# Functions of Several Variables

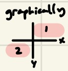
D is a set of n-tuples of real numbers $(x_1, x_2, ..x_n)$

↳ real-valued function f on D is a rule that assigns a unique single real number to each element in D

↳ if $w = f(x_1, x_2, x_3, ... x_n)$, the set D is called the function's domain (input)

↳ set of values in $w$ = range (output, answers)

Find the domain of each: (easiest to look at where domain is NOT defined)

① $f(x,y) = \sqrt{xy} \rightarrow \underset{\text{general}}{xy \geq 0} \rightarrow \underset{\text{formal}}{\{(x,y) \mid xy \geq 0\}} \rightarrow$ graphically 

② $f(x,y) = \frac{1}{\sqrt{x-y}} \rightarrow x - y > 0 \rightarrow \{(x,y) \mid x > y\}$

③ $f(x,y,z) = \frac{\sqrt{z}}{x^2 - y^2} \rightarrow z \geq 0, x^2 - y^2 \neq 0 \rightarrow \{(x,y,z) \mid z \geq 0, x^2 \neq y^2\}$ or $|x| \neq |y|$

Give domain and range of $f(x,y) = \sqrt{4-x^2} - \sqrt{9-y^2}$

↳ domain → $4 - x^2 \geq 0, 9 - y^2 \geq 0 \rightarrow x^2 \leq 4, y^2 \leq 9$

↳ $\{(x,y) \mid x^2 \leq 4, y^2 \leq 9\}$ or $\{(x,y) \mid -2 \leq x \leq 2, -3 \leq y \leq 3\}$

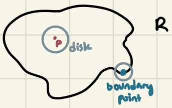
↳ range → smallest value, minimize $\sqrt{4-x^2}$, maximize $\sqrt{9-y^2}$

↳ largest value, maximize $\sqrt{4-x^2}$, minimize $\sqrt{9-y^2}$

↳ $[-3, 2]$ → will be an interval, not a set

# Boundary Points

A point $(x_0, y_0)$ is an interior point of a set/region of R in the xy-plane if it is at the center of a disk entirely in R




↳ boundary points of a set/region R have disks with points outside and inside of R (does not need to be in R)

↳ Closed Sets contains all of its boundary points

↳ Open Sets contains none of its boundary points

} Otherwise, neither

Bounded regions - lies inside a disk with finite radius

↳ Unbounded regions - are not bounded

Interior Point - point in the center of a solid sphere in R

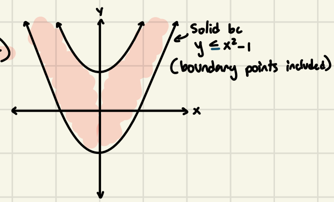
↳ boundary points - spheres with points inside/outside R

Describe domain of $f(x,y) = \cos^{-1}(y - x^2)$

↳ arccos needs argument $-1 \rightarrow 1$

↳ $-1 \leq y - x^2 \leq 1$, $x^2 - 1 \leq y \leq x^2 + 1$

↳ closed and unbounded




# Level Curves and Surfaces

If c is a value in the range of f, then we can sketch $f(x,y) = c$

↳ This is called a level curve

Level Surface - set of points in space where a function of 3 variables = a constant → $f(x,y,z) = c$

# Limits for Functions of Several Variables

Let f be a function defined at least on some deleted neighborhood of $x_0$ ⟶ Formal Definition of Limit

↳ $\lim_{x \to x_0} f(x) = L$

↳ if for every $\epsilon > 0$, $\delta > 0$ such that if $0 < \|x - x_0\| < \delta$ implies that $|f(x) - L| < \epsilon_s$

A function of 2/more variables approaches a limit L as $(x,y,z)$ approaches $(x_0, y_0, z_0)$ ⟶ Multivariable Definition of Limit

↳ $\lim_{(x,y) \to (x_0, y_0)} f(x,y) = L$

↳ if for every number $\epsilon > 0$ there exists a $\delta > 0$, such that for all $(x,y)$ in the domain of f

