Math 2552: Differential Equations

↳ Chapters 4 & 6

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Author's Note

↳ This is solely covering the topics of the lecture videos, so I recommend you read the textbook in regards to working out homework

↳ Pretty much all info regarding computer science has been omitted (i.e. functions)

↳ Lecture numbers are on the left of titles

↳ Some videos have been combined across headings

6.1 Matrix Functions and Converting n^{th} Order DEs

Matrix Functions – matrices whose elements are functions

↳ $P(t) = \begin{pmatrix} P_{11}(t) & \cdots & P_{1n}(t) \\ \vdots & & \vdots \\ P_{n1}(t) & \cdots & P_{nn}(t) \end{pmatrix}$ → Differentiation: element ij of $\frac{dP}{dt} = \frac{d}{dt} P_{ij}$

→ Integration: ij of $\int_a^b P(t)\,dt = \int_a^b P_{ij}\,dt$

Express n^{th} order DE as a linear system

↳ $y^{(4)} + y = \sin(t)$

$x_1 = y$, $x_2 = y'$, $x_3 = y''$, $x_4 = y'''$

$x_1' = y' = x_2$

$x_2' = y'' = x_3$

$x_3' = y''' = x_4$

$x_4' = y'''' = -x_1 + \sin(t)$

$\vec{x}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin(t) \end{pmatrix}$

6.2.1-2 Basic Theory of 1^{st} Order Linear Systems

If P and $\vec{g}$ are continuous on (α, β), $t_0 \in (\alpha, \beta)$, then there is a unique solution to the IVP: $\vec{x}' = P(t)\vec{x} + g(t)$, $\vec{x}(t_0) = \vec{x}_0$

↳ Example: Identify an interval on which a unique solution exists: $(t-2)y'' + 3y = t$, $y(0) = 0$, $y'(0) = 1$

↳ $x_1 = y$, $x_2 = y'$ → $x_1' = x_2$, $x_2' = \frac{t}{t-2} - \frac{3x_1}{t-2}$ → $\vec{x}' = \begin{pmatrix} 0 & 1 \\ \frac{-3}{t-2} & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ \frac{t}{t-2} \end{pmatrix} = P\vec{x} + \vec{g}$

↳ P, g continuous on $(-\infty, 2)$ – unique solution on $(-\infty, 2)$

6.2.3-6 Linear Independence of Functions / Wronskian

↳ Determine whether the functions are linearly independent

↳ $y_1 = e^t$, $y_2 = e^{-2t}$, $y_3 = 3e^t - 2e^{-2t}$ → by inspection, $y_3 = 3y_1 - 2y_2$, so dependent

↳ $y_1 = t$, $y_2 = t^2$, $y_3 = 1 - 2t^2$ → if y_1, y_2, y_3 are independent, then $c_1y_1 + c_2y_2 + c_3y_3 = 0$ for only $c_1 = c_2 = c_3 = 0$, and $c_1y_1' + c_2y_2' + c_3y_3' = 0$ (and the 2nd $\frac{d}{dt}$)

↳ $\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

↳ if $\det \begin{pmatrix} t & t^2 & 1-2t^2 \\ 1 & 2t & -4t \\ 0 & 2 & -4 \end{pmatrix} = 0$, then linearly dependent

THE WRONSKIAN!

↳ cofactor expansion later... $\neq 0$, independent

Theorem: Let $y_1, y_2, \dots y_n$ be solutions of $y^{(n)} + p_1 y^{(n-1)} + \dots + p_n y = 0$ on an interval I, in which p is continuous. If for some $t_0 \in I$ their Wronskian is non-zero, then every solution of the DE, $\phi(t)$ can be written as $\phi(t) = \sum_{i=1}^{n} c_i y_i(t)$

↳ $y_1, y_2, \dots, y_n$ form a fundamental set of solutions for the DE

↳ n solutions to n^{th} order DE

↳ these n solutions are linearly independent

↳ Any solution is a linear combination of y_i, so y_i spans the solution space

6.2.5-6 Linear Independence of Vector Functions

Example: Are the functions linearly independent?

