Matrix Addition and Scalar Multiplication

zero matrix – Matrix whose every entry is 0 $\quad O_{2\times3} = \begin{pmatrix} 0&0&0 \\ 0&0&0 \end{pmatrix} \quad O_{2\times1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

identity matrix – 1s on all diagonals $\quad I_2 = \begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix} \quad I_3 = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$

↳ All SQUARE

$$a_{i,j} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \left(\quad\quad\quad \right) \end{array}$$

$A + B = a_{i,j} + b_{i,j}$

if c is a real number, $cA = \underset{\text{entries}}{ca_{i,j}}$

Ex

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + c\begin{pmatrix} 7 & 4 & 7 \\ 0 & 0 & k \end{pmatrix} = \begin{pmatrix} 15 & 10 & 17 \\ 4 & 5 & 16 \end{pmatrix}$$

$1 + 7c = 15 \rightarrow \boxed{c = 2}$

$6 + ck = 16 \rightarrow 6 + 2k = 16 \rightarrow \boxed{k = 5}$

Matrix Multiplication

A and B are matrices $\rightarrow AB = A(\vec{b}_1 \dots \vec{b}_p) = (A\vec{b}_1 \dots A\vec{b}_p)$

$$C = AB = \begin{pmatrix} 2&0 \\ 1&1 \end{pmatrix}\begin{pmatrix} 2&0&0 \\ 3&4&0 \end{pmatrix} \rightarrow \left(\begin{pmatrix} 2&0 \\ 1&1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 2&0 \\ 1&1 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 2&0 \\ 1&1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$\rightarrow$ $2(2) + 3(0) = 4$, $2(1) + 3(1) = 5$ | $0(0) + 4(0) = 0$, $0(1) + 4(1) = 4$ | $0(2) + 1(0) = 0$, $0(1) + 1(0) = 0$

$\rightarrow \begin{pmatrix} 4&0&0 \\ 5&4&0 \end{pmatrix}$

# of elements in <u>row</u> A = # of elements in <u>column</u> B $\quad (1\ 2) \rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Row Column Rule – if A has <u>rows</u> $\vec{a}_i$ and B has <u>columns</u> $\vec{b}_j$, each element of $C = AB$ is the dot product $c_{ij} = \vec{a}_i \cdot \vec{b}_j$

$$\begin{pmatrix} 2&0 \\ 1&1 \end{pmatrix}\begin{pmatrix} 2&0&0 \\ 3&4&0 \end{pmatrix} = \begin{pmatrix} 4&0&0 \\ 5&4&0 \end{pmatrix}$$

$1(2) + 1(3) = 5$

WARNINGS (Matrix Multiplication)

1. $AB \neq BA$
2. $AB = AC$ does NOT mean $B = C$
3. $AB = 0$ does Not mean $A = 0$ and $B = 0$
4. $I_m A = A I_n$

Matrix Transpose and Powers

transpose: A^T is the matrix with columns that are the rows of A

$$\begin{pmatrix} 1&2&3&4 \\ 0&1&0&2 \end{pmatrix}^T = \begin{pmatrix} 1&0 \\ 2&1 \\ 3&0 \\ 4&2 \end{pmatrix}$$

power: For an $m\times n$ matrix and k is $\mathbb{R}$, A^k is product of k copies of A

$A^k = AA \dots A$

① $(A^T)^T = A$

② $(A+B)^T = A^T + B^T$

③ $(rA)^T = rA^T$

④ $(AB)^T = B^T A^T$ reversed

Ex

$C = \begin{pmatrix} 1&2 \\ 0&1 \end{pmatrix}$

$C^2 = \begin{pmatrix} 1&2 \\ 0&1 \end{pmatrix}\begin{pmatrix} 1&2 \\ 0&1 \end{pmatrix} = \begin{pmatrix} 1&4 \\ 0&1 \end{pmatrix}$

Ex

$A = \begin{pmatrix} 1&0 \\ 0&0 \end{pmatrix} \quad B = \begin{pmatrix} 1&0&0 \\ 0&0&8 \end{pmatrix} \quad C = \begin{pmatrix} 1&0&0 \\ 0&2&0 \\ 0&0&2 \end{pmatrix}$ → which operations are defined / what are the dimensions?

