Math 2552: Differential Equations

↳ Chapters 1-3

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Author's Note

↳ This is solely covering the topics of the lecture videos, so I recommend you read the textbook in regards to working out homework

↳ Pretty much all info regarding computer science has been omitted (i.e. functions)

↳ Lecture numbers are on the left of titles

↳ Some videos have been combined across headings

1.1.1 Differential Equations and Solutions

↳ Mathematical Models / Direction Fields
↳ Newton's Law of Cooling

Objective

↳ Apply exponential decay/growth model to solve + analyze first order diff eqns.

Example: Newton's Law of Cooling

↳ object at temp $u(t)$, put in environment with temp T

↳ $\frac{d}{dt}u = -k(u-T)$

→ u is an unknown

→ k and T are parameters of a system

↳ Differential Equation – an equation with a function and its derivatives

↳ Solution – a differentiable function that satisfies the DE on some interval

Example: Verify that $Ce^{-kt} + T$, $C \in \mathbb{R}$ is a solution to $\frac{d}{dt}u = -k(u-T)$

↳ Left-Hand side: $\frac{d}{dt}(Ce^{-kt}+T)$

$$= -Cke^{-kt}$$
$$= -k(Ce^{-kt} + T - T)$$
$$= -k(u-T)$$

1.1.2 Dynamical Systems

dynamical system → a system which behaves according to a set of laws
(Ex. Newton's law of Cooling)
↳ phenomenon may be biological, mechanical, social, etc.
→ dynamic, system evolves over time

→ Our job is to predict and characterize the long-term behaviors

1.2.1 Qualitative Methods

↳ Mathematical Models / Direction Fields
↳ Diagrams

Objective

↳ Determine / Classify equilibrium solutions
↳ Sketch direction fields + phase lines
↳ Sketch solution curves of autonomous DEs based on qualitative analysis

Direction Field – can be used to qualitatively analyse a system

Example: Newton's Law of Cooling $\left[\frac{d}{dt}u = -k(u-T)\right]$

↳ $T=1$, $k=2$, $\frac{d}{dt}u = -k(u-T) = 2-2u$

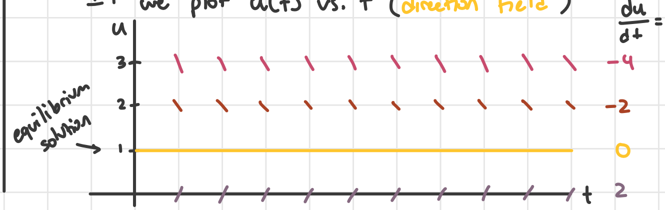
↳ $u=0$, $\frac{du}{dt} = 2$

↳ $u=1$ $\quad = 2-2 = 0$

↳ $u=2$ $\quad = 2-4 = -2$

↳ $u=3$ $\quad = 2-6 = -4$

If we plot $u(t)$ vs. t (direction field)



1.2.2 Autonomous DEs

Autonomous DE — form: $\frac{dy}{dt} = f(y)$ (Ex. $y' = 2-y$, its equilibrium sdn. is along $2-y=0$)

↳ equilibrium solution — satisfies $y = \text{constant}$

↳ also known as critical, fixed, stationary, steady-state points

↳ MUST satisfy $\frac{dy}{dt} = 0$

1.2.3 Phase Lines, Phase Portraits, and f(y) vs y

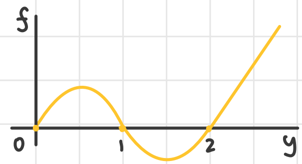
Example: Suppose $\frac{dy}{dt} = f(y) = y(y-1)(y-2)$, $y_0 \geq 0$, $t \geq 0$, $y_0 = y(0)$

↳¹ Find equil. points

$y' = 0$ at $y=0, y=1, y=2$ (By inspection)

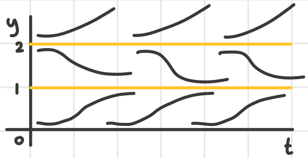
↳² Sketch $f(y)$ vs y

y	0	(0,1)	1	(1,2)	2	(2,∞)
f	0	+	0	−	0	+



↳³ Sketch phase portrait (phase line) of DE

↳⁴ Sketch a few solution curves / integral curves for DE



1.2.4 Classification of Equilibrium Points

Possible patterns of an autonomous DE $\left[\frac{dy}{dt} = f(y)\right]$ near an equilibrium point, $y = y_1$

Type	Behavior	Visual
stable	soln. curves on both sides tend to y_1	
unstable	soln. curves on both sides tend away from y_1	
semi-stable	soln. curves on 1 side tend to y_1 (other away)	

