### Quantum Topology

Sepeni Saryad

#### Motivatio

Applications
Physics Origins

## Spin and Representation

resentation Theory What is Spin?

Deriving Spin Representations of Spin

## Spin What is Topology

Spin-Statistics Theorem

Fractional Spin in 2D Topology

## Quantum Topology: {Physics} ∩ {Pure Maths}

A crash course on quantum mechanics and its surprisingly deep roots in pure mathematics.

Sepehr Saryazdi

University of Sydney

February 11, 2023

### Overview

### Quantum Topology

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Motivation

Physical
Applications

Spin and Representation Theory What is Spin? Deriving Spin Representations of Spin

Topology of Spin What is Topology Spin-Statistics Theorem Fractional Spin in 2D Topology

- 1 Motivation
  Physical Applications
  Physics Origins
- 2 Spin and Representation Theory What is Spin? Deriving Spin Representations of Spin
- 3 Topology of Spin
  What is Topology?
  Spin-Statistics Theorem
  Fractional Spin in 2D Topology

## **Physical Applications**

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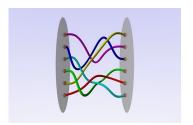
Motivatio

Physical Applications Physics Origins

Spin and Representation

What is Spin?
Deriving Spin
Representations o

Topology of Spin What is Topology Spin-Statistics Theorem  Microsoft announced earlier this year it has invested in researching Anyons (2-dimensional topological quasi-particles) as a basis for quantum computing.





## **Physical Applications**

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Motivation

Physical

Applications

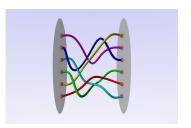
Physics Origins

Spin and Representation Theory What is Spin?

Deriving Spin
Representations o
Spin

Topology of

I opology of Spin What is Topology? Spin-Statistics Theorem Fractional Spin in 2D Topology  Microsoft announced earlier this year it has invested in researching Anyons (2-dimensional topological quasi-particles) as a basis for quantum computing.





 If pulled off, the topological nature of this qubit could make it more robust against random errors.

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Applications

Applications

Physics Origin

## Spin and Representation

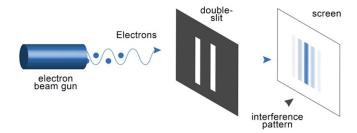
Theory What is Spin?

Deriving Spin Representations of Spin

Topology o

Spin
What is Topology
Spin-Statistics

rneorem Fractional Spin i 2D Topology Where did this all come from?



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#### Motivatior Physical

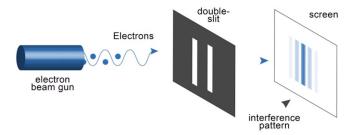
Applications
Physics Origins

# Spin and Representation

What is Spin?
Deriving Spin
Representations of Spin

Spin
What is Topology
Spin-Statistics
Theorem
Fractional Spin in

Where did this all come from?



 De Broglie: Everything must be both a wave and particle.

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Physical Applications Physics Origins

Spin and Rep resentation Theory

What is Spin?
Deriving Spin
Representations of Spin

Spin
What is Topology
Spin-Statistics
Theorem

Theorem
Fractional Spin ir
2D Topology

Born and Schrödinger's trick: Make everything probabilistic!

### Markov Chain Analogy

$$\textit{S} = \{1,2,3\}$$

 $X(t) \in S := \text{state of system at time } t$ 

$$A = \begin{pmatrix} P(1 \to 1) & P(2 \to 1) & P(3 \to 1) \\ P(1 \to 2) & P(2 \to 2) & P(3 \to 2) \\ P(1 \to 3) & P(2 \to 3) & P(3 \to 3) \end{pmatrix}$$
$$X(t+1) = AX(t)$$

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Physics Origins

### Markov Chain Analogy

$$X(t+1) = AX(t) = A^{t}X(1)$$

$$X(1) = \begin{pmatrix} P(X(1) = 1) \\ P(X(1) = 2) \\ P(X(1) = 3) \end{pmatrix}$$

$$= P(X(1) = 1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + P(X(1) = 2) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + P(X(1) = 3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow X = P(X(1) = 1)\mathbf{e}_{1} + P(X(1) = 2)\mathbf{e}_{2} + P(X(1) = 3)\mathbf{e}_{3}$$

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Motivation

Physical

Applications

Physics Origins

Spin and Rep resentation Theory

What is Spin?
Deriving Spin
Representations

Spin
What is Topology:
Spin-Statistics
Theorem
Fractional Spin in

• Now make it quantum...

