Line Integrals

If f is defined on the curve $r(t) = g(t)i + h(t)j + k(t)k$, $a \le t \le b$, then the line integral of f over C is:

↳ $\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k$

Evaluating a line integral:

1. Find a smooth parametrization of C: $r(t) = g(t)i + h(t)j + k(t)k$, $a \le t \le b$
2. Evaluate as: $\int_C f(x,y,z)\,ds = \int_a^b f(g(t), h(t), k(t))\,|v(t)|\,dt$

Evaluate $\int_C (x-y+z+1)\,ds$ where C is the straight-line segment $x=t, y=1-t, z=1$ from $(0,1,1)$ to $(1,0,1)$

↳ $0 \le t \le 1$ → $\int_0^1 (t-(1-t)+1+1)\sqrt{2}\,dt = \sqrt{2}\int_0^1 (2t+1)\,dt = 2\sqrt{2}$

↳ $r(t) = ti + (1-t)j + k$

↳ $r'(t) = i - j$, $\|r'(t)\| = \sqrt{2}$

Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ given by $C_1: r(t) = tk$ for $0 \le t \le 1$ and $C_2: r(t) = ti + tj + k$ for $0 \le t \le 1$

↳ $\int_{C_1} f\,ds + \int_{C_2} f\,ds \rightarrow \int_0^1 -t^2(1)\,ds + \int_0^1 (t + \sqrt{t} - 1)\sqrt{2}\,dt = \frac{\sqrt{2}-2}{6}$

Mass and Moment Calculations

Mass: $M = \int_C \delta\,ds$

First Moments around coordinate planes: $M_{yz} = \int_C x\,\delta\,ds$, $M_{xz} = \int_C y\,\delta\,ds$, $M_{xy} = \int_C z\,\delta\,ds$

Coordinates of center of mass: $\bar{x} = \frac{M_{yz}}{M}$, $\bar{y} = \frac{M_{xz}}{M}$, $\bar{z} = \frac{M_{xy}}{M}$

Moments of Inertia: $I_x = \int_C (y^2+z^2)\,\delta\,ds$, $I_y = \int_C (x^2+z^2)\,\delta\,ds$, $I_z = \int_C (x^2+y^2)\,\delta\,ds$

↳ $I_L = \int_C r^2\,\delta\,ds$ where $r(x,y,z)$ = distance from point (x,y,z) to line L

Line Integrals of Vector Fields

Line integral of F along path C:

↳ $\int_C F\cdot T\,ds = \int_C F\left(\frac{dr}{ds}\right)ds = \int_C F\cdot dr$ →

1. Find components of r into scalar components of F
2. Find vector $\frac{dr}{dt}$
3. Evaluate $\int_C F\cdot dr = \int_a^b F(r(t))\cdot\frac{dr}{dt}\,dt$

$\int_C M\,dx + N\,dy + P\,dz$:

↳ $\int_C M(x,y,z)\,\underline{dx} + \int_C N(x,y,z)\,\underline{dy} + \int_C P(x,y,z)\,\underline{dz}$

↳ $\int_C M(x,y,z)\,d\underset{z}{\underset{y}{x}} = \int_a^b M(g(t), h(t), k(t))\,\underset{k}{\underset{h}{g}}'(t)\,dt$

Work, Flow, and Flux

Work: $W = \int_C F\cdot T\,ds = \int_a^b F(r(t))\cdot\frac{dr}{dt}\,dt$

Flow integral: Flow $= \int_C F\cdot T\,ds$

↳ If the curve starts/ends at the same point $(A=B)$, that Flow is called a circulation around the curve

Flux: Flux of F across $C = \int_C F\cdot n\,ds$ ← outward-pointing unit vector on C

↳ Flux across smooth closed plane curve: Flux $= Mi + Nj$ across $C = \oint M\,dy - N\,dx$

↳ evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \le t \le b$, that traces C counterclockwise <u>once</u>

Find the flow and flux of the field $F = \underset{-y=M}{-yi} + \underset{N=1}{j}$ around/across the closed semicircular path $r_1(t) = 2\cos(t)i + 2\sin(t)j$, $0 \le t \le \pi$, followed by $r_2(t) = ti$, $-2 \le t \le 2$

