

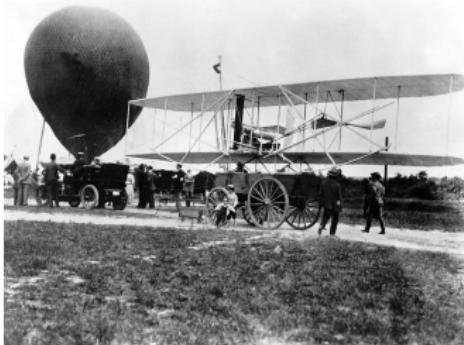
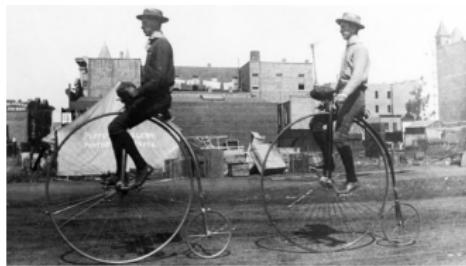
Algebra, Geometry and Robots

The two-way relationship between mathematics and robotics, with a focus on modern algebra and geometry.

Sepehr Saryazdi

University of Sydney

The Story of Humanity



ChatGPT ✓

What day of the week is it?

Today is Thursday.

Humanity's Next Tool: Robots



KUKA - Robotic Arms



Waymo - Self-Driving Cars



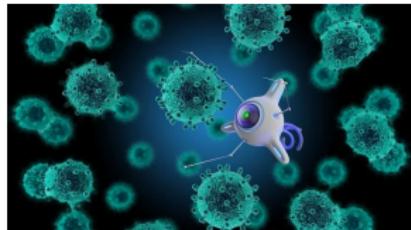
NASA - Martian Rovers



BMW -
Humanoid Robots



Boston Dynamics - Dog Robots



Nanorobots

What is Robotics?

Common Definition of Robotics

$R(t)$ = Robotics at time t

$A(t)$:= What's Theoretically Possible at time t

$P(t)$:= What's Already Possible at time t

$$R(t) := A(t) - P(t)$$

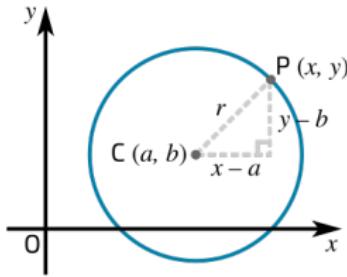
Goal of Robotics

$$\lim_{t \rightarrow \infty} R(t) = 0$$

The Story of Mathematics

$$1 + 2 = 3$$

$$ax^2 + bx + c = 0$$



$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\mathrm{GL}_n(\mathbb{R})$$

 $\mathrm{Grp} \rightarrow \mathrm{Set}$

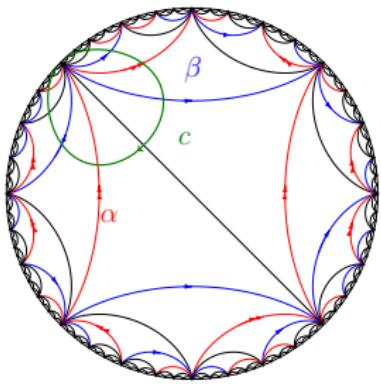
The Story of Mathematics: Today's Mathematics

Algebra

Analysis

Topology Geometry

Who am I?



Recent Master's in
Mathematics (Geometric
Topology)



Starting PhD with ACFR and
CSIRO's Robotic Perception
and Autonomy Group

Kempe - Steam Engines



Alfred Kempe
1849-1922

Steam Engine

Kempe - Tracing Curves

<https://www.youtube.com/watch?v=9NbxE4PMeyQ>

Kempe - Curves

Curves in the Plane (Algebraic Subset)

$$P(x, y) = \sum_{ij} c_{ij} x^i y^j$$

A curve $\mathcal{C} \subseteq \mathbb{R}^2$ is defined as the set of solutions to the equation $P(x, y) = 0$.



