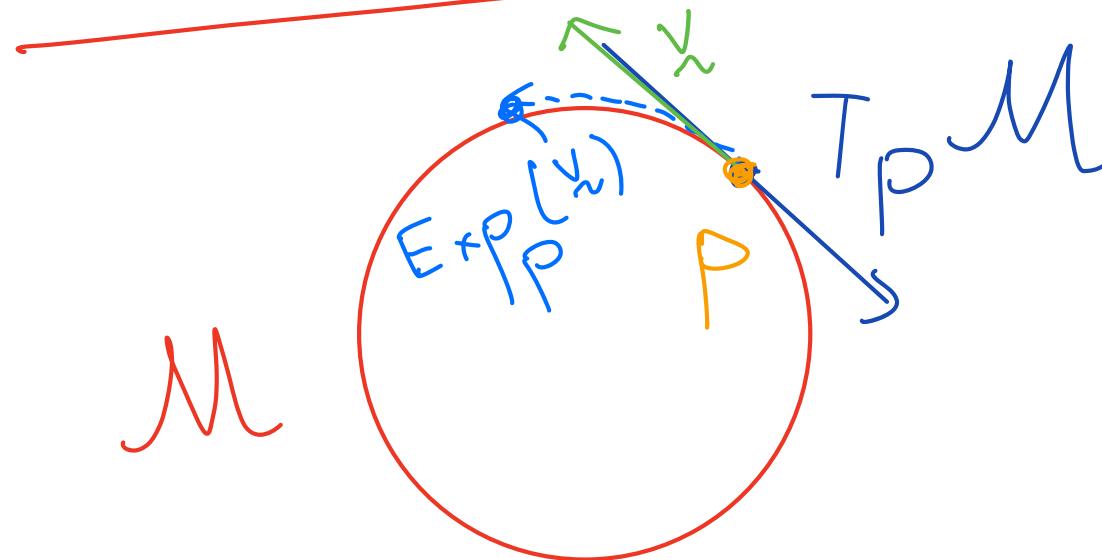


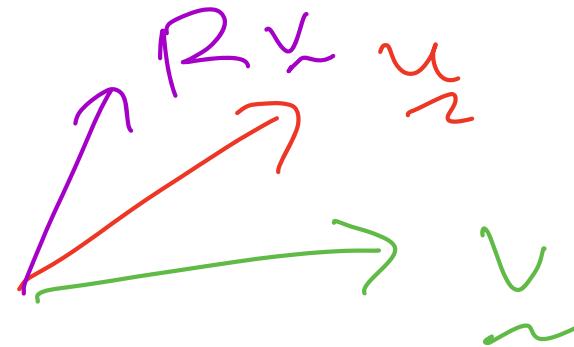
# Lie Theory for Control & Estimation



By Sepehr  
Saryazdi

Motivation:

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I \\ \det R = 1 \end{array} \right\}$$



How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \| R \underline{v} - \underline{u} \|^2 ?$$

Motivation:

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I \\ \det R = 1 \end{array} \right\}$$

Flow to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \| R \underline{x} - \underline{y} \|^2 ?$$

Motivation:

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How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \|Rv - u\|^2 ?$$

Naive method:

$$f(r_{11}, \dots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} v - u \right\|^2$$

Solve  $\frac{\partial f}{\partial r_{ij}} = 0$ . This is hard!

Motivation:

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How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \| R \underline{v} - \underline{u} \|^2 ?$$

Improved method w/ grad. descent:

$$f(r_{11}, \dots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \underline{v} - \underline{u} \right\|^2$$

$$\underline{r} := (r_{11}, \dots, r_{33}), \quad \underline{r}_{n+1} = \underline{r}_n - \gamma \nabla f(\underline{r}_n)$$

Motivation:

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I, \\ \det R' = 1 \end{array} \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \| R v_n - u_n \|^2 ?$$

Improved method w/ grad. descent:

$$f(r_{11}, \dots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} v_n - u_n \right\|^2$$

$$\tilde{r}_{n+1} = r_n - \gamma \nabla f(r_n) = (r'_{11}, \dots, r'_{33}), \quad R' \notin SO(3) !$$

Motivation:

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I, \\ \det R' = 1 \end{array} \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \|R \tilde{v} - \tilde{u}\|^2 ?$$

Improved method w/ grad. descent:

$$f(r_{11}, \dots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{pmatrix} \tilde{v} \\ \tilde{u} \end{pmatrix} \right\|^2$$

Want:  $\left( \begin{pmatrix} r \\ \tilde{r} \end{pmatrix} \right)_{n \in \mathbb{N}}$  s.t.  $\tilde{r}_n \in SO(3)$  &  $r_n \rightarrow \operatorname{argmin}_r f$

# Motivation:

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I \\ \det R' = 1 \end{array} \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \|R \tilde{v} - v\|^2 ?$$

Want:  $(r_n)_{n \in \mathbb{N}}$  s.t.  $\tilde{v}_n \in SO(3)$   
 $r_n \rightarrow \underset{r}{\operatorname{argmin}} f$

Idea: Define  $\oplus$  s.t.  
 $R \oplus dR \in SO(3)$

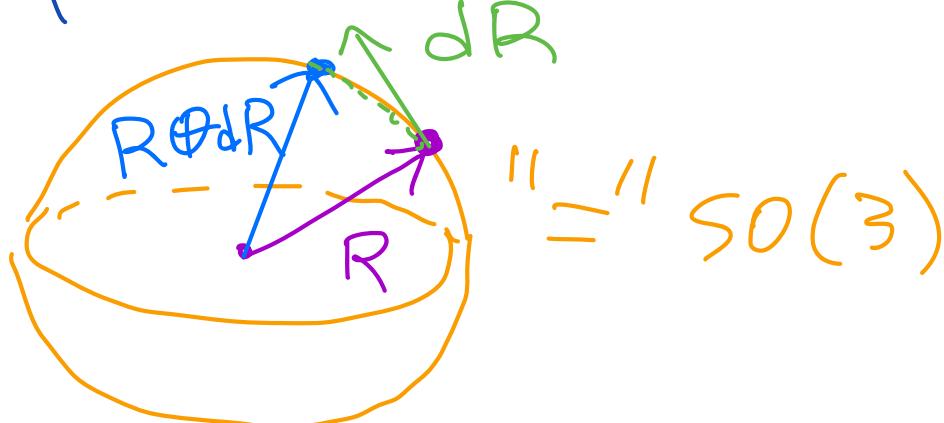
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Want:  $(r_n)_{n \in \mathbb{N}}$  s.t.  $r_n \in SO(3)$   
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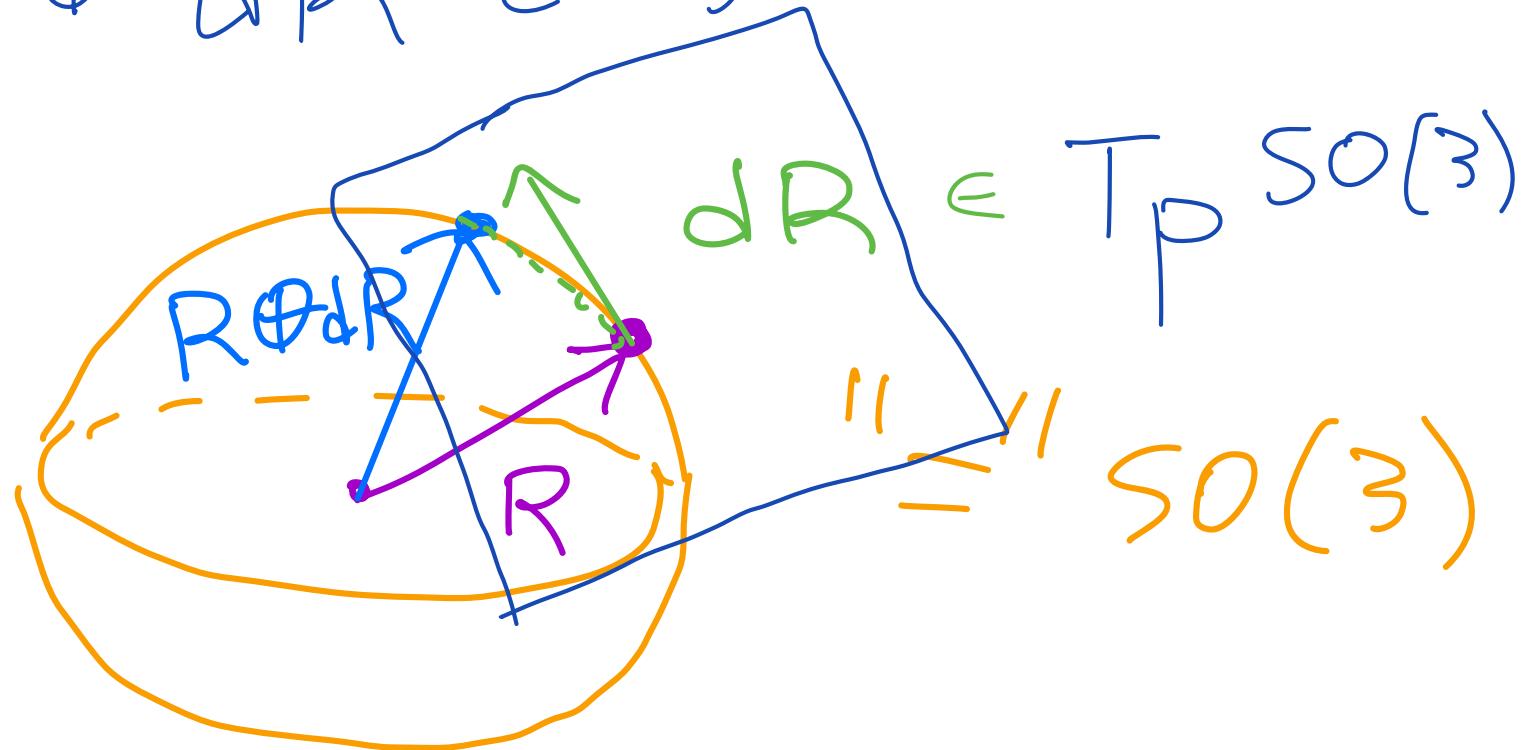