↳ $\vec{r}_1'(t) = \underset{dx}{-2\sin(t)\vec{i}} + \underset{dy}{2\cos(t)\vec{j}}$ → Flow: $\int_0^{\pi} \langle -2\sin(t), 2\cos(t)\rangle \cdot \langle -2\sin(t), 2\cos(t)\rangle\,dt = \int_0^{\pi} 4\sin^2(t) + 4\cos^2(t)\,dt = \int_0^{\pi} 4\,dt = 4\pi$

↳ $\vec{r}_2'(t) = \underset{dx}{\vec{i}} + \underset{dy}{0\vec{j}}$ → ↳ $\int_{-2}^{2} \langle 0,1\rangle \cdot \langle 1,0\rangle\,dt = \int_{-2}^{2} 0\,dt = 0$ → $4\pi + 0 = 4\pi$

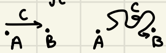
Flux: $\int_0^{\pi}$ $\overset{Mdy}{-4\sin(t)\cos(t)} - \overset{-Ndx}{-4\cos(t)\sin(t)} = 0$ → Flux $= 0$, no in/out flow, only circulation

↳ r_2: $\int_{-2}^{2} (0 - t)\,dt = 0$

Conservative Fields and Potential Functions

F is a vector field in region D, and for any points A and B the line integral $\int_C F\cdot dr$ along path C from A to B in D is the <u>same</u> over any path A → B

↳ $\int_C F\cdot dr$ is <u>path independent</u> in D and field F is <u>conservative</u> on D



$\int_C F\cdot dr = \int_C F\cdot dr$

↳ if $F = \nabla f$, f is the potential function for F

↳ $F = Mi + Nj + Pk$ is conservative ⟷ F is a gradient field ∇f for function f

↳ $F = Mi + Nj + Pk$ is conservative ⟷ $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

Which fields are conservative?

a) $F = y\vec{i} + (x+z)\vec{j} - y\vec{k} \rightarrow \frac{\partial P}{\partial y} = -1$, $\frac{\partial N}{\partial z} = 1, \neq -1$ ✗

b) $F = (yz)\vec{i} + (xz)\vec{j} + (xy)\vec{k} \rightarrow \frac{\partial P}{\partial y} = x$, $\frac{\partial N}{\partial z} = x$, ✓ $\frac{\partial M}{\partial z} = y$, $\frac{\partial P}{\partial x} = y$ ✓ $\frac{\partial N}{\partial x} = z$, $\frac{\partial M}{\partial y} = z$ ✓ conservative

c) $F = (y\sin z)\vec{i} + (x\sin z)\vec{j} + (xy\cos z)\vec{k} \rightarrow \frac{\partial P}{\partial y} = x\cos z$, $\frac{\partial N}{\partial z} = x\cos z$ ✓ $\frac{\partial M}{\partial z} = y\cos z$, $\frac{\partial P}{\partial x} = y\cos z$ ✓ $\frac{\partial N}{\partial x} = \sin z$, $\frac{\partial M}{\partial y} = \sin z$ ✓ conservative

$\nabla f = f_x\,\vec{i} + f_y\,\vec{j} + f_z\,\vec{k}$

Find a <u>potential</u> function f for $F = (y\sin z)\vec{i} + (x\sin z)\vec{j} + (xy\cos z)\vec{k}$

↳ $\frac{\partial f}{\partial x} = y\sin z \rightarrow \int y\sin z\,dx = \underbrace{xy\sin z + g(y,z)}_{\text{Potential function}} = f$ → $f(x,y,z) = xy\sin z + C$

($g(y,z)$ "C")

↳ $\frac{\partial f}{\partial y} = x\sin z + \frac{\partial g}{\partial y} = f_y \rightarrow x\sin z + \frac{\partial g}{\partial y} = x\sin z \rightarrow \frac{\partial g}{\partial y} = 0$

↳ $\frac{\partial f}{\partial z} = xy\cos x + \frac{\partial g}{\partial z} = xy\cos z \rightarrow \frac{\partial g}{\partial z} = 0$

Fundamental Theorem of Line Integrals

Let C be a smooth curve joining point A to point B, parametrized by $r(t)$. $F = \nabla f$ on a domain containing C.

↳ $\int_C \overset{\nabla f}{F}\cdot dr = f(B) - f(A)$

↳ $\oint_C F\cdot dr = 0$ around every loop (closed curve C) in D = F is <u>conservative</u> on D

(closed curve)

$F = \nabla f$ for $f(x,y,z) = \frac{2x}{y^2+z^2+1}$ — Find $\int_C F\cdot dr$, C is the curve from $(1,2,-1)$ to $(2,3,0)$.

