

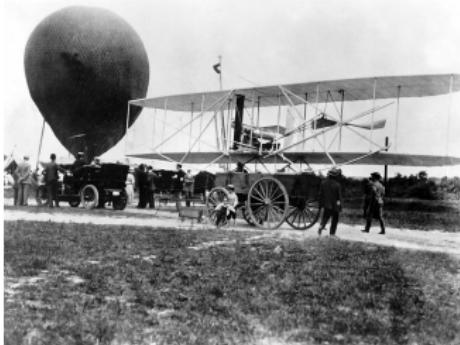
# Algebra, Geometry and Robots

The two-way relationship between mathematics and robotics, with a focus on modern algebra and geometry.

Sepehr Saryazdi

University of Sydney

# The Story of Humanity



三

ChatGPT



What day of the week is it?

Today is Thursday.



# Humanity's Next Tool: Robots



KUKA - Robotic Arms



# Waymo - Self-Driving Cars



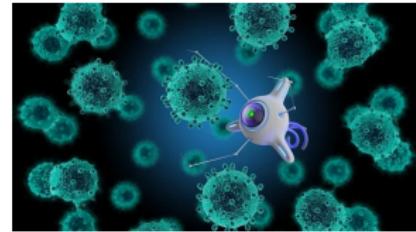
NASA - Martian Rovers



BMW -  
Humanoid Robots



## Boston Dynamics - Dog Robots



## Nanorobots

# What is Robotics?

## Common Definition of Robotics

$R(t)$  = Robotics at time  $t$

$A(t)$  := What's Theoretically Possible at time  $t$

$P(t)$  := What's Already Possible at time  $t$

$$R(t) := A(t) - P(t)$$

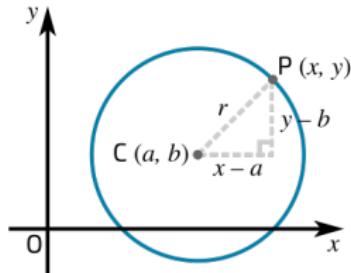
## Goal of Robotics

$$\lim_{t \rightarrow \infty} R(t) = 0$$

# The Story of Mathematics

$$1 + 2 = 3$$

$$ax^2 + bx + c = 0$$



$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\mathrm{GL}_n(\mathbb{R})$$

# Grp → Set

# The Story of Mathematics: Today's Mathematics

# Algebra

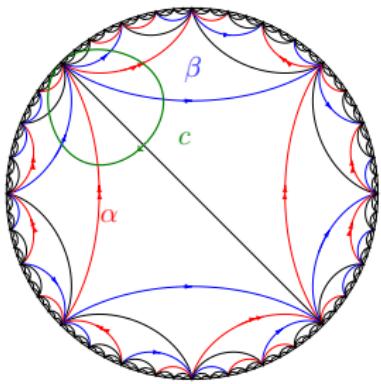
## Analysis

# Topology      Geometry

# Overview

- 1 Robotics
- 2 Mathematics
- 3 Who am I?
- 4 Kempe's Universality Theorem
- 5 Gröbner Basis
- 6 Lie Theory
- 7 Other Applications

# Who am I?



Recent Master's in  
Mathematics (Geometric  
Topology)



Started PhD with ACFR and  
CSIRO's Robotic Perception  
and Autonomy Group

# Kempe - Steam Engines



Alfred Kempe  
1849-1922

Steam Engine

# Kempe - Tracing Curves

<https://www.youtube.com/watch?v=9NbxE4PMeyQ>



# Kempe - Curves

## Curves in the Plane (Algebraic Subset)

$$P(x, y) = \sum_{ij} c_{ij} x^i y^j$$

A curve  $\mathcal{C} \subseteq \mathbb{R}^2$  is defined as the set of solutions to the equation  $P(x, y) = 0$ .