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Motivation:

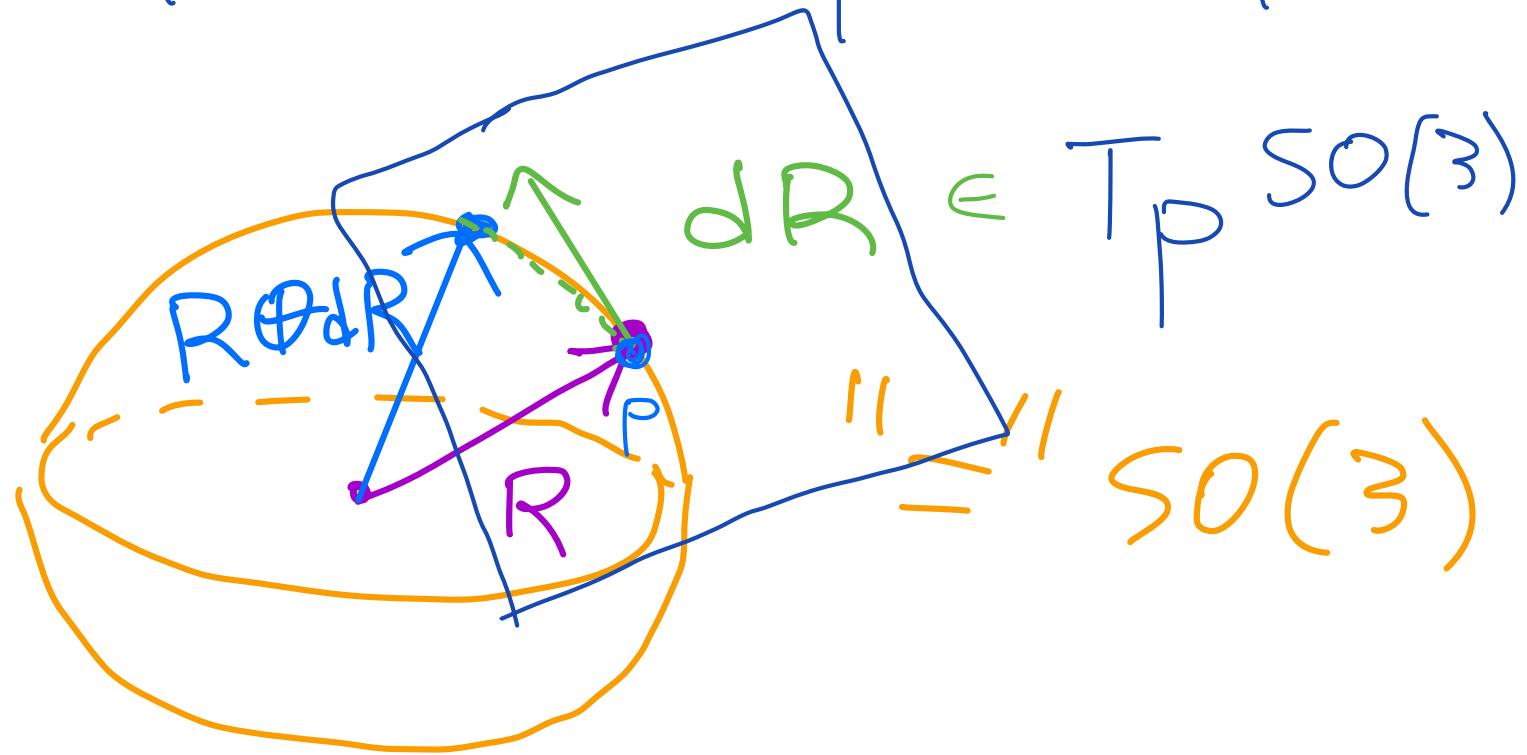
$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I \\ \det R' = 1 \end{array} \right\}$$

Idea: Define  $\oplus$  s.t.  
 $R \oplus dR \in SO(3)$

Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I \\ \det R = 1 \end{array} \right\}$$

Q: How to compute  $T_p SO(3)$ ?

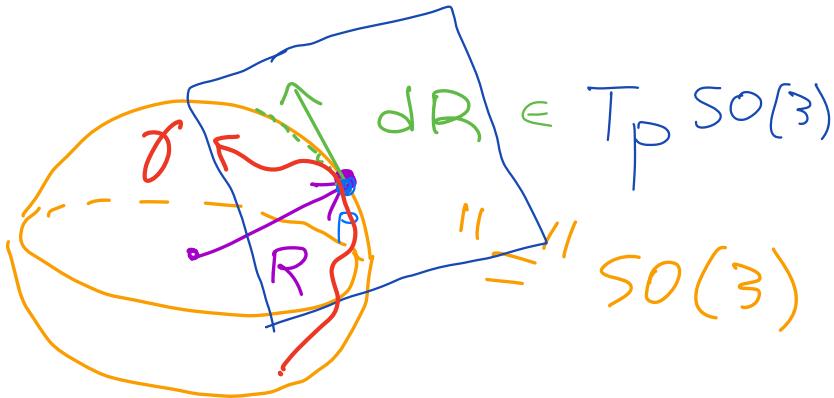


Think of "velocity" vectors  
at a point  $p \in SO(3)$ .

Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I \\ \det R' = 1 \end{array} \right\}$$

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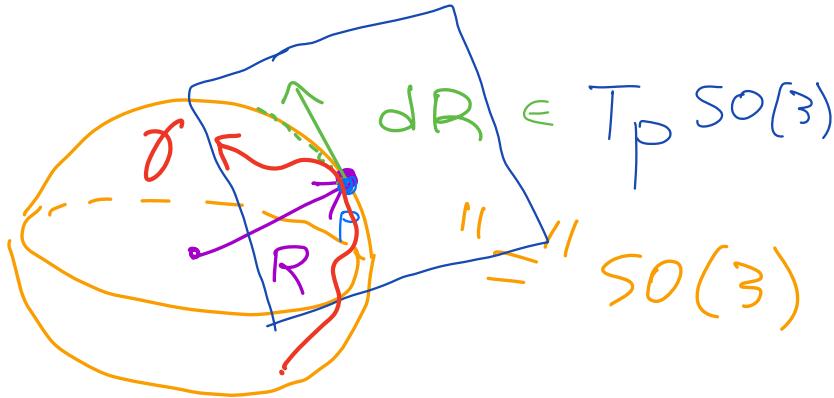
Think of "velocity" vectors of a curve at a point  $p_{SO(3)}$ .

$$\gamma: (-\epsilon, \epsilon) \rightarrow SO(3)$$

Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I \\ \det R = 1 \end{array} \right\}$$

Q: How to compute  $T_p SO(3)$ ?



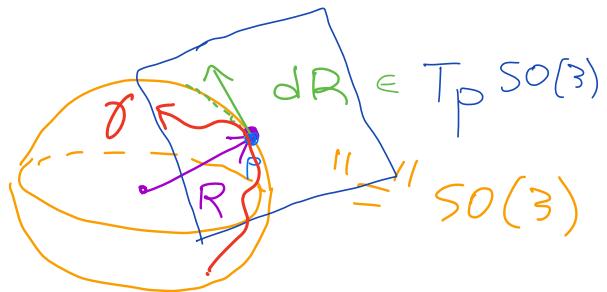
Think of "velocity" vectors of a curve at a point  $p \in SO(3)$ .

$$\gamma: (-\epsilon, \epsilon) \rightarrow SO(3), \text{ e.g. } \gamma(t) = R_z^{(t)} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$P = \gamma(0) = I$$

Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid R^T R = I, \det R' = 1 \right\}$$

Q: How to compute  $T_p SO(3)$ ?



Think of "velocity" vectors of a curve at a point  $P \in SO(3)$ .

