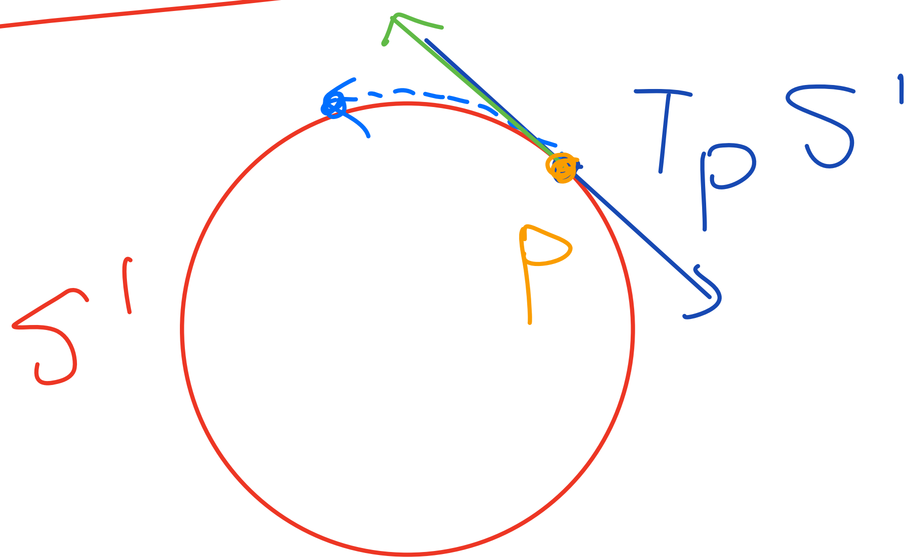
Lie Theory for Control & Estimation



By Sepehr Saryazdi

Motivation:

$$SO(3) = \left\{ R \in \mathrm{Mat}_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

How to compute

$$\underset{R \in SO(3)}{\operatorname{argmin}} \| R\underset{\sim}{v} - \underset{\sim}{u} \|^2 \quad ?$$

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Naive method:

$$f(r_{11}, \dots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \underset{\sim}{v} - \underset{\sim}{u} \right\|^2$$

Solve $\dfrac{\partial f}{\partial r_{ij}} = 0$. This is hard!

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Improved method w/ grad. descent:

$$f(r_{11}, \dots, r_{33}) = \left\| \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \underset{\sim}{v} - \underset{\sim}{u} \right\|^2$$

$$\underset{\sim}{r} := (r_{11}, \dots, r_{33}), \quad \underset{\sim}{r}_{n+1} = \underset{\sim}{r}_n - \gamma \nabla f(\underset{\sim}{r}_n)$$

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$$\underset{\sim}{r}_{n+1} = \underset{\sim}{r}_n - \gamma \nabla f(\underset{\sim}{r}_n) =: (r'_{11}, \dots, r'_{33}),\ R' \notin SO(3)!$$

Motivation:

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Want: $(\underset{\sim}{r}_n)_{n \in \mathbb{N}}$ s.t. $\underset{\sim}{r}_n \in SO(3)$ & $r_n \longrightarrow \underset{r}{\operatorname{argmin}} f$

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Idea: Define $\oplus$ s.t.

$$R \oplus dR \in SO(3)$$

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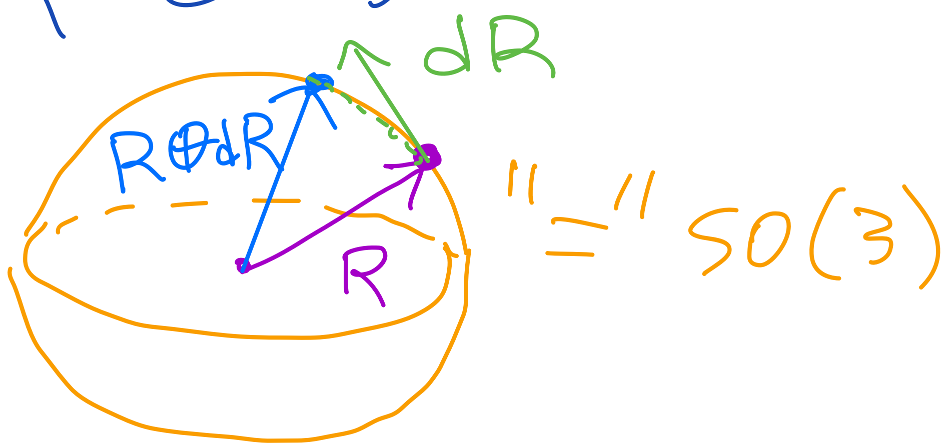

