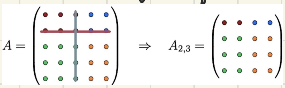
Determinant of a Matrix

If A is an $n \times n$ Matrix and has elements a_{ij}

1) If $n=1$, $A=[a_{11}]$ (1x1 matrix) and has determinant $\det A = a_{11}$

2) If $n>1$, $\det A = (a_{11})\det A_{11} - a_{12}\det A_{12} + \ldots + (-1)^{1+n}(a_{1n})\det A_{1n}$ (sum of determinants) → alternating pattern

↳ A_{ij} (← sub-matrix) is the matrix gotten by eliminating row i and column j of A



↳ Taking Out 2nd Row, 3rd Column

Notation:
$\det A = |A|$

↳ Determinant of 2x2 Matrix

$\det A$, when $A = \begin{array}{c} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \scriptstyle 1 \quad 2 \end{array} \rightarrow aA_{11} - bA_{12}$ (number of columns)

$= ad - bc$

Cofactor Expansion

↳ Determinant of 3x3 Matrix

Compute $\det\begin{pmatrix} 3 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{vmatrix} 3 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{vmatrix} \rightarrow 3\underset{}{\overset{\det}{A_{11}}} + 5\overset{\det}{A_{12}} + 0\overset{\det}{A_{13}}$

$3\overset{\det}{\begin{vmatrix} 4 & -1 \\ 2 & 0 \end{vmatrix}} + 5\overset{\det}{\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix}} + 0$

det of 2x2 is $(ad-bc)$

$3(4\cdot\overset{0}{\cancel{0}} - (-1)(2)) + \cancel{5(2\cdot 0 - (-1)\cdot 0)}^{0} + 0$

$\downarrow$

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Cofactors and Triangular Matrices

Cofactors - used for notation of determinants

↳ (i,j) cofactor for $n \times n$ matrix A → $C_{ij} = (-1)^{i+j}\det A_{ij}$

↳ Determinants of a matrix can be computed to any row/column of the matrix → Ex. determinant to jth column is $\det A = a_1C_{1j} + a_2C_{2j} + \ldots + a_{nj}C_{nj}$

Example

Calculate determinant of $A = \begin{pmatrix} 5 & 4 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{pmatrix}$ ↖ lots of 0s! → $\det A = 5C_{11} + 0C_{21} + 0C_{31} + 0C_{41}$

$= 5C_{11}$ ← 0s in the column, save time

$= 5(-1)^{1+1}\begin{vmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix}$

$= 5(-1)^2(0C_{13} + 0C_{23} + 3C_{33})$

$= 5(3C_{33})$

$= 5((-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix})$

$= 5(3(1\cdot 1 - 2(-1)))$

$= 45$

If A is a triangular matrix (e.g. echelon form), then $\det A = a_{11}a_{22}\ldots a_{nn}$

↳ b/c det of a 2x2 triangular matrix is the product of the entries on the main diagonal $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$

$ac - 0b = ac$

Determinants and Row Operations

↳ Determinants can be hard/impossible to compute for some cofactor expansions (large N)

↳ Use Row Operations!

Row Operations - let A be a square matrix

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|A| = ad - bc$

1) If a multiple of a row is added to another row to produce B, $\det B = \det A$ → ↳ $A \sim \begin{pmatrix} a & b \\ c+ka & d+kb \end{pmatrix} = B$, $|B| = a(d+\cancel{kb}) - b(c+\cancel{ka})$ (row operations are cancelled out)

$= ad - bc$

2) If rows are swapped to produce B, then $\det B = -\det A$ ⟶ $A \sim \begin{pmatrix} c & d \\ a & b \end{pmatrix} = B$, $|B| = (cb - ad)$

$= -(ad - bc)$

3) If one row is multiplied by a scalar to get B, then $\det B = k\det A$ → $A \sim \begin{pmatrix} 2a & 2b \\ c & d \end{pmatrix} = B$, $|B| = (2ad - bc)$

↳ $= 2(ad - bc)$

Example

Compute $\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \rightarrow A: \begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{pmatrix} \underset{R_3+1R_1}{\overset{R_2+2R_1}{\sim}} \begin{pmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{pmatrix} \overset{R_2 \leftrightarrow R_3}{\sim} \underset{\text{triangular}}{\begin{pmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{pmatrix}} = B$

↳ $\det A = -\det B = -(1\cdot 3\cdot -5) = +15$, $\det A = 15$

If A is reduced to echelon form by r row swaps of rows + columns

↳ $|A| = \begin{cases} (-1)^r \times (\text{product of pivots}), \text{ when } A \text{ is invertible (nonzero)} \\ 0 \longrightarrow \text{when } A \text{ is singular} \end{cases}$

Properties of Determinants

For square matrices A and B:

1. $\det A = \det A^T$
2. A is invertible if and only if $\det A \neq 0$
3. $\det(AB) = \det A \cdot \det B$
4. if A is invertible, then $\det(A^{-1}) = \frac{1}{\det A}$

Example!