Kempe - Available Linkages

Parallelogram Linkage

Peaucellier-Lipkin Linkage

<https://demonstrations.wolfram.com/PeaucelliersAndHartsInversor/>



Kempe - Free Rhombus Linkage

Rhombus Linkage

$$\text{Want: } P(x, y) = \sum_{ij} c_{ij} x^i y^j = 0$$

$$x = r \cos(\alpha) + r \cos(\beta)$$

$$y = r \sin(\alpha) + r \sin(\beta)$$

Kempe - Substitute and Trig!

Substitute

$$x = r \cos(\alpha) + r \cos(\beta)$$

$$y = r \sin(\alpha) + r \sin(\beta)$$

$$P(x, y) = \sum_{ij} c_{ij} (r \cos(\alpha) + r \cos(\beta))^i (r \sin(\alpha) + r \sin(\beta))^j = 0$$

Refactor Everything Into cos's

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos(A) \cos(B) = \frac{1}{2} (\cos(A+B) + \cos(A-B))$$

Kempe - Substitute and Trig!

Refactor Everything Into cos's

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos(A)\cos(B) = \frac{1}{2} (\cos(A+B) + \cos(A-B))$$

Collect Like Terms and Simplify

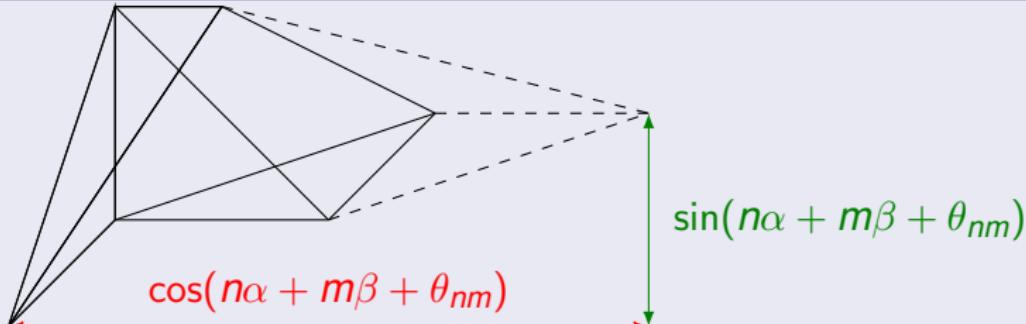
$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

Kempe - Linkage Arithmetic

Simplified Form

$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

Treat Each cos Term Like The Projection of a 'New' Linkage!

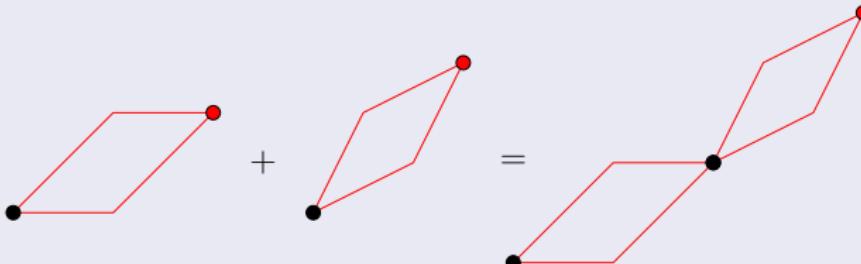


Kempe - Linkage Arithmetic

Simplified Form

$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}), \theta_{nm} \in \{0, \pm\pi/2\}$$