1.3.1 Classifying DEs

↳ Classification of ODEs

↳ Standard Form

Objectives → Classify DEs (can figure out how to solve them)

↳ Convert a first order linear ODE into standard form

* Ordinary DE — functions depend on 1 variable (Ex. Newton's Law of Cooling)

Partial DE — functions depend on >1 variable (Ex. heat equation, $\frac{\partial}{\partial t} u(x,t) = D\frac{\partial u^2}{\partial x^2}$)

Order of a DE — highest degree derivative that appears in equation

↳ Ex. $\underset{\uparrow}{u^{(3)}} + 2e^t u'' + uu' = t^4$, order is 3

3rd derivative

derivative #

n^{th} order linear ODE: $\sum_{n=0}^{n} a_k u^{(n-k)} = a_0(t)u^{(n)}(t) + a_1(t)u^{(n-1)}(t) + \ldots + a_n(t)u^0(t) = g(t)$

↳ Coefficients $a_0(t), a_1(t), \ldots a_n(t)$ and $g(t)$ are <u>given</u>; $u(t)$ is <u>unknown</u>

↳ linear ODE is homogeneous if $g(t) = 0$

↳ DE not in form? nonlinear (coefficients may be nonlinear with respect to t)

↳ Ex. $t^2y'' = y$ is linear, $t^2y'' = y^2$ is nonlinear

general 1^{st} order linear equation [*]: $a_0(t)\frac{dy}{dt} + a_1(t)y = h(t)$

↳ y is unknown; a_0, a_1, h given

Standard Form of 1^{st} order linear ODE: if $a_0(t) \neq 0$, we can divide by $a_0(t)$ and put [*] in form: $1\frac{dy}{dt} + p(t)y = g(t)$

↳ where $p(t) = a_1(t)/a_0(t)$ and $g(t) = h(t)/a_0(t)$

↳ known as Standard Form

2.1.1 Separable Equations

↳ Solving First order separable differentiable equations.

Objectives

↳ Classify DEs as separable

↳ Solve separable 1^{st} order linear ODEs

A first order differentiable equation is separable if it can be written in form: $\frac{dy}{dx} = f(x,y) = p(x)q(y)$

↳ Ex. $y' = e^{3x+y} = e^{3x}e^{y}$

Example: $\frac{dy}{dt} = (1-12t)y^2$, $y = y(t)$, $y(0) = \frac{1}{8}$ [$\frac{dy}{dt} = y'(t)$ is a limit, dy is a variable]

↳ Is it linear?

No, because includes y^2

↳ Compute $dy = \frac{dy}{dt}dt$ ← definition of a differential

$$dy = \frac{dy}{dt}dt = (1-12t)y^2\,dt$$

$$\frac{1}{y^2}dy = (1-12t)\,dt,\ y \neq 0$$

$$\int 1/y^2\,dy = \int(1-12t)dt$$

$$-\frac{1}{y} = t - 6t^2 + C$$

$$y = \frac{1}{-t + 6t^2 + C}$$

$$\frac{1}{8} = \frac{1}{-0 + 6(0)^2 + C},\ C = 8,\quad y = \frac{1}{-t^2 + 6t^2 + 8}$$

2 2.1 Solving a Linear First Order Equation

↳ Solving 1^{st} order linear ODEs using an integration factor

Objs.

↳ Convert DEs into standard form

↳ Classify DEs as linear

Reminder — Standard Form: $\underset{\uparrow \text{ coefficient} = 1}{1}\frac{dy}{dt} + p(t)y(t) = g(t)$

Example: $ty' + 2y = 4t$ for all $t \geq 0$ ⟶ Separable?

No! Cannot express as $y' = p(t)q(y)$

↳ Solve

Standard form: $y' + \frac{2}{t}y = 4,\ t \neq 0$

↳ Multiply by integrating factor: $\mu = e^{\int 2/t\,dt} = e^{2\ln(t)} = t^2$

↳ $\underbrace{t^2\frac{dy}{dt} + 2ty}_{} = 4t^2 \longrightarrow \int\frac{d}{dt}(t^2y) = \int 4t^2 \longrightarrow t^2y = \frac{4}{3}t^3 + C \rightarrow y = \frac{4}{3}t + \frac{C}{t^2}$ if $t \neq 0$

↳ if $t = 0$, $y = 0$

(looks like a derivative of a product due to integrating factor *magic*)

TL:DR

Given a linear 1^{st} order ODE in standard form: $\frac{dy}{dt} + p(t)y = g(t)$

↳ Multiply by integration factor $\mu(t) = e^{\int p\,dt}$: $\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t)$

↳ μ is made so left-hand side is derivative of a product: ↳ $\frac{d}{dt}[\mu(t)y(t)]$

2.3.1-5 A Water Tank Problem

Example: A tank initially has 40 lbs of salt in 600 gal water. Starting at $t=0$, water with ½ lbs salt per gallon is added to the tank at 4 gal/min, mixture is drained at same rate.