Use a determinant to find all values of λ such that matrix C is <u>not invertible</u>

$$C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$|C| = (5-\lambda)\, C_{11} + 0C_{12} + 0C_{13}$$

$$= (5-\lambda)\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

makes it singular

$$\hookrightarrow 0 = (5-\lambda)(\lambda^2 - 1)$$

$$\lambda = 5,\ \lambda = \pm 1$$

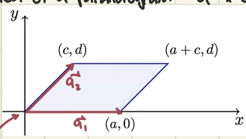
Example

Value of $\det A$, where $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & 7 & 3 \end{pmatrix}^{100}$

$\hookrightarrow |A| = \left|\begin{pmatrix} 0 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & 7 & 3 \end{pmatrix}^{100}\right| = \left|\begin{pmatrix} 0 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & 7 & 3 \end{pmatrix}\right|^{100} = (0 - (3\cdot1 - 2\cdot1) + 0)^{100} = (-1)^{100} = 1$

Determinant and the Area of a Parallelogram

area of a parallelogram = a x d



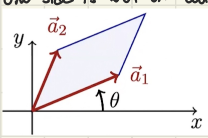
Changing c does nothing

$(c \cdot 0) = 0$

absolute value, area can't be negative

$\hookrightarrow$ Making a matrix with columns $\vec{a}_1$ and $\vec{a}_2$ yields $A = (\vec{a}_1, \vec{a}_2) = \begin{pmatrix} a & c \\ 0 & d \end{pmatrix}$, $|\det A| = |ad| =$ area of parallelogram

If one side is NOT on coordinate axis



$A' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & c \\ 0 & d \end{pmatrix}$

$\hookrightarrow$ does the determinant of the new matrix = ad?

$\det A' = 1\,(ad) = ad$ Yes! rotations do not change A!

Example

Compute area of a parallelogram determined by $\vec{0}, \vec{u}, \vec{v}$, and $\vec{u}+\vec{v}$; $\vec{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\hookrightarrow$ area $= |\det A|$ ← absolute value of $\det A$

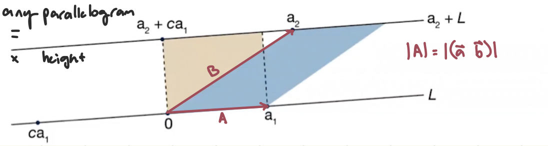
area $= \left|\det \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}\right|$

area $= |(-1 - 6)| = 7$

Area, Volume, and Determinants

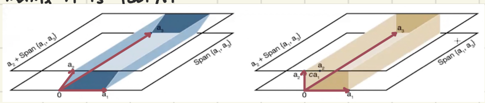
Key Geometric Fact (which works in any dimension). The area of the parallelogram spanned by two vectors $\vec{a}, \vec{b}$ is equal to the area spanned by $\vec{a}, c\vec{a} + \vec{b}$, for any scalar c.

area of any parallelogram = base x height



(3d parallelogram)

Volume of the parallelpiped spanned by columns of an n x n matrix A is $|\det A|$



Example

Calculate area of a parallelogram → $(-2,-2), (0,3), (4,-1), (6,4)$ ↗ shift to origin

$(0,0), (2,5), (6,1), (8,6)$

area $= \left|\begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}\right|$

area $= |(2-30)| = 28$

Calculate area of parallelepiped → $\underbrace{(2,0,0), (1,3,0), (0,1,4)}_{\text{adjacent vertices}}, (0,0,0)$

volume $= \left|\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}\right|$ → triangular (in echelon form)

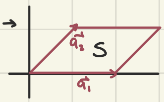
volume $= |2 \cdot 3 \cdot 4| = 24$

Determinants and Linear Transformations

If $T_A: \mathbb{R}^n \to \mathbb{R}^n$, and S is some parallelogram in $\mathbb{R}^n$, then volume $(T_A(S)) = |\det(A)| \cdot$ volume (S)

$T_A(\vec{x}) = A\vec{x}$ (scale fac)

↳ 2x2 Proof →



Any point in S is $c_1\vec{a_1} + c_2\vec{a_2}$, $0 \le c_1, c_2 \le 1$

new parallelogram

transformation x matrix (parallelogram)

Transform S using $T_A \to T_A(c_1\vec{a_1} + c_2\vec{a_2}) = A(c_1\vec{a_1} + c_2\vec{a_2}) - c_1A\vec{a_1} + c_2A\vec{a_2} \to (A\vec{a_1} + A\vec{a_2}) = A(\vec{a_1}\ \vec{a_2}) \to AA$

Volume of S

area $= \det|AA| \to \det|A| \det|A|$

Example

R is $p_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $p_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, if $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ (transformation matrix), what is the area of the image R under map $\vec{x} \to A\vec{x}$?

