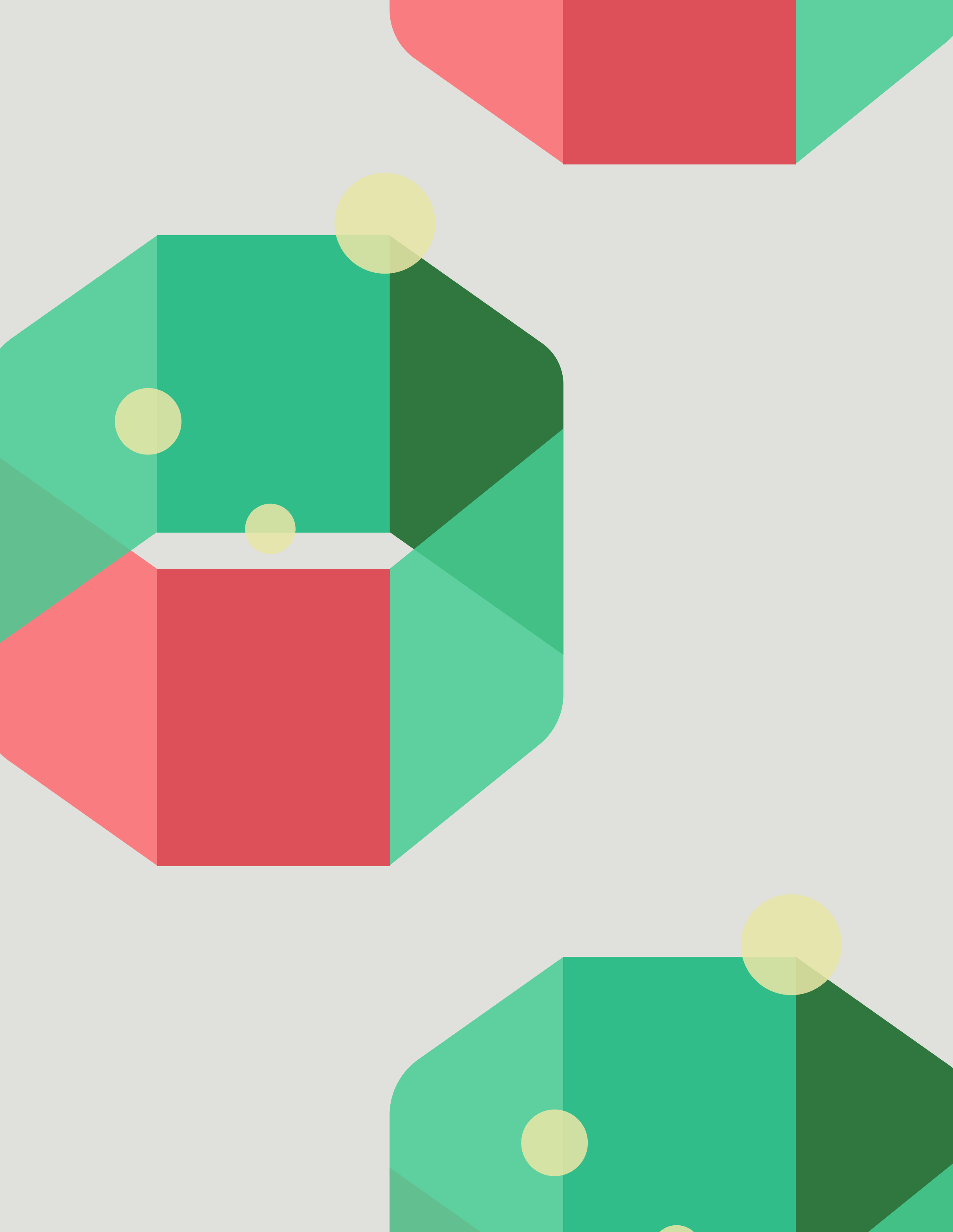


Symmetric Matrices

If matrix $A = A^T$, then A is symmetric

↳ $A = (2)$ ✓ $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ✓ $C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ ✗ $D = \begin{pmatrix} 4 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ ✗

Common Example: A^TA is always symmetric

↳ If A and C are $n \times n$ matrices, $\vec{x} \in \mathbb{R}^n$ and C is symmetric – Which of the following are symmetric?

