Math 2552: Differential Equations

↳ Chapters 5 & 7-8

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Author's Note

↳ This is solely covering the topics of the lecture videos, so I recommend you read the textbook in regards to working out homework

↳ Pretty much all info regarding computer science has been omitted (i.e. functions)

↳ Lecture numbers are on the left of titles

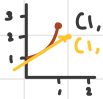
↳ Some videos have been combined across headings

8.1.1-4 Euler's Method

Chapter 8 considers IVPs of form $y'(t) = f(t,y), y(t_0)=y_0$

↳ when exact solns. can't be found, we can <u>approximate</u> solutions with procedures

ex. consider $\frac{dy}{dt} = y, y(0) = 1$



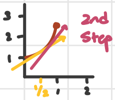
↳ find exact solution, $\phi(t) \rightarrow$ by inspection, $y = e^t$

↳ estimate $\phi(t)$ using Euler's method (tangent line) at $t=0$, from 1 step and 2 steps

↳ $y_1 = y_0 + y'(0)(t-t_0)$, $y_0 = y(0)$ and $t_0 = 0$

$= 1 + 1(t-0) = 1 + t = y_1 \rightarrow y_1(1) = 2$

↳ 2 steps:



$\rightarrow y_1 = y_0 + y'(0)\cdot(t - t_0)$

$= 1 + t$

$y_1(0.5) = 1.5 \rightarrow y_2 = y_1 + y'(t_1)(t - t_1)$

$= 1.5 + 1.5(t - \frac{1}{2})$

$y_2(1) = 1.5 + \frac{3}{2}(1 - \frac{1}{2})$

$= \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$

Euler's Method: solution to IVP $\frac{dy}{dt} = f(t,y)$, $y(0) = y_0$ can be approximated at t_0, t_1, t_2 using $y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$, $n = 0,1,2...$ <u>memorize!</u>

↳ Often step size $t_{n+1} - t_n$ has uniform value h, so it simplifies

↳ $y_{n+1} = y_n + hf(t_n, y_n)$, $n = 0,1,2...$ <u>Memorize!</u>

$t_{n+1} = t_n + h$

8.2.1-2 Accuracy of Numerical Methods

$\phi(t)$ = exact solution, y_{n+1} = approximate solution

↳ Finding error: $e_{n+1} = \phi(t_{n+1}) - y_{n+1}$

↳ Taylor Expansion: $\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \frac{1}{2}\phi''(c)h^2$, $c \in (t_n, t_n + h)$

↳ Use $\phi(t_n + h) = \phi(t_{n+1}) \rightarrow e_{n+1} = \phi(t_{n+1}) - y_{n+1}$

$= (\phi(t_n) + \phi'(t_n)h + \frac{1}{2}\phi''(c)h^2) - (y_n + hf(t_n, y_n))$

↳ For local truncation error, we assume $y_n = \phi(t_n)$ because we only want error introduced at <u>each</u> step

↳ so, if $y_n = \phi(t_n)$ then $y_n' = \phi'(t_n)$

↳ error becomes $e_{n+1} = \frac{1}{2}\phi''(c)h^2$

ex. $y' = y$, $y(0) = 1$, find local truncation error in terms of t and ϕ.

↳ $e_{n+1} = \frac{1}{2}\phi''(c)h^2$ but $\phi' = y$ (given DE)

$= \frac{1}{2} 1(c)h^2 = \frac{1}{2}y'(t)h^2$

ex. $y' = \frac{1}{2} - t + 2y$, $y(0) = 1$

↳ $e_{n+1} = \frac{1}{2}\phi''(c)h^2$, $\phi' = \frac{1}{2} - t + 2y \xrightarrow{\text{derive with } t} \phi'' = 2\phi'(t_n) - 1 \rightarrow 2(\frac{1}{2} - t + 2\phi) - 1 \rightarrow 4\phi - 2t$

$= \frac{1}{2}(4\phi - 2c)h^2$, $c \in (t_n, t_{n+1})$

8.3.1-4 Improved Euler and Runge-Kutta Methods

More general approach: $y_{n+1} = y_n + h(\underbrace{w_1 k_1 + w_2 k_2 + \dots w_n k_n}_{\text{weighted average}})$

↳ w_i are weights, sum to 1

↳ k_i is $f(x,y)$ evaluated at certain points

Improved Euler: replaces $f(t_n, y_n)$ in Euler Method with $\dfrac{f(t_n,y_n) + f(t_{n+1}, y_n + hf(t_n,y_n))}{2}$

↳ uses average of slopes at t_n and t_{n+1}

↳ solution to the IVP $y'(t) = f(t,y)$, $y(0)$ can be approximated at t_0, t_1, t_2 using:

- ↳ $y_{n+1} = y_n + \frac{h}{2}\left(f(t_n,y_n) + f(t_{n+1}, y_n + h f(t_n,y_n))\right)$, $n = 0, 1, 2\dots$ ← step size (h)
- ↳ $t_{n+1} = t_n + h$
- ↳ TL;DR – takes tangent at 2 points, averages to find slope

Runge-Kutta: replaces $f(t_n,y_n)$ with a weighted average of 4 values of $f(x,y)$

↳ solution to $y'(t) = f(t,y)$, $y(0) = y_0$ can be approximated:

- ↳ $y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})$
  - ↳ $k_{n1} = f(t_n, y_n)$
  - ↳ $k_{n2} = f(t_n + \frac{h}{2}, y_n + \frac{1}{2}hk_{n1})$
  - ↳ $k_{n3} = f(t_n + \frac{h}{2}, y_n + \frac{1}{2}hk_{n2})$
  - ↳ $k_{n4} = f(t_n + h, y_n + hk_{n3})$
- ↳ $t_{n+1} = t_n + h$

ex. Use 1 iteration of Imp. Euler and RK to estimate IVPs at a specific point.

$y' = y$, $y(0) = 1$. Estimate solution at $t=1$.