$\vec{y}_1(t) = \begin{pmatrix} t \\ 1-t \\ 0 \end{pmatrix}$, $\vec{y}_2(t) = \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix}$, $\vec{y}_3(t) = \begin{pmatrix} t \\ 3-t \\ 2t \end{pmatrix}$ $\qquad c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$

↳ by inspection, $\vec{y}_3(t) = 2y_2 + y_1$, so dependent (when $c_1 = 1$, $c_2 = 2$, and $c_3 = -1$)

Example: Do $\vec{x}_1(t) = e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{x}_2 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ form a fundamental set of solutions for $\vec{x}' = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\vec{x}$

↳ You may assume that $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are solutions

↳ $\det\begin{pmatrix} e^{5t} & -e^{-t} \\ e^{5t} & e^{-t} \end{pmatrix} = e^{5t}e^{-t}\det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \neq 0$, linearly independent, so forms a <u>fundamental set</u>

6.3.1-4 Homogeneous Systems with Constant Coefficients

Reminder: geometric multiplicity of eigenvalue — # of eigenvectors ($1 \leq$ geo $\leq$ algebraic)

↳ algebraic multiplicity — # of times the eigenvalue repeats

Consider the linear, 1^{st} order, constant coefficient system, $\vec{x}' = P\vec{x}$, $\vec{x} = \vec{x}(t)$, $P \in \mathbb{R}^{n \times n}$

↳ assume solution of form $\vec{x}(t) = e^{\lambda t}\vec{v}$

↳ 3 Cases (Section 6.3, 6.4, 6.7 ← not covered)
$\mathbb{R}$ $\quad \mathbb{C}$ $\quad \mathbb{C}$, defective

defective matrix: Sum of eigenvectors (geometric multiplicity) $\neq$ # of columns

↳ nondefective matrix: geo mult = # of columns

Case 1: Real λs, nondefective A

If $(\lambda_1, \vec{v}_1), \ldots (\lambda_n, \vec{v}_n)$ are eigenpairs for $n \times n$ matrix A, and A has n linearly independent eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$, then $\vec{x}' = A\vec{x}$ has:

↳ A general solution → $\vec{x} = \sum_{i=1}^{n} c_i e^{\lambda_i t}\vec{v}_i$

↳ A fundamental set of solutions → $\{e^{\lambda_i t}\vec{v}_i\}$

Example: Find solution to $\vec{x}' = P\vec{x}$ ⟶ $P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ $\qquad \lambda = 0, 1$

↳ $\lambda_1 = 0$: $P - 0I = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, set x_3 free, if $x_3 = -1$, $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

↳ $\lambda_2 = 1$: $P - I = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, set x_2, x_3 free → $x_2 = 1, x_3 = 0$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$x_1 = x_2 - x_3$ → $x_2 = 0$, $x_3 = 1$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

↳ General Solution: $\vec{x} = c_1 e^{0t}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

6.4.1 Nondefective Matrices with Complex Eigenvalues

Review procedure → compute eigenvalues $\lambda = \alpha \pm i\beta$

→ eigenvectors $\vec{v} = \vec{a} \pm i\vec{b}$

→ general solution: $\vec{x} = c_1 e^{\alpha t}[\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)] + c_2 e^{\alpha t}[\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)]$

note: On quizzes and tests if dimension > 2, the eigenvalues will be given

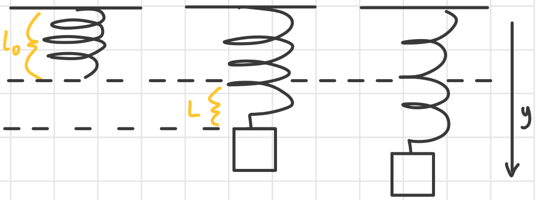
↳ Additional examples are provided for 6.4.E1-5

4.1.1-4 Spring-Mass Systems

3 cases → Spring of length L_0 attached to horizontal surface

→ Mass M attached to spring, spring length in equil. position is $L + L_0$

→ External force added to extend spring



Case 2: $F_g - F_s = 0$

↳ $mg - kL = 0$

Case 3: external force $= f(t)$

↳ $my'' = F_g + F_s + f$

$= mg + k(L+y) + f$

$my'' = -ky + f$

Case 4: damping force $= F_d = -\gamma y'$

↳ $my'' = mg - k(L+y) + f - \gamma y'$

↳ $my'' + \gamma y' + ky = f$

Example: Construct initial value problems → 2lb mass, stretches 4in, released 3in above equil.