### Markov Chain Analogy

$$X = P(X(1) = 1)\mathbf{e}_1 + P(X(1) = 2)\mathbf{e}_2 + P(X(1) = 3)\mathbf{e}_3$$
$$X \mapsto |X\rangle, \mathbf{e}_i \mapsto |i\rangle, P(X(1) = i) \mapsto |c_i|^2, c_i \in \mathbb{C}$$

$$\begin{aligned} |X\rangle &= c_1 |1\rangle + c_2 |2\rangle + c_3 |3\rangle \\ |c_1|^2 + |c_2|^2 + |c_3|^2 &= 1 \Leftrightarrow \bar{X} \cdot X = \langle X, X \rangle_{\mathbb{C}} = \langle X | X \rangle = 1 \\ &\Rightarrow |X\rangle \in \mathbb{C}^3, \langle X | X \rangle = 1 \end{aligned}$$

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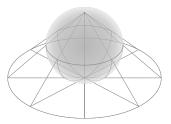
Motivatio

Physics Origins

Spin and Representation

What is Spin?
Deriving Spin
Representations o

Topology of Spin What is Topology? Spin-Statistics Theorem • Generalise to any state space  ${\mathcal H}$  with complex field  ${\mathbb C}.$ 



### **Generalised Quantum State Spaces**

$$\mathcal{S} = \mathcal{H} P := (\mathcal{H} - \{\boldsymbol{0}\})/_{\{\lambda \boldsymbol{v} | \boldsymbol{v} \neq \boldsymbol{0} \in \mathcal{H}, \lambda \neq \boldsymbol{0} \in \mathbb{C}\}}$$

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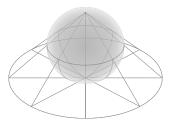
Motivatio

Physics Origins

Spin and Representation

What is Spin?
Deriving Spin
Representations o

Topology of Spin What is Topology Spin-Statistics Theorem • Generalise to any state space  ${\mathcal H}$  with complex field  ${\mathbb C}.$ 



### **Generalised Quantum State Spaces**

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•  $\lambda \mathbf{v} \sim \mathbf{v}$ 

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Motivatio

Physics Origins

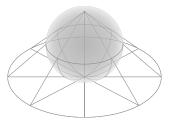
Spin and Rep

What is Spin? Deriving Spin

Representations o Spin

Spin
What is Topology
Spin-Statistics
Theorem
Fractional Spin in
2D Topology

• Generalise to any state space  ${\mathcal H}$  with complex field  ${\mathbb C}.$ 



### **Generalised Quantum State Spaces**

$$\mathcal{S} = \mathcal{H} P := (\mathcal{H} - \{\boldsymbol{0}\})/_{\{\lambda \boldsymbol{v} | \boldsymbol{v} \neq \boldsymbol{0} \in \mathcal{H}, \lambda \neq \boldsymbol{0} \in \mathbb{C}\}}$$

- $\lambda \mathbf{v} \sim \mathbf{v}$
- This is a complex projective space!