↳ $|f(x,y) - L| < \epsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

Find $\lim_{(x,y) \to (0,0)} f(x,y) = \frac{y^2}{x^2+y^2} = \frac{0}{0}$ ← indeterminant

Check Paths

↳ along x-axis $(x,y) \rightarrow (x,0)$ so $f(x,0) = \frac{0}{x^2}$

↳ $\lim_{x \to 0} \frac{0}{x^2} = \lim_{x \to 0} 0 = 0$

↳ along y-axis $(x,y) \rightarrow (0,y)$ so $f(0,y) = \frac{y^2}{y^2}$

↳ $\lim_{y \to 0} \frac{y^2}{y^2} = \lim_{y=0} 1 = 1$

≠, so DNE

Find $\lim_{(x,y) \to (0,0)} f(x,y) = \frac{2xy^2}{x^2+y^2}$

↳ $|f(x,y) - L| < \epsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2}$

↳ $\left| \frac{2xy^2}{x^2+y^2} - 0 \right| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta \rightarrow 0 < 2\sqrt{x^2+y^2} < 2\delta$

all squares are +

$\frac{2|x|y^2}{x^2+y^2} \leq 2|x| = 2\sqrt{x^2} \leq 2\sqrt{x^2+y^2} < \epsilon$, let $\delta = \frac{\epsilon}{2}$

$y^2 \leq x^2 + y^2$, so $\frac{y^2}{x^2+y^2} \leq 1$

# Continuity for Functions of Several Variables

A function of $f(x,y)$ is continuous at a point $(x_0,y_0)$ if:

① $f$ is defined at $(x_0,y_0)$
② $\lim_{(x,y)\to(0,0)} f(x,y)$ exists
③ $\lim_{(x,y)\to(0,0)} f(x,y) = f(x_0,y_0)$

$\Big\}$ $f(x,y)$ is continuous $\longleftrightarrow$ continuous at every point in domain

At what points in the plane is $f(x,y) = \frac{x+y}{2+\cos(x)}$ continuous?

↳ discontinuous when $2+\cos(x)=0$, but since $-1 \le \cos(x) \le 1$, never discontinuous

↳ Continuous at all points $(x,y)$

# Partial Derivatives I

The partial derivative of $f(x,y)$ with respect to $x$ at the point $(x_0,y_0)$ is:

↳ $\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$ provided the limit exists ⟶ treats $y$ as a constant

↳ respect to $y$ ⟶ $\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$ ⟶ treats $x$ as a constant

A function $z=f(x,y)$ is differentiable at $(x_0,y_0)$ if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and $\Delta z = f(x_0+\Delta x, y_0+\Delta y)$ satisfies an equation of the form:

↳ $\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$

↳ $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$

↳ $f$ is differentiable if it is differentiable at every point in its domain

↳ graph has a smooth surface

Find all partial derivatives of $f(x,y) = 3x^2 - 2y + xy$

$f_x = \frac{\partial f}{\partial x} = 6x+y$ , $f_y = \frac{\partial f}{\partial y} = -2+x$

# Partial Derivatives II

Higher Order Derivatives

↳ $(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

↳ $(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$ ← Mixed Partial Derivative

→ If $f(x,y)$ and all its partial derivatives $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ are defined throughout an open region containing a point $(x_0,y_0)$, and all are continuous at $(x_0,y_0)$, then $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$

Implicit Differentiation

↳ if we want to find $f_z$ with respect to $x$ or $y$, we assume $z = z(x,y)$

↳ Use chain rule → ex. adding on $\frac{\delta z}{\delta x}$ when diff. $z$ with respect to $x$

# The Chain Rule

If $w=f(x,y)$ is differentiable and $x=x(t)$, $y=y(t)$ are differentiable functions of $t$, then $w=f(x(t),y(t))$ is differentiable