# Kempe - Available Linkages

Parallelogram Linkage

Peaucellier-Lipkin Linkage

<https://demonstrations.wolfram.com/PeaucelliersAndHartsInversor/>



# Kempe - Free Rhombus Linkage

## Rhombus Linkage

$$\text{Want: } P(x, y) = \sum_{ij} c_{ij} x^i y^j = 0$$

$$x = r \cos(\alpha) + r \cos(\beta)$$

$$y = r \sin(\alpha) + r \sin(\beta)$$

# Kempe - Substitute and Trig!

## Substitute

$$x = r \cos(\alpha) + r \cos(\beta)$$

$$y = r \sin(\alpha) + r \sin(\beta)$$

$$P(x, y) = \sum_{ij} c_{ij} (r \cos(\alpha) + r \cos(\beta))^i (r \sin(\alpha) + r \sin(\beta))^j = 0$$

## Refactor Everything Into cos's

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos(A) \cos(B) = \frac{1}{2} (\cos(A+B) + \cos(A-B))$$

# Kempe - Substitute and Trig!

Refactor Everything Into cos's

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos(A)\cos(B) = \frac{1}{2} (\cos(A+B) + \cos(A-B))$$

Collect Like Terms and Simplify

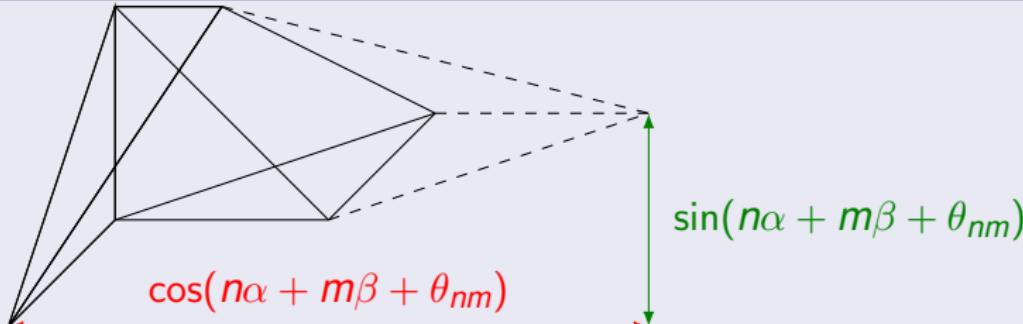
$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

# Kempe - Linkage Arithmetic

## Simplified Form

$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

Treat Each cos Term Like The Projection of a 'New' Linkage!

