Quantum Topology: $\{\text{Physics}\} \cap \{\text{Pure Maths}\}$

A crash course on quantum mechanics and its surprisingly deep roots in pure mathematics.

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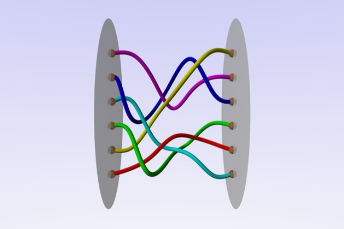
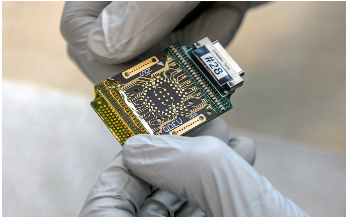
Overview

Physical Applications

- Microsoft announced earlier this year it has invested in researching Anyons (2-dimensional topological quasi-particles) as a basis for quantum computing.





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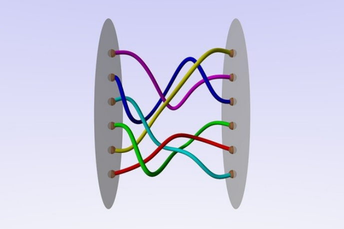
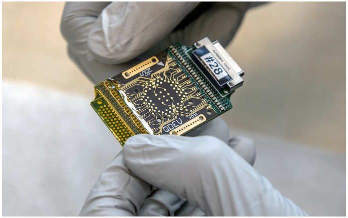
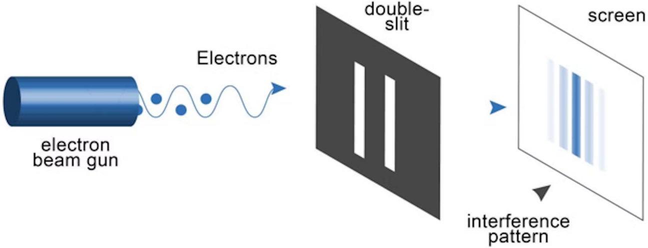




- If pulled off, the topological nature of this qubit could make it more robust against random errors.

Physics Origins

- Where did this all come from?



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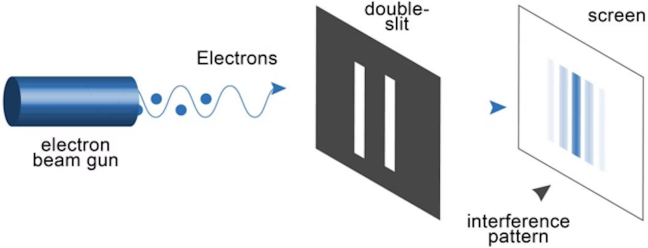


- De Broglie: Everything must be both a wave and particle.

Physics Origin

- Born and Schrödinger's trick: Make everything probabilistic!

Markov Chain Analogy

$$S = \{1, 2, 3\}$$

$$X(t) \in S := \text{state of system at time } t$$

$$A = \begin{pmatrix} P(1 \to 1) & P(2 \to 1) & P(3 \to 1) \\ P(1 \to 2) & P(2 \to 2) & P(3 \to 2) \\ P(1 \to 3) & P(2 \to 3) & P(3 \to 3) \end{pmatrix}$$

$$X(t+1) = AX(t)$$

Physics Origins

Markov Chain Analogy

$$X(t+1) = AX(t) = A^t X(1)$$

$$X(1) = \begin{pmatrix} P(X(1)=1) \\ P(X(1)=2) \\ P(X(1)=3) \end{pmatrix}$$

$$= P(X(1)=1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + P(X(1)=2)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + P(X(1)=3)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow X = P(X(1)=1)\mathbf{e}_1 + P(X(1)=2)\mathbf{e}_2 + P(X(1)=3)\mathbf{e}_3$$

Physics Origins

- Now make it quantum...

Markov Chain Analogy

$$X = P(X(1) = 1)\mathbf{e}_1 + P(X(1) = 2)\mathbf{e}_2 + P(X(1) = 3)\mathbf{e}_3$$

$$X \mapsto |X\rangle, \mathbf{e}_i \mapsto |i\rangle, P(X(1) = i) \mapsto |c_i|^2, c_i \in \mathbb{C}$$

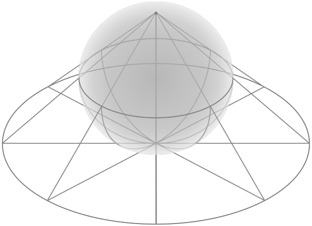
$$|X\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$$

$$|c_1|^2 + |c_2|^2 + |c_3|^2 = 1 \Leftrightarrow \bar{X} \cdot X = \langle X, X\rangle_{\mathbb{C}} = \langle X|X\rangle = 1$$

$$\Rightarrow |X\rangle \in \mathbb{C}^3, \langle X|X\rangle = 1$$

Physics Origins

- Generalise to any state space $\mathcal{H}$ with complex field $\mathbb{C}$.