↳ Imp. Euler: $y_{n+1} = y_n + \frac{h}{2}\left(f(t_n,y_n) + f(t_{n+1}, y_n + hf(t_n,y_n))\right)$

- ↳ $f(t_n,y_n) = y_n$, $t_{n+1} = t_n + h$, $h=1$
  - ↳ $y_{n+1} = y_n + \frac{h}{2}\left(y_n + f(t_n + h, y_n + 1\, y_n)\right)$

$$= y_n + \frac{h}{2}(y_n + 2y_n)$$

$$= y_n + \frac{3}{2}y_n = \frac{5}{2}y_n \text{, so } y_1 = \frac{5}{2}y_0 = \frac{5}{2} = 2.5$$

↳ RK: $y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})$, $h=1$, $y_0 = 1$, find $y_1(1)$

- ↳ $k_{n1} = f(t_0, y_0) = 1$ → t term is not in original IVP, $y' = y$
- ↳ $k_{n2} = f(t_0 + \frac{h}{2}, y_0 + \frac{1}{2}hk_{n1}) = y_0 + \frac{1}{2}hk_{n1} = 3/2$
- ↳ $k_{n3} = f(t_0 + \frac{h}{2}, y_0 + \frac{1}{2}hk_{n2}) = y_0 + \frac{1}{2}hk_{n2} = 7/4$
- ↳ $k_{n4} = f(t_0 + h, y_0 + hk_{n3}) = y_0 + hk_{n3} = 11/4$

↳ $y_1 = y_0 + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}) = 1 + \frac{1}{6}\underbrace{\left(1 + 2\cdot\frac{3}{2} + 2\cdot\frac{7}{4} + \frac{11}{4}\right)}_{\text{stop here on exam}} \sim 2.708$

7.1.1 Review of Critical Point Classification

This section focuses on non-linear autonomous 2-dimensional systems

↳ $\frac{dy}{dt} = f(x,y)$, $\frac{dx}{dt} = g(x,y)$

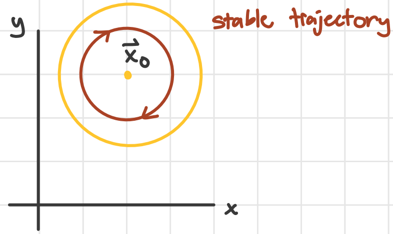
↳ ex. $\frac{dx}{dt} = 4-2y$, $\frac{dy}{dt} = y^2 - xy$

Critical Points occur when both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$

↳ stable if $\|\vec{x} - \vec{x}_0\|$ is bounded, $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

↳ asymptotically stable if $\vec{x} \to \vec{x}_0$  bowl

↳ unstable if not bounded



7.1.2-3 Phase Portrait for a Pendulum

Mass on an inflexible rod that pivots

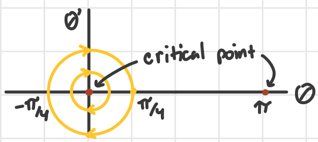
$\theta \in [-\pi, +\pi]$

↳ Under gravity, angle $\theta(t)$ satisfies

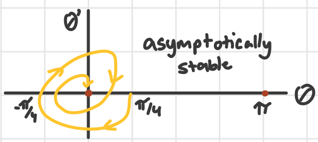
↳ $\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \sin\theta = 0$

ex. Sketch the phase portrait, θ' vs θ, for a few different initial conditions. Where are critical points?

$\gamma = 0$, no damping



$\gamma \neq 0$, some damping



7.1.4-7 Nullclines, Slope Fields, Critical Points

Determine critical points of system: $\frac{dx}{dt} = 4-2y$, $\frac{dy}{dt} = 12-3x^2$

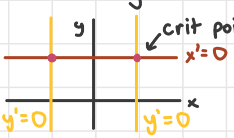
↳ centers of swirls: $(2,2)$ and $(-2,2)$ ← using a direction field

↳ $x' = 4-2y$ ①, $y' = 12-3x^2$ ②

↳ To identify critical points, set $x' = y' = 0$

① yields $0 = 4-2y$, or $y = 2$

② yields $0 = 12-3x^2$, or $x = \pm 2$



x-nullcline – line along which $\frac{dx}{dt} = 0$

y-nullcline – line along which $\frac{dy}{dt} = 0$

nullcline – a line along which $x' = y' = 0$

ex. Find all the critical points: $\frac{dx}{dt} = 2x - x^2 - xy$, $\frac{dy}{dt} = 3y - 2y^2 - 3xy$

↳① $x' = x(2-x-y) \rightarrow 0 = x(2-x-y)$, $x = 0$ or $y = 2-x$

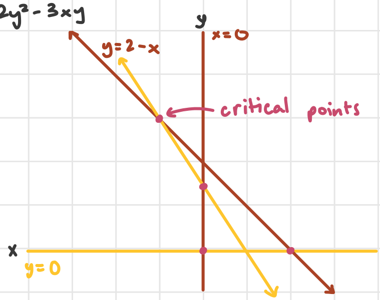
↳② $y' = y(3-2y-3x) \rightarrow 0 = y(3-2y-3x)$, $y = 0$ or $y = \frac{3}{2}(1-x)$

↳ Substitute $x=0$ into ②: $y(3-2y-0) = 0 \rightarrow y = 0$ or $y = 3/2$

↳ $(0,0)$, $(0, 3/2)$ are crit points

↳ Subs $y = 2-x$ into ②: $0 = (2-x)(3-2(2-x)-3x)$

↳ $(2-x)(-1-x)$, so $(2,0)$, $(-1,3)$ are crit points



4 straight lines = 4 critical points (assuming none of the lines are identical)

↳ # crit points ≤ 4

7.2.1-2 Almost Linear Systems

Suppose $\vec{x}' = A\vec{x} + \vec{g}$ and $\vec{x} = \vec{x}_0$ is a critical point of (1). If $\vec{g} \approx \vec{0}$ near $\vec{x}_0$, then:

↳ $\vec{x}' \approx A\vec{x}$ near $\vec{x}_0$ → we can use this idea to approximate solution to (1) around $\vec{x} = \vec{0}$

↳ if $\vec{x} = \vec{0}$ is a critical point, and $\det A \neq 0$, $\vec{x} = \vec{0}$ is an isolated critical point

Almost Linear System – if $\vec{x} = \vec{0}$ is an ICP (isolated critical point) of $\vec{x}' = A\vec{x} + \vec{g}$ AND $\frac{\|\vec{g}\|}{\|\vec{x}\|} \to \vec{0}$, as $\vec{x} \to \vec{0}$, then the system is an almost linear system near $\vec{x} = \vec{0}$