$(T_A(S)) = |\det|A| \cdot \text{volume } S|$

$= |(1+1)(3-8)|$

$= |2(-5)|$

$= 10$

$T_A = A\vec{x}$, A is 2x2 matrix, reflects vectors through $\mathbb{R}^2$ through $x_1 = x_2$, then projects them into line $x_1 = 0$; calculate determinant of A

$T_A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$T_A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\det T_A = (0 \cdot 0 - 0 \cdot 1)$

$\det T_A = 0$

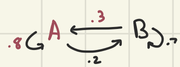
Markov Chain/Steady States

A small town 2 libraries, A and B

↳ After 1 month, in the books checked out of A

↳ 80% returned to A

↳ 20% returned to B



↳ After 1 month, in the books checked out of B

↳ 30% returned to A

↳ 70% returned to B

If both libraries have 1000 books today, how many books does each library have after 1 month / 1 year / n months?

books evenly distributed $\to x_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix} \begin{matrix} \to A \\ \to B \end{matrix}$, $x_1 = \begin{pmatrix} \text{prop. of books at } A \\ \text{prop. of books at } B \end{pmatrix} = \begin{pmatrix} .5(.8) + .5(.3) \\ .5(.2) + .5(.7) \end{pmatrix} = \underbrace{\begin{bmatrix} .8 & .3 \\ .7 & .2 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} .5 \\ .5 \end{bmatrix}}_{x_0}$

after 1 month

2 months $\to x_2 = P\vec{x_1} = P(P\vec{x_0}) = P^2 x_0$

k months $\to x_k = P^k x_0$

} Markov Chain

Probability Vector – a vector, $\vec{x}$, with non-negative elements that add to 1

Stochastic Matrix – square matrix P whose columns are probability vectors

$$\begin{array}{cc|c} \multicolumn{2}{c}{\text{From}} & \\ A & B & \\ - & - & A \\ - & - & B \end{array} \ \text{To}$$

Markov Chain – a sequence of probability vectors $\vec{x}_k$ and a stochastic matrix P such that $\vec{x}_{k+1} = P\vec{x}_k$, $k = 0, 1, 2\ldots$

Steady-State Vector – for P, a probability vector $+\vec{q}$ such that $P\vec{q} = \vec{q}$

↳ Determine $\vec{q}$ for the matrix P

$P = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}\vec{q} = \vec{q}$

$= P\vec{q} - \vec{q} = 0$

$= P\vec{q} - I\vec{q} = 0,\ I\vec{q} = \vec{q}$

$= (P - I)\vec{q} = 0$

$P - I = \begin{pmatrix} -.2 & .3 \\ .2 & -.3 \end{pmatrix} \to \left(\begin{array}{cc|c} -.2 & .3 & 0 \\ .2 & -.3 & 0 \end{array}\right)$ rows are multiples

↳ $-1(R_2) \to -.2\ \ .3\ \ 0$

$-2x_1 + 3x_2 = 0$, set $x_2 = 2$, $x_1 = 3$ } $\vec{q} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}\frac{1}{5}$

entries need to add up to 1

$2 + 3$

Markov Chain Convergence

As k in $[\vec{x}_{k+1} = P\vec{x}_k, k = 0, 1, 2\ldots] \to \infty$, does $\vec{x}_1, \vec{x}_2$ <u>converge</u> to a steady-state?

Regular stochastic matrix – if there is some k such that P^k <u>only</u> contains positive entries

↳ ex. $A = \begin{pmatrix} .1 & .7 \\ .9 & .3 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & 7 \\ 9 & 3 \end{pmatrix}$

If P is a regular stochastic matrix, then P has a <u>unique</u> steady-state vector $\vec{q}$, $\vec{x}_{k+1} = P$ <u>converges</u> to $\vec{q}$ as $k \to \infty$

A car rental company has 3 locations: A, B, and C → returned to

	A	B	C
A	.8	.1	.2
B	.2	.6	.3
C	0	.3	.5

← rented from

↳ make a stochastic matrix P

↳ What happens to the distribution over time?