1) $AA^T \rightarrow (AA^T)^T = (A^T)^T A^T = AA^T$ ✓

2) $\vec{x}\vec{x}^T \rightarrow \vec{x}^T\vec{x}$, ✓

3) $C^2 \rightarrow C^TC$, ✓

Additional Notes

↳ If a matrix is symmetric, it must be square □

↳ If a matrix is square □ and diagonal ↘, it is symmetric

Orthogonal Diagonalization

If A is symmetric, eigenvectors $\vec{v}_1$ and $\vec{v}_2$ (corresponding to 2 eigenvalues) are orthogonal

↳ eigenspaces to distinct eigenvalues are orthogonal subspaces

Diagonalize A using orthogonal matrix, P

$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\lambda = -1, 1$ $\rightarrow \lambda = -1: A - (-1)I \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ x_3 is free, $x_3 = 1$ $\} v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

↳ $\lambda = 1: A - 1(I) \longrightarrow \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ x_2, x_3 free, $x_2 = 0$, $x_3 = 1$ $\} v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

luckily orthogonal, use Gram-Schmidt if not

$\Big\} \rightarrow A = PDP^T$, $P = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

orthonormal ↓ eigenvalues ↓

If P is an orthogonal $n \times n$ matrix, then $P^{-1} = P^T$

↳ Symmetric matrices can be diagonalized as PDP^T bc $P^{-1} = P^T$

↳ Gram Schmidt may be needed if there are repeated eigenvalues (to make a full set of orthonormal eigenvectors)

↳ If $A = PDP^T$, then A is symmetric

↳ If $A = PDP^T$, then A is also diagonalizable

Spectrum of A → set of eigenvalues for a matrix

↳ Symmetric Matrices:

↳ All eigenvalues of A are real

↳ (all) Eigenspaces are mutually orthogonal

↳ $A = PDP^T$, where P is orthogonal

Spectral Decomposition of a Symmetric Matrix

Expansions to approximate matrices

↳ A is a symmetric matrix (orthogonal diagonalization)

↳ $A = PDP^T = (\vec{u}_1 \dots \vec{u}_n) \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{pmatrix} \rightarrow \underbrace{A = \lambda_1 \vec{u}_1 u_1^T + \dots + \lambda_n \vec{u}_n \vec{u}_n^T}_{\text{Spectral Decomposition}} = \sum_{i=1}^{n} \lambda_i \vec{u}_i u_i^T$

Sum of terms weighted by eigenvalues

Each term in sum $\lambda_i \vec{u}_i u_i^T$ is an $n \times n$ matrix with rank 1

↳ Each column is a multiple of u_i

Ordering eigenvalues large → small (in absolute value),

$|\lambda_i| \geq |\lambda_{i+1}|$

you are able to truncate the sum to get rid of small terms in order to approximate matrix A

Construct a spectral decomposition for A:

$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

↳ $A = \sum_{i=1}^{2} \lambda_i \vec{u}_i u_i^T = 4 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} (1/\sqrt{2} \;\; 1/\sqrt{2}) + 2 \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} (-1/\sqrt{2} \;\; 1/\sqrt{2}) \rightarrow \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

Quadratic Forms

Quadratic Form: Function $Q: \mathbb{R}^n \rightarrow \mathbb{R}^m$, given by $Q(\vec{x}) = \vec{x}^TA\vec{x} = (x_1 \; x_2 \dots x_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

↳ Uses a symmetric matrix to analyze quadratic functions

Compute quadratic form $Q = \vec{x}^TA\vec{x}$ using $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \rightarrow (x \; y) \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow 4x^2 + 3y^2$

cross-product, contains both variables

$A = \begin{pmatrix} 4 & 1 \\ 1 & -3 \end{pmatrix} \rightarrow (x \; y) \begin{pmatrix} 4 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow 4x^2 - 3y^2 + 2xy$