↳ $\int_C \vec{F}\cdot dr = f(2,3,0) - f(1,2,-1) = \frac{4}{10} - \frac{2}{6} = \frac{1}{15}$

Any expression $M(x,y,z)dz + N(x,y,z)dy + P(x,y,z)dz$ is a differential form — A differential form is <u>exact</u> on a domain space D in space if:

↳ $M\,dx + N\,dy + P\,dz = \frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy + \frac{\delta f}{\delta z}dz = df$

Show the differentiable form is <u>exact</u>, then evaluate: $\int_{(1,1,2)}^{(3,5,0)} \overset{M}{yz\,dz} + \overset{N}{xz\,dy} + \overset{P}{xy\,dz}$

↳ $\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y}$ exact ✓

↳ $f = \int y\,dx = xyz + g(y,z)$

$f_y = \underset{-xz}{xz} + \frac{\partial g}{\partial y} = \underset{-xz}{N} \rightarrow \frac{\partial g}{\partial y} = 0$

$f_z = xy + \frac{\partial g}{\partial z} = P \rightarrow \frac{\partial g}{\partial z} = 0$

$xyz + C = f$, so $f(3,5,0) - f(1,1,2) = -2$

Green's Theorem, Part 1

Circulation density at point (x,y) is: $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ [also known as the k-component of the curl, $(\text{curl } F)\cdot k$] → Use for work around a path

Flux density (divergence) of a field at point (x,y) is: $\text{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

Green's Theorem → Circulation-Curl / Tangential Form: $\oint_C F\cdot T\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

↳ Flux-Divergence / Normal Form: $\oint_C F\cdot n\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy$

Use Green's Theorem to find the counterclockwise circulation / outward flux for $F = \overset{M}{(x^2+4y)}\vec{i} + \overset{N}{(x+y^2)}\vec{j}$ over square $0 \le x \le 1,\ 0 \le y \le 1$

↳ CC: $\int_0^1\int_0^1 \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1\int_0^1 (1-4)\,dxdy = -3$

↳ OF: $\int_0^1\int_0^1 \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy = \int_0^1\int_0^1 (2x+2y)\,dxdy = 2$

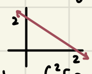
Green's Theorem, Part II

Find the work done by $F = \overset{M}{(3x-5y)}\vec{i} + \overset{N}{(5x-3y)}\vec{j}$ around the circle $(x-1)^2+(y-2)^2=9$

↳ CC: $\iint_R (5 - -5)\,dxdy = 90\pi$

Evaluate $\oint \overset{M}{y^2}dx + \overset{N}{x^2}dy$, C is the triangle bounded by $x=0,\ x+y=2,\ y=0$

↳ $\iint_R (2x-2y)\,dxdy$



$0 \le x \le 2$

$0 \le y \le 2-x$

↳ $\int_0^2\int_0^{2-x} (2x-2y)\,dydx = \int_0^2 \left[2xy - y^2\right]_0^{2-x} = 0$

Area Formula: Area of R is $\iint_R dydx = \frac{1}{2}\oint xdy - ydx$

Find a formula for an ellipse: $r(t) = a\cos(t)\vec{i} + b\sin(t)\vec{j},\ 0 \le t \le 2\pi$

↳ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $x = a\cos(t)$ and $y = b\sin(t)$

↳ $\frac{1}{2}\oint_C xdy - ydx$, $dx = -a\sin(t)dt$, $dy = b\cos(t)dt$

↳ $\frac{1}{2}\int_0^{2\pi} (ab\cos^2(t) + ab\sin^2(t))\,dt = ab\pi$

Parameterizing Surfaces

Curve parameterization: $r(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

Surface parameterization: $r(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$

↳ Domain: Set of points (u,v) that can be substituted into r

Common Surfaces:

↳ Sphere → $x^2+y^2+z^2=a^2$

↳ $r(\phi,\theta) = a\sin(\phi)\cos(\theta)\vec{i} + a\sin(\phi)\sin(\theta)\vec{j} + a\cos(\phi)\vec{k}$
$0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$

↳ Cylinder → $x^2+y^2=a^2,\ 0 \le z \le b$

↳ $r(\theta,z) = a\cos(\theta)\vec{i} + a\sin(\theta)\vec{j} + zk$
$0 \le \theta \le 2\pi,\ 0 \le z \le b$