↳ $mg - ky = 0$, $ky = mg$, $k\left(\frac{4}{12}\right) = 2(32)$, $k = 192$

↳ $my'' = mg - ky$, where $m = 2$, $k = 192$, $y(0) = -\frac{3}{12} = -\frac{1}{4}$

Example: Construct → 0.25kg mass, stretches 0.5m, damped with constant 0.01 Ns/m, released at downward velocity 4 m/s at 0.01m below equilibrium

↳ $mg - ky = 0$, $(0.25)(10) = k(0.5)$, $k = 5$

↳ $my'' = mg - ky - \gamma y'$, where $m = 0.25$, $k = 5$, $\gamma = 0.01$, $y(0) = 0.01$, $y'(0) = 4$

4.2.1-3 Second Order Linear Homogeneous Equations

Recall: Existence and Uniqueness → If $p, q,$ and g are continuous on open interval I, $t_0 \in I$, there is a unique solution to the IVP: $y'' + p(t)y' + q(t) = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_1$

Example: Find largest interval in which the solution to the IVP is certain to exist

↳ $(t-2)^2 y'' + ty' + \sqrt{t}\,y = 0$, $y(1) = 1$, $y'(1) = 2$

↳ $y'' + \frac{ty'}{(t-2)^2} + \frac{\sqrt{t}\,y}{(t-2)^2} = 0$, $t \geq 0$ and $t \neq 2$, $t = 1$ is in interval, so $t \in [0, 2)$

Recall: Fundamental Set of Solutions → Let y_1, y_2, be solutions of $y'' + py' + qy = 0$ on an interval, I, in which p is continuous. If for some $t_0 \in I$ their Wronskian is non-zero, then every solution of the DE, $\phi(t)$ can be written as $\phi(t) = c_1 y_1 + c_2 y_2$

↳ Given 1 solution, y_1, the reduction of order allows us to derive y_2 such that $\{y_1, y_2\}$ forms a fundamental set

Example: $y_1(t)$ is a solution to $y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0$ → Try $y_2 = y_1 \underset{\text{something we don't know}}{v(t)} = tv$

↳ We want to substitute into ①, need y' and y''

↳ $y_2' = \frac{d}{dt}(tv) = v + tv'$ $\xrightarrow{\text{substitute}}$ $y'' - \frac{1}{t}y' + \frac{1}{t^2}y = (2v' + tv'') - \frac{1}{t}(v + tv') + \frac{1}{t^2}(tv)$

↳ $y_2'' = \frac{d}{dt}(v + tv') = 2v' + tv''$

$= tv'' + v' = 0$

$= \int \frac{d}{dt}(v' t) = \int 0 \, dt$

$= v' t = k_1 \rightarrow \int \frac{k_1}{t} = \int v' \rightarrow k_1 \ln(t) + k_2$

↳ Solution is $y = c_1 y_1 + c_2 y_2 = c_1 t + c_2 t(k_1 \ln(t) + k_2)$

↳ k_1 and k_2 are arbitrary, so $y = c_1 t + c_2 t \ln t$

4.3.1-2 Linear Homologous Equations with Constant Coefficients

Consider $ay'' + by' + cy = 0$ (constant coefficients)

↳ Find / solve $\vec{x}' = A\vec{x} \rightarrow x_1 = y \quad x_1' = x_2$

$x_2 = y', \quad x_2' = -\frac{b}{a}x_2 - \frac{c}{a}x_1 \rightarrow \vec{x} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \vec{x}$

↳ eigenvalues: $|A - \lambda I| = -\lambda\left(-\lambda - \frac{b}{a}\right) + \frac{c}{a} = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0$

$a\lambda^2 + b\lambda + c = 0$

$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

↳ eigenvectors: $A - \lambda_{1,2} I = \begin{pmatrix} -\lambda_{1,2} & 1 \\ \cdot & \cdot \end{pmatrix} \rightarrow -\lambda_{2,1} x_1 + x_2 = 0$

unecessary

set $x_1 = 1$, then $x_2 = \lambda_{1,2}$

↳ $\vec{v}_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

↳ general solution: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} = c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