Adding Linkages



Kempe - Linkage Arithmetic

<https://demonstrations.wolfram.com/KempesTranslator/>

Translating Linkages

$$a + \cos(\theta), a, \theta \in \mathbb{R}$$

Kempe - Linkage Arithmetic

<https://demonstrations.wolfram.com/KempesMultiplicator/>

Multiplying Angles

$$\cos(n\alpha), n \in \mathbb{Z}$$

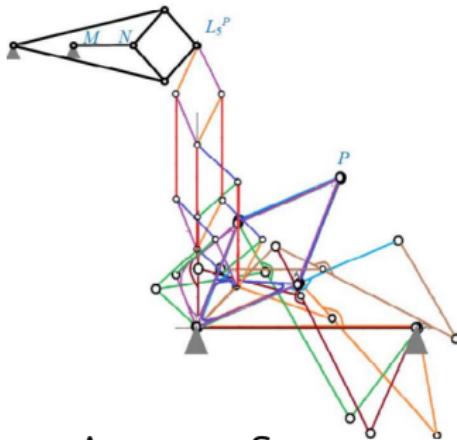
Kempe - Linkage Arithmetic

<https://demonstrations.wolfram.com/KempesAngleAdder/>

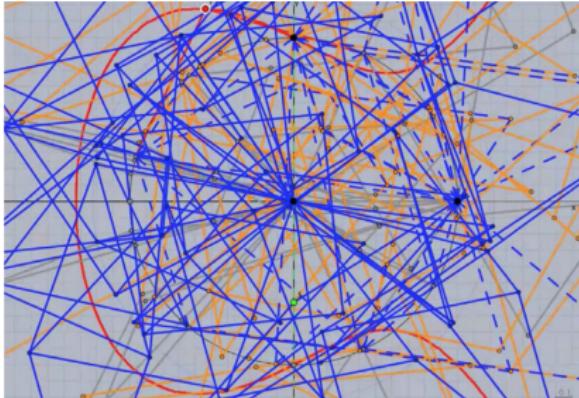
Adding Angles

$$\cos(\theta_1 + \theta_2), \theta_1, \theta_2 \in \mathbb{R}$$

Kempe - Combining Linkages



Anupam Saxena
 $(x - y)(x + y + 1/\sqrt{2}) = 0$



Alexander Kobel
 $x^3 - y^2 - x + 1 = 0$

Simplified Form

$$P(x, y) = \sum_{n,m \in \mathbb{Z}} c'_{nm} \cos(n\alpha + m\beta + \theta_{nm}) = 0$$



Kempe's Universality Theorem

Kempe's Universality Theorem (Statement made rigorous by Thurston & Kapovich-Millson)

Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a curve. Let $f : [a, b] \rightarrow \mathcal{C}$ be a map such that $f(t) = (p(t), q(t))$ where $p, q \in \mathbb{R}[x]$ are polynomials. Then there is a linkage \mathcal{L} with some pinned vertices, and a vertex v of \mathcal{L} so that v traces out $f([a, b])$ over all the linkage's configurations.



Kempe's Universality Theorem

Corollary - (Thurston) There is a linkage which signs your name.

Break up your name into k segments, each of which is the image of a smooth function $f_k : [a_k, b_k] \rightarrow \mathbb{R}^2$. Apply the Stone-Weierstrass Approximation Theorem to approximate your curves by polynomials. Apply the previous result and carefully combine linkages, or apply the result to Lagrange polynomial interpolation.



Kempe's Universality Theorem - Generalisations

- The proofs were rigorously completed in 2002 by Kapovich and Millson.
- (Kapovich-Millson) The result generalises to higher-dimensional algebraic sets, i.e. simultaneous solutions to $P_1(x_1, \dots, x_n) = 0, P_2(x_1, \dots, x_n) = 0, \dots, P_m(x_1, \dots, x_n) = 0$.
- (Kapovich-Millson) Let M be any smooth compact manifold. Then there is a linkage \mathcal{L} whose configuration space is diffeomorphic to a disjoint union of a number of copies of M .

Gröbner Basis - Polynomials

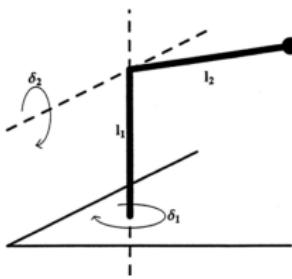


Bruno Buchberger
1942 - Now

■ Example: Systems of Polynomial Equations

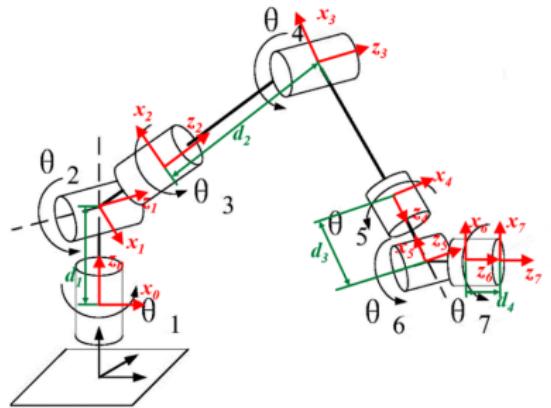
(This example is taken from (Buchberger, Kutzler 1986)).