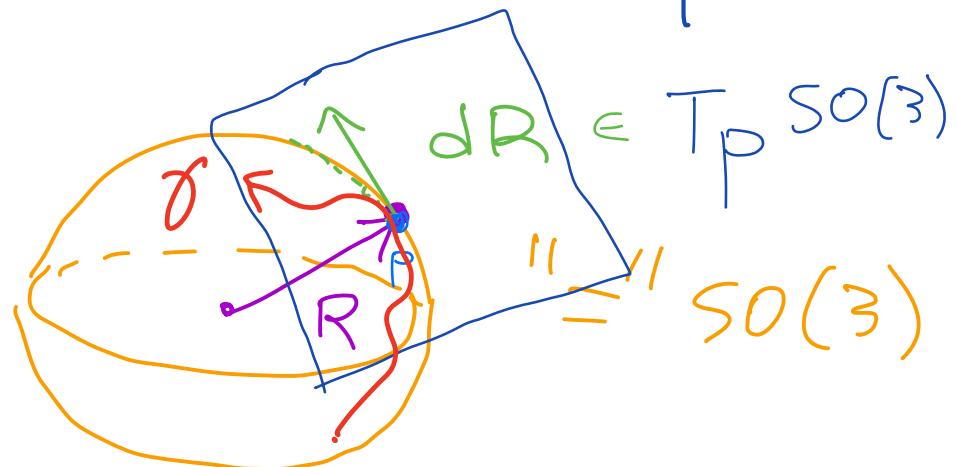
$$\gamma: (a, b) \rightarrow SO(3), \text{ e.g. } \gamma(t) = R_z^{(t)} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, P = \gamma(0)$$

$$\gamma'(t) = \begin{bmatrix} -\sin t & -\cos t & 0 \\ \cos t & -\sin t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \gamma'(0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid R^T R = I, \det R = 1 \right\}$$

Q: How to compute  $T_p SO(3)$ ?



$$\gamma: (a, b) \rightarrow SO(3), \text{ e.g. } \gamma(t) = R_z^{(t)} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P = \gamma(0) = I$$

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$dR$

# Computing $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I \\ \det R = 1 \end{array} \right\}$$

Q: How to compute  $T_p SO(3)$ ?

$$\mathcal{J}_Z(t) = R_Z^{(t)} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P = \mathcal{J}(0) = I, \quad \mathcal{J}'_Z(0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{J}_Y(t) = R_Y(t) = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix}, \quad \mathcal{J}'_Y(0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\mathcal{J}_X(t) = R_X^{(t)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}, \quad \mathcal{J}'_X(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

# Computing $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid R^T R = I, \det R = 1 \right\}$$

Q: How to compute  $T_p SO(3)$ ?

$$\gamma_z(t) = R_z(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P = \gamma_z(0) = I, \quad \gamma'_z(0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\gamma_y(t) = R_y(t) = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix}, \quad \gamma'_y(0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\gamma_x(t) = R_x(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}, \quad \gamma'_x(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

what about any  $\gamma$ ?

$$\gamma(t) = R_x(\theta_1(t)) R_y(\theta_2(t)) R_z(\theta_3(t))$$

Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I, \\ \det R = 1 \end{array} \right\}$$

Q: How to compute  $T_p SO(3)$ ?

what about any  $\mathcal{T}$  passing through  $P = I$ ?

$$\mathcal{T}(t) = R_x(\theta_1(t)) R_y(\theta_2(t)) R_z(\theta_3(t))$$

$$\mathcal{T}(t) = \mathcal{T}_x(\theta_1(t)) \mathcal{T}_y(\theta_2(t)) \mathcal{T}_z(\theta_3(t))$$

$$\begin{aligned} \mathcal{T}'(t) &= \mathcal{T}_x'(\theta_1(t)) \cdot \mathcal{T}_y(\theta_2(t)) \cdot \mathcal{T}_z(\theta_3(t)) \cdot \theta_1'(t) \\ &\quad + \mathcal{T}_x(\theta_1(t)) \cdot \mathcal{T}_y'(\theta_2(t)) \cdot \mathcal{T}_z(\theta_3(t)) \cdot \theta_2'(t) \end{aligned}$$

$$+ \mathcal{T}_x(\theta_1(t)) \cdot \mathcal{T}_y(\theta_2(t)) \cdot \mathcal{T}_z'(\theta_3(t)) \cdot \theta_3'(t)$$

# Computing $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid R^T R = I, \det R = 1 \right\}$$

Q: How to compute  $T_p SO(3)$ ?

what about any  $\gamma$  passing through  $p = I$ ?

$$\gamma(t) = \gamma_x(\theta_1(t)) \gamma_y(\theta_2(t)) \gamma_z(\theta_3(t))$$

$$\gamma'(0) = \cancel{\gamma'_x(0)} \cdot \cancel{\gamma_x(0)} \overset{I}{\cancel{\gamma_y(0)}} \cdot \cancel{\gamma_z(0)} \overset{I}{\cancel{\cdot \theta'_1(0)}} +$$

$$+ \cancel{\gamma_x(0)} \cdot \cancel{\gamma'_y(0)} \cdot \cancel{\gamma_y(0)} \overset{II}{\cancel{\gamma_z(0)}} \overset{II}{\cancel{\cdot \theta'_2(0)}}$$

$$+ \cancel{\gamma_x(0)} \overset{II}{\cancel{\cdot \gamma'_y(0)}} \overset{II}{\cancel{\cdot \gamma_y(0)}} \cancel{\gamma_z(0)} \overset{II}{\cancel{\cdot \theta'_3(0)}}$$

$$\gamma'(0) = \gamma'_x(0) \theta'_1(0) + \gamma'_y(0) \theta'_2(0) + \gamma'_z(0) \theta'_3(0)$$

Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I, \\ \det R = 1 \end{array} \right\}$$

Q: How to compute  $T_p SO(3)$ ?  
what about any  $\mathcal{T}$  passing through  $p = I$ ?

$$\begin{aligned}\mathcal{T}'(0) &= \mathcal{T}_x'(0)\theta_1'(0) + \mathcal{T}_y'(0)\theta_2'(0) + \mathcal{T}_z'(0)\theta_3'(0) \\ &= \mathcal{T}_x'(0)\omega_x + \mathcal{T}_y'(0)\omega_y + \mathcal{T}_z'(0)\omega_z \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_z \\ &= \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}\end{aligned}$$

Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid R^T R = I, \det R = 1 \right\}$$

Q: How to compute  $T_p SO(3)$ ?

As a vector space:

$$\mathcal{J}'(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_z$$

$\text{Lie Algebra of } SO(3) := T_p \mathbb{R}^3$

$$SO(3) = \text{span}_{\mathbb{R}} \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$
$$= \text{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial \alpha}(I), \frac{\partial}{\partial \beta}(I), \frac{\partial}{\partial \gamma}(I) \right\}$$
$$= \text{span}_{\mathbb{R}} \left\{ E_1(I), E_2(I), E_3(I) \right\}$$

Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I, \\ \det R = 1 \end{array} \right\}$$

Q: How to compute  $T_p SO(3)$ ?  
what about any  $\gamma$  passing through  $p = I$ ?

Representing vectors

$$\chi = (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$$

$$\tilde{\gamma} := v_1 E_1 + v_2 E_2 + v_3 E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_z$$

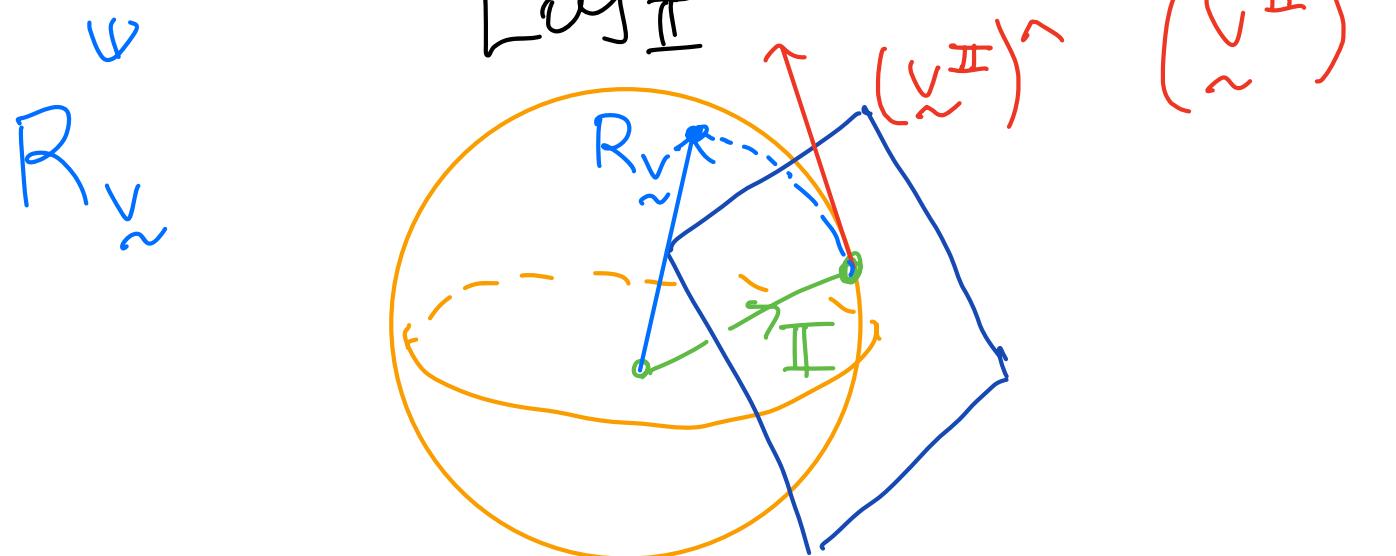
$$(\tilde{\gamma})^\vee = \chi \in T_{\tilde{\gamma}} SO(3)$$

Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid R^T R = I, \det R = 1 \right\}$$

$$SO(3) \xrightarrow{\frac{d}{dt} \mid I} T_I SO(3)$$

$$SO(3) \xleftarrow[\text{LOG } I]{\text{EXP}_I} T_I SO(3)$$



# Computing $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I, \\ \det R = 1 \end{array} \right\}$$

$$SO(3) \xleftarrow{Exp_{I\!\!I}} T_{I\!\!I} SO(3)$$

$R_{\tilde{v}}$        $\tilde{v}$   
 $Exp_{I\!\!I}^{(I\!\!I)\tilde{v}} := \gamma_{\tilde{v}}(1), \quad \gamma_{\tilde{v}}^{(I\!\!I)\tilde{v}}$ ,  $\gamma_{\tilde{v}}^{(I\!\!I)\tilde{v}}$  a geodesic  
 w/ const. velocity  $\tilde{v}$

$$I\!\!I \oplus I\!\!I \tilde{v} := Exp_{I\!\!I}^{(I\!\!I)\tilde{v}} = R_{\tilde{v}}$$

Computing  $T_p SO(3)$ :