↳ $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

↳ $w=f(x(t),y(t),z(t)) \to \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial y}\frac{dz}{dt}$

# Implicit Differentiation

If $F(x,y)$ is differentiable and $F(x,y)=0$ describes $y$ as a differentiable function of $x$,

↳ $\frac{dy}{dx} = -\frac{F_x}{F_y}$ if $F_y \neq 0$

↳ $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$ if $F_z \neq 0$

# Directional Derivatives

Directional Derivative — $f'_u(x_0,y_0)$ or $D_u f(P_0) \rightarrow f$ in the direction of $u$ at the point $P_0 = (x_0,y_0)$

↳ The rate of change of f in the u-direction

If $u = u_1 i + u_2 j$, then:

↳ $D_u f(P_0) = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$

$\nabla f(x,y)\big|_{P_0(x_0,y_0)} \cdot u$

Find the directional derivative of $f(x,y) = x^2 + xy^2$ at $P(1,1)$ in the direction $i-j$

↳ $|i-j| = \sqrt{2}$, so $u = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$

$D_u f(1,1) = \lim_{s\to 0} \frac{f(1+s/\sqrt{2},\ 1+s/\sqrt{2}) - f(1,1)}{s} \rightarrow \lim_{s\to 0} \frac{(1+s/\sqrt{2})^2 + (1+s/\sqrt{2})(1+s/\sqrt{2})^2 - 2}{s} \rightarrow \lim_{s\to 0} \frac{s/\sqrt{2} + s^3/\sqrt{2}}{s} = \lim_{s\to 0} \frac{1}{\sqrt{2}} + \frac{s^2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$

# Gradient

Gradient of a function is vector:

↳ $\nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

Nabla (Operator) → $\nabla(f(x) + g(x)) = \nabla f(x) + \nabla g(x)$

↳ $\nabla(\alpha f(x)) = \alpha \nabla f(x)$

↳ $\nabla(f(x) g(x)) = f(x)\nabla g(x) + \nabla f(x) g(x)$

Find the gradient of $f(x,y) = 2e^x \sin(x^2+y)$

$\nabla f(x,y) = (4e^x \cos(x^2+y)x + 2e^x \sin(x^2+y))i + (2e^x \cos(x^2+y))j$

Directional Derivative → $f'_u(P_0) = \nabla f(x_0,y_0) \cdot u$

↳ f increases most rapidly when $\cos\theta = 1$ (when $\theta = 0$, $u$ in direction of $\nabla f$) → at each point P in domain, f increases most rapidly in direction of gradient vector at P and the derivative is $D_u f = \|\nabla f\|$

↳ f decreases most rapidly in $-\nabla f$ and the derivative in this direction is $D_u f = -\|\nabla f\|$

↳ Any direction u orthogonal to a gradient $\nabla f \neq 0$ is a direction of 0 change in f

Find a unit vector in the direction in which f increases most rapidly at P and give a rate of change

↳ $f(x,y) = y^{-2} e^{2x}$ at $P(0,1) \rightarrow \nabla f = \left(\frac{2e^{2x}}{y^2}\right) i + \left(\frac{-2e^{2x}}{y^3}\right) j$

↳ $\nabla f(0,1) = 2i - 2j \rightarrow \|\nabla f(0,1)\| = \sqrt{8} = 2\sqrt{2}$

↳ $u_{\nabla f} = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$

# Tangent Lines to Level Curves

Tangent Line to Level Curve $f(x,y) = c$ at point $(x_0,y_0)$ is: $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$

Derivative along a path — if $r(t) = x(t)i + y(t)j + z(t)k$ is a smooth path, C, and $w = f(r(t))$ is a scalar function along C, then:

↳ derivative, $\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$

Find the rate of change of f with respect to t along the curve: $f(x,y) = x^2 y$ and $r(t) = e^t i + e^{-t} j$

↳ $\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$ $\quad r'(t) = e^t i - e^{-t} j$