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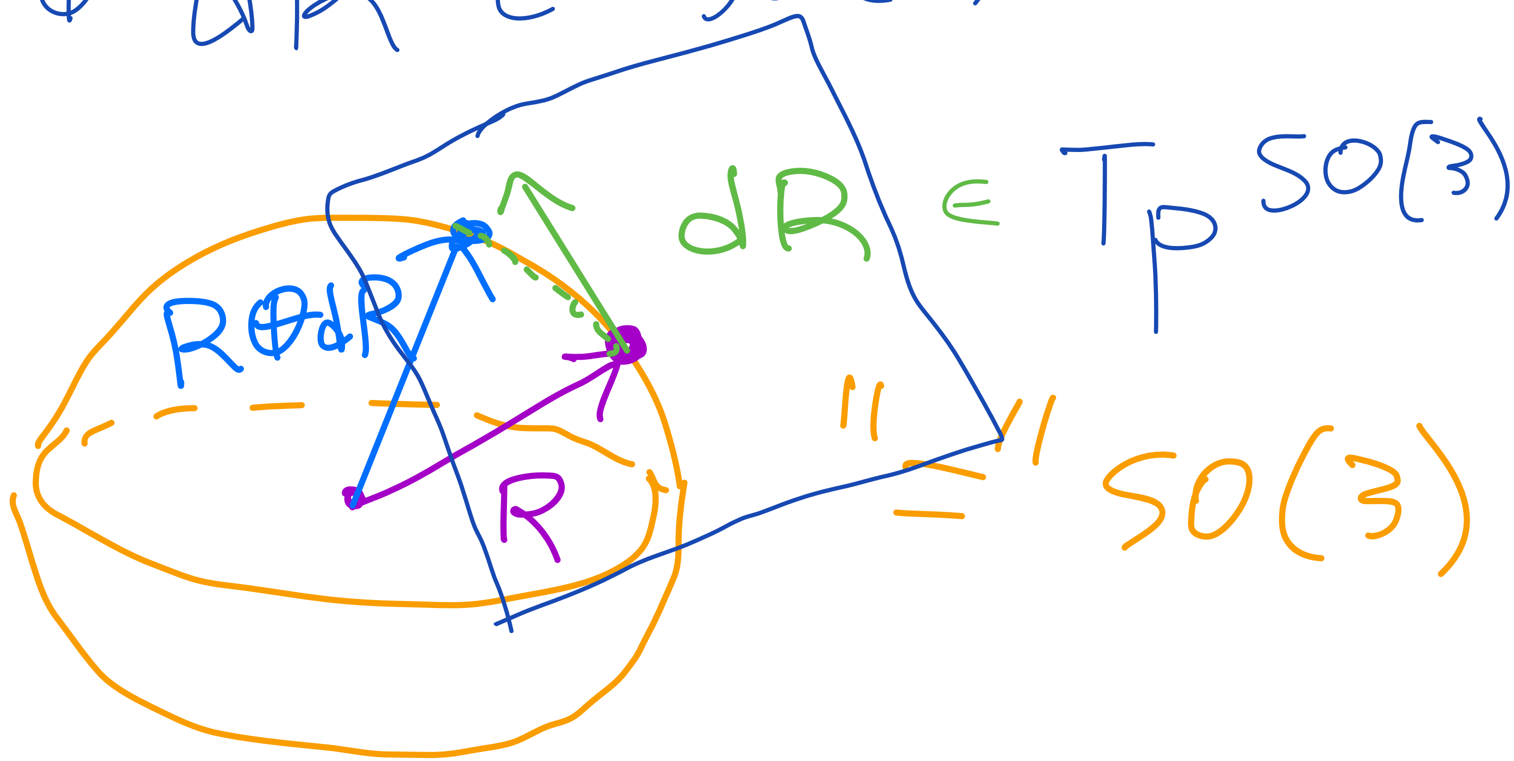

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Idea: Define $\oplus$ s.t.

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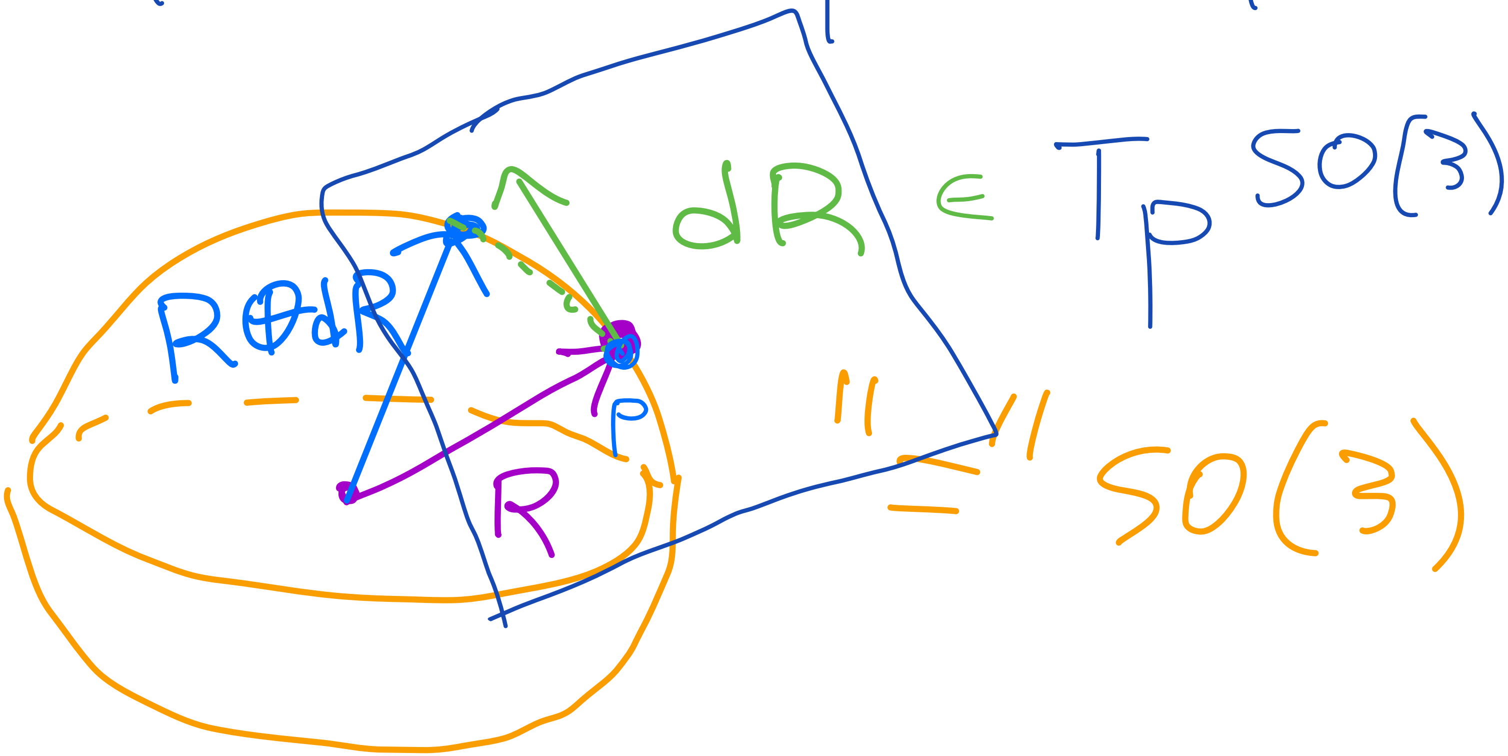
Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

