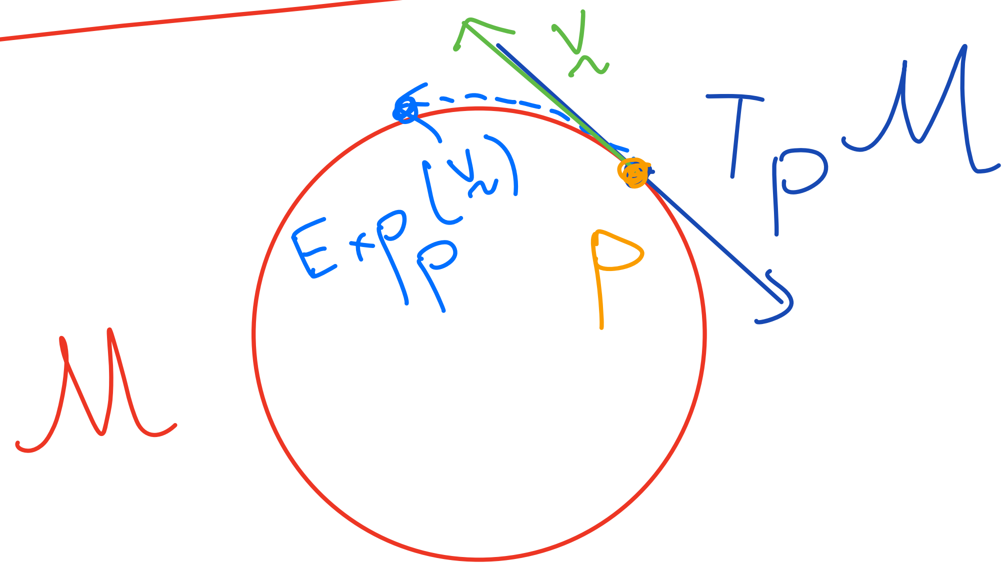
Lie Theory for Control & Estimation



By Sepehr Saryazdi

Motivation:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \; \| R\underset{\sim}{v} - \underset{\sim}{u} \|^2 \quad ?$$

Motivation:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \| R \underset{\sim}{v} - \underset{\sim}{u} \|^2 \quad ?$$

Motivation:

$$SO(3) = \left\{ R \in \mathrm{Mat}_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \| R\underset{\sim}{v} - \underset{\sim}{u} \|^2 \;?$$

Naive method:

$$f(r_{11}, \dots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \underset{\sim}{v} - \underset{\sim}{u} \right\|^2$$

Solve $\dfrac{\partial f}{\partial r_{ij}} = 0$. This is hard!

Motivation:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \| R \underset{\sim}{v} - \underset{\sim}{u} \|^2 \;?$$

Improved method w/ grad. descent:

$$f(r_{11}, \dots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \underset{\sim}{v} - \underset{\sim}{u} \right\|^2$$

$$\underset{\sim}{r} := (r_{11}, \dots, r_{33}), \quad \underset{\sim}{r}_{n+1} = \underset{\sim}{r}_n - \gamma \nabla f(\underset{\sim}{r}_n)$$

Motivation:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \| R\underset{\sim}{v} - \underset{\sim}{u} \|^2 \quad ?$$

Improved method w/ grad. descent:

$$f(r_{11}, \dots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \underset{\sim}{v} - \underset{\sim}{u} \right\|^2$$

$$\underset{\sim}{r}_{n+1} = \underset{\sim}{r}_n - \gamma \nabla f(\underset{\sim}{r}_n) =: (r'_{11}, \dots, r'_{33}),\ R' \notin SO(3)!$$

Motivation:

$$SO(3) = \left\{ R \in \mathrm{Mat}_3(\mathbb{R}) \;\middle|\; \begin{matrix} R^T R = \mathbb{I}, \\ \det R = 1 \end{matrix} \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \| R\underset{\sim}{v} - \underset{\sim}{u} \|^2 \quad ?$$

Improved method w/ grad. descent:

$$f(r_{11}, \dots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \underset{\sim}{v} - \underset{\sim}{u} \right\|^2$$

Want: $(\underset{\sim}{r}_n)_{n \in \mathbb{N}}$ s.t. $\underset{\sim}{r}_n \in SO(3)$ & $r_n \longrightarrow \underset{r}{\operatorname{argmin}} f$

Motivation:

$$SO(3) = \left\{ R \in \mathrm{Mat}_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \| R\underset{\sim}{v} - \underset{\sim}{u} \|^2 \quad ?$$

Want: $(\underset{\sim}{r}_n)_{n \in \mathbb{N}}$ s.t. $\underset{\sim}{r}_n \in SO(3)$ & $r_n \longrightarrow \underset{r}{\operatorname{argmin}} f$

Idea: Define $\oplus$ s.t.

$$R \oplus dR \in SO(3)$$

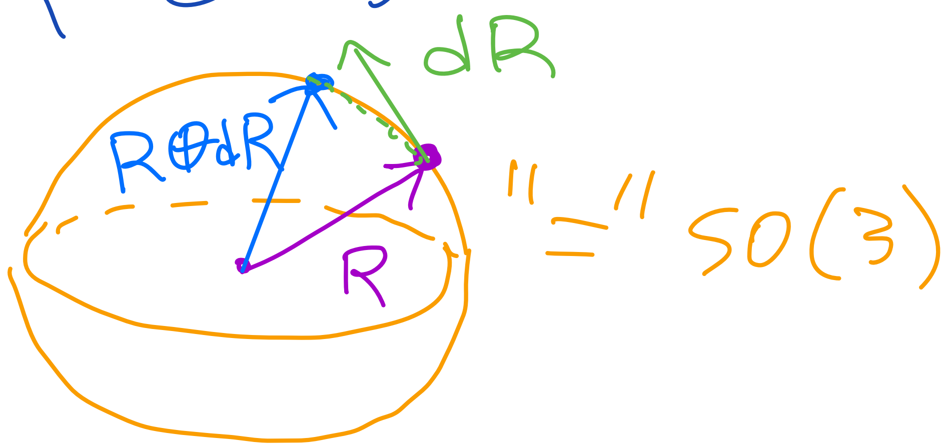
Motivation:

$$SO(3) = \left\{ R \in \mathrm{Mat}_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

Want: $\left(\underset{\sim}{r}_n\right)_{n \in \mathbb{N}}$ s.t. $\underset{\sim}{r}_n \in SO(3)$ & $r_n \longrightarrow \underset{r}{\operatorname{argmin}} f$

Idea: Define $\oplus$ s.t.

$$R \oplus dR \in SO(3)$$

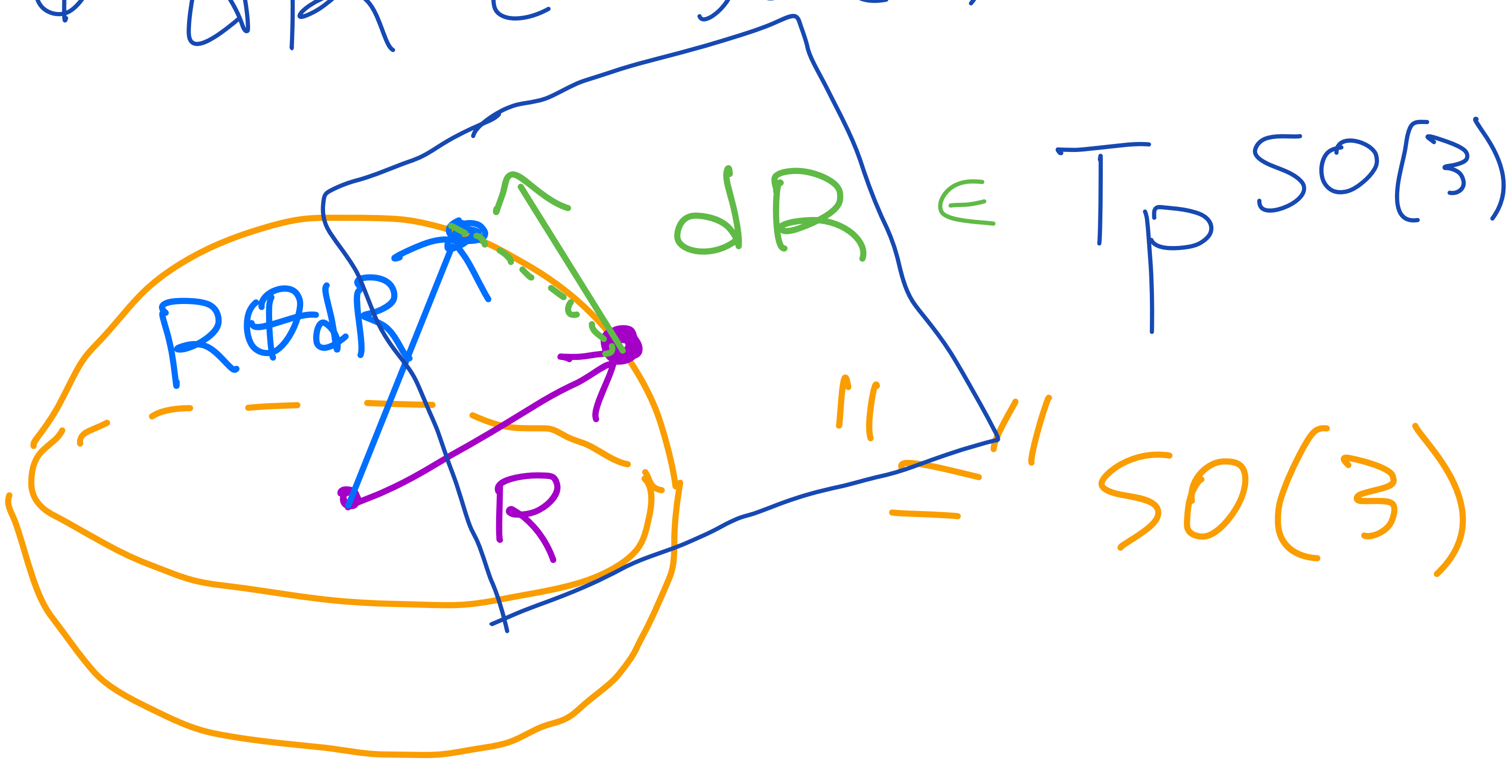


Motivation:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \,\middle|\, R^T R = \mathbb{I},\ \det R = 1 \right\}$$