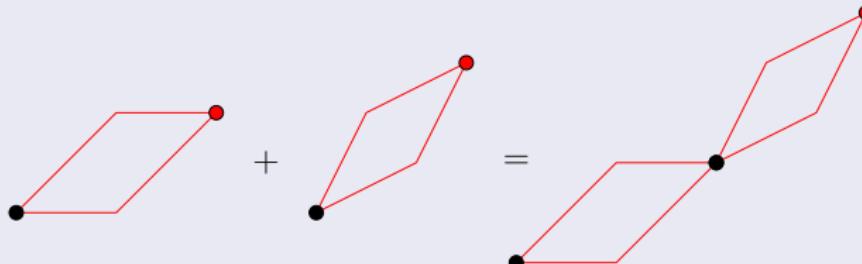


# Kempe - Linkage Arithmetic

## Simplified Form

$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

## Adding Linkages



# Kempe - Linkage Arithmetic

<https://demonstrations.wolfram.com/KempesTranslator/>

Translating Linkages

$$a + \cos(\theta), a, \theta \in \mathbb{R}$$

# Kempe - Linkage Arithmetic

<https://demonstrations.wolfram.com/KempesMultiplicator/>

Multiplying Angles

$$\cos(n\alpha), n \in \mathbb{Z}$$

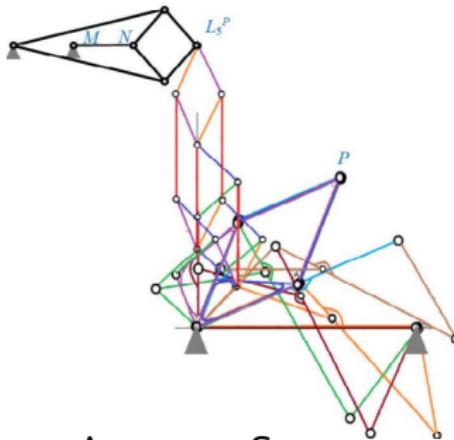
# Kempe - Linkage Arithmetic

<https://demonstrations.wolfram.com/KempesAngleAdder/>

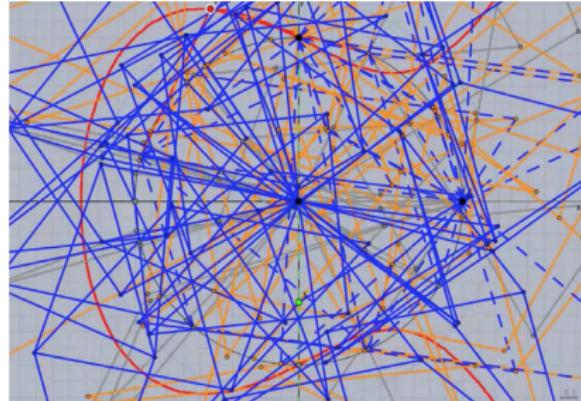
## Adding Angles

$$\cos(\theta_1 + \theta_2), \theta_1, \theta_2 \in \mathbb{R}$$

# Kempe - Combining Linkages



Anupam Saxena  
 $(x - y)(x + y + 1/\sqrt{2}) = 0$



Alexander Kobel  
 $x^3 - y^2 - x + 1 = 0$

## Simplified Form

$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}) = 0$$



# Kempe's Universality Theorem

Kempe's Universality Theorem (Statement made rigorous by Thurston & Kapovich-Millson)

Let  $\mathcal{C} \subseteq \mathbb{R}^2$  be a curve. Let  $f : [a, b] \rightarrow \mathcal{C}$  be a map such that  $f(t) = (p(t), q(t))$  where  $p, q \in \mathbb{R}[x]$  are polynomials. Then there is a linkage  $\mathcal{L}$  with some pinned vertices, and a vertex  $v$  of  $\mathcal{L}$  so that  $v$  traces out  $f([a, b])$  over all the linkage's configurations.



# Kempe's Universality Theorem

Corollary - (Thurston) There is a linkage which signs your name.

Break up your name into  $k$  segments, each of which is the image of a smooth function  $f_k : [a_k, b_k] \rightarrow \mathbb{R}^2$ . Apply the Stone-Weierstrass Approximation Theorem to approximate your curves by polynomials. Apply the previous result and carefully combine linkages, or apply the result to Lagrange polynomial interpolation.



# Kempe's Universality Theorem - Generalisations

- The proofs were rigorously completed in 2002 by Kapovich and Millson.
- (Kapovich-Millson) The result generalises to higher-dimensional algebraic sets, i.e. simultaneous solutions to  $P_1(x_1, \dots, x_n) = 0, P_2(x_1, \dots, x_n) = 0, \dots, P_m(x_1, \dots, x_n) = 0$ .
- (Kapovich-Millson) Let  $M$  be any smooth compact manifold. Then there is a linkage  $\mathcal{L}$  whose configuration space is diffeomorphic to a disjoint union of a number of copies of  $M$ .

# Gröbner Basis - Polynomials

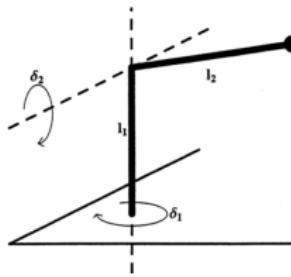


Bruno Buchberger  
1942 - Now

## ■ Example: Systems of Polynomial Equations

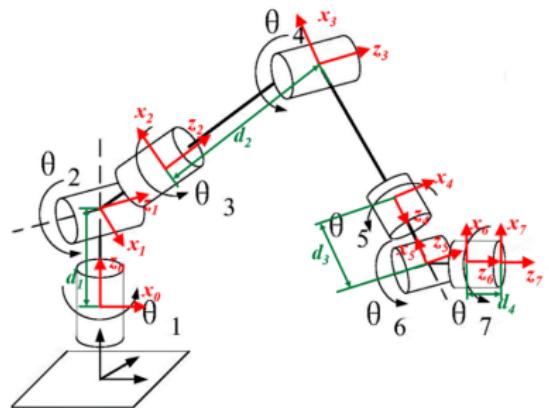
(This example is taken from (Buchberger, Kutzler 1986)).