① $A + 3C^2 \rightarrow$ not defined, cant add 2×2 to 3×3

② $A(AB)^T \rightarrow AB$ is 2×3, so $(AB)^T$ is 3×2, not defined, cant do $(2\times\underset{\text{has to match}}{2 \cdot 3}\times2)$

③ $A + ABCB^T \rightarrow \underset{2\times2}{A} \cdot \underset{2\times3}{B} \cdot \underset{\rightarrow 3\times3}{C} \cdot \underset{3\times2}{B^T} \rightarrow \underset{2\times2}{ABCB^T} + \underset{2\times2}{A}$, defined at 2×2

Inverse Matrix

$\overset{\text{matrix}}{A}$ is invertible (or non-singular) if there is a $\overset{\text{matrix}}{C}$ so that $AC = CA = I_n \rightarrow C = A^{-1}$

NOT invertible = singular

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular IF $ad - bc \neq 0$ AND $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Ex

$$\begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}^{-1} = \frac{1}{\underset{-14 \quad +15}{2(-7) - 5(-3)}}\begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} = \frac{1}{1}\begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}$$

↳ Solve using inverses

$3x_1 + 4x_2 = 7$

$5x_1 + 6x_2 = 7$

$A\vec{x} = \vec{b}$, $A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = \frac{1}{18-20}\begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$\vec{x} = \frac{1}{-2}\begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$\vec{x} = -\frac{1}{2}\begin{pmatrix} 35 - 28 \\ -35 + 21 \end{pmatrix}$$

$$\vec{x} = -\frac{1}{2}\begin{pmatrix} 14 \\ -14 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} -7 \\ 7 \end{pmatrix}$$

Inverse of nxn Matrix

$A\vec{x} = \vec{b}$ has a unique solution, $\vec{x} = A^{-1}\vec{b}$ } needs to have 1 solution for A to have an inverse

↳ method to solve n equations with n variables

IF A^{-1} exists, multiply $A\vec{x} = \vec{b}$ by A^{-1}:

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

algorithm: ① Row Reduce $(A \mid I_n) \rightarrow$ RREF

② If reduction has form $(I_n \mid B)$ then A is invertible and $B = A^{-1}$ → otherwise, A is NOT invertible

Ex

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad (A \mid I) = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = (I_3 \mid A^{-1})$$

Elementary Matrices

A and B are invertible $n \times n$ matrices

$(A^{-1})^{-1} = A$

$(AB)^{-1} = B^{-1}A^{-1} \rightarrow (ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$(A^T)^{-1} = (A^{-1})^T$

Elementary Matrix – matrix "E" differs from I_n by 1 row operation

- Swap rows
- multiply row by non-zero scalar
- add a multiple of 1 row to another

} each operation can be represented by an elementary matrix

① Every elementary matrices are invertible

② Every "E" is square

$E\begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ → 2nd row is MODIFIED

E must be 3x3, 1 row operation

$E = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow R_2 + 2R_1$

(can check by multiplying)

Invertible Matrix Theorem

Equivalent Expressions – all the following are equivalent – if 1 is true, all are true; if 1 is false, all are false

a) A is invertible.
b) A is row equivalent to I_n.
c) A has n pivotal columns (all columns are pivotal).
d) $A\vec{x} = \vec{0}$ has only the trivial solution.
e) The columns of A are linearly independent.
f) The equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$.
g) The columns of A span $\mathbb{R}^n$.
h) There is a $n \times n$ matrix C so that $CA = I_n$ (A has a left inverse.)
i) There is a $n \times n$ matrix D so that $AD = I_n$ (A has a right inverse.)
j) A^T is invertible.