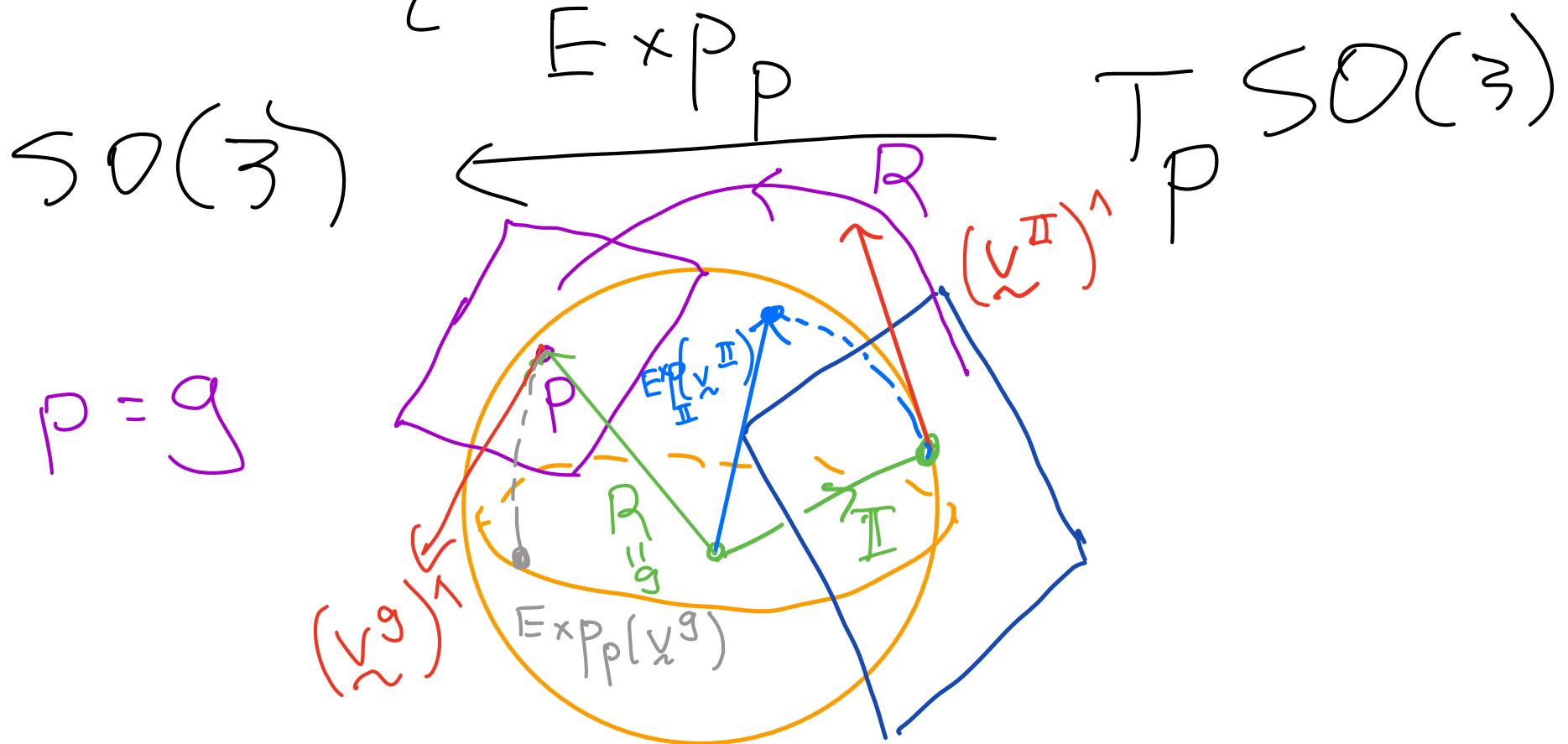
$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid \begin{array}{l} R^T R = I \\ \det R = 1 \end{array} \right\}$$

$$SO(3) \xleftarrow{E \times P_I} T_{\tilde{v}}^{SO(3)} \quad \begin{matrix} \tilde{v} \\ \sim \end{matrix} \quad \begin{matrix} \tilde{v} \\ \sim \end{matrix}^{I^{\oplus 2}}$$

$$E \times_P (\tilde{v}) = \gamma_{\tilde{v}}(1) \quad \tilde{v} \text{ a geodesic const. } w/ \text{ velocity } \dot{v}$$

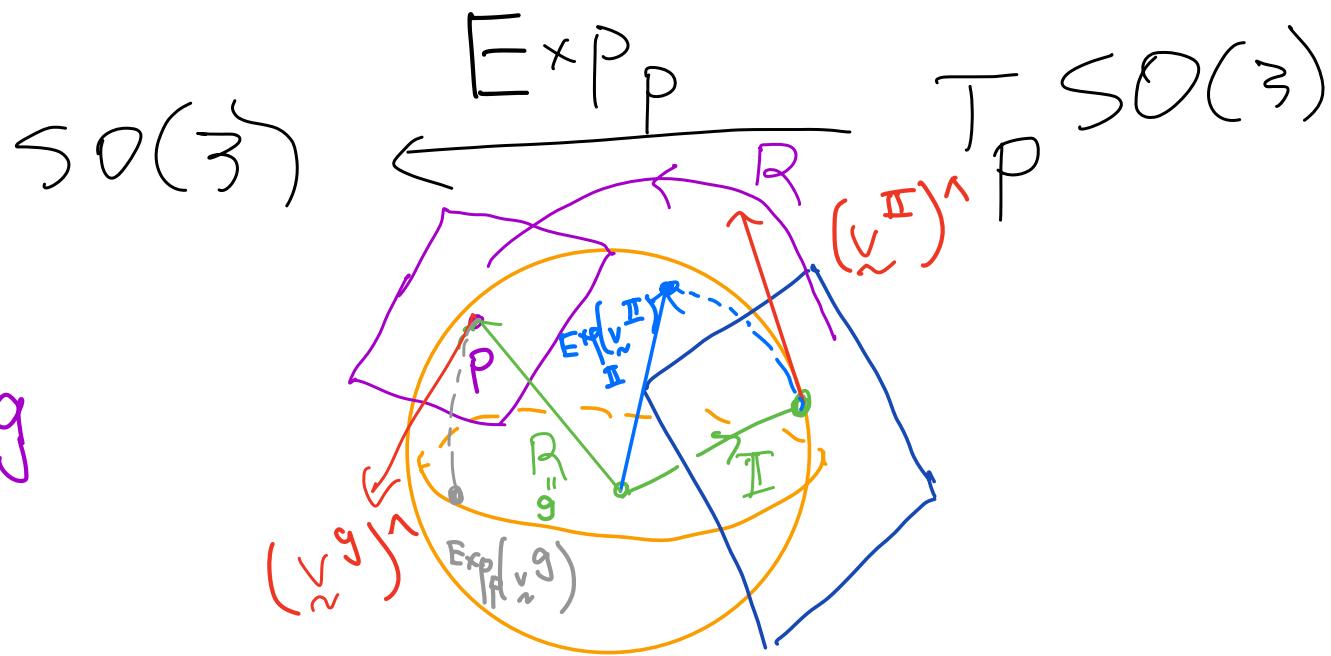
Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid R^T R = I, \det R = 1 \right\}$$



Computing  $T_p SO(3)$ :

$$SO(3) = \left\{ R \in \text{Mat}_3(\mathbb{R}) \mid R^T R = I, \det R' = 1 \right\}$$



$$R \oplus \tilde{v}^R := R \cdot \exp_{\frac{R}{2}}(\tilde{v}^{\text{II}}) \stackrel{(*)}{=} \exp_R(\tilde{v}^R)$$

why? we'll  
return to this later...

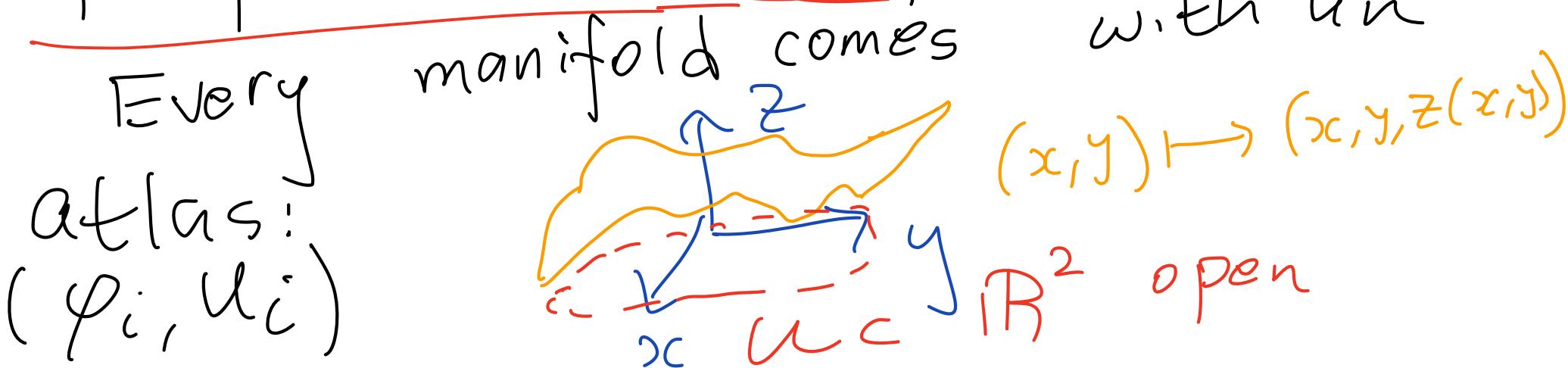
# Properties of $\text{EXP}_p : T_p \overset{SO(3) \rightarrow SO(3)}{\longrightarrow}$

$$\text{EXP}_p(\underline{v^p}) := \underline{x}_{\underline{v^p}}(1)$$

$\underline{x}$  a geodesic

How do we actually  
compute  $\text{EXP}_p(\underline{v^p})$  for  
any manifold  $M$ ?