↳ ① Make a DE for $Q(t)$: # lbs of salt at $t>0$

$\frac{dQ}{dt}$ = rate in − rate out

rate in $\rightarrow \left(\frac{4\text{ gal}}{\text{min}}\right)\left(\frac{1/2\text{ lbs salt}}{1\text{ gal water}}\right) = 2$ lbs/min

rate out $= \left(\frac{Q(t)}{600\text{ gal}}\right)\left(\frac{4\text{ gal}}{\text{min}}\right) = \frac{1}{150}Q(t)$ lbs/min

$\rightarrow \boxed{Q' = 2 - \frac{1}{150}Q(t)}$

↳ ② Solve DE

$Q' = 2 - \frac{1}{150}Q(t) \rightarrow Q' + \frac{1}{150}Q(t) = 2$ (standard form)

$\mu = e^{\int \frac{1}{150}dt} = e^{\frac{t}{150}} \longrightarrow Q'e^{\frac{t}{150}} + \frac{1}{150}e^{\frac{t}{150}}Q(t) = 2e^{\frac{t}{150}} \rightarrow \int \frac{d}{dt}\left(e^{t/150}Q(t)\right) = \int 2e^{\frac{t}{150}}$

$e^{t/150}Q(t) = 300e^{t/150} + C \rightarrow Q = 300 + Ce^{-t/150}$

but when $t=0$, $Q=40 \rightarrow C = -260 \rightarrow \boxed{Q = 300 - 260e^{-t/150}}$

↳ ③ What happens to conc. of salt in tank after a long time?

$-260^{-t/150} \rightarrow 0$ as $t \rightarrow \infty$, left with 300 lbs of salt with 600 gal water

2.4.1 Uniqueness of an IVP

IVP - initial value problem

↳ existence - does an IVP have a solution / where?

↳ uniqueness - is this solution unique?

→ both for nonlinear + linear

Ex. $\frac{dy}{dt} = 8ty^{1/5}$, $y(0) = 0$

↳ linear? No! $\rightarrow y^{1/5}$

↳ Solve for $y(t)$

$y^{-1/5}\frac{dy}{dt} = 8t$

$\int y^{-1/5}\,dy = \int 8t\,dt$

$\frac{5}{4}y^{4/5} = 4t^2 + C$

$\rightarrow y^{4/5} = \frac{16}{5}t^2 + C$, but $y(0)=0$, so $C=0$

$y = \left(\frac{16}{5}t^2\right)^{5/4}$

↳ Any other solutions?

$y = -\left(\frac{16}{5}t^2\right)^{5/4}$, but if $y=0$, then $y=0$ is another solution

$y = \pm\sqrt[4]{\left(\frac{16}{5}t^2\right)^5}$, so both ±

2.4.2 Existence and Uniqueness of a Linear IVP

If p and g are continuous on (α, β), $t_0 \in (\alpha, \beta)$, then there is a unique solution to IVP:

↳ $y' + p(t)y = g(t)$, $\underline{y(t_0) = y_0}$

Ex. $(9-t^2)y' + 5ty = 3t^2$, $y(-1) = 1$ → Determine <u>an</u> interval where a solution exists

$t_0 = -1$, $y_0 = 1$

↳ Standard Form: $y' + \frac{5ty}{9-t^2} = \frac{3t^2}{9-t^2}$ interval: $(-3, +3)$

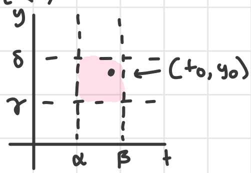
↳ $\frac{5ty}{(3-t)(3+t)}$

2.4.3 Existence and Uniqueness of a Nonlinear IVP

If f and $\frac{\partial f}{\partial y}$ are continuous over $\alpha < t < \beta$, and $\gamma < y < \delta$ which contains point (t_0, y_0), then there is a <u>unique</u> solution to IVP $y' = f(t,y)$, $y(t_0) = y_0$ on interval $\alpha < t < \beta$

↳ Ex. does $\frac{dy}{dt} = 8ty^{1/5}$; $y(0) = 0$ satisfy these conditions?

No! because... $\left\{ \begin{array}{l} f \text{ continuous everywhere} \\ \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(8ty^{1/5}) = \frac{8t}{5} y^{-4/5}, \text{ not continuous at } y=0 \end{array} \right\}$



For 1st order ODE $y' + p(t)y = g(t)$:

1) if p and g are continuous, there is a general solution with an arbitrary constant (representing all solutions)
2) There is an explicit expression for the ODE
3) Points where solutions are NOT continuous are identified from coefficients

2.5.1-4 Autonomous DEs

↳ Bifurcation points / diagrams

↳ Concavity

Objs.