Inverse A^{-1} transforms $A\vec{x}$ back into $\vec{x}$

↳ IF $AB = I$, $B = A^{-1}$ and $A = B^{-1}$

IMT – set of equivalent statements, dividing them into invertible or not

If a square matrix has identical colums, the columns are dependent → not invertible

Ex

1) Is this matrix invertible? $\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ 0 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$ ← every column pivotal, a is invertible

2) $A^2 + 4A = I \rightarrow AA + 4A = I \rightarrow A(A+4) = I \rightarrow A^{-1}A(A+4) = A^{-1} \rightarrow A + 4 = A^{-1}$

3) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

not possible, x is always pivotal

Partioned Matrices and Matrix Multiplication

$$A = \begin{pmatrix} 3 & 1 & 4 & 1 & 0 \\ 1 & 6 & 1 & 0 & 1 \\ 0 & 0 & 0 & 4 & 2 \end{pmatrix} \rightarrow A = \begin{pmatrix} {}^{1}\begin{pmatrix} 3 & 1 & 4 \\ 1 & 6 & 1 \end{pmatrix} & \overset{2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \\ {}^{4}\begin{pmatrix} 0 & 0 & 0 \end{pmatrix} & {}^{4}\begin{pmatrix} 4 & 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

blocks – what a matrix is partioned into, giving a succint representation (summarize the structure)

row column method – $AB \rightarrow \text{row}_i A \cdot \text{col}_j B$

↳ Partioned Matrices, treat each block as a scalar

$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_2 & X \end{pmatrix} \begin{pmatrix} U \\ Y \end{pmatrix}$$

Where $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, U = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \end{pmatrix}$.

$$\rightarrow I_2 U + XY = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

Inverse of $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$ ↙ Identity

$$\left.\begin{aligned} &0W + CY = 0 \rightsquigarrow CY = 0 \rightsquigarrow C^{-1}CY = C^{-1}0 \rightsquigarrow Y = 0 \\ &0X + CZ = I_n \rightsquigarrow CZ = I_n \rightsquigarrow C^{-1}CZ = C^{-1} \rightsquigarrow Z = C^{-1} \\ &AW + BY = I_n \rightsquigarrow AW + B0 = I_n \rightsquigarrow A^{-1}AW = A^{-1}I_n \rightsquigarrow W = A^{-1} \\ &AX + BZ = 0 \rightsquigarrow AX = -BZ = -BC^{-1} \rightsquigarrow AX = -BC^{-1} \rightsquigarrow X = -A^{-1}BC^{-1} \end{aligned}\right\} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

Inverse of Partitioned Matrices

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \frac{1}{ac}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$$

A, B, and C are real $n \times n$ matrices, $n > 1$ → $\frac{1}{A}$ is undefined

LU Factorization

$\vec{x} = A^{-1}\vec{b}$ But for large $n \times n$ matrices, computing the inverse is hard

Matrix factorization/decomposition - factorization of a matrix into product of matrices

↳ useful for solving for $Ax = b$, or understanding products

↳ factor into lower and upper matrices

Rectangular matrix A is upper triangular if $a_{i,j} = 0$ for $i > j$ (every element below diagonal starting from $a_{1,1}$ has to be 0)

$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 2 & 4 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ ↳ lower triangular (every element above diagonal starting from $a_{1,1}$ is 0)

elements below diagonal, is 0

↳ Invertible IF all numbers on diagonal ARE nonzero

I_n, O matrices are both lower and upper triangular matrices

Theorem: If A is a matrix that can be reduced to echelon without row exchanges, then $A = LU$

↳ L is a lower triangular matrix with 1's on the diagonal, and U is echelon form of A

↳ $\overset{3\times 2}{A} = LU = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix}$

1) Construct LU decomposition of A to get L and U

2) Set $U\vec{x} = \vec{y}$, solve for y in $Ly = b$

3) Solve for x in $Ux = y$

Ex

Solve $Ax = b$

$$A = LU = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}_{U} = \vec{b} \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