Systems of multivariate polynomial equations are pervasive in all areas of engineering. For example, consider the following simple robot:



Robotics Example  
Bruno Buchberger

# Gröbner Basis - Polynomials in Kinematics



General Kinematics Problem  
Xuanming Zhang et. al.

Simple Kinematics Example

# Gröbner Basis - Polynomials in Kinematics

## Kinematics As A Map

$$F : [0, 2\pi)^3 \rightarrow \mathbb{R}^2$$

$$(\theta_1, \theta_2, \theta_3) \mapsto (x, y)$$



# Gröbner Basis - Polynomials in Kinematics

## Kinematics Map

$$(x, y) = F(\theta_1, \theta_2, \theta_3)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \end{pmatrix}$$



# Gröbner Basis - Polynomials in Kinematics

## Kinematics Map

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \end{pmatrix}$$
$$= \begin{pmatrix} L_1 c_1 + L_2 c_{12} + L_3 c_{123} \\ L_1 s_1 + L_2 s_{12} + L_3 s_{123} \end{pmatrix}$$

## Kinematics Polynomials

$$\begin{aligned} L_1 c_1 + L_2 c_{12} + L_3 c_{123} - x &= 0 \\ L_1 s_1 + L_2 s_{12} + L_3 s_{123} - y &= 0 \\ c_1^2 + s_1^2 - 1 &= 0 \\ c_{12}^2 + s_{12}^2 - 1 &= 0 \\ c_{123}^2 + s_{123}^2 - 1 &= 0 \end{aligned}$$



# Gröbner Basis - Polynomials in Kinematics

## Kinematics Polynomials

$$L_1 c_1 + L_2 c_{12} + L_3 c_{123} - x = 0$$

$$L_1 s_1 + L_2 s_{12} + L_3 s_{123} - y = 0$$

$$c_1^2 + s_1^2 - 1 = 0$$

$$c_{12}^2 + s_{12}^2 - 1 = 0$$

$$c_{123}^2 + s_{123}^2 - 1 = 0$$

## Abstracted Polynomials

$$P_1(x_i, y_i) = L_1 x_1 + L_2 x_2 + L_3 x_3 - x = 0$$

$$P_2(x_i, y_i) = L_1 y_1 + L_2 y_2 + L_3 y_3 - y = 0$$

$$P_3(x_i, y_i) = x_1^2 + y_1^2 - 1 = 0$$

$$P_4(x_i, y_i) = x_2^2 + y_2^2 - 1 = 0$$

$$P_5(x_i, y_i) = x_3^2 + y_3^2 - 1 = 0$$



# Gröbner Basis - Extending Gauss-Jordan

## Linear Equations

$$\begin{array}{l} ax_1 + x_2 = 1 \\ x_2 + x_3 = 2 \\ bx_1 + x_2 + x_3 = 1 \end{array} \Leftrightarrow \begin{pmatrix} a & 1 & 0 \\ 0 & 1 & 1 \\ b & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

## Gauss-Jordan Elimination

$$R_3 \mapsto aR_3 - bR_1$$

# Gröbner Basis - Extending Gauss-Jordan

## Linear Equations

$$\begin{array}{l} ax_1 + x_2 = 1 \\ x_2 + x_3 = 2 \\ bx_1 + x_2 + x_3 = 1 \end{array} \Leftrightarrow \begin{pmatrix} a & 1 & 0 \\ 0 & 1 & 1 \\ b & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