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Physical Applications

Physics Origins

## Spin and Representation

What is Spin?
Deriving Spin
Representations of

Topology of Spin What is Topology? Spin-Statistics

Spin-Statistics Theorem Fractional Spin in 2D Topology • Modelling particles in space is as easy as  $L^p(\mathbb{R}^n, \mathbb{C})$ ...

$$\mathcal{H} = L^{2}(\mathbb{R}^{3}, \mathbb{C}) = \left\{ \psi : \mathbb{R}^{3} \to \mathbb{C} \middle| \int_{\mathbb{R}^{3}} |\psi(x, y, z)|^{2} dx dy dz < \infty \right\}$$

Inner product:

$$\langle \psi, \phi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \psi^*(x, y, z) \phi(x, y, z) dx dy dz$$

Or as physicists write it:

$$|\psi\rangle, |\phi\rangle \in \mathcal{HP}$$

$$\langle \psi | \phi \rangle := \langle \psi, \phi \rangle_{\mathcal{H}}$$

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Physics Origins

resentation Theory

Deriving Spin
Representations of Spin

Topology of Spin What is Topology Spin-Statistics Theorem

Theorem
Fractional Spin is

Physical Interpretation of Wavefunctions.

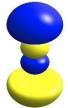
### Probability of Observation in the Continuum

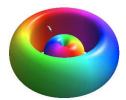
If  $|\psi\rangle \in \mathcal{HP}$  is the wavefunction of a particle p, then

$$\int_{\mathbb{R}^3} |\psi(x,y,z)|^2 dx dy dz = 1$$

$$P(p \in (x, y, z)) \approx |\psi(x, y, z)|^2 dx dy dz$$







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Applications

Physics Origin

Spin and Representation

### Theory

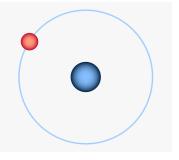
What is Spin?

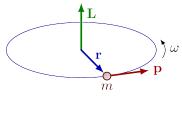
Deriving Spin
Representations

Representations Spin

Spin
What is Topology?
Spin-Statistics

Spin-Statistics Theorem Fractional Spin in 2D Topology • First consider angular momentum.





### Angular Momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

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## Spin and Representation

Theory
What is Spin?

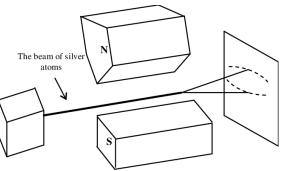
Deriving Spin Representati

Topology

What is Topology Spin-Statistics

> Fractional Spin in 2D Topology

 Shoot neutral silver atoms through a magnetic field, it should not be deflected.



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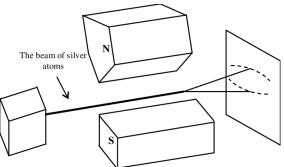
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## Spin and Representation

What is Spin?
Deriving Spin
Representations of

### Topology Spin

What is Topology? Spin-Statistics Theorem Fractional Spin in 2D Topology  Shoot neutral silver atoms through a magnetic field, it should not be deflected.



However, it actually gets deflected!

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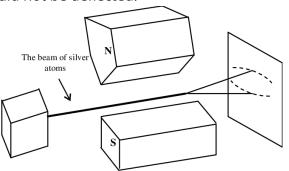
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## Spin and Rep resentation

What is Spin?
Deriving Spin
Representations of
Spin

Spin
What is Topology
Spin-Statistics
Theorem

 Shoot neutral silver atoms through a magnetic field, it should not be deflected.



- However, it actually gets deflected!
- This is due to the electrons being like a little magnet.

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Applications
Physics Origins

Spin and Rep resentation Theory

What is Spin?
Deriving Spin

Representations Spin

Spin What is Topology

Fractional Spin i 2D Topology  This 'magnetism' comes from the electron having an intrinsic angular momentum.





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Motivatio

Applications
Physics Origins

resentation Theory

What is Spin? Deriving Spin Representations o Spin

Spin
What is Topology
Spin-Statistics

Theorem Fractional Spin i 2D Topology  This 'magnetism' comes from the electron having an intrinsic angular momentum.





• This is what we mean by spin.

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Physical

Applications
Physics Origins

Theory
What is Spin?
Deriving Spin
Representations of Spin

Topology of Spin What is Topology? Spin-Statistics Theorem Fractional Spin in 2D Topology • This 'magnetism' comes from the electron having an intrinsic angular momentum.





- This is what we mean by spin.
- It's not actually spinning! It's just our angular momentum models are the most accurate!