↳ assume P is regular

a) $P = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ 0 & .3 & .5 \end{bmatrix}$

b) $(P - I)\vec{q} = \vec{0}$

$= \begin{bmatrix} -.2 & .1 & .2 \\ .2 & -.4 & .3 \\ 0 & .3 & -.5 \end{bmatrix}\vec{q}$

$= \frac{1}{10}\begin{bmatrix} -2 & 1 & 2 \\ 2 & -4 & 3 \\ 0 & 3 & -5 \end{bmatrix}\vec{q}$

$\vec{q}$ is probability vector in NULL SPACE of $P-I$ (solution set)

$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 2 & -4 & 3 & 0 \\ 0 & 3 & -5 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 6 & 0 & -11 & 0 \\ 0 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, x_3 is free, if $x_3 = 6$, then $x_1 = 11$ AND $x_2 = 10$ $\Big\} \vec{q} = \begin{bmatrix} 11 \\ 10 \\ 6 \end{bmatrix}\frac{1}{27}$

After a long time, the distribution of cars is given by $\vec{q}$

Eigenvalues / Eigenvectors

Consider $T_A = \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}^{A \cdot \vec{x}} \to$ Can you state a non-zero vector $\vec{v} \in \mathbb{R}^2$ that satisfies $A\vec{v} = \lambda\vec{v}$, where $\lambda \in \mathbb{R}$?

↳ reflects over $x_2 = x_1$

↳ Any vector ON the line (reflected) → ex. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, or $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ — $T(\vec{v}_1) = A(\vec{v}_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda\vec{v}_1$, $\lambda = 1$

↳ Scalar multiplication to negative $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ → $T(\vec{v}_1) = A(\vec{v}_1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda\vec{v}_1$, $\lambda = -1$

If $Ax = \lambda x$ has a nontrivial solution, λ is an eigenvalue

If $A \in \mathbb{R}^{n \times n}$ (square) and there is a $\vec{v} \neq 0$ in $\mathbb{R}^n$, AND $A\vec{v} = \lambda\vec{v}$

↳ Eigenvector – $\vec{v}$ for A (can't be $\vec{0}$)

↳ Eigenvalue – $\lambda \in \mathbb{C}$ (could be complex, makes $A - \lambda I$ singular)

↳ If $\lambda \in \mathbb{R}$, then $\begin{cases} \lambda > 0, A\vec{v} \text{ and } \vec{v} \text{ same direction} \\ \lambda < 0, A\vec{v} \text{ and } \vec{v} \text{ different direction} \end{cases}$

Which of the following are eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

a) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $Av = \lambda v$, $Av = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\lambda = 2$ ✓

b) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $Av = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$, $\lambda = 2$ ✓

c) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $Av = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\lambda = 0$ ✓

d) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $Av = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, ✗

e) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $Av = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, ✗ eigenvectors can't be 0

Would HAVE to be singular, b/c $\vec{v}$ CANNOT be 0 vector

Determine whether $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$ → $A\vec{v} = \lambda\vec{v}$ → $A\vec{v} - 3\vec{v} = 0$ → $(A - 3I)\vec{v} = 0$

↳ $\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$ singular, columns are dependent → $\lambda = 3$ is an eigenvalue

Eigenspaces

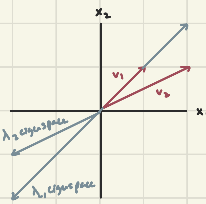
$A \in \mathbb{R}^{n \times n}$, eigenspace for a given λ (eigenvalue) span a subspace of $\mathbb{R}^n$ called λ-eigenspace of A

↳ λ-eigenspace for matrix A is $\text{Null}(A - \lambda I)$

Sketch eigenvectors / eigenspaces:

$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$, $\lambda = -1, 2$ → $\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix}$ → $1x_1 - x_2 = 0$ → $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix}$ → $1x_1 - 2x_2 = 0$ → $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$



basis for λ_1 eigenspace → $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

basis for λ_2 eigenspace → $\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$

Construct a basis for eigenspaces of the matrix:

$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, $\lambda = 1, 3$ → $(A - \underset{(1)}{\lambda_1} I \mid 0) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$, $x_1 = 0$, $x_2 = -x_3$, x_3 is free, $v_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

→ $(A - \underset{(3)}{\lambda_2} I \mid 0) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$, $x_2 = -x_2$, x_1, x_3 is free, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

↳ basis for λ_1 eigenspace: $\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$

↳ basis for λ_2 eigenspace: $\{v_2, v_3\}$

Eigenvalue Theorems

If A is a real (all values are $\mathbb{R}$) $n \times n$ matrix

↳ If triangular, diagonal elements are eigenvalues (CANNOT ROW REDUCE TO TRIANGULAR)

↳ If A is not invertible, 0 is an eigenvalue (and vice versa)

↳ Stochastic Matrices have an eigenvalue of 1

↳ If $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues, then $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_k$ are linearly independent

Give 2 eigenvalues

1. $A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ → stochastic, $1 = \lambda_1$
↳ linearly dependent, not invertible, $0 = \lambda_2$

2. $B = \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$ → triangular, $\lambda_1 = 2$, $\lambda_2 = 5$