↳ Use concavity to sketch solution curves of a DE

↳ Sketch bifurcation diagram for 1st order autonomous DE

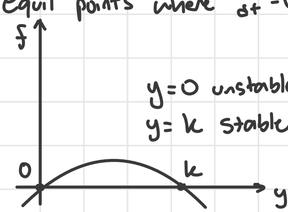
Autonomous (Check 1.2.2) — if $\frac{dy}{dt} = f(y)$, then $\frac{d^2y}{dt^2} = \frac{d}{dt} f(y) = \frac{df}{dy} \frac{dy}{dt}$

↳ y is concave up when f' and y' have the same sign

Ex. $\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right)$, $r > 0$, $K > 0$

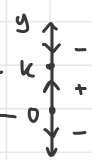
↳ Sketch f vs. y, identify/classify equilibrium points of y

equil points where $\frac{dy}{dt} = 0$, when $y = 0$ or $y = K$



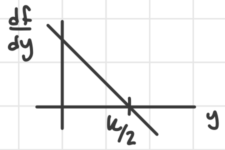
$y = 0$ unstable (negative slope to positive slope)

$y = K$ stable (positive slope to negative slope)



↳ For $y \in \mathbb{R}$ determine whether y is concave up / down

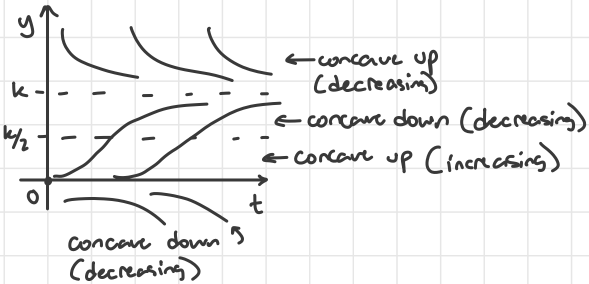
$\frac{d^2y}{dt^2} \to \frac{df}{dy}\frac{dy}{dt} \to \frac{df}{dy} = r - \frac{2ry}{K} = 0$ for $y = K/2$



using f vs y and $\frac{df}{dy}$ vs y

	$(-\infty, 0)$	$(0, K/2)$	$(K/2, K)$	(K, ∞)
df/dy	+	+	−	−
dy/dt	−	+	+	−
concave	down	up	down	up

↳ Sketch a few integration curves



2.5.5–8 Autonomous DE Example 2

$y' = a - y^2$ for parameter a

↳ Find equilibrium points (3 cases)

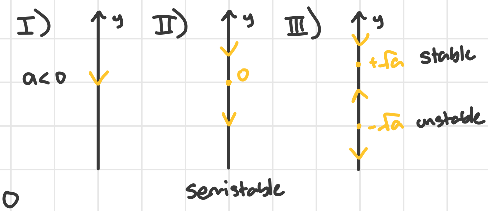
$0 = y' = a - y^2, \quad a = y^2$

I) $a < 0$, no equilibrium points

II) $a = 0$, 1 equil. pt when $y = 0$

III) $a > 0$, 2 equil. pts when $y = \pm\sqrt{a}$

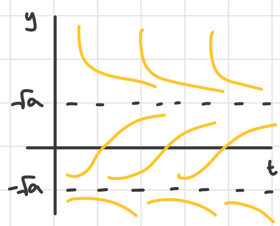
↳ Sketch phase lines for each case and classify critical points



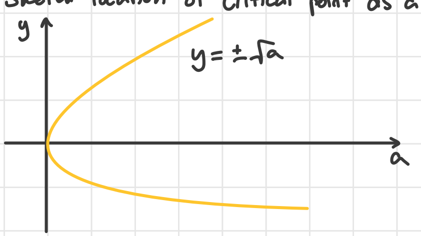
↳ Sketch a few solution curves when $a > 0$

$\frac{d^2y}{dt^2} = \frac{df}{dy}\frac{dy}{dt}, \quad \begin{cases} \frac{df}{dy} = 0 - 2y = -2y \\ \frac{dy}{dt} = a - y^2 \end{cases}$

	$(-\infty, -\sqrt{a})$	$(-\sqrt{a}, 0)$	$(0, \sqrt{a})$	$(\sqrt{a}, \infty)$
df/dy	+	+	−	−
dy/dt	−	+	+	−
concave	down	up	down	up



↳ Sketch location of critical point as a function of a in ay-plane (bifurcation diagram)



