Math 3012: Combinatorics

↳ Lectures 1-9

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Author's Note

↳ This is solely covering the topics of the lecture videos, so I recommend you read the textbook in regards to working out homework

↳ Pretty much all info regarding computer science has been omitted (i.e. functions)

↳ Lecture numbers are on the left of titles

1.1 Introduction to Sequences

Thought experiment: If you wanted to claim 2 sequences as identical or different

↳ identical: would have to check every position 1 by 1 (HARDER)

↳ different: would just have to point out 1 position as different (EASIER)

hash (MD5 sums, SHA 5-12 sums, etc.) — to hash a big file into 1 string

↳ not exact, works in probabalistic setting

↳ if 2 DVDs hash to the same string, they are ALMOST CERTAINLY identical

↳ nobody tests every bit of a DVD

Example: Given the numbers 12 17 22 31 48, $12+22+31 = 17+48$

↳ Can you find a fair division of [insert lots of numbers here]?

↳ EASIER: prove there is an answer — test 1

↳ HARDER: prove there isn't an answer — test all

How can we find if a number is a prime number or not?

↳ Systematically: you could go number by number to divide

↳ Used for public key cryptography (factoring integers for large numbers)

How can we add large-number fractions?

↳ Systematically: give an explicit rule: do $\frac{ad+bc}{bd}$

↳ known as an algorithm

↳ we want to carry out tasks explicity, via algorithms

1.2 Strings

String — let n be a positive integer and $[n] = \{1, 2 \ldots n\}$; a sequence of length n such as $(a_1, a_2 \ldots a_n)$ is a string

↳ also known as words, arrays, vectors

↳ Entries: characters, letters, coordinates, etc.

↳ Set of possible entries: alphabet

↳ Bit String: 10110110...

↳ Ternary String: 20100120...

↳ Word from 4-letter alphabet: abccdcdaa...

↳ AND a word from a 5-letter alphabet, and 6, and 7

The First Principle of Enumeration

Enumeration — counting

Multiplication Principle — if each possibility is independent, then the total number of possibilities (combinations of different states) is each number of states multiplied

↳ 5 shirts × 4 pants × 3 shoes = 60 total combinations

\# of bit strings with length $n = 2^n$

\# of words with length n from a m letter alphabet is m^n (for English, 26^n)

\# of Georgia License plates is $26^3 10^4$ (3 letters, 4 numbers 0-9)

Permutations and Combinations

Permutation – Strings where there is NO repetition

↳ # of permutations of length n from m letter alphabet: $P(m,n) = m(m-1)(m-2)\dots(m-n+1)$

↳ ex. 12 7 8 6 4 9 11 ✓

↳ non ex. 5 6 <u>7</u> 2 4 9 A <u>7</u> 6 ✗

↳ Example: How many permutations of 68 objects taken 23 at a time?

↳ $P(68, 23)$ ← leave it as is

Example: A group of 250 students holds elections to identify a class president, vice president, and treasurer.

↳ permutations: $250 \times 249 \times 248 = P(250, 3)$

Example: A group of 250 students hold elections for a leadership committee of 3 people.

↳ combination: $\frac{250 \times 249 \times 248}{1 \times 2 \times 3} = C(250, 3)$

L.3 Binomial Coefficients, Making Precise Definitions, Recursion

In Line Notation – $C(38,17) = P(38,17)/17! = 38!/(21!\,17!)$

↳ Graphic Notation – preferred notation ⟶ $\binom{38}{17} = \frac{38!}{17!\,21!}$

↳ "38 choose 17"

Rather than …, write explicitly → 1, 4, 9, 25… ✗

↳ $a_n = n^2$ ✓

Factorials: $0! = 1$, and when $n > 1$, $n! = n * (n-1)!$

Combinatorial Identities and Pascal's Triangle

Example: How many bit strings of length 38 have exactly 17 ones?

↳ How many subsets of size 17 are in a set of size 30?

↳ Answer to both: $C(38,17) = P(38,17)/17! = 38!/(21!\,17!)$

ground set: $X = \{a,b,c,d,e,f,g,h\}$ $\quad X = \{a,b,c,d,e,f,g,h\}$

↳ sub set: $S = \{b,c,f\}$ → bit string $= \{0\,1\,1\,0\,0\,1\,0\,0\}$

↳ associate a string whether an element of a ground set relate to subset

Complement: $\binom{n}{k} = \binom{n}{n-k}$ → why? (value of k = $0 \le k \le n$)

↳ eliminating multiplication $= \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$

↳ why? → bit string of length n: $\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\_}\ \dots\ \underset{n-1}{\_}\ \underset{n}{0}$ — there is k 1's in the set $n-1$

↳ $\binom{n}{k}$ → exactly k 1's in sequence — if position $n = 0$,

↳ vice versa, if $\_\ \_\ \_\ \_\ \dots\ \underset{n-1}{\_}\ \underset{n}{1}$ — there is $k-1$ 1's in the set $n-1$

Position $n = 1$,

Pascal's Triangle:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \vdots & & & & \end{array}$$

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

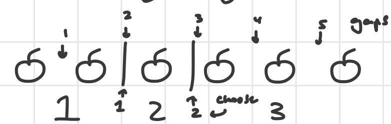
↳ Example: $\binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = \frac{56 \cdot \cancel{6}}{\cancel{6}} = 56$

↳ you don't have to do any division / multiplication!

Enumerating Distributions

Example: Given a set of m objects and n cells (boxes, bins, etc.) – how many ways can they be distributed?

