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1. 다음 벡터들의 집합이 (1) $\mathbb{R}^2$, (2) $\mathbb{R}^3$을 생성 (span)할 수 있는 지를 보이세요. (각 1점)

(1) $\{(1,3),(1,-1)\}$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha+\beta \\ 3\alpha-\beta \end{pmatrix}$$

$\alpha+\beta = a$

$3\alpha-\beta = b$

$\rightarrow \beta = \frac{3a-b}{4}$

$\alpha = \frac{a+b}{4}$

$$= \frac{a+b}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{3a-b}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ $\Rightarrow$ 생성 O

(2) $\{(1,1,1),(1,2,3),(2,-1,1)\}$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + r \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha+\beta+2r \\ \alpha+2\beta-r \\ \alpha+3\beta+r \end{pmatrix}$$

역행렬 존재 O,

α, β, r 값 존재

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ r \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$\Rightarrow$ 생성 O

2. 다음 벡터공간에 대하여 차원을 구해보자. (각 1점)

(1) $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a = -b \right\}$

3

$$\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

(2) $\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x + y + z + w = 0,\ x, y, z, w\text{는 실수} \right\}$

3

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$$

3. 다음 행렬의 rank를 구해보자. (각 1점)

(1)

$$\begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & -1 & 5 \\ 2 & 4 & -6 & -2 & 4 \\ 1 & 7 & -13 & -5 & -3 \end{pmatrix}$$

$\boxed{\text{rank}=3}$

• 2행기준.

$$\begin{pmatrix} 0 & 1 & -2 & -2 & -7 \\ 1 & 3 & -5 & -1 & 5 \\ 0 & -2 & 4 & 0 & -6 \\ 0 & 4 & -8 & -4 & -8 \end{pmatrix}$$

$$\begin{array}{ccccc} 2 & 6 & -10 & -2 & 10 \\ -2 & -5 & 8 & 0 & -17 \\ \hline 0 & 1 & -2 & -2 & -7 \end{array}$$

$$\begin{array}{ccccc} -2 & -6 & 10 & 2 & -10 \\ 2 & 4 & -6 & -2 & 4 \\ \hline 0 & -2 & 4 & 0 & -6 \end{array}$$

$$\begin{array}{ccccc} -1 & -3 & 5 & 1 & -5 \\ 1 & 7 & -13 & -5 & -3 \\ \hline 0 & 4 & -8 & -4 & -8 \end{array}$$

$$\begin{array}{ccccc} 0 & 2 & -4 & -4 & -14 \\ 0 & -2 & 4 & 0 & -6 \\ \hline 0 & 0 & 0 & -4 & -20 \end{array}$$

$$\begin{array}{ccccc} 0 & -4 & 8 & 8 & 28 \\ 0 & 4 & -8 & -4 & -8 \\ \hline 0 & 0 & 0 & 4 & 20 \end{array}$$

• 1행기준

$$\begin{pmatrix} 0 & 1 & -2 & -2 & -7 \\ 1 & 3 & -5 & -1 & 5 \\ 0 & 0 & 0 & -4 & -20 \\ 0 & 0 & 0 & 4 & 20 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -2 & -2 & -7 \\ 1 & 3 & -5 & -1 & 5 \\ 0 & 0 & 0 & -4 & -20 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 1 & 5 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 1 & 5 \\ 2 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\text{rank}=3}$

4. $\mathbb{R}^4$의 부분공간 W의 기저 $\{u_1, u_2, u_3\}$, $u_1 = (1,0,1,1)$, $u_2 = (1,0,2,3)$, $u_3 = (2,-1,1,0)$ 일 때, W의 직교기저 $\{v_1, v_2, v_3\}$ 에 대하여 $||v_1||||v_2||||v_3||$를 구해보자. (1점)

(단위벡터)

$V_1 = (2,-1,1,0) \rightarrow \vec{E_1} = \frac{1}{\sqrt{6}}(2,-1,1,0),$

$V_2 = u_1 - (u_1 \cdot \vec{E_1})\vec{E_1}$ $\qquad ||V_1|| = \sqrt{6}$

$= (1,0,1,1) - \left((1,0,1,1)\cdot\frac{1}{\sqrt{6}}(2,-1,1,0)\right)\frac{1}{\sqrt{6}}(2,-1,1,0)$