1) Set $Ux = y$, Solve $Ly = b$

$$\underbrace{\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & 2 & 1 & \\ & & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}}_{y} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} y_1 = 2 \\ 2 + y_2 = 3,\ y_2 = 1 \\ 2 + y_3 = 2,\ y_3 = 0 \\ 0 + y_4 = 0,\ y_4 = 0 \end{matrix} \rightarrow y = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

2) Solve $Ux = y$

$$\underbrace{\begin{pmatrix} 1 & & \\ & 2 & 1 \\ & & 2 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{y} \rightarrow \begin{matrix} x_1 = 2 \\ 2x_3 = 0,\ x_3 = 0 \\ 2x_2 + x_3 = 1,\ x_2 = 1/2 \end{matrix} \rightarrow x = \begin{pmatrix} 2 \\ 0 \\ 1/2 \end{pmatrix}$$

Computing LU Factorization

why/how?

If A can be row reduced to echelon U without changing rows, $E_p \cdots E_1 A = U \rightarrow L L^{-1} A = LU$ ⟶ $\underbrace{E_1 E_2}_{\text{elementary row operations}} E_3 A = U$

↳ E_j matrices perform elementary row operations - b/c rows were not swapped, E_j is lower triangular/invertible

↳ Ex. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

↳ So, $A = \underbrace{E_1^{-1} \cdots E_p^{-1}}_{L} U = LU$

$L^{-1} A = U$

$A = LU$

Why? ⟶ AND $L^{-1}L = I$, then $E_3 E_2 E_1 L = I$

algorithm

1) Reduce A to echelon form U by row replacement

2) Place entries in L, so same sequence of row operations reduces L to I

$$A = \begin{pmatrix} 4 & -3 & -1 & 5 \\ -16 & 12 & 2 & -17 \\ 8 & -6 & -12 & 22 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 + 4R_1} \begin{pmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & -10 & 12 \end{pmatrix} \xrightarrow{R_3 - 5R_2} \begin{pmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -3 \end{pmatrix} = U$$

$L = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix}$ → could check by doing LU, should $= A$

Leontif Input-Output Model

An "economy" with N sectors with outputs as a vector with $\mathbb{R}^N$ ← rows in 1 column, variables

↳ $\vec{x}$ = output vector

↳ x_i = entry i of vector $\vec{x}$

= number of units produced by sector i

consumption matrix C – describes how units are consumed by sectors to provide an output

↳ Defining C – Sector i sends a proportion of units to sector j, called $c_{i,j}(x_i)$

↳ sector j requires a proportion of units from sector i, called $c_{i,j}(x_i)$

↳ entries of C are between 0 and 1 and $C\vec{x}$ = units consumed | $\vec{x} - C\vec{x}$ = units left after internal consumption

Ex

An economy has 3 sectors: E, W, M

↳ for every 100 units of output,

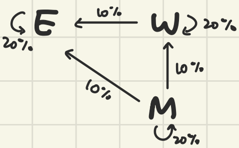
↳ E needs 20 from E, 10 from W, 10 from M

↳ W needs 0 from E, 20 from W, 10 from M

↳ M needs 0 from E, 0 from W, 20 from M

↳ output vector is $\vec{x} = \begin{pmatrix} x_E \\ x_W \\ x_M \end{pmatrix}$, construct a consumption matrix

Example Economy (internal consumption)



Units consumed $= C\vec{x} = x_E(\;) + x_W(\;) + x_M(\;)$ (← matrix, ← vector)

$x_E\begin{pmatrix} .2 \\ .1 \\ .1 \end{pmatrix} + x_W\begin{pmatrix} 0 \\ .2 \\ .1 \end{pmatrix} + x_M\begin{pmatrix} 0 \\ 0 \\ .2 \end{pmatrix} = C\vec{x}$