↳ in 2 dimensions, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, so $\vec{x}' = A\vec{x} + \vec{g}$ can be written as:

$$\left.\begin{aligned} &↳ \frac{g_1}{\|\vec{x}\|} = \frac{g_1}{r} \to 0 \\ &↳ \frac{g_2}{\|\vec{x}\|} = \frac{g_2}{r} \to 0 \end{aligned}\right\} \text{as } r \to 0,\ r = \sqrt{x^2 + y^2} \Big\} \text{ i.e. } \|\vec{g}\| \text{ is tending to } 0 \text{ faster than } \|\vec{x}\|$$

7.2.3-4 Tests for Almost Linear Systems

Consider autonomous system $\vec{x}' = F(x,y)$, $\vec{y}' = G(x,y)$

↳ assume this system has an ICP at (x_0, y_0)

$F_x = \frac{\partial F}{\partial x}$, $F_y = \frac{\partial F}{\partial y}$

↳ Taylor expansions about (x_0, y_0), using multivariable calculus:

↳ $F(x,y) = \underbrace{F(x_0,y_0)}_{0 \text{ order}} + \underbrace{F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)}_{\text{1st order}} + \eta$ ← remainder

↳ $\eta/r \to$ as $\vec{x} \to \vec{x}_0$, η = eta = remainder

↳ But, at (x_0,y_0), $F(x_0,y_0) = 0$

$$\left.\begin{aligned} &↳ F(x,y) = \begin{pmatrix} F_x & F_y \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \eta \\ \text{Similarly, } &↳ G(x,y) = \begin{pmatrix} G_x & G_y \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \eta \end{aligned}\right\} \text{A}$$

$$\left.\begin{aligned} &↳ \frac{dx}{dt} = \frac{d}{dt}(x - x_0) = F \\ &↳ \frac{dy}{dt} = \frac{d}{dt}(y - y_0) = G \end{aligned}\right\} \text{B} \longrightarrow \vec{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \vec{\eta}$$

↳ $\vec{x}' = J\vec{x}$

Summary: suppose system $x' = F(x,y)$ and $y' = G(x,y)$ has an ICP at (x_0,y_0). If F and G are twice differentiable in a region around (x_0,y_0), the system is almost linear

↳ corresponding **Jacobian Matrix** of system:

↳ $J = \begin{pmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{pmatrix}$ can be used to construct the almost linear system

7.2.5-9 Examples of Almost Linear Systems

ex. For each system, a) find all critical points, b) construct linear systems for each critical point, and c) classify critical points according to stability

1) $\frac{dx}{dt} = x + x^2 + y^2$, $\frac{dy}{dt} = y - xy$

↳ $x' = x^2 + x + y^2 \to (x^2 + x + 1/4) + y^2 - 1/4 \to (x - 1/2)^2 + y^2 - 1/4$ ← circle

↳ $y' = y(1-x)$

a) $y' = 0 = y(1-x) \to y = 0$ or $x = 1$

$$\left.\begin{aligned} &y = 0:\ x' = 0 = x^2 + x = x(x+1),\ x = 0, -1 \\ &x = 1:\ x' = 0 = 1^2 + 1 + y^2 \to y^2 = -2 \end{aligned}\right\} \text{crit points: } (0,0) \text{ and } (-1,0)$$

b) $\vec{x}' = J\vec{x} = \begin{pmatrix} 1+2x & 2y \\ -y & 1-x \end{pmatrix}$

↳ at $(0,0)$, $\vec{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} \rightarrow \lambda = 1$, so unstable node at $(0,0)$

↳ at $(-1,0)$, $\vec{x}' = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\vec{x} \rightarrow \lambda = -1, \lambda = 2$, unstable saddle at $(-1,0)$

ex. 2) $\frac{dx}{dt} = (2+x)(y-x)$, $\frac{dy}{dt} = (4-x)(y+x)$

↳ $x' = 0 = (2+x)(y-x) \rightarrow x = -2, x = y$

↳ along $x=-2$: $y' = 0 = (4+2)(y-2) \longrightarrow$ crit point at $(-2,2)$

↳ along $x=y$: $y' = 0 = (4-x)(x+x) \longrightarrow$ crit point at $(4,4)$ and $(0,0)$

↳ $y' = 0 = (4-x)(y+x) \rightarrow x=4, y=-x$

↳ along $x=4$: $x' = 0 = (2+4)(y-4) \longrightarrow$ crit point at $(4,4)$

↳ along $y=-x$: $y' = 0 = (2+x)(x-x) \rightarrow$ crit point at $(-2,2)$

b) $\left.\begin{array}{l} x' = (2+x)(y-x) = 2y - 2x + xy - x^2 \\ y' = (4-x)(y+x) = 4y + 4x - xy - x^2 \end{array}\right\} \vec{x} = J\vec{x} = \begin{pmatrix} -2+y-2x & 2+x \\ 4-y-2x & 4-x \end{pmatrix}$

↳ at $(0,0)$, $J = \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix}$, $0 = (\lambda+2)(\lambda-4) - 8$, $\lambda = 1 \pm \sqrt{68} \longrightarrow$ unstable node
(1 pos, 1 neg)

↳ at $(4,4)$, $J = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix}$, $0 = (-6-\lambda)(-\lambda) + 48$, $\lambda = -3 \pm \frac{1}{2}\sqrt{36 - 4 \cdot 48} \longrightarrow$ stable, spiral sink
(imaginary, neg real part)

↳ at $(-2,2)$, $J = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix}$, $\lambda = 4, 6 \rightarrow$ unstable node

Affect of Perturbations – consider 2-dimensional autonomous system $\vec{x}' = A\vec{x} + \vec{g}$ (not $\vec{0}$) and $\lambda = 2i$ is an eigenvalue of A near ICP (x_0, y_0)

↳ how does linear system $\vec{x}' = A\vec{x}$ behave near ICP?

↳ eigenvalues are $\pm 2i$, stable center

↳ how does the <u>almost</u> linear system behave?

