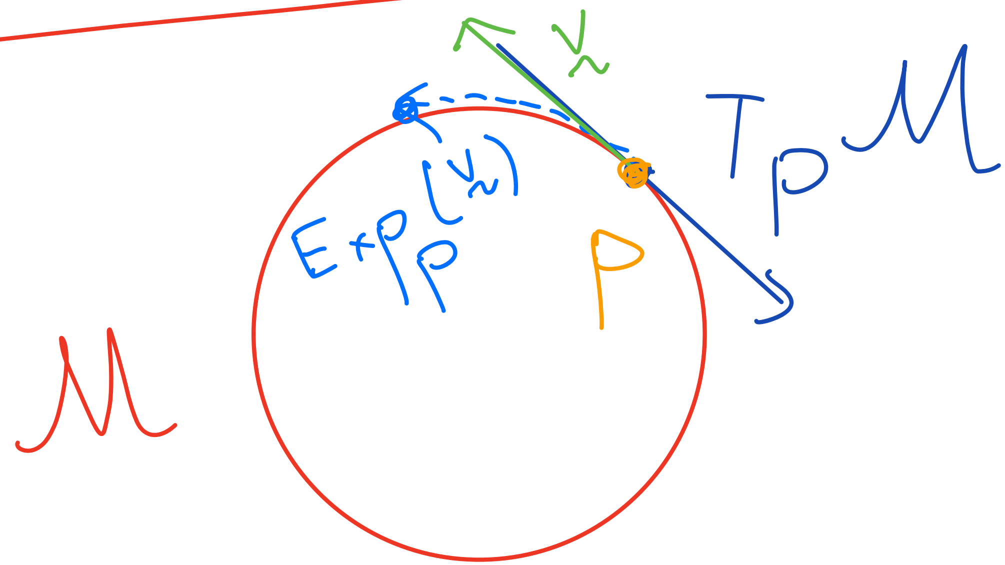
Lie Theory for Control & Estimation



By Sepehr Saryazdi

Motivation:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; \begin{matrix} R^T R = \mathbb{I}, \\ \det R = 1 \end{matrix} \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \; \| R\underset{\sim}{v} - \underset{\sim}{u} \|^2 \quad ?$$

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Naive method:

$$f(r_{11}, \ldots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \underset{\sim}{v} - \underset{\sim}{u} \right\|^2$$

Solve $\dfrac{\partial f}{\partial r_{ij}} = 0$. This is hard!

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How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \; \| R \underset{\sim}{v} - \underset{\sim}{u} \|^2 \;?$$

Improved method w/ grad. descent:

$$f(r_{11}, \ldots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \underset{\sim}{v} - \underset{\sim}{u} \right\|^2$$

$$\underset{\sim}{r} := (r_{11}, \ldots, r_{33}), \quad \underset{\sim}{r}_{n+1} = \underset{\sim}{r}_n - \gamma \nabla f(\underset{\sim}{r}_n)$$

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$$\underset{\sim}{r}_{n+1} = \underset{\sim}{r}_n - \gamma \nabla f(\underset{\sim}{r}_n) =: (r'_{11}, \dots, r'_{33}), \quad R' \notin SO(3)!$$

Motivation:

$$SO(3) = \left\{ R \in \mathrm{Mat}_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

How to compute

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Want: $\left(\underset{\sim}{r}_n \right)_{n \in \mathbb{N}}$ s.t. $\underset{\sim}{r}_n \in SO(3)$ & $r_n \longrightarrow \underset{r}{\operatorname{argmin}} f$

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Idea: Define $\oplus$ s.t.

$$R \oplus dR \in SO(3)$$

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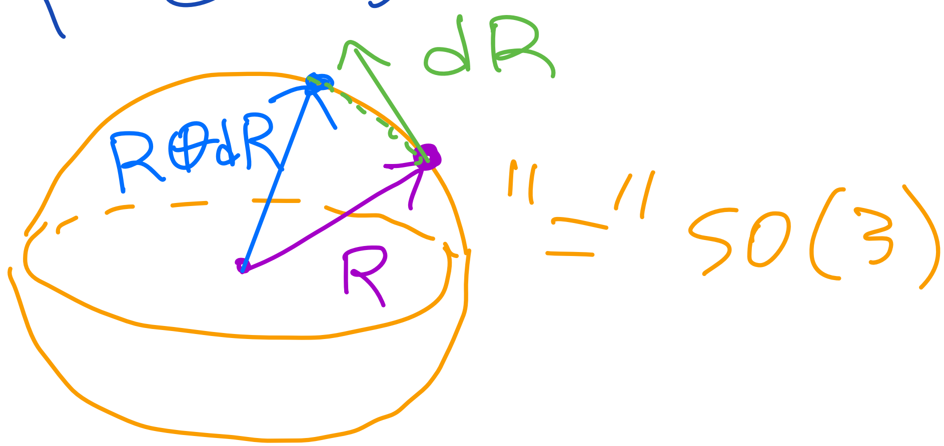

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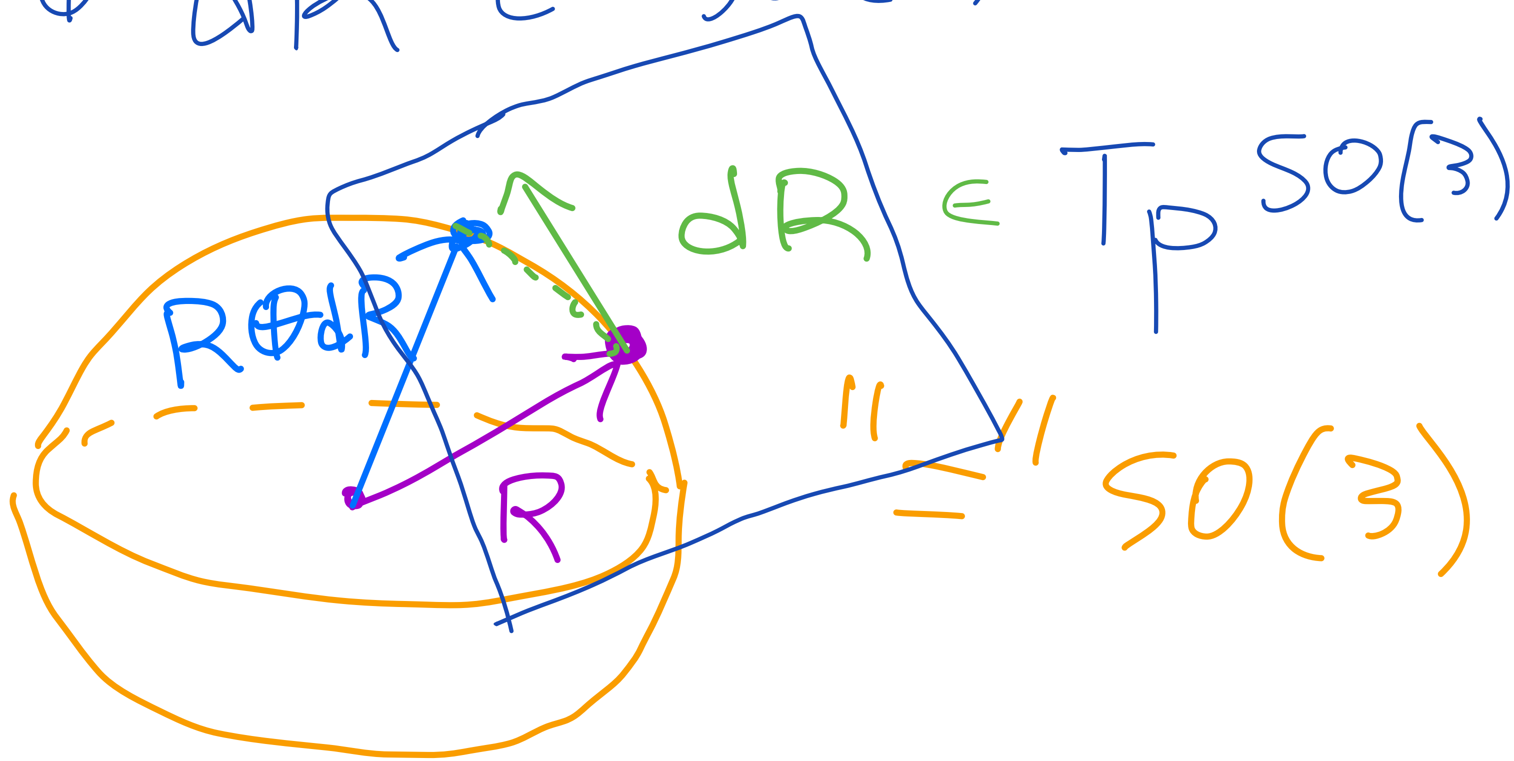

