

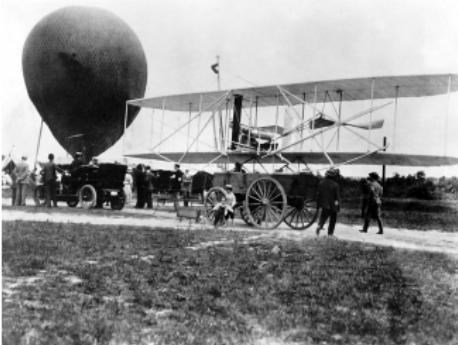
Algebra, Geometry and Robots

The two-way relationship between mathematics and robotics, with a focus on modern algebra and geometry.

Sepehr Saryazdi

University of Sydney

The Story of Humanity



ChatGPT ▾



What day of the week is it?

Today is Thursday.



Humanity's Next Tool: Robots



KUKA - Robotic Arms



Waymo -
Self-Driving Cars



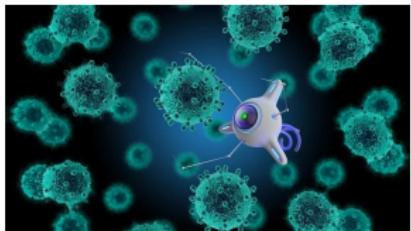
NASA - Martian Rovers



BMW -
Humanoid Robots



Boston Dynamics -
Dog Robots



Nanorobots

What is Robotics?

Common Definition of Robotics

$R(t)$ = Robotics at time t

$A(t)$:= What's Theoretically Possible at time t

$P(t)$:= What's Already Possible at time t

$$R(t) := A(t) - P(t)$$

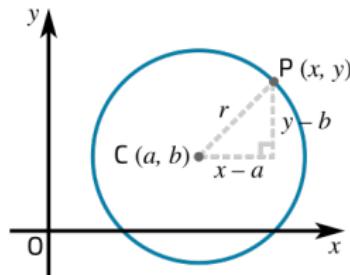
Goal of Robotics

$$\lim_{t \rightarrow \infty} R(t) = 0$$

The Story of Mathematics

$$1 + 2 = 3$$

$$ax^2 + bx + c = 0$$



$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\mathrm{GL}_n(\mathbb{R})$$

$$\mathrm{Grp} \rightarrow \mathrm{Set}$$

The Story of Mathematics: Today's Mathematics

Algebra

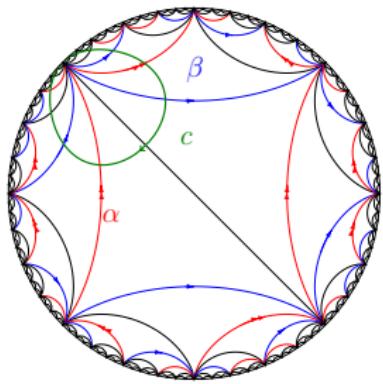
Analysis

Topology Geometry

Overview

- 1 Robotics
- 2 Mathematics
- 3 Who am I?
- 4 Kempe's Universality Theorem
- 5 Gröbner Basis
- 6 Lie Theory
- 7 Other Applications

Who am I?



Recent Master's in
Mathematics (Geometric
Topology)



Started PhD with ACFR and
CSIRO's Robotic Perception
and Autonomy Group

Kempe - Steam Engines



Alfred Kempe
1849-1922

Steam Engine

Kempe - Tracing Curves

<https://www.youtube.com/watch?v=9NbxE4PMeyQ>



Kempe - Curves

Curves in the Plane (Algebraic Subset)

$$P(x, y) = \sum_{ij} c_{ij} x^i y^j$$

A curve $\mathcal{C} \subseteq \mathbb{R}^2$ is defined as the set of solutions to the equation $P(x, y) = 0$.



Kempe - Available Linkages

Parallelogram Linkage

Peaucellier-Lipkin Linkage

<https://demonstrations.wolfram.com/PeaucelliersAndHartsInversor/>



Kempe - Free Rhombus Linkage

Rhombus Linkage

$$\text{Want: } P(x, y) = \sum_{ij} c_{ij} x^i y^j = 0$$

$$x = r \cos(\alpha) + r \cos(\beta)$$

$$y = r \sin(\alpha) + r \sin(\beta)$$

Kempe - Substitute and Trig!