Properties of  $\text{EXP}_p: T_p \xrightarrow{\text{SO}(3) \rightarrow \text{sq}(3)}$



$\text{SO}(3)$  as a manifold has a chart given by

$$\text{SO}(3)$$

$$\varphi: (0, 2\pi)^3 \rightarrow \text{SO}(3)$$

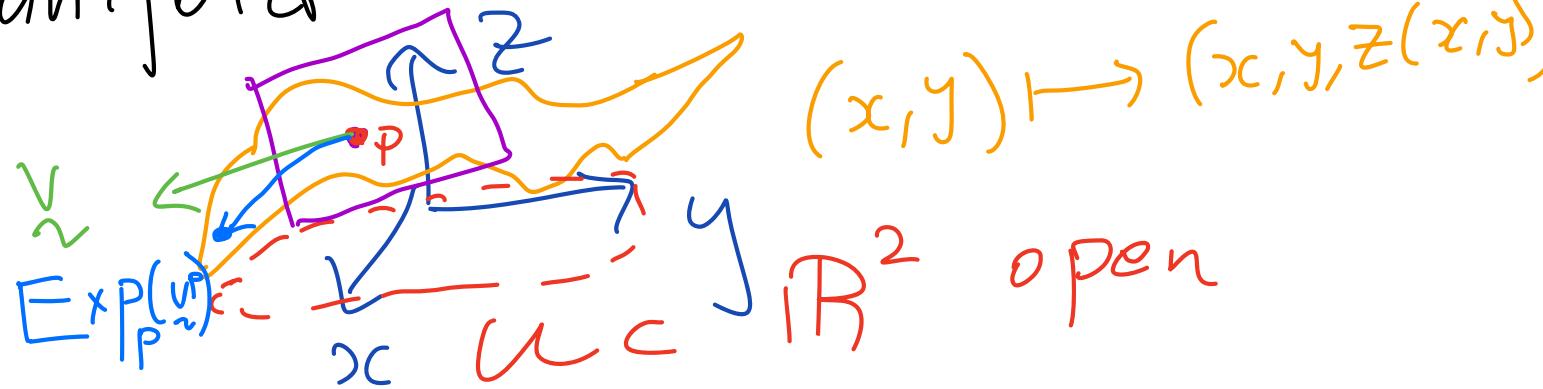
$$\varphi(\alpha, \beta, \gamma) = R_x(\alpha) R_y(\beta) R_z(\gamma)$$

Properties of  $\text{EXP}_P$ :  $T_P \xrightarrow{\text{SO}(3) \rightarrow \text{SO}(3)}$

Every manifold  $M$  comes with an

atlas:

$$(\varphi_i, U_i)$$



If  $\gamma$  is given, then

$\text{EXP}_P(\gamma_P)$  can be computed

using the geodesic equation:

$$\begin{aligned} p &= \dot{\varphi}(u'(0), \dots, u^n(0)) \\ \gamma &= [v^1, \dots, v^n] \end{aligned}$$

initial condition of ODE Christoffel Symbols

ODE

$$\frac{d^2 u^k}{dt^2} + \sum_{ij} \Gamma_{ij}^k \frac{du^i}{dt} \frac{du^j}{dt} = 0,$$

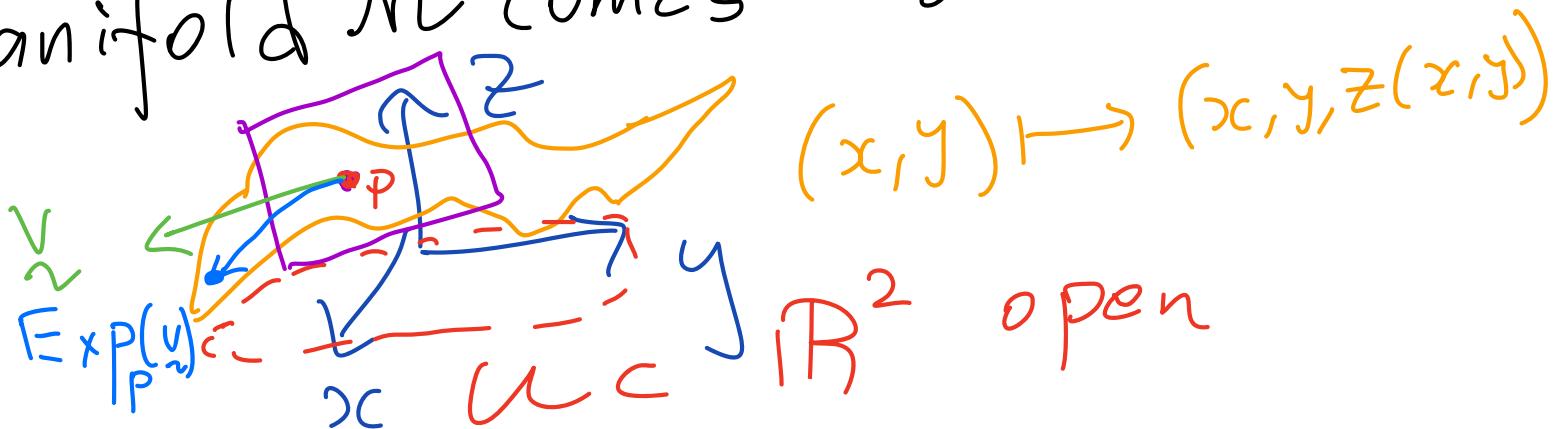
def<sup>n</sup> of  $\Gamma_{ij}^k$

$$\begin{aligned} \varphi_{u^i u^j} &= \sum_k \Gamma_{ij}^k E_k \\ T_P M &= \text{span}_{\mathbb{R}} \{E_1, \dots, E_n\} \end{aligned}$$

Properties of  $\text{EXP}_P$ :  $T_P \xrightarrow{\text{SO}(3) \rightarrow \text{SO}(3)}$  with an

Every manifold  $M$  comes

atlas:  
 $(\varphi_i, U_i)$



# Properties of EXP<sub>P</sub>: $T_P \xrightarrow{SO(3) \rightarrow SO(3)}$

How to compute  $\text{EXP}_{\mathbb{P}}((\tilde{v}^{\mathbb{I}})^{\wedge})$  in an easier way?

$$\tilde{v}^{\mathbb{I}} = (\omega_x, \omega_y, \omega_z)$$

$$(\tilde{v}^{\mathbb{I}})^{\wedge} = \omega_x E_1 + \omega_y E_2 + \omega_z E_3 \in SO(3)$$

want:  $R(t) \approx \mathbb{I} + \frac{dR(0)}{dt} t$  (linearising  $SO(3)$  at  $\mathbb{I}$ )

$$= \mathbb{I} + (\tilde{v}^{\mathbb{I}})^{\wedge} t$$

The ODE  $\dot{R} = R (\tilde{v}^{\mathbb{I}})^{\wedge}$  w/  $R(0) = \mathbb{I}$  has unique solution given by:

$$\exp((\tilde{v}^{\mathbb{I}})^{\wedge}) = \sum_{k=0}^{\infty} \frac{((\tilde{v}^{\mathbb{I}})^{\wedge})^k}{k!} \in SO(3) \quad (\text{Regular exponential})$$

for small  $\|\tilde{v}^{\mathbb{I}}\|$

# Properties of $\text{Exp}_P$ : $T_P \xrightarrow{\text{SO}(3) \rightarrow \text{SO}(3)}$

$$\exp((\tilde{v}^{\text{II}})) = \sum_{k=0}^{\infty} \frac{((\tilde{v}^{\text{II}}))^k}{k!} \quad (\text{easy to calculate})$$

Equivalence of  $\text{Exp}$  &  $\text{Exp}_{\text{II}}$

If  $G$  is a Lie group w/ bi-invariant Riemannian metric, then

$$\exp((\tilde{v}^{\text{II}})) = \text{Exp}_{\text{II}}(\tilde{v}^{\text{II}})$$

Milnor, 1976

A Lie group  $G$  admits a group  
bi-invariant metric iff  $G \cong K \times H$   
 where  $K$  is a compact group  
 &  $H$  abelian.

# Properties of EXP<sub>P</sub>: $T_P \xrightarrow{SO(3) \rightarrow SO(3)}$

compactness

$SO(3)$  is compact &  $\det R = 1$

$$R^T R = \mathbb{I}$$

$R$  has bounded matrix norm  
 $\Rightarrow SO(3)$  is bounded.

$r := (r_{11}, \dots, r_{33})$   
equations  
 $\Rightarrow SO(3)$

satisfies  $SO(3)$  is closed & bounded  $\Rightarrow$  compact.  
is closed. polynomial

# Properties of $\text{Exp}_p : T_p \xrightarrow{\text{SO}(3) \rightarrow \text{SO}(3)}$

For Lie groups  $G$  w/ bi-invariant Riemannian metrics (e.g.  $\text{SO}(3)$ ),  
the geodesic equation (Michael Greis,  
has the solution Notes on Riemannian Geom  
of Lie Groups)

$$\gamma(t) = \underbrace{\text{Exp}_g^{(t\tilde{v})}}_{\text{geodesics starting at } g} = g \circ \underbrace{\text{Exp}_e^{(t\tilde{v}^{\text{II}})}}_{\text{geodesics starting at II}}$$

$$g \oplus g_{\tilde{v}} := g \circ \text{Exp}_e^{(\tilde{v}^{\text{II}})} \quad (\text{before}) \\ = \text{Exp}_g^{(\tilde{v}^g)}$$

# Properties of $\text{Exp}_P : T_P \xrightarrow{\text{SO}(3)} \text{sq}(3)$

$$\gamma(t) = \text{Exp}_g^{(t\tilde{v})} = g \cdot \text{Exp}_e^{(t\tilde{v}^I)}$$

*geodesics starting at  $g$*

$$\text{Exp}_e^{(t\tilde{v}^I)}$$

*geodesics starting at  $e$*

How is  $\tilde{v}^I$  related to  $\tilde{v}^g$ ?  
 As elements related by  $d(Lg^{-1})_g$ ,  
 they are related by  $T_g \text{SO}(3)$ ,  
 i.e.  $(\tilde{v}^I)^g = d(Lg^{-1})_g(\tilde{v}^g)$ .