Characteristic Polynomial

λ is an eigenvalue of A WHEN $(A-\lambda I)$ is not invertible

↳ To calculate eigenvalues, solve $\det(A-\lambda I)=0$

$\det(A-\lambda I)$ is the characteristic polynomial of A

$\det(A-\lambda I)=0$ is the characteristic equation of A

roots of the characteristic polynomial are the eigenvalues of A

trace of a matrix – sum of diagonal elements

↳ $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, char equation $\rightarrow 0=\det(M-\lambda I)=(a-\lambda)(d-\lambda)-bc$

$= \lambda^2-(a+d)\lambda+(ad-bc)$

$= \lambda^2-(\text{trace } M)\lambda+(\det M)$

Calculate Eigenvalues

$A=\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

Characteristic Polynomial $=\det(A-\lambda I)$

$=\det\begin{pmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$

$=(4-\lambda)(1-\lambda)-(4)$

$=\lambda^2-5\lambda+4-4=\lambda^2-5\lambda$

$=\lambda(\lambda-5)$

eigenvalues $=0, 5$
(when $\lambda(\lambda-5)=0$)

Calculate Eigenvalues

$A=\begin{pmatrix} 6 & 18 \\ 3 & 9 \end{pmatrix}$ → bc $3v_1=v_2$, multiples of columns

singular

$0=\lambda^2-\text{trace}A\,\lambda+\det A$

$=\lambda^2-15\lambda+(\overset{(0)}{54-54})$

$=\lambda(\lambda-15)$

eigenvalues $=0, 15$

Algebraic and Geometric Multiplicities

algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial

↳ The number of times that eigenvalue repeats in that characteristic polynomial

↳ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow (1-\lambda)(0-\lambda)(-1-\lambda)(0-\lambda)$

0 repeats 2 times

λ	alg. mult:
1	1
0	2
-1	1

geometric multiplicity of an eigenvalue λ is the dimension of $\text{Null}(A-\lambda I)$

↳ Always at LEAST 1; can be smaller than algebraic multiplicity

↳ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow (0-\lambda)(0-\lambda)$

λ	am
0	2

→ $\lambda=0$ is ONLY eigenvalue, am $=2$, gm $=1$

↳ $\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$, $x_2=0$, x_1 is free, $v_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

only vector for eigenvalues

A is $m\times n$ matrix, a_i is am for λ_i, g_i is gm for λ_i

↳ $1 \le a_i \le n$

↳ $1 \le g_i \le a_i$

Construct a 4x4 matrix with $\lambda=0$ the only eigenvalue, but the gm of λ_i is one

$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow (0-\lambda)(0-\lambda)(0-\lambda)(0-\lambda)$

λ	am	g
0	4	1

g → 3 pivots, 1 free variable

↳ $\text{Null}(A-0I)$ → 1

1 real eigenvalue, algebraic 2

$\begin{pmatrix} -3 & k \\ 2 & -6 \end{pmatrix} \rightarrow \det(A-\lambda I)$

$=\det\begin{pmatrix} -3-\lambda & k \\ 2 & -6-\lambda \end{pmatrix}$

$=(-3-\lambda)(-6-\lambda)-2k$

$=\lambda^2+9\lambda+18-2k$

complete the square ↳ $\left(\frac{9}{2}\right)^2=\frac{81}{4}$

$18-2k=\frac{81}{4}$

$72-8k=81$
$-72 \qquad -72$

$-8k=9$
$\div -8$

$k=-\frac{9}{8}$

Markov Chains and Eigenvectors

$\vec{x}_{k+1}=P\vec{x}_k$, $k=0,1,2\ldots$ as $k\rightarrow\infty$

↳ If P is a regular stochastic matrix, there will be a unique steady-state

↳ If P is NOT regular, how would you find $\vec{x}_{k+1}$?

$\vec{x}_{k+1}=P\vec{x}_k=\begin{pmatrix} 1 & .1 \\ 0 & .9 \end{pmatrix}\vec{x}_k$, $k=0,1\ldots$ $\vec{x}_0=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A \xleftarrow{.1} B$ 5.9 ← Use eigenvalues

$\lambda_1=1$, $\vec{v}_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\lambda_2=.9$, $\vec{v}_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$x_{k+1}=Px_k$

$x_0=c_1v_1+c_2v_2$ basis for R^2

$x_1=Px_0$

$Pv_1=\lambda v_1$

$x_1=P(c_1v_1+c_2v_2)=c_1Pv_1+c_2Pv_2$

→ $x_1=c_1\lambda_1v_1+c_2\lambda_2v_2$

$x_2=Px_1=P(c_1\lambda_1v_1+c_2\lambda_2v_2)$

$x_2=c_1\lambda_1^2v_1+c_2\lambda_2^2v_2$

$x_3=c_1\lambda_1^3v_1+c_2\lambda_3^3v_3$

$x_k=Px_{k-1}$

$x_k=c_1\lambda_1^kv_1+c_2\lambda_2^kv_2$

$\lambda_1=1$, $\lambda_2=.9$, so as $x\rightarrow\infty$, $x_k\rightarrow c_1v_1$

$1^\infty=1$, $.9^\infty\rightarrow 0$

Eigenvalues of a stochastic matrix $\overset{3\times 3}{A}$ are
$\lambda_1=1$, $\lambda_2=\frac{1}{4}$, $\lambda=\frac{1}{8}$

If $\vec{p}$ is in R^3, what does $A^k\vec{p}$ converge to?