↳ Side Constraints:
1) Distinct / Non-distinct objects
2) Distinct / Non-distinct cells (bins)
3) Empty cells allowed / Not allowed
4) Upper / Lower bounds on # of objects in a cell



3 bins, 6 objects, 5 gaps, choose 2

n bins, m objects, $m-1$ gaps, $n-1$ chosen

distinct bins, non-distinct obj., non-empty cells → $\binom{m-1}{n-1}$

Binomial Coefficients and Distributing Objects

foundational enumeration problem – given a set of m identical objects and n distinct cells, the # of ways they can be distributed (and no cells are empty) is: $\binom{m-1}{n-1}$

↳ Restatement: how many solutions in positive integers to equation: $x_1 + x_2 + x_3 + \dots x_n = m$ (cells)

↳ given m identical objects and n distinct cells, the # of ways they can be distributed (with 0 empty)

↳ $\binom{m-1}{n-1}$

Distributing Objects with Different Restrictions

How many solutions in (allowing 0, empty cells) non-negative integers to: $x_1 + x_2 + x_3 + \dots x_n = m$

↳ $\binom{m+n-1}{n-1}$ → Add n artificial elements, 1 for each variable

How many solutions in non-negative integers to: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 142$

↳ Subject to: $x_1, x_2, x_5, x_7 \geq 0$; $x_3 \geq 8$; $x_4 > 0$; $x_6 > 19$

- $x_1, x_2, x_5, x_7 \geq 0$: artificially add 4 n to m and subtract 1
- $x_3 \geq 8$: add a stack of 7 to x_3 (set aside, $142 - 7$)
- $x_4 > 0$: do nothing to distribution
- $x_6 > 19$: add a stack of 19 to x_6 (set aside, $142 - 19$)

$$\binom{142 + (\overset{n}{4} - 1) - 7 - 19}{7 - 1} \rightarrow \binom{119}{6}$$

Tip: Good = All – Bad (may be easier)

How many integer solutions in non-negative integers : $x_1 + x_2 + x_3 + x_4 = 63$

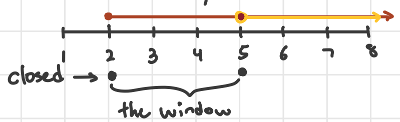
↳ Subject to $x_1, x_2 \geq 0$; $2 \leq x_3 \leq 5$; $x_4 > 0$

$x_1, x_2 \geq 0$: add 2 to m, subtract 1

$2 \leq x_3 \leq 5$: split

$x_4 > 0$: nothing ⍨

set aside 1 ← $x \geq 2$, $x > 5$ → set aside 5



$$\binom{63 + (2-1) - 1}{3} = \binom{63}{3}$$

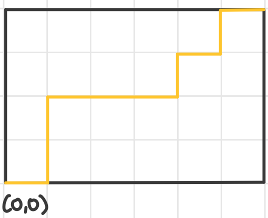
$$\binom{63 + (2-1) - 5}{3} = \binom{59}{3}$$

$$\binom{63}{3} - \binom{59}{3}$$

Lattice Paths

More applications: Lattice Paths

↳ Restriction: Walk on the edges of a grid: Only move R (right) and U (up)



↖ Choices

RUURRRURUR

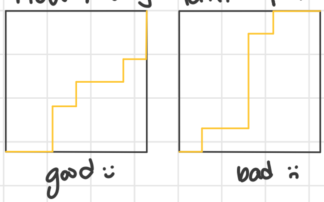
↳ The # of lattice paths from (0,0) to (m,n) is $\binom{m+n}{m}$

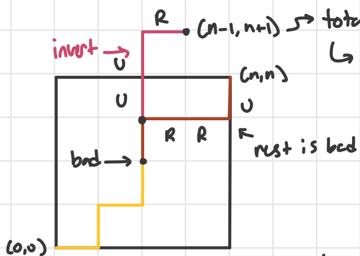
↳ Lattice paths correspond to a choice of m horizontal and n vertical moves in a sequence of m + n moves

↳ binomial coefficient because either up or right (2 choices) U ↑ → R

Catalan Numbers: Lattice Paths Not Above the Diagonal

How many lattice paths from (0,0) to (n,n) never go above the diagonal?





(n-1, n+1) → total number of paths = $\binom{n-1+n+1}{n-1}$

↳ corresponds to the # of bad paths

↳ so, total − # bad = # good

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{\binom{2n}{n}}{n+1}$$

Catalan Number

↳ (0,0) → (n,n), not over diagonal

↳ 1, 1, 2, 5, 14, 42

0 1 2 3 4 5

Applications of Catalan Numbers

↳ How many ways to parenthesize an expression:

↳ $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \ \ldots \ x_n$

↳ for example, n=4, we have 5 ways: → when n=5, there are 14 (and so on)

↳ $x_1 \cdot (x_2 \cdot (x_3 \cdot x_4))$

↳ $x_1 \cdot ((x_2 \cdot x_3) \cdot x_4)$

↳ $(x_1 \cdot x_2) \cdot (x_3 \cdot x_4)$

↳ $((x_1 \cdot x_2) \cdot x_3) \cdot x_4$

↳ $(x_1 \cdot (x_2 \cdot x_3)) \cdot x_4$

Deriving Recurrence Relations

How many regions are determined by n lines that <u>intersect</u> at general position?
↳ d_n = number of regions you get
↳ d_1 = 1 line in plane, so 2 regions →

↳ d_2 = 2 lines in plane, 4 regions →
↳ $d_{n+1} = d_n + n + 1$ when $n \geq 0$
↳ $d_5 = d_4 + 4 + 1$
↳ $d_5 = 11 + 5 = 16$
↳ $d_6 = d_5 + 5 + 1$
↳ $d_5 = 22$

↳ Related: Intersecting Circles?
↳ $d_1 = 2$

↳ $d_{n+1} = d_n + 2n$
↳ $d_2 = 2 + 2 \cdot 1 = 4$

Additional: How many ternary sequences do not contain 01 in consecutive positions?
↳ $t_1 = 3$: 0, 1, 2
↳ $t_2 = 8$: 00, 10, 11, 12, 21, 20 . . . 01 ×
↳ $t_3 = 21$: 010, 100, 012, 102, 210, 110 × → 3·3·3 total = 27 − 6 bad = 21
↳ $t_n = 3t_{n-1} - t_{n-2}$ when $n \geq 2$ OR $t_{n+2} = 3t_{n+1} - t_n$
doesn't matter btw $n-2$ or n

1.4 Mathematical Induction

If S is a set of positive integers, 1 is in S, and <u>$k+1$</u> is in S whenever <u>k</u> is in S, then S is the <u>set of all positive integers</u>
↳ to prove S_n is true for all n 1) Show S_n holds when $n=1$
2) Assume S_n is true when $n=k$ and show it is true when $n=k+1$