For  $\text{SO}(3)$ ,  $d(Lg^{-1}) = R^{-1}$ .

# Properties of $\text{Exp}_P : T_p \overset{SO(3) \rightarrow SO(3)}{\longrightarrow}$

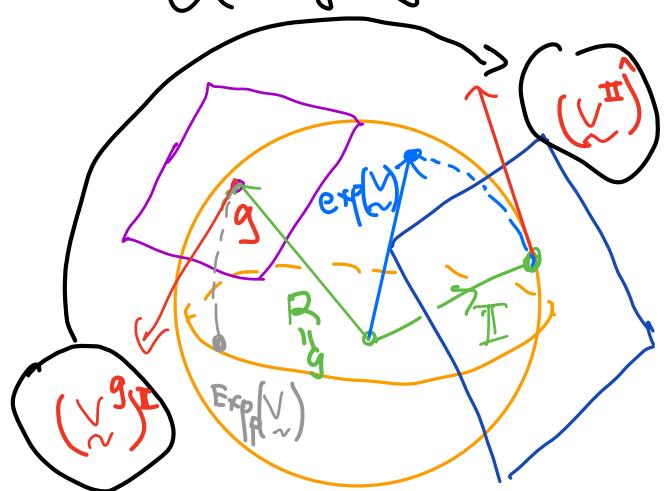
$$g(t) = \text{Exp}_g^{(t\tilde{v})} = g \circ \text{Exp}_e^{(t\tilde{v}^{\text{II}})}$$

$\tilde{v}$        $\tilde{v}^{\text{II}}$

geodesic  $\hookrightarrow$   
starting at  $g$

geodesics  
starting at  $\text{II}$

How is  $\tilde{v}^{\text{II}}$  related to  $\tilde{v}^g$ ?  
 As  $\tilde{v}^{\text{II}}, \tilde{v}^g \in \mathbb{R}^3$ , they coincide  
 for bi-invariant Lie Groups.



i.e.

$$\begin{aligned}\tilde{v}^g &= (\omega_x, \omega_y, \omega_z) \\ &= \tilde{v}^{\text{II}}\end{aligned}$$

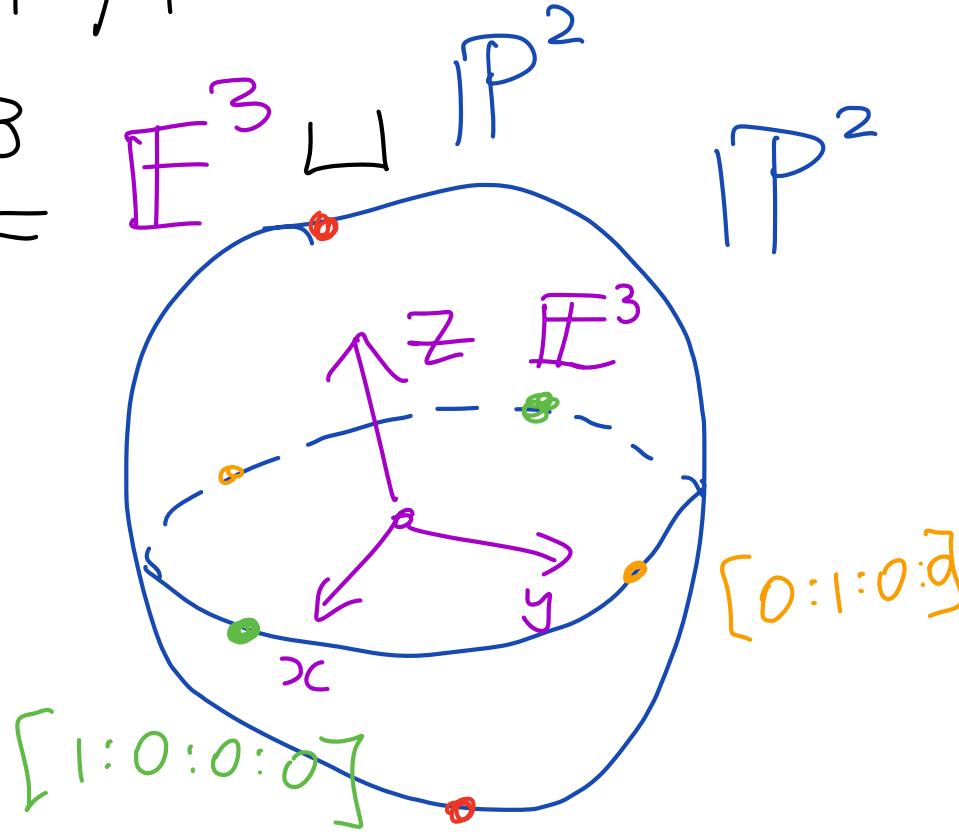
## Other Examples: SE(3)

$$SE(3) = \left\{ \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \mid R \in SO(3), t \in \mathbb{R}^3 \right\}$$

$$SE(3) \curvearrowright \mathbb{P}^3 = \mathbb{E}^3 \sqcup \mathbb{P}^2$$

$$\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \cdot [x:y:z:1]$$

$$\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$



$$\lim_{x \rightarrow \infty} [x:y:z:1] = \lim_{x \rightarrow \infty} \left[ 1: \frac{y_0}{x}: \frac{z_0}{x}: \frac{1}{x} \right] = [1:0:0:0]$$

Other Examples: SE(3)

$$SE(3) = \left\{ \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \mid \begin{array}{l} R \in SO(3), \\ t \in \mathbb{R}^3 \end{array} \right\}$$

$$SE(3) \curvearrowright \mathbb{P}^3 = \mathbb{E}^3 \sqcup \mathbb{P}^2$$

Other Examples: SE(3)

$$SE(3) \cong SO(3) \times \underbrace{\mathbb{R}^3}_{\text{abelian}}$$

$SE(3)$

$$\begin{aligned} T_{\mathbb{R}}^{SE(3)} &= \text{span}_{\mathbb{R}} \left\{ \frac{d}{dt} \Big|_{t=0} \begin{pmatrix} R_X^{(t)} & 0 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt} \Big|_{t=0} \begin{pmatrix} R_Y^{(t)} & 0 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt} \Big|_{t=0} \begin{pmatrix} R_Z^{(t)} & 0 \\ 0 & 1 \end{pmatrix}, \right. \\ &\quad \left. \frac{d}{dt} \Big|_{t=0} \begin{pmatrix} I & t e_1 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt} \Big|_{t=0} \begin{pmatrix} I & t e_2 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt} \Big|_{t=0} \begin{pmatrix} I & t e_3 \\ 0 & 1 \end{pmatrix} \right\} \\ &= \text{span} \left\{ \frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta}, \frac{\partial}{\partial \gamma}, \frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta}, \frac{\partial}{\partial \gamma} \right\} \end{aligned}$$

$$= \text{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

## Other Examples: SE(3)

A twist  $\zeta$  is given as:

$$\zeta := \omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z}$$

$$+ v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

$$= \begin{pmatrix} 0 & -\omega_z & \omega_y & v_x \\ \omega_z & 0 & -\omega_x & v_y \\ -\omega_y & \omega_x & 0 & v_z \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

# Properties of Lie Algebras:

Matrix group multiplication can be "reproduced" with Lie Algebra multiplication:

## Baker-Campbell-Hausdorff:

$$x \circ y \longleftrightarrow X + Y + \frac{1}{2}[X, Y] + \frac{1}{12} [X, [X, Y]] + \frac{1}{12} [Y, [Y, X]] + \dots$$

w/  $X = \log x, Y = \log y$ ,  $[X, Y] := XY - YX$

# Properties of Lie Algebras:

## Hopf-Rinow

If  $M$  is a connected Riemannian manifold, then  $\text{Exp} : T_p M \rightarrow M$  is surjective for every  $p \in M$ .

## Corollary

If  $G$  is a connected Lie Group with a bi-invariant Riemannian metric, then  $\exp : \mathfrak{g} \rightarrow G$  is surjective. Hence we can "globally" linearise  $G$  by working with its Lie Algebra  $\mathfrak{g}$ .

# Changes of Reference Frames

Q: How can we compute change of reference frame?