↳ Cone → $z = \sqrt{x^2+y^2},\ 0 \le z \le b$

↳ $r(r,\theta) = r\cos(\theta)\vec{i} + r\sin(\theta)\vec{j} + rk$
$0 \le \theta \le 2\pi,\ 0 \le r \le b$

Find a parameterization for the paraboloid $z = 4 - x^2 - y^2,\ z \ge 0$

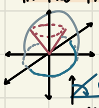
↳ $4 - r^2 = z,\ 0 \le r \le 2$

↳ $r(r,\theta) = \overset{x}{r\cos\theta}\vec{i} + \overset{y}{r\sin\theta}\vec{j} + \overset{z}{(4-r^2)}\vec{k}$, $0 \le \theta \le 2\pi$, $0 \le r \le 2$

Find a parameterization for the portion of the sphere $x^2+y^2+z^2=4$ in the first octant between the xy-plane and cone $z = \sqrt{x^2+y^2}$

$\rho = 2$

$z = 2\cos\phi$



z-axis

$\phi = \frac{\pi}{4}$ 2 $\frac{\pi}{2}$

↳ sphere $\vec{r}(\phi,\theta) = 2\sin\phi\cos\theta\vec{i} + 2\sin\phi\sin\theta\vec{j} + 2\cos\phi\vec{k}$

↳ restrict: $0 \le \theta \le \frac{\pi}{2}$, $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$

Surface Area

A parameterized surface is smooth if r_u and r_v (Partials) are continuous and $r_u \times r_v$ is never zero on the interior of the parameter domain

↳ area: $a \le u \le b,\ c \le v \le d \rightarrow A = \iint_R |r_u \times r_v|\,dA = \int_c^d\int_a^b |r_u \times r_v|\,dudv$

↳ Surface area differential: $d\sigma = |r_u \times r_v|\,dudv$

Find the area of the surface formed by the portion of the plane $y + 2z = 2$ inside the cylinder $x^2+y^2=9$

↳ $\vec{r}(u,v) = \overset{x \text{ can be anything}}{u\vec{i}} + v\vec{j} + (1 - \frac{v}{2})\vec{k}$ ($z = 1 - y/2 \rightarrow 1 - v/2$) → $\vec{r}_u = \vec{i}$, $\vec{r}_v = \vec{j} - \frac{1}{2}\vec{k}$

↳ (inside cylinder) $0 \le u^2+v^2 \le 9$

↳ $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -1/2 \end{vmatrix} = \frac{1}{2}\vec{j} + \vec{k}$, $|\vec{r}_u \times \vec{r}_v| = \frac{\sqrt{5}}{2}$

→ $A = \iint_R \frac{\sqrt{5}}{2}\,dA = \frac{\sqrt{5}}{2}(9\pi) = \frac{9\sqrt{5}}{2}\pi$

$\iint_R dA = \pi r^2 = \pi(3)^2$

area of circle, $r = 3$

↳ Implicit Surface Area: Area of $F = (x,y,c) = c \rightarrow \iint_R \frac{|\nabla F|}{|\nabla F \cdot p|}\,dA$ — p is normal to R

Find the area of the region cut by the plane $x + 2y + 2z = 5$ by the cylinder whose sides are $x = y^2$ and $x = 2 - y^2$

R is in xy-plane, p is in z-axis, $\vec{k}$

↳ $\nabla F = \vec{i} + 2\vec{j} + 2\vec{k}$ → $\nabla F \cdot p \rightarrow \nabla F \cdot \vec{k} = 2$

↳ $\|\nabla F\| = 3$

↳ $\iint_3 \frac{3}{2}\,dA = \int_{-1}^1\int_{y^2}^{2-y^2} \frac{3}{2}\,dxdy = 4$

↳ Graph Surface Area $[z = f(x,y)]$: $A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\,dxdy$

Find the area of the surface cut from $z = \overset{f}{x^2+y^2}$ by the plane $z = 2$ → intersection when $x^2+y^2=2$

↳ $f_x = 2x,\ f_y = 2y \rightarrow \iint_R \sqrt{4x^2+4y^2+1}\,dxdy$

↳ $\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\,r\,drd\theta = \frac{13\pi}{3}$

