross)

↳ elements are <u>ordered pairs</u>

Ground Set — set of elements within a poset

↳ ex. P is reflexive: contains all of the forms of (x, x)

Poset Notation — $x < y$ in P same as $x \leq y$ in P

↳ $y < x$ in P same as $x < y$ in P

↳ ex. P is a collection of sets, set $x \leq y$ in P when x is a subset of y

↳ $\{2, 5\} < \{2, 5, 7, 8\}$ and $\{2, 5\} < \{2, 5\}$

Linear Order (Total Order) — posets where all comparisons can be made

↳ for all x, y either $x \leq y$ or $y \leq x$ in P

Covers in Posets

x is covered by y when $x < y$ in P and there is no point z with $x < z < y$ in P

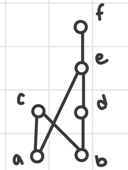
↳ y covers x

Cover graph – vertices are points of P, adjacent points cover each other (any orientation)

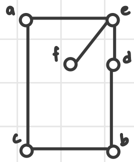
↳ Order Diagram – drawing of cover graph where y is higher than x if $y \geq x$ in P (1 orientation)

$P = \{(a,a), (b,b), (c,c), (d,d), (e,e), (f,f)$
$(a,c), (a,e), (b,c), (b,d), (d,e), (e,f)$
$(a,f), (b,e), (b,f), (d,f)\}$ →

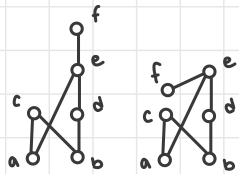
$a \leq f$



order diagram



cover graph



Two different posets, same cover graph

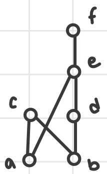
Comparability

Two points are comparable when either $x < y$ or $y < x$ in P

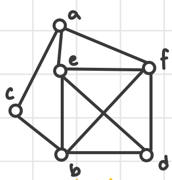
↳ If not, they are incomparable

Comparability Graph – elements x and y are adjacent if comparable in P

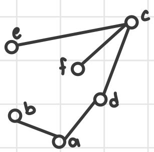
Incomparability Graph – elements adjacent if incomparable in P



poset



comparability graph



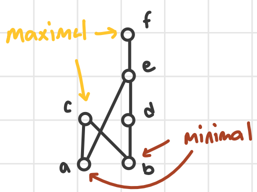
incomparability graph

height doesn't matter

Maximal and Minimal Points

Maximal Points – a point x where there is no $y > x$ in P (nothing bigger)

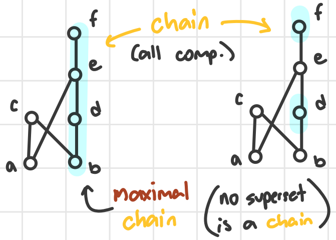
Minimal Points – a point z where there is no $z > y$ in P (nothing smaller)



Chains and Antichains

Chains – a subset where every pair is comparable

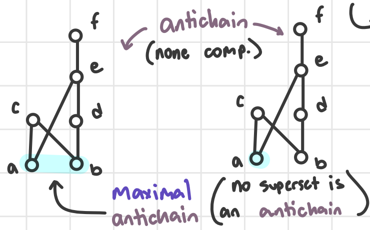
↳ **Maximal chain** – no superset is also a chain (can't add more points and keep it a chain)



↳ **height** – max size of a chain in P

Antichain – a subset where every pair is incomparable

↳ **Maximal antichain** – no superset is also an antichain



↳ **width** – Max size of an antichain in P

(A 1-element set is both a chain and an antichain)

6.14 Partitioning Posets

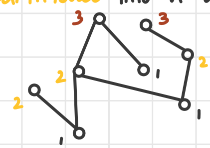
TIP: Sets of minimal elements / maximal elements form antichains, so width ≥ set of mins / maxes

If P can be partitioned into t antichains, height ≤ t
If P can be partitioned into s chains, width ≤ s
} Pigeon-Hole Principle

↳ Can be used to provide a <u>certificate</u> for assertions height ≤ t or width ≤ s

↳ **Mirsky's Theorem** – a poset of height h can be partitioned into h antichains

