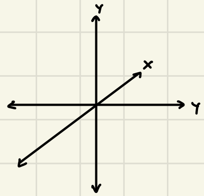
Three Dimensional Coordinate System

→ Denoted by $\mathbb{R}^3$

→ 3 axes: x-axis, y-axis, z-axis

→ Points are ordered triples: $P = (x, y, z)$

→ Origin located at intersection



Distance formula → $|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

Sphere – a 3D object where every point is the same distance from the center

↳ The center is at (x_0, y_0, z_0)

↳ Distance from center is the radius

$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = a^2$

Vectors

Vector – an object with a direction and length

↳ represented with a directed line segment ↗

↳ Ex. $\overrightarrow{PQ}$ starts at P, ends at Q

↳ If P is at (x_1, y_1, z_1) and Q is at (x_2, y_2, z_2) then $\overrightarrow{PQ} = \langle x_2-x_1, y_2-y_1, z_2-z_1 \rangle$

↳ Length is $|\overrightarrow{PQ}|$, the magnitude of the vector — the norm, denoted as $\|\overrightarrow{PQ}\|$

↳ Can also be written $\langle v_1, v_2, v_3 \rangle = \mathbf{v}$

Vector Applications

Vector Addition – if $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$, $u + v = \langle u_1+v_1, u_2+v_2, u_3+v_3 \rangle$

Vector Multiplication – $kv = \langle kv_1, kv_2, kv_3 \rangle$

Unit Vector – vector with length 1: if $v \neq 0$, $\frac{v}{\|v\|}$ is a unit vector in the v direction

↳ Any vector can be written as a linear combination of the 3 unit vectors: $i = \langle 1,0,0 \rangle$, $j = \langle 0,1,0 \rangle$, and $k = \langle 0,0,1 \rangle$

Dot Product

Dot Product of two vectors u and v, $u \cdot v$, is the scalar value $u_1v_1 + u_2v_2 + u_3v_3$

↳ Also equal to $\|u\|\|v\| \cos\theta$

Angle between u and v is $\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\|\|v\|}\right)$

Orthogonal – u and v are orthogonal if $u \cdot v = 0$

Vector Projection – a onto b is $\text{proj}_b a = \left(\frac{a \cdot b}{\|b\|^2}\right) b = \underbrace{\left(\frac{a \cdot b}{\|b\|}\right)} \frac{b}{\|b\|}$

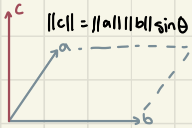
scalar component of a in direction of b

↳ $\text{comp}_b a = \frac{a \cdot b}{\|b\|} = a \cdot \frac{b}{\|b\|} = \|a\| \cos\theta$

Work – done by a constant force F over displacement $D = \overrightarrow{PQ}$ is $W = F \cdot D$

Cross Product

Cross Product – vector perpendicular to both vector a and b whose length = area of the parallelogram



$\|c\| = \|a\|\|b\|\sin\theta$

↳ $a \times b = \|a\|\|b\|\sin\theta\, n$, n being the unit vector perpendicular to the plane containing a and b

↳ Two vectors are parallel $\longleftrightarrow a \times b = 0$

↳ Cross Products are anti-commutative $\longrightarrow u \times v = -v \times u$

Cross Product as a determinant – if $u = u_1 i + u_2 j + u_3 k$ and $v = v_1 i + v_2 j + v_3 k$

↳ $u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$

Find $a \times b$ given – $a = 4i + 3j$, $b = i - 3j + 2k$

↳ $a \times b = \begin{vmatrix} i & j & k \\ 4 & 3 & 0 \\ 1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ -3 & 2 \end{vmatrix} i - \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} j + \begin{vmatrix} 4 & 3 \\ 1 & -3 \end{vmatrix} k = 6i - 8j - 15k$

Applications of the Cross Product

Triple Scalar Product – $(u \times v) \cdot w$ → The absolute value of this quantity is the volume of the parallelepiped

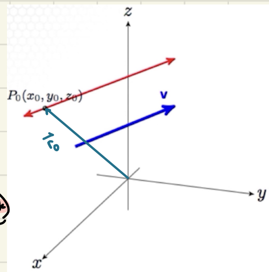
↳ $(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