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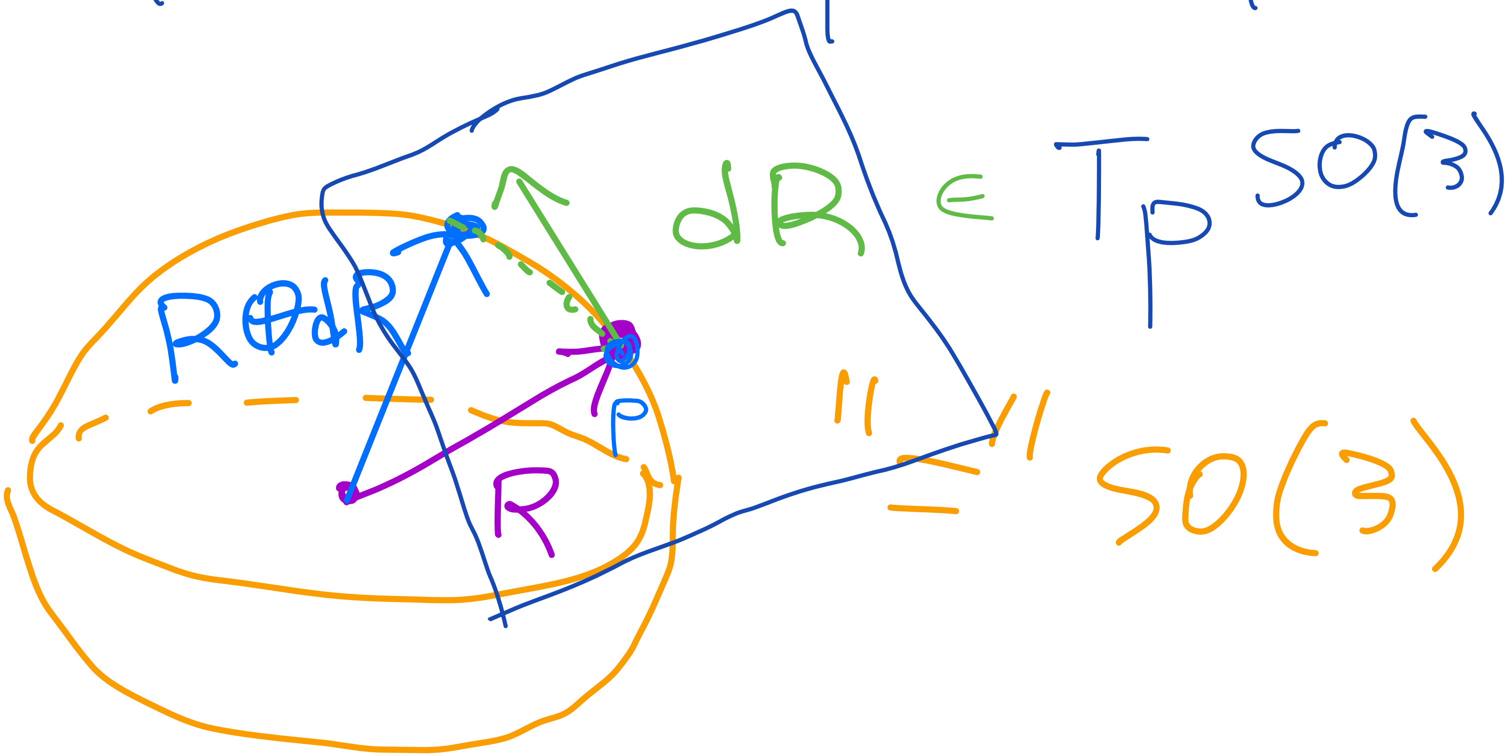
Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

Q: How to compute $T_p SO(3)$?



Think of "velocity" vectors at a point $p \in SO(3)$.

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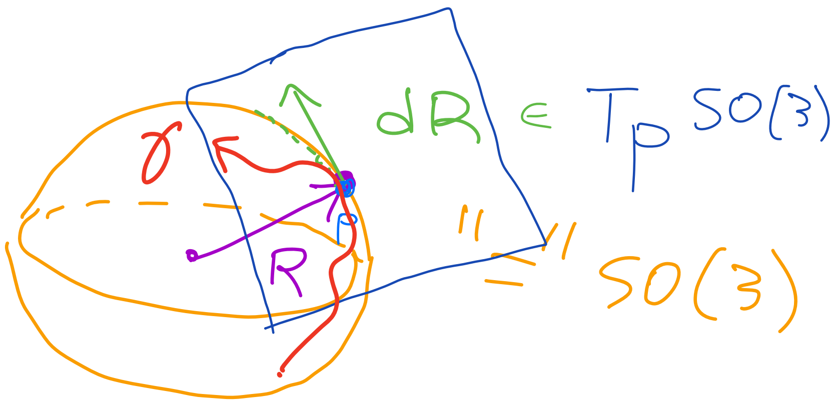

Think of "velocity" vectors of a curve at a point $p \in SO(3)$.

$$\gamma : (-\epsilon, \epsilon) \longrightarrow SO(3)$$

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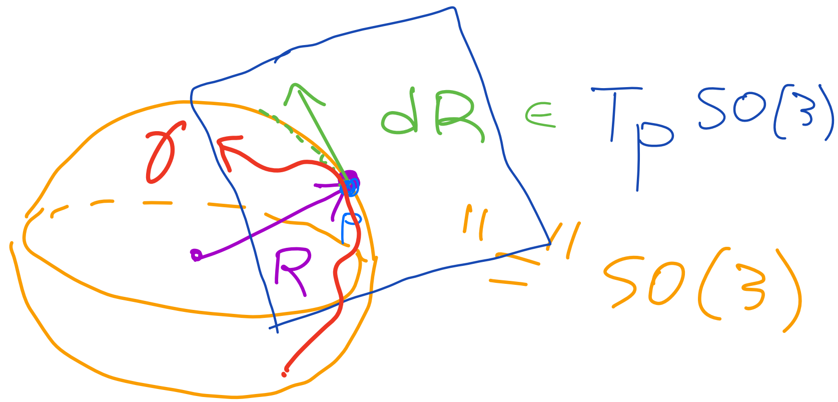

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$$p = \gamma(0) = \mathbb{I}$$

Computing $T_p SO(3)$:

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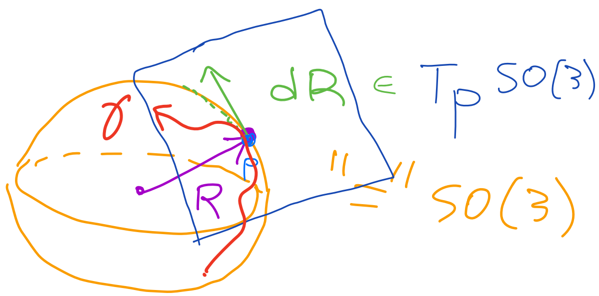

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$$\gamma'(t) = \begin{bmatrix} -\sin t & -\cos t & 0 \\ \cos t & -\sin t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \gamma'(0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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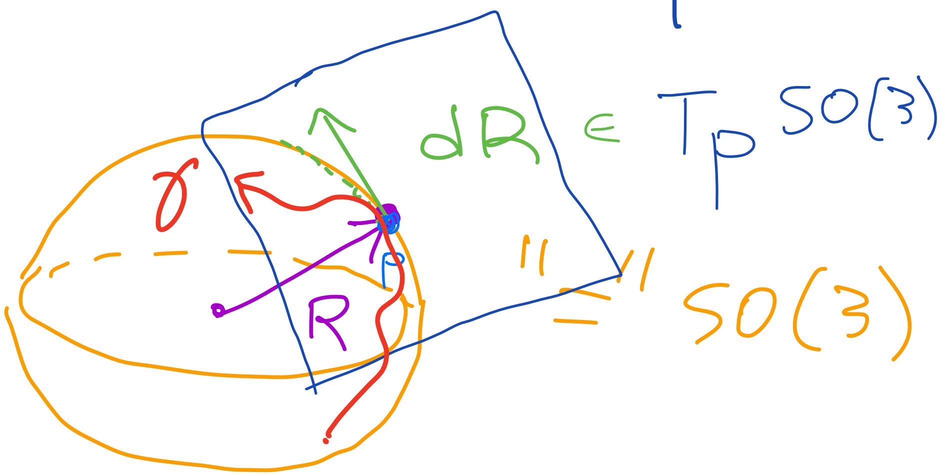

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Computing $T_p SO(3)$:

$$SO(3) = \{ R \in Mat_3(\mathbb{R}) \mid R^T R = \mathbb{I}, \ \det R = 1 \}$$

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$$\gamma_z(t) = R_z(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad p = \gamma(0) = \mathbb{I}, \quad \gamma_z'(0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\gamma_x(t) = R_x(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}, \quad \gamma_x'(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Computing $T_p SO(3)$:

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what about any γ?