↳ DIFFERENTLY

↳ Types of critical points/stability of linear system and almost linear systems are the <u>same</u>, except when <u>eigenvalues of A are pure imaginary or repeated</u>

↳ Table for classification in Table 3.5.5

7.3.1–3 Competing Species Model

Logistic Equation: $\frac{dx}{dt} = x(\epsilon - \sigma x)$, ϵ = growth rate, $\frac{\sigma}{\epsilon}$ = saturation level

$(\epsilon - \sigma x)$ = an environmental capacity for a species (e.g. food supply)

↳ $x(t)$ represents population of 1 species at time t; we can extend this to 2 species

↳ 2 species $\begin{cases} \frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x) \\ \frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y) \end{cases} \longrightarrow \begin{cases} \frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x - \alpha_1 y) \\ \frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y - \alpha_2 x) \end{cases}$

↳ sharing the same food supply

ex. Find crit points, construct linear systems, classify crit points according to stability

1) $\frac{dx}{dt} = x(1.5 - x - 0.5y)$, $\frac{dy}{dt} = y(2 - 0.5y - 1.5x)$

$\left.\begin{array}{l} ↳\ x' = 0 = x(\frac{3}{2} - x - \frac{1}{2}y) \rightarrow x = 0 \text{ or } y = 3 - 2x \\ ↳\ y' = 0 = y(2 - \frac{1}{2}y - \frac{3}{2}x) \rightarrow y = 0 \text{ or } y = 4 - 3x \end{array}\right\}$ From graph, critical points at $(0,0)$, $(0,4)$, $(3/2, 0)$, and $(1,1)$

↳ $x' = \frac{3}{2}x - x^2 - \frac{1}{2}xy$, $y' = 2y - \frac{1}{2}y^2 - \frac{3}{2}xy$

↳ $J = \begin{pmatrix} 3/2 - 2x - y/2 & -x/2 \\ -3y/2 & 2 - y - \frac{3}{2}x \end{pmatrix} \to (0,0),\ J = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}$, unstable node

↳ $(3/2, 0) \to J = \begin{pmatrix} -3/2 & -3/4 \\ 0 & 2 - 9/4 \end{pmatrix}$, stable node

↳ $(0,4) \to J = \begin{pmatrix} -1/2 & 0 \\ 0 & -2 \end{pmatrix}$, stable node

↳ $(1,1) \to J = \begin{pmatrix} -1 & -1/2 \\ -3/2 & -1/2 \end{pmatrix}$, $0 = (-1-\lambda)(-1/2 - \lambda) + (-3/4)$, $\lambda = -\frac{3}{4} \pm \frac{1}{4}\sqrt{\frac{9}{4}+1}$, unstable saddle

7.4.1 Predator – Prey Equations

Modeling Predator–Prey relationships: y preys on x

↳ in absence of x, y dies out, so $y' = -Cy$, $C > 0$

↳ in absence of y, $x' = Ax$, $A > 0$

↳ number of encounters is proportional to xy

$$\left.\begin{aligned} \frac{dx}{dt} &= Ax - \alpha xy \\ \frac{dy}{dt} &= -Cy + \gamma xy \end{aligned}\right\}$$

Lotka–Volterra equations
↳ not necessary to learn

ex. Find crit points, system, classify: $\frac{dx}{dt} = x - 0.5xy$, $\frac{dy}{dt} = -0.75y + 0.25xy$

↳ $x' = x(1 - \frac{y}{2}) \to x = 0$ or $y = 2$

↳ $y' = y(-\frac{3}{4} + \frac{x}{4}) \to y = 0$ or $x = 3$

$\to$ $(0,0)$ and $(3,2)$ are critical points

↳ $J = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}$ $\to (0,0) \to \begin{pmatrix} 1 & 0 \\ 0 & -3/4 \end{pmatrix}$, unstable node, λs are + and −

$\to (3,2) \to \begin{pmatrix} 0 & -3/2 \\ 1/2 & 0 \end{pmatrix}$, $\lambda = \pm \frac{\sqrt{3}}{2} i$, center

5.1.1–6 Definition of the Laplace Transform

Suppose $f(t)$ is defined on $[0, \infty)$. The Laplace Transform is given by:

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

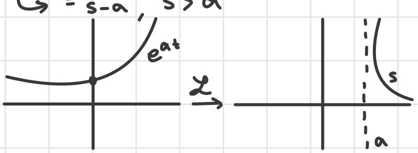
↳ domain of $\mathcal{L}[f]$ is the set of values where the integral exists

ex. Compute $\mathcal{L}[1]$ and $\mathcal{L}[e^{at}]$

↳ $\mathcal{L}[1] = \int_0^\infty e^{-st}\, 1\, dt \to \lim_{b\to\infty} \int_0^b e^{-st} dt \to \lim_{b\to\infty} -\frac{1}{s} e^{-st}\Big|_0^b \to \lim_{b\to\infty} -\frac{1}{s}(e^{-sb} - 1) \to -\frac{1}{s}(0-1) = \frac{1}{s}$

↳ $\mathcal{L}[e^{at}] = \int_0^\infty e^{-st}(e^{at})\, dt = \int_0^\infty e^{(a-s)t} dt = \lim_{b\to\infty} \frac{1}{a-s} e^{(a-s)t}\Big|_0^b \to \frac{1}{a-s}(0-1)$, but only for $a < s$

↳ $= \frac{1}{s-a}$, $s > a$



The Laplace Transform is a linear operator because:

↳ $\mathcal{L}[f + g] = \mathcal{L}[f] + \mathcal{L}[g]$

↳ $\mathcal{L}[kf] = k\mathcal{L}[f]$

ex. Use the linearity of $\mathcal{L}$ to find $\mathcal{L}[11 + 5e^{4t} + \sin at]$

↳ $\mathcal{L}[11] = 11\mathcal{L}[1] = \frac{11}{s}$

↳ $\mathcal{L}[5e^{4t}] = 5\mathcal{L}[e^{4t}] = \frac{5}{s-4}$

↳ $\mathcal{L}[\sin at] = ?$

↳ $\sin at = \frac{1}{2i}(e^{iat} - e^{-iat})$

↳ $\mathcal{L}[\sin at] = \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) \to \frac{1}{2i}\left(\frac{s+ia}{(s-ia)(s+ia)} - \frac{s-ia}{(s+ia)(s-ia)}\right) = \frac{a}{s^2+a^2}$

↳ $\mathcal{L}[11+5e^{4t}+\sin at] = \frac{11}{s} + \frac{5}{s-4} + \frac{a}{s^2+a^2}$

Laplace Transforms can also be used for piecewise functions

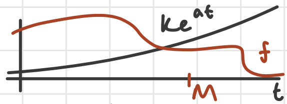
ex. Let $f(t) \begin{cases} e^{2t} & 0\le t<1 \\ 4 & 1\le t \end{cases}$, compute $\mathcal{L}[f]$