↳ Proof: recursively remove the set of minimal elements



1. Start with mins
2. remove mins
3. label new mins
4. numbers are antichains ($h \leq 3$)

Dilworth's Theorem

Dilworth's Theorem – A poset of width w can be partitioned into w chains

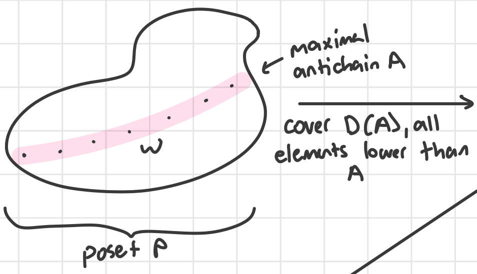
↳ Proof via induction: True when width = 1 and thus when $|P| = 1$

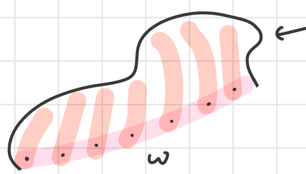
↳ Assume true when $|P| \leq k$

↳ Consider $|P| = k+1$

For each maximum antichain A, let $D(A) = \{x : x < a \text{ for some } a \text{ in } A\}$ and $U(A) = \{x : x > a \text{ for some } a \text{ in } A\}$ → $P = A \cup D(A) \cup U(A)$ is a partition into piecewise disjoint sets

↳ Case 1: There is a maximal antichain A where both D(A) and U(A) are NOT empty

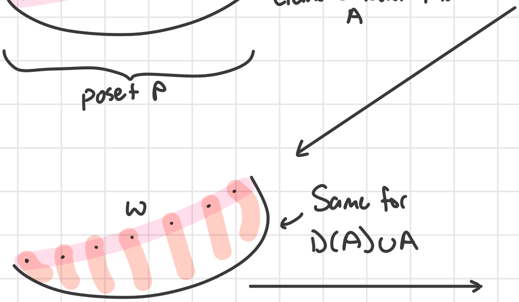


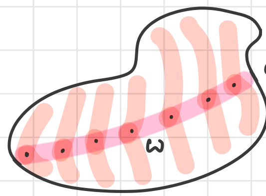


2nd step of Ind.
Assume true when $|P| \le k$, can be partitioned into w chains

A ∪ U(A) forms another poset, original poset had k+1 points, took away D(A) which had >0 elements, so <u>new poset has ≤ k elements</u>

↳ width = same as original poset





Glue back together: w chains, k+1 points

↳ Case 2: At least one of D(A) or U(A) is empty

↳ Choose a maximal element y, then choose a minimal element x with $x \le y$ in P (x could = y)
C = {x, y} is a chain - one of either 1 or 2 points, and width of P − C is w−1. Partition P−C into w−1 chains, and then add chain C to obtain the desired chain partition of P

6.15 Types of Graphs

A graph is a cover graph when there is a poset P with the same ground set so that G is the cover graph of P → Is it a cover graph is NP (easily testable certificate)

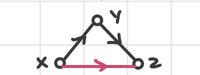
↳ Can be P in certain situations, like if it has a triangle

Alternative Observation: If the edges can be <u>oriented</u> such that there is no cycle $C = \{x_1, x_2, \dots, x_n\}$ in G for which edge $x_i x_{i+1}$ is oriented from x_i to x_{i+1} for all $i = 1, 2, \dots, n-1$

Comparability Graphs → Problem is in P

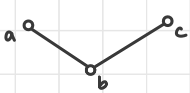
↳ Can be P in certain scenarios, such as when contains the odd cycle C_5

↳ Alt definition: If G can be transitively oriented (if there is a directed edge from x to y and from y to z, then xz is an edge in G and directed x to z), then G is a comparability graph



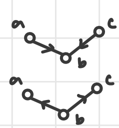
x → z MUST exist

P_3 Rule (Vee Rule) − In a transitive orientation of G, when {a, b, c} <u>induces</u> a path P_3 as shown, then either:

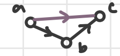


↳ Both edges are oriented towards b, or

↳ Both edges are oriented away from b



→ If not, there is an <u>induced</u> edge



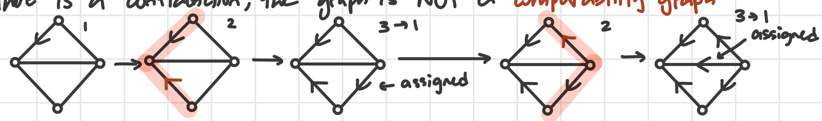
↳ Algorithm: Choose an unoriented edge and assign a direction ①