Express $Q = x^2 - 6xy + 9y^2$ in the form $Q = \vec{x}^TA\vec{x}$, $x \in \mathbb{R}^2$ and $A = A^T$ (symmetric)

$Q = (x \; y) \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, $Q = 1x^2 \underbrace{- 6xy}_{\text{divide by } 2} + 9y^2 \rightarrow A = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$

Write $Q(\vec{x}) = 5x_1^2 - x_2^2 + 3x_3^2 + 6x_1x_3 - 12x_2x_3$ in terms of $\vec{x}^TA\vec{x}$

$Q = (x_1 \; x_2 \; x_3) \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $A = \begin{pmatrix} 5 & 0 & 3 \\ 0 & -1 & -6 \\ 3 & -6 & 3 \end{pmatrix}$

Transform $\vec{x} \rightarrow A\vec{x}$; Squared length of vector $A\vec{x}$ is a quadratic form: $\|A\vec{x}\|^2 = (A\vec{x}) \cdot (A\vec{x}) = \vec{x}^TA^TA\vec{x}$

↳ Symmetric matrices can be used to characterize linear transforms

Change of Variable

Given $Q = \vec{x}^T A\vec{x}$, where $\vec{x} \in R^n$ is a variable vector and A is a $n \times n$ symmetric matrix, $A = PDP^T$

↳ A change of variable can be represented as $\vec{x} = P\vec{y}$, or $\vec{y} = P^{-1}\vec{x}$

↳ Quadratic form becomes
$$Q = \vec{x}^T A\vec{x} = (P\vec{y})^T A (P\vec{y}) = \vec{y}^T P^T A P\vec{y} = \vec{y}^T D\vec{y}$$

If A is a symmetric matrix then there exists an orthogonal change of variable $\vec{x} = P\vec{y}$ that transforms $\vec{x}^T A\vec{x}$ to $\vec{y}^T D\vec{y}$ with no cross-product terms

Compute the quadratic form $Q = \vec{x}^T A\vec{x}$ for $A = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$ and identify a change of variable that removes the cross-product term

$\lambda_1 = 9$, $\lambda_2 = 4$, $\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ → $\vec{x} = P\vec{y}$, $P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

↳ $Q = \vec{x}^T A\vec{x} = \vec{y}^T D\vec{y} = 9y_1^2 + 4y_2^2$

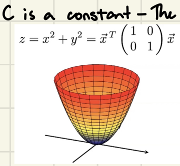
$$(y_1 \;\; y_2)\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

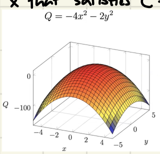
Quadratic Surfaces

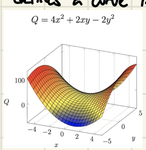
Suppose $Q(\vec{x}) = \vec{x}^T A\vec{x}$, where $A \in R^{2\times2}$ is symmetric, and C is a constant – The set of $\vec{x}$ that satisfies $C = \vec{x}^T A\vec{x}$ defines a curve in R^2

↳ $Q = x^T\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\vec{x} = 2x^2 + 2y^2 + 2xy = 1$

C, changing C changes the size of ellipse







↳ When C is varied continuously, a surface is generated in R^3

↳ $z = Q(\vec{x}) = \vec{x}^T A\vec{x}$, where $A \in R^{2\times2}$ is symmetric — the set of points that satisfy $z = \vec{x}^T A\vec{x}$ defines a surface in R^3

A quadratic form Q is positive definite if $Q > 0$ for all $\vec{x} \neq \vec{0}$

↳ Also when all eigenvalues are positive

↳ negative definite if $Q < 0$ for all $\vec{x} \neq \vec{0}$

↳ when all eigenvalues are negative

→ positive semidefinite if $Q \geq 0$ for all $\vec{x}$

→ negative semidefinite if $Q \leq 0$ for all $\vec{x}$

→ indefinite if Q takes on positive and negative values for $\vec{x} \neq \vec{0}$