Systems of multivariate polynomial equations are pervasive in all areas of engineering. For example, consider the following simple robot:



Robotics Example
Bruno Buchberger

Gröbner Basis - Polynomials in Kinematics



General Kinematics Problem
Xuanming Zhang et. al.

Simple Kinematics Example

Gröbner Basis - Polynomials in Kinematics

Kinematics As A Map

$$F : [0, 2\pi)^3 \rightarrow \mathbb{R}^2$$

$$(\theta_1, \theta_2, \theta_3) \mapsto (x, y)$$



Gröbner Basis - Polynomials in Kinematics

Kinematics Map

$$(x, y) = F(\theta_1, \theta_2, \theta_3)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \end{pmatrix}$$



Gröbner Basis - Polynomials in Kinematics

Kinematics Map

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \end{pmatrix}$$
$$= \begin{pmatrix} L_1 c_1 + L_2 c_{12} + L_3 c_{123} \\ L_1 s_1 + L_2 s_{12} + L_3 s_{123} \end{pmatrix}$$

Kinematics Polynomials

$$\begin{aligned} L_1 c_1 + L_2 c_{12} + L_3 c_{123} - x &= 0 \\ L_1 s_1 + L_2 s_{12} + L_3 s_{123} - y &= 0 \\ c_1^2 + s_1^2 - 1 &= 0 \\ c_{12}^2 + s_{12}^2 - 1 &= 0 \\ c_{123}^2 + s_{123}^2 - 1 &= 0 \end{aligned}$$



Gröbner Basis - Polynomials in Kinematics

Kinematics Polynomials

$$L_1 c_1 + L_2 c_{12} + L_3 c_{123} - x = 0$$

$$L_1 s_1 + L_2 s_{12} + L_3 s_{123} - y = 0$$

$$c_1^2 + s_1^2 - 1 = 0$$

$$c_{12}^2 + s_{12}^2 - 1 = 0$$

$$c_{123}^2 + s_{123}^2 - 1 = 0$$

Abstracted Polynomials

$$P_1(x_i, y_i) = L_1 x_1 + L_2 x_2 + L_3 x_3 - x = 0$$

$$P_2(x_i, y_i) = L_1 y_1 + L_2 y_2 + L_3 y_3 - y = 0$$

$$P_3(x_i, y_i) = x_1^2 + y_1^2 - 1 = 0$$

$$P_4(x_i, y_i) = x_2^2 + y_2^2 - 1 = 0$$

$$P_5(x_i, y_i) = x_3^2 + y_3^2 - 1 = 0$$



Gröbner Basis - Extending Gauss-Jordan

Linear Equations

$$\begin{array}{l} ax_1 + x_2 = 1 \\ x_2 + x_3 = 2 \\ bx_1 + x_2 + x_3 = 1 \end{array} \Leftrightarrow \begin{pmatrix} a & 1 & 0 \\ 0 & 1 & 1 \\ b & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Gauss-Jordan Elimination

$$R_3 \mapsto aR_3 - bR_1$$

Gröbner Basis - Extending Gauss-Jordan

Linear Equations

$$\begin{array}{l} ax_1 + x_2 = 1 \\ x_2 + x_3 = 2 \\ bx_1 + x_2 + x_3 = 1 \end{array} \Leftrightarrow \begin{pmatrix} a & 1 & 0 \\ 0 & 1 & 1 \\ b & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Gauss-Jordan Elimination

$$R_3 \mapsto aR_3 - bR_1$$

 \Leftrightarrow

$$R_3 \mapsto \frac{ab}{b}R_3 - \frac{ab}{a}R_1$$



Gröbner Basis - Triangularisation!