$$\gamma(t) = R_x(\theta_1(t)) R_y(\theta_2(t)) R_z(\theta_3(t))$$

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$$\gamma(t) = R_x(\theta_1(t))\, R_y(\theta_2(t))\, R_z(\theta_3(t))$$

$$\gamma(t) = \gamma_x(\theta_1(t))\, \gamma_y(\theta_2(t))\, \gamma_z(\theta_3(t))$$

$$\begin{aligned}
\gamma'(t) &= \gamma_x'(\theta_1(t)) \cdot \gamma_y(\theta_2(t)) \cdot \gamma_z(\theta_3(t)) \cdot \theta_1'(t) \\
&+ \gamma_x(\theta_1(t)) \cdot \gamma_y'(\theta_2(t)) \cdot \gamma_z(\theta_3(t)) \cdot \theta_2'(t) \\
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\end{aligned}$$

Computing $T_p\, SO(3)$:

$$SO(3) = \left\{ R \in \mathrm{Mat}_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

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what about any γ passing through $p = \mathbb{I}$?

$$\gamma(t) = \gamma_x(\theta_1(t))\, \gamma_y(\theta_2(t))\, \gamma_z(\theta_3(t))$$

$$\begin{aligned} \gamma'(0) = {} & \gamma_x'(0) \cdot \cancelto{\mathbb{I}}{\gamma_y(0)} \cdot \cancelto{\mathbb{I}}{\gamma_z(0)} \cdot \theta_1'(0) \\ & + \cancelto{\mathbb{I}}{\gamma_x(0)} \cdot \gamma_y'(0) \cdot \cancelto{\mathbb{I}}{\gamma_z(0)} \cdot \theta_2'(0) \\ & + \cancelto{\mathbb{I}}{\gamma_x(0)} \cdot \cancelto{\mathbb{I}}{\gamma_y(0)} \cdot \gamma_z'(0) \cdot \theta_3'(0) \end{aligned}$$

$$\gamma'(0) = \gamma_x'(0)\, \theta_1'(0) + \gamma_y'(0)\, \theta_2'(0) + \gamma_z'(0)\, \theta_3'(0)$$

Computing $T_p SO(3)$:

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$$\gamma'(0) = \gamma_x'(0)\,\theta_1'(0) + \gamma_y'(0)\,\theta_2'(0) + \gamma_z'(0)\,\theta_3'(0)$$

$$= \gamma_x'(0)\,\omega_x + \gamma_y'(0)\,\omega_y + \gamma_z'(0)\,\omega_z$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_z$$

$$= \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \,\middle|\, \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

Q: How to compute $T_p SO(3)$?

As a vector space:

$$\gamma'(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_z$$

$$\mathfrak{so}(3) := T_{\mathbb{I}} SO(3) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$= \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial \alpha}(\mathbb{I}), \frac{\partial}{\partial \beta}(\mathbb{I}), \frac{\partial}{\partial \gamma}(\mathbb{I}) \right\}$$

$$= \operatorname{span}_{\mathbb{R}} \left\{ E_1(\mathbb{I}), E_2(\mathbb{I}), E_3(\mathbb{I}) \right\}$$

Lie Algebra of $SO(3)$

Computing $T_p SO(3)$:

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what about any γ passing through $p = \mathbb{I}$?

Representing vectors

$$\underset{\sim}{v} = (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$$

$$\underset{\sim}{v}^\wedge := v_1 E_1 + v_2 E_2 + v_3 E_3 = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_z}_{\in T_{\mathbb{I}} SO(3)}$$

$$(\underset{\sim}{v}^\wedge)^\vee = \underset{\sim}{v}$$

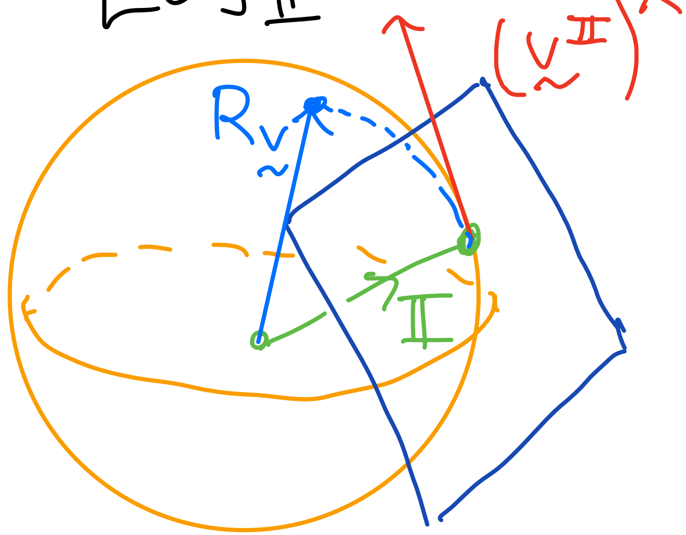
Computing $T_p\, SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

$$SO(3) \xrightarrow{\left.\frac{d}{dt}\right|_{\mathbb{I}}} T_{\mathbb{I}}\, SO(3)$$

$$SO(3) \underset{\mathrm{Log}_{\mathbb{I}}}{\overset{\mathrm{Exp}_{\mathbb{I}}}{\rightleftarrows}} T_{\mathbb{I}}\, SO(3)$$

$$\begin{array}{ccc} \Psi & & \Psi \\ R_{\underset{\sim}{v}} & & (\underset{\sim}{v}^{\mathbb{I}})^{\wedge} \end{array}$$



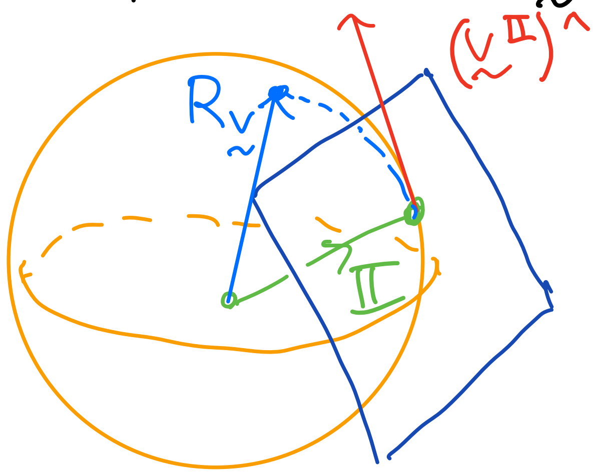
Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \,\middle|\, \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

$$SO(3) \xleftarrow{\;Exp_{\mathbb{I}}\;} T_{\mathbb{I}} SO(3)$$

$$\underset{R_{\underset{\sim}{v}}}{\cup} \qquad \underset{(\underset{\sim}{v}^{\mathbb{I}})^\wedge}{\cup}$$

$Exp_{\mathbb{I}}(\underset{\sim}{v}^{\mathbb{I}}) := \gamma_{\underset{\sim}{v}}(1)$, $\gamma_{\underset{\sim}{v}}$ a geodesic w/ const. velocity $\underset{\sim}{v}$



$$\mathbb{I} \oplus^{\mathbb{I}} \underset{\sim}{v} := Exp_{\mathbb{I}}(\underset{\sim}{v}^{\mathbb{I}}) = R_{\underset{\sim}{v}}$$

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

$$\begin{array}{ccc} SO(3) & \xleftarrow{\ Exp_{\mathbb{I}}\ } & T_{\mathbb{I}} SO(3) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ R_{\underset{\sim}{v}} & & \left(\underset{\sim}{v}^{\mathbb{I}}\right)^{\wedge} \end{array}$$

$Exp_{\mathbb{I}}(\underset{\sim}{v}) = \gamma_{\underset{\sim}{v}}(1)$ $\quad \gamma_{\underset{\sim}{v}}$ a geodesic w/ const. velocity $\underset{\sim}{v}$

Computing $T_p SO(3)$:

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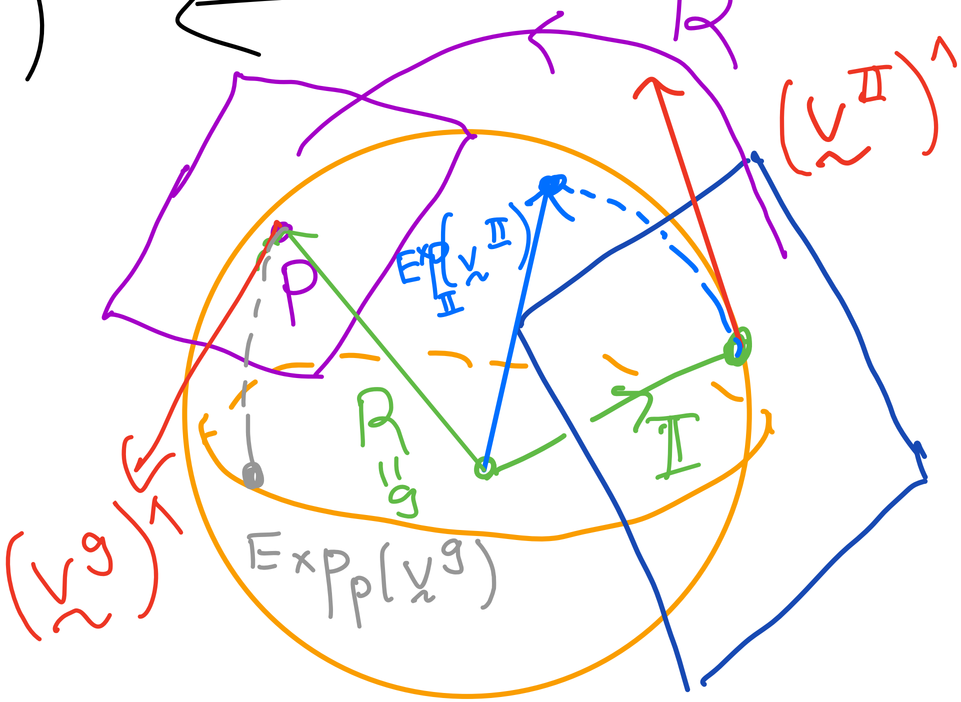
$$SO(3) \xleftarrow{\;Exp_p\;} T_p SO(3)$$

$p = g$



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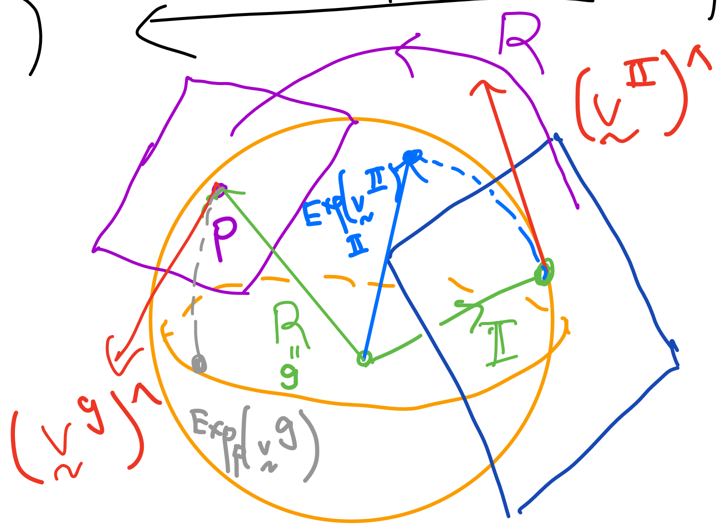


$p = g$

$$R \oplus \underset{\sim}{v}^R := R \cdot Exp_{\mathbb{I}}(\underset{\sim}{v}^{\mathbb{I}}) \overset{(*)}{=} Exp_R(\underset{\sim}{v}^R)$$

why? we'll return to this later...

Properties of $\mathrm{Exp}_p : T_p SO(3) \to SO(3)$

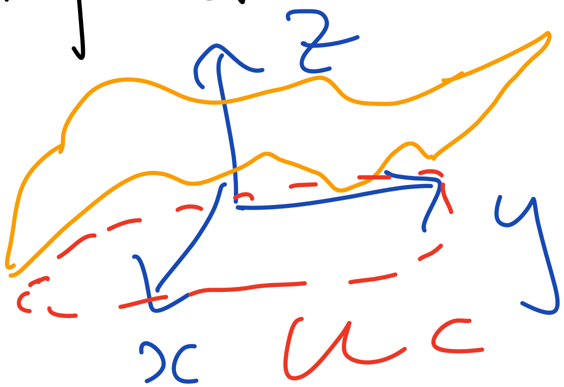
$$\mathrm{Exp}_p(\underset{\sim}{v}^p) := \underbrace{\gamma_{\underset{\sim}{v}^p}(1)}$$

γ a geodesic

How do we actually compute $\mathrm{Exp}_p(\underset{\sim}{v}^p)$ for any manifold $\mathcal{M}$?

Properties of $\mathrm{Exp}_p : T_p SO(3) \to so(3)$

Every manifold comes with an atlas: (φ_i, U_i)



$(x,y) \longmapsto (x,y,z(x,y))$

$U \subset \mathbb{R}^2$ open

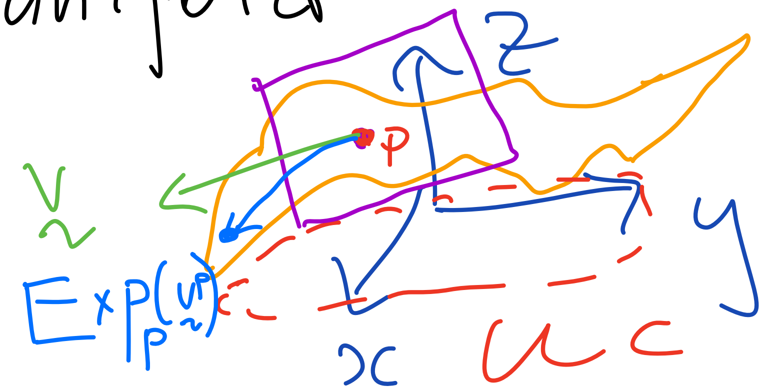
$SO(3)$ as a manifold has a chart given by

$$\varphi : (0,2\pi)^3 \longrightarrow SO(3)$$

$$\varphi(\alpha,\beta,\gamma) = R_x(\alpha)\, R_y(\beta)\, R_z(\gamma)$$

Properties of $\mathrm{Exp}_p : T_p SO(3) \to SO(3)$

Every manifold M comes with an atlas: (φ_i, U_i)



$U \subset \mathbb{R}^2$ open

$(x,y) \longmapsto (x,y,z(x,y))$

If φ is given, then $\mathrm{Exp}_p(\underset{\sim}{v}^p)$ can be computed using the geodesic equation:

def^n of Γ^k_{ij}

$$p = \varphi(u^1(0), \dots, u^n(0))$$

$$\underset{\sim}{v} = (v^1, \dots, v^n)$$

initial condition of ODE

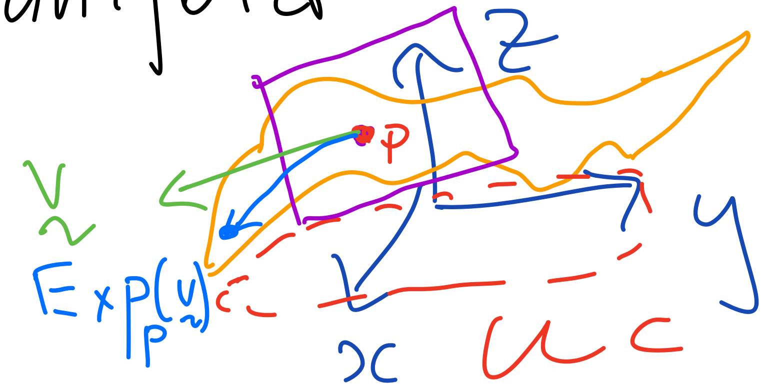
$$\overbrace{\frac{d^2 u^k}{dt^2} + \sum_{ij} \underbrace{\Gamma^k_{ij}}_{\text{Christoffel Symbols}} \frac{du^i}{dt}\frac{du^j}{dt} = 0}^{\text{ODE}},$$

$$\varphi_{u^i u^j} = \sum_k \Gamma^k_{ij} E_k,$$

$$T_p M = \mathrm{span}_{\mathbb{R}}\{E_1, \dots, E_n\}$$

Properties of $\mathrm{Exp}_p : T_p SO(3) \to \mathfrak{so}(3)$

Every manifold M comes with an atlas: (φ_i, U_i)



$\mathbb{R}^2$ open

$(x, y) \longmapsto (x, y, z(x,y))$

Properties of $\mathrm{Exp}_p : T_p SO(3) \to so(3)$

How to compute $\mathrm{Exp}_{\mathbb{I}}((\underset{\sim}{v}^{\mathbb{I}})^\wedge)$ in an easier way?

$$\underset{\sim}{v}^{\mathbb{I}} = (\omega_x, \omega_y, \omega_z)$$

$$(\underset{\sim}{v}^{\mathbb{I}})^\wedge = \omega_x E_1 + \omega_y E_2 + \omega_z E_3 \in so(3)$$

want: $R(t) \approx \mathbb{I} + \frac{dR(0)}{dt} t$ (linearising $SO(3)$ at $\mathbb{I}$)

$$= \mathbb{I} + (v^{\mathbb{I}})^\wedge t$$

The ODE $\dot{R} = R\,(v^{\mathbb{I}})^\wedge$ w/ $R(0) = \mathbb{I}$ has unique solution given by:

$$\exp((\underset{\sim}{v}^{\mathbb{I}})^\wedge) = \sum_{k=0}^{\infty} \frac{((\underset{\sim}{v}^{\mathbb{I}})^\wedge)^k}{k!} \in SO(3)$$

(Regular exponential)

for small $\|\underset{\sim}{v}^{\mathbb{I}}\|$

Properties of $\mathrm{Exp}_p : T_p SO(3) \to so(3)$

$$\exp((\underset{\sim}{v}^{\mathrm{II}})^{\wedge}) = \sum_{k=0}^{\infty} \frac{((\underset{\sim}{v}^{\mathrm{II}})^{\wedge})^{k}}{k!} \quad \text{(easy to calculate)}$$

Equivalence of exp & $\mathrm{Exp}_{\mathrm{II}}$

If G is a Lie Group w/ bi-invariant Riemannian metric, then

$$\exp((\underset{\sim}{v}^{\mathrm{II}})^{\wedge}) = \mathrm{Exp}_{\mathrm{II}}(\underset{\sim}{v}^{\mathrm{II}})$$

Milnor, 1976

A Lie Group G admits a bi-invariant metric iff $G \underset{\text{group}}{\cong} K \times H$ where K is a compact group & H abelian.