### Spin as a "limit"

$$\boldsymbol{S} = \lim_{\boldsymbol{r} \rightarrow \boldsymbol{0}, |\boldsymbol{v}| \rightarrow \infty} \boldsymbol{r} \times \boldsymbol{v}$$

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### Intivatio

Physical Applications Physics Origins

#### Spin and Rep resentation Theory

### Deriving Spin

Spin

## Topology of

What is Topology? Spin-Statistics Theorem Fractional Spin in We need to use group theory.

### **Group Theory**

Let X be a set and ()  $\cdot$  () :  $X \times X \rightarrow X$  a function.

$$G=(X,\cdot)$$

Example 1:

$$X = \mathbb{R} \setminus \{0\}, a \cdot b := ab$$
  
$$a^{-1} \cdot a = \frac{1}{a}a = 1$$

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#### Motivatic

Physical Applications Physics Origins

# Spin and Representation Theory

Deriving Spin
Representations o

Topology of Spin

What is Topology Spin-Statistics Theorem Fractional Spin in 2D Topology • Groups encode symmetries about your set *X*.



### **Group Theory**

$$G=(X,\cdot)$$

### Example 2:

 $X = \{$ all ways to change a Rubik's cube $\}$ 

$$() \cdot () = a$$
 composition of actions

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Motivatio

Physical Applications

Physics Origins

Spin and Rep resentation Theory

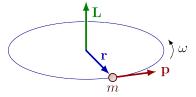
Deriving Spir

Representations of

Topology of

What is Topology
Spin-Statistics

Fractional Spin ir 2D Topology  Let's think about what symmetries we want spin/angular momentum to have.



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### Motivatio

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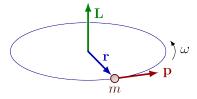
Spin and Rep resentation

# Theory What is Spin? Deriving Spin

Representations of Spin

Spin
What is Topology
Spin-Statistics
Theorem
Fractional Spin in

 Let's think about what symmetries we want spin/angular momentum to have.



 If we rotate everything about L, the spin should not change!

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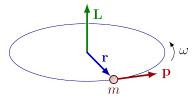
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Spin and Rep resentation Theory

Deriving Spin
Representations of

Spin
What is Topology?
Spin-Statistics
Theorem
Fractional Spin in
2D Topology

 Let's think about what symmetries we want spin/angular momentum to have.



- If we rotate everything about L, the spin should not change!
- This means the symmetry we want to impose are rotations.

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Motivatio

Applications

Physics Origin

resentation
Theory

Deriving Spir

Representations o

Topology o

Spin
What is Topology
Spin-Statistics

Theorem
Fractional Spin ir

• Rotations should preserve orientation, so  $\det R > 0$ .

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#### Motivatior

Physical Applications Physics Origins

Spin and Representation
Theory

### Deriving Spin

Representations o

## Spin What is Topology Spin-Statistics

Theorem
Fractional Spin is

- Rotations should preserve orientation, so  $\det R > 0$ .
- Rotations should not change lengths of vectors, so  $||R\mathbf{v}|| = ||\mathbf{v}||$ .

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## Motivation Physical Applications

Physical Applications Physics Origins

Spin and Representation Theory

Deriving Spin Representation

Topology of Spin What is Topology? Spin-Statistics Theorem

- Rotations should preserve orientation, so  $\det R > 0$ .
- Rotations should not change lengths of vectors, so  $||R\mathbf{v}|| = ||\mathbf{v}||$ .

### Rotations

$$||R\mathbf{v}|| = ||\mathbf{v}|| \Leftrightarrow (R\mathbf{v})^T (R\mathbf{v}) = \mathbf{v}^T \mathbf{v}$$

$$\Leftrightarrow \mathbf{v}^T (R^T R) \mathbf{v} = \mathbf{v}^T \mathbb{I} \mathbf{v} \Leftrightarrow \mathbf{v}^T (R^T R - \mathbb{I}) \mathbf{v} = \mathbf{0}$$

$$\Rightarrow R^T R = \mathbb{I}$$

$$\Rightarrow \det(R^T R) = 1 \Rightarrow (\det R)^2 = 1 \Rightarrow \det R = \pm 1$$

$$\Rightarrow \det R = 1$$

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Physical Applications Physics Origins

Spin and Representation
Theory

What is Spin?