Idea: Define $\oplus$ s.t.

$$R \oplus dR \in SO(3)$$



Motivation:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

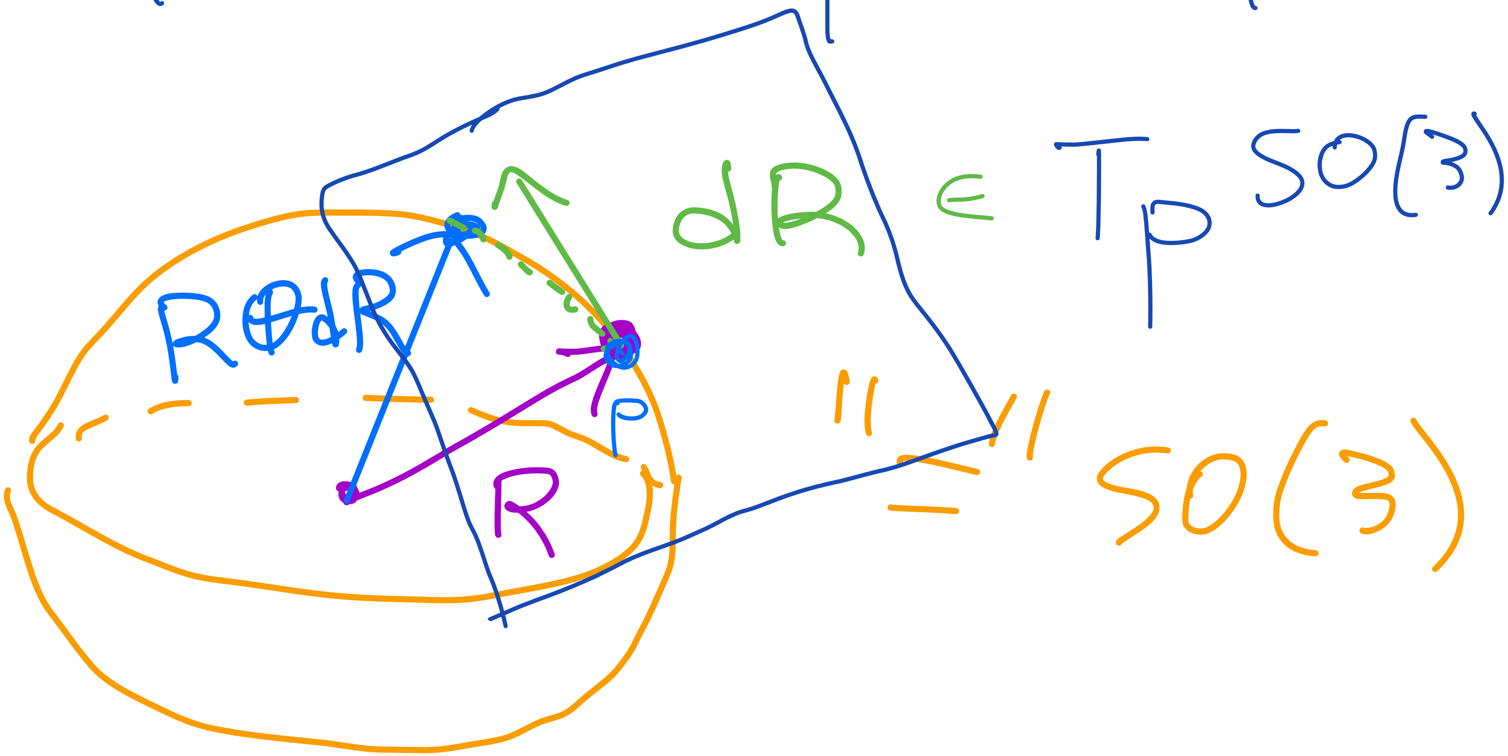
Idea: Define $\oplus$ s.t.

$$R \oplus dR \in SO(3)$$

Computing $T_p\, SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

Q: How to compute $T_p\, SO(3)$?

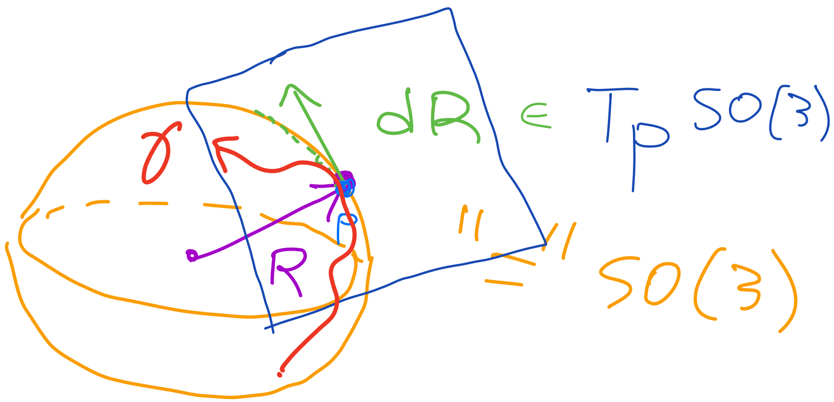


Think of "velocity" vectors at a point $p \in SO(3)$.

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

Q: How to compute $T_p SO(3)$?



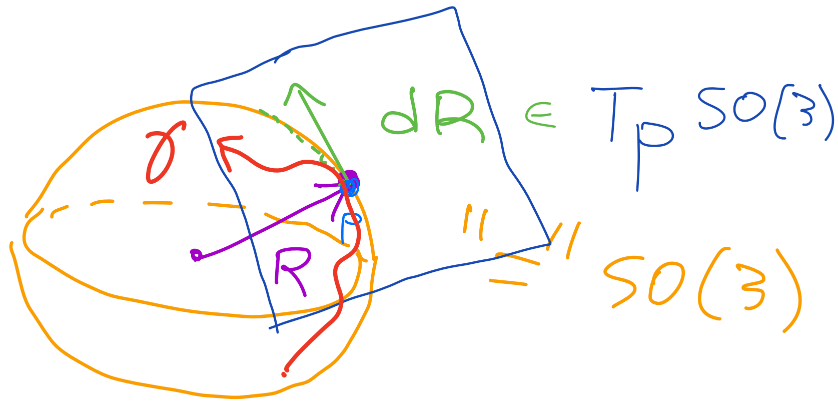
Think of "velocity" vectors of a curve at a point $p \in SO(3)$.

$$\gamma : (-\epsilon, \epsilon) \longrightarrow SO(3)$$

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

Q: How to compute $T_p SO(3)$?



Think of "velocity" vectors of a curve at a point $p \in SO(3)$.

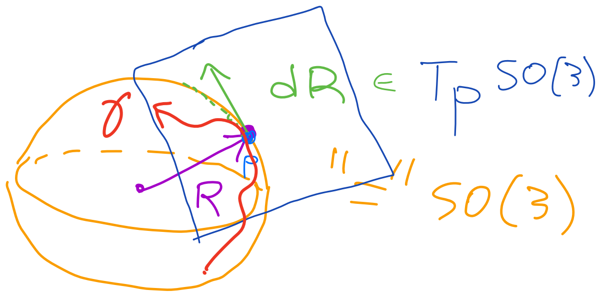
$\gamma : (-\epsilon, \epsilon) \longrightarrow SO(3)$, e.g. $\gamma(t) = R_z(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$p = \gamma(0) = \mathbb{I}$

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

Q: How to compute $T_p SO(3)$?



Think of "velocity" vectors of a curve at a point $p \in SO(3)$.