↳ $\mathcal{L}[f] = \int_0^1 e^{2t}e^{-st}\,dt + \int_1^\infty e^{-st}\cdot 4\,dt = \frac{1}{2-s}e^{(2-s)t}\Big|_0^1 + 4\left(\frac{-1}{s}\right)e^{-st}\Big|_1^\infty$

↳ $\frac{1}{2-s}(e^{2-s}-1) - \frac{4}{s}(0-e^{-s})$

5.1.7-8 Exponential Order

Conditions needed for the Laplace Transform to exist:



↳ f is of exponential order:

↳ there exists constants k, a, and M such that $|f(t)| \le ke^{at}$ for all $t > M$

↳ to check, show that $\frac{f(t)}{e^{at}}$ is bounded for sufficiently large t

ex. determine whether $\cos at, t^2, e^{t^2}$ are of exponential order

↳ $\cos at$, $|\cos at| \le 1$, so $|\cos at| < 1e^{at}$, $t > 1$ ← any positive number ✓

↳ t^2, note $e^{2t} = 1 + 2t + \frac{(2t)^2}{2!} + \dots$, $t^2 < 1 + 2t + \frac{(2t)^2}{2!} + \dots$, so $t^2 < e^{2t}$ ✓

↳ e^{t^2}, for any a, the ratio $\frac{e^{t^2}}{e^{at}} = e^{t^2-at}$, which always $\to \infty$ ✗

5.2.1-6 Properties of the Laplace Transform

Suppose $F(s) = \mathcal{L}[f(t)]$

↳ Theorem: Translation in s-domain: $\mathcal{L}[e^{at}f(t)] = F(s-c)$ } proofs in the textbook

↳ Theorem: First Derivative: $\mathcal{L}[f'(t)] = sF(s) - f(0)$

↳ For higher order derivatives, repeat application

ex. Compute the Laplace Transform of $y' = e^{-6t}\sin(t)$, $y(0) = 2$

↳ $\mathcal{L}[y'] = sY - y(0) = sY - 2$

↳ $\mathcal{L}[e^{-6t}\sin(t)] = \frac{1}{(s+6)+1}$ } $Y = \frac{2}{s} + \frac{1/s}{(s+6)+1}$

Theorem: Second Order Derivatives: $\mathcal{L}[f''(t)] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0)$

Theorem: Higher Order Derivatives: $\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) \dots - f^{(n-1)}(0)$

ex. Laplace Transform for $y'' + 9y = e^{-t}\sin(4t)$, $y(0) = 0$, $y'(0) = 1$

↳ $s^2Y - s\underset{\uparrow \atop s(0)=0}{y(0)} - \underset{\uparrow \atop y'(0)=1}{y'(0)} + 9Y = \frac{4}{(s+1)^2+4^2}$

↳ $(s^2+9)Y = 1 + \frac{4}{(s+1)^2+4^2}$

Derivatives in the s-Domain: Suppose $F(s) = \mathcal{L}[f(t)]$

↳ $\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s) = (-1)^n \frac{d^nF}{ds^n}$

↳ It follows from this theorem: $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

ex. Compute Laplace Transform: $y^{(4)} - y = 0$, $y(0) = y''(0) = y'''(0) = 0$, $y'(0) = 4$

↳ $s^4Y - s^3\underset{\uparrow \atop 0}{y(0)} - s^2\underset{\uparrow \atop 4}{y'(0)} - s\underset{\uparrow \atop 0}{y''(0)} - \underset{\uparrow \atop 0}{y'''(0)} - Y = 0$

↳ $s^4Y - s^2 - Y = 0 \rightarrow (s^4-1)Y = 4s^2$

ex. Compute Laplace Transform: $y' + ty = t^2$, $y(0) = 1$

↳ $(sY - y(0)) + \left(-1 \frac{dY}{ds}\right) = \frac{1}{s^2}$

↳ $Y' + sY = \frac{1}{s^2} + 1$

5.3.1 – 4 Inverse Laplace Transform Examples

ex. Determine $\mathcal{L}^{-1}\{Y\}$, where:

1) $Y_1(s) = \frac{4}{s^3} = 2 \frac{2!}{s^3} = 2\mathcal{L}\{t^2\}$, $\mathcal{L}^{-1}\{Y_1\} = 2t^2$

2) $Y_2(s) = \frac{3}{s^2+9} = 3\frac{1}{(s+3)^2} = \mathcal{L}\{\sin(3t)\}$, $\mathcal{L}^{-1}\{Y_2\} = \sin 3t$

3) $Y_3(s) = \frac{s-1}{s^2-2s+5} = \frac{s-1}{(s-1)^2+2^2} = \mathcal{L}\{e^t \cos 2t\}$

complete the square

ex. Calculate $\mathcal{L}^{-1}[Y]$ for the following:

1) $Y_1(s) = \frac{5}{(s+2)^4}$

↳ $Y_1(s) = \frac{5}{6} \cdot \frac{3!}{(s+2)^4} = \mathcal{L}^{-1}\{\frac{5}{6} t^3 e^{-2t}\}$, $y_1(t) = \frac{5}{6} t^3 e^{-2t}$

2) $Y_2(s) = \frac{7s-1}{(s+1)(s+2)(s-3)}$

↳ $Y_2(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}$ } so $y(t) = Ae^{-t} + Be^{-2t} + Ce^{3t}$
but, what are A, B, C?

↳ multiply everything by $(s+1)(s+2)(s-3)$:

↳ $7s-1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$

↳ To obtain A, B, C $\xrightarrow{\text{Method 1}}$ expand, collect, compare, make linear system, solve

↳ Method 2: guess and check (choose numbers for s)

↳ set $s=-2$: $7(-2)-1 = 0 + B(-1)(-5) + 0 \longrightarrow B = -3$

↳ set $s=3$: $7(3)-1 = 20 = 0 + 0 + C(3+1)(3+2) \longrightarrow C = 1$

↳ set $s=-1$: $7(-1)-1 = -8 = A(-1+2)(-1-3) + 0 + 0 \longrightarrow A = 2$

↳ $Y_2(s) = \frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}$

↳ $\mathcal{L}^{-1}\{Y_2\} = 2e^{-t} - 3e^{-2t} + e^{3t}$

3) $Y_3 = \frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$

↳ multiply by $(s-1)^2(s+3)$: $s^2 + 9s + 2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2$