## Gauss-Jordan Elimination

$$R_3 \mapsto aR_3 - bR_1$$

 $\Leftrightarrow$ 

$$R_3 \mapsto \frac{ab}{b}R_3 - \frac{ab}{a}R_1$$



# Gröbner Basis - Triangularisation!

## Abstracted Polynomials

$$P_1(x_i, y_i) = L_1x_1 + L_2x_2 + L_3x_3 - x = 0$$

$$P_2(x_i, y_i) = L_1y_1 + L_2y_2 + L_3y_3 - y = 0$$

$$P_3(x_i, y_i) = x_1^2 + y_1^2 - 1 = 0$$

$$P_4(x_i, y_i) = x_2^2 + y_2^2 - 1 = 0$$

$$P_5(x_i, y_i) = x_3^2 + y_3^2 - 1 = 0$$

## Compute S-Polynomial

$$S(P_i, P_j) = \frac{\text{lcm}(\text{LT}(P_i), \text{LT}(P_j))}{\text{LT}(P_i)} P_i - \frac{\text{lcm}(\text{LT}(P_i), \text{LT}(P_j))}{\text{LT}(P_j)} P_j$$

# Gröbner Basis - Triangularisation!

## Abstracted Polynomials

$$P_1(x_i, y_i) = L_1x_1 + L_2x_2 + L_3x_3 - x = 0$$

$$P_2(x_i, y_i) = L_1y_1 + L_2y_2 + L_3y_3 - y = 0$$

$$P_3(x_i, y_i) = x_1^2 + y_1^2 - 1 = 0$$

$$P_4(x_i, y_i) = x_2^2 + y_2^2 - 1 = 0$$

$$P_5(x_i, y_i) = x_3^2 + y_3^2 - 1 = 0$$

## S-Polynomial Example

$$S(P_1, P_2) = \frac{\text{lcm}(\text{LT}(P_1), \text{LT}(P_2))}{\text{LT}(P_2)} P_1 - \frac{\text{lcm}(\text{LT}(P_1), \text{LT}(P_2))}{\text{LT}(P_2)} P_2$$

$$= \cancel{\frac{L_3x_3y_3}{L_3x_3}} P_1 - \cancel{\frac{L_3x_3y_3}{L_3y_3}} P_2 = y_3(L_1x_1 + L_2x_2 + L_3\cancel{x_3} - x) - x_3(L_1y_1 + L_2y_2 + L_3\cancel{y_3} - y)$$



# Gröbner Basis - Triangularisation!

```

1 import sympy as sp
2 x1,x2,x3,y1,y2,y3 = sp.Symbol("x1"), sp.Symbol("x2"), sp.Symbol("x3"), sp.Symbol("y1"), sp.Symbol("y2"), sp.Symbol("y3")
3 L1,L2,L3 = 1,1,1
4 x,y = sp.Symbol("x"), sp.Symbol("y")
5 F = [L1*x1+L2*x2+L3*x3 - x,
6      L1*y1+L2*y2+L3*y3 - y,
7      x1**2 + y1**2 - 1,
8      x2**2 + y2**2 - 1,
9      x3**2 + y3**2 - 1]
10
11 basis = sp.GroebnerBasis(F, x1, x2, x3, y1, y2, y3, order='lex')
12 print(basis)

```

# Gröbner Basis - Triangularisation!

## Triangular Form

$$-x + x_1 + x_2 + x_3 = 0$$

$$x_2^2 + y_2^2 - 1 = 0$$

$$x^2 - 2x\textcolor{violet}{x}_2 - 2x\textcolor{blue}{x}_3 + 2\textcolor{violet}{x}_2\textcolor{blue}{x}_3 + y^2 - 2y\textcolor{violet}{y}_2 - 2y\textcolor{blue}{y}_3 + 2\textcolor{violet}{y}_2\textcolor{blue}{y}_3 + 1 = 0$$

•  
•  
•

# Gröbner Basis - Triangularisation!

## Result

$$\cos(\theta_1) = x_1(x, y)$$

$$\cos(\theta_1 + \theta_2) = x_2(x, y)$$

$$\cos(\theta_1 + \theta_2 + \theta_3) = x_3(x, y)$$

$$(\theta_1, \theta_2, \theta_3) \in F^{-1}[(x, y)]$$



# Lie Theory

# Other Applications of Mathematics