↳ 1st component: $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

Alternative Approach: Try $y = e^{\lambda t}$, λ could be complex $\rightarrow y'' = \lambda^2 e^{\lambda t}, \; y' = \lambda e^{\lambda t}, \; y = e^{\lambda t}$

↳ $\frac{a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + c e^{\lambda t} = 0}{e^{\lambda t}} \rightarrow a\lambda^2 + b\lambda + c = 0$, roots are (again) $\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

↳ Same solution, $\vec{x} = c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

Theorem: General Solution — if λ_1 and λ_2 are the roots of $ay'' + by' + cy = 0$

↳

λ_1, λ_2	general solution
real distinct	$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$
real repeated, $\lambda_1 = \lambda_2$	$c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_2 t}$
complex, $\lambda_1 = \alpha + i\beta$	$e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$

4.3.3-4 Complex λ Case

Consider the spring-mass system with no forcing, no damping, $k = 4$

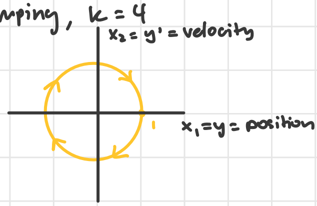
↳ $y'' + 4y = 0, \quad y(0) = 1, \; y'(0) = 0$

↳ $\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\vec{x} = \vec{x}'$

↳ Phase portrait can be sketched without solving

↳ system eigenvalues: $\lambda^2 + 4 = 0, \quad \lambda = \pm 2i$

↳ eigenvectors: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i\begin{pmatrix} 0 \\ 2 \end{pmatrix}$



↳ $\vec{x}(t) = c_1\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ 2 \end{pmatrix}\sin 2t\right] + c_2\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 2t + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\cos 2t\right] = c_1 \cos 2t + c_2 \sin 2t$

↳ alternate method, $y=e^{\lambda t} \to \dfrac{\lambda^2 e^{\lambda t} + 4e^{\lambda t} = 0}{e^{\lambda t}} \to \lambda^2 + 4 = 0$

Theorem 4.3.2

↳ $\alpha = 0$, $\beta = 2$, so $\vec{x} = e^{0t}(c_1 \cos 2t + c_2 \sin 2t) = c_1 \cos 2t + c_2 \sin 2t$

↳ Solve IVT: $y(0) = 1 = c_1 \cos 0 + c_2 \sin 0$, $c_1 = 1$

↳ $y'(0) = 0 = -2c_1 \sin 0 + 2c_2 \cos 0$, $c_2 = 0$ → $y = \cos(2t)$

4.3.5 Cauchy-Euler Equation

A DE of form $\left[ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0\right]$ is a 2nd-order Cauchy-Euler Equation

↳ Solve: use solution of form $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

↳ Substitution yields $am^2 + (b-a)m + c = 0$ (divide by x^m at the end)

↳ let roots be λ_1, λ_2 → distinct, real: $y = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}$

→ repeated: $y = c_1 x^{\lambda_1} + c_2 x^{\lambda_1} \ln x$

→ complex: $y = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right]$, $\lambda = \alpha + i\beta$

4.4.1 Mechanical Vibrations

Consider an unforced, undamped system: $my'' + ky = 0$

↳ $y'' + \frac{k}{m}y = 0$, $\omega_0 = \sqrt{\frac{k}{m}} > 0$, let $y = e^{\lambda t} \to \lambda^2 + \frac{k}{m} = 0$, so $\lambda = \pm\sqrt{\frac{k}{m}}\, i$ or $\pm \omega_0 i$

↳ $y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

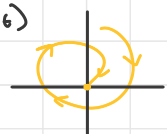
↳ let $c_1 = R\cos\delta$, $c_2 = R\sin\delta \to y = R\cos\delta \cos\omega_0 t + R\sin\delta\sin\omega_0 t$

↳ $y = R\cos(\omega_0 t - \delta)$, R = Max amplitude, ω_0 = natural frequency, δ = phase shift