↳ $A^k\vec{p}=A^k(c_1v_1+c_2v_2+c_3v_3)$

↳ $A^k\vec{p}=c_1\lambda_1^{k\rightarrow 1}v_1+c_2\lambda_2^{k\rightarrow 0}v_2+c_3\lambda_3^{k\rightarrow 0}v_3$

↳ $A^k\vec{p}=c_1v_1$

→ What is c_1/c_2?

$x_0=c_1v_1+c_2v_2$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}=c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}+c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

$c_2=0$,
$c_1=1$

Similar Matrices

If A is an $m \times n$ matrix, A^k for large k is hard

↳ Especially if n is large and a lot of the elements $\neq 0$

① $n \times n$ matrices A and B are similar if there is a P so that $A = PBP^{-1}$

↳ if $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, then $P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \to A$ is similar to B

② if A and B are similar, they have the same characteristic polynomial

↳ if same characteristic polynomial, same eigenvalues

↳ BUT same eigenvalues $\underset{\text{(not always)}}{\neq}$ similar

true or false?

a) If A is similar to I, then $A = I$ ✓

$A = PIP^{-1}$, $A = PP^{-1}$, $A = I$

b) If A is similar to B, and $A = PBP^{-1}$, then $A^2 = PB^2P^{-1}$

$AA = (PBP^{-1})(PBP^{-1}) \to A^2 = PBBP^{-1} \to A^2 = PB^2P^{-1}$ ✓

c) If A and B has the same eigenvalues, A and B are similar ✗

(not necessarily)

Diagonalize

Why? → Hard to find large A^k

↳ More efficient way for A^k

diagonal - only non-zero elements are in main diagonal

↳ I_n, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

↳ $A = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, $A^2 = \begin{pmatrix} 3^2 & 0 \\ 0 & 5^2 \end{pmatrix}$, $A^k = \begin{pmatrix} 3^k & 0 \\ 0 & 5^k \end{pmatrix}$

diagonizable – If A is similar to diagonal matrix D, $A = PDP^{-1}$

↳ $A = \underbrace{(\vec{v}_1, \vec{v}_2 \dots)}_{\text{linearly independent eigenvectors}} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \dots \end{bmatrix}}_{\text{corresponding eigenvalues (in order)}} (\vec{v}_1, \vec{v}_2 \dots)^{-1}$

A is diagonizable $\underset{\text{if and only if}}{\longleftrightarrow}$ A has n linearly independent eigenvectors

Diagonalize if possible

$\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix} \to (2-\lambda)(-1-\lambda)$

$\lambda_1 = 2 \to \left(\begin{array}{cc|c} 0 & 6 & 0 \\ 0 & -3 & 0 \end{array}\right) \to$ x_1 is free, $6x_2 = 0$ $\quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -1 \to \left(\begin{array}{cc|c} 3 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right) \to$ x_2 is free, $x_1 = -2x_2$ $\quad v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$A = PDP^{-1} \to \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

↳ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \lambda_1 = 1 \to \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ x_1 is free, $x_2 = 0$ $\quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

↳ NOT Diagonalizable

Diagonalization Theorems

If A is $n \times n$ and has n distinct eigenvalues, A is diagonalizable

↳ For an $n \times n$ matrix to be diagonalizable, it must have n linearly independent eigenvectors

↳ It is NOT necessary to have n distinct eigenvalues (I is diagonalizable)

Non-Zero Square matrix in RREF, diagonalizable, singular

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

↳ Just b/c a matrix is NOT invertible $\neq$ NOT diagonalizable

If A has repeated eigenvalues, how can it be diagonalizable?

→ A is $n \times n$

→ A has distinct eigenvalues $\lambda_1, \dots \lambda_k$, $\underset{\text{distinct}}{k} \leq n$

→ a_i = algebraic multiplicity of λ_i

→ g_i = dimension of λ_i eigenspace

↳ geometric multiplicity

— A is diagonalizable if and only if $g_i = a_i \longleftrightarrow \Sigma g_i = n$

— A is diagonalizable $\longleftrightarrow$ the eigenvectors, for all eigenvalues, form a basis for R^n

For what values of k is $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ diagonalizable?