↳ $s=1$: $12 = 0 + 4B + 0$, $B = 3$

↳ $s=-3$: $-16 = 0 + 0 + 16C$, $C = -1$

↳ compare s^2 terms:

↳ lefthand side: s^2

↳ righthand side: $As^2 + Cs^2 = As^2 - s^2 = s^2$, so $A = 2$ } $y(t) = 2e^t + 3te^t - e^{3t}$

5.4.1 Solving DEs with the Laplace Transform

Process:

1) Compute the LP of our IVP
2) solve for $Y(s) = \mathcal{L}[y(t)]$
3) Compute the inverse LP of Y to determine $y(t)$

Helpful formulas: $\mathcal{L}[e^{at}] = \frac{1}{s-a}$, $\mathcal{L}[\sin kt] = \frac{k}{s^2+k^2}$, $\mathcal{L}[\cos kt] = \frac{s}{s^2+k^2}$

ex. 1) $y' + 3y = 13\sin 2t$, $y(0) = 6$

↳ $\mathcal{L}\{y' + 3y\} = \mathcal{L}\{13\sin 2t\} \rightarrow sY - y(0) + 3Y = \frac{26}{s^2+4} \rightarrow sY + 3Y = 6 + \frac{26}{s^2+4}$

↳ $Y(s+3) = 6 + \frac{26}{(s^2+4)(s+3)} \rightarrow Y = \frac{6}{s+3} + \frac{26}{(s^2+4)(s+3)}$

↳ $Y = \frac{6(s^2+4)}{(s+3)(s^2+4)} + \frac{26}{(s^2+4)(s+3)} = \frac{6s^2+50}{(s^2+4)(s+3)} \rightarrow \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$

↳ multiply by $(s+3)(s^2+4) \longrightarrow 6s^2 + 50 = A(s^2+4) + (Bs+C)(s+3)$

↳ $s=-3$: $6(-3)^2 + 50 = A(9+4) + 0$, $A = 8$

↳ $s=0$: $0 + 50 = 8(0+4) + (0+C)(0+3)$, $C = 6$

↳ group s^2 terms: $6s^2 = As^2 + Bs^2 \rightarrow 6s^2 = 8s^2 + Bs^2$, $B = -2$

↳ $Y = \frac{A}{s+3} + \frac{Bs+C}{s^2+4} \rightarrow Y = \frac{8}{s+3} + \frac{-2s+6}{s^2+4} \rightarrow Y = 8\left(\frac{1}{s+3}\right) - 2\left(\frac{s}{s^2+2^2}\right) + 3\left(\frac{2}{s^2+2^2}\right)$ $\frac{k}{s^2+k^2}$

↳ $y = 8e^{-3t} - 2\cos(2t) + 3\sin(2t)$

ex. 2) $y'' - 3y' + 2y = e^{-4t}$, $y(0) = 1$, $y'(0) = 5$

↳ $\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{e^{-4t}\} \rightarrow (s^2Y - sy(0) - y'(0)) - 3(sY - y(0)) = \frac{1}{s+4}$

↳ $(s^2 - 3s + 2)Y - s - 5 + 3 = \frac{1}{s+4} \rightarrow (s^2 - 3s + 2)Y = s + 2 + \frac{1}{s+4}$

↳ $(s-1)(s-2)Y = \frac{(s+2)(s+4)}{(s+4)} + \frac{1}{s+4} \longrightarrow Y = \frac{(s+2)(s+4)+1}{(s-1)(s-2)(s+4)}$

↳ $Y = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4} \leftarrow$ soln eventually becomes $Ae^{t} + Be^{2t} + Ce^{-4t} = y(t)$

↳ $(s+2)(s+4) + 1 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$

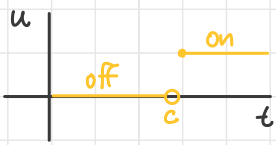
↳ $B = 25/6$, $C = 1/30$, $A = -16/5$

↳ $y(t) = Ae^{t} + Be^{2t} + Ce^{-4t} \rightarrow y(t) = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$

5.5.1 - 7 Discontinuous Functions

unit step function

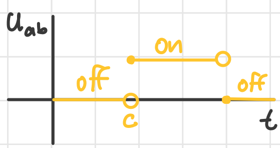
↳ $u_c(t) = \begin{cases} 0 & \leq t < c \\ 1 & c \leq c \end{cases}$

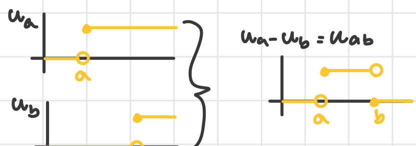


Question: How can we express the indicator function in terms of step functions?

indicator function

↳ $u_{bc}(t) = \begin{cases} 0 & 0 \leq t < b \\ 1 & b \leq t < c \\ 0 & c \leq t \end{cases}$





ex. Express the following functions in terms of step functions

1) $f(t) = \begin{cases} 2 & 0 \leq t < 3 \\ -2 & t \geq 3 \end{cases}$

↳ $f = 2\underbrace{u_{03}}_{\text{only on for } t \in (0,3)} - 2\underbrace{u_3}_{\text{only on for } t \in (3,\infty)} = 2(u_0 - u_3) - 2u_3$

$= 2u_0 - 4u_3$

2) $g(t) = \begin{cases} t & 0 \leq t < 2 \\ t^2 & 2 \leq t < 4 \\ t^3 & 4 \leq t \end{cases}$

↳ $g(t) = tu_{02} + t^2u_{24} + t^3u_4$

↳ $g(t) = t(u_0 - u_2) + t^2(u_2 - u_4) + t^3u_4$

↳ $g(t) = tu_0 - tu_2 + t^2u_2 - t^2u_4 + t^3u_4$

↳ $g(t) = tu_0 + (t^2 - t)u_2 + (t^3 - t^2)u_4$

Theorem of a step function : $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$

ex. $y' + y = f$, $y(0) = 0$, $f = \begin{cases} 1 & 0 \le t < 1 \\ -1 & t \ge 1 \end{cases}$

↳ $f = 1 \cdot u_{01} + (-1)u_1 = (u_0 - u_1) - u_1 = u_0 - 2u_1$

↳ $y' + y = f \to \mathcal{L}\{y' + y\} = \mathcal{L}\{f\} \to sY - y(0) + Y = \frac{1}{s} - \frac{2e^{-s}}{s}$