↳ Use the P_3 rule to force additional orientations ②

↳ If no additional forces / conflicts, go back to first step ③

↳ Conflict - if there is a contradiction, the graph is NOT a comparability graph

↳ Example:

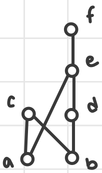


↳ There exists a transitive orientation, so this could be a comparability graph

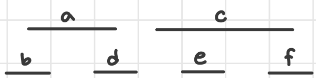
Gallai's Theorem - a list of graphs that Gallai determined cannot be found within comparability graphs

6.16 Interval Orders

A poset P is an interval order ⟷ there exists a function I assigning to each x in P a closed interval $I(x) = [a_x, b_x]$ of the real line R such that $x < y$ in P ⟷ $b_x < a_y$ in R



poset P

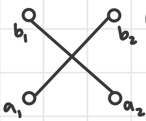


$b < d$, but b is incomparable with a

$a < e$ and $a < f$

I(P)

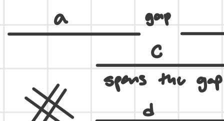
↳ Fishburne's Theorem - a poset is an interval order ⟷ it does not contain the standard example S_2



$S_2 = 2+2$

↳ Proof: P is an interval order ⇒ No 2+2

↳ If poset is an interval order, what would the intervals be?

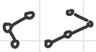


$a < b$

$c < d$

c incomp. with a and b

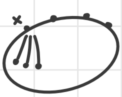
also spans the gap, but $c < d$, so there must be a gap btw c and d

↳ Fishburne's Theorem also applies to chains incomparable with each other  ← 2+2

↳ Proof via induction → base case: 1-element poset → a

↳ Assume valid when $|P| = k$, $k \geq 1$

↳ Suppose $|P| = k+1$, and P has no 2+2



$D(x) = \{y \in P : y < x \text{ in } P\}$

Choose x with $|D(x)|$ maximum

↳ Case 1: x is comparable with all

$(|P| = k)$

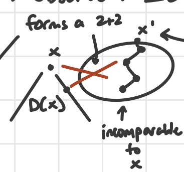
→ Remove x, Poset P-x is in interval order

↳ Add x to the right, P is still in interval order

↳ Case 2: Some maximals are incomparable with x → $I(x) = \{u : y \text{ is incomparable to } x\}$

↳ $I(x) \neq 0$

↳ Observe: I(x) is an antichain → Why?

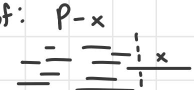


← over at least 1 element

but since D(x) is maximum, there is at least 1 point that x is over that x' is not

Therefore, I(x) has to be an antichain of maximal elements

↳ Continuing the proof: P-x

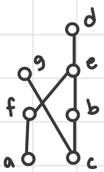


maximal elements (1 antichain)

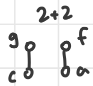
Interval Order Algorithm

1. Compute down sets for each element in P (including null sets)
↳ If 2 down sets are not comparable by inclusion (1 a subset of another), P contains a 2+2
2. Label D(x) from small → large 1, 2, …, m
3. Compute up sets for each element
4. Label U(x) from large → small 1, 2, …, m
5. Assign x to interval [i, j] where D(x) gets label i and U(x) label j

Example 1



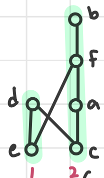
D(a) = 0
D(b) = c
D(c) = 0
D(d) = e b c f a
D(e) = b c f a
D(f) = a
D(g) = c



D(f) is not a subset of D(g)
D(g) is not a subset of D(f)

Poset P not interval order ← Not comparable by inclusion

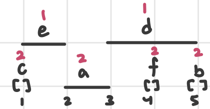
Example 2 (Is in interval order)