Q: How to compute $T_p SO(3)$?



Think of "velocity" vectors at a point $p \in SO(3)$.

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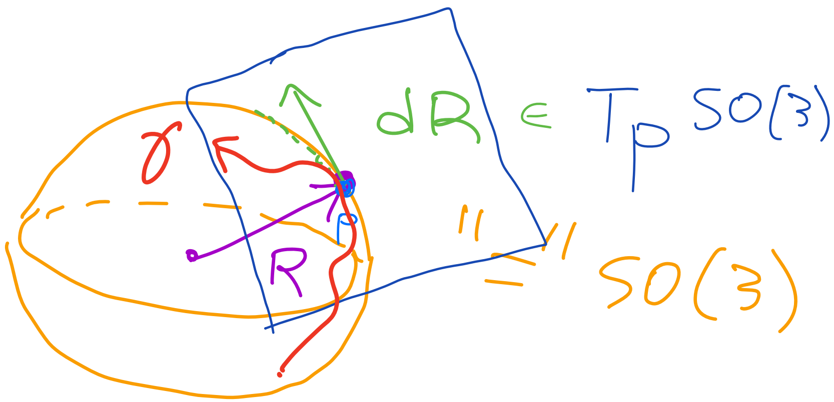

Think of "velocity" vectors of a curve at a point $p \in SO(3)$.

$$\gamma : (-\epsilon, \epsilon) \longrightarrow SO(3)$$

Computing $T_p SO(3)$:

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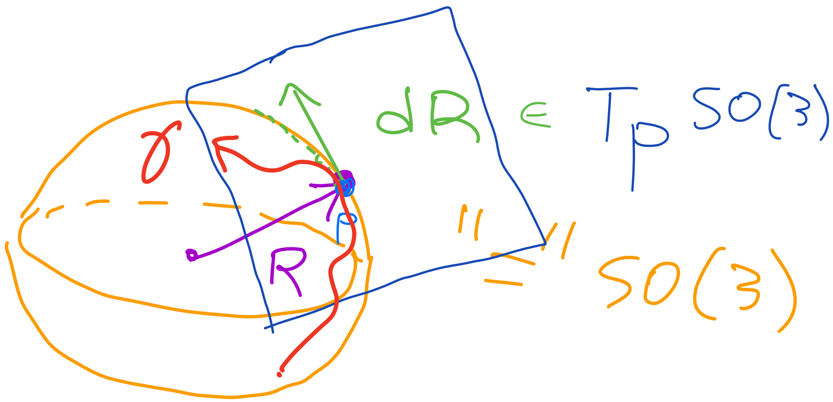

Think of "velocity" vectors of a curve at a point $p \in SO(3)$.

$$\gamma : (-\epsilon, \epsilon) \longrightarrow SO(3), \text{ e.g. } \gamma(t) = R_z(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p = \gamma(0) = \mathbb{I}$$

Computing $T_p SO(3)$:

$$SO(3) = \{ R \in Mat_3(\mathbb{R}) \mid R^T R = I, \ \det R = 1 \}$$

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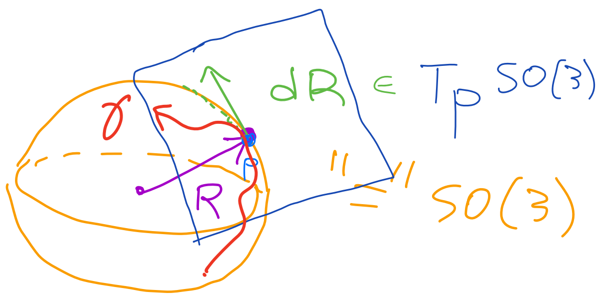

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$$\gamma'(t) = \begin{bmatrix} -\sin t & -\cos t & 0 \\ \cos t & -\sin t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \gamma'(0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Computing $T_p SO(3)$:

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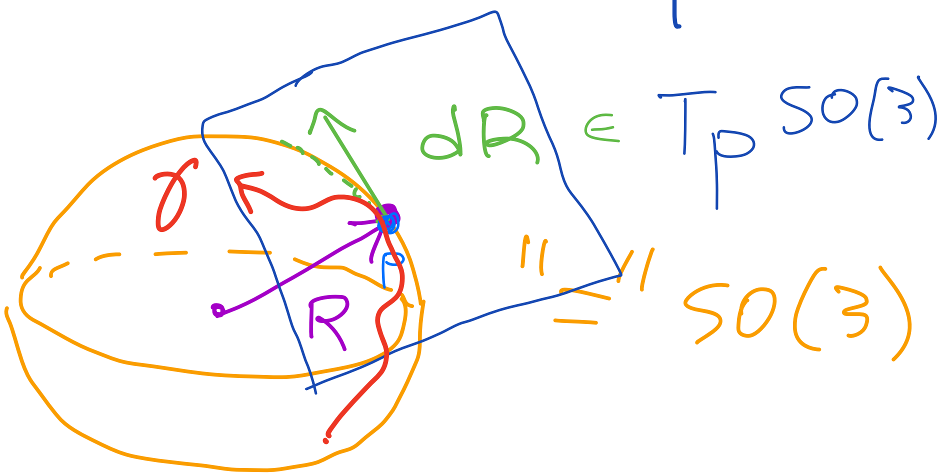

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Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; R^T R = \mathbb{I},\ \det R = 1 \right\}$$

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$$\gamma_z(t) = R_z(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad p = \gamma(0) = \mathbb{I}, \quad \gamma_z'(0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\gamma_y(t) = R_y(t) = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix}, \quad \gamma_y'(0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\gamma_x(t) = R_x(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}, \quad \gamma_x'(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Computing $T_p SO(3)$:

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what about any γ?

$$\gamma(t) = R_x(\theta_1(t))\, R_y(\theta_2(t))\, R_z(\theta_3(t))$$

Computing $T_p SO(3)$:

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Q: How to compute $T_p SO(3)$?

what about any γ passing through $p = \mathbb{I}$?

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$$\begin{aligned}
\gamma'(t) = {} & \gamma_x'(\theta_1(t)) \cdot \gamma_y(\theta_2(t)) \cdot \gamma_z(\theta_3(t)) \cdot \theta_1'(t) \\
& + \gamma_x(\theta_1(t)) \cdot \gamma_y'(\theta_2(t)) \cdot \gamma_z(\theta_3(t)) \cdot \theta_2'(t) \\
& + \gamma_x(\theta_1(t)) \cdot \gamma_y(\theta_2(t)) \cdot \gamma_z'(\theta_3(t)) \cdot \theta_3'(t)
\end{aligned}$$

Computing $T_p SO(3)$:

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what about any γ passing through $p = \mathbb{I}$?

$$\gamma(t) = \gamma_x(\theta_1(t))\,\gamma_y(\theta_2(t))\,\gamma_z(\theta_3(t))$$

$$\begin{aligned}\gamma'(0) = \;& \gamma_x'(0)\cdot\cancelto{\mathbb{I}}{\gamma_y(0)}\cdot\cancelto{\mathbb{I}}{\gamma_z(0)}\cdot\theta_1'(0) \\ &+ \cancelto{\mathbb{I}}{\gamma_x(0)}\cdot\gamma_y'(0)\cdot\cancelto{\mathbb{I}}{\gamma_z(0)}\cdot\theta_2'(0) \\ &+ \cancelto{\mathbb{I}}{\gamma_x(0)}\cdot\cancelto{\mathbb{I}}{\gamma_y(0)}\cdot\gamma_z'(0)\cdot\theta_3'(0)\end{aligned}$$

$$\gamma'(0) = \gamma_x'(0)\,\theta_1'(0) + \gamma_y'(0)\,\theta_2'(0) + \gamma_z'(0)\,\theta_3'(0)$$

Computing $T_p SO(3)$:

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Q: How to compute $T_p SO(3)$?

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$$= \gamma_x'(0)\,\omega_x + \gamma_y'(0)\,\omega_y + \gamma_z'(0)\,\omega_z$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_z$$

$$= \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Computing $T_p SO(3)$:

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Q: How to compute $T_p SO(3)$?

As a vector space:

$$\gamma'(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_z$$

$$\mathfrak{so}(3) := T_{\mathbb{I}} SO(3) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$= \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial \alpha}(\mathbb{I}), \frac{\partial}{\partial \beta}(\mathbb{I}), \frac{\partial}{\partial \gamma}(\mathbb{I}) \right\}$$

$$= \operatorname{span}_{\mathbb{R}} \left\{ E_1(\mathbb{I}), E_2(\mathbb{I}), E_3(\mathbb{I}) \right\}$$

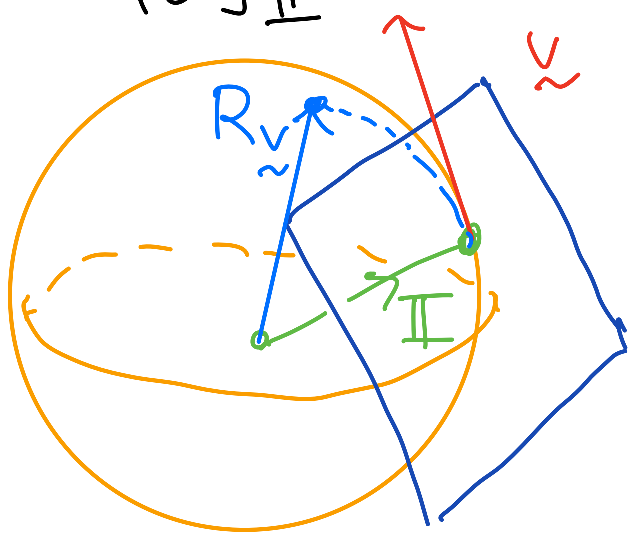
$\mathfrak{so}(3)$: Lie Algebra of $SO(3)$