Substitute

$$x = r \cos(\alpha) + r \cos(\beta)$$

$$y = r \sin(\alpha) + r \sin(\beta)$$

$$P(x, y) = \sum_{ij} c_{ij} (r \cos(\alpha) + r \cos(\beta))^i (r \sin(\alpha) + r \sin(\beta))^j = 0$$

Refactor Everything Into cos's

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos(A) \cos(B) = \frac{1}{2} (\cos(A+B) + \cos(A-B))$$

Kempe - Substitute and Trig!

Refactor Everything Into cos's

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos(A)\cos(B) = \frac{1}{2} (\cos(A+B) + \cos(A-B))$$

Collect Like Terms and Simplify

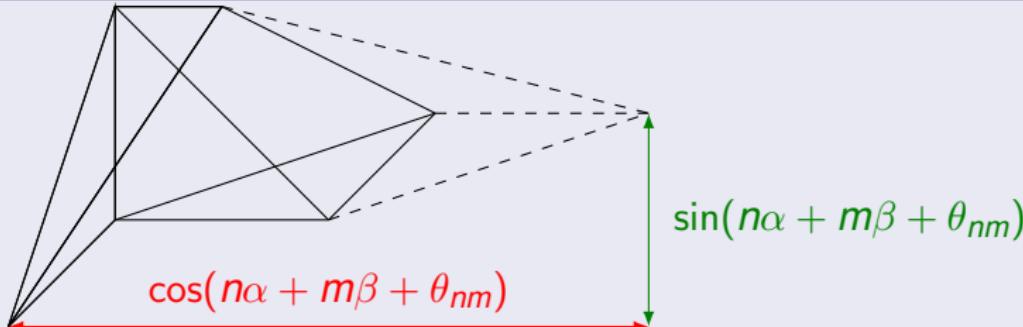
$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

Kempe - Linkage Arithmetic

Simplified Form

$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

Treat Each cos Term Like The Projection of a 'New' Linkage!