2 D(a) = c
5 D(b) = e c a f
1 D(c) = 0
3 D(d) = e c
1 D(e) = 0
4 D(f) = e c a

3 U(a) = b f
5 U(b) = 0
1 U(c) = b f d a
5 U(d) = 0
2 U(e) = b f d
4 U(f) = b

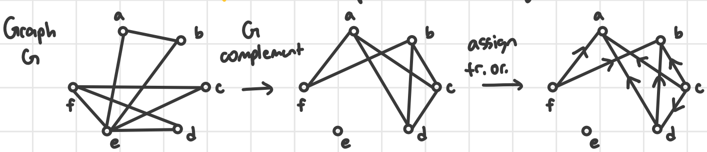
2 (minimum) interval order → coloring



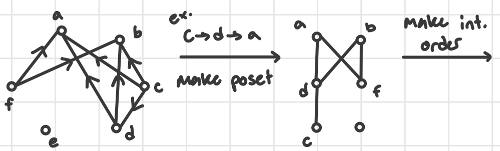
Recognizing Interval Graphs

Step 1: Given graph G, first let H be the complement (edges → non-edges) of G, then test H to see if it is a comparability graph (test if H can be transitively oriented) If no, G is NOT an interval graph
↳ If yes, Step 2

Step 2: Let P be the poset associated with the transitive order of H – test P to see if it is an interval graph → If no, G is NOT an int. graph
↳ If yes, G is an interval graph and interval order for P = interval graph for G
↳ Interval Graphs – overlaps indicate edges, so b over a means a is adjacent to b

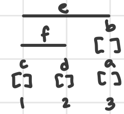


Step 1: G could be int. graph ✓
↳ Move onto Step 2



3 D(a) = c d f
3 D(b) = c d f
1 D(c) = 0
2 D(d) = c
1 D(e) = 0
1 D(f) = 0

3 U(a) = 0
3 U(b) = 0
1 U(c) = a b d
2 U(d) = a b
3 U(e) = 0
2 U(f) = a b



Any two sets comp. = G is int. graph

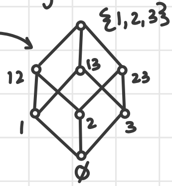
Int order for P = Int graph for G

Dilworth Problem – If poset P is an interval order, this algorithm finds an interval rep. for P – these same intervals are the interval graph for the incomparability graph G of P. Using First Fit to color G (left → right), then we solve the Dilworth Problem for P, i.e. the width and minimum chain partition of P (width = # of colors used)

Subset Lattices

For an integer $n \geq 1$, the poset consisting of all subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion is a subset lattice

↳ Denoted by $\mathbf{2}^n$, Example: $\mathbf{2}^3$
bold face



↳ Subsets as bit strings (cubes)

↳ Size of $\mathbf{2}^n$ is 2^n

↳ Unique maximal element: $\{1, 2, \ldots, n\}$

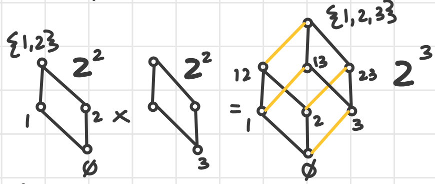
↳ Unique minimal element: $\emptyset$ ← empty set

↳ Height of $\mathbf{2}^n = n+1$ (all maximal chains are maximum)

↳ Subsets are built in an inductive manner

↳ Subset $\mathbf{2}^{n+1}$ could be viewed as $\mathbf{2} \times \mathbf{2}^n$

↳ Example: $\mathbf{2}^2 \to \mathbf{2}^3$ ⟶



For $n \geq 2$, the n-cube subset $\mathbf{2}^n$ is Hamiltonian (all vertices visited once)

Sperner's Theorem

If A is a set with $|A| = k$, then the # of maximal chains in $\mathbf{2}^n$ containing A is $k!(n-k)!$

↳ Width of $\mathbf{2}^n \geq C(n,k)$ where $0 \leq k \leq n$

↳ Largest $C(n,k)$ is when $k = [n/2]$ → When n is even, there is 1 value of k when C is max

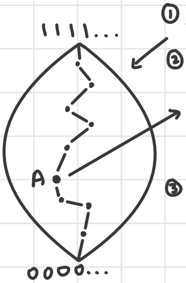
↳ n is odd, there are two ↓ even

↳ Example: width of $\mathbf{2}^{13} \geq C(13,6) = C(13,7)$ but width of $\mathbf{2}^{14} \geq C(14,7)$

Sperner's Theorem – the width of $\mathbf{2}^n = C(n, \lfloor n/2 \rfloor)$ floor / ceiling

↳ There is no antichain any bigger than the size of a binomial coefficient — Proof is shown below

↳ DETAIL 1: Proving $|A|!(n-|A|)!$ or $k!(n-k)!$:



① Subset lattice of all 0's to 1's (bit strings)

② Pick any set (bit string), call it A → How many maximal chains go through A?