Properties of $\mathrm{Exp}_p : T_p SO(3) \to SO(3)$

Compactness

$SO(3)$ is compact as:

$$R^T R = \mathbb{I} \quad \& \quad \det R = 1$$

$\Downarrow$

R has bounded matrix norm, so $SO(3)$ is bounded.

$\underset{\sim}{r} := (r_{11}, \dots, r_{33})$ satisfies polynomial equations $\Rightarrow SO(3)$ is closed.

so $SO(3)$ is closed & bounded $\Rightarrow$ compact.

Properties of $\text{Exp}_p : T_p SO(3) \to SO(3)$

For Lie Groups G w/ bi-invariant Riemannian metrics (e.g. $SO(3)$),

the geodesic equation (Michael Greis, Notes on Riemannian Geom of Lie Grps) has the solution

$$\gamma_{\underset{\sim}{v}}(t) = \underbrace{\text{Exp}_g(t\underset{\sim}{v}^g)}_{\text{geodesics starting at } g} = g \circ \underbrace{\text{Exp}_e(t\underset{\sim}{v}^{I})}_{\text{geodesics starting at } I}$$

$$g \oplus g_{\underset{\sim}{v}} := g \circ \text{Exp}_e(\underset{\sim}{v}^{I}) \quad \text{(before)}$$

$$= \text{Exp}_g(\underset{\sim}{v}^g)$$

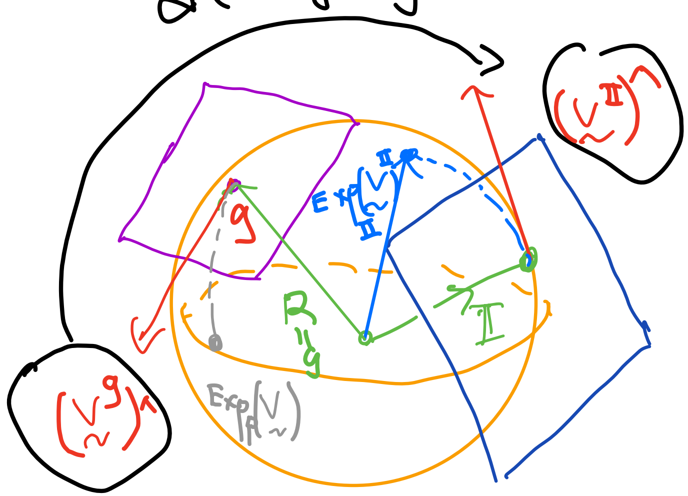
Properties of $\mathrm{Exp}_p : T_p SO(3) \to SO(3)$

$$\gamma_{\underset{\sim}{v}}(t) = \underbrace{\mathrm{Exp}_g(t\underset{\sim}{v}^g)}_{\text{geodesics starting at } g} = g \circ \underbrace{\mathrm{Exp}_e(t\, \underset{\sim}{v}^{\mathbb{I}})}_{\text{geodesics starting at } \mathbb{I}}$$

How is $\underset{\sim}{v}^{\mathbb{I}}$ related to $\underset{\sim}{v}^g$?

As elements $(\underset{\sim}{v}^{\mathbb{I}})^\wedge \in T_{\mathbb{I}} SO(3)$, $(\underset{\sim}{v}^g)^\wedge \in T_g SO(3)$, they are related by the differential of $L_{g^{-1}} : x \mapsto g^{-1}x \;\; \forall x \in G$, at g

$d(L_{g^{-1}})_g$



i.e. $(\underset{\sim}{v}^{\mathbb{I}})^\wedge = d(L_{g^{-1}})_g(\underset{\sim}{v}^g)^\wedge$

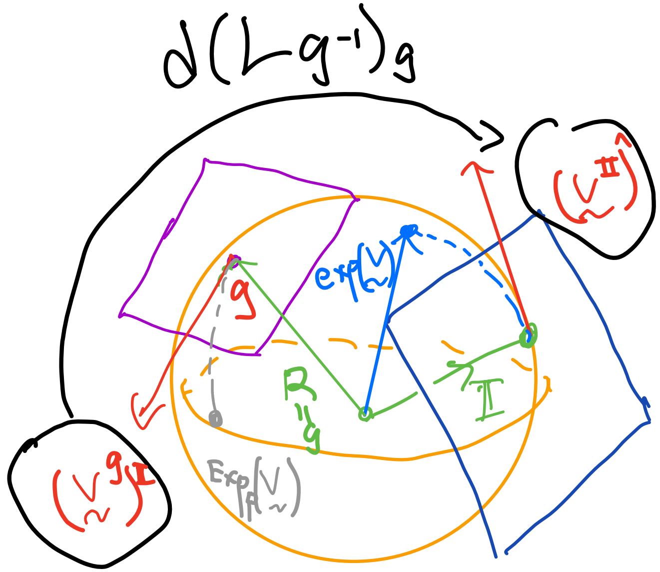
For $SO(3)$, $d(L_{g^{-1}}) = R^{-1}$.

Properties of $\mathrm{Exp}_p : T_p SO(3) \to SO(3)$

$$\gamma_{\underset{\sim}{v}}(t) = \underbrace{\mathrm{Exp}_g(t\underset{\sim}{v}^g)}_{\text{geodesics starting at } g} = g \circ \underbrace{\mathrm{Exp}_e(t\underset{\sim}{v}^{\mathbb{I}})}_{\text{geodesics starting at } \mathbb{I}}$$

How is $\underset{\sim}{v}^{\mathbb{I}}$ related to v^g?

As $\underset{\sim}{v}^{\mathbb{I}}, \underset{\sim}{v}^g \in \mathbb{R}^3$, they coincide for bi-invariant Lie Groups.



i.e.

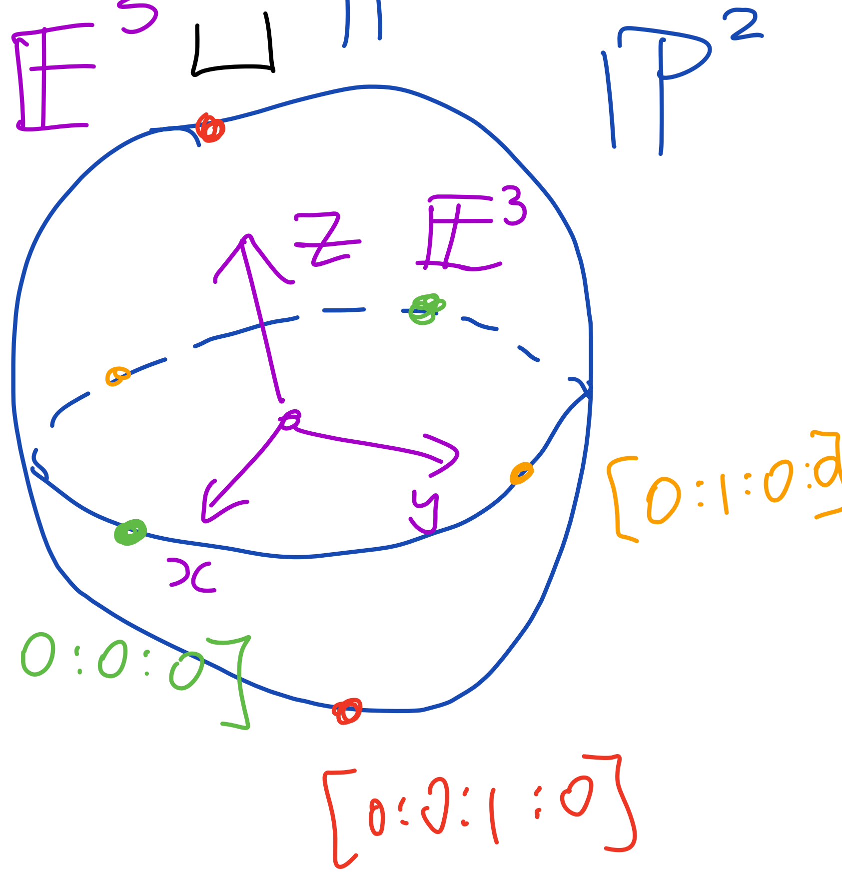
$$\underset{\sim}{v}^g = (\omega_x, \omega_y, \omega_z) = \underset{\sim}{v}^{\mathbb{I}}$$