↳ $\nabla f(x,y) = (2xy)i + (x^2)j \rightarrow \nabla f(e^t, -e^{-t}) = 2(e^t e^{-t})i + (e^t)^2 j$

↳ $\nabla f(r(t)) = 2i + e^{2t} j$, $\nabla f(r(t)) \cdot r'(t) = 2e^t - e^t = e^t$

# Tangent Planes and Normal Lines

Tangent Plane to level surface $f(x,y,z)=c$ of a differentiable function $f$ at point $P_0(x_0,y_0,z_0)$ where the gradient is not 0 is the plane through $P_0$ normal to $\nabla f(x_0,y_0,z_0)$

↳ $f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0$

↳ Tangent Plane to surface $z=f(x,y)$ at $(x_0,y_0,z_0)$: $f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0$

Normal Line to level surface $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$ is the line through $P_0$ parallel to $\nabla f(x_0,y_0,z_0)$

↳ $x=x_0+f_x(P_0)t,\ y=y_0+f_y(P_0)t,\ z=z_0+f_z(P_0)t$

Find parametric equations for the line tangent to curve at intersection of surfaces: $\underbrace{xyz=1}_{f(x,y,z)}$, $\underbrace{x^2+2y^2+3z^2=6}_{g(x,y,z)}$ at $(1,1,1)$

↳ $\nabla f(x,y,z)\times\nabla g(x,y,z)=\vec{v}$

↳ $\nabla f = yzi+xzj+xyk$ $\quad\nabla f(1,1,1)=\langle 1,1,1\rangle$

↳ $\nabla g = 2xi+4yj+6zk$ $\quad\nabla g(1,1,1)=\langle 2,4,6\rangle$

$\times = \langle 2,-4,2\rangle \rightarrow \vec{v}=\langle 1,-2,1\rangle$

↳ $x(t)=1+t,\ y(t)=1-2t,\ z(t)=1+t$

# Differentials

Linearization of a function $f(x,y)$ at $(x_0,y_0)$ is: $L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

↳ Standard linear approximation of $f$ at point: $f(x,y)\approx L(x,y)$

Total Differential – moving from $(x_0,y_0)$ to $(x_0+dx,\ y_0+dy)$

change in x

↳ $df=f_x(x_0,y_0)dx+f_y(x_0,y_0)dy$

Standard Error – $|E|\leq\frac{1}{2}M(|x-x_0|+|y-y_0|)^2$

Upper bound of $|f_{xx},f_{yy},f_{xy}|$ on rectangle centered at P

small distance from P

Differentials – estimating change in a certain direction: $df=(\nabla f(P_0)\cdot u)\,ds$

By how much will $f(x,y,z)=x+y+x\cos z-y\sin z$ change if a point moves from $P_0(2,-1,0)$ to $P_1(0,1,2)$ at a distance of $\overset{ds}{.2}$?

↳ $\vec{u}_{\vec{P_0P_1}}=\frac{\langle -2,2,2\rangle}{2\sqrt{3}}=\langle -\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\rangle$

↳ $\nabla f=\langle 1+\cos z,\ 1-\sin z,\ -x\sin z-y\cos z\rangle\rightarrow\nabla f(2,-1,0)=\langle 2,1,1\rangle$

↳ $df=(\nabla f(2,-1,0)\cdot\vec{u})ds=0$

# Extreme Values and Saddle Points

Local Maximum – if $f(a,b)\geq f(x,y)$ for all domain points in an open disk centered at $(a,b)$

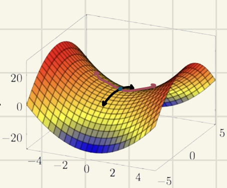
Local Minimum – if $f(a,b)\leq f(x,y)$ for all domain points in an open disk centered at $(a,b)$

↳ involves looking for points where the tangent plane is horizontal

First Derivative Test – if $f(x,y)$ has a local min/max at an interior point in domain, then $f_x$ and $f_y=0$