Lines in Space

vector equation for the line L through point (x_0, y_0, z_0) parallel to vector v is $r(t) = r_0 + tv$, $-\infty < t < \infty$

↳ r is position vector of point $P(x, y, z)$ on L and r_0 is the position vector of $P_0(x_0, y_0, z_0)$

← Parameter



Standard parametrization of the line L through the point $P_0(x_0, y_0, z_0)$ parallel to vector $v = v_1 i + v_2 j + v_3 k$

↳ $x(t) = x_0 + tv_1$, $y(t) = y_0 + tv_2$, $z(t) = z_0 + tv_3$, $-\infty < t < \infty$

Find a vector equation for the line passing through $P(3, 2, 3)$ and parallel to $r(t) = (i + j + k) + t\underbrace{(2i + 3j - 3k)}_{\vec{v}}$

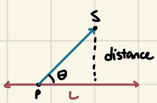
↳ $\vec{r}(t) = (3\vec{i} + 2\vec{j} + 3\vec{k}) + t(2\vec{i} + 3\vec{j} - 3\vec{k})$

Find a set of scalar parametric equations for the line through $P(2, 2, 1)$ and $Q(3, -2, -2)$

↳ $\vec{PQ} = \langle 1, -4, -3 \rangle = \vec{r}$

↳ $x(t) = 2 + t$, $y(t) = 2 - 4t$, $z(t) = 1 - 3t$

Distance from point to a line: point S to line L passing through P and parallel to v



↳ $d = \dfrac{\|\vec{PS} \times v\|}{\|v\|}$

Find distance from $S(2, 0, 2)$ to line through $P(3, -1, 1)$ parallel to vector $v = i - 2j - 2k$

↳ $d = \dfrac{\|\vec{PS} \times v\|}{\|v\|}$ $\quad \vec{PS} = \langle -1, 1, 1 \rangle$ $\quad \vec{PS} \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -2 & -2 \end{vmatrix} = (-2+2)\hat{i} - (2-1)\hat{j} + (2-1)\hat{k} = -\hat{j} + \hat{k}$

↳ $\dfrac{\sqrt{0+1+1}}{\sqrt{1+4+4}} = \dfrac{\sqrt{2}}{\sqrt{9}} = \dfrac{\sqrt{2}}{3}$

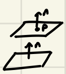
Planes in Space

Plane through the point $P_0(x_0, y_0, z_0)$ normal (perpendicular) to $n = Ai + Bj + Ck$ is given by

↳ vector equation $n \cdot \vec{P_0P} = 0$ and component equation $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Find the equation of a plane which passes through the point $P(3, -2, 3)$ and parallel to $4x + 3y + 4z + 6 = 0$

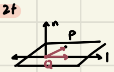
↳ $\vec{n} = \langle 4, 3, 4 \rangle \rightarrow 4(x-3) + 3(y+2) + 4(z-3) = 0 \rightarrow 4x + 3y + 4z - 18 = 0$



Parallel planes

Find the equation for the plane that passes through point $P(1, 3, 4)$ and contains $l: x(t) = 3t, y(t) = 4t, z(t) = 2 + 2t$

↳ $\underbrace{Q(0,0,2)}_{t=0}$, $\vec{QP} = \langle 1, 3, 2 \rangle$, $\underbrace{\vec{v} = \langle 3, 4, 2 \rangle}_{\text{coefficients of } l}$, $\vec{n} = \vec{QP} \times \vec{v} = -2i + 4j - 5k \rightarrow 2x - 4y + 5z - 10 = 0$



Angle between 2 intersecting planes is the acute angle between their normal vectors

↳ $\cos\theta = \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\|\vec{n}_2\|}$

Find the angle between 2 planes: $4x+4y-2z=3$, $2x+y+z=-1$

↳ $n_1 = \langle 4,4,-2\rangle$, $\|n_1\| = 6$ $\quad \vec{n}_2 = \langle 2,1,1\rangle$, $\|\vec{n}_2\| = \sqrt{6}$