$C = \begin{pmatrix} .2 & 0 & 0 \\ .1 & .2 & 0 \\ .1 & .1 & .2 \end{pmatrix}\vec{x}$

or

$C = \frac{1}{10}\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

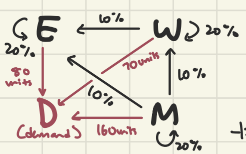
$$C = \frac{1}{10}\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{matrix} \to E \text{ (output is 2)} \\ \to W \text{ (output is 3)} \\ \to M \text{ (output is 4)} \end{matrix}$$

1) total output for each sector is sum of outgoing edges for each sector

2) Elements of C have no units; they are percentages

3) output vector has units $(\vec{x})$

Example Economy (external demand)



Solve $(I-C)\vec{x} = \vec{d}$

$\frac{1}{10}\begin{pmatrix} 8 & 0 & 0 \\ -1 & 8 & 0 \\ -1 & -1 & 8 \end{pmatrix}\vec{x} = \begin{pmatrix} 80 \\ 70 \\ 160 \end{pmatrix}$

$8x_1 = 800$ | $x_1 = 100$

$-1x_1 + 8x_2 = 700$ | $x_2 = 100$

$-1x_1 - 1x_2 + 8x_3 = 1600$ | $x_3 = \frac{1800}{8} = 225$

$\vec{x} = \begin{pmatrix} 100 \\ 100 \\ 225 \end{pmatrix}$ (output model)

external demand – is there an $\vec{x}$ such that $\vec{x} - C\vec{x} = \vec{d}$?

↳ $(I - C)\vec{x} = \vec{d}$ (Leontif Input-Output Model)

↳ Solving for $\vec{x}$ gives output that meets external demand exactly

$\vec{x} = (I - C)^{-1}\vec{d}$

Homogeneous Coordinates

homogeneous coordinates – used to model translations using matrix multiplication

↳ each point (x,y) in $\mathbb{R}^2$ can be identified with point (x, y, H), $H \neq 0$ (usually, $H=1$) on the plane in $\mathbb{R}^3$ that lies H units above the xy-plane

↳ translation of form $(x,y) \to (x+h, y+k)$ can be represented by matrix multiplication:

$$\underset{\text{translation}}{\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}} \underset{\text{Point}}{\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}} = \underset{\text{representation}}{\begin{pmatrix} x+h \\ y+k \\ 1 \end{pmatrix}}$$

Ex

Triangle S has $(1,1), (2,4), (3,1)$

↳ Transform T rotates points by $\frac{\pi}{2}$ radians counterclockwise around $(0,1)$

1) Rep data in a matrix, D
2) Use matrix multiplication to find image of S
3) Sketch S and image under T

Points ↓

$$D = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} D \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix} = D'$$

h ↗; T; (shift down 1), rotation around (0,1)

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \underset{D''}{\begin{pmatrix} 0 & -3 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}}$$

← rotation

(rotation around 0,0)

(shift up 1)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} D'' = \begin{pmatrix} 0 & -3 & 0 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{pmatrix} = TD$$

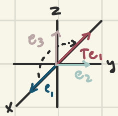
3D Transformations

$(X, Y, Z, 1)$ are homogeneous representation for (x, y, z) in R^3

$$\begin{pmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \\ z+l \\ 1 \end{pmatrix}$$

$(x, y, z) \to (x+h, y+k, z+l)$

Example



a) Translation in R^3 by vector $\vec{p} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$ $\longrightarrow$ $\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

b) Rotation in R^3 around x_2-axis by π radians $\longrightarrow$ $T = A\vec{x}$, $A = (a_1, a_2, a_3)$, $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\to T(e_1) = Ae_1 = 1a_1 + 0a_2 + 0a_3 = \underline{a_1} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$T(e_2) = Ae_2 = a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$T(e_3) = Ae_3 = a_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ $\quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

c) Projection onto plane $x_3 = 4$

↳ use homogeneous coordinates: $\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Shift (up 4)}}\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{projection } (x_3=0 \text{ plane})}\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Shift (down 4)}} = \underbrace{T}_{\text{Projection onto } x_3=4}$