↳ Critical Point – An interior point of domain $f(x,y)$ where BOTH $f_x$ and $f_y$ are 0 / do not exist

↳ Saddle Point – If a critical point is neither a max nor min (a point that is the max in one direction and a min in the other direction)




Second Partials Test – $A=f_{xx}(x_0,y_0),\ B=f_{xy}(x_0,y_0),\ C=f_{yy}(x_0,y_0)$, form discriminant $D=AC-B^2$

↳ If $D<0$, $(x_0,y_0)$ is a saddle point

↳ If $D>0$, then look at sign of A; If $A>0$, local minimum at $(x_0,y_0)$. If $A<0$, local maximum at $(x_0,y_0)$

↳ If $D=0$, test is inconclusive

Find critical points / local extremes of $f(x,y) = x^2 + 2xy + 3y^2 + 2x + 10y + 1$

↳ $f_x = 2x + 2y + 2 \rightarrow 0 = 2x + 2y + 2 \rightarrow -2x = 2y + 2$

↳ $f_y = 2x + 6y + 10 \rightarrow 0 = 2x + 6y + 10 \rightarrow -2x = 6y + 10$

$\rightarrow 2y + 2 = 6y + 10$, $-2y - 10$, $-2y - 10$ $\rightarrow \frac{-8 = 4y}{4} \rightarrow -2 = y$ $\rightarrow$ Critical point at $(1,-2)$

↳ $2x + 2y + 2 = 0 \rightarrow 2x + 2(-2) + 2 = 0 \rightarrow 2x - 2 = 0 \rightarrow x = 1$

↳ $f(1,-2) = (1)^2 + 2(1)(-2) + 3(-2)^2 + 2(1) + 10(-2) + 1 = -8$

↳ $A = f_{xx} = 2$, $B = f_{xy} = 2$, $C = f_{yy} = 6$

↳ $D = AC - B^2 = 2(6) - 4 = 8$, $D > 0$ AND $A > 0$, local minimum of $-8$ at $(1,-2)$

## Extreme Values and Saddle Points II

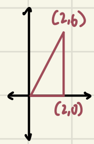
Absolute maximum – a point $\geq f(x,y)$ for all $(x,y) \in$ domain $D$

Absolute minimum – a point $\leq f(x,y)$ for all $(x,y) \in$ domain $D$

Steps:

1. Find critical points in interior of $D$
2. Find extreme points on boundary of $D$ (include endpoints)
3. Find $f$ at points found in 1 and 2
4. Largest – max, Smallest – min

Find absolute values taken on by $f(x,y) = 4xy - x^2 - y^2 - 6x$ on $D = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 3x\}$




↳ $\nabla f = (4y - 2x - 6)i + (4x - 2y)j$

↳ $0 = 4y - 2x - 6 \rightarrow 3 = 2y - x$

$0 = 4x - 2y \rightarrow 0 = -2y + 4x$

$\rightarrow 3 = 3x$, $x = 1$, $y = 2$

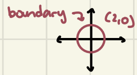
↳ check boundary:

| $y = 0$, $0 \leq x \leq 2$ | $x = 2$, $0 \leq y \leq 6$ | $y = 3x$, $0 \leq x \leq 2$ |
|---|---|---|
| $f(x) = -x^2 - 6x$ | $f(y) = 8y - 4 - y^2 - 24$ | $f(x) = 4(3x)x - x^2 - (3x)^2 - 6x = 2x^2 - 6x$ |
| $f'(x) = -2x - 6 = 0$, $x = -3$ (lower than 0, no good) | $f'(y) = -2y + 8$, $y = 4$, $x = 2$ | $f'(x) = 4x - 6$, $x = \frac{3}{2}$, $y = \frac{9}{2}$ |