Generalised Quantum State Spaces

$$S = \mathcal{H}P := (\mathcal{H} - \{\mathbf{0}\})/_{\{\lambda\mathbf{v} | \mathbf{v} \neq \mathbf{0} \in \mathcal{H}, \lambda \neq 0 \in \mathbb{C}\}}$$

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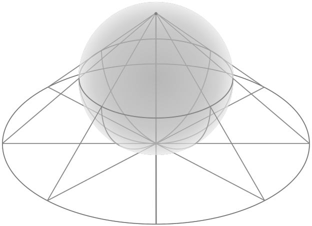

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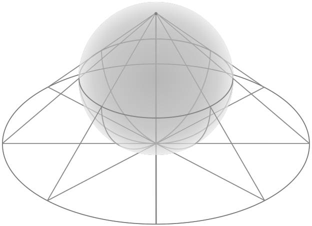

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- $\lambda\mathbf{v} \sim \mathbf{v}$
- This is a complex projective space!

Physics Origins

- Modelling particles in space is as easy as $L^p(\mathbb{R}^n, \mathbb{C})$...

$$\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}) = \left\{ \psi : \mathbb{R}^3 \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}^3} |\psi(x, y, z)|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z < \infty \right\}$$

Inner product:

$$\langle \psi, \phi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \psi^*(x, y, z)\phi(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Or as physicists write it:

$$|\psi\rangle, |\phi\rangle \in \mathcal{H}P$$

$$\langle \psi | \phi \rangle := \langle \psi, \phi \rangle_{\mathcal{H}}$$

Physics Origins

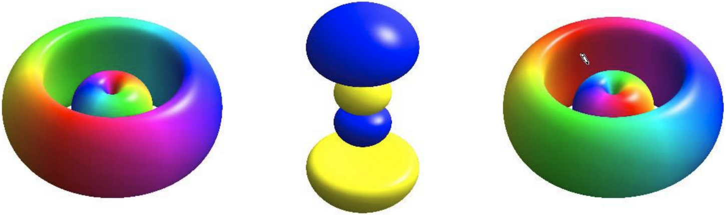
- Physical Interpretation of Wavefunctions.

Probability of Observation in the Continuum

If $|\psi\rangle \in \mathcal{H}P$ is the wavefunction of a particle p, then

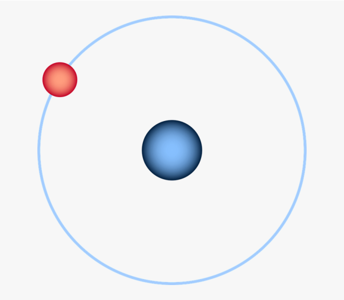
$$\int_{\mathbb{R}^3} |\psi(x,y,z)|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z = 1$$

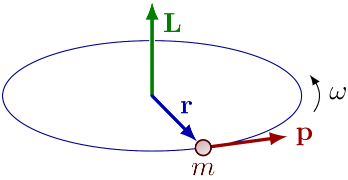
$$P(p \in (x,y,z)) \approx |\psi(x,y,z)|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$$



What is Spin?

- First consider angular momentum.



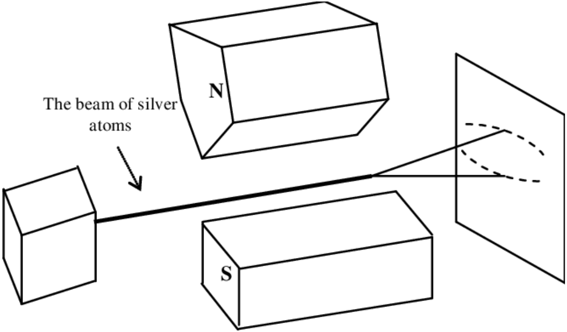


Angular Momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

What is Spin?

- Shoot neutral silver atoms through a magnetic field, it should not be deflected.