For a pose  $B_T_A \in SE(3)$ , we

can just do

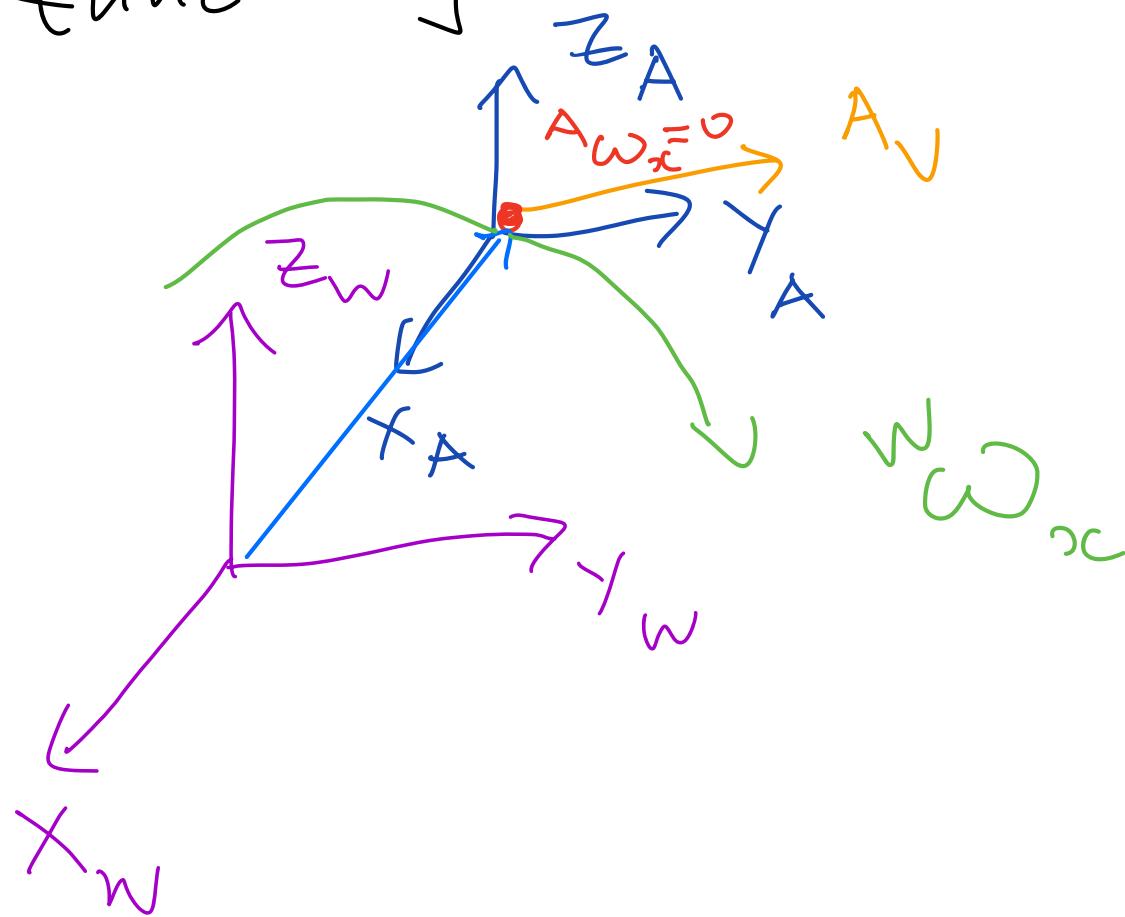
$$B_p = T_A P$$

What about twists  $\zeta$ ?

$$B_\zeta = ? \cdot A_\zeta$$

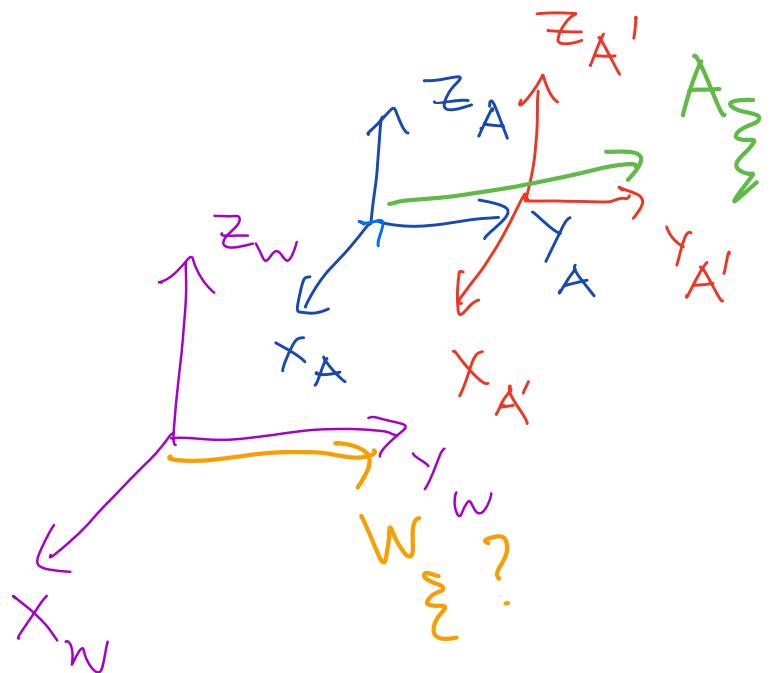
## Changes of Reference Frames

In frame A, it is not rotating. However, it appears to "rotate" within world frame W. instantaneously



Changes of Reference Frames

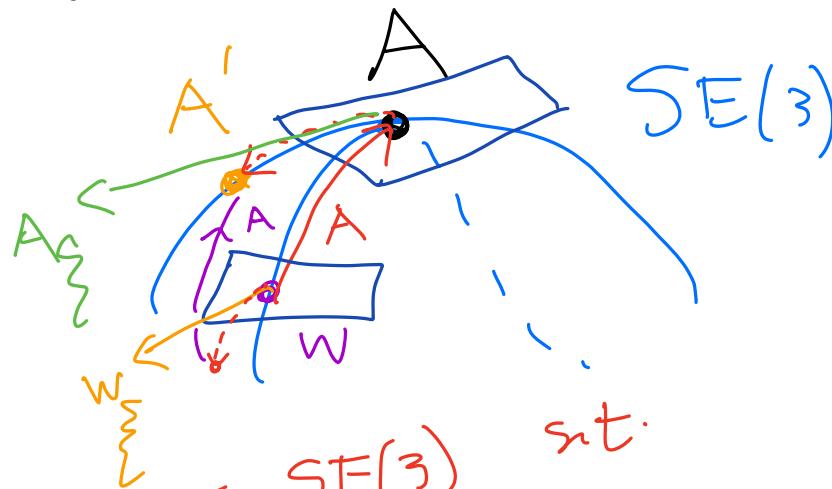
Frame  $A'$  = Frame A after  $dt$   
 $\in \text{SE}(3)$



Goal: Find  ${}^W_E E, {}^T_W T$  s.t.  $A' = A + {}^W_S S$

first increment by  ${}^W_S S$  in  $W$ .      first move by  $A$  in  $W$ .

# Changes of Reference Frames

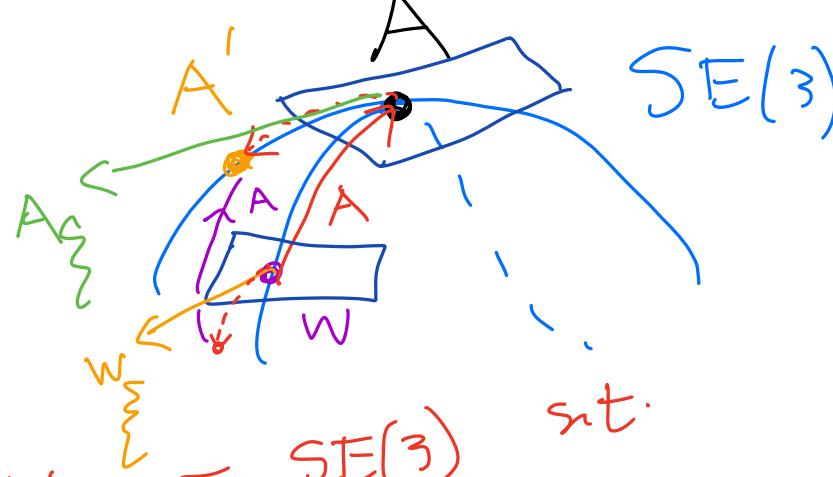


Goal: Find  ${}^W_E \tau_W$  s.t.  
 ${}^W_E \oplus A = A' = A \oplus A_S$   
 first increment by  ${}^W_E$  in  $W$       first move by  $A$  in  $W$ .

$\mathbb{R}^3$

$$\begin{aligned} \text{Exp}_W({}^W_E w) \cdot A &= {}^W_E \oplus A = A \oplus A_E = \text{Exp}_A({}^A_E A) \\ &\quad A \cdot \text{Exp}_W({}^A_E w) \\ \text{exp}({}^W_E w) \cdot A &= A \text{exp}({}^A_E w) \in \hat{\mathbb{R}}^3 \\ \Rightarrow \text{exp}({}^W_E w) &= A \text{exp}({}^A_E w) A^{-1} \end{aligned}$$

# Changes of Reference Frames



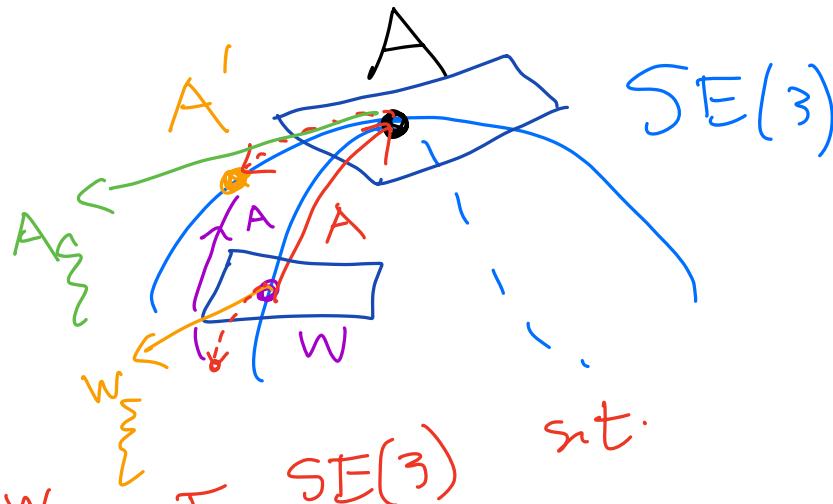
Goal: Find  ${}^W \xi \in {}^T_W \text{SE}(3)$  s.t.