Surface Integrals, Part I

Surface Integral of G over S: $\iint_S G(x,y,z)\,d\sigma = \lim_{n\to\infty}\sum_{k=1}^{n} G(x_k,y_k,z_k)\Delta\sigma_k$

↳ Formulas:

↳ Given Parametrically ($r(u,v)=f(u,v)\vec{i}+g(u,v)\vec{j}\ldots$) ⟶ $\iint_S G(x,y,z)\,d\sigma = \iint_R G(f(u,v),g(u,v),h(u,v))\,|r_u\times r_v|\,du\,dv$ (continuous function)

↳ Given Implicitly ($F(x,y,z)=c$) ⟶ $\iint_S G(x,y,z)\,d\sigma = \iint_R G(x,y,z)\,\frac{|\nabla f|}{|\nabla F\cdot p|}\,dA$ — unit vector ⊥ to R

↳ Given Explicitly ($z=f(x,y)$) ⟶ $\iint_S G(x,y,z)\,d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x^2+f_y^2+1}\,dx\,dy$

Evaluate $\iint_S 2xy\,d\sigma$ over the surface $x+2y+2z=4$ in the first octant

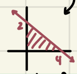
↳ Explicit: $z = 2-y-\frac{x}{2}$ ($f(x,y)$) ⟶ $\iint_S 2xy\sqrt{\frac{1}{4}+1+1}\,dx\,dy = \iint_S 3xy\,dx\,dy$

↳ $f_x=-\frac{1}{2},\ f_y=-1$

$x+2y+2z=4$

↳ on xy-plane, $z=0$

↳ so, $x=4-2y$



$0\le y\le 2$

$0\le x\le 4-2y$

⟶ $\int_0^2\int_0^{4-2y} 3xy\,dx\,dy = 8$

Integrate the function $G(x,y,z)=x^2\sqrt{5-4z}$ over the surface of the parabolic dome $z=1-x^2-y^2,\ z\ge 0$

↳ $\iint_S x^2\sqrt{5-4z}\,d\sigma$ ⟶ $\iint_S x^2\sqrt{5-4z}\sqrt{4x^2+4y^2+1}\,dx\,dy$ ($z = 1-x^2-y^2$) ⟶ $\iint_S x^2(1+4x^2+4y^2)\,dx\,dy$ ⟶ $\int_0^{2\pi}\int_0^1 r^2\cos^2(\theta)(1+4r^2)\,r\,dr\,d\theta = \pi$

↳ Explicit: $z = 1-x^2-y^2$ ($f(x,y)$)

↳ $f_x=-2x,\ f_y=-2y$

bounds? ↳ $z=1-x^2-y^2$, when xy-plane, a circle forms with $r=1$

Surface Integrals, Part II

Surface Integral of F over S: $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ = Flux (F: vector field; n: unit vectors ⊥ S)

↳ if written as $g(x,y,z)=c$, $n=\pm\frac{\nabla g}{\|\nabla g\|}$

↳ if written as $r(u,v)=f(u,v)\vec{i}\ldots$, Flux $=\iint_S F\,(r_u\times r_v)\,du\,dv$

Find the flux of $F=-x\vec{i}-y\vec{j}+z^2\vec{k}$ outward through the portion of a cone $z=\sqrt{x^2+y^2}$ between $z=1$ and $z=2$

↳ cone, so can be written in terms of θ and r (Polar Parameterization) ⟶ $\iint_S \vec{F}\cdot(v\cos u\,\vec{i}+v\sin u\,\vec{j}-v\vec{k})\,du\,dv$ ⟶ $\iint_S(-v^2\cos^2 u - v^2\sin^2 u - v^3)\,du\,dv$

needs to be in u and v → $F=\underbrace{-v\cos u\,\vec{i}}_{-x}-\underbrace{v\sin u\,\vec{j}}_{-y}+\underbrace{v^2}_{z^2}$ ↳ bounds? $\int_0^{2\pi}\int_1^2$ " " $= -\frac{73}{6}\pi$

↳ Parameterized Surface: $\vec{r}(u,v) = v\cos u\,\vec{i} + v\sin u\,\vec{j} + v\vec{k}$ (θ, r; $x\vec{i}$, $y\vec{j}$, $z\vec{k}$)

↳ $\vec{r}_u = -v\sin u\,\vec{i}+v\cos u\,\vec{j}$, $\vec{r}_v=\cos u\,\vec{i}+\sin u\,\vec{j}+\vec{k}$, $\vec{r}_u\times\vec{r}_v = v\cos u\,\vec{i}+v\sin u\,\vec{j}-v\vec{k}$