↳ $(1+s)Y = \frac{1 - 2e^{-s}}{s} \to Y = \frac{1}{s(s+1)} - \frac{2}{s(s+1)}e^{-s}$

↳ $\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \to 1 = A(s+1) + B(s) \to A = 1,\ B = -1$

↳ $Y = \left(\frac{1}{s} - \frac{1}{s+1}\right) - 2\left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-s} = \frac{1}{s} - \frac{1}{s+1} - \frac{2}{s}e^{-s} + \frac{2}{s+1}e^{-s}$

↳ for the last term, we want to use $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$ with $c = -1$

Note, $e^{-s+1-1} = e\,e^{-(s+1)}$

↳ $y = 1 - e^{-t} - 2u(t-1) + \mathcal{L}^{-1}\left\{2e\,\frac{e^{-(s+1)}}{s+1}\right\}$

↳ $y = 1 - e^{-t} - 2u(t-1) + 2e\,e^{-t}u_1$

Shift in the t-domain: $\mathcal{L}\{u_c f(t-c)\} = e^{-cs}F(s)$

ex. Compute the following: 1) $\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = u_2 f(t-2) = u_2\, e^{4(t-2)}$

↳ $c = 2$ in theorem

2) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\}$

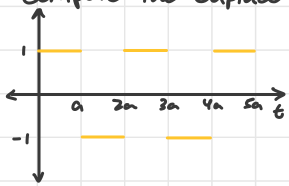
↳ $c = \frac{\pi}{2}$, $F(t) = \frac{s}{s^2+9}$, so $f(t) = \cos(3t)$, $\mathcal{L}^{-1} = u_{\frac{\pi}{2}} \cos[3(t - \frac{\pi}{2})]$

5.5.8-11 Periodic Functions

Suppose $f(t)$ is periodic with period T. Then $\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-Ts}}\int_0^T e^{-st}f(t)\,dt$

ex. Compute the Laplace Transform of the following functions:

1)



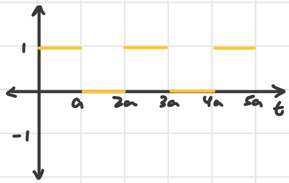
→ period: 2a

↳ $\mathcal{L}\{y_1\} = \frac{1}{1 - e^{-2as}}\int_0^{2a} e^{-st}y_1(t)\,dt$

↳ $= \frac{1}{1 - e^{-2as}}\int_0^a e^{-st}dt + \int_a^{2a} e^{-st}dt$

↳ $= \frac{1}{1 - e^{-2as}}\left(-\frac{1}{s}(e^{-as} - 1) + \frac{1}{s}(e^{-2as} - e^{-as})\right)$

↳ $\mathcal{L}\{y_1\} = \frac{1}{1 - e^{-2as}}\ \frac{1}{s}\left(1 - 2e^{-sa} + e^{-2as}\right)$



→ $\mathcal{L}\{y_2\} = \frac{1}{1 - e^{-2as}}\int_0^{2a} y_2 e^{-st}dt$

↳ $= \frac{1}{1 - e^{-2as}}\left(\int_0^a y_2 e^{-st} + \int_a^{2a} 0\,dt\right)$

↳ $= \frac{1}{1 - e^{-2as}}\left(-\frac{1}{s}e^{-st}\Big|_0^a\right)$

↳ $\mathcal{L}\{y_2\} = \frac{1}{s(1 - e^{-2as})}\left(e^{-as} - 1\right)$

ex. evaluate $\mathcal{L}\{\cos(t)u_\pi\}$

↳ we can use $\mathcal{L}\{u_c f(t-c)\} = e^{-cs}F(s)$

↳ $\mathcal{L}\{\cos t\ u_\pi\} = \int_0^\infty e^{-st}(\cos t)u_\pi\,dt$, $\cos t = -\cos(t - \pi)$

$= -\int_0^\infty e^{-st}\cos(t-\pi)u(t-\pi)\,dt \longrightarrow\ = -e^{-\pi s}\frac{s}{s^2+1}$

5.6.1-4 DEs with Discontinuous Forcing Functions

ex. Solve using Laplace Transform

1) $y'' + 4y = g,\ y(0) = y'(0) = 0,\ g(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & 1 \le t < 2 \\ 0 & \text{else} \end{cases}$

$\hookrightarrow s^2Y - sy(0) - y'(0) + 4Y = \mathcal{L}\{1u_{01} - 1 \cdot u_{12} + 0u_2\}$

$\hookrightarrow Y(s^2+4) = \mathcal{L}\{u_0 - u_1 - (u_1 - u_2) + 0u_2\}$

$\hookrightarrow Y(s^2+4) = \mathcal{L}\{u_0 - 2u_1 + u_2\}$

$\hookrightarrow Y = \frac{1}{s^2+4}\mathcal{L}\{u_0 - 2u_1 + u_2\}$

$\hookrightarrow Y = \frac{1}{s^2+4}\left(\frac{e^{-0s}}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}\right)$

$\rightarrow Y = \frac{1}{s(s^2+4)}\left(1 - 2e^{-s} + e^{-2s}\right)$, let $F(s) = \frac{1}{s(s^2+4)}$

$\hookrightarrow Y = F - 2Fe^{-s} + Fe^{-2s}$

$\hookrightarrow y(t) = f(t) - 2f(t-1)u_1 + f(t-2)u_2$

$\hookrightarrow F = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

$\hookrightarrow s(s^2+4) = A(s^2+4) + Bs^2 + Cs$

$\hookrightarrow$ constant terms: $1 = 4A \rightarrow A = 1/4$

$\hookrightarrow$ s terms: $Cs = 0,\ C = 0$

$\hookrightarrow$ s^2 terms: $0 = As^2 + Bs^2 \rightarrow B = -1/4$

$F = \frac{1}{4s} - \frac{1}{4}\left(\frac{s}{s^2+4}\right)$

$f(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$

ex. $y'' + 4y' + 4y = g,\ y(0) = y'(0) = 0,\ g = u_\pi - u_{2\pi}$

$\hookrightarrow (s^2Y - sy(0) - y'(0)) + 4(sY - y(0)) + 4Y = \frac{e^{-\pi s}}{s} - \frac{e^{-2s}}{s}$

