Double Integrals Over Rectangles, Part I

If $f = f(x,y)$ is continuous on rectangle $R: a \le x \le b,\ c \le y \le d$ and P is a partition of R
↳ and M_{ij} are minimums/maximums of f in R_{ij}
↳ Lower Sum: $L_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta x_i \Delta y_j m_{ij}$
↳ Upper Sum: $U_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta x_i \Delta y_j M_{ij}$
↳ double integral of f over P is I that satisfies $L_f(P) \le I \le U_f(P)$ for all partitions P
↳ $I = \iint_R f(x,y)\,\underbrace{dx\,dy}_{dA}$

Fubini's Theorem → First Form: $\iint_R f(x,y)\,dA = \int_d^c \int_b^a f(x,y)\,dx\,dy = \int_b^a \int_d^c f(x,y)\,dy\,dx$

Evaluate $\int_0^3 \int_{-2}^0 (x^2y - 2xy)\,dy\,dx$ $\quad -2 \le y \le 0$, $\ 0 \le x \le 3$
↳ $x^2\frac{y^2}{2} - xy^2 \Big|_{-2}^{0} = [0-(2x^2-4x)]$
↳ $\int_0^3 [-2x^2+4x]\,dx = -\frac{2}{3}x^3 + 2x^2\Big|_0^3 = -18+18 = 0$

Double Integrals over Rectangles, Part II

Volume of solid region over xy-plane bounded by R and above by $f(x,y)$ is
↳ $V = \iint_R f(x,y)\,dA$

Double Integrals over General Regions, Part I

Fubini's Theorem (Stronger Form) — if R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$
↳ $\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$
↳ If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$
↳ $\iint_R f(x,y)\,dA = \int\int_{h_2(x)}^{h_1(x)} f(x,y)\,dx\,dy$

Evaluate $\iint_R x^3y\,dA$ where $0 \le x \le 1$, $0 \le y \le x$
↳ $\int_0^1\int_0^x x^3y\,dy\,dx = \int_0^1 \frac{1}{2}x^3y^2\Big|_0^x = \int_0^1 \frac{1}{2}x^3x^2 = \int_0^1 \frac{1}{2}x^5 = \frac{x^6}{6}\Big|_0^1 = \frac{1}{12}$

Double Integrals over General Regions, Part II

To find limits of integration, there are 2 ways:

1. Vertical cross sections:
↳ $\int_{x=a}^{x=b}\int_{y=g(x)}^{y=h(x)} f(x,y)\,dy\,dx$
2. Horizontal cross sections:
↳ $\int_{y=c}^{y=d}\int_{x=g(y)}^{x=h(y)} f(x,y)\,dx\,dy$

Evaluate $\int_0^2\int_{x^2}^4 2x\cos(y^2)\,dy\,dx$ by changing order of integration
↳ $x^2 \le y \le 4$, $0 \le x \le 2$
↳ $0^2 \le y \le 4$, $0 \le x \le \sqrt{y}$ → $\int_0^4\int_0^{\sqrt{y}} 2x\cos(y^2)\,dx\,dy = \int_0^4 x^2\cos(y^2)\Big|_0^{\sqrt{y}}$ → $\frac{1}{2}\sin(16)$

Find volume of solid bounded by coordinate planes + plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$
↳ $f(x,y) = z = 4 - 2x - \frac{4}{3}y$
↳ $0 \le x \le 2$, $0 \le y \le 3 - \frac{3}{2}x$ → $V = \int_0^2\int_0^{3-3/2x} (4 - 2x - \frac{4}{3}y)\,dy\,dx \to 4$

Double Integrals over General Regions, Part III

If $f(x,y)$ and $g(x,y)$ are continuous over the region R:

1) Constant Multiple

↳ $\iint cf(x,y)\,dA = c\iint f(x,y)\,dA$ for any number c

2) Sum and Difference

↳ $\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

3) Domination

↳ $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R

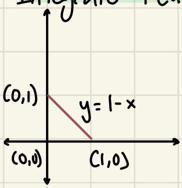
↳ $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on R

4) Additivity (If R is the union of two nonoverlapping regions R_1 and R_2)

↳ $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

Integrate $f(u,v) = v - 2\sqrt{u}$ over the triangular region cut from 1st Quadrant of uv-plane and line $u+v=1$

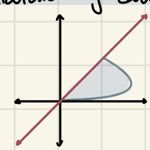
↳ $\int_0^1\int_0^{1-x} (y - 2\sqrt{x})\,dy\,dx \rightarrow \int_0^1 \left[\frac{y^2}{2} - 2yx^{1/2}\right]_0^{1-x} \rightarrow \frac{1}{2}x - \frac{x^2}{2} + \frac{1}{6}x^3 - \frac{4}{3}x^{3/2} + \frac{4}{3}x^{5/2} \Big|_0^1 = -\frac{11}{30}$