↳ $\cos\theta = \dfrac{|n_1\cdot n_2|}{\|n_1\|\|n_2\|} = \dfrac{|8+4-2|}{6\cdot\sqrt{6}} = \dfrac{10}{6\sqrt{6}} = \dfrac{5}{3\sqrt{6}} \rightarrow \theta = \cos^{-1}\left(\dfrac{5}{3\sqrt{6}}\right)$

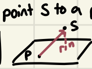
Find distance from point $S(1,-2,2)$ to plane $2x-y+2z=-3$

↳ Find a point on the plane $P(\overset{x=0}{0},3,\overset{z=0}{0})$

↳ $\vec{PS} = \langle 1,-4,2\rangle$, $\vec{n} = \langle 2,-1,2\rangle$, $\|\vec{n}\| = 3$ ↳ $d = \left|\dfrac{2+4+4}{3}\right| = \dfrac{10}{3}$

Distance between point S to a plane that contains point P and has a normal vector n is:

↳ $d = \left|\vec{PS}\cdot\dfrac{n}{\|n\|}\right|$



Intersecting Lines and Planes in Space

Determine whether the following vectors L_1 and L_2 are parallel, coincident, skew, or intersecting

d parallel $\left|\vec{PS}\times\dfrac{\vec{n}}{|\vec{n}|}\right|$ $\quad$ d skew $\left|\vec{PS}\cdot\dfrac{L_1\times L_2}{|L_1\times L_2|}\right|$

↳ $L_1 = r(t) = (-i+2j+k) + t(i-3j+2k)$

$L_2 = R(s) = (2i-j) + s(-2i+6j-4k)$

① Check for parallel ✓ $\rightarrow$ direction vectors $\vec{v}_1 = \langle 1,-3,2\rangle$ $\vec{v}_2 = \langle -2,6,-4\rangle \rightarrow -2\vec{v}_1 = v_2$, $v_1 \parallel v_2$

② Check for coincident $\rightarrow \vec{r}(0) = (-1,2,1) \rightarrow$ is this on ℓ_2? $\rightarrow x = 2-2s = -1 \rightarrow 3 = 2s \rightarrow s = 3$, but $\vec{R}(3/2) \neq r(0)$

Two planes are parallel if their normal vectors are parallel — if two vectors are not parallel, they intersect in a line

↳ Direction vector is found by the cross product of the normal vectors from the two planes

Find a set of scalar parametric equations for the line formed by the two intersecting planes

↳ $P_1: x+2y+3z=5$, $P_2: 3x-4y-z=1$

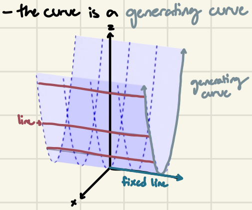
direction vector ↳ $\vec{n}_1 = \langle 1,2,3\rangle$, $\vec{n}_2 = \langle 3,-4,-1\rangle$ (not parallel) $\longrightarrow \vec{n}_1\times\vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & -4 & -1\end{vmatrix} = (-2+12)i - (-1-9)j + (-4-6)k \rightarrow 10i+10j-10k$

direction vector of $\vec{v}_1 = \langle 1,1,-1\rangle$

Point ↳ let $x=0$: $\left.\begin{matrix}2y+3z=5 \\ -4y-z=1\end{matrix}\right\}$ $y=-4/5$, $z=11/5 \rightarrow P(0,-4/5,11/5)$, $\vec{v}_1 = \langle 1,1,-1\rangle$, $x(t) = 1$, $y(t) = -4/5+t$, $z(t) = 11/5 - t$

Cylinders and Quadric Surfaces

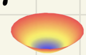
Cylinder is a surface generated by moving a straight line along a given planar curve while that line is parallel to a fixed line — the curve is a generating curve



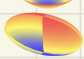
Quadric Surface is the graph in space of a 2nd-degree equation in x, y, and z

↳ $Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Jz+K=0$

↳ Elliptical Paraboloid $\rightarrow \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = \dfrac{z}{c}$

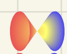


↳ Ellipsoid $\rightarrow \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} = 1 \longrightarrow$



↳ Hyperbolic Paraboloid $\rightarrow \dfrac{y^2}{b^2}-\dfrac{x^2}{a^2} = \dfrac{z}{c}$, $c>0$