↳ Test $k = 0 \to A = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, ✓

↳ Test $k \neq 0 \to A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \to (A - I \mid 0) = \left(\begin{array}{cc|c} 0 & k & 0 \\ 0 & 0 & 0 \end{array}\right) \to \underset{\text{ONLY VECTOR}}{\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$, ✗

How to Diagonalize a matrix with repeated eigenvalue?

$\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 3$, Construct $AP = PD$ when $A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$

↳ $\lambda_1 = 1$, $A - \lambda_1 I = \begin{pmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to$ Null Space $= v_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

↳ $\lambda_2 = 2$, $A - \lambda_1 I = \begin{pmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to v_2$, $x_1 = -x_2 - 4x_3$

↳ $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 2 & -1 & -4 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

Can a matrix be diagonalized?

↳ To diagonalize, you need D, P, and P^{-1}

↳ D is from <u>eigenvalues</u>, always able to make

↳ P MUST be $n \times n$ <u>and</u> invertible (linearly independent eigenvectors)

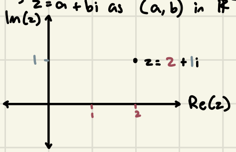
↳ NOT every A will have linearly independent eigenvecto

Review of Complex Numbers

$x^2+1=0$, $x=\pm\sqrt{-1}$ or $\pm i$

↳ $\mathbb{C}=\{a+bi \mid a,b \text{ in } \mathbb{R}\}$

↳ $z=a+bi$ as (a,b) in $\mathbb{R}^2$



conjugate complex numbers: $\overline{a+bi}=a-bi$

absolute value of complex: $|a+bi|=\sqrt{a^2+b^2}$

polar form of complex: $a+ib=r(\cos\phi+i\sin\phi)$

↳ $r=|a+bi|$ $\tan\phi=\frac{b}{a}$

If x and y are complex numbers, $\vec{v}\in\mathbb{C}^n$

↳ $\overline{(x+y)}=\bar{x}+\bar{y}$

↳ $A\in\mathbb{R}^{n\times n}$, $\overline{A\vec{v}}=A\bar{\vec{v}}$

↳ $\underset{\text{imaginary}}{\mathrm{Im}}(x\bar{x})=0$

Complex Roots of Characteristic Polynomial

Every Polynomial of degree n has exactly n complex (real included) roots, counting multiplicity

↳ $(x-2)^2$ is 2nd order polynomial, 2 roots

If $\lambda\in\mathbb{C}$ is a root of a real polynomial $p(x)$, then conjugate $\bar{\lambda}$ is <u>also</u> a root of $p(x)$

↳ if λ is an eigenvalue of a real matrix A w/ eigenvector $\vec{v}$, $\bar{\lambda}$ is an eigenvalue of A w/ eigenvector $\bar{\vec{v}}$

If A has eigenvector $\vec{v}=\begin{pmatrix}1\\i\end{pmatrix}$, give another eigenvector of A

$\vec{v}_2=\begin{pmatrix}1\\-i\end{pmatrix}$

4 of eigenvectors of 7×7 matrix are $-2, 4+i, -4-i,$ and i

other 3: $4-i, -4+i, -i$

Rotation Dilation Matrices

rotation-dilation matrix → $\begin{pmatrix}a & -b\\ b & a\end{pmatrix}=C$

↳ eigenvalues of C → $0=\det(C-\lambda I)=(a-\lambda)^2+b^2=\lambda^2-2a\lambda+a^2+b^2$

↳ $\lambda=\frac{2a}{2}\pm\frac{1}{2}\sqrt{(-2a)^2\pm4(a^2+b^2)}=a\pm ib \to r(\cos\phi\pm\sin\phi)$

↳ where $r^2=a^2+b^2$, $\tan\phi=\frac{b}{a}$

↳ $\lambda=a\pm bi$

determine eigenvalues of $\begin{pmatrix}2 & -3\\ 3 & 2\end{pmatrix}$

$\lambda=a\pm bi$, $\lambda=2\pm3i$

PCP^{-1} Decomposition

If A is a real 2x2 matrix with eigenvalue $\lambda=a-bi$ $(b\neq0)$ and associated eigenvector $\vec{v}$, $A=PCP^{-1}$ where $P=(\underset{\text{real}}{\mathrm{Re}\,\vec{v}}\ \ \underset{\text{imaginary}}{\mathrm{Im}\,\vec{v}})$ and $C=\underbrace{\begin{pmatrix}a & -b\\ b & a\end{pmatrix}}_{\text{rotation-dilation matrix}}$

Construct matrices P and C such that $AP=PC$

$A=\begin{pmatrix}1 & -2\\ 1 & 3\end{pmatrix}$, $\lambda=2\pm i$ → picked $2-i$ as eigenvalue