↳ check points:

| Critical Points | Boundary Points |
|---|---|
| $f(1,2) = -3$ | $f(0,0) = -4$ |
| $f(2,4) = 0$ (max) | $f(2,0) = 0$ (max) |
| $f(3/2, 9/2) = -16$ (min) | $f(2,6) = -9/2$ |

Find absolute extreme values of $x^2 + y^2 + 3xy + 2$ on $D: \{(x,y) : x^2 + y^2 \leq 4\}$




↳ $\nabla f = (2x + 3y)i + (2y + 3x)j$

↳ $2x + 3y = 0 \rightarrow -6x - 9y = 0$

$3x + 2y = 0 \rightarrow 6x + 4y = 0$

$\rightarrow -5y = 0$, $(0,0)$ (within boundary)

↳ check boundary $\rightarrow x = 2\cos(t)$, $y = 2\sin(t)$, $0 < t < 2\pi$

↳ $f(2\cos t, 2\sin t) = 4\cos^2 t + 4\sin^2 t + 12\sin t\cos t + 2 = 6\sin 2t + 2 = f(t)$

(2,6) ↳ $f'(t) = 12\cos(2t) = 0$, $2t = \pi/2, \frac{5\pi}{2}, \frac{7\pi}{2} \rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

↳ $(\pm\sqrt{2}, \pm\sqrt{2})$, absolute max at $(\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2}) = 12$

↳ absolute min at $(-\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2}) = 0$

## Lagrange Multipliers

Orthogonal Gradient Theorem – if $f(x,y,z)$ is differentiable in a region with a smooth curve $C: r(t) = x(t)i + y(t)j + z(t)k$

↳ If $P_0$ is a point on $C$ where $f$ has a local minimum/maximum relative to its values in $C$, then $\nabla f$ is orthogonal to $C$ at $P_0$

↳ Corollary → At the points on a smooth curve $r(t) = x(t)i + y(t)j$ where a differentiable function $f(x,y)$ takes on its local min/max relative to values on the curve, $\nabla f \cdot r' = 0$

Method of Lagrange Multipliers – if $f(x,y,z)$ and $g(x,y,z)$ are differentiable AND $\nabla g \neq 0$ when $g(x,y,z) = 0$, find $x, y, z$ for:

↳ $\nabla f = \lambda \nabla g$ and $g(x,y,z) = 0$

↳ Max / Min of the first function subject to constraint of second function $(g(x,y,z) = 0)$

Maximize $xy$ on ellipse $4x^2+9y^2=36$

↳ $f(x,y)=xy$ , $g(x,y)=4x^2+9y^2-36 \longrightarrow \nabla f(x,y)=(y)i+(x)j \longrightarrow \nabla f=\lambda\nabla g$

function being maximized   set function $=0$, so $-36 \longrightarrow \nabla g(x,y)=(8x)i+(18y)j \longrightarrow g=0$

↳ $y=\lambda 8x$ , $x=\lambda 18y$ , $4x^2+9y^2-36=0$

3 equations, 3 variables

$y=\lambda(8x) \rightarrow xy=\lambda(8x^2) \longrightarrow \lambda(8x^2)=\lambda(18y^2)$ , $\lambda\neq 0$

$x=\lambda(18y) \rightarrow xy=\lambda(18y^2)$     $4x^2=9y^2$

$4x^2+9y^2-36=0 \longrightarrow 4x^2+4x^2-36=0$ , $16x^2=36$, $x=\pm\frac{3}{\sqrt{2}} \rightarrow$ if $x=+\frac{3}{\sqrt{2}}$, $y=\pm\sqrt{2}$

↳ $(\frac{3}{\sqrt{2}},\sqrt{2})$ and $(\frac{3}{\sqrt{2}},-\sqrt{2})$

Maximize $xy \rightarrow (\frac{3}{\sqrt{2}},\sqrt{2})$ and $(-\frac{3}{\sqrt{2}},-\sqrt{2})=3$

$\rightarrow$ if $x=-\frac{3}{\sqrt{2}}$, $y=\pm\sqrt{2}$

↳ $(-\frac{3}{\sqrt{2}},\sqrt{2})$ and $(-\frac{3}{\sqrt{2}},-\sqrt{2})$

# Taylor's Formula for One and Two Variables

Taylor Polynomial - if a function $f$ has $n$ derivatives at $x=a$, then:

↳ $P_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n$ is the nth-degree Taylor Polynomial for $f$ at $a$.