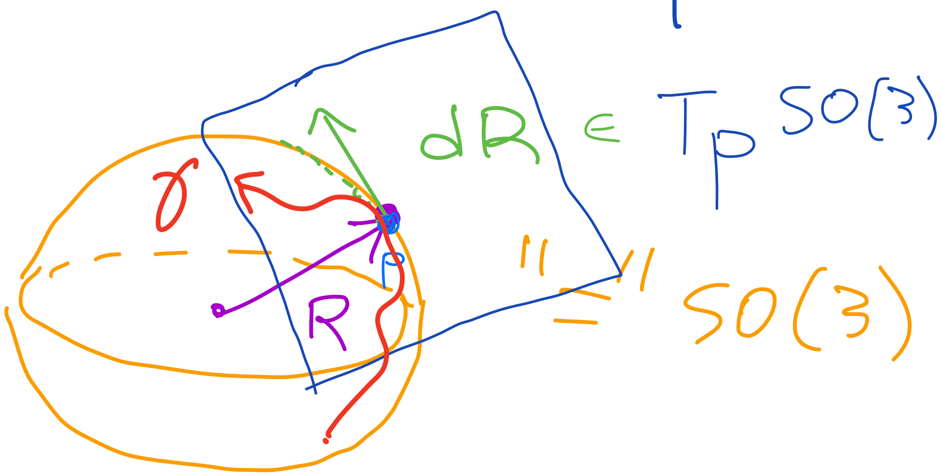
$\gamma : (a,b) \to SO(3)$, e.g. $\gamma(t) = R_z(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $p = \gamma(0) = \mathbb{I}$

$$\gamma'(t) = \begin{bmatrix} -\sin t & -\cos t & 0 \\ \cos t & -\sin t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \gamma'(0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

Q: How to compute $T_p SO(3)$?



$$\gamma : (a,b) \to SO(3), \text{ e.g. } \gamma(t) = R_z(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad p = \gamma(0) = \mathbb{I}$$

$$\gamma'(t) = \begin{bmatrix} -\sin t & -\cos t & 0 \\ \cos t & -\sin t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underbrace{\gamma'(0)}_{= dR} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

Q: How to compute $T_p SO(3)$?

$$\gamma_z(t) = R_z(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad p = \gamma(0) = \mathbb{I}, \quad \gamma_z'(0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\gamma_y(t) = R_y(t) = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix}, \quad \gamma_y'(0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\gamma_x(t) = R_x(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}, \quad \gamma_x'(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Computing $T_p SO(3)$:

$$SO(3) = \{ R \in Mat_3(\mathbb{R}) \mid R^T R = I, \ \det R = 1 \}$$

Q: How to compute $T_p SO(3)$?

$$\gamma_z(t) = R_z(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad p = \gamma(0) = I, \quad \gamma_z'(0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\gamma_y(t) = R_y(t) = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix}, \quad \gamma_y'(0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\gamma_x(t) = R_x(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}, \quad \gamma_x'(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

what about any γ?

$$\gamma(t) = R_x(\theta_1(t)) R_y(\theta_2(t)) R_z(\theta_3(t))$$

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in \mathrm{Mat}_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

Q: How to compute $T_p SO(3)$?

what about any γ passing through $p = \mathbb{I}$?

$$\gamma(t) = R_x\big(\theta_1(t)\big)\, R_y\big(\theta_2(t)\big)\, R_z\big(\theta_3(t)\big)$$

$$\gamma(t) = \gamma_x\big(\theta_1(t)\big)\, \gamma_y\big(\theta_2(t)\big)\, \gamma_z\big(\theta_3(t)\big)$$

$$\begin{aligned}
\gamma'(t) = {} & \gamma_x'\big(\theta_1(t)\big) \cdot \gamma_y\big(\theta_2(t)\big) \cdot \gamma_z\big(\theta_3(t)\big) \cdot \theta_1'(t) \\
& + \gamma_x\big(\theta_1(t)\big) \cdot \gamma_y'\big(\theta_2(t)\big) \cdot \gamma_z\big(\theta_3(t)\big) \cdot \theta_2'(t) \\
& + \gamma_x\big(\theta_1(t)\big) \cdot \gamma_y\big(\theta_2(t)\big) \cdot \gamma_z'\big(\theta_3(t)\big) \cdot \theta_3'(t)
\end{aligned}$$

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in \mathrm{Mat}_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

Q: How to compute $T_p SO(3)$?

what about any γ passing through $p = \mathbb{I}$?

$$\gamma(t) = \gamma_x\underline{(\theta_1(t))}\,\gamma_y\underline{(\theta_2(t))}\,\gamma_z\underline{(\theta_3(t))}$$

$$\begin{aligned}
\gamma'(0) = {} & \gamma_x'(0)\cdot\cancelto{\mathbb{I}}{\gamma_y\underline{(0)}}\cdot\cancelto{\mathbb{I}}{\gamma_z\underline{(0)}}\cdot\theta_1'(0) \\
& + \cancelto{\mathbb{I}}{\gamma_x\underline{(0)}}\cdot\gamma_y'(0)\cdot\cancelto{\mathbb{I}}{\gamma_z\underline{(0)}}\cdot\theta_2'(0) \\
& + \cancelto{\mathbb{I}}{\gamma_x\underline{(0)}}\cdot\cancelto{\mathbb{I}}{\gamma_y\underline{(0)}}\cdot\gamma_z'(0)\cdot\theta_3'(0)
\end{aligned}$$

$$\gamma'(0) = \gamma_x'(0)\,\theta_1'(0) + \gamma_y'(0)\,\theta_2'(0) + \gamma_z'(0)\,\theta_3'(0)$$

Computing $T_p SO(3)$:

$$SO(3) = \{ R \in Mat_3(\mathbb{R}) \mid R^T R = \mathbb{I},\ \det R = 1 \}$$

Q: How to compute $T_p SO(3)$?

what about any γ passing through $p = \mathbb{I}$?

$$\gamma'(0) = \gamma_x'(0)\,\theta_1'(0) + \gamma_y'(0)\,\theta_2'(0) + \gamma_z'(0)\,\theta_3'(0)$$

$$= \gamma_x'(0)\,\omega_x + \gamma_y'(0)\,\omega_y + \gamma_z'(0)\,\omega_z$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_z$$

$$= \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in \mathrm{Mat}_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

Q: How to compute $T_p SO(3)$?

As a vector space:

$$\gamma'(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_z$$

$$\mathfrak{so}(3) := T_{\mathbb{I}} SO(3) = \mathrm{span}_{\mathbb{R}} \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$= \mathrm{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial \alpha}(\mathbb{I}), \frac{\partial}{\partial \beta}(\mathbb{I}), \frac{\partial}{\partial \gamma}(\mathbb{I}) \right\}$$

$$= \mathrm{span}_{\mathbb{R}} \left\{ E_1(\mathbb{I}), E_2(\mathbb{I}), E_3(\mathbb{I}) \right\}$$

Lie Algebra of $SO(3)$

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

Q: How to compute $T_p SO(3)$?

what about any γ passing through $p = \mathbb{I}$?

Representing vectors

$$\underset{\sim}{v} = (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$$

$$\underset{\sim}{v}^\wedge := v_1 E_1 + v_2 E_2 + v_3 E_3 = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_z}_{\in T_{\mathbb{I}} SO(3)}$$

$$(\underset{\sim}{v}^\wedge)^\vee = \underset{\sim}{v}$$

Computing $T_p SO(3)$:

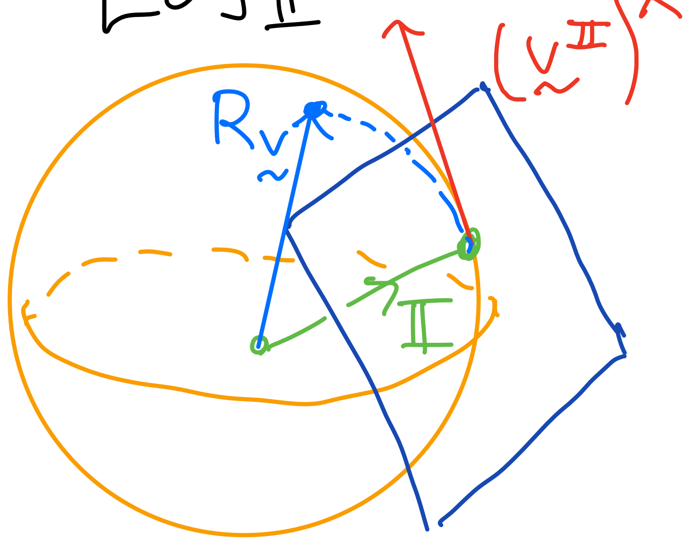
$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

$$SO(3) \xrightarrow{\left.\frac{d}{dt}\right|_{\mathbb{I}}} T_{\mathbb{I}}\, SO(3)$$

$$SO(3) \underset{\mathrm{Log}_{\mathbb{I}}}{\overset{\mathrm{Exp}_{\mathbb{I}}}{\rightleftarrows}} T_{\mathbb{I}}\, SO(3)$$

$$\rotatebox{90}{$\in$} \qquad\qquad\qquad\qquad \rotatebox{90}{$\in$}$$

$$R_{\tilde{v}} \qquad\qquad\qquad\qquad (\tilde{v}^{\mathbb{I}})^{\wedge}$$

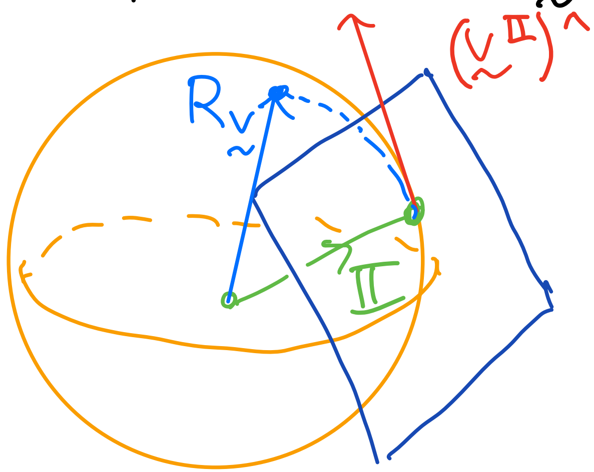


Computing $T_p SO(3)$:

$$SO(3) = \{R \in Mat_3(\mathbb{R}) \mid R^T R = \mathbb{I},\ \det R = 1\}$$

$$\begin{array}{ccc} SO(3) & \xleftarrow{\ Exp_{\mathbb{I}}\ } & T_{\mathbb{I}} SO(3) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ R_{\tilde{v}} & & (\tilde{v}^{\mathbb{I}})^{\wedge} \end{array}$$