① Find eigenvectors (for P) → $(A-\lambda I \mid 0)$ → $\left(\begin{array}{cc|c}1-\lambda & -2 & 0\\ 1 & 3 & \end{array}\right)$ → $(1-\lambda)x_1-2\underset{\text{free}}{x_2}=0$

don't need 2nd row, $A-\lambda I$ HAS to be singular

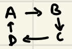
$v_1=\begin{pmatrix}2\\1-\lambda\end{pmatrix}=\begin{pmatrix}2\\1-(2-i)\end{pmatrix}=\begin{pmatrix}2\\-1+i\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix}+i\begin{pmatrix}0\\1\end{pmatrix}$

② $P=(\text{Real}\ \ \text{Imaginary})=\begin{pmatrix}2 & 0\\ -1 & 1\end{pmatrix}$, $C=\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\to\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}$ $2-i$

Long-Term Markov Chain

Is the corresponding stochastic matrix regular?

$$P=\begin{array}{c}\\ \\ \\ \\ \text{from}\end{array}\begin{array}{c}\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}\\ \begin{matrix}A & B & C & D\end{matrix}\end{array}\begin{array}{c}A\\B\\C\\D\\ \\ \end{array}$$ to



, $\vec{x}_0=\frac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ steady state, chain with transition matrix

If P is a regular stochastic $m\times m$ matrix with $m\geq2$, then:

↳ For any initial probability vector $\vec{x}_0$, $\lim_{n\to\infty}P^n\vec{x}_0=\vec{q}$

↳ P has a unique eigenvector $\vec{q}$ with eigenvalue $\lambda=1$

↳ Stochastic Matrix II such that $\lim_{n\to\infty}P^n=II$

↳ Eigenvalues of P satisfy $|\lambda|\leq1$

To investigate long-term behavior

↳ compute steady-state vector, solve $(P-I)\vec{q}=0$

↳ compute $P^n\vec{x}_0$ for large n (requires a computer)

↳ compute P^n for large n, each column of resulting matrix is steady-state

A steady-state vector for a stochastic matrix is an eigenvector true; $P\vec{q}=\vec{q}$, eigenvector

Example of a 2x2 stochastic matrix in echelon form; a steady-state vector is $\vec{q}=\begin{pmatrix}1\\0\end{pmatrix}$

↳ $A\begin{pmatrix}1\\0\end{pmatrix}=1\begin{pmatrix}1\\0\end{pmatrix} \rightarrow A\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \rightarrow A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

P is a regular stochastic matrix $\vec{r}=\frac{1}{6}\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $x_0=\frac{1}{10}\begin{pmatrix}9\\0\\1\end{pmatrix}$, where does $P^k\vec{x}_0$ converge to?

↳ $\vec{x}_k$ converges to $\vec{r}$, P is a regular stochastic matrix

$P=\begin{pmatrix}.8&.2\\.2&.8\end{pmatrix}$, $\vec{x}_0=\begin{pmatrix}0\\1\end{pmatrix}$, find $\vec{x}_k$ as $k\to\infty$, $\lambda=1$ and $.6$

↳ $\begin{pmatrix}-.2&.2\\.2&-.2\end{pmatrix}$, $\lambda=1 \rightarrow \begin{pmatrix}1&-1\\0&0\end{pmatrix}$, $x_1=x_2$, x_2 is free $\rightarrow \vec{q}=\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{2}$

Page Rank

Page Rank (PR) - algorithm to order search results

↳ Collection of webpages with links to each other

↳ Users who are navigating the web

↳ Set of rules that govern how users navigate web

↳ A user on a webpage is equally likely to go to any page their webpage links to

↳ If a user does not link to other pages, the user stays on that page

↳ Distribution is modeled using $\vec{x}_{k+1}=P\vec{x}_k$ ← $n \times n$ stochastic; n ↑ number of pages in a web

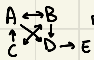
Limitations:

↳ Will our transition be regular stochastic?

↳ How can we make a model that is regular stochastic?

↳ What about pages that dont link to other pages?

Construct a Markov Chain



$$\begin{array}{c|c}
\text{From} & \begin{array}{ccccc} A & B & C & D & E \end{array} & \\
& \begin{bmatrix} 0 & .33 & .5 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & .33 & 0 & 0 & 0 \\ .5 & .33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \text{ To}
\end{array}$$

↳ transition matrix

$\vec{q} \rightarrow$ long-term behavior of users on webpage

importance are entries of $\vec{q}$

Page Rank – ranking of each page based on importance

↳ Two pages with same importance has the same Page Rank

Computing Page Rank

Adjustments

1) If a page does not link to another page, the user will choose any page on that web with equal probability $\rightarrow P_*$

2) A user at any page will navigate to any links in their page with = probability p, and any page in the web with equal probability $1-p$

↳ Transition matrix $G = pP_* + (1-p)K$ ← all elements $= \frac{1}{n}$

has to be regular stochastic when $0 \le p < 1$

damping factor ↑ (Google uses $.85 = p$)