Example: The sum of the first n odd integers is n^2: $1+3+5+7+\ldots+(2n-1) = n^2 \leftarrow S_n$
↳ 1) verify if $n=1$, so $(2(1)-1) = 1^2$
↳ 2) assume true when $n=k$, so assume $1+3+5+7+\ldots+(2k-1) = k^2$
↳ next, do for $k+1$, $1+3+5+7+\ldots+(2k-1)+(2k+1) = k^2 + (2k+1)$
$= k^2 + 2k + 1$
$= (k+1)^2$
next term $= 2(k+1) - 1$
$2k + 2 - 1$
$2k + 1$
next term, $n^2 \rightarrow k^2, (k+1)^2$ next

dangerous!
Ambiguity: $1+3+5+7+\ldots+(2n-1)\ n^2$
↳ Instead, $\sum_{i=1}^{1} a_i = a_1$ and $\sum_{i=1}^{k+1} a_i = a_{k+1} + \sum_{i=1}^{k} a_i$
↳ So for example → $\sum_{i=1}^{n} 2i-1 = n^2$
1) $\sum_{i=1}^{1} 2i-1 = 2(1) - 1 = 1 = 1^2$
2) assume $\sum_{i=1}^{k} 2i-1 = k^2$
↳ $\sum_{i=1}^{k+1} 2i-1 = k^2 + 2(k+1) - 1$
$= k^2 + 2k + 1 = (k+1)^2$

TIP!
Combinatorial proofs are usually preferable than formal inductive proofs

Show that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

↳ 1) $\frac{1(1+1)(2(1)+1)}{6} = 1 = 1^2$

2) Assume $1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

↳ Show $1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$$= \frac{(2k^3 + 3k^2 + k) + (6k^2 + 12k + 6)}{6}$$

$$= \frac{2k^3 + 9k^2 + 13k + 6}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Example: for all $n \geq 1$, $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

1) When $n=1$, $1^3 + (1+1)^3 + (1+2)^3 = 1 + 8 + 27 = 36$ ✓

2) assume true when $n=k$

↳ $(k+1)^3 + (k+2)^3 + (k+3)^3$

$= (k^3 + 9k^2 + 27k + 27) + (k+1)^3 + (k+2)^3$

$= [(k^3 + (k+1)^3 + (k+2)^3] + [9k^2 + 27k + 27]$

Example: Show that for all $n \geq 2$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$

1) When $n=2$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{1+\sqrt{2}}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}+2}{2} = \left(1 + \frac{\sqrt{2}}{2}\right)^2 > (\sqrt{2})^2$

↓

$1 + \sqrt{2} + \frac{1}{2} > 2$

$\sqrt{2} + \frac{3}{2} > 2$

$-3/2 \quad -3/2$

$(\sqrt{2})^2 > \left(\frac{1}{2}\right)^2$

$2 > \frac{1}{4}$

2) assume true when $n=k$ $\quad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} > \sqrt{k}$

Show that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$

↳ So prove $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

$k + \frac{1}{k+1} + \frac{2\sqrt{k}}{\sqrt{k+1}} > k+1$

$\frac{1}{k+1} + \frac{2\sqrt{k}}{\sqrt{k+1}} > 1$

if this is true, then previous true

$\frac{2\sqrt{k}}{\sqrt{k+1}} > 1$

$\frac{4k}{k+1} > 1$

$4k > k+1$

$-k$

$3k > 1$, ✓

k is <u>at least</u> 2, so this is true

Basis for Long Division & Greatest Common Dividers

Long Division Basis: $n = qm + r$ ← remainder

↳ Requires a proof by induction (more complicated bc its a statement)

1) If $n=1$, if $m=1$, then $1 \cdot 1 + 0 = 1$ (so S_1 is true)

2) Assume S_k is true, and q and r are integers

↳ $k+1 = mq + (r+1)$ <u>unless</u> $r+1 = m$, in which case $k+1 = (q+r)m + 0$

↳ remainder increases OR the number is cleanly divided by m

Euclidean Algorithm

m and n are positive integers, $n \geq m$ — choose q and r with $q \geq 0$ and $0 \leq r < m$ so that $n = qm + r$

↳ if $r = 0$, then greatest common divisor $\gcd(n,m) = m$

↳ if $r > 0$, then $\gcd(n,m) = \gcd(m,r)$

↳ $\frac{n}{d} = \frac{(qm+r)}{d} = q\left(\frac{m}{d}\right) + \frac{r}{d}$

Example: Find $\gcd(10262736, 85470) \to 10262736 \% 85470 = 6336 \to 85470 \% 6336 = 3102$

↳ $6336 \% 3102 = 132 \to 3102 \% 132 = 66 \to 132 \% \overset{\text{answer}}{66} = 0$

Find integers a and b so that $an + bm = \gcd(n,m)$ | $n = 10262736$, $m = 85470$

↳ $66 = 3102 - 23 \cdot 132 \leftarrow 132 = 6336 - 2 \cdot 3102$

$= -23 \cdot (6336 - 2 \cdot 3102)$

$= -23 \cdot 6336 + 46 \cdot 3102 \leftarrow 3102 = 85470 - 13 \cdot 6336$

$= -23 \cdot 6336 + 46 \cdot (85470 - 13 \cdot 6336)$

$= -634 \cdot 6336 + 46 \cdot 85470 \leftarrow 6336 = 1026736 - 120 \cdot 85470$

$= -634 \cdot (1026736 - 120 \cdot 85470) + 46 \cdot 85470$

$63 = -634 \cdot 1026736 + 76127 \cdot 85470 \to a = -634,\ b = 76127$

6.5 Multinomial Coefficients

How many different rearrangements of: $\overset{1\,2\,3\,4\,5\,6\,7\,8\,9\,10}{AABBCCEEEF}$?

↳ $\binom{10}{2,2,2,3,1} = \frac{10!}{2!\,2!\,2!\,3!\,1!} \to$ informally known as the "Mississippi" Problem

Only two parts = a multinomial coefficient known as a binomial coefficient

↳ $\binom{26}{7,19} = \binom{26}{7} \leftarrow$ only use the binomial notation when writing

Binomial Theorem

↳ $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

(think like a choice)

↳ $(x+y)^n = (x+y)(x+y)\ldots(x+y) \to$ from each n term, you either take x <u>or</u> y

↳ if k is the # of times you take y, you take x $n-k$ times

Example: What is the <u>coefficient</u> of $a^{14} b^{18}$ in $(3a^2 - 5b)^{25}$?