Deriving Spin

Representations of Spin

Topology of

Spin
What is Topology:
Spin-Statistics
Theorem
Fractional Spin in

### Rotations

$$R^T R = \mathbb{I}, \det R = 1$$

This is the special orthogonal group!

### Special Orthogonal Group

$$\mathsf{SO}(3,\mathbb{R}) = \left\{ R \in \mathsf{Mat}_{3,3}(\mathbb{R}) \middle| R^\mathsf{T} R = \mathbb{I}, \det R = 1 
ight\}$$

 It turns out that this is also a topological group and a manifold!

Quantum Topology

How can we view this as a manifold?

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Motivation

Physical
Applications

Applications
Physics Origins

resentation Theory

What is Spin?

Deriving Spin

Representations

Topology of Spin

Spin
What is Topology:
Spin-Statistics
Theorem
Fractional Spin in
2D Topology

- How can we view this as a manifold?
- Think of it as a set of solutions to a function.

## Manifold Structure of $SO(3, \mathbb{R})$

$$f: \mathbb{R}^9 \cong \mathsf{Mat}_{3,3}(\mathbb{R}) o \mathsf{Mat}_{3,3}(\mathbb{R})$$

$$f(\mathbf{x}) = M^T M - \mathbb{I}$$

Our solution set is then:

$$\mathsf{SO}(3,\mathbb{R})\cong\{\mathbf{x}\in\mathbb{R}^9\mid f(\mathbf{x})=0\}$$

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#### Motivatio (

Physical Applications Physics Origin

Spin and Rep resentation Theory

### Deriving Spin Representations of

Representations of Spin

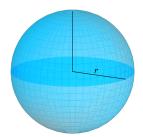
### Spin What is Topolog

Theorem
Fractional Spin ir

Manifold Structure of  $SO(3, \mathbb{R})$ 

$$\mathsf{SO}(3,\mathbb{R})\cong\{\mathbf{x}\in\mathbb{R}^9\mid f(\mathbf{x})=0\}$$

- An example for your mind to imagine is the sphere.
- $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 1 = 0\}$



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Physical Applications Physics Origins

Spin and Rep resentation Theory

Deriving Spin
Representations of

Spin
Topology of

What is Topology Spin-Statistics Theorem

Theorem
Fractional Spin |
2D Topology

### Manifold Structure of SO(3, $\mathbb{R}$ )

$$\mathsf{SO}(3,\mathbb{R})\cong\{\mathbf{x}\in\mathbb{R}^9\mid f(\mathbf{x})=0\}$$

• We can parameterise the sphere using some coordinates, like  $\phi(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ .



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#### lotivatio

Physical Applications Physics Origins

Spin and Rep resentation Theory

## Deriving Spin Representations of

Topology of Spin What is Topology Spin-Statistics Theorem

Spin-Statistics Theorem Fractional Spin in 2D Topology

### Manifold Structure of $SO(3, \mathbb{R})$

$$\mathsf{SO}(3,\mathbb{R})\cong\{\mathbf{x}\in\mathbb{R}^9\mid f(\mathbf{x})=0\}$$

• We can parameterise the sphere using some coordinates, like  $\phi(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ .



• We can similarly parameterise this  $SO(3, \mathbb{R})$  space with some function

$$\phi: \mathbb{R}^3 \subseteq \mathbb{R}^9 \cong \mathsf{Mat}_{3,3}(\mathbb{R}) \to \mathsf{SO}(3,\mathbb{R}).$$

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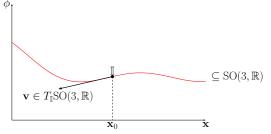
### Deriving Spin

## Manifold Structure of $SO(3, \mathbb{R})$

$$\mathsf{SO}(3,\mathbb{R})\cong\{\boldsymbol{x}\in\mathbb{R}^9\mid \mathit{f}(\boldsymbol{x})=0\}$$

