Algebra, Geometry and Robots

The two-way relationship between mathematics and robotics, with a focus on modern algebra and geometry.

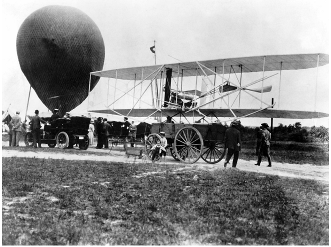
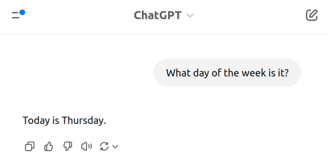
Sepehr Saryazdi

University of Sydney

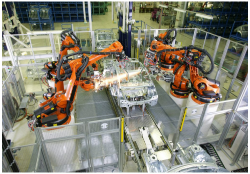
The Story of Humanity



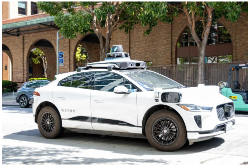


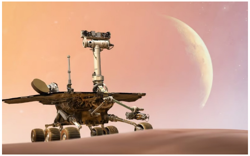
Humanity's Next Tool: Robots



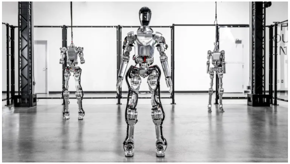
KUKA - Robotic Arms



Waymo - Self-Driving Cars

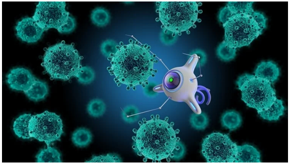


NASA - Martian Rovers



BMW - Humanoid Robots

Boston Dynamics - Dog Robots



Nanorobots

What is Robotics?

Common Definition of Robotics

$$R(t) = \text{Robotics at time } t$$

$$A(t) := \text{What's Theoretically Possible at time } t$$

$$P(t) := \text{What's Already Possible at time } t$$

$$R(t) := A(t) - P(t)$$

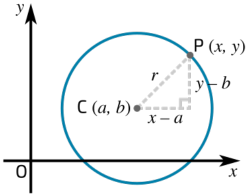
Goal of Robotics

$$\lim_{t \to \infty} R(t) = 0$$

The Story of Mathematics

$$1 + 2 = 3$$

$$ax^2 + bx + c = 0$$



$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$GL_n(\mathbb{R})$$

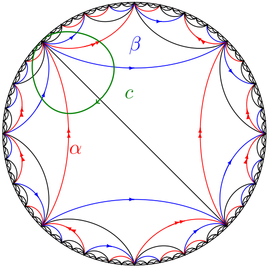
$$\mathrm{Grp} \to \mathrm{Set}$$

The Story of Mathematics: Today's Mathematics

Algebra	Analysis
Topology	Geometry

Overview

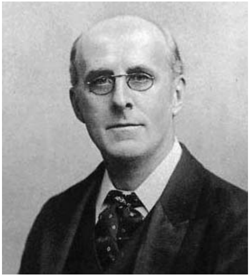
Who am I?



Recent Master's in Mathematics (Geometric Topology)

Started PhD with ACFR and CSIRO's Robotic Perception and Autonomy Group

Kempe - Steam Engines



Alfred Kempe
1849-1922

Steam Engine

Kempe - Tracing Curves

https://www.youtube.com/watch?v=9NbwE4PMeyQ

Kempe - Curves

Curves in the Plane (Algebraic Subset)

$$P(x, y) = \sum_{ij} c_{ij} x^i y^j$$

A curve $\mathcal{C} \subseteq \mathbb{R}^2$ is defined as the set of solutions to the equation $P(x, y) = 0$.

Kempe - Available Linkages

Parallelogram Linkage

Peaucellier–Lipkin Linkage

https://demonstrations.wolfram.com/PeaucelliersAndHartsInversor/

Kempe - Free Rhombus Linkage

Rhombus Linkage

$$\text{Want: } P(x, y) = \sum_{ij} c_{ij} x^i y^j = 0$$

$$x = r\cos(\alpha) + r\cos(\beta)$$

$$y = r\sin(\alpha) + r\sin(\beta)$$

Kempe - Substitute and Trig!

Substitute

$$x = r\cos(\alpha) + r\cos(\beta)$$

$$y = r\sin(\alpha) + r\sin(\beta)$$

$$P(x, y) = \sum_{ij} c_{ij}(r\cos(\alpha) + r\cos(\beta))^i (r\sin(\alpha) + r\sin(\beta))^j = 0$$

Refactor Everything Into cos's

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A + B) + \cos(A - B))$$

Kempe - Substitute and Trig!

Refactor Everything Into cos's

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

Collect Like Terms and Simplify

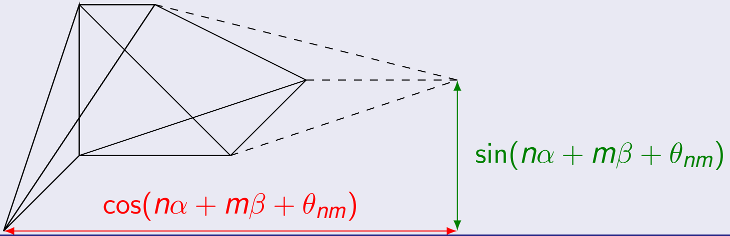
$$P(x,y) = \sum_{n,m\in\mathbb{Z}} c'_{nm}\cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

Kempe - Linkage Arithmetic

Simplified Form

$$P(x,y) = \sum_{n,m\in\mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

Treat Each cos Term Like The Projection of a 'New' Linkage!