↳ Elliptical Cone $\rightarrow \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = \dfrac{z^2}{c^2} \longrightarrow$



↳ Hyperboloid (1 Sheet) $\rightarrow \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2} = 1 \longrightarrow$

↳ Hyperboloid (2 Sheets) $\rightarrow \dfrac{z^2}{c^2}-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2} = 1 \longrightarrow$

Vector Functions

If f_1, f_2, and f_3 are real-valued functions on some interval I, then each t in the interval I, we can form the vector function

↳ $f(t) = f_1(t)i + f_2(t)j + f_3(t)k \rightarrow f_1, f_2,$ and f_3 are components of $f \rightarrow$ scalar form: $x(t) = f_1(t)$, $y(t) = f_2(t)$, $z(t) = f_3(t)$

↳ a point t is in the domain of $f \longleftrightarrow$ it is in each component of f

Find the domain of $f(t) = \cos(t)i + \ln(4-t)j + \sqrt{t+1}k$

$\left.\begin{matrix}\cos(t)\text{: domain } (-\infty,\infty) \\ \ln(t)\text{: domain } (-\infty,4) \\ \sqrt{t+1}\text{: domain } [-1,\infty)\end{matrix}\right\}$ domain of $\vec{f}(t)$: $[-1,4)$

Limits and Derivatives of Vector Functions

$r(t) = f(t)i + g(t)j + h(t)k$ has domain D, and L is a vector – r has limit L as t approaches to

↳ $\lim_{t\to t_0} r(t) = L$ if for every $\epsilon > 0$, there is a $\delta > 0$ so that for all $t \in D \rightarrow \|r(t) - L\| < \epsilon$ whenever $0 < |t - t_0| < \delta$

Limit Rules

↳ Let f and g be vector functions and u a real-value function – if $t \to t_0$ and $f(t) = L$, $g(t) = M$, and $u(t) \to U$

↳ ① $f(t) + g(t) = L + M$ ② $af(t) \to aL$ ③ $u(t)f(t) \to UL$ ④ $f(t) \cdot g(t) \to L \cdot M$ ⑤ $f(t) \times g(t) \to L \times M$

Find $\lim_{t\to 0} f(t)$ given that $f(t) = \frac{\sin t + t}{3t} i + e^{2t} j + \sin(t - \pi) k$

$$\lim_{t\to 0} \frac{\sin t + t}{3t} = \lim_{t\to 0} \frac{\sin t}{3t} + \frac{t}{3t} = \lim_{t\to 0} \frac{\cos t}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\lim_{t\to 0} e^{2t} = e^{2(0)} = 1$$

$$\lim_{t\to 0} \sin(t - \pi) = \sin(0 - \pi) = \sin(-\pi) = 0$$

$$\lim_{t\to 0} f(t) = 2/3 i + j$$

vector $f(t)$ is continuous at a point $t = t_0$ if $\lim_{t\to 0} f(t) = f(t_0)$

↳ $f(t)$ is continuous ⟷ each of its components is continuous at t_0

↳ A function is continuous if it is continuous at every point in its domain

Derivative → $r'(t) = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}$

↳ $(f \cdot g)'(t) = [f(t) \cdot g'(t)] + [f'(t) \cdot g(t)]$

↳ $(f \times g)'(t) = [f(t) \times g'(t)] + [f'(t) \times g(t)]$ order matters!

Tangent Lines, Velocity, and Acceleration

Curve traced by $r(t) = f(t)i + g(t)j + h(t)k$ is smooth if $\frac{dr}{dt}$ is continuous and never 0

Tangent line to a smooth curve at $t = t_0$ is the line parallel to $r'(t_0)$ and through $(f(t_0), g(t_0), h(t_0))$

↳ if $r(t)$ is the position of a particle in a smooth curve, $v(t) = \frac{dr}{dt}$ is the velocity vector tangent to the curve

↳ speed: $\|v\|$

↳ direction of motion: direction of v, usually as a unit vector

↳ acceleration vector: $a(t) = \frac{dv}{dt}$

Integrals of Vector Functions

Differentiable vector function $R(t)$ is the antiderivative of $r(t)$ if $\frac{dR}{dt} = r$ at each point of I