$Exp_{\mathbb{I}}({}^{\mathbb{I}}\tilde{v}) := \gamma_{\tilde{v}}(1)$, $\gamma_{\tilde{v}}$ a geodesic w/ const. velocity $\tilde{v}$



$$\mathbb{I} \oplus {}^{\mathbb{I}}\tilde{v} := Exp_{\mathbb{I}}(\tilde{v}^{\mathbb{I}}) = R_{\tilde{v}}$$

Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

$$\begin{array}{ccc} SO(3) & \xleftarrow{\;Exp_{\mathbb{I}}\;} & T_{\mathbb{I}} SO(3) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ R_{\underset{\sim}{v}} & & (\underset{\sim}{v}^{\mathbb{I}})^{\wedge} \end{array}$$

$$Exp_{\mathbb{I}}(\underset{\sim}{v}) = \gamma_{\underset{\sim}{v}}(1)$$

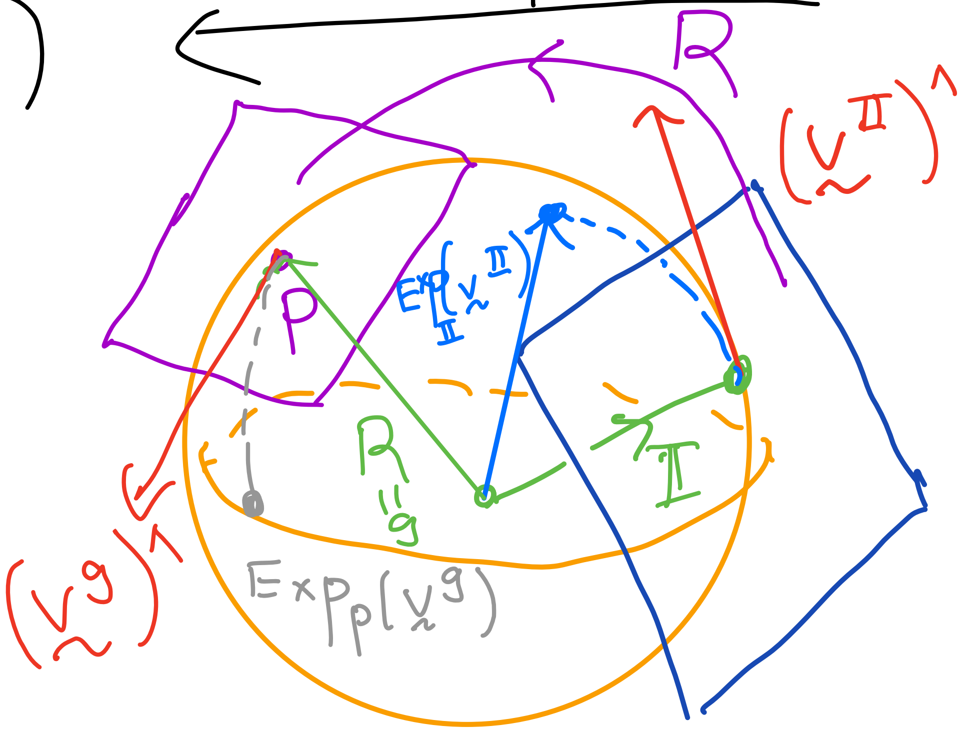
$\gamma_{\underset{\sim}{v}}$ a geodesic w/ const. velocity $\underset{\sim}{v}$

Computing $T_p\, SO(3)$:

$$SO(3) = \left\{ R \in \mathrm{Mat}_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

$$SO(3) \xleftarrow{\ \mathrm{Exp}_p\ } T_p\, SO(3)$$

$p = g$

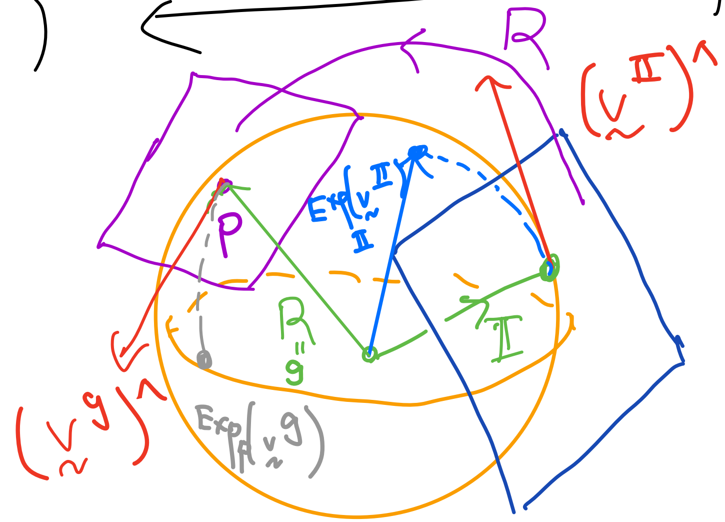


Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

$$SO(3) \xleftarrow{Exp_p} T_p SO(3)$$

$p = g$



$$R \oplus v^R := R \cdot Exp_{\mathbb{I}}(v^{\mathbb{I}}) \overset{(*)}{=} Exp_R(v^R)$$

why? we'll return to this later...

Properties of $\mathrm{Exp}_p : T_p SO(3) \to SO(3)$

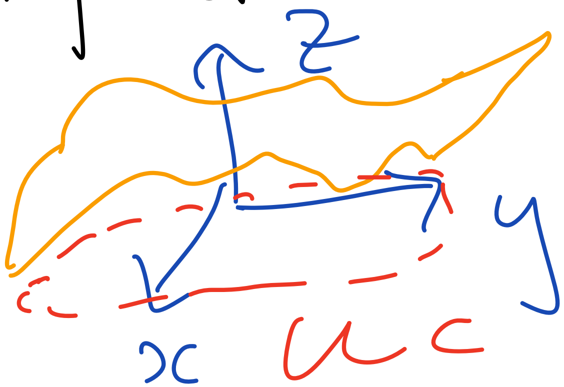
$$\mathrm{Exp}_p(\underset{\sim}{v}^p) := \underbrace{\gamma_{\underset{\sim}{v}^p}(1)}$$

γ a geodesic

How do we actually compute $\mathrm{Exp}_p(\underset{\sim}{v}^p)$ for any manifold $\mathcal{M}$?

Properties of $\mathrm{Exp}_p : T_p \, SO(3) \to so(3)$

Every manifold comes with an atlas: (φ_i, U_i)



$\mathbb{R}^2$ open

$(x,y) \longmapsto (x,y,z(x,y))$

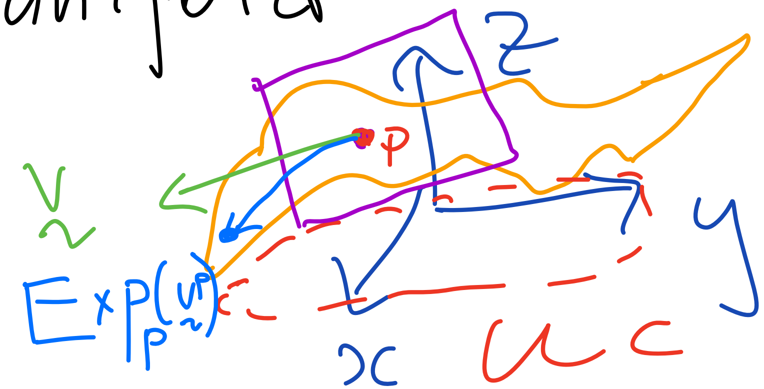
$SO(3)$ as a manifold has a chart given by

$$\varphi : (0,2\pi)^3 \longrightarrow SO(3)$$

$$\varphi(\alpha,\beta,\gamma) = R_x(\alpha)\, R_y(\beta)\, R_z(\gamma)$$

Properties of $\mathrm{Exp}_p : T_p SO(3) \to so(3)$

Every manifold M comes with an atlas: (φ_i, U_i)