Area by Double Integration

Area of a closed/bounded region R is ⟶ $A = \iint_R dA$

Calculate by double integration the area bounded by curves $y = x$ and $x = 4y - y^2$



↳ $y = 4y - y^2 \rightarrow y^2 - 3y = 0 \rightarrow y = 0,\ y = 3$

↳ $0 \leq y \leq 3,\ y \leq x \leq 4y - y^2$

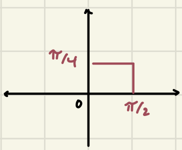
↳ $A = \int_0^3\int_y^{4y-y^2} dx\,dy \rightarrow \int_0^3 3y - y^2\,dy \rightarrow 9/2$

Average Value by Double Integration

Average Value of f over $R = \frac{1}{\text{area of } R}\iint_R f\,dA$

Find average value of $\cos(x+y)$ over $0 \leq x \leq \pi/2,\ 0 \leq y \leq \pi/4$

↳ area of R: $\pi^2/8 \rightarrow A = \frac{1}{(\pi^2/8)}\int_0^{\pi/4}\int_0^{\pi/2} \cos(x+y)\,dx\,dy \rightarrow \frac{8}{\pi^2}(\sqrt{2} - 1)$



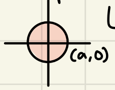
Double Integrals in Polar Form, Part I

Reminder: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $x^2 + y^2 = r^2$, $\theta = \arctan\left(\frac{y}{x}\right)$

Area of a sector: $\frac{\theta}{2\pi}\cdot\pi r^2 = \frac{1}{2}r^2\theta$

↳ $\int_\alpha^\beta \frac{1}{2}r^2\theta \longrightarrow \int_\alpha^\beta\int_a^b r\,dr\,d\theta$ ⟶ Area of a closed/bounded region R in polar plane is $A = \iint_R \overbrace{r\,dr\,d\theta}^{dA = dx\,dy}$

Use polar coordinates/double integration to find a formula for the area of a circle with radius a



↳ $\overset{\text{polar}}{r = a}$ or $\overset{\text{coordinate}}{x^2 + y^2 = a}$

↓

$0 \leq \theta \leq 2\pi$, $0 \leq r \leq a$ ⟶ $A = \int_0^{2\pi}\int_0^a r\,dr\,d\theta \rightarrow \int_0^{2\pi}\left(\frac{1}{2}r^2\right)_0^a d\theta \rightarrow \frac{1}{2}a^2\theta\Big|_0^{2\pi} = a^2\pi$

Double Integrals in Polar Form, Part II

Given $F=F(r,\theta)$ is continuous on $\Gamma: a \le r \le b, \alpha \le \theta \le \beta$

↳ $\iint_\Gamma F(r,\theta)\,r\,dr\,d\theta = \int_\alpha^\beta \int_a^b F(r,\theta)\,r\,dr\,d\theta$

Double integral over polar region $\Omega: \alpha \le \theta \le \beta,\ \rho_1(\theta) \le r \le \rho_2(\theta)$:

↳ $\iint_\Omega F(r,\theta)\,dr\,d\theta = \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} F(r,\theta)\,r\,dr\,d\theta$

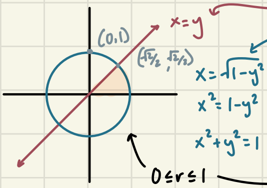
Volume of solid with R as base / bounded by $F(r,\theta)$: $V = \iint_R F(r,\theta)\,r\,dr\,d\theta$

Evaluate $\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} (x^2+y^2)^{3/2}\,dx\,dy$ by changing to polar form

↳ $x^2+y^2=r^2$

$0 \le y \le \sqrt{2}/2$

$y \le x \le \sqrt{1-y^2}$



$0 \le r \le 1$

$0 \le \theta \le \pi/4$

$\int_0^{\pi/4} \int_0^1 r^3 \cdot r\,dr\,d\theta \to \int_0^{\pi/4} \left[\tfrac{1}{5} r^5\right]_0^1 d\theta \to \int_0^{\pi/4} \tfrac{1}{5}\,d\theta$

$\to \tfrac{1}{5}\theta \Big|_0^{\pi/4} = \pi$

Find volume of solid bounded above by paraboloid $z=9-x^2-y^2$ and below xy-plane ($z=0$)

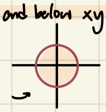
↳ $0 = 9-x^2-y^2 \to 9 = x^2+y^2 \to 9 = r^2 \to r=3$