Abstracted Polynomials

$$P_1(x_i, y_i) = L_1x_1 + L_2x_2 + L_3x_3 - x = 0$$

$$P_2(x_i, y_i) = L_1y_1 + L_2y_2 + L_3y_3 - y = 0$$

$$P_3(x_i, y_i) = x_1^2 + y_1^2 - 1 = 0$$

$$P_4(x_i, y_i) = x_2^2 + y_2^2 - 1 = 0$$

$$P_5(x_i, y_i) = x_3^2 + y_3^2 - 1 = 0$$

Compute S-Polynomial

$$S(P_i, P_j) = \frac{\text{lcm}(\text{LT}(P_i), \text{LT}(P_j))}{\text{LT}(P_i)} P_i - \frac{\text{lcm}(\text{LT}(P_i), \text{LT}(P_j))}{\text{LT}(P_j)} P_j$$

Gröbner Basis - Triangularisation!

Abstracted Polynomials

$$P_1(x_i, y_i) = L_1x_1 + L_2x_2 + L_3x_3 - x = 0$$

$$P_2(x_i, y_i) = L_1y_1 + L_2y_2 + L_3y_3 - y = 0$$

$$P_3(x_i, y_i) = x_1^2 + y_1^2 - 1 = 0$$

$$P_4(x_i, y_i) = x_2^2 + y_2^2 - 1 = 0$$

$$P_5(x_i, y_i) = x_3^2 + y_3^2 - 1 = 0$$

S-Polynomial Example

$$S(P_1, P_2) = \frac{\text{lcm}(\text{LT}(P_1), \text{LT}(P_2))}{\text{LT}(P_2)} P_1 - \frac{\text{lcm}(\text{LT}(P_1), \text{LT}(P_2))}{\text{LT}(P_2)} P_2$$

$$= \cancel{\frac{L_3x_3y_3}{L_3x_3}} P_1 - \cancel{\frac{L_3x_3y_3}{L_3y_3}} P_2 = y_3(L_1x_1 + L_2x_2 + L_3\cancel{x_3} - x) - x_3(L_1y_1 + L_2y_2 + L_3\cancel{y_3} - y)$$



Gröbner Basis - Triangularisation!

```

1 import sympy as sp
2 x1,x2,x3,y1,y2,y3 = sp.Symbol("x1"), sp.Symbol("x2"), sp.Symbol("x3"), sp.Symbol("y1"), sp.Symbol("y2"), sp.Symbol("y3")
3 L1,L2,L3 = 1,1,1
4 x,y = sp.Symbol("x"), sp.Symbol("y")
5 F = [L1*x1+L2*x2+L3*x3 - x,
6      L1*y1+L2*y2+L3*y3 - y,
7      x1**2 + y1**2 - 1,
8      x2**2 + y2**2 - 1,
9      x3**2 + y3**2 - 1]
10
11 basis = sp.GroebnerBasis(F, x1, x2, x3, y1, y2, y3, order='lex')
12 print(basis)

```

Gröbner Basis - Triangularisation!

Triangular Form

$$-x + x_1 + x_2 + x_3 = 0$$

$$x_2^2 + y_2^2 - 1 = 0$$

$$x^2 - 2x\textcolor{violet}{x}_2 - 2x\textcolor{violet}{x}_3 + 2\textcolor{violet}{x}_2\textcolor{violet}{x}_3 + y^2 - 2y\textcolor{blue}{y}_2 - 2y\textcolor{blue}{y}_3 + 2\textcolor{blue}{y}_2\textcolor{blue}{y}_3 + 1 = 0$$

•
•
•

Gröbner Basis - Triangularisation!

Result

$$\cos(\theta_1) = x_1(x, y)$$

$$\cos(\theta_1 + \theta_2) = x_2(x, y)$$

$$\cos(\theta_1 + \theta_2 + \theta_3) = x_3(x, y)$$

$$(\theta_1, \theta_2, \theta_3) \in F^{-1}[(x, y)]$$



Lie Theory - Algebra to Geometry

General Form of a Line

$$\ell : ax + by = c$$

Point Inclusion

$$P = (x_0, y_0) \in \ell \Leftrightarrow ax_0 + by_0 = c$$

Lie Theory - Algebra to Geometry

General Form of a Line

$$\ell : ax + by + c = 0$$

Point Inclusion

$$P = (x_0, y_0) \in \ell \Leftrightarrow ax_0 + by_0 + c = 0$$

Modified Point Inclusion

$$\tilde{P} = (x_0, y_0, 1) \in \ell \Leftrightarrow (a, b, c) \cdot \tilde{P} = 0$$