${}^W \xi \oplus A = A' = \underbrace{A}_{\text{first increment by } {}^W \xi \text{ in } W} \oplus A_g$

$= A'' = \underbrace{A'}_{\text{first move by } A \text{ in } W} \oplus A_g$

$$\begin{aligned}
 \exp(({}^W \xi w)^\wedge) &= A \exp((A \xi w)^\wedge) A^{-1} \\
 &= A \sum_{k=0}^{\infty} \frac{(A \xi w)^\wedge}{k!} A^{-1} \\
 &= \sum_{k=0}^{\infty} \frac{(A (\xi w)^\wedge)^k}{k!} \\
 &= \exp(A (\xi w)^\wedge A^{-1})
 \end{aligned}$$

# Changes of Reference Frames

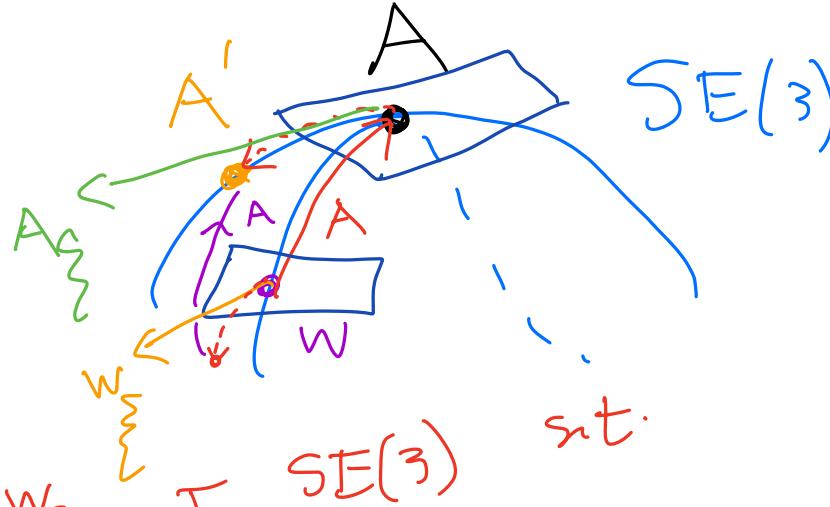


Goal: Find  $\overset{w}{\underset{\varepsilon}{\in}} E^T \overset{SE(3)}{T}_W$  s.t.  
 $\overset{w}{\underset{\varepsilon}{\in}} \oplus A = A' = \underbrace{A}_{\text{first increment by } \overset{w}{\underset{\varepsilon}{\in}} \text{ in } W} \oplus \overset{A_g}{\underset{\varepsilon}{\in}}$   
first increment by  $\overset{w}{\underset{\varepsilon}{\in}}$  in  $W$       first move by  $A$  in  $W$ .

$$\left(\overset{w}{\underset{\varepsilon}{\in}} \overset{w}{\underset{\varepsilon}{\in}}\right)^n = A \left(\overset{A}{\underset{\varepsilon}{\in}} \overset{w}{\underset{\varepsilon}{\in}}\right)^n A^{-1}$$
$$T_W^n SE(3)$$

$$T_W SE(3)$$

# Changes of Reference Frames



Goal: Find  $\overset{w}{\varepsilon} \overset{T}{E}_W = A' = \underbrace{A}_{\text{first increment by } \overset{w}{\varepsilon} \text{ in } W} + \underbrace{\overset{w}{\varepsilon}}_{\text{first move by } A \text{ in } W.}$

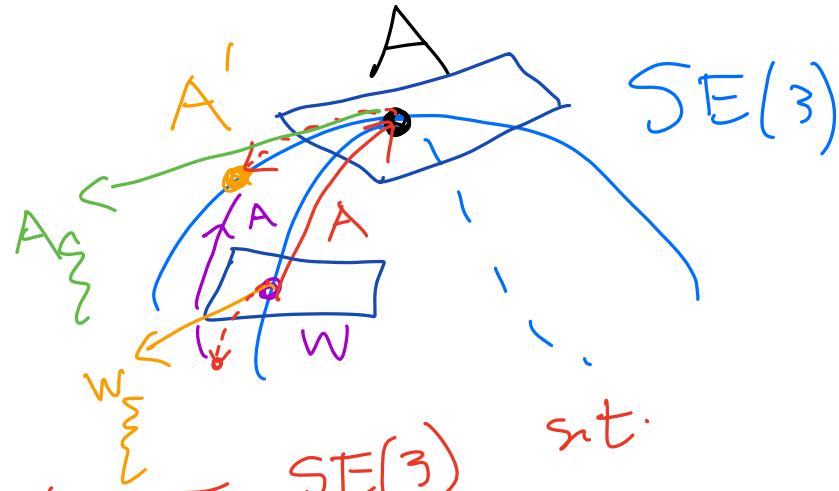
$$= \left( A (\overset{A}{\varepsilon}^w)^\wedge A^{-1} \right)^v$$

$$\overset{A}{\varepsilon}^w \in \mathbb{R}^3 \quad \overset{v}{\varepsilon} \in \mathbb{R}^3$$

Define adjoint matrix of  $A$ :

$$\text{Ad}_A : \overset{A}{\varepsilon}^w \mapsto \left( A (\overset{A}{\varepsilon}^w)^\wedge A^{-1} \right)^v$$

# Changes of Reference Frames



Goal: Find  $\overset{w}{\xi} \in \overset{T}{SE(3)}$  s.t.  
 $\overset{w}{\xi} \oplus A = A' = \underbrace{A}_{\text{first move by } A \text{ in } W.} \oplus \overset{A}{\xi}$   
first increment by  $\overset{w}{\xi}$  in W

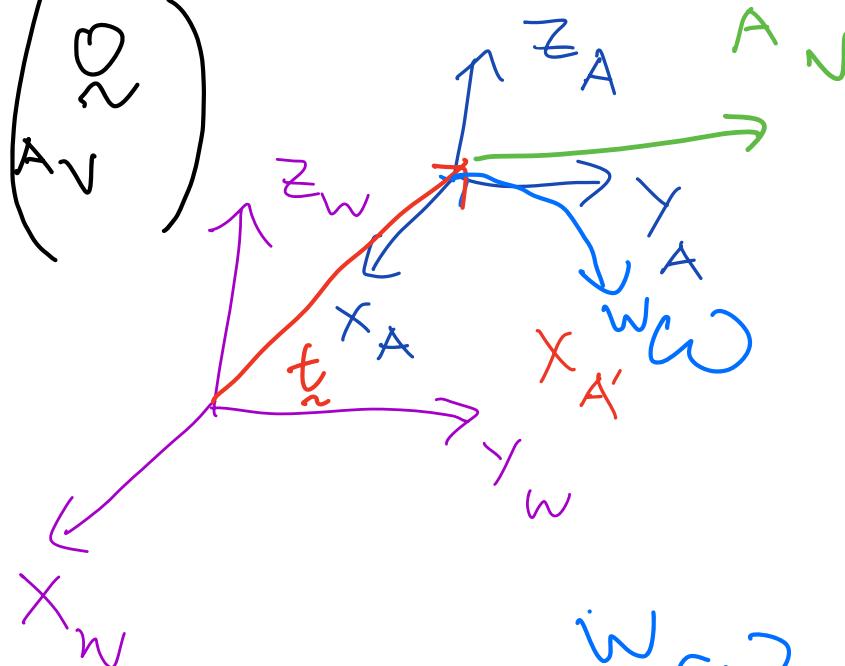
$$\overset{w}{\xi} = {}^{(w)}_{\overset{A}{\xi}} Ad_A$$

the  
reference  
frame "correct" factor for velocity  
"change of  
vectors"

# Changes of Reference Frames

$$A = \begin{pmatrix} R & \sim t \\ 0 & 1 \end{pmatrix}^A \xi = \begin{pmatrix} 0 \\ A_v \\ A_w \end{pmatrix}$$

$$= \begin{pmatrix} I & \sim t \\ 0 & 1 \end{pmatrix}$$



$$\omega_\xi = Ad_A^A \xi = \begin{pmatrix} R & [\dot{x}]_x R \\ 0 & R \end{pmatrix} \begin{pmatrix} 0 \\ A_v \\ A_w \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{v} \\ \dot{w}_w \end{pmatrix}$$

(Lie theory)

# Adjoint Representation

The adjoint representation is defined as:

$$\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$$
$$\text{Ad}_g(x) = g x g^{-1}$$

For any Lie Group  $G$ ,

the map "reproduces" the group structure.

$g \mapsto \text{Ad}_g$

# Adjoint Representation

The adjoint representation is defined as:

$$\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$$

$$\text{Ad}_g(X) = g X g^{-1}$$

$$\text{Ad}_{g \cdot h} = \text{Ad}_g \circ \text{Ad}_h$$

$$\text{Ad}_{g^{-1}} = \text{Ad}_g^{-1}$$

and by construction:

$$g \oplus \mathbb{V} = (\text{Ad}_g \mathbb{V}) \oplus g$$

## Summary

- Geometrically,  $g \oplus \tilde{v}$  gives the "closest" element  $g' \in G$  after moving away from  $g$  in direction  $\tilde{v}$
- Since  $\text{Exp}_P$  is "hard" to compute, can instead use the fact that bi-invariant Lie groups satisfy  $\exp = \text{Exp}_I$  and  $\exp$  is easy to compute.
- To change reference frames for velocity vectors, we use the adjoint matrix  $\text{Ad } g$ .