$U \subset \mathbb{R}^2$ open

$(x,y) \mapsto (x,y,z(x,y))$

If φ is given, then $\mathrm{Exp}_p(v^p)$ can be computed using the geodesic equation:

$$p = \varphi(u^1(0), \dots, u^n(0))$$

$$v = \left(v^1, \dots, v^n\right)$$

initial condition of ODE

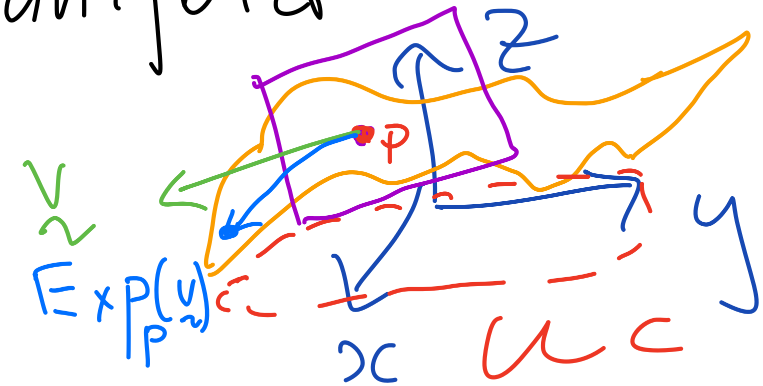
$$\overbrace{\frac{d^2 u^k}{dt^2} + \sum_{ij} \underbrace{\Gamma^k_{ij}}_{\text{Christoffel Symbols}} \frac{du^i}{dt}\frac{du^j}{dt} = 0}^{\text{ODE}},$$

$$\varphi_{u^i u^j} = \overbrace{\sum_k \Gamma^k_{ij} E_k}^{\text{def}^n \text{ of } \Gamma^k_{ij}},$$

$$T_p M = \operatorname{span}_{\mathbb{R}}\{E_1, \dots, E_n\}$$

Properties of $\mathrm{Exp}_p : T_p SO(3) \to so(3)$

Every manifold M comes with an atlas: (φ_i, U_i)



$U \subset \mathbb{R}^2$ open

$(x, y) \longmapsto (x, y, z(x,y))$

Properties of $\mathrm{Exp}_p : T_p SO(3) \to so(3)$

How to compute $\mathrm{Exp}_{\mathbb{I}}\left((\underset{\sim}{v}^{\mathbb{I}})^{\wedge}\right)$ in an easier way?

$$\underset{\sim}{v}^{\mathbb{I}} = (\omega_x, \omega_y, \omega_z)$$

$$(\underset{\sim}{v}^{\mathbb{I}})^{\wedge} = \omega_x E_1 + \omega_y E_2 + \omega_z E_3 \in so(3)$$

want: $R(t) \approx \mathbb{I} + \frac{dR(0)}{dt} t$ (linearising $SO(3)$ at $\mathbb{I}$)

$$= \mathbb{I} + (v^{\mathbb{I}})^{\wedge} t$$

The ODE $\dot{R} = R\,(v^{\mathbb{I}})^{\wedge}$ w/ $R(0) = \mathbb{I}$ has unique solution given by:

$$\exp((\underset{\sim}{v}^{\mathbb{I}})^{\wedge}) = \sum_{k=0}^{\infty} \frac{\left((\underset{\sim}{v}^{\mathbb{I}})^{\wedge}\right)^k}{k!} \in SO(3)$$

(Regular exponential)

for small $\|\underset{\sim}{v}^{\mathbb{I}}\|$

Properties of $\mathrm{Exp}_p : T_p SO(3) \to \mathfrak{so}(3)$

$$\exp\left((\underset{\sim}{v}^{I})^{\wedge}\right) = \sum_{k=0}^{\infty} \frac{\left((\underset{\sim}{v}^{I})^{\wedge}\right)^{k}}{k!} \quad \text{(easy to calculate)}$$

Equivalence of exp & Exp_{I}

If G is a Lie Group w/ bi-invariant Riemannian metric, then

$$\exp\left((\underset{\sim}{v}^{I})^{\wedge}\right) = \mathrm{Exp}_{I}(\underset{\sim}{v}^{I})$$

Milnor, 1976

A Lie Group G admits a bi-invariant metric iff $G \overset{\text{group}}{\cong} K \times H$ where K is a compact group & H abelian.

Properties of $\mathrm{Exp}_p : T_p SO(3) \to SO(3)$

Compactness

$SO(3)$ is compact as:

$R^T R = \mathbb{I}$ & $\det R = 1$

$\Downarrow$

R has bounded matrix norm, so $SO(3)$ is bounded.

satisfies polynomial equations $\Rightarrow SO(3)$ is closed.

$\underset{\sim}{r} := (r_{11}, \ldots, r_{33})$

so $SO(3)$ closed & bounded $\Rightarrow$ compact.

Properties of $\mathrm{Exp}_p : T_p SO(3) \to SO(3)$

For Lie Groups G w/ bi-invariant Riemmanian metrics (e.g. $SO(3)$), the geodesic equation (Michael Greis, Notes on Riemannian Geom of Lie Grps) has the solution

$$\gamma_{\underset{\sim}{v}}(t) = \underbrace{\mathrm{Exp}_g(t\underset{\sim}{v}^g)}_{\text{geodesics starting at } g} = g \circ \underbrace{\mathrm{Exp}_e(t\underset{\sim}{v}^{I})}_{\text{geodesics starting at } I}$$

$$g \oplus g_{\underset{\sim}{v}} := g \circ \mathrm{Exp}_e(\underset{\sim}{v}^{I}) \quad (\text{before})$$

$$= \mathrm{Exp}_g(\underset{\sim}{v}^g)$$

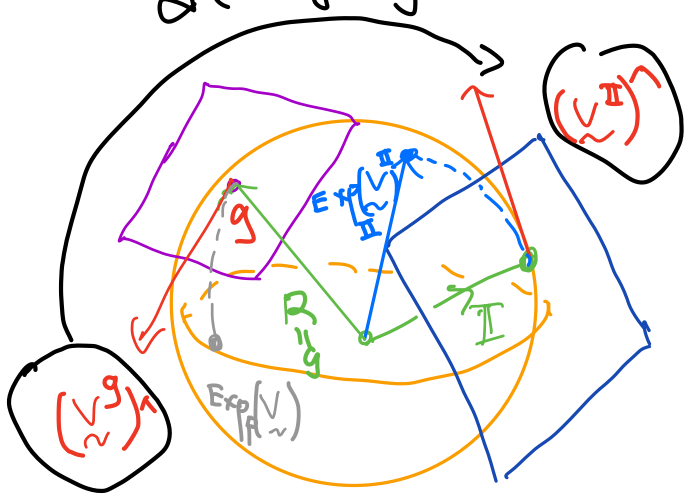
Properties of $Exp_p : T_p SO(3) \to so(3)$

$$\gamma_{\underset{\sim}{v}}(t) = \underbrace{Exp_g(t\underset{\sim}{v}^g)}_{\text{geodesics starting at } g} = g \circ \underbrace{Exp_e(t \underset{\sim}{v}^{\mathbb{I}})}_{\text{geodesics starting at } \mathbb{I}}$$

How is $\underset{\sim}{v}^{\mathbb{I}}$ related to $\underset{\sim}{v}^{g}$?

As elements $(\underset{\sim}{v}^{\mathbb{I}})^\wedge \in T_{\mathbb{I}} SO(3)$, $(\underset{\sim}{v}^{g})^\wedge \in T_g SO(3)$, they are related by the differential of $L_{g^{-1}} : x \mapsto g^{-1}x \;\; \forall x \in G$, at g

$d(L_{g^{-1}})_g$



i.e. $(\underset{\sim}{v}^{\mathbb{I}})^\wedge = d(L_{g^{-1}})_g (\underset{\sim}{v}^{g})^\wedge$

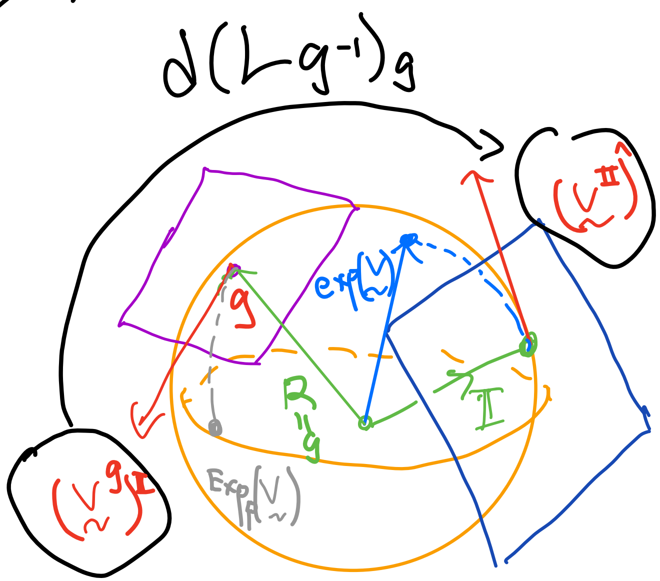
For $SO(3)$, $d(L_{g^{-1}}) = R^{-1}$.

Properties of $\mathrm{Exp}_p : T_p SO(3) \to so(3)$

$$\gamma_{\tilde{v}}(t) = \underbrace{\mathrm{Exp}_g(t\, \tilde{v}^g)}_{\text{geodesics starting at } g} = g \circ \underbrace{\mathrm{Exp}_e(t\, \tilde{v}^{\mathrm{I}})}_{\text{geodesics starting at } \mathrm{I}}$$

How is $\tilde{v}^{\mathrm{I}}$ related to v^g?