↳ other words, How many different ways to go through A, from empty to 1111...?

A = 001101000101000 (for example)

↳ 5 1's, 10 0's

③ Start from 0000...0, and go up 1 level by changing a 0 to a 1, to get closer to becoming A

000000000000000 ← ① empty set (5 possibilities to switch 0→1)
001000000000000 4 possibilities
001100000000000 3 possibilities
001101000000000 2 possibilities
001101000100000 1 possibilities
001101000101000 ← A

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ possibilities (chains) from $\emptyset$ to A

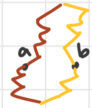
② Same step from A to 1111... (10 0's → 1's)

$10 \cdot 9 \cdot 8 \ldots = 10!$ from A → 1111...

③ $5!\,10!$ → If 1's represent inclusion, then $|A| = 5$, so $5!\,10! = |A|!(n-|A|)!$

↳ DETAIL 2: Total number of maximal chains: $n!$ oooo...

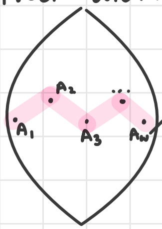
↳ DETAIL 3: 2 sets a and b, neither is a subset of the other



↳ Forms a 2-element antichain

↳ No maximal chains can go through both a and b

Proof: width = size of largest binomial coefficient



↳ Envision biggest antichain

w = width, distinct sets that are all incomparable

↳ For each i, let t_i count the # of maximal chains through A_i

$\sum_{i=1}^{w} t_i \le n!$ ← Detail 2

← Detail 1

$\sum_{i=1}^{w} |A_i|!(n-|A_i|)! \le n!$

$\sum_{i=1} \frac{|A_i|!(n-|A_i|)!}{n!} \le 1$

$\sum_{i=1}^{w} \frac{1}{C(n,|A_i|)} \le 1$

$\frac{w}{C(n,\lfloor n/2 \rfloor)} \le \sum_{i=1}^{w} \frac{1}{C(n,|A_i|)} \le 1$

$w \le C(n,\lfloor n/2 \rfloor)$

Ranked Posets and Symmetric Chains

A poset is ranked if all maximal chains are maximum (all maximal chains have = size)

↳ Middle level – line separating elements – does not go through any element IF height is even

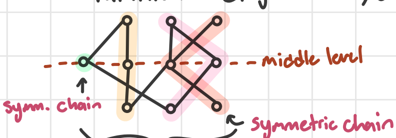
↳ goes through central elements IF height is odd

Symmetric Chain – goes the same distance above/below middle level and doesn't skip levels

↳ applies to both even + odd heights (single elements on level count if h is odd)

↳ Symmetric Chain Partition – covering an entire poset with symmetric chains

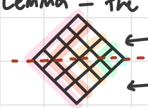
↳ Partition – disjoint sets, each set is a chain (sets can be connected)



Symmetric Chain Partition
(each vertex colored once)

Goal: Show 2^n has a Symmetric Chain Partition

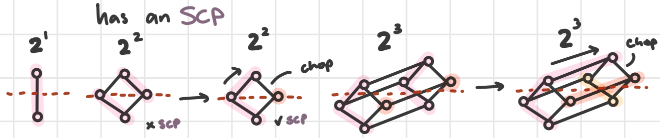
↳ Lemma – the product of 2 chains has an SCP



← Vertex at each intersection

← Any product of two chains (subset lattice, 2×2^n) has an SCP

Solves Dilworth Problem – SCP identifies maximal antichains (1 or 2) + chain partition



L.18 # Inclusion / Exclusion

X is a set of objects, and suppose that for every element i in $\{1,2,..,n\}$ we have a property P_i so for all x in X x (not ambiguous) either satisfies P_i or doesn't (true/false)

↳ Subset S of $\{1,2,\dots,n\}$, let $N(S)$ be the subset of X which has all of the x's in X that satisfies P_i for all i in S

↳ $N(\emptyset) = X$, N_0 is the subset of X with all the elements that DON'T satisfy P_i

↳ $N_0 = \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} N(S)$ → Translation: N_0 = The number of elements in the universe that satisfy NONE of the properties

↳ Example: When $n=2$, $N_0 = N(\emptyset) - N(1) - N(2) + N(1 \cap 2)$

↳ $n=3$, (3 properties) $N_0 = N(\emptyset) - [N(1) + N(2) + N(3)] + [N(12) + N(13) + N(23)] - N(123)$ $1 \cap 3$