↳ Also when at least 1 eigenvalue is negative and at least 1 is positive

Constrained Optimization Problem

Surface of a unit sphere: $1 = x_1^2 + x_2^2 + x_3^2 = \|\vec{x}\|^2$

↳ Q is a quantity (such as a temp) that is optimized → $Q = 9x_1^2 + 4x_2^2 + 3x_3^2$

↳ Identify largest/smallest values of Q / where they are located

1) Identify largest value of Q

$$Q = \vec{x}^T\begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}\vec{x} = 9x_1^2 + 4x_2^2 + 3x_3^2$$

eigenvalue

$$\leq 9x_1^2 + 9x_2^2 + 9x_3^2 = 9(x_1^2 + x_2^2 + x_3^2) = 9\|\vec{x}\|^2 = 9 \leftarrow \text{max value}$$

$\max\{Q(\vec{x}) : \|\vec{x}\| = 1\} = 9$, and max occurs at $\vec{x} = \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix}$

$\min\{Q(\vec{x}) : \|\vec{x}\| = 1\} = 3$, and min occurs at $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ \pm1 \end{pmatrix}$

↳ Max and Min values of Q were eigenvalues of A, corresponding eigenvectors were the locations

Constrained Optimization Problem – Find the min/max values of $Q(\vec{x}) = \vec{x}^T A\vec{x}$, $\vec{x} \in R^n$, $A \in R^{n\times n}$ $\|\vec{x}\| = 1$

↳ If $Q = \vec{x}^T A\vec{x}$, A is a real $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$ and associated <u>normalized</u> eigenvectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$

↳ Max of $Q(\vec{x})$ is λ_1 (largest), when $\vec{x} = \pm\vec{u}_1$

↳ Min of $Q(\vec{x})$ is λ_n (smallest), when $\vec{x} = \pm\vec{u}_n$

Constrained Optimization with a Repeated Eigenvalue

Calculate max and min of $Q(\vec{x}) = \vec{x}^T A\vec{x}$, $\vec{x} \in R^3$, subject to $\|\vec{x}\| = 1 \rightarrow Q(x) = x_1^2 + 2x_2x_3$

↳ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\lambda_1 = 1 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, max value = 1 in span of v_1 and v_2

$\lambda_1 = -1 \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \vec{v}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, min value = -1 at $\pm v_3$

Orthogonality Constraints

If A is symmetric and has eigen values $\lambda_1 \geq \dots \lambda_n$ and eigenvectors $\vec{u}_1 \dots \vec{u}_n$, subject to constraints $\|\vec{x}\| = 1$ and $\vec{x} \cdot \vec{u}_1 = 0$

↳ Max value of $Q(\vec{x}) = \lambda_2$, attained at $\vec{x} = \vec{u}_2$

Calculate max value of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in R^3$, $\|\vec{x}\| = 1$ and $\vec{x} \cdot \vec{u}_1 = 0$, identify a max point

↳ $Q(\vec{x}) = x_1^2 + 2x_2x_3$, $\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

↳ $Q = x^T A x = x^T \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x$, $\lambda = \pm 1 \rightarrow$ if $\lambda = 1$, $A - I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$, $u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} \Rightarrow$ max value of Q is +1, at $\pm u_2$

Singular Values

Singular Values – square roots of the eigenvalues of A^TA

↳ Linear Transformations – standard matrix is $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$

↳ What unit vector maximizes $\|Av\|$? What is the max?

↳ $\max_{\|v\|=1} \|Av\| \rightarrow$ max also occurs $\|Av\|^2 = v^T \underbrace{A^TA}_{\text{symmetric}} \vec{v}$

eigenvalues

↳ $A^TA = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix} \rightarrow \lambda = 8, 2$

location

$\max \|Av\|^2 = 8$, so $\max \|Av\| = \sqrt{8} \rightarrow \vec{v}$ is eigenvector for $\lambda = 8$, $A^TA - \lambda I = \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\min \|Av\|^2 = 2$, so $\min \|Av\| = \sqrt{2} \rightarrow \vec{v}_2$ is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

distances btw origin to pts

↳ $\sigma_1 = \sqrt{\lambda_1} = \sqrt{8}$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$

All eigenvalues of A^TA are NON-NEGATIVE!

