

# BSVD summary

July 10, 2023

## 1 Updating Procedure for Fixed Rank Model

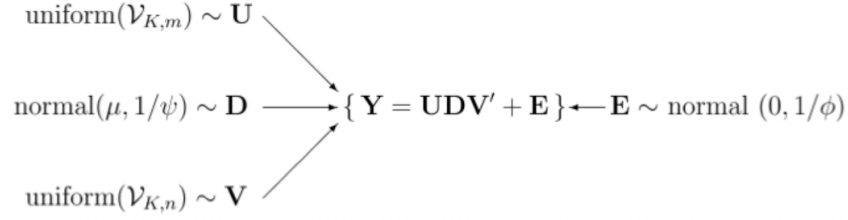


Figure 1: Graphical representation of the Model

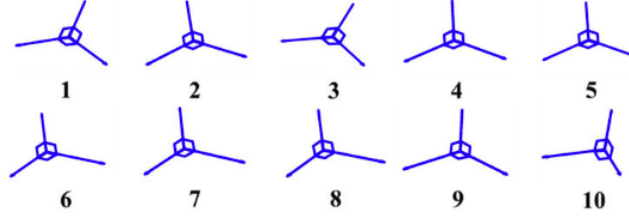


Figure 2: Steifel manifold for 3-dim

1. For  $j \in \{1, \dots, K\}$ ,

- Sample  $(\mathbf{U}_{[:,j]} \mid \mathbf{Y}, \mathbf{U}_{[:, -j]}, \mathbf{D}, \mathbf{V}, \phi) = \mathbf{N}_{\{-j\}}^u \mathbf{u}_j$ , where  $\mathbf{u}_j \sim \text{vMF}(\phi d_j \mathbf{N}_{\{-j\}}^{u'} \mathbf{E}_{-j} \mathbf{V}_{[:,j]})$
- Sample  $(\mathbf{V}_{[:,j]} \mid \mathbf{Y}, \mathbf{U}, \mathbf{D}, \mathbf{V}_{[:, -j]}, \phi) = \mathbf{N}_{\{-j\}}^v \mathbf{v}_j$ , where  $\mathbf{v}_j \sim \text{vMF}(\phi d_j \mathbf{U}_{[:,j]}' \mathbf{E}_{-j} \mathbf{N}_{\{-j\}}^v)$
- Sample  $(d_j \mid \mathbf{Y}, \mathbf{U}, \mathbf{D}_{[-j, -j]}, \mathbf{V}, \phi, \mu, \psi) \sim \text{normal}\left(\frac{\mathbf{U}_{[:, -j]}' \mathbf{E}_{-j} \mathbf{V}_{[:,j]} \phi + \mu \psi}{\phi + \psi}, \frac{1}{\phi + \psi}\right)$

2. Sample  $(\phi \mid \mathbf{Y}, \mathbf{U}, \mathbf{D}, \mathbf{V}) \sim \text{Gamma}\left(\frac{\nu_0 + mn}{2}, \frac{\nu_0 \sigma_0^2 + \|\mathbf{Y} - \mathbf{U} \mathbf{D} \mathbf{V}'\|^2}{2}\right)$

3. Sample  $(\mu \mid \mathbf{D}, \psi) \sim N\left(\frac{\psi \sum_{j=1}^K d_j + \mu_0 / v_0^2}{\psi K + 1 / v_0^2}, \frac{1}{\psi K + 1 / v_0^2}\right)$

4. Sample  $(\psi \mid \mathbf{D}, \mu) \sim \text{Gamma}\left(\frac{\eta_0 + K}{2}, \frac{\eta_0 \tau_0^2 + \sum_{j=1}^K (d_j - \mu)^2}{2}\right)$