Mass (Thin Shell): $M=\iint_S \delta\,d\sigma$

First Moments about coordinate plane: $M_{yz}=\iint_S x\,\delta\,d\sigma$, $M_{xz}=\iint_S y\,\delta\,d\sigma$, $M_{xy}=\iint_S z\,\delta\,d\sigma$

Center of Mass coordinates: $\bar{x}=\frac{M_{yz}}{M}$, $\bar{y}=\frac{M_{xz}}{M}$, $\bar{z}=\frac{M_{xy}}{M}$

Moments of Inertia: $I_x=\iint_S(y^2+z^2)\,\delta\,d\sigma$, $I_y=\iint_S(x^2+z^2)\,\delta\,d\sigma$, $I_z=\iint_S(x^2+y^2)\,\delta\,d\sigma$, and $I_L=\iint_S r^2\,\delta\,d\sigma$ where $r(x,y,z)$ = distance from point (x,y,z) to L (line)

Find the centroid for a portion of the sphere $x^2+y^2+z^2=4$ that lies in the first octant

↳ $M=\iint_S d\sigma$ = area of S ($r=2$)

↳ $4\pi r^2$ = SA of sphere; One octant $=\frac{1}{8}(4\pi r^2)=\frac{1}{2}\pi r^2$ ⟶ 2π = Mass

↳ $M_{yz}=\iint_S x\,d\sigma$ ($\frac{|\nabla F|}{|\nabla F\cdot p|}$) ↳ $\nabla f = 2x\vec{i}+2y\vec{j}+2z\vec{k}$ ⟶ $|\nabla F| = 2\sqrt{\underbrace{x^2+y^2+z^2}_{r=4}}=4$

↳ $p=\vec{i}$ because M_{yz} is in yz-plane, x is ⊥ ↳ $\nabla F\cdot p = 2x$

↳ $\iint_S x\left(\frac{4}{2x}\right)dA = \iint_S 2\,dA = 2\iint_S dA = 2\pi$ (2D area; ¼ a circle, 2D)

↳ coordinates: $\left(\frac{2\pi}{2\pi},\ ,\ \right) = (1,1,1)$ (sphere, so equal)

The Curl

del (∇) is an operator: $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

div – divergence of a vector field: div $F = \nabla \cdot F \longrightarrow$ (scalar)

curl – curl $F = \nabla \times F \longrightarrow$ (vector)

↳ curl grad $f = 0$, $\nabla \times \nabla f = 0$

Find the div and curl for $F = (x^2 - yz)\vec{i} + ye^x\vec{j} + (xy+z)\vec{k}$

↳ div $= \nabla \cdot F \rightarrow \frac{\partial}{\partial x}(x^2-yz)\vec{i} + \frac{\partial}{\partial y}(ye^x)\vec{j} + \frac{\partial}{\partial z}(xy+z)\vec{k}$

$= 2x + e^x + 1$

↳ curl $= \nabla \cdot F \rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & ye^x & xy+z \end{vmatrix} = \left[\frac{\partial}{\partial y}(xy+z) - \frac{\partial}{\partial z}(ye^x)\right]\vec{i} - \left[\frac{\partial}{\partial x}(xy+z) - \frac{\partial}{\partial z}(x^2-yz)\right]\vec{j} + \left[\frac{\partial}{\partial x}(ye^x) - \frac{\partial}{\partial y}(x^2-yz)\right]\vec{k}$

$= (x)\vec{i} - (y+y)\vec{j} + (ye^x + z)\vec{k}$

Stokes' Theorem

S is a piecewise smooth-oriented surface with boundary curve C; the circulation of F around C is:

↳ $\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot n\, d\sigma$

Use Stoke's Theorem to calculate the circulation of the field $F = y\vec{i} + xz\vec{j} + x^2\vec{k}$ where C is the triangle cut by the plane $x+y+z=1$ by the 1st octant

↳ calculate curl $(\nabla \times F) = -x\vec{i} - 2x\vec{j} + (z-1)\vec{k}$

↳ calculate unit normal for the plane $\rightarrow \vec{n} = \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$

$(\nabla \times \vec{F}) \cdot \vec{n} = -\frac{x}{\sqrt{3}} - \frac{2x}{\sqrt{3}} + \frac{z-1}{\sqrt{3}}$