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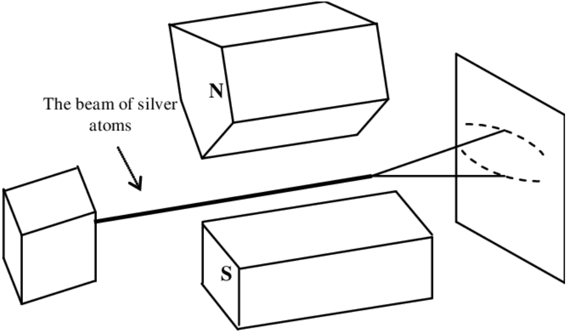

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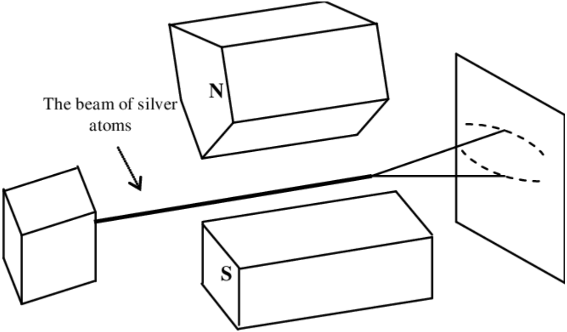

- However, it actually gets deflected!
- This is due to the electrons being like a little magnet.

What is Spin?

- This 'magnetism' comes from the electron having an intrinsic angular momentum.

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- This is what we mean by spin.

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- This 'magnetism' comes from the electron having an intrinsic angular momentum.

- This is what we mean by spin.
- It's not actually spinning! It's just our angular momentum models are the most accurate!

Spin as a "limit"

$$\mathbf{S} = \lim_{\mathbf{r}\to\mathbf{0}, |\mathbf{v}|\to\infty} \mathbf{r} \times \mathbf{v}$$

Deriving Spin

- We need to use group theory.

Group Theory

Let X be a set and $() \cdot () : X \times X \to X$ a function.

$$G = (X, \cdot)$$

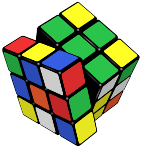
Example 1:

$$X = \mathbb{R} \setminus \{0\}, a \cdot b := ab$$

$$a^{-1} \cdot a = \frac{1}{a} a = 1$$

Deriving Spin

- Groups encode symmetries about your set X.



Group Theory

$$G = (X, \cdot)$$

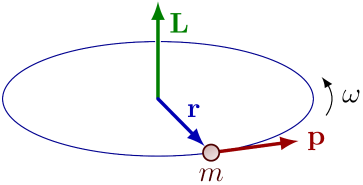
Example 2:

$$X = \{\text{all ways to change a Rubik's cube}\}$$

$$() \cdot () = \text{a composition of actions}$$

Deriving Spin

- Let's think about what symmetries we want spin/angular momentum to have.



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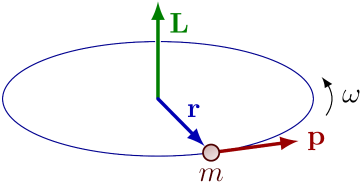

- If we rotate everything about **L**, the spin should not change!

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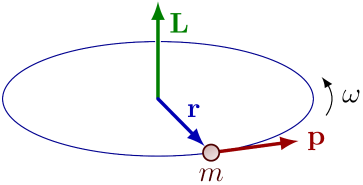

- If we rotate everything about **L**, the spin should not change!
- This means the symmetry we want to impose are rotations.

Deriving Spin

- Rotations should preserve orientation, so $\det R > 0$.

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- Rotations should not change lengths of vectors, so $||R\mathbf{v}|| = ||\mathbf{v}||$.

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Rotations

$$||R\mathbf{v}|| = ||\mathbf{v}|| \Leftrightarrow (R\mathbf{v})^T(R\mathbf{v}) = \mathbf{v}^T\mathbf{v}$$
$$\Leftrightarrow \mathbf{v}^T(R^TR)\mathbf{v} = \mathbf{v}^T\mathbb{I}\mathbf{v} \Leftrightarrow \mathbf{v}^T(R^TR - \mathbb{I})\mathbf{v} = \mathbf{0}$$
$$\Rightarrow R^TR = \mathbb{I}$$
$$\Rightarrow \det(R^TR) = 1 \Rightarrow (\det R)^2 = 1 \Rightarrow \det R = \pm 1$$
$$\Rightarrow \det R = 1$$

Deriving Spin

Rotations

$$R^T R = \mathbb{I}, \det R = 1$$

- This is the special orthogonal group!