↳ $\binom{25}{7} 3^7 (-5)^{18} \leftarrow$ coefficient of $b = y \longleftarrow \underbrace{\overset{1}{a^2}\overset{2}{a^2}\overset{3}{a^2}\overset{4}{a^2}\overset{5}{a^2}\overset{6}{a^2}\overset{7}{a^2}}_{a^{14}} \overbrace{bbbb\ldots}^{18}$

or 18, they are = ; x = coefficient of a

Multinomial Theorem

↳ $(\overset{\text{any number of } x}{x_1 + x_2 + x_3})^n = \sum_{k_1+k_2+k_3=n} \binom{n}{k_1,k_2,k_3} x_1^{k} x_2^{k} x_3^{k}$

Example: What is the coefficient of $a^6 b^8 c^6 d^6$ in $(4a^3 - 5b + 9c^2 + 7d)^{19}$?

↳ $\binom{19}{2,8,3,6} 4^2 (-5)^8 9^3 7^6$

The Pigeon Hole Principle

If you need to put $n+1$ pigeons into n holes, you will need to put some 2 pigeons into the same hole

↳ Putting $mn+1$ pigeons into n holes → put some $m+1$ pigeons into the same hole

Erdős - Szekers Theorem

↳ any sequence of $mn+1$ distinct real numbers has either an increasing subsequence of length $m+1$ or a decreasing subsequence of $n+1$

Example: $m=3$ and $n=5$ → $2, 3, -5, 0, \pi, 9, -4, -3, 7, 8, 5, 1, -6, 10, -8, -1$

↳ Find the longest increasing subsequence/decreasing subsequence

↳ Sequence $(a_1, a_2, a_3 \ldots a_t)$ (distinct list) $t = mn+1$, for each i place pigeon a_i in pigeon hole (inc, dec) where inc is the length of longest ↑ subsequence, dec is length of longest ↓ subsequence starting with a_i

↳ Then there are $mn+1$ pigeons and only mn holes

↳ 2 pigeons assigned to the same hole (a_i and a_j, where $i<j$), if $a_i < a_j$ then inc for $a_i >$ inc for a_j and vice versa

Example: Show that if S is a subset of size 6 from $\{1, 2, 3, \ldots 9\}$, then there is 2 distinct elements of S whose sum $= 10$

↳ pigeons: $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

↳ holes should add to 10: $H_1 = \{1, 9\}$, $H_2 = \{2, 8\}$, $H_3 = \{3, 7\}$, $H_4 = \{4, 6\}$, $H_5 = \{5\}$

↳ Pick 6 pigeons (from 9 total), place them into holes

↳ PHP: you have to put two pigeons into 1 hole - 6 pigeons, 5 holes

↳ If you select a number from each of the holes (1, 8, 3, 4, 5), the next number will ALWAYS end up as a pair to add up to 10 $\{1, \_\}, \{\_ 8\}, \{3, \_\}, \{4, \_\}, \{5\}$

↳ So a pigeon will ALWAYS end up in a hole with a pigeon already in it

↳ So the pair ends up adding to 10

Complexity, Problem Size, and Running Time

A problem size $= n$ when data for the problem $= n$ packets of info readable at some constant time interval

↳ ex. when integers are at most MAX_INTEGER, all integers are readable at some constant time

↳ reading a small vs. large integers is the same amnt of data

An algorithm accepts data size n, carries out $f(n)$ steps - function $f(n)$ = running time (typically imprecise)

↳ important to compare running times

Functions that $\to \infty$ as $n \to \infty$, in order

↳ $\log^* n, \log\log\log n, \log\log n, \log n, n^{.001}, \sqrt{n}, n, n\log n, n^{3/2}, n^2, n^3, n^{\log\log n}, n^{\log n}, 2^n, (\log n)^n, (\sqrt{n})^n, n^n, 2^{2^n}, 2^{2^{2^n}}$

↳ any 2 functions, 1^{st} is $o(2^{nd})$ (more on Little-Oh Notation below)

Big-Oh and Little-Oh Notation

Big-Oh Notation — $f(n)$ and $g(n)$ are two functions, notation $f \overset{f>g}{=} O(g)$ or $f(n) = O(g(n))$ means that some constant $C>0$ exists where $f(n) \le C g(n)$ for all n

↳ runtimes are hard to say exactly, so Big-Oh Notation bounds comparisons btw functions

Little-Oh Notation — $f \overset{f<g}{=} o(g)$ or $f(n) = o(g(n))$ means the ratio $\frac{f(n)}{g(n)}$ tends to 0 ; $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = 0$

Example: algorithms A_1 and A_2 have runtimes of $30n^2 + 150n\log n$ and n^3+n ; Which is faster?

↳ When n is small, then the 2nd is faster, but when n is large, the 1st is faster

↳ $f(n) = 30n^2 + 150n\log n$, $g(n) = n^3+n$

↳ $f = O(g)$ — any number that makes $g(n) \ge f(n)$, such as 1,000,000 or 1,000

↳ $f = o(g) \leftarrow \frac{f(n)}{g(n)}$ goes to 0 as $n \to \infty$

Four Motivating Problems

Given a list S of n distinct + integers

easy

1) Largest integer in S? → n steps (1 step = 1 read and 1 compare) = $|S|$
2) If first integer = a, is there b and c such that $a = b+c$? → $\binom{n-1}{2}$ steps ($n-1$ other numbers)
3) 3 integers a,b,c where $a = b+c$? → $\binom{n}{3}$ steps, picking up 3 numbers and checking

hard

4) Fair division (1 group sum = 2nd group sum)? → $n2^n$ steps, each step = choose subset T and S, add all # in T and $S-T$, then check if sums are =

Sorting and Stirling's Approximation

Sorting Problem: An unknown linear order of integers in $\{1,2,\ldots n\}$, must find order by asking in the form: Is $i<j$ in L?

↳ Example: Say the order is $L = (2,5,3,1,4)$

↳ Is $2<1$ in L? Yes
↳ Is $3<1$ in L? Yes
↳ Is $4<3$ in L? No

} eventually obtain order of L

UGA Sorting Algorithm — for each 2-element subset $\{i,j\}$ of $\{1,2,\ldots n\}$, ask Is $i<j$ in L?