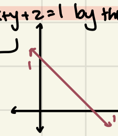
$= -\frac{3x}{\sqrt{3}} + \frac{z-1}{\sqrt{3}} \longrightarrow \iint(-3x + z - 1)\,dxdy$

$= \iint(-3x + 1 - x - y - 1)\,dxdy$

$= \iint(-4x - y)\,dydx \rightarrow \int_0^1 \int_0^{1-x} (-4x-y)\,dydx = -5/6$

↳ calculate $d\sigma \rightarrow z = 1 - x - y$, expressed explicitly

↳ $d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\,dxdy \rightarrow \sqrt{1^2+1^2+1^2}\,dxdy = \sqrt{3}\,dxdy$



$0 \le x \le 1$

$0 \le y \le 1-x$ (when $z=0$)

Use Stokes' Theorem to calculate the flux of the curl of $F = 2z\vec{i} + 3x\vec{j} + 5y\vec{k}$ across $S: r(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + (4-r^2)\vec{k}$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$

↳ calculate unit normal : $\frac{\vec{r}_r \times \vec{r}_\theta}{|\vec{r}_r \times \vec{r}_\theta|}$ ↳ $2r^2\cos\theta\,\vec{i} + 2r^2\sin\theta\,\vec{j} + r\vec{k}$

↳ calculate $d\sigma = |\vec{r}_r \times \vec{r}_\theta|$

↳ $\iint (\nabla \times \vec{F}) \cdot \underbrace{n\, d\sigma}_{\text{cancels}} \longrightarrow \iint (\nabla \times F) \cdot (\vec{r}_r \times \vec{r}_\theta)\,dr d\theta \longrightarrow \int_0^{2\pi}\int_0^2 10r^2\cos\theta + 4r^2\sin\theta + 3r\,dr d\theta = 12\pi$

↳ calculate curl : $\nabla \times F = 5\vec{i} + 2\vec{j} + 3\vec{k}$

Closed-Loop Property – if $\nabla \times F = 0$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D, $\oint F \cdot dr = 0$

The Divergence Theorem

divergence over region S : $\underbrace{\iint_S F \cdot n\, d\sigma}_{\text{Flux}} = \underbrace{\iiint_D \nabla \cdot F\, dV}_{\text{Divergence}}$

↳ Outward Flux = 0 if F has 0 divergence at every point

↳ $\text{div}(\text{curl}) = \nabla \cdot (\nabla \times F) = 0$

Calculate the outward flux of $F = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ across the cube cut from the 1st octant by planes $x=1, y=1, z=1$

↳ calculate div: $\nabla \cdot \vec{F} = 2x + 2y + 2z$

↳ calculate triple integral: $\int_0^1\int_0^1\int_0^1 2x+2y+2z\,dxdydz = 3$

Calculate the outward flux of $F = (5x^3 + 12xy^2)\vec{i} + (y^3 + e^y\sin(z))\vec{j} + (5z^3 + e^y\cos(z))\vec{k}$ across the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=2$

↳ calculate div: $\nabla \cdot \vec{F} = (15x^2 + 12y^2) + (3y^2 + e^y\sin(z)) + (15z^2 - e^y\sin(z)) = 15x^2 + 15y^2 + 15z^2$

↳ calculate triple integral: $\iiint \underbrace{15(x^2+y^2+z^2)}_{\text{spherical coordinates}}\,dV \rightarrow$

$1 \le \rho \le \sqrt{2}$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$x^2+y^2+z^2 = \rho^2$

$\int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}} (15\rho^2\rho^2\sin\phi)\,d\rho\, d\phi\, d\theta = (48\sqrt{2} - 12)\pi$

Generalization of Green's Theorem

Summary

Tangential Form → $\oint_C F \cdot T\,ds = \iint_R (\nabla \times F) \cdot k\,dA$ ⟶ Unifying Fundamental Theorem

Stokes' Theorem ⟶ $= \iint_S (\nabla \times F) \cdot n\,d\sigma$

Normal Form ⟶ $= \iint_R (\nabla \cdot F)\,dA$

Divergence Theorem $\iint_S F \cdot n\,d\sigma = \iiint_D (\nabla \cdot F)\,dV$

↳ Integral of a differential operator (∇) acting on a field over a region = sum of field components appropriate to operator over the boundary of the region