↳ Singular Values, σ_j, are all real and positive – so they are ordered $\sigma_1 = \sqrt{\lambda_1} \geq \dots \sigma_n = \sqrt{\lambda_n}$

↳ Singular Values are lengths of vectors in R^n, $\|A\vec{v}\| = \sigma_i$

Singular Vectors

For any $A \in R^{m \times n}$, orthogonal complement of Row A is Nul A, orthogonal complement of Col A is Nul A^T

↳ if v_i are the n orthogonal eigenvectors of A^TA, ordered greatest to least, and r has singular values $r \leq n$

→ $\{\vec{v}_{r+1}, \vec{v}_{r+2}, \dots \vec{v}_n\}$ is an orthogonal basis for Nul A, and $\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_r\}$ is an orthogonal basis for Row A; rank A = r

→ $\{A\vec{v}_1, A\vec{v}_2, \dots A\vec{v}_n\}$ are an orthogonal basis for Col A

If $\vec{i}$ are orthonormal eigenvectors for A^TA, and $\vec{u}_i = \frac{1}{\sigma_i}A\vec{v}_i$ for $i \leq r = \text{rank} A$, $\sigma_i = \|A\vec{v}_i\|$

↳ $\vec{v}_1, \dots \vec{v}_n$ — is an orthonormal basis for → Row A

↳ $\vec{v}_{r+1}, \dots \vec{v}_n$ — is an orthonormal basis for → Nul A

↳ $\vec{u}_1, \dots \vec{u}_r$ — is an orthonormal basis for → Col A

A is a 12x4 Matrix with 3 singular values, (T/F)

1) Basis for Col A: $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ ✓

2) Basis for Nul A: $\vec{v}_1, \vec{v}_2, \vec{v}_3$ ✗

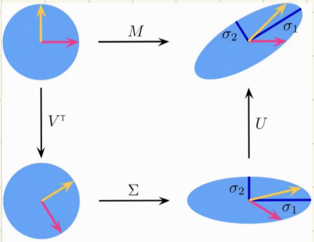
left singular vectors → vectors $\{\vec{u}_i\}$ for $i \leq m$

right singular vectors → vectors $\{\vec{v}_i\}$ for $i \leq m$

SVD

A has $\sigma_1 \leq .. \sigma_n$ and $m \geq n$ → Then A has the singular value decomposition $A = U\Sigma V^T$ where:

↳ $\Sigma = \begin{pmatrix} D \\ 0_{m-n,n} \end{pmatrix}$, $D = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \sigma_n \end{pmatrix}$ → U is an $m \times n$ orthogonal matrix → V is an $n \times n$ orthogonal matrix



↳ If $m < n$, then $\Sigma = \begin{pmatrix} D & 0_{m,n-m} \end{pmatrix}$ with everything else the same

Procedure - Suppose A is $m \times n$ and has rank r:

1) Compute squared singular values of A^TA, σ_i^2, construct Σ
2) Compute unit singular values of A^TA, $\vec{v}_i$, construct V
3) Compute orthonormal basis for Col A using $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$, $i = 1, 2, 3 \ldots r$

↳ If needed, expand $\{\vec{u}_i\}$ to form orthonormal basis for R^m and use the basis to construct U

Construct the SVD for $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

① ↳ Find Singular Values for A^TA:

$A^TA = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \rightarrow \lambda_1 = 9, \lambda_2 = 4$, so $\sigma_1 = 3$, $\sigma_2 = 2$ (largest!)