$= (1,0,1,1) - \frac{3}{\sqrt{6}}\frac{1}{\sqrt{6}}(2,-1,1,0)$

$= (0, \frac{1}{2}, \frac{1}{2}, 1)$ $\qquad ||V_2|| = \sqrt{\frac{3}{2}}$

$\rightarrow t_2 = \frac{(0,\frac{1}{2},\frac{1}{2},1)}{\sqrt{\frac{3}{2}}} = \sqrt{\frac{2}{3}}(0,\frac{1}{2},\frac{1}{2},1)$

$V_3 = u_2 - (u_2 \cdot \vec{E_1})\vec{E_1} - (u_2 \cdot \vec{E_2})\vec{t_2}$

$= (1,0,2,3) - \left((1,0,2,3)\cdot\frac{1}{\sqrt{6}}(2,-1,1,0)\right)\frac{1}{\sqrt{6}}(2,-1,1,0)$

$- \left((1,0,2,3)\cdot\sqrt{\frac{2}{3}}(0,\frac{1}{2},\frac{1}{2},1)\right)\sqrt{\frac{2}{3}}(0,\frac{1}{2},\frac{1}{2},1)$

$||V_3|| = \frac{\sqrt{6}}{3}$

$= \left(-\frac{1}{3}, -\frac{2}{3}, 0, \frac{1}{3}\right) = \frac{1}{3}(-1,-2,0,1)$

$\therefore \sqrt{6} \times \sqrt{\frac{3}{2}} \times \frac{\sqrt{6}}{3} = \sqrt{6}$

5. 다음과 같이 주어지는 행렬 A의 영공간 (null space)의 차원을 m, A^2의 영공간의 차원을 n이라 할 때, $m+n$의 값을 구해보자. (1점)

$$A = \overset{4\times 4}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

1. $m = \text{nullity}(A) = n - \text{rank}(A)$

$= 4 - 3$

$= 1$

2. $A^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(차원)

$n = \text{nullity}(A^2) = \overset{(m\times n)}{n} - \text{rank}(A^2)$

$= 4 - 0$

$= 4$

$\therefore m + n = 5$

6. 다음 3×3 행렬 A의 고유값 $(\lambda_1, \lambda_2, \lambda_3)$에 대응하는 고유벡터를 각각 $\vec{a}, \vec{b}, \vec{c}$라 할 때, $\lambda_1, \lambda_2, \lambda_3$, $\vec{a}$, $\vec{b}$, $\vec{c}$를 모두 구해보자. (단, $\lambda_1 < \lambda_2 < \lambda_3$) (1점)

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

① 고유값 구하기

$$|A - \lambda E| = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix}$$

$$= ((1-\lambda)((4-\lambda)(1-\lambda)+2)$$

$$= (1-\lambda)(\lambda^2 - 5\lambda + 6)$$

$$\rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\boxed{\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3}$$

6. 다음 3×3 행렬 A의 고유값 $(\lambda_1, \lambda_2, \lambda_3)$에 대응하는 고유벡터를 각각 $\vec{a}, \vec{b}, \vec{c}$라 할 때, $\lambda_1, \lambda_2, \lambda_3$, $\vec{a}$, $\vec{b}$, $\vec{c}$를 모두 구해보자. (단, $\lambda_1 < \lambda_2 < \lambda_3$) (1점)

$$\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

② 고유벡터 구하기

$$(A - \lambda E)\underline{x} = 0$$

1) $\lambda_1 = 1$

$$\begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2x = 0 \; \dots \; x = 0$

$3x + z = 0. \quad z = 0.$

$$\rightarrow \vec{a} = \alpha\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

2) $\lambda_2 = 2$

$$\begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2x + z = 0$

$-2x - y = 0$

$$\rightarrow \vec{b} = \beta\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$\Rightarrow z = y = -2x$

3) $\lambda_3 = 3$

$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x + z = 0$

$x + y = 0.$

$$\rightarrow \vec{c} = \gamma\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow x = -y = -z$

7. 다음 행렬의 고유값을 구해보자. (1점)

$$A = \begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{pmatrix}$$

$$|A - \lambda E| = \begin{vmatrix} a-\lambda & b & c & d \\ 0 & e-\lambda & f & g \\ 0 & 0 & h-\lambda & i \\ 0 & 0 & 0 & j-\lambda \end{vmatrix}$$

$$- \quad + \quad - \quad +$$

$$= (j-\lambda) \begin{vmatrix} a-\lambda & b & c \\ 0 & e-\lambda & f \\ 0 & 0 & h-\lambda \end{vmatrix}$$

$$+ \quad - \quad +$$

$$= (j-\lambda)(h-\lambda)(a-\lambda)(e-\lambda) = 0$$

↳ 고유값 : 특성방정식의 해

$$\boxed{\lambda = j, h, a, e}$$

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