Other Examples: SE(3)

$$SE(3) = \left\{ \begin{pmatrix} R & \underset{\sim}{t} \\ 0 & 1 \end{pmatrix} \middle| \begin{array}{l} R \in SO(3), \\ t \in \mathbb{R}^3 \end{array} \right\}$$

$$SE(3) \curvearrowright \mathbb{P}^3 = \mathbb{E}^3 \sqcup \mathbb{P}^2$$

$$\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \cdot [x:y:z:1] = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$



$$\lim_{x\to\infty} [x:y_0:z_0:1] = \lim_{x\to\infty} \left[1 : \frac{y_0}{x} : \frac{z_0}{x} : \frac{1}{x}\right] = [1:0:0:0]$$

Other Examples: $SE(3)$

$$SE(3) = \left\{ \begin{pmatrix} R & \underset{\sim}{t} \\ 0 & 1 \end{pmatrix} \,\middle|\, \begin{array}{l} R \in SO(3), \\ t \in \mathbb{R}^3 \end{array} \right\}$$

$$SE(3) \curvearrowright \mathbb{P}^3 = \mathbb{E}^3 \sqcup \mathbb{P}^2$$

Other Examples: SE(3)

$$SE(3) \cong \underbrace{SO(3) \ltimes \mathbb{R}^3}_{\text{semi-direct product}} \neq \underbrace{SO(3) \times \mathbb{R}^3}_{\text{direct product}}$$

& $\mathrm{Exp}_p \neq \exp$.

Basics of Classical Lie Groups, Chp 14

$\exp: \mathfrak{se}(3) \longrightarrow SE(3)$
is well-defined & surjective.

so we can still use exp as a way to "linearise" SE(3), but lose geometric meaning.

Other Examples: $SE(3)$

$$\mathfrak{se}(3)$$
$$\shortparallel$$
$$T_{\mathbb{I}}SE(3) = \operatorname{span}_{\mathbb{R}}\left\{ \frac{d}{dt}\Big|_{t=0}\begin{pmatrix} R_x(t) & 0 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0}\begin{pmatrix} R_y(t) & 0 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0}\begin{pmatrix} R_z(t) & 0 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0}\begin{pmatrix} \mathbb{I} & t\underset{\sim}{e}_1 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0}\begin{pmatrix} \mathbb{I} & t\underset{\sim}{e}_2 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0}\begin{pmatrix} I & t\underset{\sim}{e}_3 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \operatorname{span}\left\{ \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}, \frac{\partial}{\partial\gamma}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$$

$$= \operatorname{span}_{\mathbb{R}}\left\{ \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ -1&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix} \right\}$$

Other Examples: SE(3)

A twist ξ is given as:

$$\xi := \omega_x \frac{\partial}{\partial \alpha} + \omega_y \frac{\partial}{\partial \beta} + \omega_z \frac{\partial}{\partial \gamma} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

$$= \begin{pmatrix} 0 & -\omega_z & \omega_y & v_x \\ \omega_z & 0 & -\omega_x & v_y \\ -\omega_y & \omega_x & 0 & v_z \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Properties of Lie Algebras:

Matrix group multiplication can be "reproduced" with Lie Algebra multiplication:

Baker-Campbell-Hausdorff:

$$x \cdot y \longleftrightarrow X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\left[X, [X, Y]\right] + \frac{1}{12}\left[Y, [Y, X]\right] + \dots$$

w/ $X = \log x$, $Y = \log y$, $[X, Y] := XY - YX$

Properties of Lie Algebras:

Baker-Campbell-Hausdorff:

$$x \cdot y \longleftrightarrow X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \dots$$

w/ $X = \log x$, $Y = \log y$, $[X, Y] := XY - YX$

Corollary

if $\exp : \mathfrak{g} \to G$ is surjective, we can "globally" linearise G by working purely with its Lie Algebra $\mathfrak{g}$.

Changes of Reference Frames

Q: How can we compute a change of reference frame?

For a pose ${}^{B}T_{A} \in SE(3)$, we can just do

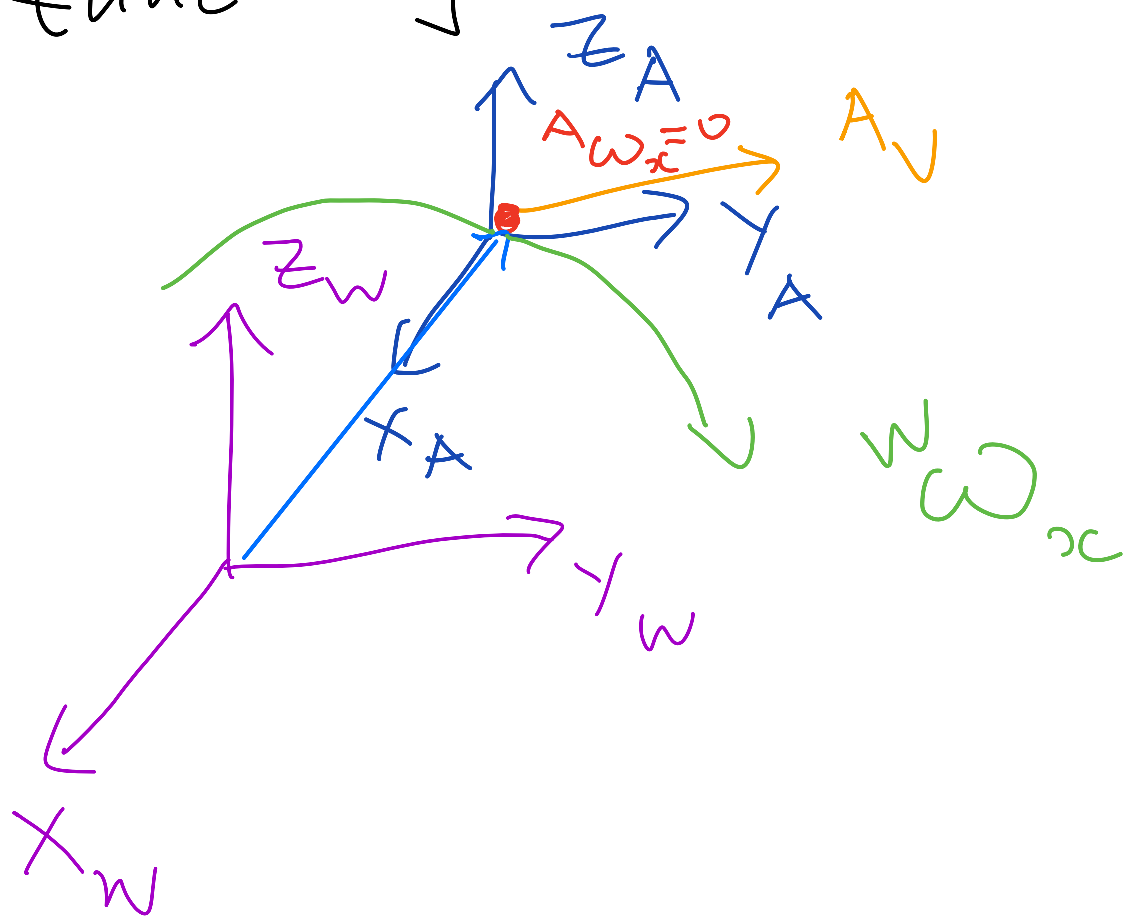
$${}^{B}p = {}^{B}T_{A} \, {}^{A}p$$

What about twists ξ?