↳ $F(r,\theta) = 9-r^2$

↳ $V = \int_0^{2\pi} \int_0^3 (9-r^2)\,r\,dr\,d\theta \to \int_0^{2\pi} \tfrac{81}{4}\,d\theta = \tfrac{81\pi}{2}$

$0 \le \theta \le 2\pi$

$0 \le r \le 3$



Triple Integrals

$\iiint_D f(x,y,z)\,dV = \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x,y,z)\,dx\,dy\,dz$

Calculate $\iiint_T z\,dV$, where T is the tetrahedron in the first octant bounded by the plane $x+y+z=1$

↳ $0 \le z \le 1-x-y$, $0 \le y \le 1-x$, $0 \le x \le 1$

↳ $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx = 1/24$

Volume with Triple Integrals

Volume of a closed / bounded region D is $V = \iint_D dV$

Find the volume of the solid bounded above by the plane $y+z=2$, below by the xy-plane, $x=6$, and $y=\sqrt{x}$

↳ $0 \le y \le 2$, $y^2 \le x \le 6$, $0 \le z \le 2-y$

↳ $V = \int_0^2 \int_{y^2}^6 \int_0^{2-y} dz\,dx\,dy = \frac{32}{3}$

Find volume of the solid bounded above by hemisphere $z=\sqrt{4-x^2-y^2}$ and below by cone $z=\sqrt{3x^2+3y^2}$

↳ $4-x^2-y^2 = 3x^2+3y^2 \to 4-(x^2+y^2) = 3(x^2+y^2) \to x^2-y^2=1$

↳ $-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, $\sqrt{3x^2+3y^2} \le z \le \sqrt{4-x^2-y^2}$

↳ $V = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx \rightsquigarrow V$

Average Value with Triple Integrals

Average Value of function F over region D: $\frac{1}{\text{volume } D} \iiint_D F\,dV$

Find average value of $F(x,y,z) = x^2+9$ over the rectangular solid in the first octant bounded by coordinate planes and $x=3, y=2, z=2$

↳ Volume of $D = 2(3)2 = 12$

↳ $\frac{1}{12} \int_0^3 \int_0^2 \int_0^2 (x^2+9)\,dz\,dy\,dx = 12$

Applications Using Double/Triple Integrals, Part I

Mass: $M=\iiint_D \delta \, dV$

First Moments around coordinate planes: $M_{yz}=\iiint_D x\,\delta \, dV$, $M_{xz}=\iiint_D y\,\delta \, dV$, $M_{xy}=\iiint_D z\,\delta \, dV$

Center of Mass: $\bar{x}=\frac{M_{yz}}{M}$, $\bar{y}=\frac{M_{xz}}{M}$, $\bar{z}=\frac{M_{xy}}{M}$

Moments of Inertia: $\underset{\text{(about x-axis)}}{I_x=\iiint(y^2+z^2)\,\delta \, dV}$, $\underset{\text{(about y-axis)}}{I_y=\iiint(x^2+z^2)\,\delta \, dV}$, $\underset{\text{(about z-axis)}}{I_z=\iiint(x^2+y^2)\,\delta \, dV}$, $\underset{\text{(about line L)}}{I_L \iiint r^2(x,y,z)\,\delta \, dV}$

Centroid: When density is constant, center of mass (centroid) is $\delta=1$

Applications using Double/Triple Integrals, Part II

Joint Probability Density Function must fufill 3 criteria:

① $f(x,y)\geq 0$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1 \xrightarrow{\text{proves}}$

③ $P((X,Y)\in R)=\iint_R f(x,y)\,dx\,dy$

$\longrightarrow$ Mean / Expected Value $\rightarrow \mu_x=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy$

↳ $\mu_y=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\,dx\,dy$

Verify that f gives a joint probability density function. Then find expected μ_x and μ_y $\rightarrow f(x,y)\begin{cases}6x^2y & 0\leq x\leq 1,\ 0\leq y\leq 1\\ 0 & \text{otherwise}\end{cases}$

↳ $\int_0^1\int_0^1(6x^2y)\,dx\,dy \rightarrow \int_0^1 2x^3y\,\big|_0^1=1$

↳ $\mu_x=\int_0^1\int_0^1 x(6x^2y)\,dx\,dy \rightarrow \int_0^1 \frac{3}{4}y\,dy=\frac{3}{4}$

↳ $\mu_y=\int_0^1\int_0^1 y(6x^2y)\,dy\,dx \rightarrow \int_0^1 2y^2\,dy=\frac{2}{3}$

Triple Integrals with Cylindrical Coordinates

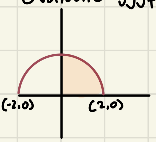
Cylindrical coordinates – a point P in space by ordered triples (r,θ,z) in which $r\geq 0$ (similar to polar)