Lie Theory - Algebra to Geometry

General Form of a Line

$$\ell : ax + by + c = 0$$

Modified Point Inclusion

$$\tilde{P} = (x_0, y_0, 1)^T \in \ell \Leftrightarrow (a, b, c) \cdot P^* = 0$$

Projective Points $P \in \mathbb{P}^2 = (\mathbb{R}^3 \setminus \{\mathbf{0}\}) / \sim$

$$[x : y : z]^T = [\lambda x : \lambda y : \lambda z]^T, \lambda \in \mathbb{R} \setminus \{0\}$$

Projective Point Inclusion

$$\tilde{P} = [x_0 : y_0 : 1]^T \in \ell \Leftrightarrow [a : b : c] \cdot \tilde{P} = 0$$

Lie Theory - Polynomials to Geometry

Duality of Lines and Points

$$\ell = [a : b : c]$$

$$P = \ell^* = [a : b : c]^T = \left[\frac{a}{c} : \frac{b}{c} : 1 \right]^T \text{ if } c \neq 0$$



Lie Theory - Algebra to Geometry

Intersection of Two Lines

$$\ell_1 = [a : b : c]$$

$$\ell_2 = [a' : b' : c']$$

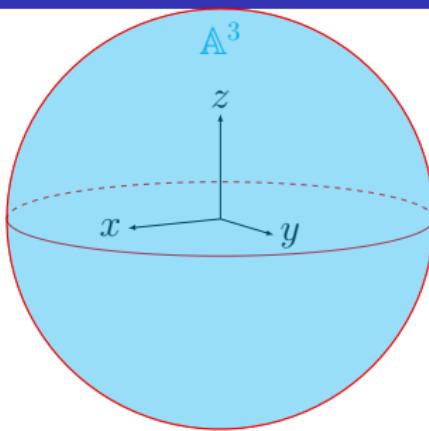
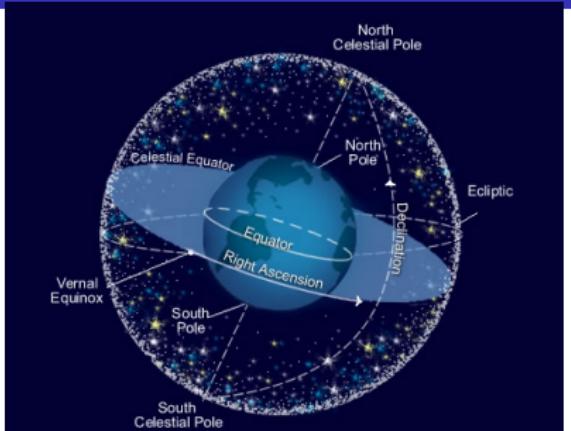
$$P = (\ell_1^* \times \ell_2^*)^*$$

Lie Theory - Algebra to Geometry

Points at Infinity

$$\mathbb{P}^2 = \underbrace{\mathbb{A}^2}_{[x:y:1]} \sqcup \underbrace{\mathbb{P}^1}_{[x:y:0]}$$

Lie Theory - Algebra to Geometry



Points at Infinity

$$\mathbb{P}^3 = \underbrace{\mathbb{A}^3}_{[x:y:z:1]} \sqcup \underbrace{\mathbb{P}^2}_{[x:y:z:0]}$$

$$\lim_{x \rightarrow \infty} [x : y : z : 1] = \lim_{x \rightarrow \infty} \left[1 : \frac{y}{x} : \frac{z}{x} : \frac{1}{x} \right] = [1 : 0 : 0 : 0]$$