Special Orthogonal Group

$$\mathrm{SO}(3, \mathbb{R}) = \left\{ R \in \mathrm{Mat}_{3,3}(\mathbb{R}) \middle| R^T R = \mathbb{I}, \det R = 1 \right\}$$

- It turns out that this is also a topological group and a manifold!

Deriving Spin

- How can we view this as a manifold?

Deriving Spin

- How can we view this as a manifold?
- Think of it as a set of solutions to a function.

Manifold Structure of $SO(3, \mathbb{R})$

$$f : \mathbb{R}^9 \cong \mathrm{Mat}_{3,3}(\mathbb{R}) \to \mathrm{Mat}_{3,3}(\mathbb{R})$$

$$f(\mathbf{x}) = M^T M - \mathbb{I}$$

Our solution set is then:

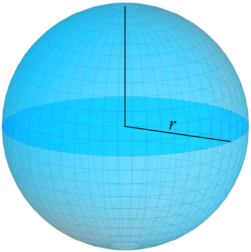
$$SO(3, \mathbb{R}) \cong \{\mathbf{x} \in \mathbb{R}^9 \mid f(\mathbf{x}) = 0\}$$

Deriving Spin

Manifold Structure of $SO(3,\mathbb{R})$

$$SO(3,\mathbb{R}) \cong \{\mathbf{x} \in \mathbb{R}^9 \mid f(\mathbf{x}) = 0\}$$

- An example for your mind to imagine is the sphere.
- $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 - 1 = 0\}$

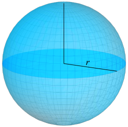


Deriving Spin

Manifold Structure of $\mathrm{SO}(3, \mathbb{R})$

$$\mathrm{SO}(3, \mathbb{R}) \cong \{\mathbf{x} \in \mathbb{R}^9 \mid f(\mathbf{x}) = 0\}$$

- We can parameterise the sphere using some coordinates, like $\phi(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.



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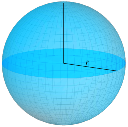

- We can similarly parameterise this $SO(3, \mathbb{R})$ space with some function $\phi : \mathbb{R}^3 \subseteq \mathbb{R}^9 \cong \mathrm{Mat}_{3,3}(\mathbb{R}) \to SO(3, \mathbb{R})$.

Deriving Spin

Manifold Structure of $\mathrm{SO}(3, \mathbb{R})$

$$\mathrm{SO}(3, \mathbb{R}) \cong \{\mathbf{x} \in \mathbb{R}^9 \mid f(\mathbf{x}) = 0\}$$

Parameterisation around $\mathbb{I} \in \mathrm{SO}(3, \mathbb{R})$:

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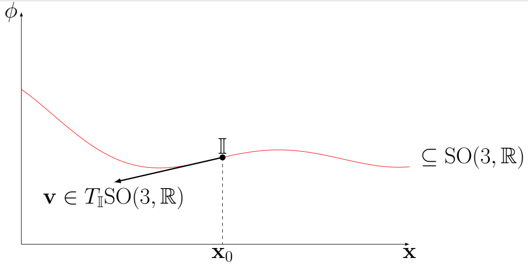

Deriving Spin

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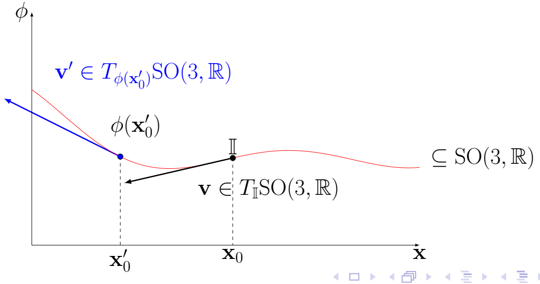

Deriving Spin

Manifold Structure of $SO(3,\mathbb{R})$

$$SO(3,\mathbb{R}) \cong \{\mathbf{x} \in \mathbb{R}^9 \mid f(\mathbf{x}) = 0\}$$

- The tangent space is the solution to our problem!

Tangent Space of $SO(3,\mathbb{R})$

$$\mathbf{S} := \text{Spin of a particle}$$

$$\mathbf{S} \in \mathfrak{so}(3) := T_{\mathbb{I}} SO(3,\mathbb{R})$$

Deriving Spin

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Tangent Space of $SO(3, \mathbb{R})$

$$\mathbf{S} := \text{Spin of a particle}$$

$$\mathbf{S} \in \mathfrak{so}(3) := T_{\mathbb{I}} SO(3, \mathbb{R})$$

- This space $\mathfrak{so}(3, \mathbb{R})$ is called the Lie Algebra of $SO(3, \mathbb{R})$ and it's where our spins live.