↳ indefinite integral of r with respect to t is the set of all antiderivatives of r, $\int r\,dt \to \int r\,dt = \underbrace{R(t) + C}_{}$ constant vector

Definite Integral: $\int_a^b r(t)\,dt = \int_a^b f(t)\,dt\, i + \int_a^b g(t)\,dt\, j + \int_a^b h(t)\,dt\, k$

↳ $\int_a^b [c \cdot f(t)\,dt] = c \cdot \int_a^b f(t)\,dt$ vector, dot product

↳ $\left\| \int_a^b f(t)\,dt \right\| \leq \int_a^b \|f(t)\|\,dt$

Projectile Motion

Ideal Projectile Motion – $r = (v_0 \cos \alpha)t\, i + [(v_0 \sin \alpha)t - \frac{1}{2} g t^2]\, j$ (α: launch angle, v_0: initial speed, g: gravity)

Maximum height – $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g}$

Flight Time – $t = \frac{2 v_0 \sin \alpha}{g}$

Range – $R = \frac{v_0^2}{g} \sin 2\alpha$

Arc Length of a Curve

length of a curve $r(t) = x(t)i + y(t)j + z(t)k$ for $a \leq t \leq b$ is $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

↳ $L = \int_a^b \|v\|\, dt$

arc length parameter – directed distance along a curve from $P(t_0)$ to any point $P(t)$ is given by function s

↳ $s(t) = \int_{t_0}^{t} \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2}\, d\tau = \int_{t_0}^{t} \|v(\tau)\|\, d\tau$

Speed and the Unit Tangent Vector

$s(t) = \int_{t_0}^{t} \|v(\tau)\| \, d\tau$, so $\frac{ds}{dt} = \|v(t)\|$

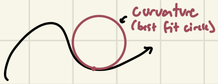
↑ arc length

↳ Speed is magnitude of velocity

Unit Tangent Vector to smooth curve $r(t) \longrightarrow T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{v(t)}{\|v(t)\|} = \frac{dr/dt}{ds/dt} = \frac{d\vec{r}}{ds}$

Curvature

If T is a unit vector of a smooth curve, the curvature function of the curve is $\kappa = \|\frac{dT}{s}\| \rightarrow \frac{\|T'(t)\|}{\|v(t)\|}$



↳ $T(t)$ is the unit tangent vector to r

Circle of Curvature , Osculating Circle at point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve

↳ is tangent to the curve at P

↳ has same curvature

↳ has a center that lies towards concave side of curve

Radius of curvature at point P is the radius of the circle of curvature $\longrightarrow r = \frac{1}{\kappa}$

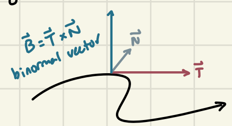
↳ along a straight line, curvature $= 0$ and along a circle of radius $= r$, curvature $= \frac{1}{r}$

Planes in Space – $\kappa = \|\frac{dT}{ds}\|$

Principal Normal Vector

If T is the unit tangent vector of a smooth curve r and if $T'(t) = 0$, then it does not change direction

↳ If $T'(t) \neq 0$, then the principal normal vector is $N(t) = \frac{T'(t)}{\|T'(t)\|}$



Osculating Plane – plane determined by unit tangent vector + principal normal vector

↳ Binormal Vector – normal vector to osculating plane , $B(t) = T(t) \times N(t)$

Tangential/Normal Components of Acceleration

$T(t) = \frac{v(t)}{\|v(t)\|} = \frac{v(t)}{ds/dt}$, so we can find velocity in terms of the unit tangent vector : $\vec{v}(t) = \vec{T}(t) \cdot \frac{ds}{dt}$

↳ $a(t) = v'(t)$, so $a(t) = T'(t)\frac{ds}{dt} + T(t)\frac{d}{dt}\left(\frac{ds}{dt}\right)$ (speed)

↳ $N = \frac{T'(t)}{\|T'(t)\|}$, so $\vec{N}(t)\|T'(t)\| = T'(t)$

$\rightarrow a(t) = \underbrace{\vec{N}(t) \cdot \|T'(t)\| \frac{ds}{dt}}_{\text{normal component}} + T(t) \underbrace{\frac{d}{dt}\left(\frac{ds}{dt}\right)}_{\text{tangential component}}$