$\hookrightarrow Y(s^2 + 4s + 4) = \frac{1}{s}(e^{-\pi s} - e^{-2s})$

$\hookrightarrow (s+2)^2 Y = \frac{1}{s}(e^{-\pi s} - e^{-2s})$

$\hookrightarrow Y = \frac{1}{s(s+2)^2}(e^{-\pi s} - e^{-2\pi s}) = F(s)e^{-\pi s} - F(s)e^{-2\pi s}$

$\hookrightarrow y = f(t-\pi)u_\pi - f(t-2\pi)u_{2\pi}$

$F = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$

$1 = A(s+2)^2 + Bs(s+2) + Cs$

$\hookrightarrow$ if $s = 0,\ A = 1/4$

$\hookrightarrow$ if $s = -2,\ C = -1/2$

$\hookrightarrow$ if $s = 1,\ 1 = 1(1+2)^2 + B \cdot 1(1+2) + C \cdot 1$

$\hookrightarrow B = -1/4$

$\hookrightarrow F = \frac{1/4}{s} + \frac{-1/4}{s+2} + \frac{-1/2}{(s+2)^2} \rightarrow f = \frac{1}{4} - \frac{1}{2}e^{-2t} - \frac{t}{2}e^{-2t}$

5.7.1-4 Impulse Functions

Dirac Delta Function — used to model functions with a large value over a short interval

$\hookrightarrow$ (is not a function)

$\hookrightarrow$ For any function $f(t)$ that is continuous over an interval containing t_0

$\hookrightarrow \delta(t-t_0)\begin{cases} 0, & t \ne t_0 \\ \text{undefined}, & t = t_0 \end{cases}$ $\hookrightarrow \int_a^b f(t)\,\delta(t-t_0)\,dt = f(t_0)$

$\hookrightarrow$ Laplace Transform: $e^{-st_0} = \mathcal{L}\{\delta(t-t_0)\}$

ex. A mass attached to a string is released from rest 1m below equilibrium. After π seconds the mass is struck by a hammer exerting an impulse. The system is governed by: $y'' + 9y = 3\delta(t-\pi),\ y(0) = 1,\ y'(0) = 0$

$\hookrightarrow \mathcal{L}\{y'' + 9y\} = \mathcal{L}\{3\delta(t-\pi)\}$

$\hookrightarrow s^2Y - sy(0) - y'(0) + 9Y = 3e^{-s\pi}$

$\hookrightarrow (s^2+9)Y = 3e^{-\pi s} + s$

$\hookrightarrow Y = \frac{s}{s^2+9} + 3\frac{e^{-\pi s}}{s^2+3^2}$

$\hookrightarrow y = \cos(3t) + \sin(3(t-\pi))u_\pi$

ex. $y'' + y = -\delta(t-\pi) + \delta(t-2\pi)$, $y(0)=0$, $y'(0)=1$

$\hookrightarrow \mathcal{L}\{y''+y\} = \mathcal{L}\{-\delta(t-\pi) + \delta(t-2\pi)\}$

$\hookrightarrow s^2Y - sy(0) - y'(0) + Y = -e^{-s\pi} + e^{-2\pi s}$

$\hookrightarrow (s^2+1)Y = -e^{-\pi s} + e^{-2\pi s} + 1$

$\hookrightarrow Y = F(s)(1 - e^{-\pi s} + e^{-2\pi s})$

$\rightarrow F(s) = \frac{1}{1+s^2} = \mathcal{L}\{\sin(t)\} \rightarrow f(t) = \sin(t)$

$\hookrightarrow y(t) = \sin(t) - \sin(t-\pi)u_\pi + \sin(t-2\pi)u_{2\pi}$

$\hookrightarrow y(t) = \sin t \cdot (1 + u_\pi + u_{2\pi})$

$\hookrightarrow 1 + u_\pi + u_{2\pi} = \begin{cases} 1 & 0 \le t < \pi \\ 2 & \pi \le t < 2\pi \\ 3 & 2\pi < t \end{cases}$

ex. $y'' + 2y' + 2y = \delta(t-\pi)$, $y(0)=1$, $y'(0)=1$

$\hookrightarrow s^2Y - sy(0) - y'(0) + 2sY - 2y(0) + 2Y = e^{-\pi s}$

$\hookrightarrow (s^2+2s+2)Y = e^{-\pi s} + 3 + s$

$\hookrightarrow Y = \frac{1}{s^2+2s+1-1+2}(e^{-\pi s} + 3 + s)$

$\hookrightarrow Y = \frac{e^{-\pi s}}{(s+1)^2+1} + \frac{3}{(s+1)^2+1} + \frac{s}{(s+1)^2+1}$

$\hookrightarrow \quad = e^{-(t-\pi)}\sin(t-\pi)u_\pi + 3e^{-t}\sin t + e^{-t}\cos t$

5.8.1-4 Convolution Integrals

Alternative to Variation of Parameters (when $g(t)$ is unknown):

The convolution between piecewise continuous functions f and g is denoted $f * g$ and is:

$$f * g = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

convolution has several useful properties:

1) $f * g = g * f$
2) $f * (g_1 + g_2) = f * g_1 + f * g_2$

Suppose f and g have Laplace Transforms $F(s)$ and $G(s)$ respectively, then:

$\hookrightarrow \mathcal{L}\{f * g\} = F(s)G(s)$

ex. Find $\mathcal{L}\{F(s)\}^{-1}$

1) $\frac{1}{(s^2+1)^2} = \frac{1}{s^2+1}\frac{1}{s^2+1}$, $\mathcal{L}^{-1}\{\frac{1}{s^2+1}\frac{1}{s^2+1}\} = \int_0^t \sin(t-\tau)\sin\tau\,d\tau$
2) $\mathcal{L}^{-1}\{\frac{14}{(s+2)(s-6)}\} = 14\int_0^t e^{-2(t-\tau)}e^{6\tau}d\tau = 14e^{-2t}\int_0^t e^{8\tau}d\tau = 14e^{-2t}(e^{8t}-1)/8$

ex. Find $\mathcal{L}\{f(t)\}$

1) $\int_0^t (t-\tau)e^{3\tau}d\tau = \mathcal{L}\{t\}\mathcal{L}\{e^{3t}\} = \frac{1}{s^2}\frac{1}{s-3}$
2) $\int_0^t \sin(t-\tau)e^{\tau}d\tau = \mathcal{L}\{\sin t\}\mathcal{L}\{e^t\} = \frac{1}{s^2+1}\frac{1}{s-1}$