Deriving Spin

Special Orthogonal Group

$$SO(3,\mathbb{R}) = \left\{ R \in \mathrm{Mat}_{3,3}(\mathbb{R}) \middle| R^T R = \mathbb{I}, \det R = 1 \right\}$$

$$\mathfrak{so}(3) = T_{\mathbb{I}} SO(3,\mathbb{R})$$

Special Unitary Group

$$SU(2,\mathbb{C}) = \left\{ U \in \mathrm{Mat}_{2,2}(\mathbb{C}) \middle| \bar{U}^T U = \mathbb{I}, |\det U| = 1 \right\}$$

$$\mathfrak{su}(2,\mathbb{C}) = T_{\mathbb{I}} SU(2,\mathbb{C})$$

- It turns out that $\mathfrak{so}(3) \cong \mathfrak{su}(2)$!

Deriving Spin

- Since $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, physicists work with $\mathfrak{su}(2)$ instead.

Basis for $\mathfrak{su}(2)$

$$\mathfrak{su}(2) = \text{span}_{\mathbb{R}} \left\{ \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$\mathfrak{su}(2) = \text{span}_{\mathbb{R}} \left\{ \frac{\sigma_x}{2}, \frac{\sigma_y}{2}, \frac{\sigma_z}{2} \right\}$$

Representations of Spin

$\mathfrak{su}(2)$

$$\mathfrak{su}(2) = \text{span}_{\mathbb{R}} \left\{ \frac{\sigma_x}{2}, \frac{\sigma_y}{2}, \frac{\sigma_z}{2} \right\}$$

- Since $\mathfrak{su}(2)$ is a Lie Algebra, it has a Lie Bracket $[\cdot,\cdot] : \mathfrak{su}(2) \times \mathfrak{su}(2) \to \mathfrak{su}(2)$.

Lie Bracket of $\mathfrak{su}(2)$

$$[A, B] := AB - BA$$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}$$

Representations of Spin

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- No matter the representation of $SO(3, \mathbb{R})$, we will always have a non-abelian Lie Algebra with a Lie Bracket of this form.
- This Lie Bracket forces the eigenvalues of any $\mathbf{S} \in \mathfrak{su}(2)$ to be half-integer or integer multiples of a constant c in quantum mechanics.

Representations of Spin

- This means that $\mathcal{H} = \mathbb{C}^n$ for some n for modelling spin.

Spin-1/2 Particle

$$\mathcal{H} = \mathbb{C}^2 = \mathrm{span}_{\mathbb{C}}\{|+z\rangle, |-z\rangle\}$$

$$\frac{\sigma_z}{2}|+z\rangle = \frac{1}{2}|+z\rangle, \frac{\sigma_z}{2}|-z\rangle = -\frac{1}{2}|-z\rangle,$$

What is Topology?

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- How do we even know our model for spin actually reflects reality?

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- Our derivation predicts that only half-integer or integer spins are possible.
- How do we even know our model for spin actually reflects reality?
- To understand this, we need to understand topology.

Source: Henry Segerman and Keenan Crane

Spin-Statistics Theorem

- This theorem creates a connection between our model of spin and the real behaviour of particles.
- What are the statistics?

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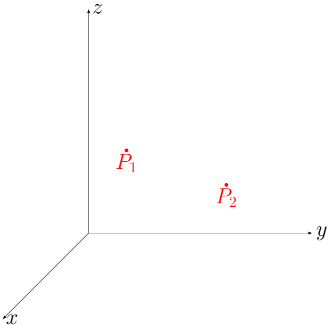
Statistics

Let $\mathcal{H}_1, \mathcal{H}_2$ be identical Hilbert Spaces.

$$\mathcal{H}_{1,2} := \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$|\psi_1, \psi_2\rangle \in \mathcal{H}_{1,2} P$$

Spin-Statistics Theorem



Statistics

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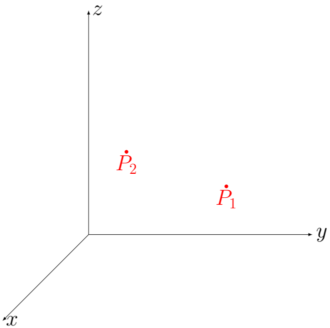
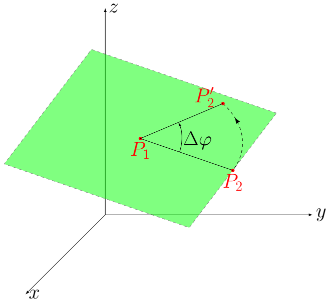
Spin-Statistics Theorem