3.1.1-5 Linear Algebra Review

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Characteristic polynomial: $ad-bc$, so roots of $(a-\lambda)(d-\lambda)-bc$ finds eigenvalues

eigenvectors — want vectors in Null $(B-\lambda I)$

↳ $(B-\lambda I)v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

3.2.1-2 A Linear System of DEs

$dx_1/dt = ax_1 + bx_2 - k$

$dx_2/dt = cx_1 + dx_2$

→ rewrite as matrix equation $\vec{x}(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -k \\ 0 \end{pmatrix}$

$\frac{d\vec{x}}{dt} = P(t)\vec{x} + g(t)$

↳ Linear system is a first order system of dimension two, $\vec{x}$ has 2 elements

↳ non-homogeneous because $\vec{g}(t) \neq \vec{0}$

↳ If $\vec{g}(t) = \vec{0}$ for all t, system is homogeneous

3.2.3-4 Solution to a System

Solution (to $\frac{d\vec{x}}{dt} = P(t)\vec{x} + g(t)$) is a set of 2 functions, $x_1(t)$ and $x_2(t)$ that satisfy the system for all t on some interval

↳ Ex. $\frac{d\vec{x}}{dt} = \begin{pmatrix} 20 & 0 \\ -10 & 30 \end{pmatrix}\vec{x}$

↳ Solutions: $\vec{u}_1(t) = e^{\lambda_1(t)}\vec{v}_1(t)$, $\vec{u}_2 = e^{\lambda_2(t)}\vec{v}_2(t)$ Task: Find 1 solution

↳ Find eigenvalues:

$(20-\lambda)(30-\lambda) = 0$

$\lambda=20, \lambda=30$

→ Find Eigenvectors

$\lambda_1 = 20$, then $P-\lambda I = \begin{pmatrix} 0 & 0 \\ -10 & 10 \end{pmatrix}$, $x_1 = x_2$ so $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{u}_1 = e^{20t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

↳ Prove the solution works ($e^{\lambda_1 t}v_1$ is a solution to $\vec{x}'(t) = P\vec{x}$)

Lefthand side: $\frac{d}{dt}\vec{x} = \frac{d}{dt}(e^{\lambda_1 t}\vec{v}_1) = \lambda_1 e^{\lambda_1 t}\vec{v}_1$

Right hand side: $P\vec{x} = P(e^{\lambda_1 t}\vec{v}) = e^{\lambda_1 t}P\vec{v}_1$

$= e^{\lambda_1 t}(\lambda_1\vec{v}_1)$ because $Pv_1 = \lambda_1 v_1$

proof!

Critical points (equilibrium points): where $\frac{d\vec{x}}{dt} = \vec{0}$

↳ Ex. $\frac{d\vec{x}}{dt} = \begin{pmatrix} 20 & 0 \\ -10 & 30 \end{pmatrix}\vec{x}$, critical point is the $\vec{0}$

3.2.6-7 Second Order Equations

We can convert 2^{nd} order linear DEs to 1^{st} order

↳ Ex. $\frac{d^2y}{dt^2} - \sin(t)\frac{dy}{dt} + 7y = e^t\cos(t) + 1$

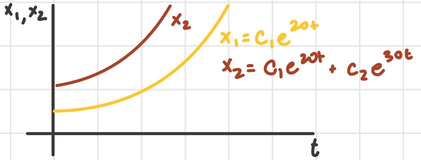
set $x_1 = y$ then $x_1' = y' = x_2$

$x_2 = y'$ $\underbrace{x_2' = y'' = e^t\cos t + 1 - 7\overset{y}{x_1} + \sin(t)x_2}_{\text{system}}$

→ $\vec{x} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -7 & \sin t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t\cos t + 1 \end{pmatrix}$

Component plots — u_1 vs t and u_2 vs t

↳ Ex. $\frac{d\vec{x}}{dt} = \begin{pmatrix} 20 & 0 \\ -10 & 30 \end{pmatrix}\vec{x}$, solutions are $\vec{u}(t) = c_1\vec{u}_1 + c_2\vec{u}_2 = c_1e^{20t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{30t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



These are solution curves (also label the axes)

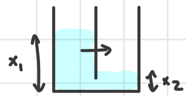
3.3.1-5 Homogeneous Linear Systems

↳ Systems of 2 first order DEs

Objs → Solve 1^{st} order linear DEs

↳ Sketch component plots / phase portraits

Example: Compartment Model — a tank is divided into two cells



↳ What happens after a long period of time?

as $t \to \infty$, $x_1 = x_2$

↳ Construct / Solve a linear system

↳ $x_1' = k_1(x_1 - x_2)$, $x_2' = k_2(x_2 - x_1)$ } assume $k_1 = k_2 = k_3$ → $\vec{x}'(t) = k\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\vec{x}$

↳ Solve: set $\vec{x} = e^{\lambda t}\vec{v}$. Is this a solution?