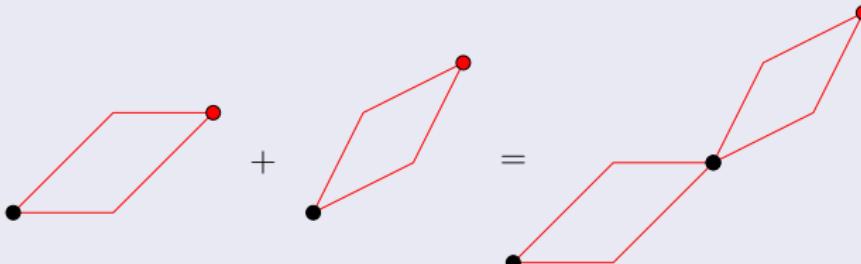


Kempe - Linkage Arithmetic

Simplified Form

$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

Adding Linkages



Kempe - Linkage Arithmetic

<https://demonstrations.wolfram.com/KempesTranslator/>

Translating Linkages

$$a + \cos(\theta), a, \theta \in \mathbb{R}$$

Kempe - Linkage Arithmetic

<https://demonstrations.wolfram.com/KempesMultiplicator/>

Multiplying Angles

$$\cos(n\alpha), n \in \mathbb{Z}$$

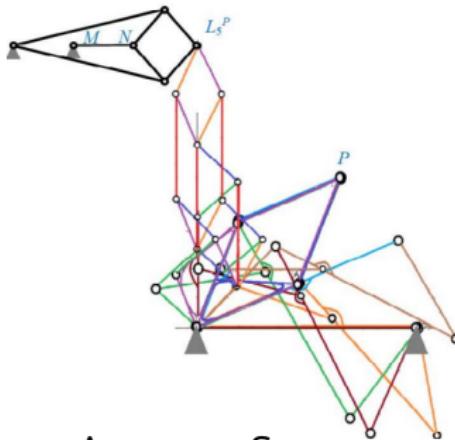
Kempe - Linkage Arithmetic

<https://demonstrations.wolfram.com/KempesAngleAdder/>

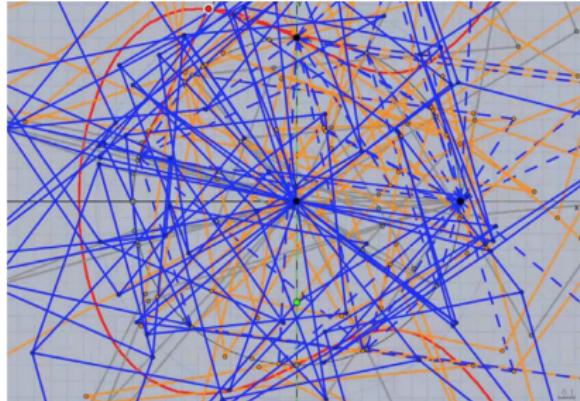
Adding Angles

$$\cos(\theta_1 + \theta_2), \theta_1, \theta_2 \in \mathbb{R}$$

Kempe - Combining Linkages



Anupam Saxena
 $(x - y)(x + y + 1/\sqrt{2}) = 0$



Alexander Kobel
 $x^3 - y^2 - x + 1 = 0$

Simplified Form

$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}) = 0$$



Kempe's Universality Theorem

Kempe's Universality Theorem (Statement made rigorous by Thurston & Kapovich-Millson)

Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a curve. Let $f : [a, b] \rightarrow \mathcal{C}$ be a map such that $f(t) = (p(t), q(t))$ where $p, q \in \mathbb{R}[x]$ are polynomials. Then there is a linkage \mathcal{L} with some pinned vertices, and a vertex v of \mathcal{L} so that v traces out $f([a, b])$ over all the linkage's configurations.



Kempe's Universality Theorem

Corollary - (Thurston) There is a linkage which signs your name.

Break up your name into k segments, each of which is the image of a smooth function $f_k : [a_k, b_k] \rightarrow \mathbb{R}^2$. Apply the Stone-Weierstrass Approximation Theorem to approximate your curves by polynomials. Apply the previous result and carefully combine linkages, or apply the result to Lagrange polynomial interpolation.



Kempe's Universality Theorem - Generalisations

- The proofs were rigorously completed in 2002 by Kapovich and Millson.
- (Kapovich-Millson) The result generalises to higher-dimensional algebraic sets, i.e. simultaneous solutions to $P_1(x_1, \dots, x_n) = 0, P_2(x_1, \dots, x_n) = 0, \dots, P_m(x_1, \dots, x_n) = 0$.
- (Kapovich-Millson) Let M be any smooth compact manifold. Then there is a linkage \mathcal{L} whose configuration space is diffeomorphic to a disjoint union of a number of copies of M .

Gröbner Basis - Polynomials

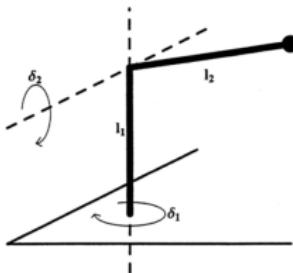


Bruno Buchberger
1942 - Now

■ Example: Systems of Polynomial Equations

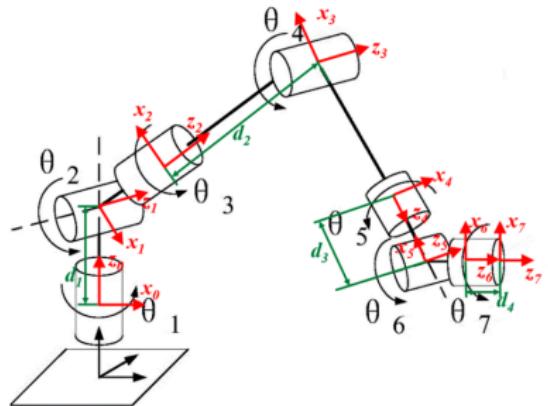
(This example is taken from (Buchberger, Kutzler 1986)).

Systems of multivariate polynomial equations are pervasive in all areas of engineering. For example, consider the following simple robot:



Robotics Example
Bruno Buchberger

Gröbner Basis - Polynomials in Kinematics



General Kinematics Problem
Xuanming Zhang et. al.

Simple Kinematics Example

Gröbner Basis - Polynomials in Kinematics

Kinematics As A Map

$$F : [0, 2\pi)^3 \rightarrow \mathbb{R}^2$$

$$(\theta_1, \theta_2, \theta_3) \mapsto (x, y)$$



Gröbner Basis

Lie Theory

Other Applications of Mathematics