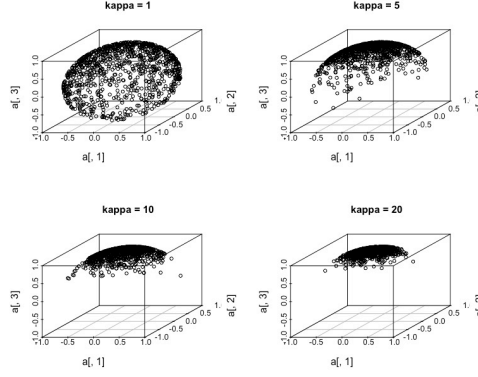


Figure 3: Sampling from vMF

## 2 Updating Procedure for Variable Rank Model

### Step A. Variable dimension sampler

- (1) Sample  $(\{d_j = 0\}, \{d_j \neq 0\})$  conditional on  $\mathbf{Y}, \Theta_{-j}, \phi, \mu, \psi$

$$\text{odds}(d_j \neq 0 | \mathbf{Y}, \Theta_{-j}, \phi, \mu, \psi) = \frac{P(d_j \neq 0 | \Theta_{-j})P(\mathbf{Y} | \Theta_{-j}, d_j \neq 0, \phi, \mu, \psi)}{P(d_j = 0 | \Theta_{-j})P(\mathbf{Y} | \Theta_{-j}, d_j = 0, \phi, \mu, \psi)}$$

- (2) Sample  $d_j$  conditional on  $\mathbf{Y}, \Theta_{-j}, \phi, \mu, \psi, d_j \neq 0$

$$p(d_j | \mathbf{Y}, \Theta_{-j}, \phi, \mu, \psi, \{d_j \neq 0\}) \propto \exp\left(-\frac{1}{2}(d_j - \mu)^2\psi\right) \exp\left(-\frac{1}{2}d_j^2\psi\right) \sum_{l=0}^{\infty} \|\tilde{\mathbf{E}}\|^{2l} \phi^{2l} d_j^{2l} a_l$$

- (3) Sample  $(\mathbf{U}_{[j]}, \mathbf{V}_{[j]})$  conditional on  $\mathbf{Y}, \Theta_{-j}, \phi, \mu, \psi, d_j$

$$P(\mathbf{u}, \mathbf{v} | \mathbf{Y}, \Theta_{-j}, \phi, \mu, \psi, d_j) = c(\mathbf{A}) \cdot e^{\mathbf{u}' \mathbf{A} \mathbf{v}}, \quad \text{where } \mathbf{A} = \phi d_j \tilde{\mathbf{E}},$$

$$c(\mathbf{A}) = [c_{\tilde{m}}(0)^{-1} c_{\tilde{n}}(0)^{-1} \sum_{l=0}^{\infty} \|\mathbf{A}\|^{2l} a_l]^{-1}, \quad c_p(0)^{-1} = \frac{2\pi^{p/2}}{\Gamma(p/2)}$$

$$\mathbf{U}_{[j]} = \mathbf{N}_{-j}^{\mathbf{u}} \mathbf{u} \quad \text{and} \quad \mathbf{V}_{[j]} = \mathbf{N}_{-j}^{\mathbf{v}} \mathbf{v}$$

### 3 Simulation Study

#### 1. Generate dataset

- $\mathbf{U} \sim \text{uniform}(\mathcal{V}_{5,m})$  and  $\mathbf{V} \sim \text{uniform}(\mathcal{V}_{5,n})$
- $\mathbf{D} = \text{diag}\{d_1, \dots, d_5\} \sim \text{iid uniform}(\frac{1}{2}\mu_{mn}, \frac{3}{2}\mu_{mn})$   
 where  $\mu_{mn} = \sqrt{n+m+2\sqrt{nm}}$
- $\mathbf{Y} = \mathbf{UDV}' + \mathbf{E}$  and  $\mathbf{E}_{m \times n} \sim \text{N}(0, 1)$