As $\tilde{v}^{\mathrm{I}}, \tilde{v}^g \in \mathbb{R}^3$, they coincide for bi-invariant Lie Groups.



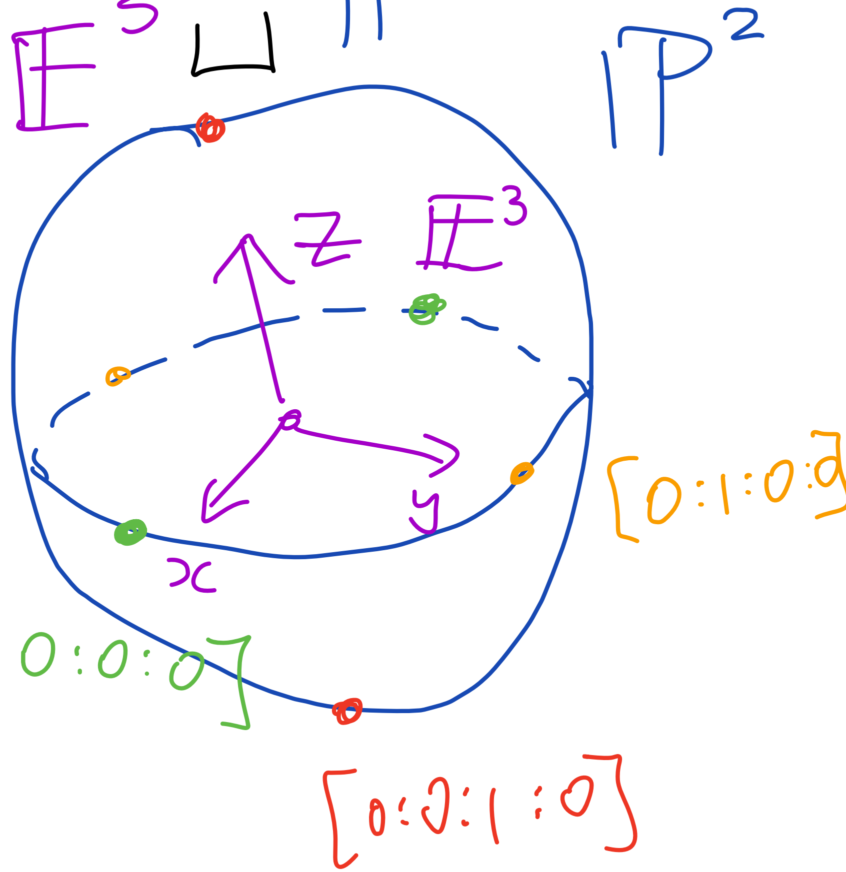
i.e.

$$\tilde{v}^g = (\omega_x, \omega_y, \omega_z) = \tilde{v}^{\mathrm{I}}$$

Other Examples: $SE(3)$

$$SE(3) = \left\{ \begin{pmatrix} R & \underset{\sim}{t} \\ 0 & 1 \end{pmatrix} \middle| \begin{array}{l} R \in SO(3), \\ t \in \mathbb{R}^3 \end{array} \right\}$$

$$SE(3) \curvearrowright \mathbb{P}^3 = \mathbb{E}^3 \sqcup \mathbb{P}^2$$



$$\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \cdot [x:y:z:1] = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\lim_{x\to\infty} [x:y_0:z_0:1] = \lim_{x\to\infty} \left[1 : \frac{y_0}{x} : \frac{z_0}{x} : \frac{1}{x}\right] = [1:0:0:0]$$

Other Examples: SE(3)

$$SE(3) = \left\{ \begin{pmatrix} R & \underset{\sim}{t} \\ 0 & 1 \end{pmatrix} \middle| \begin{array}{l} R \in SO(3), \\ t \in \mathbb{R}^3 \end{array} \right\}$$

$$SE(3) \curvearrowright \mathbb{P}^3 = \mathbb{E}^3 \sqcup \mathbb{P}^2$$

Other Examples: SE(3)

$$SE(3) \cong \underbrace{SO(3) \ltimes}_{\text{semi-direct product}} \mathbb{R}^3 \neq SO(3) \underbrace{\times}_{\text{direct product}} \mathbb{R}^3$$

& $\mathrm{Exp}_p \neq \exp$.

Basics of Classical Lie Groups, Chp 14

$\exp : \mathfrak{se}(3) \longrightarrow SE(3)$

is well-defined & surjective.

so we can still use exp as a way to "linearise" SE(3), but lose geometric meaning.

Other Examples: SE(3)

$$SE(3) \ni \mathbb{I}$$

$$T_{\mathbb{I}} SE(3) = \operatorname{span}_{\mathbb{R}} \left\{ \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} R_x(t) & 0 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} R_y(t) & 0 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} R_z(t) & 0 \\ 0 & 1 \end{pmatrix}, \right.$$

$$\left. \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} \mathbb{I} & t\underset{\sim}{e}_1 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} \mathbb{I} & t\underset{\sim}{e}_2 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} I & t\underset{\sim}{e}_3 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \operatorname{span} \left\{ \frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta}, \frac{\partial}{\partial \gamma}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$$

$$= \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

Other Examples: SE(3)

A twist ξ is given as:

$$\xi := \omega_x \frac{\partial}{\partial \alpha} + \omega_y \frac{\partial}{\partial \beta} + \omega_z \frac{\partial}{\partial \gamma} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

$$= \begin{pmatrix} 0 & -\omega_z & \omega_y & v_x \\ \omega_z & 0 & -\omega_x & v_y \\ -\omega_y & \omega_x & 0 & v_z \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Properties of Lie Algebras:

Matrix group multiplication can be "reproduced" with Lie Algebra multiplication:

Baker-Campbell-Hausdorff:

$$x \cdot y \longleftrightarrow X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \ldots$$

w/ $X = \log x$, $Y = \log y$, $[X,Y] := XY - YX$

Properties of Lie Algebras:

Baker-Campbell-Hausdorff:

$$x \cdot y \longleftrightarrow X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \dots$$

w/ $X = \log x$, $Y = \log y$, $[X, Y] := XY - YX$

Corollary

if $\exp : \mathfrak{g} \to G$ is surjective, we can "globally" linearise G by working purely with its Lie Algebra $\mathfrak{g}$.

Changes of Reference Frames

Q: How can we compute a change of reference frame?

For a pose ${}^{B}T_{A} \in SE(3)$, we can just do

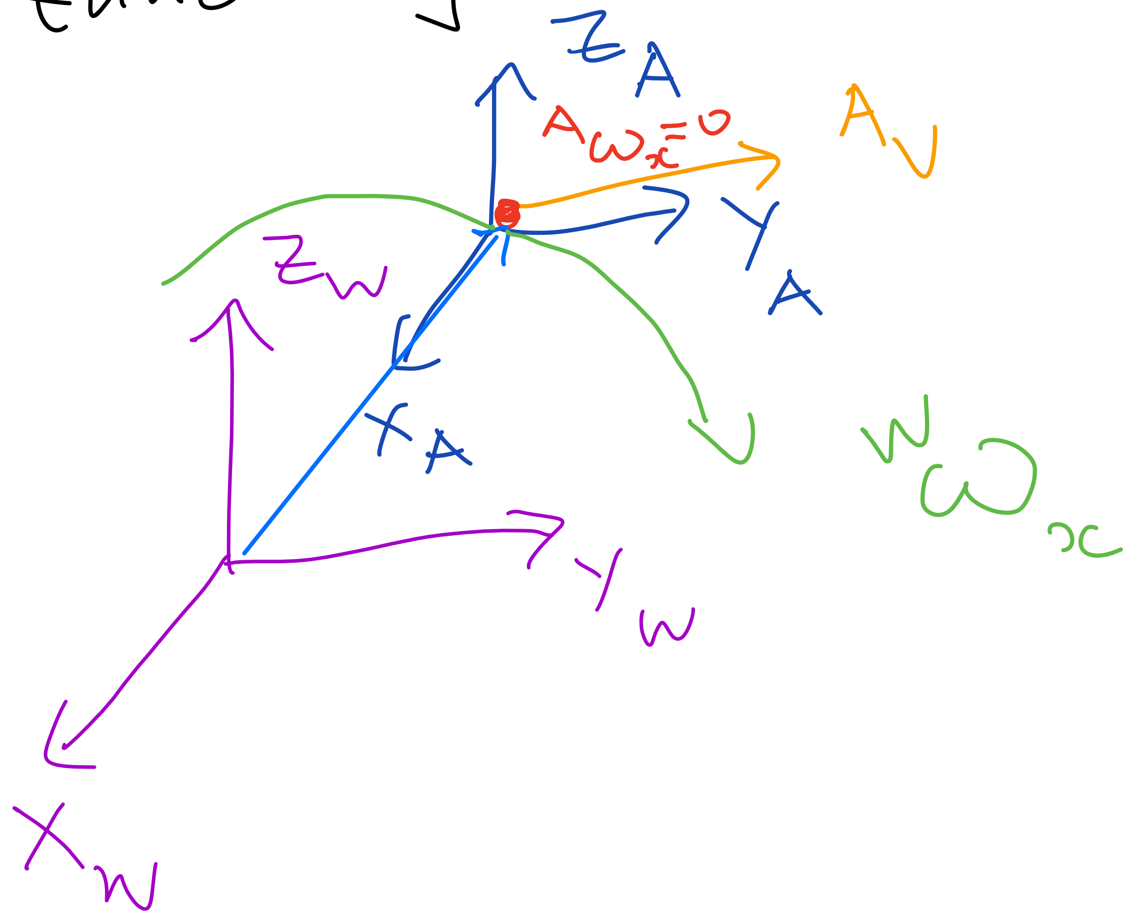
$${}^{B}p = {}^{B}T_{A}\,{}^{A}p$$

What about twists ξ?

$${}^{B}\xi = {}^{B}?_{A}\,{}^{A}\xi$$

Changes of Reference Frames

In frame A, it is not rotating. However, it appears to "rotate" instantaneously within world frame W.