Shift

Subsets and Subspaces

Subset of R^n is any collection of vectors in R^n ($n \times m$ matrix)

↳ span of 3×4 matrices is a subset of R^3

↳ set of all vectors of the form $\begin{pmatrix} 1 \\ k \end{pmatrix}$ is a subset of R^2

(H of R^n)

Subspace is a type of subset that is closed under scalar multiplication / vector addition

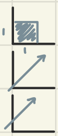
↳ for any c (real number) and for $\vec{u}$ and $\vec{v} \in H$, (vectors in subset)

(H must include 0 vector)

↳ $c\vec{u}$ is in subset/subspace H AND $\vec{u} + \vec{v}$ is in subset/subspace H

Example

Which is a subspace of R^2?



a) unit square — No! — if $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{u} \cdot 100 = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$, outside of H

b) line through origin — Yes!

c) line not through origin — No! MUST include 0 vector!

Set Builder Notation $\to$ $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in R^2 \mid ab = 0 \right\}$

a) 2 vectors in $V \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) 2 vectors not in $V \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

c) Is 0 vector in V? $\to$ Yes, $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \to 0 \cdot 0 = 0$

d) Is V a subspace? $\to$ No, $\underset{\vec{u}}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ and $\underset{\vec{v}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ are in V, but $\vec{u} + \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, not in V

Column and Null Spaces

for $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_p$ in R^n, Span $\{\vec{v}_1 \ldots \vec{v}_p\}$ is the set of all possible linear combinations $\vec{v}_j$

↳ This is a subspace spanned by $\vec{v}_1 \ldots \vec{v}_p$

Given $m \times n$ matrix $A = [\vec{a}_1 \ldots \vec{a}_n]$

↳ column space (Col A) of A is the subspace of R^m spanned by $\vec{a}_1 \ldots \vec{a}_n$

↳ null space (Null A) of A is the subspace of R^n spanned by all vectors $\vec{x}$ that solve $A\vec{x} = 0$

Vector in column space of A and a vector in null space of A

$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} -4 \\ -3 \\ 1 \\ 1 \end{pmatrix}$ ← make every row = 0

↖ any column is in column space of A

Matrix with column space spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and null space $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\longrightarrow$ $A\vec{x} = 0$

$A \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Example — is $\vec{b}$ in column space of A?

$A = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$ $\to$ $\underline{b}$ has to be in span of $\{a_1, a_2, a_3\}$

$c_1\begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} + c_2\begin{pmatrix} -3 \\ 6 \\ 7 \end{pmatrix} + c_3\begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$

$\left(\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right) \to$ if consistent, $\vec{b}$ is in column space

↳ if not, No!

$\begin{pmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Consistent!

is $\vec{v}$ in null space of A?

$A = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \end{pmatrix}$, λ is real $\Big\}$ if $\vec{v}$ is in null space, $A\vec{v} = 0$

$-5 + 9 - 4 = 0$
$20 - 18 - 2 = 0$
$15 - 21 + 6 = 0$, so $\vec{v}$ is in null space of A

Basis of a Subspace

basis of a subspace is the set of linearly independent* vectors in *H that span *H

↳ Ex. $H = \{\vec{x} \in \mathbb{R}^4 \mid x_1 - 3x_2 - 5x_3 + 7x_4 = 0\}$ is a subspace — H is a null space for (a) what matrix A? (b) Construct a basis for H

$A\vec{x} = 0$ $\quad \overset{A}{(1\ -3\ -5\ 7)}\overset{\vec{x}}{(\vec{x})} = 0$ $\quad x_1 = 3x_2 + 5x_3 - 7x_4$