↳ Yes: $\vec{x}' = \lambda e^{\lambda t}\vec{v}$ and $P\vec{x} = P(e^{\lambda t}\vec{v}) = e^{\lambda t}P\vec{x} = \lambda e^{\lambda t}\vec{v}$, eigenvalues $= \lambda = 0, -2k$

↳ eigenvectors, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, P is symmetric ($P = P^T$) so $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

↳ Solution: $\vec{x} = c_1e^0\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-2kt}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

↳ Is solution unique?

↳ Consider any $\vec{x}_0 = \vec{x}(t_0)$

↳ Then $\vec{x}_0 = c_1\vec{x}_1(t_0) + c_2\vec{x}_2(t_0)$, a vector equation

$= (\vec{x}_1\, \vec{x}_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, a matrix equation

↳ Unique solution: we need $(\vec{x}_1\, \vec{x}_2)$ to be linearly independent (invertible)

↳ unique c_1 c_2 for, any t_0

3.3A1-5 Additional Examples

1) $y'' + 2y' + \alpha y = 0$, $\alpha \in \mathbb{R}$

Let $x_1 = y$, $x_2 = y'$ → $x_1' = x_2$, $x_2' = -x_2 - \alpha x_1$ → $\vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

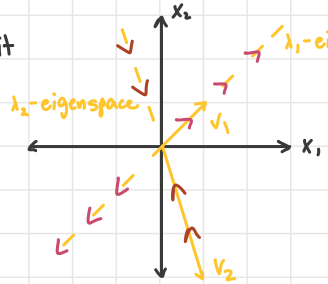
eigenvalues: $\lambda = 1 \pm \sqrt{1-\alpha}$

2) Set $\alpha = -3$, $A = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$ $\lambda_1 = 1$: $\begin{pmatrix} -1 & 1 \\ \cdot & \cdot \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -3$: $\begin{pmatrix} 3 & 1 \\ \cdot & \cdot \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

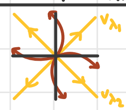
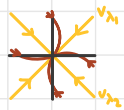
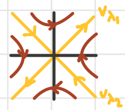
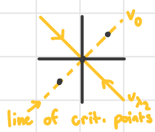
↳ Solution: $\vec{x} = c_1e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

3) Phase Portrait



↳ If $c_1 = 0$, $x = c_2 x_2 = c_2 e^{-3t}\begin{pmatrix}1\\-3\end{pmatrix}$, then $\vec{x} \to \vec{0}$ along λ_2-eigenspace

↳ If $c_2 = 0$, $x = c_1 x_1 = c_1 e^{t}\begin{pmatrix}1\\1\end{pmatrix}$, then moves along line $\vec{x}_1 = \vec{x}_2$

Eigenvalues	Phase Portrait	Type of Critical Point	Stability
$\lambda_1 \neq \lambda_2$ $+$ $\quad +$		nodal source	Unstable
$\lambda_1 \neq \lambda_2$ $-$ $\quad -$		nodal sink	Asymptotically Stable
$\lambda_1 \neq \lambda_2$ $+$ $\quad -$		saddle	Unstable
$\lambda_1 = 0, \lambda_2 > 0$ (if <0, arrows on v_{λ_2} flipped)		n/a	n/a

3.3.W1-W3 The Wronskian

Consider $\vec{x}' = A\vec{x}$, with eigenvalues λ_1, λ_2 and eigenvectors $\vec{v}_1, \vec{v}_2$

↳ Write a solution: $\vec{x} = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}\vec{v}_2$

↳ If for some $t_0 \in \mathbb{R}$, that $\vec{x}(t_0) = \begin{pmatrix}b_1\\b_2\end{pmatrix}$. Construct a matrix equation to solve for c_1 and c_2

↳ $\begin{pmatrix}b_1\\b_2\end{pmatrix} = c_1 e^{\lambda_1 t_0}\vec{v}_1 + c_2 e^{\lambda_2 t_0}\vec{v}_2 = c_1 e^{\lambda_1 t_0}\begin{pmatrix}x_{11}\\x_{21}\end{pmatrix} + c_2 e^{\lambda_2 t_0}\begin{pmatrix}x_{12}\\x_{22}\end{pmatrix}$

↳ $$\begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}e^{\lambda_1 t_0}x_{11} & e^{\lambda_2 t_0}x_{12}\\ e^{\lambda_1 t_0}x_{21} & e^{\lambda_2 t_0}x_{22}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

↳ Is this matrix invertible? ⟷ Is there a unique soln. to this system?