$${}^{B}\xi = {}^{B}?_{A} \, {}^{A}\xi$$

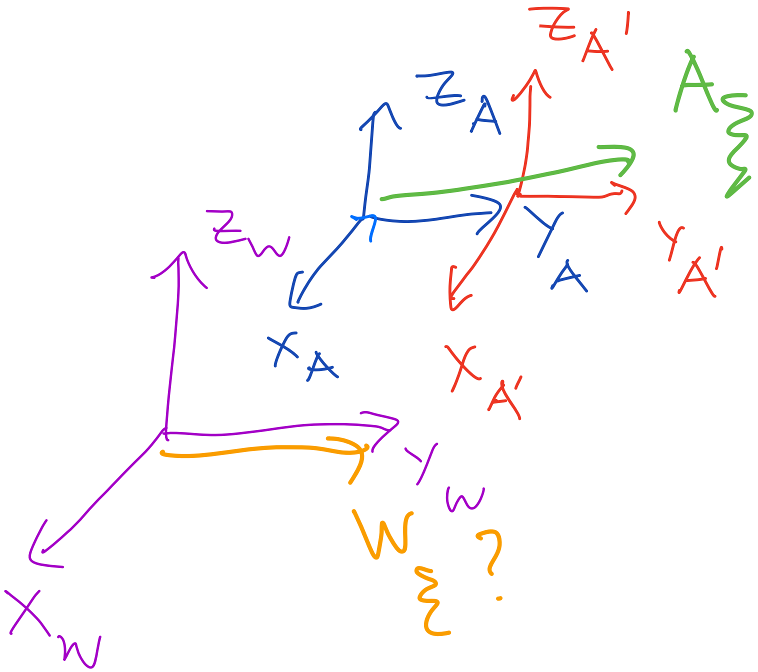
Changes of Reference Frames

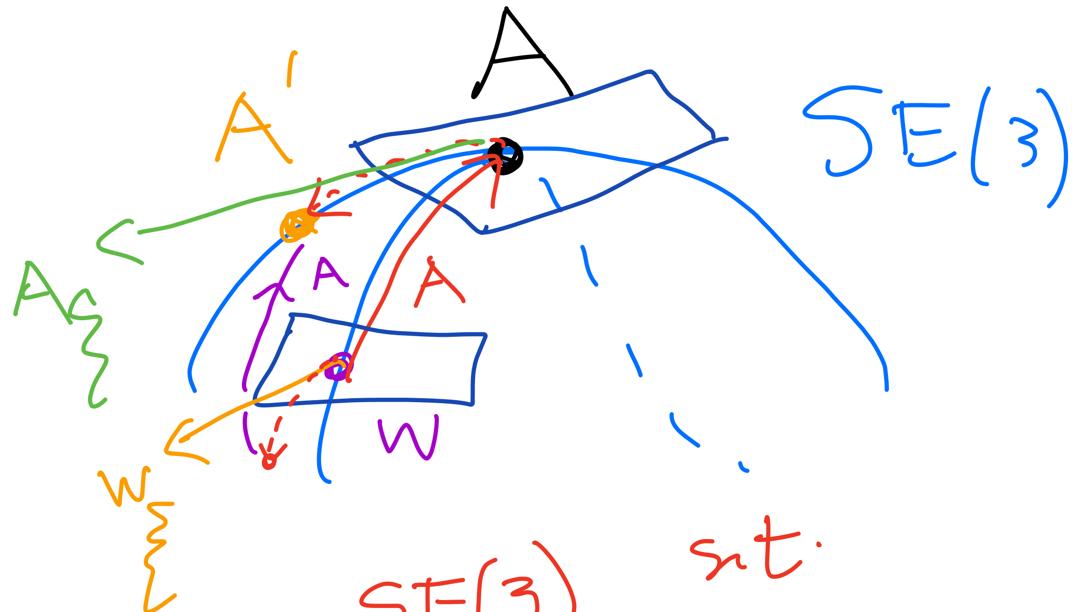
In frame A, it is not rotating.
However, it appears to "rotate" instantaneously within world frame W.



Changes of Reference Frames

$$\underset{\cup\!\!\!\!\!\!\in\; SE(3)}{\text{Frame } A'} = \underset{\in\; SE(3)}{\text{Frame } A \text{ after } dt}$$

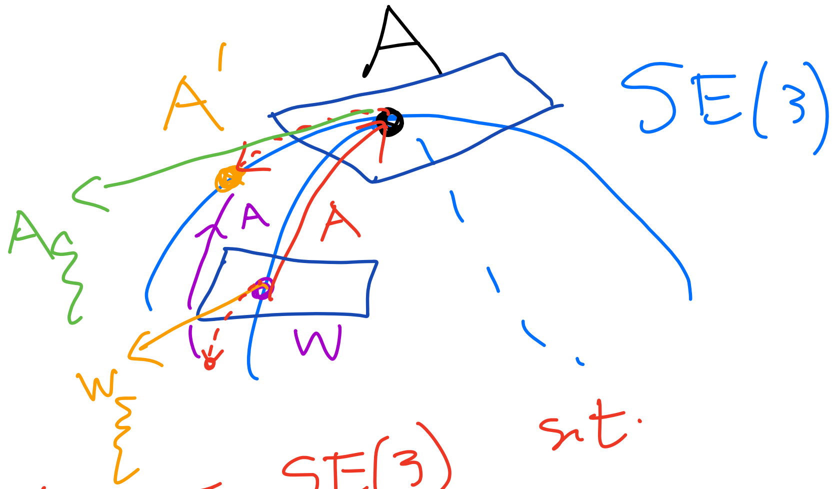




Goal: Find ${}^W\xi \in T_W SE(3)$ s.t.

$$\underbrace{{}^W\xi \oplus}_{\text{first increment by } {}^W\xi \text{ in } W} A = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus {}^A\xi$$

Changes of Reference Frames



Goal: Find ${}^{W}\xi \in T_W SE(3)$

$$\underbrace{{}^{W}\xi}_{\text{first increment by } {}^{W}\xi \text{ in } W} \oplus A = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus {}^{A}\xi$$

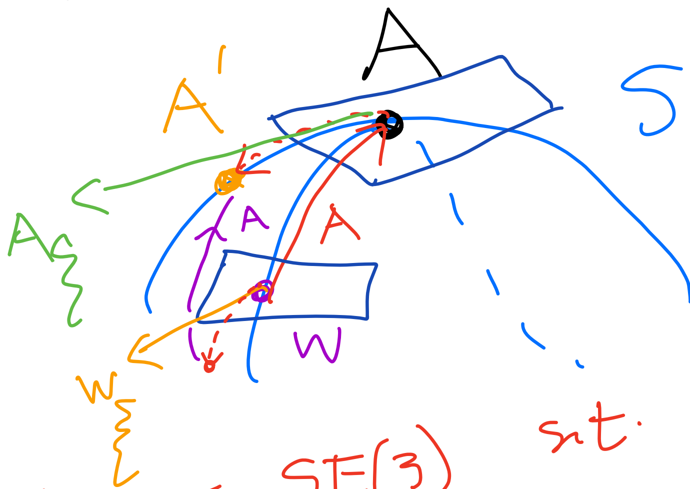
$$\mathrm{Exp}_W({}^{W}\xi\, w) \cdot A = {}^{W}\xi \oplus A = A \oplus {}^{A}\xi = \mathrm{Exp}_A\big(\underbrace{{}^{A}\xi\, A}_{\mathbb{R}^3}\big)$$

$$= A \cdot \mathrm{Exp}_W(\underbrace{{}^{A}\xi\, w}_{\sim})$$

$$\exp\big(({}^{W}\xi\, w)^\wedge\big) \cdot A \qquad\qquad A\exp\big(({}^{A}\xi\, w)^\wedge\big) \quad \hat{\mathbb{R}}^3$$

$$\Rightarrow \exp\big(({}^{W}\xi\, w)^\wedge\big) = A\exp\big(({}^{A}\xi\, w)^\wedge\big)A^{-1}$$

Changes of Reference Frames



s.t.

Goal: Find $^W\xi \in T_W SE(3)$

$$\underbrace{^W\xi \oplus A}_{\text{first increment by } ^W\xi \text{ in } W} = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus {^A\xi}$$

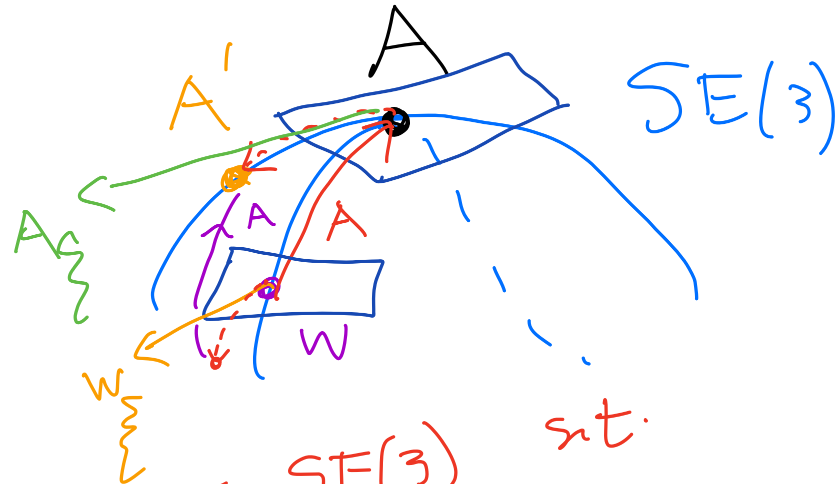
$$\exp\left(\left({^W\xi}^w\right)^\wedge\right) = A\exp\left(\left({^A\xi}^w\right)^\wedge\right)A^{-1}$$