Parameterisation around  $\mathbb{I} \in SO(3, \mathbb{R})$ :

$$\phi: \mathbb{R}^3 \subseteq \mathbb{R}^9 \cong \mathsf{Mat}_{3,3}(\mathbb{R}) \to \mathsf{SO}(3,\mathbb{R})$$



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Sepeni

#### Motivatio

Applications
Physics Origin

Physics Origins

#### Spin and Rep resentation Theory

## Deriving Spin

Representations o Spin

### Spin What is Topolog

Spin-Statistics Theorem

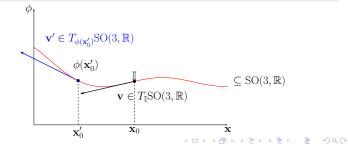
Fractional Spin in 2D Topology

# Manifold Structure of $SO(3, \mathbb{R})$

$$\mathsf{SO}(3,\mathbb{R})\cong\{oldsymbol{x}\in\mathbb{R}^9\mid f(oldsymbol{x})=0\}$$

Parameterisation around  $\mathbb{I} \in SO(3, \mathbb{R})$ :

$$\phi: \mathbb{R}^3 \subseteq \mathbb{R}^9 \cong \mathsf{Mat}_{3,3}(\mathbb{R}) \to \mathsf{SO}(3,\mathbb{R})$$



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#### /otivatio

Physical Applications Physics Origins

Spin and Rep resentation Theory

## Deriving Spin

Representations of Spin

# Spin What is Topology Spin-Statistics

Theorem
Fractional Spin i

## Manifold Structure of $SO(3, \mathbb{R})$

$$\mathsf{SO}(3,\mathbb{R})\cong\{\mathbf{x}\in\mathbb{R}^9\mid f(\mathbf{x})=0\}$$

The tangent space is the solution to our problem!

## Tangent Space of $SO(3, \mathbb{R})$

$$\mathbf{S} \in \mathfrak{so}(3) := T_{\mathbb{I}}\mathsf{SO}(3,\mathbb{R})$$

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#### Activatio

Physical Applications Physics Origins

Spin and Representation
Theory

# Deriving Spin Representations of

Spin

Topology of

Spin
What is Topology
Spin-Statistics
Theorem
Fractional Spin in

## Manifold Structure of $SO(3, \mathbb{R})$

$$\mathsf{SO}(3,\mathbb{R})\cong\{\mathbf{x}\in\mathbb{R}^9\mid f(\mathbf{x})=0\}$$

The tangent space is the solution to our problem!

## Tangent Space of $SO(3, \mathbb{R})$

**S** := Spin of a particle

$$\mathbf{S} \in \mathfrak{so}(3) := T_{\mathbb{I}}\mathsf{SO}(3,\mathbb{R})$$

• This space  $\mathfrak{so}(3,\mathbb{R})$  is called the Lie Algebra of  $SO(3,\mathbb{R})$  and it's where our spins live.

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#### /otivatio

Physical Applications Physics Origins

Spin and Representation
Theory

What is Spin?

Deriving Spin

Representations of Spin

Spin
What is Topology
Spin-Statistics
Theorem

## Special Orthogonal Group

$$\mathsf{SO}(3,\mathbb{R}) = \left\{ R \in \mathsf{Mat}_{3,3}(\mathbb{R}) \middle| R^T R = \mathbb{I}, \det R = 1 
ight\}$$
  $\mathfrak{so}(3) = T_{\mathbb{I}} \mathsf{SO}(3,\mathbb{R})$ 

## Special Unitary Group

$$\mathsf{SU}(2,\mathbb{C}) = \left\{ U \in \mathsf{Mat}_{2,2}(\mathbb{C}) \middle| ar{U}^T U = \mathbb{I}, |\det U| = 1 
ight\}$$
  $\mathfrak{su}(2,\mathbb{C}) = T_{\mathbb{I}} \mathsf{SU}(2,\mathbb{C})$ 

• It turns out that  $\mathfrak{so}(3) \cong \mathfrak{su}(2)!$ 

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#### Motivatio

Physical Applications Physics Origins

Spin and Representation
Theory

## What is Spin? Deriving Spin

Representations of Spin

# Spin What is Topology? Spin-Statistics Theorem

Spin-Statistics Theorem Fractional Spin ir 2D Topology • Since  $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ , physicists work with  $\mathfrak{su}(2)$  instead.