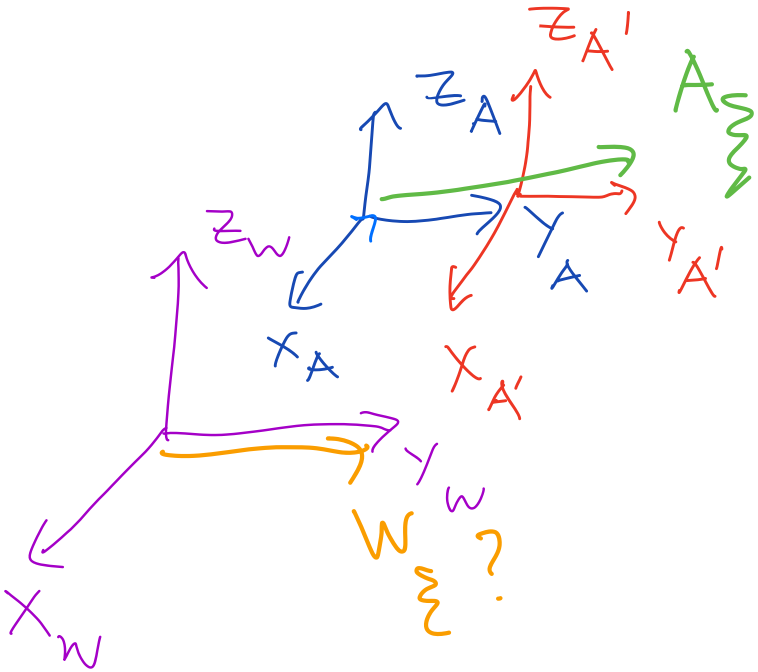


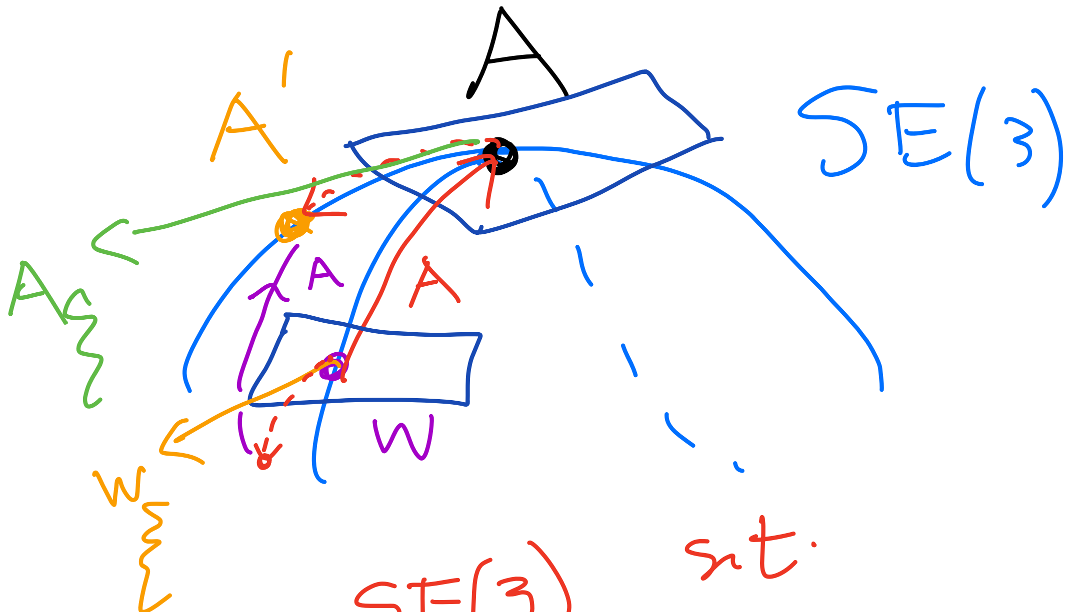
Changes of Reference Frames

Frame A' = Frame A after dt

$\cap$ $\qquad\qquad\qquad\qquad$ $\cap$

$SE(3)$ $\qquad\qquad\qquad$ $SE(3)$

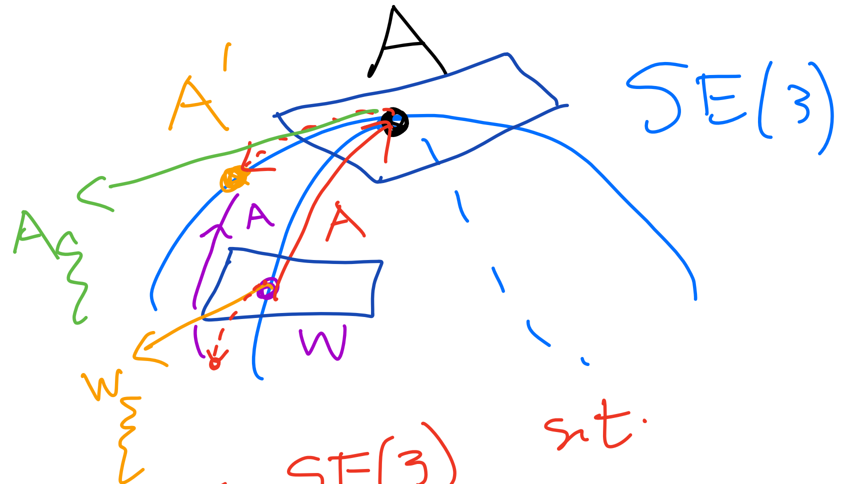




Goal: Find ${}^W\xi \in T_W SE(3)$ s.t.

$$\underbrace{{}^W\xi}_{\text{first increment by } {}^W\xi \text{ in } W} \oplus A = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus {}^A\xi$$

Changes of Reference Frames



Goal: Find ${}^{W}\xi \in T_W SE(3)$ s.t.

$$\underbrace{{}^{W}\xi \oplus A}_{\text{first increment by } {}^{W}\xi \text{ in } W} = A' = \underbrace{A}_{\text{first move by } A \text{ in } W} \oplus\, {}^{A}\xi$$

$$\mathrm{Exp}_W({}^{W}\xi\, W)\cdot A = {}^{W}\xi \oplus A = A \oplus {}^{A}\xi = \mathrm{Exp}_A\big(\overbrace{{}^{A}\xi\, A}^{\mathbb{R}^3}\big)$$

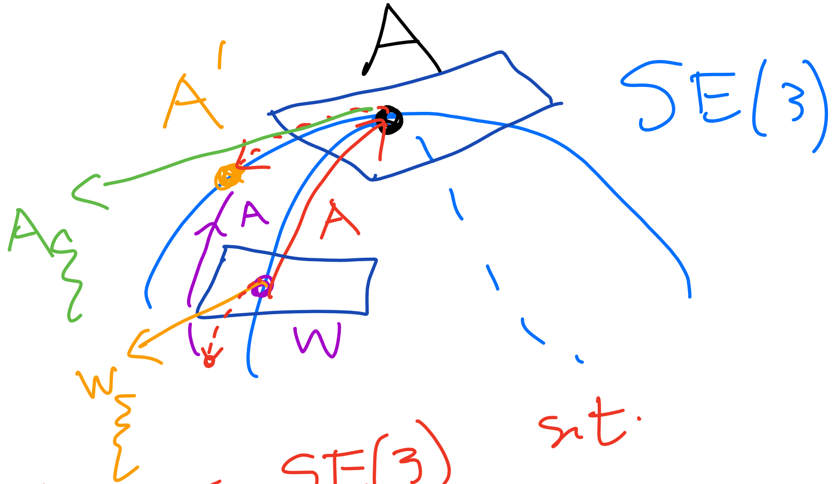
$$= A\cdot \mathrm{Exp}_W\big(\underbrace{{}^{A}\xi\, W}\big)$$

$$\mathrm{Exp}_W({}^{W}\xi\, W)\cdot A = \exp\big(({}^{W}\xi\, W)^\wedge\big)\cdot A$$

$$A\cdot \mathrm{Exp}_W({}^{A}\xi\, W) = A\exp\big(({}^{A}\xi\, W)^\wedge\big) \qquad \hat{\mathbb{R}}^3$$

$$\Rightarrow \exp\big(({}^{W}\xi\, W)^\wedge\big) = A\exp\big(({}^{A}\xi\, W)^\wedge\big)A^{-1}$$