Tangential component – $a_T = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}(\|\vec{r}'(t)\|)$

Normal component – $a_N = \|T'(t)\| \frac{ds}{dt} = \kappa \cdot \frac{ds}{dt} \cdot \frac{ds}{dt} = \kappa \left(\frac{ds}{dt}\right)^2 = \sqrt{|a|^2 - a_T^2}$

Curvature and Torsion

Torsion function of a smooth curve is $\tau = -\frac{dB}{ds} \cdot N$ (Positive, Negative, or 0)

↳ $B = T \times N$

↳ measures how the binormal vector changes with arclength (twist in curve)

↳ alternative formula: $\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\|v \times a\|^2}$ assuming $\neq 0$

If curvature $\neq 0$, a_N is a multiple of the square of the speed

↳ $T \cdot a = a_T(T \cdot T) + a_N(T \cdot N) = a_T$

↳ $a_T = \frac{v \cdot a}{ds/dt}$ and $a_N = \frac{\|v \times a\|}{ds/dt} = \kappa\left(\frac{ds}{dt}\right)^2$

↳ $\kappa = \frac{\|v \times a\|}{(ds/dt)^3}$

Find the curvature, k, and torsion, τ for helix: $r(t) = 3\cos(t)i + 3\sin(t)j + 4(t)k$

$v(t) = -3\sin(t)i + 3\cos(t)j + 4k$ $\quad$ $a(t) = -3\cos(t)i - 3\sin(t)j + 0k$

$|v| = \sqrt{9\sin^2(t) + 9\cos^2(t) + 16}$ $\quad$ $a(t) = -3\cos(t)i - 3\sin(t)j$

$|v| = \sqrt{9(1) + 16}$

$$v \times a = \begin{vmatrix} i & j & k \\ -3\sin t & 3\cos t & 4 \\ -3\cos t & -3\sin t & 0 \end{vmatrix} = (0 + 12\sin t)i - (12\cos t)j + (9\sin^2 t + 9\cos^2 t)k = 12\sin(t)i - 12\cos(t)j + 9k$$

$|v| = 5 = \frac{ds}{dt}$

$\|v \times a\| = \sqrt{144(1) + 81}$

$\|v \times a\| = 15$

$$k = \frac{\|v \times a\|}{(ds/dt)^3} = \frac{15}{125} = \frac{3}{25}$$

$$\tau = \frac{\begin{vmatrix} -3\sin t & 3\cos t & 4 \\ -3\cos t & -3\sin t & 0 \\ 3\sin t & -3\cos t & 0 \end{vmatrix}}{225} = \frac{0 - 0 + 36}{225} = \frac{4}{25}$$

Motion in Polar/Cylindrical Coordinates

Particle moving along polar coordinate plane

↳ coordinates $P(r, \theta)$ ← changes with respect to t

↳ position, velocity, acceleration in u_r, the unit vector in direction of $\overrightarrow{OP}$, and u_θ, unit vector pointing in $\theta\uparrow$

↳ $u_r = (\cos\theta)i + (\sin\theta)j$ and $u_\theta = -\sin\theta j + \cos\theta j$ (u_θ and u_r are orthogonal)

↳ u_r is in direction of $\underset{\text{origin} \to \text{point } P}{\overrightarrow{OP}}$, so $r = r u_r$

↳ $r(t) = r(\cos\theta)i + r(\sin\theta)j$

↳ $v(t) = (\dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta})i + (\dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta})j$

↳ $v(t) = \dot{r}u_r + r\dot{\theta}u_\theta$

↳ $a(t) = (\ddot{r} - r\dot{\theta}^2)u_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})u_\theta$

Formulas

general vectors:

Angle between u and v is $\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\|\|v\|}\right)$ ↳ Vector Projection — a onto b is $\text{proj}_b a = \left(\frac{a \cdot b}{\|b\|^2}\right) b = \left(\underbrace{\frac{a \cdot b}{\|b\|}}\right)\frac{b}{\|b\|}$