Consider a damped system: $my'' + \gamma y' + ky = 0$

↳ Let $y = e^{\lambda t}$, DE becomes $m\lambda^2 + \gamma\lambda + k = 0$, so $\lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

roots	motion	occurs when	motion
$\mathbb{C}$	underdamped	$\gamma^2 - 4km < 0$	$e^{\alpha t}(A\cos\beta t + B\sin\beta t)$
$\mathbb{R}$, repeated	critically damped	$\gamma^2 - 4km = 0$	$(A + Bt)e^{\lambda_1 t}$
$\mathbb{R}$, distinct	overdamped	$\gamma^2 - 4km > 0$	$Ae^{\lambda_1 t} + Be^{\lambda_2 t}$



reaches equil.

↳ $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \text{position} \\ \text{velocity} \end{pmatrix}$

4.5.1-4 Nonhomogeneous Equations

We seek solutions to nonhomogeneous problem: $y'' + p(t)y' + q(t)y = g(t)$ *

↳ corresponding homogeneous problem: $y'' + p(t)y' + q(t)y = 0$

↳ if Y_1 and Y_2 are solutions to *, then difference is: $Y_2 - Y_1$ (this is <u>a</u> solution to homogeneous problem)

↳ so, $Y_1'' + pY_1 + qY_1 = g$ and $Y_2'' + pY_2' + qY_2 = g$

↳ general solution: $Y_2 - Y_1 = 0$ is a solution to homogeneous

↳ $Y_2 - Y_1 = c_1 y_1 + c_2 y_2 \to Y_2 = \underbrace{c_1 y_1 + c_2 y_2}_{\text{solution to homog.}} + Y_1$ ← any particular solution to nonhomog.

Steps to solve a Nonhomogeneous DE

① Solve homogeneous problem

② Find <u>particular</u> solution using → method of undetermined coefficients
↳ variation of parameters

③ Add functions found in previous 2 steps to find general solution

Example: Find a solution for $y'' + 3y' + 2y = 10e^{3t}$

↳ does $y_p = e^{3t}$ work? ×, $\frac{d^2}{dt^2}(e^{3t}) + 3\frac{d}{dt}(e^{3t}) + 2(e^{3t}) = 20e^{3t} \neq 10e^{3t}$

↳ try $y_p = \underline{A}e^{3t}$, $\underline{A}$ will $= \frac{1}{2}$, solving the equation

Example: $y'' + 3y' + 2y = \sin t$

↳ try $y_p = A\sin t$ ×, $\sin t(-A+2A) + \cos t(3A) \neq \sin t$

↳ try $y_p = A\cos t + B\sin t$, A will $= -\frac{3}{10}$, $B = \frac{1}{10}$

Example: $y'' - 6y' + 9y = e^{3t}$

↳ try $y = Ae^{3t}$ ×, does not work because e^{3t} is a solution to homog. problem

↳ try $y = tAe^{3t}$ ×, does not work because

↳ try $y = t^2Ae^{3t}$, works because consider $y' + y = e^{3t}$, integrating factor is $\mu = e^t$, $\frac{d}{dt}(e^t y) = 1$, so $e^t y = t + C$, $y = te^{-t} + Ce^{-t}$

particular ↗ ↖ solution to homog.

General Strategy (for method of undetermined coefficients)

1) Solve general solution for homog.
2) Can undetermined coefficients be used?
3) If $g(t) = \Sigma_i^n g_i$, then consider each of n sub-problems separately
4) Solve 1^{st} sub-problem: assume particular form, determine coefficients
5) Repeat 4) for each sub-problem
6) Form general solution
7) Solve IVP

Table of Particular Solutions

s is the smallest non-negative integer so that Y is a solution

$g(t)$	Particular solution $Y(t)$
$P_n(t)$	$t^s Q_n$
$P_n(t)e^{\alpha t}$	$t^s e^{\alpha t} Q_n$
$P_n(t)e^{\alpha t}\sin(\beta t)$	$t^s e^{\alpha t}(\cos(\beta t)Q_n + \sin(\beta t)R_n)$
$P_n(t)e^{\alpha t}\cos(\beta t)$	$t^s e^{\alpha t}(\cos(\beta t)Q_n + \sin(\beta t)R_n)$

$P_n(t) = a_0t^n + a_1t^{n-1} + \ldots + a_n$

$Q_n(t) = A_0t^n + A_1t^{n-1} + \ldots + A_n$

$R_n(t) = B_0t^n + B_1t^{n-1} + \ldots + B_n$

(when in doubt, multiply by t!)