Lie Theory - Special Euclidean Group

Special Orthogonal Group

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid R^T R = \mathbb{I}, \det R = 1 \right\}$$

$$SO(3) \curvearrowright \mathbb{E}^3$$

Special Euclidean Group

$$SE(3) = \left\{ \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \in \text{Mat}_4(\mathbb{R}) \mid R \in SO(3), t \in \mathbb{R}^3 \right\}$$

$$SE(3) \curvearrowright \mathbb{P}^3 = \mathbb{E}^3 \sqcup \mathbb{P}^2$$

Lie Theory - Special Euclidean Group

Special Euclidean Group

$$\text{SE}(3) = \left\{ \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \in \text{Mat}_4(\mathbb{R}) \mid R \in \text{SO}(3), t \in \mathbb{R}^3 \right\}$$

$$\text{SE}(3) \curvearrowright \mathbb{P}^3 = \mathbb{E}^3 \sqcup \mathbb{P}^2$$



Lie Theory - Special Euclidean Group Lie Algebra

Special Euclidean Group

$$\text{SE}(3) = \left\{ \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \in \text{Mat}_4(\mathbb{R}) \mid R \in \text{SO}(3), t \in \mathbb{R}^3 \right\}$$

$$\text{SE}(3) \curvearrowright \mathbb{P}^3 = \mathbb{E}^3 \sqcup \mathbb{P}^2$$

$$T_g \text{SE}(3) = \text{span} \left\{ \frac{\partial}{\partial \alpha}(g), \frac{\partial}{\partial \beta}(g), \frac{\partial}{\partial \gamma}(g), \frac{\partial}{\partial x}(g), \frac{\partial}{\partial y}(g), \frac{\partial}{\partial z}(g) \right\}$$

Lie Algebra of Special Euclidean Group

$$\mathfrak{se}(3) := T_{\mathbb{I}} \text{SE}(3)$$

Lie Theory - Special Euclidean Group Lie Algebra

Lie Algebra of Special Euclidean Group

$$\mathfrak{se}(3) := T_{\mathbb{I}} \text{SE}(3)$$

Velocity Vectors

$$V \in \mathfrak{se}(3)$$

$$V = c_1 \frac{\partial}{\partial \alpha}(\mathbb{I}) + c_2 \frac{\partial}{\partial \beta}(\mathbb{I}) + c_3 \frac{\partial}{\partial \gamma}(\mathbb{I}) + c_4 \frac{\partial}{\partial x}(\mathbb{I}) + c_5 \frac{\partial}{\partial y}(\mathbb{I}) + c_6 \frac{\partial}{\partial z}(\mathbb{I})$$

Lie Theory - Special Euclidean Group Lie Algebra

Velocity Vectors

$$V \in \mathfrak{se}(3)$$

$$V = c_1 \frac{\partial}{\partial \alpha} + c_2 \frac{\partial}{\partial \beta} + c_3 \frac{\partial}{\partial \gamma} + c_4 \frac{\partial}{\partial x} + c_5 \frac{\partial}{\partial y} + c_6 \frac{\partial}{\partial z}$$

Exponential Map

$$\exp : \mathfrak{se}(3) \rightarrow SE(3)$$

$$\exp(V) = \gamma_V(1)$$

Lie Theory - Lie Algebras for Motion Planning

Exponential Map

$$\exp : \mathfrak{se}(3) \rightarrow \text{SE}(3)$$

$\exp(V) = \gamma_V(1)$, γ a geodesic with initial velocity V at \mathbb{I}



Lie Theory - Lie Algebras for Motion Planning

<https://www.youtube.com/watch?v=6Wmw4IUHIX8>

Others

- Category Theory for Robots (<http://ames.caltech.edu/A%20categorical%20theory.pdf>)
- Path Planning with Homotopy (<https://ieeexplore.ieee.org/document/10598326>)
- Optimisation

Summary

- Kempe was inspired by robotics to pursue questions about curve-producing linkages. This led to more mathematics as his work was carried forward by Thurston and completed by Kapovich and Millson.
- Buchburger was solving problems in computational algebra with potential applications for robotics in mind. His work later became an important modern tool in robotics and mathematics.
- Projective geometry was invented to simplify proofs in mathematics. With improved numerical stability, it became an essential part of robotics. Lie Theory then generalised derivatives to manifolds, commonly present within a robot's state space.