↳ In general, there are 2^n terms

Derangements

Permutation σ of $\{1,2,\dots,n\}$ is a derangement ⟷ $\sigma(i) \neq i$ for all $i = 1,2,\dots,n$

↳ 38754126 (12345678 ✓) and 21436587 (12345678 ✓) are derangements

↳ 57314682 (12345678) and 72568572 (12345678) are NOT

↳ d_n denotes number of derangements of $\{1,2,\dots,n\}$

↳ [Historical Context]: Hatcheck Problem (if you want more details, Lecture 7.5-7.6)

$$d_n = \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} N(S) = \sum_{0 \le k \le n} (-1)^k C(n,k)(n-k)!$$

↳ When S is a subset of $\{1,2,\dots,n\}$ and $|S| = k$, $|N(S)| = (n-k)!$

↳ If σ satisfies P_i and i belongs to S, then $\sigma(i) = i$ – Positions corresponding to elements of S are determined, and other n-k positions are an arbitrary permutation of remaining elements

↳ $P_i = \sigma(i) = i$, so derangement = objects that DON'T fulfill the properties

↳ Example: $d_2 = (-1)^0\binom{2}{0}(2-0)! + (-1)^1\binom{2}{1}(2-1)! + (-1)^2\binom{2}{2}(2-2)!$ → $(1)(1)(2) + (-1)(2)(1) + (1)(1)(1)$

$2 + -2 + 1$

$d_2 = 1$

Counting Surjections

For an integer n, let [n] denote $\{1,2,\dots,n\}$. Also, let $S(n,m)$ denote the number of surjections ("onto" functions) from [n] to [m]

↳ Surjection – mapping objects to other objects (distinct objects to distinct cells, no empty cells)

↳ Example: Find $S(5,3)$: 1, 2, 3, 4, 5

1, 2, 3 (3, 1, 1) one orientation (1,2,3) ↳ 6 permutations (2,3,1 | 3,1,2 | etc.)

1, 2, 3 (2, 2, 1) another orientation ↳ 6 permutations

↳ Of 5 elements, choose 3 to go to 1 → $\binom{5}{3} \cdot 2 \to 20 \cdot 6 = 120$

↳ 2 elements left, 2 choices

$\binom{5}{2}\binom{5-2}{2}$

$10 \cdot 3 \cdot 6 = 180$

→ 300 ×

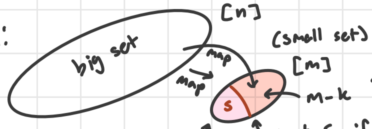
(there are overlaps counted)

↳ Surjection Formula

$$S(n,m) = \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} N(S) = \sum_{0 \le k \le m} (-1)^k C(m,k)(m-k)^n$$

If P_i is satisfied, i is NOT in domain

Explanation: [back to Lecture 2 topics]



subset S, if size of S = k (|S| = k), then

a function satisfies those properties when those elements are NOT in domain

↳ How many functions from m-element set to [m-k]?

↳ $(m-k)^n$

↳ Example Application: $S(5,3)$

$$
\begin{aligned}
&= (-1)^0\binom{3}{0}(3-0)^5 = 3^5 \\
&+ (-1)^1\binom{3}{1}(3-1)^5 = -3\cdot 2^5 \\
&+ (-1)^2\binom{3}{2}(3-2)^5 = 3\cdot 1^5 \\
&+ (-1)^3\binom{3}{3}(3-3)^5 = 0
\end{aligned}
\longrightarrow 243 - 96 + 3 = \underline{150} \checkmark
$$

Euler ϕ-Function

For integer $n \geq 2$, let $\phi(n)$ denote the number of elements in $[n]$ which are relatively prime to n

↳ Relatively Prime – numbers with greatest common divisor = 1

↳ Example: $\phi(12) = 4 \longleftarrow 1, 5, 7,$ and 11 are relatively prime to 12

Inclusion / Exclusion Formula → if prime factors of n are: $p_1, p_2, \ldots, p_k$

↳ Then: $\phi(n) = n(1 - 1/p_1)(1 - 1/p_2)\ldots(1 - 1/p_k)$

↳ Example: Factors of $324481700624 = 2^4(109)(727)(255923)$

↳ $\phi(n) = 324481700624(1-1/2)(1-1/109)(1-1/727)(1-1/255923)$