Changes of Reference Frames



Goal: Find ${}^{W}\xi \in T_W SE(3)$ s.t.

$$\underbrace{{}^{W}\xi \oplus A}_{\text{first increment by } {}^{W}\xi \text{ in } W} = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus {}^{A}\xi$$

$$\exp\left(\left({}^{W}\xi\, w\right)^\wedge\right) = A \exp\left(\left({}^{A}\xi\, w\right)^\wedge\right) A^{-1}$$

$$= A \sum_{k=0}^{\infty} \frac{\left(\left({}^{A}\xi\, w\right)^\wedge\right)^k}{k!} A^{-1}$$

$$= \sum_{k=0}^{\infty} \frac{\left(A \left({}^{A}\xi\, w\right)^\wedge A^{-1}\right)^k}{k!}$$

$$= \exp\left(A \left({}^{A}\xi\, w\right)^\wedge A^{-1}\right)$$

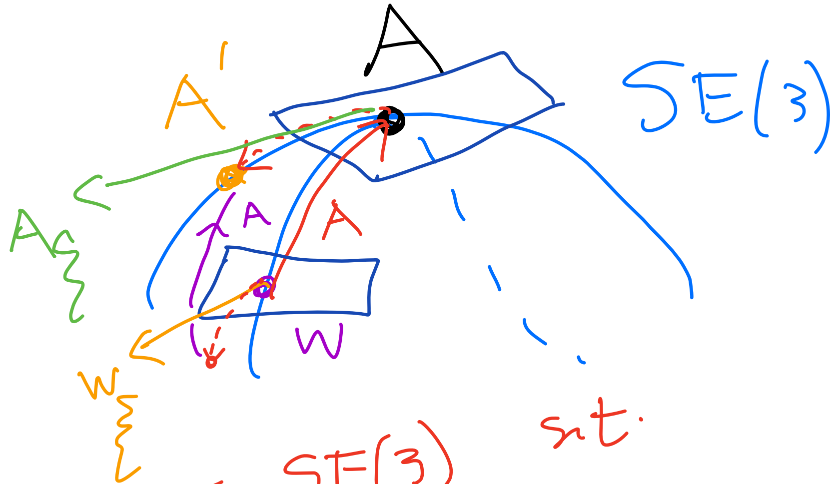
Changes of Reference Frames

Goal: Find ${}^W\xi \in T_W SE(3)$ s.t.

$$\underbrace{{}^W\xi \oplus A}_{\text{first increment by } {}^W\xi \text{ in } W} = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus {}^A\xi$$

$$\underbrace{\left({}^W\xi^W\right)^\wedge}_{\in\, T_W SE(3)} = A \underbrace{\left({}^A\xi^W\right)^\wedge}_{\in\, T_W SE(3)} A^{-1}$$

Changes of Reference Frames



Goal: Find ${}^W\xi \in T_W SE(3)$ s.t.

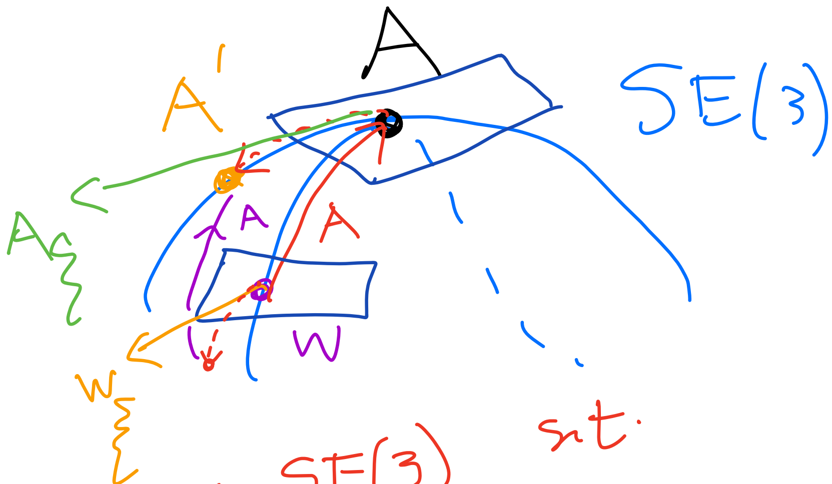
$$\underbrace{{}^W\xi \oplus A}_{\text{first increment by } {}^W\xi \text{ in } W} = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus {}^A\xi$$

$$\underbrace{{}^W\xi^W}_{\in \mathbb{R}^3} = \underbrace{\left(A\,({}^A\xi^W)^\wedge A^{-1}\right)}_{\in \mathbb{R}^3}{}^\vee$$

Define adjoint matrix of A:

$$Ad_A : {}^A\xi^W \longmapsto \left(A\,({}^A\xi^W)^\wedge A^{-1}\right)^\vee$$

Changes of Reference Frames



Goal: Find $^W\xi \in T_W SE(3)$ s.t.

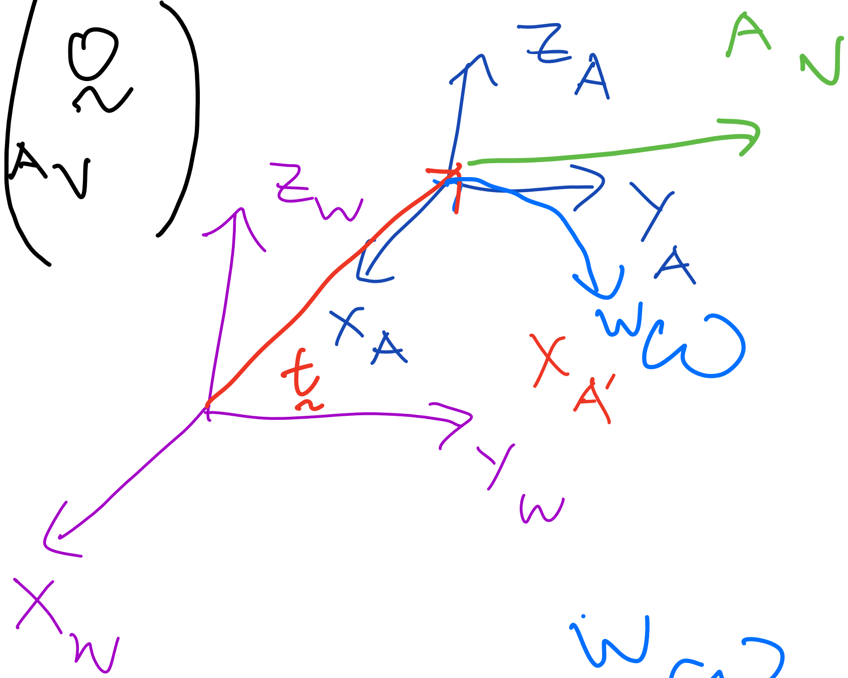
$$\underbrace{{}^W\xi \oplus A}_{\text{first increment by } {}^W\xi \text{ in } W} = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus {}^A\xi$$

$$ {}^W\xi = \underbrace{{}^{(W)}\mathrm{Ad}_A}_{\text{the "correct" change of reference frame factor for velocity vectors}} {}^A\xi $$

Changes of Reference Frames

$$A = \begin{pmatrix} R & \underset{\sim}{t} \\ 0 & 1 \end{pmatrix} \qquad {}^{A}\xi = \begin{pmatrix} \underset{\sim}{0} \\ {}^{A}v \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{I} & \underset{\sim}{t} \\ 0 & 1 \end{pmatrix}$$



$${}^{w}\underset{\sim}{\omega}$$

(micro Lie Theory)

$${}^{w}\xi = \mathrm{Ad}_A\,{}^{A}\xi = \begin{pmatrix} R & [\underset{\sim}{t}]_\times R \\ 0 & R \end{pmatrix}\begin{pmatrix} \underset{\sim}{0} \\ {}^{A}v \end{pmatrix} = \begin{pmatrix} \underset{\sim}{t} \times \underset{\sim}{v} \\ v \end{pmatrix}$$

Adjoint Representation

The adjoint representation is defined as:

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$$

$$\mathrm{Ad}_g(X) = gXg^{-1}$$

For any Lie Group G, the map

$$g \longmapsto \mathrm{Ad}_g$$

"reproduces" the group structure.

Adjoint Representation

The adjoint representation is defined as:

$$Ad_g : \mathfrak{g} \to \mathfrak{g}$$

$$Ad_g(X) = g X g^{-1}$$

$$Ad_{g \cdot h} = Ad_g \circ Ad_h$$

$$Ad_{g^{-1}} = Ad_g^{-1}$$

and by construction:

$$g \oplus \underset{\sim}{v} = (Ad_g \underset{\sim}{v}) \oplus g$$

Summary

- Geometrically, $g \oplus \underset{\sim}{v}$ gives the "closest" element $g' \in G$ after moving away from g in direction $\underset{\sim}{v}$ if G has a bi-invariant metric
- Since Exp_p is "hard" to compute, can instead use the fact that bi-invariant Lie Groups satisfy $\exp = \mathrm{Exp}_I$, and exp is easy to compute.
- If only interested in "linearising" G, can instead work w/ its Lie Algebra $\mathfrak{g}$ and exp.
- To change reference frames for velocity vectors, we use the adjoint matrix Ad_g.