#### 2. Set prior and hyperparameters

- $\mu \sim \text{N}(\mu_0, v_0^2)$ 
  - $\mu_0 = \frac{1}{n+1} \sum_{j=0}^n \hat{\mu}_j$
  - $v_0^2 = \frac{1}{n} \sum_{j=0}^n (\hat{\mu}_j - \bar{\mu})^2$
- $\psi \sim \text{Gamma}(\eta_0/2, \eta_0\tau_0^2/2)$ 
  - $\eta_0 = 2$
  - $\tau_0^2 = \frac{1}{n+1} \sum_{j=0}^n \hat{\tau}_j^2$
- $\phi \sim \text{Gamma}(\nu_0/2, \nu_0\sigma_0^2/2)$ 
  - $\nu_0 = 2$
  - $\sigma_0^2 = \frac{1}{n+1} \sum_{j=0}^n \hat{\sigma}_j^2$
- prior for the rank  $K$  was taken to be uniform on  $\{1, \dots, n\}$

#### 3. Simulation

- initial setting
  - niter = 20,000; nburn = 10,000; nthin = 10
- Starting values
  - $K = 0$
  - $\{\phi, \mu, \psi\}$  : Prior mode

#### 4. Simulation Result

- Trace plot
- ACF

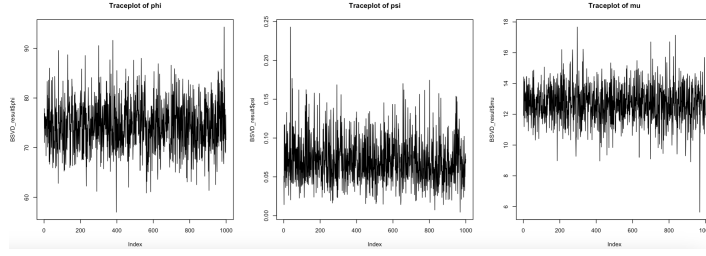


Figure 4: Trace plot of  $\phi, \psi, \mu$

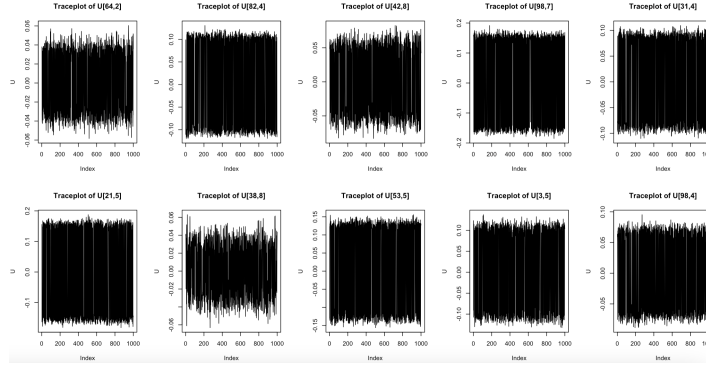


Figure 5: Trace plot of  $U$

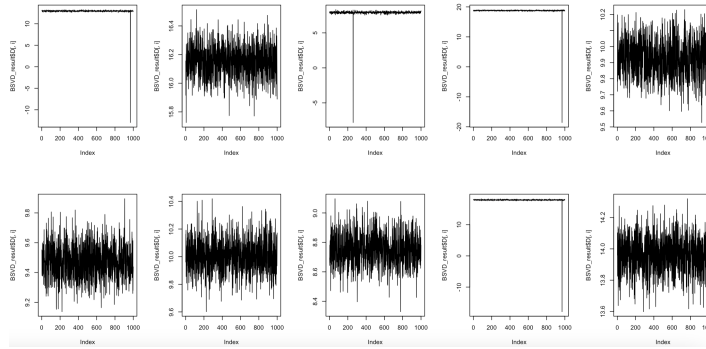


Figure 6: Trace plot of  $d_j$

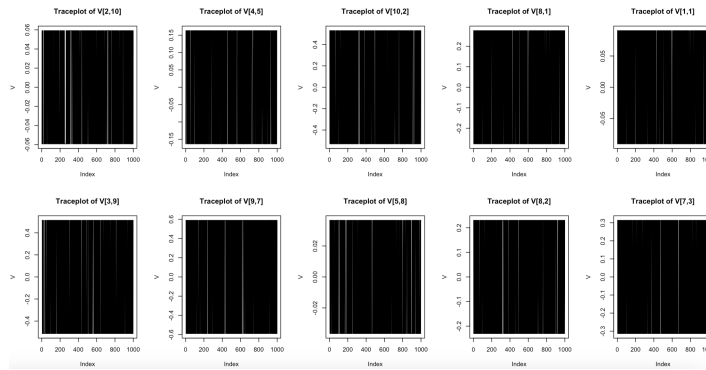


Figure 7: Trace plot of  $V$

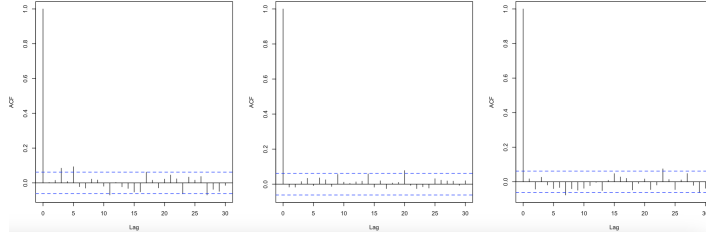


Figure 8: ACF of  $\phi, \psi, \mu$

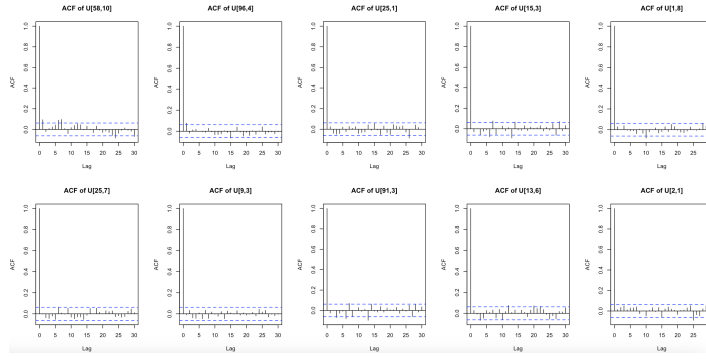


Figure 9: ACF of  $U$

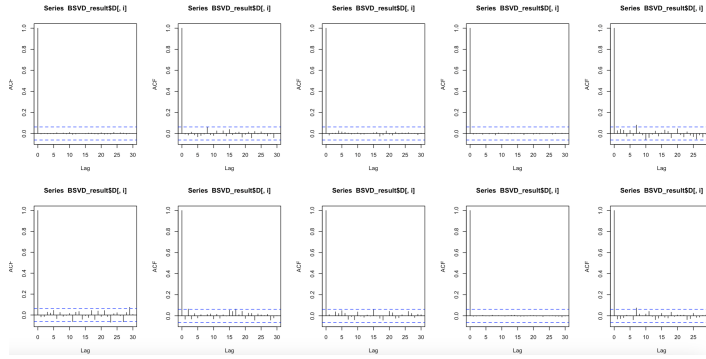


Figure 10: ACF of  $d_j$

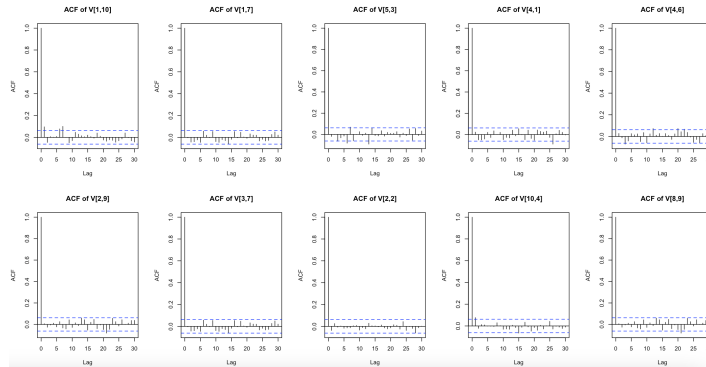


Figure 11: ACF of  $V$