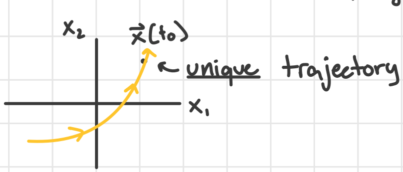
↳ take determinant, if $\det \neq 0$, then there is a unique soln.

↳ $$\begin{vmatrix}e^{\lambda_1 t_0}x_{11} & e^{\lambda_2 t_0}x_{12}\\ e^{\lambda_1 t_0}x_{21} & e^{\lambda_2 t_0}x_{22}\end{vmatrix} = \underbrace{e^{\lambda_1 t_0} e^{\lambda_2 t}}_{\text{always} > 0}\underbrace{\begin{vmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{vmatrix}}_{?} = \underbrace{e^{(\lambda_1+\lambda_2)t_0}\,|\vec{v}_1\ \vec{v}_2|}_{\text{never zero}}$$

$\vec{v}_1$ and $\vec{v}_2$ correspond to distinct λ_1, λ_2

↳ linearly independent

↳ unique c_1, c_2



If $\vec{x}_1$ and $\vec{x}_2$ are 2 linearly independent solutions to $\vec{x}' = Ax$, where A is a real 2 × 2 matrix, then $\vec{x}_1$ and $\vec{x}_2$ form a fundamental set

Ex. The vectors $\vec{x}_1 = e^{t}\begin{pmatrix}1\\1\end{pmatrix}$ and $\vec{x}_2 = e^{2t}\begin{pmatrix}2\\3\end{pmatrix}$ are a fundamental set of solutions to $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix}-1 & 2\\-3 & 4\end{pmatrix}$

↳ Summary: when A = 2x2, λs are real + distinct, solution to $\vec{x}' = A\vec{x}$, $x(t_0) = x_0$ is unique for any t_0

3.4.1-5 Complex Eigenvalues

Solve $\vec{x}' = \begin{pmatrix}-1 & 2\\-1 & -3\end{pmatrix}\vec{x}$

↳ eigenvalues: $\lambda^2 + 4\lambda + 5 = 0$, $\lambda = 2 \pm i$

↳ eigenvectors: $\lambda_1 = 2 - i$, $\begin{pmatrix}1-i & 2\\ \cdot & \cdot\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \vec{0}$, $(1-i)v_1 + 2v_2 = 0$, set $v_1 = 2$, $v_2 = i-1$

↳ $\vec{v}_1 = \begin{pmatrix}2\\i-1\end{pmatrix}$, v_2 is the complex conjugate, $v_2 = \begin{pmatrix}2\\-i-1\end{pmatrix}$

↳ $\vec{x} = c_1 e^{(-2+i)t}\begin{pmatrix}2\\i-1\end{pmatrix} + c_2 e^{(2-i)}\begin{pmatrix}2\\-i-1\end{pmatrix}$

BUT we want <u>Real valued</u> solutions

↳ $\lambda = \alpha \pm \beta i$, $v = \vec{a} \pm \vec{b} i$

↳ $\alpha = 2$, $\beta = 1$ ↳ $\vec{a} = \begin{pmatrix}2\\-1\end{pmatrix}$, $\vec{b} = \begin{pmatrix}0\\1\end{pmatrix}$

$$\vec{x}(t) = c_1 e^{\alpha t}\left[\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right] + c_2 e^{\alpha t}\left[\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)\right]$$ memorize!

↳ $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}$ where

$x_1 = e^{-2t}\left[\begin{pmatrix}2\\-1\end{pmatrix}\cos(t) - \begin{pmatrix}0\\1\end{pmatrix}\sin t\right]$

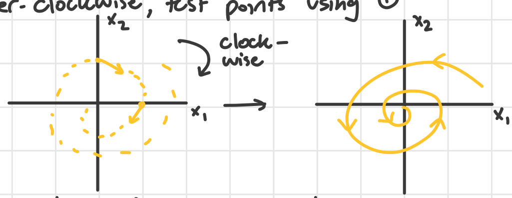
$x_2 = e^{-2t}\left[\begin{pmatrix}2\\-1\end{pmatrix}\sin(t) + \begin{pmatrix}0\\1\end{pmatrix}\cos t\right]$ ①

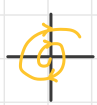
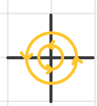
to determine clockwise or counter-clockwise, test points using ①

↳ $\vec{x}' = \begin{pmatrix}-1 & 2\\-1 & -3\end{pmatrix}\vec{x}$

↳ if $\vec{x} = \begin{pmatrix}1\\0\end{pmatrix}$, then $\vec{x}' = \begin{pmatrix}-1\\-1\end{pmatrix}$