4.6.1-4 Forced Vibrations

↳ The damped, forced spring system: $y'' + 3y' + 2y = \sin t$
↳ has solution: $y = \underbrace{c_1 e^{-2t} + c_2 e^{-t}}_{\text{homogeneous solution}} + \underbrace{0.1\sin t - 0.3\cos t}_{\text{particular solution}}$

homogeneous solution (transient solution) goes to 0 as $t \to \infty$ bc of damping ($3y'$)

particular solution (steady-state) does not go to 0 as $t \to \infty$

Undamped Oscillator — consider $y'' + \omega_0^2 y = F_0 \cos(\omega t)$, $y(0) = 0, y'(0) = 0, F_0 > 0$
↳ 2 Cases: $\omega \neq \omega_0$ (beats) or $\omega = \omega_0$ (resonance) $\omega_0^2 = \frac{k}{m}$, ω characterizes forcing
↳ Case 1, $\omega \neq \omega_0$ → homogeneous: $y_h = e^{\lambda t}$, $\lambda = 0 \pm i\omega_0$, so $y_h = c_1\cos(\omega_0 t) + c_2 \sin(\omega_0 t)$
↳ particular: $y_p = A\cos(\omega t) + B\sin(\omega t)$ → $-A\omega^2\cos(\omega t) - B\omega^2 \sin(\omega t) + \omega_0^2(y_p) = F_0\cos(\omega t)$
$y_p' = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$ ↳ B must be 0, $A = \frac{F_0}{\omega_0^2 - \omega_0^2}$
$y_p'' = -A\omega^2\cos(\omega t) - B\omega^2 \sin(\omega t)$ ↳ $y_p = \frac{F_0}{\omega_0^2 - \omega_0^2}\cos(\omega t)$
↳ general solution: $y = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2}\cos(\omega t)$
IVP ↳ $0 = y(0) = c_1\cos(0) + \frac{F_0}{\omega_0^2-\omega^2}\cos(0)$, so $c_1 = -\frac{F_0}{\omega_0^2 - \omega^2}$
↳ $0 = y'(0) = c_1 \cdot 0 + c_2\omega_0$, $c_2 = 0$
↳ Final solution: $y = -\frac{F_0}{\omega_0^2-\omega^2}\cos(\omega_0 t) + \frac{F_0}{\omega_0^2-\omega^2}\cos(\omega t)$ → as frequencies approach one another, the amplitude increases to ∞
↳ Case 2: $\omega_0 = \omega$, $y_h = c_1\cos(\omega t) + c_2\sin(\omega t)$ ← same as case 1
↳ particular, $y_p = tA\cos(\omega t) + tB\sin(\omega t)$, because DE is $y'' + \omega y = F\cos(\omega t)$ → $A\sin(\omega t) + B\cos(\omega t)$ is already y_h
↳ general solution: $y = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{F}{2\omega} t\sin(\omega t)$
↳ as $t \to \infty$, $|y| \to \infty$, resonance

4.7.1-3 Variation of Parameters for 2nd Order DEs

Why? → undetermined coefficients doesn't give an explicit solution
↳ can only be used when $g(t)$ is a combo of sin, cos, exponential, polynomial

Variation of Parameters → we seek solns. to nonhomogeneous: $y'' + p(t)y' + q(t)y = g(t)$
↳ solution to homogeneous: $y_h = c_1 y_1(t) + c_2 y_2(t)$
↳ to find particular, replace c_1 and c_2 with functions: $y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$
↳ goal: find $v_1(t)$ and $v_2(t)$

explicit formula (textbook)

$$v_1 = -\int \frac{g(t)y_2(t)}{W(y_1,y_2)}, \quad v_2 = \int \frac{g(t)y_1(t)}{W(y_1,y_2)}$$