↳ total of $\binom{n}{2} = \frac{n(n-1)}{2}$ questions asked, now can assemble L

Lower bound in sorting — worst case: asks at least $\log_2 n!$ questions

↳ $n!$ different linear permutations of $\{1,2,\ldots,n\}$

↳ each time a question is asked, worst case: # of possibilities ↓ $\frac{1}{2}$ at most

↳ t questions asked, $2^t \ge n!$

(advanced calculus)

Stirling Approximation — $\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n} = 1$, so $\log_2 n! \sim n\log_2 n$

↳ A sorting algorithm is optimal if its running time is $O(n\log n)$

Informal Discussion of Merge Sort

To find an unknown linear order, divide the problem into 2 equal size subproblems
↳ find restriction consisting of first $n/2$ integers $\{1, 2, \dots n/2\}$
↳ find restriction to last $n/2$ integers $\{n/2+1, n/2+2, \dots n\}$
  ↳ Merge answers to find full linear order
  ↳ 2 sorted lists of $n/2$ can be merged in running time n
  ↳ Merge Sort has running time $r(n)$, satisfying recurrence $r(n) = 2r(n/2) + n$, so $r(n) = O(n \log n)$

1.6 Induction Exercises and a Little-o Proof

Example: Show that $n^2 > 5n + 13$ when $n \geq 7$ ← true from some point on
↳ Base Case: $7^2 = 49 > 5 \cdot 7 + 13 = 48$
↳ Assume $k^2 > 5k + 13$
  ↳ Show $(k+1)^2 = k^2 + \underbrace{2k+1}_{k+1 \text{ term}}$
$> (5k + 13) + (2k + 1)$ (−8, +8)
$= \underbrace{(5k+5)}_{k^2 \text{ term}} + (2k + 9)$

Show $(k+1)^2 > 5(k+1) + 13 = (5k+5) + 13$
↳ Is $2k + 9 \geq 13$? ← true if $k \geq 2$
  ↳ If $k \geq 2$, then $2k \geq 4$, so $2k + 9 \geq 13$
    ↳ $k \geq 2$ bc $n \geq 7$, so $k \geq 7$

ratio → 0   stronger result

Example: Show that $5n + 13 = o(n^2)$
↳ Let $\varepsilon > 0$, set n_0 as the least positive integer so that $n_0 > 10/\varepsilon$ and $(n_0)^2 > 26/\varepsilon$
  ↳ If $n \geq n_0$, then: $(5n + 13)/n^2 \to 5/n + 13/n^2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$
$\varepsilon = \varepsilon/2 + \varepsilon/2$

Alternative Forms of Induction

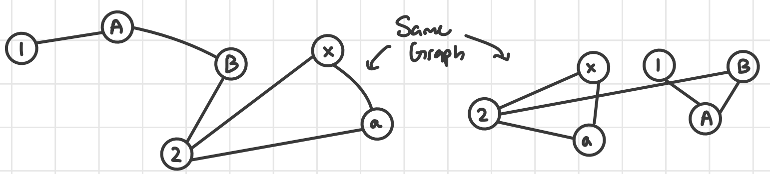
↳ argue by contradiction — if a statement S_n is NOT true for all $n \geq 1$, there is a least positive integer for which S_n fails

↳ strong induction — prove S_n holds for all $n \geq 1$ — 2 Steps:
same as induction ⟶ ↳ Verify S_1 is valid
different than induction (base case → k) ⟶ ↳ Assume that for some $k \geq 1$, statement S_m is valid for all m with $1 \leq m \leq k$
  ↳ Show S_{k+1} is valid

Introduction to Graph Theory

Graph G is a pair (V, E) where V is a finite set and E is a set of 2-element subsets of V
↳ V is the vertex set of G
↳ E is the edge set of G
} → needs to be explicitly defined

Example: $G = (V, E)$ where $V = \{1, 2, A, x, B, a\}$ (order doesn't matter) and $E = \{\{1, A\}, \{2, x\}, \{x, a\}, \{A, B\}, \{B, 2\}, \{2, a\}\}$



Vertices – known as nodes, points, locations, stations, etc.

Edges – known as arcs, lines, links, pipes, connectors, etc.

↳ edge $\{x,y\}$ known as xy or yx (Consider the edge xy)

↳ If the edge is an integer, be careful! Use $\{\ \}$ notation

↳ x and y are adjacent in G / are neighbors in G

$N_G(x)$ – the set of all neighbors of vertex x in G

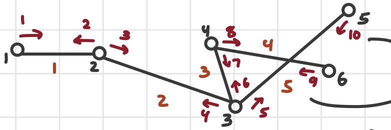
↳ if G is fixed in discussion, shortened to $N()$

↳ Degree of x in G: $|N_G(x)|$, size of $N_G(x)$

↳ also known as $\deg_G(x)$

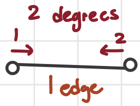
First Theorem in Graph Theory

Let $G=(V,E)$ and q = number of edges in G → Then $\sum_{x \in V} \deg_G(x) = 2q$



$q = 5$

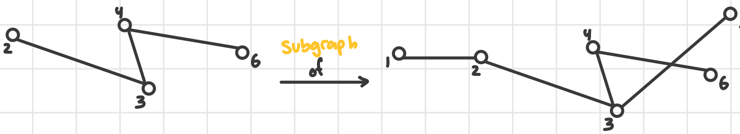
degrees $= 10 = 2(5)$



Corollary – For any graph, the number of vertices with an odd degree is even

Notion of a Subgraph

A graph $G' = (V', E')$ is a subgraph of $G = (V,E)$ when V' is contained in V and E' is contained in E



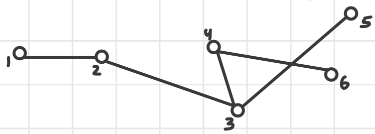
Paths, Hamiltonian Paths, Size of Paths

When $n \geq 1$, a sequence $P = (x_1, x_2, \dots x_n)$ of n distinct vertices in G is a path from x_1 to x_n IF x_i is adjacent to x_{i+1} in G whenever $1 \leq i < n$

↳ any sequence of 1 vertex is still a path

↳ Must have DISTINCT vertices

↳ All CONSEQUTIVE vertices must have edges



← (1, 2, 3, 5) is a path in G

Hamiltonian Path – visits each vertex exactly once

↳ Hamiltonian Cycle – last vertex is adjacent to the first vertex

Length – measure how big a path is by # of edges

↳ Path (a,b,c,d) has length 3

Size – measure by # of vertices ← Prefered (for now)

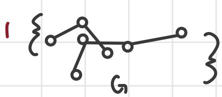
↳ Path (a,b,c,d) has size 4

Connected Graphs

G is connected if for all x,y in V with $x \neq y$, there is a path from x to y

Let G be a disconnected graph

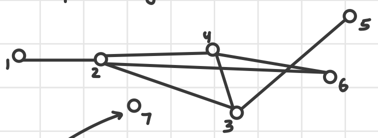
↳ Subgraph $H = (V', E')$ is a component of G if H is connected and any subgraph that contains H is disconnected