Statistics

$$|\psi_2, \psi_1\rangle \propto |\psi_1, \psi_2\rangle \in \mathcal{H}_{1,2} P \Rightarrow |\psi_2, \psi_1\rangle = e^{i\theta} |\psi_1, \psi_2\rangle, \theta \in \mathbb{R}$$

Spin-Statistics Theorem



- For complicated physics reasons I will not go into, the phase shift is related to the winding angle $\Delta\varphi$ and the statistic is ν.

Statistics

$$|\psi_1, \psi_2\rangle \mapsto e^{i\nu\Delta\varphi}|\psi_1, \psi_2\rangle$$

Spin-Statistics Theorem

Statistics

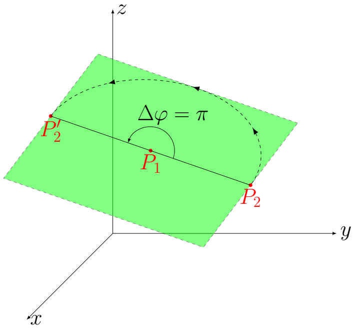
$$|\psi_1, \psi_2\rangle \mapsto e^{i\nu\Delta\varphi}|\psi_1, \psi_2\rangle$$

$$|\psi_2, \psi_1\rangle = e^{i\theta}|\psi_1, \psi_2\rangle$$

Theorem (Spin-Statistics)

If the space is Lorentz invariant and causality holds, then $e^{i\theta} \in \{-1, 1\}$ if and only if particles P_1, P_2 have half-integer or integer spin.

Spin-Statistics Theorem



Statistics

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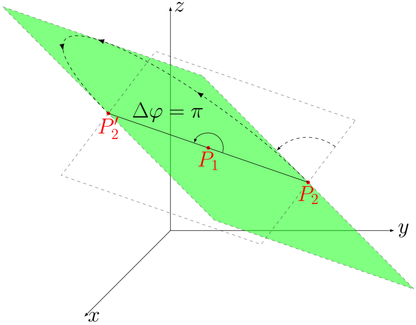
Spin-Statistics Theorem



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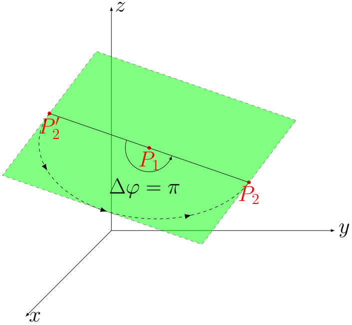

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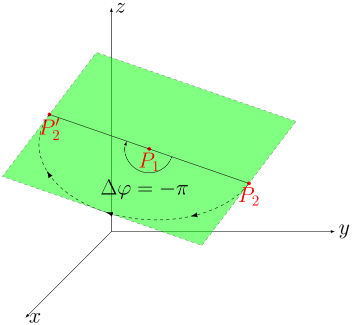

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- Hence we know our spins being half-integer/integer is a topological property for $n \geq 3$ dimensions.
- If we tried to do this in a 2D topology, then the rotation argument wouldn't work.
- Indeed, the Spin-Statistics theorem breaks down in a 2D topological space.

Fractional Spin in 2D Topology

- The spin representations in 2D come from $SO(2, \mathbb{R})$.
- This group is abelian, so its Lie Algebra is abelian and it has a trivial Lie Bracket.

2D Representations

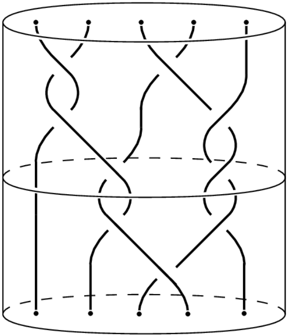
$$A, B \in \mathfrak{so}(2, \mathbb{R}) \Rightarrow AB = BA$$

$$\Rightarrow [A, B] = AB - BA = 0$$

- This means any $r \in \mathbb{Q}$ is a possible eigenvalue, giving a fractional spectrum for spin states.

Fractional Spin in 2D Topology

- This gives rise to Anyons, which are quasi-particles of this type.



- Studying these objects is possible through braid groups and HOMFLY polynomials.