Computing $T_p SO(3)$:

$$SO(3) = \{ R \in Mat_3(\mathbb{R}) \mid R^T R = \mathbb{I},\ \det R = 1 \}$$

$$SO(3) \xrightarrow{\left.\frac{d}{dt}\right|_{\mathbb{I}}} T_{\mathbb{I}} SO(3)$$

$$\begin{array}{ccc} SO(3) & \underset{\log_{\mathbb{I}}}{\overset{\exp_{\mathbb{I}}}{\rightleftarrows}} & T_{\mathbb{I}} SO(3) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ R_{\underset{\sim}{v}} & & \underset{\sim}{v} \end{array}$$



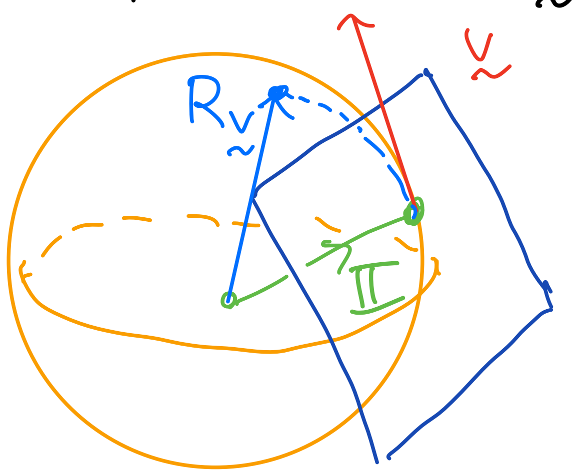
Computing $T_p SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

$$SO(3) \xleftarrow{\exp_{\mathbb{I}}} T_{\mathbb{I}} SO(3)$$

$$\begin{array}{cc} \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\ R_{\underset{\sim}{v}} & \underset{\sim}{v} \end{array}$$

$\exp_{\mathbb{I}}(\underset{\sim}{v}) = \gamma_{\underset{\sim}{v}}(1)$, $\gamma_{\underset{\sim}{v}}$ a geodesic w/ const. velocity $\underset{\sim}{v}$



$$\mathbb{I} \oplus \underset{\sim}{v} := \exp_{\mathbb{I}}(\underset{\sim}{v}) = R_{\underset{\sim}{v}}$$

Computing $T_p\, SO(3)$:

$$SO(3) = \left\{ R \in Mat_3(\mathbb{R}) \;\middle|\; \begin{array}{l} R^T R = \mathbb{I}, \\ \det R = 1 \end{array} \right\}$$

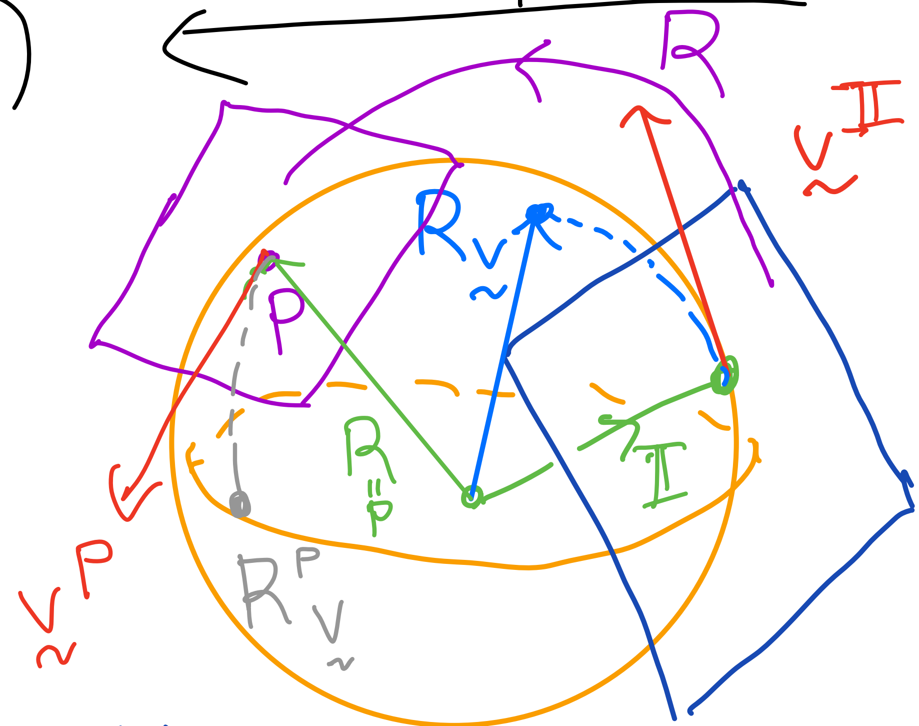
$$\begin{array}{ccc} SO(3) & \xleftarrow{\exp_{\mathbb{I}}} & T_{\mathbb{I}}\, SO(3) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ R_{\underset{\sim}{v}} & & \underset{\sim}{v} \end{array}$$

$$\exp_{\mathbb{I}}(\underset{\sim}{v}) = \gamma_{\underset{\sim}{v}}(1) \qquad \gamma_{\underset{\sim}{v}} \text{ a geodesic w/ const. velocity } \underset{\sim}{v}$$