2 components within G

Cycles and Cliques

When $n \geq 3$, a sequence $P = (x_1, x_2, \ldots x_n)$ of n distinct vertices in G is called a cycle of length n in G if x_i is adjacent to x_{i+1} in G whenever $1 \leq i < n$ and x_n is adjacent to x_1 in G



← $(2, 3, 4, 6)$ is a cycle of length 4

adjacent

↳ length = size, because # of edges = # of vertices in a cycle

Loose Points - vertices with 0 neighbors, $\deg_G(x) = 0$

When $n \geq 1$, a set S of vertices in G is a clique if any <u>2 distinct vertices</u> in S are <u>adjacent</u>

↳ When looking at graph above, $\{7\}, \{2,4,3\}, \{3,5\}$ are cliques

↳ A one-element set is <u>always</u> a clique, because you cannot deny 2 distinct vertices are adjacent

↳ Every member of a clique must be <u>adjacent</u> to every other vertex

Questions for Thought

Which of these questions are hard to prove "yes" / "no"?

1) Is G connected?

Does G have → 2) a path on at least $n/2$ vertices?

↳ 3) a cycle of size at least $n/2$?

↳ 4) a clique of size at least $n/2$?

} Proving "yes" is easier because you need to prove 1 case true

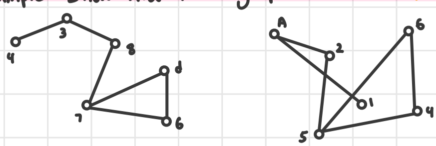
How to defend a "no" answer?

Isomorphic Graphs

Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic when there is a bijection $f: V_1 \rightarrow V_2$ so that $\{x, y\}$ is an edge in G_1 $\longleftrightarrow$ $\{f(x), f(y)\}$ is an edge in G_2

↑ if and only if

Example: Show how these graphs are isomorphic



→ Vertex 4 in G_1 maps to Vertex 1 in G_2

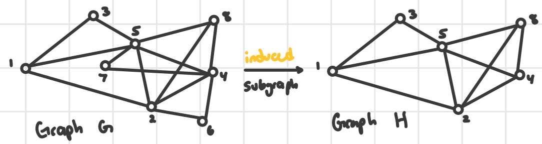
→ Vertex 3 in G_1 maps to Vertex A in G_2

etc.

4.7 Induced Subgraphs and Cut Vertices

A graph $H=(V',E')$ is an induced subgraph of $G=(V,E)$ if $V' \subseteq V$ (belongs to) AND xy is an edge in H whenever x and y are distinct vertices in V' + xy is an edge in G

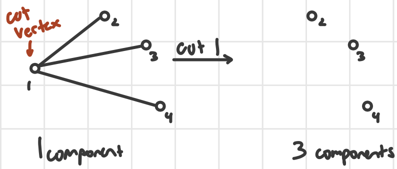
↳ every possible edge that COULD be present IS present
(from G)



← 6 and 7 are missing, but remaining vertices have all edges

↳ An induced subgraph is denoted by its vertex set

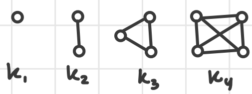
↳ So $H = G - \{6,7\}$

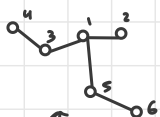


A vertex is a cut vertex if the induced subgraph $G-x$ has more components than G

Special Classes of Graphs

For $n \geq 1$, K_n denotes a complete graph (clique) on n vertices (all vertices connected via at least 1 edge)





A graph G is a tree if G is connected and has no cycles

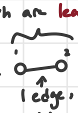
↳ Vertex of degree 1 = leaf (The right graph has 3: 2, 4, and 6)

↳ Other vertices are cut vertices

When T is a tree on n vertices and $n \geq 2$, then T has AT LEAST two leaves

↳ n vertices, n-1 edges

Proof:

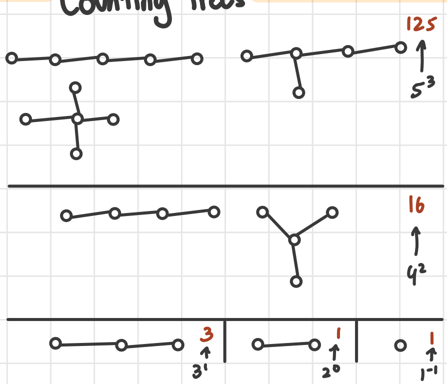
1) Base Case: A tree on 2 vertices:  so true

2) Take any leaf of a tree and remove it

↳ Left with another tree with at least 2 leaves

} Bootstrap Principle

Counting Trees



Unlabeled Trees - given n vertices, how many unlabeled trees can be made?

↳ If we label them 1→n, how many combinations?

↳ n^{n-1} number of labeled combinations

Trails and Circuits

A sequence $(x_1, x_2 \dots x_t)$ of vertices is called a trail (or walk) if for every $i = 1, 2 \dots t-1$, $x_i x_{i+1}$ is an edge

↳ Vertices do NOT have to be distinct (just that any two consecutive vertices are adjacent)

A trail is a circuit if the last vertex is adjacent to the first

A trail is a Euler trail if the edges are distinct (for every edge e of G there is a unique i, $1 \leq i \leq t$)

↳ Applies for a Euler circuit

↳ A graph has an Euler circuit ⟷ if has only 1 component which is NOT a loose vertex and every vertex has an even degree

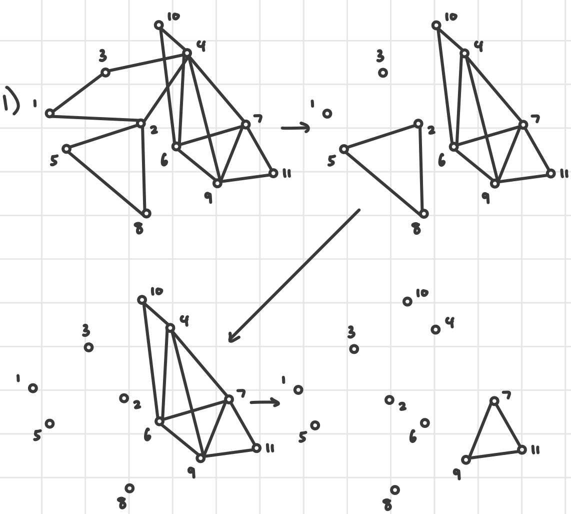
Algorithm → 1) Choose a root vertex r and start with a trivial circuit that starts/ends at r