↳ Use when there is an axis of symmetry, integrand involves x^2+y^2, integrating over a circle / part of a circle in xy-plane

① r and θ are polar coordinates for vertical projection of P on the xy-plane

② z is the rectangular coordinate

Evaluate $\iiint_T dV$ ($dV \rightarrow r\,dr\,d\theta\,dz$) where T is the solid formed by $0\leq x\leq 2$, $0\leq y\leq\sqrt{4-x^2}$, $0\leq z\leq\sqrt{16-x^2-y^2}$



↳ $0\leq r\leq 2$, $0\leq\theta\leq\frac{\pi}{2}$, $0\leq z\leq\sqrt{16-r^2}$

↳ $\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta \rightarrow \int_0^{\pi/2}\int_0^2[rz]_0^{\sqrt{16-r^2}} \rightarrow \int_0^{\pi/2}\left(\frac{64}{3}-8\sqrt{3}\right)d\theta \rightarrow \frac{32\pi}{3}-4\pi\sqrt{3}$

Triple Integrals with Spherical Coordinates

Spherical coordinates – a point P in space by ordered triples (ρ,ϕ,θ)

① ρ is the distance from P to the origin ($\rho\geq 0$)

② ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis ($0\leq\phi\leq\pi$)

③ θ is the angle from cylindrical coordinates

↳ $\iiint_V dV=\iiint_T \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

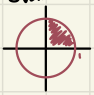
Equations:

$r=\rho\sin\phi$ $\quad x=r\cos\theta=\rho\sin\phi\cos\theta$

$z=\rho\cos\phi$ $\quad y=r\cos\theta=\rho\sin\phi\sin\theta$

$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{r^2+z^2}$

Evaluate $\iiint_T dV$ using spherical coordinates where T is the solid formed by $0\leq x\leq 1$, $0\leq y\leq\sqrt{1-x^2}$, $\sqrt{x^2+y^2}\leq z\leq\sqrt{2-x^2-y^2}$



$\rightarrow 0\leq\theta\leq\pi/2$

↳ $r=\sqrt{2-r^2}$

↳ $r=1$

$z=1$, $\rho=\sqrt{1^2+1^2}=\sqrt{2}$

↳ $z=\rho\cos\phi$, $1=\sqrt{2}\cos\phi \rightarrow 0\leq\phi\leq\frac{\pi}{4}$

$\rightarrow \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=-\pi/3+\pi\sqrt{2}/3$

Substitution with Double Integrals

Jacobian of transform $x=g(u,v)$, $y=h(u,v)$ is:

↳ $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

If $f(x,y)$ is continuous over region R, G is the preimage of R under transformation $x=g(u,v)$, $y=h(u,v)$, one-to-one on interior of G:

↳ $\iint_R f(x,y)\,dx\,dy = \iint_R f(g(u,v),h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

Transform/evaluate the integral $\iint_R (x+y)\cos(\pi(x-y))\,dx\,dy$ where R is bounded by $x-y=0$, $x-y=\frac{1}{2}$, $x+y=0$, $x+y=\frac{1}{2}$

↳ $x-y=u \rightarrow 0 \le u \le \frac{1}{2}$

↳ $x+y=v \rightarrow 0 \le v \le \frac{1}{2}$

↳ $2x = u+v \rightarrow x = \frac{1}{2}u + \frac{1}{2}v$

↳ $2y = u+v \rightarrow y = \frac{1}{2}u - \frac{1}{2}v$

$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\int_0^{1/2}\int_0^{1/2} \frac{1}{2} v\cos(\pi u)\,du\,dv = \frac{1}{16\pi}$

Substitution with Triple Integrals

Jacobian $(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$

If G is the preimage of R under transformation $x=g(u,v,w)$, $y=h(u,v,w)$, $z=k(u,v,w)$, then:

$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_G f(g(u,v,w),h(u,v,w),k(u,v,w)) \left|\frac{\partial(u,v,w)}{\partial(u,v,w)}\right| du\,dv\,dw$

Let R be the region in xyz-space defined by $1 \le x \le 2, 0 \le xy \le 2, 0 \le z \le 1$. Find $\iiint_R (x^2y + 3xyz)\,dx\,dy\,dz$ using $u=x$, $v=xy$, $w=3z$

↳ $x=u$, $y=\frac{v}{x} \rightarrow \frac{v}{u}$, $z=\frac{1}{3}w$

↳ $1 \le u \le 2$, $0 \le v \le 2$, $0 \le w \le 3$

↳ $x^2y + 3xyz \rightarrow x(xy) + 3(xy)z \rightarrow uv + vw$

$J = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u} \rightarrow \int_0^3\int_0^2\int_1^2 (uv+wv)\frac{1}{3u}\,du\,dv\,dw = 2+3\ln(2)$