$$= A \sum_{k=0}^{\infty} \frac{\left(\left({^A\xi}^w\right)^\wedge\right)^k}{k!} A^{-1}$$

$$= \sum_{k=0}^{\infty} \frac{\left(A\left({^A\xi}^w\right)^\wedge A^{-1}\right)^k}{k!}$$

$$= \exp\left(A\left({^A\xi}^w\right)^\wedge A^{-1}\right)$$

Changes of Reference Frames

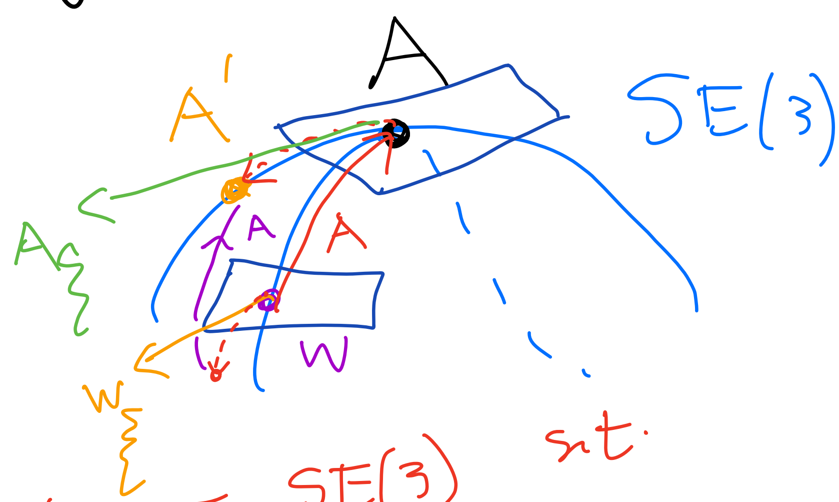


Goal: Find $^W\xi \in T_W SE(3)$ s.t.

$$\underbrace{^W\xi \oplus A}_{\text{first increment by } ^W\xi \text{ in } W} = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus {}^A\xi$$

$$\underbrace{\left({}^W\xi^W\right)^\wedge}_{\in\, T_W SE(3)} = A \underbrace{\left({}^A\xi^W\right)^\wedge}_{\in\, T_W SE(3)} A^{-1}$$

Changes of Reference Frames



Goal: Find $^W\xi \in T_W SE(3)$ s.t.

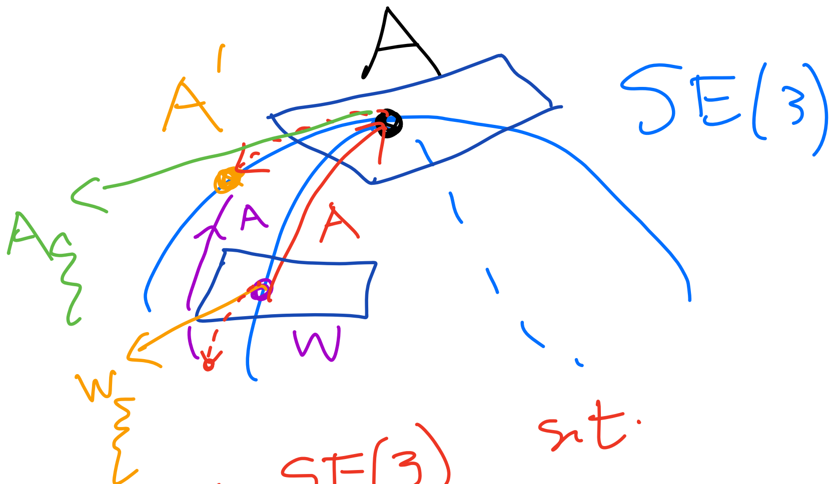
$$\underbrace{{}^W\xi \oplus A}_{\text{first increment by } {}^W\xi \text{ in } W} = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus {}^A\xi$$

$$\underbrace{{}^W\xi^W}_{\in\ \mathbb{R}^3} = \underbrace{\left(A\,({}^A\xi^W)^\wedge A^{-1}\right)^\vee}_{\in\ \mathbb{R}^3}$$

Define adjoint matrix of A:

$$\mathrm{Ad}_A : {}^A\xi^W \longmapsto \left(A\,({}^A\xi^W)^\wedge A^{-1}\right)^\vee$$

Changes of Reference Frames



Goal: Find ${}^W\xi \in T_W SE(3)$ s.t.

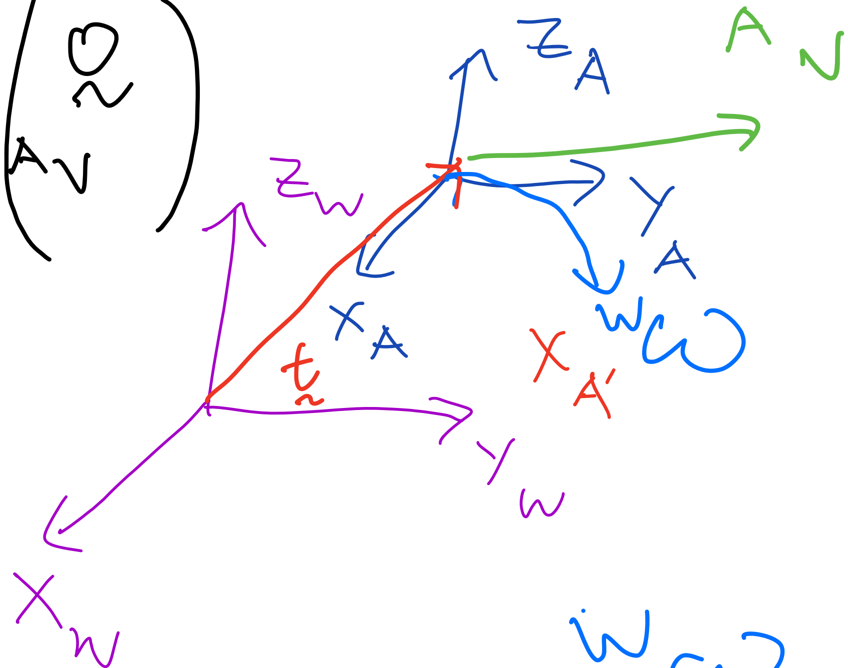
$$\underbrace{{}^W\xi \oplus A}_{\text{first increment by } {}^W\xi \text{ in } W} = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus {}^A\xi$$

$${}^W\xi = \underbrace{{}^{(W)}\mathrm{Ad}_A}_{\text{the "correct" change of reference frame factor for velocity vectors}} {}^A\xi$$

Changes of Reference Frames

$$A = \begin{pmatrix} R & \underset{\sim}{t} \\ 0 & 1 \end{pmatrix} \qquad {}^{A}\xi = \begin{pmatrix} \underset{\sim}{0} \\ {}^{A}v \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{I} & \underset{\sim}{t} \\ 0 & 1 \end{pmatrix}$$



$\underset{\sim}{{}^{W}\omega}$

$${}^{W}\xi = Ad_A\, {}^{A}\xi \overset{\text{(micro Lie theory)}}{=} \begin{pmatrix} R & [\underset{\sim}{t}]_{\times} R \\ 0 & R \end{pmatrix} \begin{pmatrix} \underset{\sim}{0} \\ {}^{A}v \end{pmatrix} = \begin{pmatrix} \underset{\sim}{t} \times \underset{\sim}{v} \\ v \end{pmatrix}$$

Adjoint Representation

The adjoint representation is defined as:

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$$

$$\mathrm{Ad}_g(X) = gXg^{-1}$$

For any Lie Group G, the map

$$g \longmapsto \mathrm{Ad}_g$$

"reproduces" the group structure.

Adjoint Representation

The adjoint representation is defined as:

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$$

$$\mathrm{Ad}_g(X) = g X g^{-1}$$

$$\mathrm{Ad}_{g \cdot h} = \mathrm{Ad}_g \circ \mathrm{Ad}_h$$

$$\mathrm{Ad}_{g^{-1}} = \mathrm{Ad}_g^{-1}$$

and by construction:

$$g \oplus \underset{\sim}{v} = (\mathrm{Ad}_g \underset{\sim}{v}) \oplus g$$

Summary

- If G has a bi-invariant metric, $g \oplus \underset{\sim}{v}$ gives the closest element $g' \in G$ after moving away from g in direction $\underset{\sim}{v}$.
- Since Exp_p is "hard" to compute, can instead use the fact that bi-invariant Lie Groups satisfy $\exp = \mathrm{Exp}_I$, and exp is easy to compute.
- If only interested in "linearising" G, can instead work w/ its Lie Algebra $\mathfrak{g}$ and exp.
- To change reference frames for velocity vectors, we use the adjoint matrix Ad_g.