2) Given a partial circuit $(r = x_0, x_1, \dots, x_t = r)$ that traverses some but not all of the edges of G containing r, remove these edges from G. Let i be the least integer for which x_i is incident with 1 of the remaining edges. Form a "greedy" partial circuit among the remaining edges of form $(x_i = y_0, y_i, \dots y_s = x_i)$

3) Expand the original circuit by setting $r = (x_0, x_1, \dots, x_{i-1}, x_i = y_0, y_1, \dots, y_s = x_i, x_{i+1}, \dots, x_t = r)$

1) Start with circuit (1)

2) Partial circuit (1,2,4,3,1) is formed



3) Remove edges

4) Vertex incident with remaining edge: 2

5) New greedy approach: (2,5,8,2)

6) New circuit: (1,2,5,8,2,4,3,1)

7) Remove edges

8) First vertex incident: 4

9) New greedy approach: (4,6,7,4,9,6,10,4)

10) New circuit: (1,2,5,8,2,4,6,7,4,9,6,10,4,3,1)

11) Last greedy approach: (7,9,11,7)

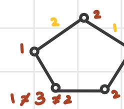
Maximum Clique Size and Graph Coloring

Maximum Clique Size – $\omega(G)$

Chromatic Number – $\chi(G)$

↳ Fewest Chromatic Number (or Colorings) ≥ Largest Clique

↳ $\chi(G) \geq \omega(G)$



← $\omega(G) = 2$
$\chi(G) = 3$

L.8 Hamiltonian Paths and Cycles

Hamiltonian Path – $n \geq 3$ vertices, a path $P = (x_1, x_2, \ldots, x_n)$ that visits each vertex once

↳ **Hamiltonian Cycle** – a cycle $C = (x_1, x_2, \ldots, x_n)$ that returns to start

→ Answering "yes" to if a graph has a hamiltonian path can be validated by a certificate in the form of a permutation of the vertex set of G

→ Answering "no" seems to not have a certificate (that can be checked quickly, in general)

↳ However, every vertex MUST have ≥ 2 degree, so if a vertex does NOT have ≥ 2 degrees

↳ The certificate can be found in SOME cases

Computational Complexity

Informal Perspective: Class P consists of all yes-no questions for which the answer can be determined with an algorithm (provably correct) and has a running time which is polynomial in the input size

↳ Ex. Is 2388643 in the list n? (yes/no)

↳ Ex. Given a graph G, does it have a Euler circuit? (yes/no)

⟶ Class NP consists of all yes-no questions for which there is a yes certificate, whose "correctness" can be certified with an algorithm whose running time is polynomial in the input size

↳ Ex. Given a list of n numbers, is there a fair division? (yes, and is correct)

↳ Ex. Given a graph G, is there a Hamiltonian Cycle? (yes, and is correct)

↳ Any answer in P is in NP

↳ But does this mean NP = P ? Nobody knows!

Revisiting Euler Circuits and Bipartite Graphs

A "no" answer to a Euler Circuit can have a certificate

↳ If graph is not connected

↳ If there is a vertex of odd degree

Bipartite Graphs – a triple (A, B, E) where A and B are disjoint finite sets and E is a collection of 2-element sets, 1 of A and 1 of B (E shows the vertex pairs/edges)

Complete Bipartite Graphs – $m, n \geq 1$, Graph $K_{m,n}$ has $m+n$ vertices, m on one side and n on the other, with mn edges – each vertex is adjacent to every vertex on the other side

If a bipartite graph has a Hamiltonian Cycle, then it is connected and $\overset{\text{size}}{|A|} = \overset{\text{size}}{|B|}$

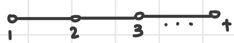
↳ The complete bipartite graph $K_{n,n+1}$ does NOT have a Hamiltonian cycle bc $|A| \neq |B|$

Dirac's Theorem – if G is a graph on n vertices and every vertex has at least n/2 neighbors, then G has a Hamiltonian cycle

↳ Graph $K_{n,n+1}$ has $2n+1$ vertices but vertices in $B = n+1$ have only n neighbors, and $n < (2n+1)/2$

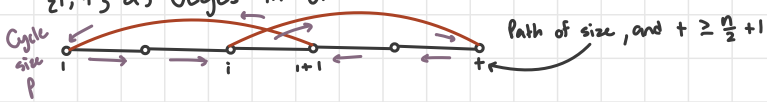
Hamiltonian Algorithm → 1) Grab any vertex, and build a long path (as long as possible)

Initialization { ↳ all neighbors of the ends is on the path (otherwise there would be a longer path), and $t > (1+n)/2$

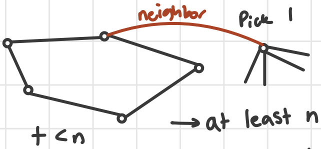


2) Phase 1: Turn long path into a cycle of same size

↳ Pigeon-hole principle: there are consecutive vertices i and $i+1$ on the path with $\{1, i+1\}$ and $\{i, t\}$ as edges in G



3) If $t = n$, done; if not, then $t < n$ → Pick any vertex not on path



→ at least $n/2$ neighbors,

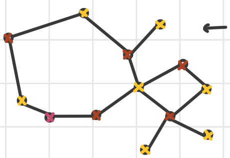
↳ but path $t > n/2$, so there MUST be 1 neighbor from vertex to path

↳ continues for all vertices in cycle

L9 **Chromatic Numbers (cont'd)**

$\chi(G)$ is the chromatic number of a graph

↳ fewest amount of colors to add to adjacent vertices to ensure no adjacent vertex has the same color



← $\chi(G) = 3$, Odd Cycles have to be at least 3

↳ (Yes/No) Questions: $\chi(G) \leq 2$? } Class NP, check that same-colored

↳ $\chi(G) \leq 3$? } vertices are NOT adjacent

↳ Is this in class P? Who knows?

Is $\chi(G) \leq 2$? → class P

1) Take vertex, color 1, take neighbors, color 2

2) Check: if any colored 2 have edges with each other, there is an odd cycle (Answer is No!)