Computing $T_P SO(3)$:

$$SO(3) = \{ R \in Mat_3(\mathbb{R}) \mid R^T R = \mathbb{I},\ \det R = 1 \}$$

$$SO(3) \xleftarrow{Exp_P} T_P SO(3)$$



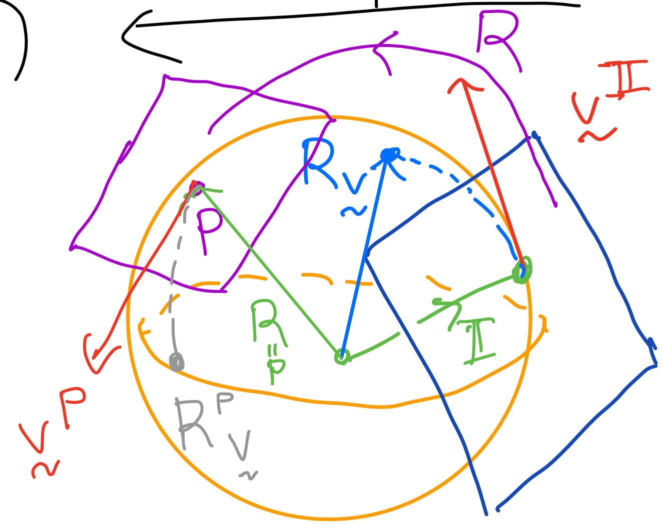
$$T_P SO(3) = R \cdot T_{\mathbb{I}} SO(3) \ !$$

why? we'll return to this later...

Computing $T_p SO(3)$:

$$SO(3) = \{ R \in Mat_3(\mathbb{R}) \mid R^T R = \mathbb{I}, \det R = 1 \}$$

$$SO(3) \xleftarrow{Exp_p} T_p SO(3)$$



$$T_p SO(3) = R \cdot T_{\mathbb{I}} SO(3) \;!$$

$$R \oplus \underset{\sim}{v} := Exp_R(\underset{\sim}{v}^p) = R \cdot \exp_{\mathbb{I}}(\underset{\sim}{v}^{\mathbb{I}}) = R \cdot (\mathbb{I} \oplus \underset{\sim}{v}^{\mathbb{I}})$$

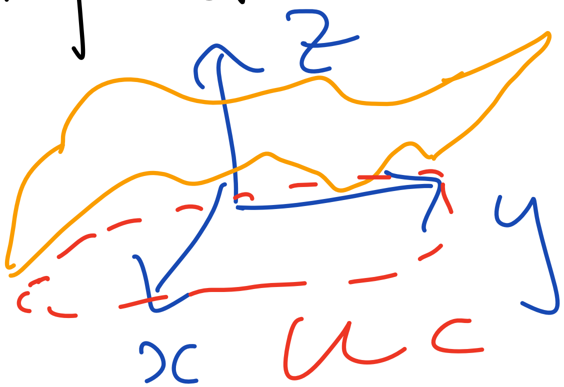
Properties of $\mathrm{Exp}_p : T_p SO(3) \to SO(3)$

$$\mathrm{Exp}_p(\underset{\sim}{v}) = \underbrace{\gamma_{\underset{\sim}{v}}(1)}_{\gamma \text{ a geodesic}}$$

How do we actually compute $\mathrm{Exp}_p(\underset{\sim}{v})$ for any manifold M?

Properties of $\mathrm{Exp}_p : T_p SO(3) \to so(3)$

Every manifold comes with an atlas: (φ_i, U_i)



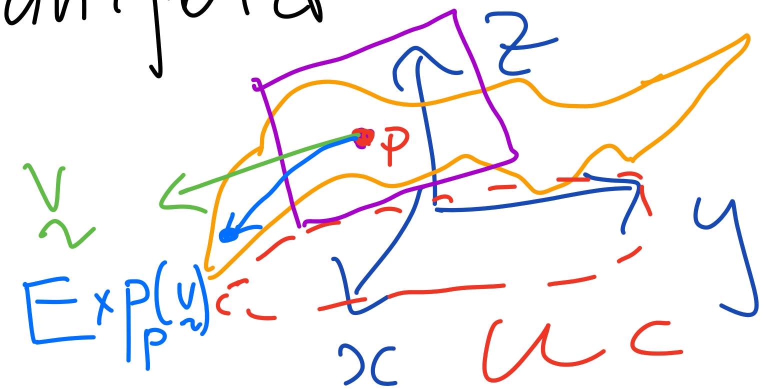
$SO(3)$ as a manifold has a chart given by

$$\varphi : (0, 2\pi)^3 \longrightarrow SO(3)$$

$$\varphi(\alpha, \beta, \gamma) = R_x(\alpha) R_y(\beta) R_z(\gamma)$$

Properties of $\exp_p : T_p SO(3) \to so(3)$

Every manifold M comes with an atlas: (φ_i, U_i)



$U \subset \mathbb{R}^2$ open

$(x,y) \mapsto (x,y,z(x,y))$

If φ is given, then $\operatorname{Exp}_p(\underset{\sim}{v})$ can be computed using the geodesic equation:

$p = \varphi(u^1(0), \dots, u^n(0))$

$\underset{\sim}{v} = (v^1, \dots, v^n)$

initial condition of ODE

$$\underbrace{\overbrace{\frac{d^2 u^k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{du^i}{dt}\frac{du^j}{dt} = 0}^{\text{ODE}}}_{}$$