Suppose $g$ has continuous derivatives, and $g(2)=3$, $g'(2)=-4$, $g''(2)=7$, and $g'''(2)=-5$. Write the 3rd-degree Taylor Polynomial centered at $x=2$

↳ $P_n(x)=3-4(x-a)+\frac{7}{2}(x-a)^2+-\frac{5}{6}(x-a)^3$

Taylor's Theorem $- f(x)-P_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some value between a and x – absolute value of this difference is called the standard error: error $= |f(x)-P_n(x)|=\frac{|f^{(n+1)}(c)|}{(n+1)!}|x-a|^{n+1}$

Estimate error for approximation of $\cos(2x)$ by $P_{10}(x)$ for $x$ between $0$ and $\pi/4$ centered at $x=0$

$n+1=11$

↳ error $\leq \frac{|f^{(n+1)}(c)|}{(n+1)!}(x-a)^{n+1}$

↳ $|f''(c)|=2^{11}\cdot|(\sin(2x))| \rightarrow f''(c)\leq 2048(1) \rightarrow$ error $\leq \frac{2048}{11!}(\pi/4)^{11}$

# Taylor's Formula for Two Variables

Taylor's Formula for $f(x,y)$ at Point $(a,b)$ – if $f(x,y)$ is continuous through open rectangular region $R$ at $(a,b)$:

↳ $f(a+h,b+k)=f(a,b)+(hf_x+kf_y)|_{(a,b)}+\ldots+\frac{1}{n!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^n f|_{(a,b)}+\frac{1}{(n+1)!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^{n+1} f|_{(a,b)}$

Use Taylor's Formula for $f(x,y)$ to find quadric approximation of $f$ near the origin for $f(x,y)=e^x\sin(y)$

↳ $f(0,0)=0$

↳ $f_x=\sin(y)e^x=0$, $f_y=\cos(y)e^x=1$,

↳ $f_{xx}=\sin(y)e^x=0$, $f_{xy}=\cos(y)e^x=1$, $f_{yy}=-\sin(y)e^x=0$

↳ $Q(x,y)=\underbrace{[0+x(0)+y(1)]}_{\text{Linear Term}}+\frac{1}{2!}\underbrace{[x^2(0)+2xy(1)+y^2(0)]}_{\text{Quadric Term}}$

↳ $Q(x,y)=y+xy$

# Partial Derivatives With Constraints

Steps:

① Decide which variables are dependent and which are independent

② If possible, eliminate the other dependent variable

③ Differentiate + Solve

Find $\left(\frac{\partial w}{\partial y}\right)_x$ if $w = x^2y^2 + yz - z^3$ and $\underbrace{x^2 + y^2 + z^2 = 6}$

$\hookrightarrow \frac{\partial w}{\partial y} = 2x^2y + y\frac{\partial z}{\partial y} + z - 3z^2\frac{\partial z}{\partial y}$ $\quad\hookrightarrow 0 + 2y + 2z\frac{\partial z}{\partial y} = 0$

$\quad\hookrightarrow 2x^2y - \frac{y^2}{z} + z + 3yz$ $\quad\hookrightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$

If $w = x^2 + y - z + \sin(t)$ and $x+y = t$, ← dependent ①

① dependent find $\left(\frac{\partial w}{\partial y}\right)_{z,t}$ independent ①

② $x = t - y$, $w = (t-y)^2 + y - z + \sin(t)$

③ $\frac{\partial w}{\partial y} = 2(t-y)(-1) + 1 = -2t + y + 1 = w$