② ↳ Construct Σ:

$\sigma_1 = 3, \sigma_2 = 2 \rightarrow \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ σ on diagonal; $0_{4-2,2} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

③ ↳ Construct right-singular vectors $\{\vec{v}_i\}$ and V:

$A^TA - \lambda_1 I = \begin{pmatrix} -5 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A^TA - \lambda_2 I = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigenvector

$\Bigg\}$ $V = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ → normalize (already normalized)

④ ↳ Construct left-singular vectors $\{\vec{u}_i\}$ using $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$:

$\vec{u}_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$\vec{u}_2 = \frac{1}{\sigma_2} Av_2 = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Bigg\}$ $U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

↑↑ have to be orthogonal to u_1 and u_2 AND be unit length

↳ Construct the final SVD:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \underset{U}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \cdot \underset{\Sigma}{\begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}} \cdot \underset{V}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

SVD of 3×2 Matrix with Rank 1

Construct SVD for $A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$

↳ $A^TA = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix} \rightarrow \lambda_1 = 18, \lambda_2 = 0 \rightarrow \sigma_1 = \sqrt{18}, \sigma_2 = 0$

↳ $\sigma_1 = \sqrt{18}, \sigma_2 = 0 \rightarrow \begin{pmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \Sigma$

↳ $A^TA - \lambda_1 I = \begin{pmatrix} -9 & -9 \\ -9 & -9 \end{pmatrix} \Rightarrow \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A^TA - \lambda_2 I = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix} \Rightarrow \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Bigg\}$ $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

↳ $\vec{u}_1 = \frac{1}{\sigma_i} A\vec{v}_i \rightarrow \vec{u}_1 = \frac{1}{\sqrt{18}}\begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

↳ What about u_2 and u_3? → two orthogonal vectors are: $\vec{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{x}_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

↳ $\vec{u}_1 \in \text{Col} A$, so these two vectors are in $(\text{Col} A)^\perp$

↳ U is an orthogonal matrix, so how can we create an orthogonal basis for $(\text{Col} A)^\perp$

↳ Solution: Gram-Schmidt!

↳ $\vec{u}_2 = \vec{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{u}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2$

↳ $u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $u_3 = \frac{1}{\sqrt{45}}\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ → $U = \begin{pmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & -5/\sqrt{45} \end{pmatrix}$

SVD and the Condition Number of Matrix

Condition Number – If A is an invertible $n \times n$ matrix, the ratio $\frac{\sigma_1}{\sigma_n}$ is the condition number of A

↳ Describes the sensitivity that <u>any</u> approach to solutions to $A\vec{x}=\vec{b}$ has errors to A

↳ The larger the condition number, the more sensitive the system is to errors

SVD and Spectral Decomposition

Spectral Decomposition for any matrix with rank r (To approximate non-symmetric matrices)

↳ $A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$

A has the following SVD: ⟶ Spectral Decomposition:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = \sum_{s=1}^{r} \sigma_s \vec{u}_s \vec{v}_s^T = \underset{\sigma_1}{3} \underset{\vec{u}_1}{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}} \underset{\vec{v}_s^T}{\begin{pmatrix} 0 & 1 \end{pmatrix}} + 2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = 3 \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

SVD and the Four Fundamental Subspaces

If $\vec{v}_i$ are orthonormal eigenvectors for A^TA and $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$ for $i \le r = \text{rank} A$, $\sigma_i = \|A\vec{v}_i\|$

↳ Then, we have the following bases for any $m \times n$ matrix:

$\vec{v}_1 \dots \vec{v}_r$ is an orthonormal basis for Row A

$\vec{v}_{r+1}, \dots \vec{v}_n$ is an orthonormal basis for Nul A

} Right-Singular vectors

$\vec{u}_1, \dots \vec{u}_r$ is an orthonormal basis for Col A

$\vec{u}_{r+1}, \dots \vec{u}_n$ is an orthonormal basis for Nul A^T

} Left-Singular vectors

↳ b/c U is an orthogonal basis

Given SVD of A, find rank(A), and bases for Nul A and Col A

$$A = U\Sigma V^T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 1 & 2 & 3 & & \end{matrix} \end{array} \begin{pmatrix} 0 & 0 & \sqrt{0.8} & 0 & -\sqrt{0.2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{0.8} & 0 & \sqrt{0.2} \end{pmatrix}$$

↳ 3 non-zero singular values → rank A = 3

↳ First 3 rows of V^T = Row A = Nul $A^{\perp}$

↳ Last 2 rows of V^T = Nul A

↳ First 3 columns of U = Col A

↳ Last 2 columns of U = Nul A^T