$A\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{\vec{x}} = 0$

$x = \begin{pmatrix} 3x_2 + 5x_3 - 7x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \rightarrow x_2\underbrace{\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + x_3\underbrace{\begin{pmatrix} 5 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{v_2} + x_4\underbrace{\begin{pmatrix} -7 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_3}$ ← has to be linearly independent

v_1, v_2, v_3 are IN H AND independent

basis of A is the set $\{v_1, v_2, v_3\}$

↳ Ex. Construct a basis for Null A and Col A

$A = \begin{pmatrix} -3 & 6 & -1 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & 5 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow$ Null A $\rightarrow$ $A\vec{x} = 0$ $\begin{pmatrix} 1 & -2 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ $\quad x_1 - 2x_2 = 0$, $x_3 = 0$ $\quad x = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$x_1 - 2x_2 = 0 \rightarrow x_1 = 2x_2$
x_2 is free
$x_3 = 0 \rightarrow x_3 = 0$
x_4 is free

$x = x_2\overset{v_1}{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}} + x_4\overset{v_2}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$ $\quad v_1, v_2$ is basis for Null A $\{v_1, v_2\}$

basis for Col A are columns for pivot points ← Col A → $\left\{\underbrace{\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}}_{v_2}\right\}$

Coordinate Systems

Many possible choices for basis for a subspace! (Different Properties)

↳ if $B = \{\vec{b}_1, \dots \vec{b}_p\}$ is a basis for subspace H, and $\vec{x}$ is in H, coordinates of $\vec{x}$ relative B are scalars $c_1, \dots c_p$ so that $\vec{x} = c_1\vec{b}_1 + \dots c_p\vec{b}_p$

AND ↳ $\underset{\text{"x relative to B"}}{[x]_B} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ = coordinate vector of x relative B (B-coordinate vector)

Example

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$ → verify that $\vec{x}$ is in span of $B = \{\vec{v}_1, \vec{v}_2\}$ and calculate $[\vec{x}]_B$

$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad c_1 = 2, c_2 = 3,$ $\vec{x}$ is in span $\{v_1, v_2\}$ $= [\vec{x}]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Dimension of a Subspace

dimension (or cardinality) of a non-zero subspace H, dim H, is the # number of vectors in a basis of H, $\dim\{\vec{0}\} = 0$ (0 vector cannot be a basic vector)

↳ Ex. dimensions of the basis of Col A for each is 2

pivot columns $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

1) $\dim \mathbb{R}^n = n$

2) $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\dim(\text{Col } A) = 0$

3) dim (Null A) is the # of free variables

4) dim (Col A) is the # of basic variables / pivots

5) $H = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ has dimension 2

x_1 is represented by x_2 and x_3, so basis is 2 vectors, x_2 and x_3

↳ e.g. $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \dots \rightarrow \underbrace{(1, 1, 1)}_{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A\vec{x} = 0$, $\vec{x} = x_2(\vdots) + x_3(\vdots)$

H is a p-dimensional subspace of $\mathbb{R}^n$, any set of p-independent vectors that are in H are automatically a basis for H.

↳ Ex. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

two bases for Col A are:

$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$

b/c Col A has 2 dimensions

Rank and Invertibility

rank – dimension of its column space

↳ Ex. $A = \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{pmatrix} \sim \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

3 pivot columns, rank = 3
2 free variables, Null A = 2 (dim)
} 5 columns

If A has n columns, rank A + dim (Null A) = n

Let A be a $n \times n$ matrix. These conditions are equivalent.

1. A is invertible.
2. The columns of A are a basis for $\mathbb{R}^n$.
3. $\text{Col } A = \mathbb{R}^n$. (all columns are pivotal)
4. $\text{rank } A = \dim(\text{Col } A) = n$.
5. $\text{Null } A = \{\vec{0}\}$. (no free variables)

Example

2x3 matrix in RREF

a) rank = 3 $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ not possible

b) rank = 2 $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ possible

c) dim(Null A) = 2 $\quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ possible

d) Null A = $\{\vec{0}\}$ not possible, bc rank would HAVE to be 3

e) A is 2x2, invertible, rank = 1 — not possible, rank has to be 2 b/c all columns need to have a pivot

f) A is 4x4, invertible, rank = dim (Null A) — not possible, not all columns are pivotal