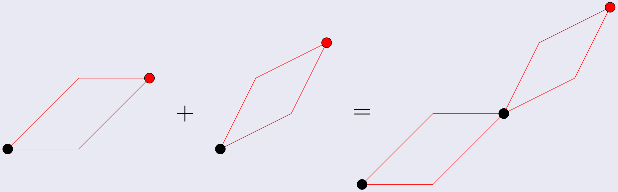


Kempe - Linkage Arithmetic

Simplified Form

$$P(x,y) = \sum_{n,m\in\mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

Adding Linkages



Kempe - Linkage Arithmetic

https://demonstrations.wolfram.com/KempesTranslator/

Translating Linkages

$$a + \cos(\theta), a, \theta \in \mathbb{R}$$

Kempe - Linkage Arithmetic

https://demonstrations.wolfram.com/KempesMultiplicator/

Multiplying Angles

$$\cos(n\alpha), n \in \mathbb{Z}$$

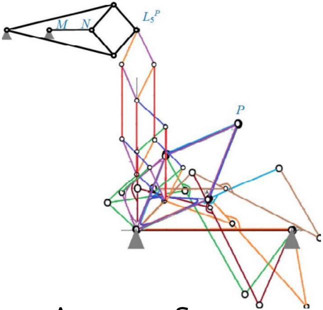
Kempe - Linkage Arithmetic

https://demonstrations.wolfram.com/KempesAngleAdder/

Adding Angles

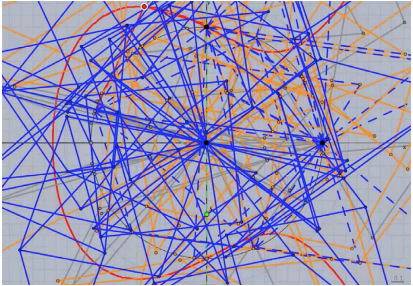
$$\cos(\theta_1 + \theta_2), \theta_1, \theta_2 \in \mathbb{R}$$

Kempe - Combining Linkages



Anupam Saxena

$(x-y)(x+y+1/\sqrt{2})=0$



Alexander Kobel

$x^3-y^2-x+1=0$

Simplified Form

$$P(x,y)=\sum_{n,m\in\mathbb{Z}} c'_{nm}\cos(n\alpha+m\beta+\theta_{nm})=0$$

Kempe's Universality Theorem

Kempe's Universality Theorem (Statement made rigorous by Thurston & Kapovich-Millson)

Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a curve. Let $f : [a, b] \to \mathcal{C}$ be a map such that $f(t) = (p(t), q(t))$ where $p, q \in \mathbb{R}[x]$ are polynomials. Then there is a linkage $\mathcal{L}$ with some pinned vertices, and a vertex v of $\mathcal{L}$ so that v traces out $f([a, b])$ over all the linkage's configurations.

Kempe's Universality Theorem

Corollary - (Thurston) There is a linkage which signs your name.

Break up your name into k segments, each of which is the image of a smooth function $f_k : [a_k, b_k] \to \mathbb{R}^2$. Apply the Stone-Weierstrass Approximation Theorem to approximate your curves by polynomials. Apply the previous result and carefully combine linkages, or apply the result to Lagrange polynomial interpolation.

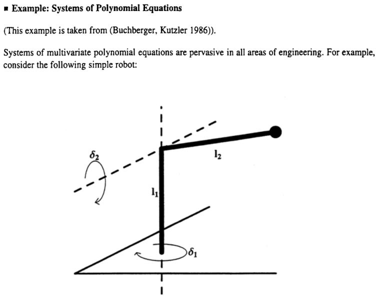
Kempe's Universality Theorem - Generalisations

- The proofs were rigorously completed in 2002 by Kapovich and Millson.
- (Kapovich-Millson) The result generalises to higher-dimensional algebraic sets, i.e. simultaneous solutions to $P_1(x_1, \ldots, x_n) = 0, P_2(x_1, \ldots, x_n) = 0, \ldots, P_m(x_1, \ldots, x_n) = 0.$
- (Kapovich-Millson) Let M be any smooth compact manifold. Then there is a linkage $\mathcal{L}$ whose configuration space is diffeomorphic to a disjoint union of a number of copies of M.

Gröbner Basis - Polynomials

Bruno Buchberger

1942 - Now



■ **Example: Systems of Polynomial Equations**

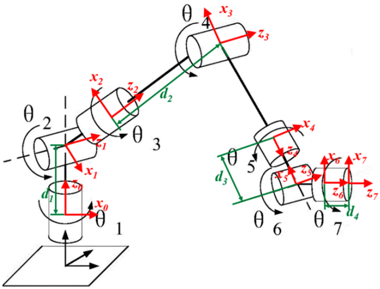
(This example is taken from (Buchberger, Kutzler 1986)).

Systems of multivariate polynomial equations are pervasive in all areas of engineering. For example, consider the following simple robot:

Robotics Example

Bruno Buchberger

Gröbner Basis - Polynomials in Kinematics



General Kinematics Problem
Xuanming Zhang et. al.

Simple Kinematics Example

Gröbner Basis - Polynomials in Kinematics

Kinematics As A Map

$$F : [0, 2\pi)^3 \to \mathbb{R}^2$$
$$(\theta_1, \theta_2, \theta_3) \mapsto (x, y)$$

Gröbner Basis - Polynomials in Kinematics

Kinematics Map

$$(x, y) = F(\theta_1, \theta_2, \theta_3)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \end{pmatrix}$$

Gröbner Basis

Lie Theory

Other Applications of Mathematics