## Basis for $\mathfrak{su}(2)$

$$\begin{split} \mathfrak{su}(2) &= \mathsf{span}_{\mathbb{R}} \left\{ \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \\ \mathfrak{su}(2) &= \mathsf{span}_{\mathbb{R}} \left\{ \frac{\sigma_{x}}{2}, \frac{\sigma_{y}}{2}, \frac{\sigma_{z}}{2} \right\} \end{split}$$

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Physical Applications Physics Origins

# Spin and Representation Theory What is Spin?

Deriving Spin Representations of Spin

Topology of Spin What is Topology Spin-Statistics Theorem  $\mathfrak{su}(2)$ 

$$\mathfrak{su}(2) = \mathsf{span}_{\mathbb{R}}\left\{ rac{\sigma_{\mathsf{X}}}{2}, rac{\sigma_{\mathsf{y}}}{2}, rac{\sigma_{\mathsf{z}}}{2} 
ight\}$$

• Since  $\mathfrak{su}(2)$  is a Lie Algebra, it has a Lie Bracket  $[\cdot,\cdot]:\mathfrak{su}(2)\times\mathfrak{su}(2)\to\mathfrak{su}(2).$ 

## Lie Bracket of su(2)

$$[A, B] := AB - BA$$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk} \frac{\sigma_k}{2}$$

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Representations of

## Lie Bracket of $\mathfrak{su}(2)$

$$\mathfrak{su}(2) = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\sigma_{x}}{2}, \frac{\sigma_{y}}{2}, \frac{\sigma_{z}}{2} \right\}$$

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• No matter the representation of  $SO(3,\mathbb{R})$ , we will always have a non-abelian Lie Algebra with a Lie Bracket of this form.

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Physical Applications Physics Origins

Spin and Representation
Theory
What is Spin?
Deriving Spin
Representations of

Topology of Spin What is Topology Spin-Statistics Theorem Fractional Spin in

## Lie Bracket of su(2)

$$egin{aligned} \mathfrak{su}(2) &= \mathsf{span}_{\mathbb{R}} \left\{ rac{\sigma_{\mathsf{X}}}{2}, rac{\sigma_{\mathsf{y}}}{2}, rac{\sigma_{\mathsf{z}}}{2} 
ight\} \ &[\mathsf{A}, \mathsf{B}] := \mathsf{A}\mathsf{B} - \mathsf{B}\mathsf{A} \ &\left[ rac{\sigma_{\mathsf{i}}}{2}, rac{\sigma_{\mathsf{j}}}{2} 
ight] = i\epsilon_{\mathsf{i}\mathsf{j}\mathsf{k}} rac{\sigma_{\mathsf{k}}}{2} \end{aligned}$$

- No matter the representation of  $SO(3,\mathbb{R})$ , we will always have a non-abelian Lie Algebra with a Lie Bracket of this form.
- This Lie Bracket forces the eigenvalues of any
   S ∈ su(2) to be half-integer or integer multiples of a constant c in quantum mechanics.

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Physical Applications

Spin and Representation

What is Spin?
Deriving Spin
Representations of

Spin Topology of

What is Topology? Spin-Statistics Theorem Fractional Spin in • This means that  $\mathcal{H} = \mathbb{C}^n$  for some n for modelling spin.

## Spin-1/2 Particle

$$\mathcal{H} = \mathbb{C}^2 = \operatorname{span}_{\mathbb{C}}\{|+z\rangle, |-z\rangle\}$$
$$\frac{\sigma_z}{2}|+z\rangle = \frac{1}{2}|+z\rangle, \frac{\sigma_z}{2}|-z\rangle = -\frac{1}{2}|-z\rangle,$$

# What is Topology?

Quantum **Topology** 

 Our derivation predicts that only half-integer or integer spins are possible.