↳ $\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$

↳ solve system of nonlinear equations: $y_1 v_1' + y_2 v_2' = 0$
$y_1' v_1' + y_2' v_2' = g$
↳ integrate v_1' and v_2' to find v_1 and v_2
↳ set $y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$

Proof via derivation: $y_p = v_1 y_1 + v_2 y_2 \to y_p' = v_1 y_1' + v_2 y_2' + \overbrace{v_1' y_1 + v_2' y_2}^{\text{set to } 0}$
↳ $y_p'' = (v_1' y_1' + v_2' y_2') + v_1 y_1'' + v_2 y_2'' \to$ Substitute into $y_p'' + p y_p' + q y_p$

$\hookrightarrow v_1\underbrace{(y_1''+py_1'+qy_1)}_{=0} + v_2\underbrace{(y_2''+py_2'+qy_2)}_{=0} + \underbrace{v_1'y_1' + v_2'y_2' = g(t)}_{\text{becomes part of the system}}$

$=0 \leftarrow$ bc homogeneous $\rightarrow =0$

Ex. find y_p of $t^2y''-4ty'+6y = 4t^3$, $t>0$, given that $y_1=t^2$ and $y_2=t^3$ are y_h solns.

$\hookrightarrow y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 4t = g(t)$

$\hookrightarrow$ via system: $y_1v_1' + y_2v_2' = 0 \longrightarrow t^2v_1' + t^3v_2' = 0$

$y_1'v_1' + y_2'v_2' = g \longrightarrow 2tv_1' + 3t^2v_2' = 4t$

$\rightarrow \left(\begin{array}{cc|c} t^2 & t^3 & 0 \\ 2t & 3t^2 & 4t \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & t & 0 \\ 2 & 3t & 4 \end{array}\right)$

$\left(\begin{array}{cc|c} 1 & t & 0 \\ 2 & 3t & 4 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & 3t & 4 \\ 1 & t & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 0 & t & 4 \\ 1 & t & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & t & 4 \end{array}\right)$, so $v_1' = -4$, $tv_2' = 4$ $\Big\}$ $v_1 = -4t$, $v_2 = 4\ln t$

$\hookrightarrow v_2' = \frac{4}{t}$

$\hookrightarrow y_p = v_1y_1 + v_2y_2 = -4t\cdot t^2 + 4\ln t \cdot t^3$

4.7.4-6 Variation of Parameters for 1st Order System

Seek solutions to $\vec{x}' = P\vec{x} + \vec{g}(t)$, $P = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}$, $\vec{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$

$\hookrightarrow$ Solution is $\vec{x}_h = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

$\hookrightarrow$ replace c_1 and c_2 with $v_1(t)$ and $v_2(t)$

$\hookrightarrow$ Suppose solutions are $\vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$ $\rightarrow$ then introduce fundamental matrix $X(t)$

$\hookrightarrow X(t) = [\vec{x}_1, \vec{x}_2] = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$

derivation in textbook / 4.7.6

Particular Solution Formula: for $\vec{x}' = P\vec{x} + g(t)$, $\vec{x}_p = X(t)\int X^{-1}(t)\vec{g}(t)\,dt$

Ex. Find $\vec{x}_p$ given $\vec{x}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\vec{x}(t) + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}$, solns. to y_h are $\vec{x}_1 = e^t\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\vec{x}_2 = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\hookrightarrow X(t) = \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix}$, $\det X = 3e^te^{-t} - e^{-t}e^t = 3-1=2$

$\hookrightarrow X^{-1}(t) = \frac{1}{\det X}\begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 3e^t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 3e^t \end{pmatrix}$

$\hookrightarrow \vec{x}_p = X\int X^{-1}g\,dt = \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix}\int \frac{1}{2}\begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 3e^t \end{pmatrix}\begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}dt = \frac{1}{2}\begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix}\int \begin{pmatrix} e^t - e^{-t} \\ -e^{3t} + 3e^t \end{pmatrix}dt$

$\hookrightarrow$ eventually $\rightarrow \vec{x}_p = \frac{1}{2}\begin{pmatrix} 8/3\, e^{2t} + 6 \\ 2/3\, e^{2t} + 4 \end{pmatrix}$