Christoffel Symbols

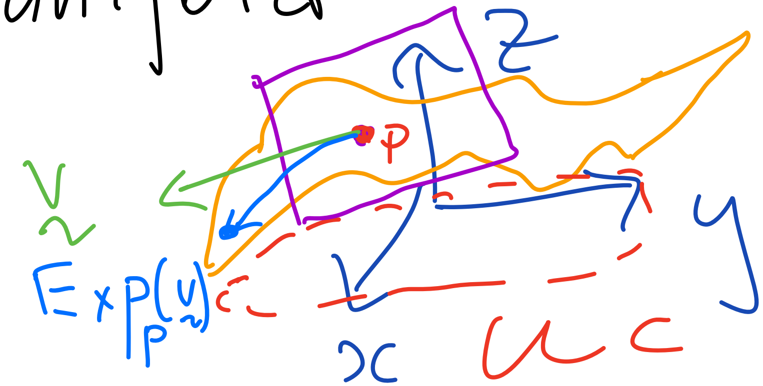
def^n of Γ^k_{ij}

$$\varphi_{u^i u^j} = \sum_k \Gamma^k_{ij} E_k,$$

$T_p M = \operatorname{span}_{\mathbb{R}}\{E_1, \dots, E_n\}$

Properties of $\mathrm{Exp}_p : T_p SO(3) \to so(3)$

Every manifold M comes with an atlas: (φ_i, U_i)



$(x, y) \longmapsto (x, y, z(x,y))$

$U \subset \mathbb{R}^2$ open

Properties of $Exp_p : T_p SO(3) \to so(3)$

For Lie Groups G w/ bi-invariant Riemmanian metrics (e.g. $SO(3)$),

the geodesic equation has the solution (Ken's Thm)

$$\gamma_{\underset{\sim}{v}}(t) = g \cdot \exp_e(t g^{-1} \underset{\sim}{v})$$

$$g^{-1}v := d(L_{g^{-1}})_g(v) \in \mathfrak{g}$$

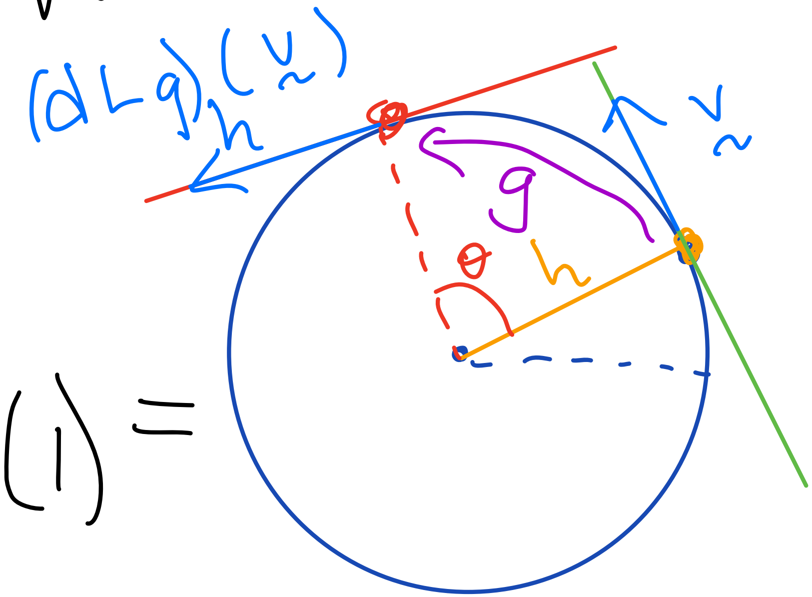
$L_g : x \mapsto gx \ \forall x \in G$ (left-action)

$(dL_g)_h :=$ the differential of L_g at $h \in G$.

Properties of $\exp_p : T_p SO(3) \to so(3)$

What is dL_g?

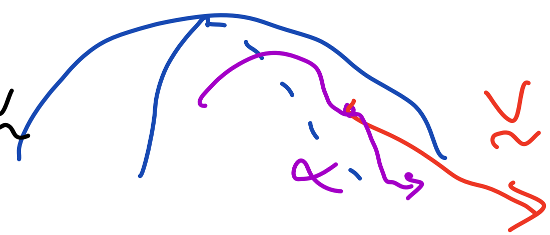
$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

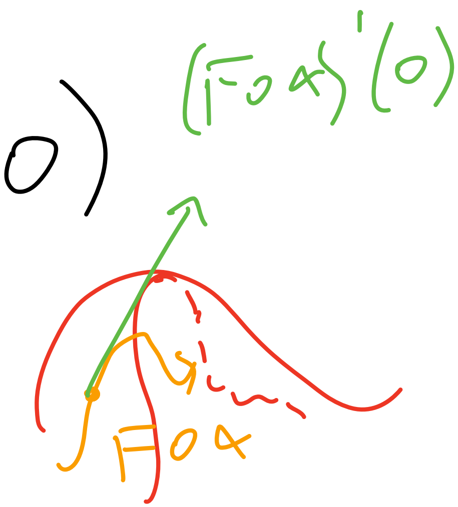
$SO(1) =$ 

Formally:

$$(dF)_p(\underset{\sim}{v}) := (F\circ\alpha)'(0)$$

$\alpha(0) = p$, $\alpha'(0) = \underset{\sim}{v}$





Properties of $\exp_p : T_p SO(3) \to so(3)$

What is $\exp_{I}(\underset{\sim}{v})$?