↳ If not, then good

3) Continue with neighbor vertices, color, then check

t-coloring – an assignment of integers (colors) from $\{1, 2, \ldots, t\}$ to vertices of G so that adjacent vertices have distinct colors

↳ A chromatic number $\chi(G)$ is the least number which is a valid t-coloring

↳ Recall: Max clique size $\omega(G) \leq \chi(G)$

Triangle-Free Graph Theorem

Theorem: for every $t \geq 3$, there is a graph G with $\chi(G) = t$ and $\omega(G) = 2$

↳ Triangle – a type of clique with size 3

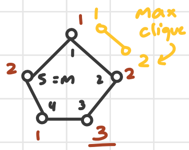
↳ Graphs with $\omega(G) \leq 2$ are triangle-free

↳ So the theorem = there are triangle-free graphs $[\omega(G) \leq 2]$ with LARGE chromatic numbers $[\chi(G) = \text{large!}]$

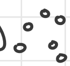
1. Pigeon-Hole Principle: $t = 3$, G = odd cycle C_5. Suppose for some $t \geq 3$, there is a triangle-free graph G with $\chi(G) = t$. How you build a new triangle-free graph with chromatic number $t+1$: G has vertices $x_1, x_2, \ldots, x_m$ → Start with a "large" independent set Y. For each m-element subset

$\{y_1, y_2, \ldots, y_m\}$ of Y, attach a copy of G with x_i adjacent to y_i for each $i = 1, 2, \cdots, n$. This works if Y has size at least $t(m-1)+1$ by the Pigeon-Hole Principle

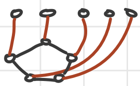
1) Odd Cycle C_5
2) $\chi(G) = t = 3$
3) $\omega(G) = 2$



4) Graph C_5 has m vertices (5)
5) Create a big graph with NO triangles and $\chi(G) = t+1$
6) Create a large independent set (no edges btw vertices)

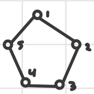


7) Pick up a subset of size m (5 in this case)
8) Attach a copy of graph G to subset
9) Repeat for all subsets of size m in Y
10) Suppose that for the large graph Y, the coloring is the same as the small graph G (t)

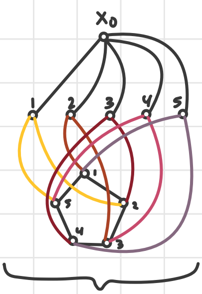


↳ Pigeon-hole principle – putting elements of set Y into t colorings (3 in this case), puts a lot of pigeons into very few holes, there will be several pigeons in 1 hole

2. Mycielski Construction: $t=3$, $G=$ odd cycle C_5, suppose for some $t \geq 3$, there is a triangle-free graph G with $\chi(G) = t$. Creating a graph with $\chi(G) = t+1$: Start with copy of G, add independent set Y containing a "mate" y_x for every vertex x of G, so the mate y_x has exactly the same # of neighbors in G as does x. Then add one new vertex x_0 which is adjacent to every vertex in Y but to none of the vertices in G



$\leftarrow G$, $\chi(G) = t$ ①
$t \geq 3$



← arbitrary vertex addition ④

← independent set = size of G ②

← Mates, same neighbors as corresponding vertex ③

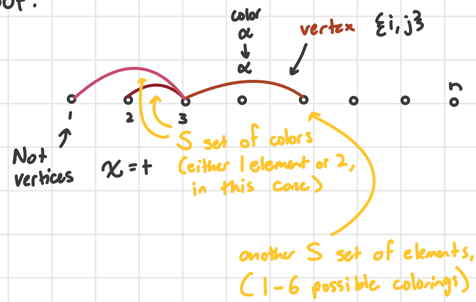
Next graph, $t=4$ $(\chi(G)=4)$ ⑤, has to be $t+1$ of previous because the new vertex is adjacent to all t colors (from set Y)

Shift Graph

When $n \geq 2$, shift graph S_n has $C(n,2)$ vertices and these are the 2-element subsets of $\{1, 2, \ldots, n\}$
For each 3-element subset $\{i, j, k\}$ of $\{1, 2, \ldots, n\}$, with $i < j < k$, the vertex $\{i, j\}$ is adjacent to the vertex $\{j, k\}$ in S_n

Theorem: $\chi(S_n)$ is the least positive integer t so that $S^t \geq n$ (when $n \geq 2$)

Proof:



↳ The sequence of sets is distinct

↳ How many subsets are there?

↳ $2^t \geq n$ if all sets are distinct

1.10 Girths and Forests

Forest - graphs with no cycles

↳ every component is a tree

↳ girth = ∞ ↳ $\chi(G) = 1$ or 2, always

Girth - Size of the smallest cycle in G

↳ Constructions for triangle-free graphs have small girths

↳ Theorem: for every pair (g, t) of positive integers $g, t \geq 3$, there is a graph with girth G and $\chi = t$

Perfect Graphs & Complements

Perfect Graphs - if $\chi(G) = \omega(G)$ for every induced subgraph (subgraph that is a cycle)

↳ Any graph with an odd cycle as an induced subgraph is NOT perfect

↳ with odd cycles on 5 or more is NOT perfect

Complements (G^c) — the graphs have the same vertex set of G with edges not present in G

↳ pair xy of distinct vertices have an edge in G^c ⟷ it does not have an edge in G

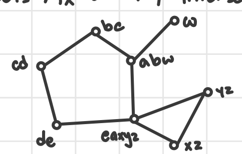
↳ If G^c has an odd cycle within an induced subgraph, G is NOT perfect

Berge's Perfect Graph Conjecture – A graph is perfect ⟷ both G and G^c DO NOT have an odd cycle as an induced subgraph

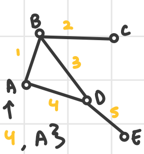
Intersection Graphs

Let $F = \{A_x : x \in X\}$ be a family of sets. We associate with F an intersection graph G where the vertices of G are the elements of X and xy is an edge in G when the sets A_x and A_y intersect

↳ 2 vertices are adjacent ⟷ their sets intersect ⟶



(ex.)



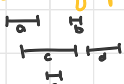
$\{1, 4, A\}$

edges

differentiate

↳ Applications of intersection graphs - interval graphs

↳ Interval graphs : intersection graphs of a family of closed intervals



} c and a intersect, but b and e don't

↳ To color: Apply Greedy Algorithm and color in order of left endpoints →