↳ if $\vec{x} = \begin{pmatrix}0\\1\end{pmatrix}$, then $\vec{x}' = \begin{pmatrix}2\\-3\end{pmatrix}$



Eigenvalues	Phase Portrait	Type of Critical Point	Stability
$\lambda = \mu \pm iv$ $\mu < 0$		spiral sink	Asymptotically Stable
$\mu > 0$		spiral source	Unstable
$\mu = 0$		center	Stable

3.5.1-3 Repeated Eigenvalues

Example: The motion of an object is given by $\vec{r}(t)$, where $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

↳ $\frac{dx}{dt} = -x + ky$ ↳ at $t=0$, $r(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

↳ $\frac{dy}{dt} = -y$

↳ $\vec{r}' = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix} \end{array} \vec{r} = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

↳ $(-1-\lambda)(-1-\lambda) = (-1-\lambda)^2$, $\lambda = -1$ multiplicity 2

↳ Eigenvectors $(A - \lambda I)\vec{v} = 0$, $\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

↳ x is free, and if $\begin{cases} k = 0,\ y \text{ is free} \to \text{case 1} \\ k \neq 0,\ y = 0 \to \text{case 2} \end{cases}$

$k=0$, then $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, both x, y free, can pick:

↳ $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{r}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= c_1 \vec{x}_1 + c_2 \vec{x}_2$

↳ Case 2. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \to y' = -y \to y = c_2 e^{-t}$

↳ $x' = -x + ky \to x' + x = kc_2 e^{-t}$

↳ $\mu = e^{\int 1\,dt} = e^t \to e^t x' + e^t x = kc_2 \to \frac{d}{dt}(xe^t) = kc_2 \to \int \frac{d}{dt}(xe^t) = \int kc_2\,dt$

↳ $\frac{xe^t = kc_2 t + c_1}{e^t} \to x = kc_2 e^{-t} + c_1$

↳ $r(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} kt + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

↳ $c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(\vec{v}t + \vec{w})$ repeated eigenvalues have this form!

(Method works well if we have a triangular matrix

3.5.4-5 Approach for Degenerate Matrices

$\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\vec{x}$

↳ $(1-\lambda)(3-\lambda) + 1 = 3 - 3\lambda - \lambda + \lambda^2 + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, $\lambda = 2$ mult 2

↳ eigenvector: $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\vec{v} = \vec{0}$, $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

↳ a solution to ① is $\vec{x}_1 = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

↳ Is there another that is linearly independent?

↳ Try $\vec{x}_2 = te^{2t}\vec{v} + e^{2t}\vec{w}$, $\vec{w}$ is an unknown constant vector

↳ What is $\vec{w}$? Substitute $\vec{x}_2$ into ①

$\vec{x}_2' = e^{2t}\vec{v} + 2te^{2t}\vec{v} + 2e^{2t}\vec{w}$

↳ $A\vec{x}_2 = A(te^{2t}\vec{v} + e^{2t}\vec{w})$

Divide by e^{2t} and equate: $\vec{v} + 2t\vec{v} + 2\vec{w} = A(t\vec{v}) + A\vec{w}$

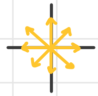
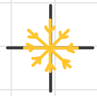
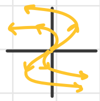
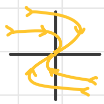
↳ $(A - 2I)\vec{w} = \vec{v}$

$\left[\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]\vec{w} = \vec{v}$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \to \vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

General Rule

↳ $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(\vec{v}t + \vec{w})$, find $\vec{w}$ by $(A - \lambda I)\vec{w} = \vec{v}$

Eigenvalues	Phase Portrait	Type of Critical Point	Stability
$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda > 0$		unstable proper (or star) node	Unstable
$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda < 0$		stable proper (or star) node	Asymptotically Stable
A is not diagonal, $\lambda > 0$		unstable improper (degenerate) node	Unstable
A is not diagonal, $\lambda < 0$		stable improper (degenerate) node	Asymptotically Stable
General Table			
$\lambda_1 > \lambda_2 > 0$		Node	Unstable
$\lambda_1 < \lambda_2 < 0$			A. Stable
$\lambda_2 < 0 < \lambda_1$		Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$		Proper / Improper Node	Unstable
$\lambda_1 = \lambda_2 < 0$			A. Stable
$\lambda = \mu \pm i\nu$: $\mu > 0$		Spiral Point	Unstable
$\lambda = \mu \pm i\nu$: $\mu < 0$			A. Stable
$\lambda = \mu \pm i\nu$: $\lambda_1 = i\nu$, $\lambda_2 = -i\nu$		Center	Stable