e.g. $\underset{\sim}{v} = \omega_x E_1 + \omega_y E_2 + \omega_z E_3 \in so(3)$

$$\exp_{I}(\underset{\sim}{v}) = \sum_{k=0}^{\infty} \frac{\underset{\sim}{v}^k}{k!} \quad \text{(Regular exponential)}$$

so can now compute

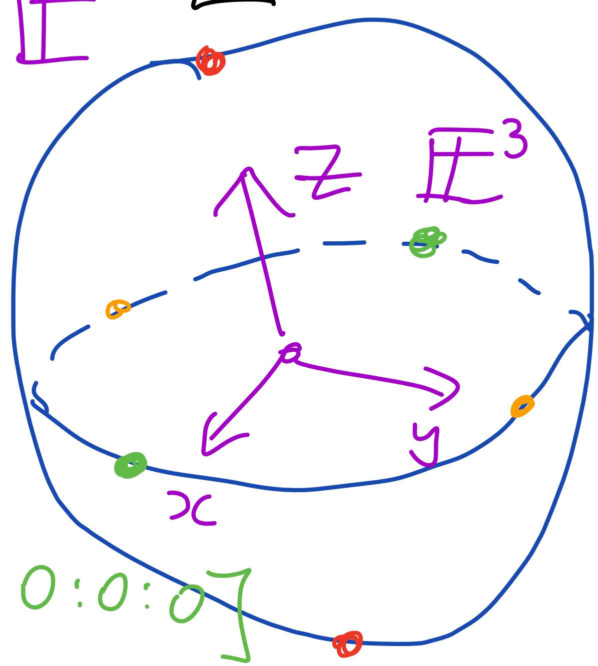
i.e. exp anywhere, $g \oplus \underset{\sim}{v}$ for any "nice" Lie Group G & $T_g G$.

Other Examples: SE(3)

$$SE(3) = \left\{ \begin{pmatrix} R & \tilde{t} \\ 0 & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} R \in SO(3), \\ t \in \mathbb{R}^3 \end{array} \right\}$$

$$SE(3) \circlearrowright \mathbb{P}^3 = \mathbb{E}^3 \sqcup \mathbb{P}^2$$

$\mathbb{P}^2$



[0:1:0:0]

[1:0:0:0]

[0:0:1:0]

$$\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \cdot [x:y:z:1] = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\lim_{x \to \infty} [x : y_0 : z_0 : 1] = \lim_{x \to \infty} \left[1 : \frac{y_0}{x} : \frac{z_0}{x} : \frac{1}{x} \right] = [1:0:0:0]$$

Other Examples: SE(3)

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$$SE(3) \curvearrowright \mathbb{P}^3 = \mathbb{E}^3 \sqcup \mathbb{P}^2$$

Other Examples: SE(3)

$$SE(3) \cong \underbrace{SO(3)}_{\text{compact}} \times \mathbb{R}^3 \text{ so } \mathrm{Exp}_{I} = \exp$$

(Milnor, 1976)

$$\mathfrak{se}(3) = T_{I} SE(3) = \mathrm{span}_{\mathbb{R}} \Big\{ \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} R_x(t) & 0 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} R_y(t) & 0 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} R_z(t) & 0 \\ 0 & 1 \end{pmatrix},$$

$$\frac{d}{dt}\Big|_{t=0} \begin{pmatrix} I & t\underset{\sim}{e}_1 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} I & t\underset{\sim}{e}_2 \\ 0 & 1 \end{pmatrix}, \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} I & t\underset{\sim}{e}_3 \\ 0 & 1 \end{pmatrix} \Big\}$$

$$= \mathrm{span} \left\{ \frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta}, \frac{\partial}{\partial \gamma}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$$

$$= \mathrm{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ -1&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \end{pmatrix} \right\}$$

Other Examples: SE(3)

A twist ξ is given as:

$$\xi := \omega_x \frac{\partial}{\partial \alpha} + \omega_y \frac{\partial}{\partial \beta} + \omega_z \frac{\partial}{\partial \gamma} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

$$= \begin{pmatrix} 0 & -\omega_z & \omega_y & v_x \\ \omega_z & 0 & -\omega_x & v_y \\ -\omega_y & \omega_x & 0 & v_z \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Properties of Lie Algebras:

Matrix group multiplication can be "reproduced" with Lie Algebra multiplication:

Baker-Campbell-Hausdorff:

$$x \cdot y \longleftrightarrow X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\left[X, [X, Y]\right] + \frac{1}{12}\left[Y, [Y, X]\right] + \ldots$$

w/ $X = \log x$, $Y = \log y$, $[X, Y] := XY - YX$

Properties of Lie Algebras:

The adjoint representation is defined as:

$$Ad_g : \mathfrak{g} \to \mathfrak{g}$$

$$Ad_g(X) = gXg^{-1}$$

which is the "velocity" version of the transformations between frames.

As a linear map on vector spaces, Ad_g can be thought of as a matrix.

Properties of Lie Algebras:

The adjoint representation is defined as:

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$$

$$\mathrm{Ad}_g(X) = g X g^{-1}$$

which is the "velocity" version of transformations between frames

$$\mathbb{I}\, V = [\mathrm{Ad}_g]^{g} \underset{\sim}{V}$$

$\Rightarrow$