↳ scalar component of a in direction of b $\text{comp}_b a = \|a\| \cos\theta$

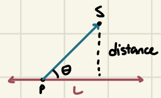
Cross Product — vector perpendicular to both vector a and b whose length = area of the parallelogram

↳ $a \times b = \|a\|\|b\| \sin\theta\, n$, n being the unit vector perpendicular to the plane containing a and b

vector equation for the line L through point (x_0, y_0, z_0) parallel to vector v is $v(t) = r_0 + \overset{\text{Parameter}}{t} v$, $-\infty < t < \infty$

↳ r is position vector of point $P(x, y, z)$ on L and v_0 is the position vector of $P_0(x_0, y_0, z_0)$

Distance from point to a line: point S to line L passing through P and parallel to v: $\vec{d} = \frac{\|\vec{PS} \times v\|}{\|v\|}$



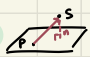
planes:

Plane normal vector is $n = \vec{QP} \times \vec{v}$, if Q is any point on the plane given by parametric form $(x = at, y = bt, z = ct, \text{etc})$, P is a given point,

and $\vec{v}$ is composed of the coefficients of t

Angle between 2 intersecting planes is the acute angle between their <u>normal</u> vectors: $\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\|\vec{n}_2\|}$

Distance between point S to a plane that contains point P and has a normal vector n is: $d = \left|\vec{PS} \cdot \frac{n}{\|n\|}\right|$

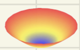


d parallel $\left|\vec{PS} \times \frac{\vec{n}}{|n|}\right|$

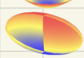
d skew $\left|\vec{PS} \cdot \frac{L_1 \times L_2}{|L_1 \times L_2|}\right|$

quadric surfaces:

↳ Elliptical Paraboloid → $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



↳ Ellipsoid → $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ⟶



↳ Hyperbolic Paraboloid → $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$, $c > 0$

↳ Elliptical Cone → $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ ⟶

↳ Hyperboloid (1 Sheet) → $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ ⟶

↳ Hyperboloid (2 Sheets) → $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ⟶

projectile motion:

Ideal Projectile Motion — $r = (v_0 \cos\overset{\text{launch angle}}{\alpha}) t\, i + \left[(\overset{\text{initial speed}}{v_0} \sin\alpha) t - \frac{1}{2}\underset{\text{gravity}}{g} t^2\right] j$

Maximum height — $y_{max} = \frac{(v_0 \sin\alpha)^2}{2g}$

Flight Time — $t = \frac{2 v_0 \sin\alpha}{g}$

Range — $R = \frac{v_0^2}{g} \sin 2\alpha$

arc length

length of a curve: $L = \int_a^b \|v\|\, dt$

arc length parameter — <u>directed distance</u> along a curve from $P(t_0)$ to any point $P(t)$: $s(t) = \int_{t_0}^{t} \|v(\tau)\|\, d\tau$

Speed is magnitude of velocity: $\frac{ds}{dt} = \|v(t)\|$ ↳ Unit Tangent Vector to smooth curve $r(t)$ ⟶ $T(t) = \frac{v(t)}{\|v(t)\|}$

curvature:

curvature function of the curve is $\kappa = \frac{\|T'(t)\|}{\|v(t)\|} = \frac{\|v \times a\|}{(ds/dt)^3}$ (useful for torsion)

Radius of curvature of the circle of curvature: $r = \frac{1}{\kappa}$

torsion:

Torsion: $\tau = -\frac{dB}{ds} \cdot N = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\|v \times a\|^2}$

principal normal vector:

principal normal vector is $N(t) = \frac{T'(t)}{\|T'(t)\|}$

Binormal Vector — normal vector to osculating plane, $B(t) = T(t) \times N(t)$

polar coordinates

$P(r, \theta)$ ← changes with respect to t

$v(t) = \dot{r} u_r + r\dot{\theta} u_\theta$

$a(t) = (\ddot{r} - r\dot{\theta}^2) u_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) u_\theta$

components of acceleration:

$$\left.\begin{array}{l}\text{Tangential component} - a_T = \frac{d}{dt}(\|\vec{r}'(t)\|) \\ \text{Normal component} - a_N = \sqrt{|a|^2 - a_T^2}\end{array}\right\} a(t) = N(t) a_N + T(t) a_T$$