# What is Topology?

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Motivation

Physical Applications Physics Origins

Spin and Representation

Theory
What is Spin?

Deriving Spin Representations o Spin

Spin
What is Topology

What is Topology Spin-Statistics

Theorem
Fractional Spin is
2D Topology

- Our derivation predicts that only half-integer or integer spins are possible.
- How do we even know our model for spin actually reflects reality?

# What is Topology?

Quantum Topology

What is Topology?

- Our derivation predicts that only half-integer or integer spins are possible.
- How do we even know our model for spin actually reflects reality?
- To understand this, we need to understand topology.



Source: Henry Segerman and Keenan Crane

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#### Spin and Rep resentation Theory

What is Spin?
Deriving Spin
Representations

Topology of

What is Topolog
Spin-Statistics
Theorem

Fractional Spin ii 2D Topology

- This theorem creates a connection between our model of spin and the real behaviour of particles.
- What are the statistics?

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## Motivatio

Physical Applications Physics Origins

# Spin and Representation Theory

What is Spin?
Deriving Spin
Representations of
Spin

Spin
What is Topology
Spin-Statistics

Spin-Statistics Theorem Fractional Spin ir 2D Tonology

- This theorem creates a connection between our model of spin and the real behaviour of particles.
- What are the statistics?

## **Statistics**

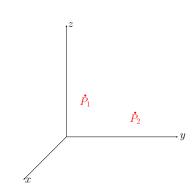
Let  $\mathcal{H}_1, \mathcal{H}_2$  be identical Hilbert Spaces.

$$\mathcal{H}_{1,2} := \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$|\psi_1,\psi_2\rangle\in\mathcal{H}_{1,2}P$$

### Quantum Topology

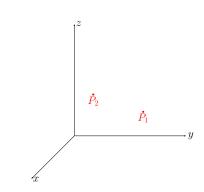
Theorem



$$|\psi_1,\psi_2\rangle\in\mathcal{H}_{1,2}P$$

## Quantum **Topology**

Theorem



$$|\psi_2,\psi_1\rangle \propto |\psi_1,\psi_2\rangle \in \mathcal{H}_{1,2}P \Rightarrow |\psi_2,\psi_1\rangle = e^{i\theta}|\psi_1,\psi_2\rangle, \theta \in \mathbb{R}$$

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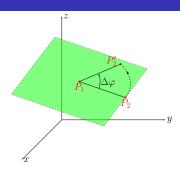
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What is Spin?
Deriving Spin
Representations or Spin

Spin
What is Topology
Spin-Statistics
Theorem
Fractional Spin in



• For complicated physics reasons I will not go into, the phase shift is related to the winding angle  $\Delta \varphi$  and the statistic is  $\nu$ .

$$|\psi_1,\psi_2\rangle\mapsto e^{i\nu\Delta\varphi}|\psi_1,\psi_2\rangle$$

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# Spin and Representation

resentation Theory What is Spin?

Deriving Spin Representations o Spin

Spin What is Topology
Spin-Statistics
Theorem
Fractional Spin in

## **Statistics**

$$|\psi_1,\psi_2\rangle\mapsto e^{i\nu\Delta\varphi}|\psi_1,\psi_2\rangle$$

$$|\psi_2,\psi_1\rangle=e^{i heta}|\psi_1,\psi_2\rangle$$

## Theorem (Spin-Statistics)

If the space is Lorentz invariant and causality holds, then  $e^{i\theta} \in \{-1,1\}$  if and only if particles  $P_1,P_2$  have half-integer or integer spin.

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# Spin and Representation

What is Spin?

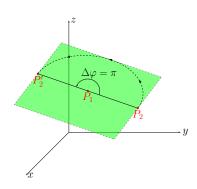
Representations o

## Topology of

What is Topology Spin-Statistics

Theorem
Fractional Spin

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$$\Rightarrow |\psi_2, \psi_1\rangle = e^{i\pi\nu} |\psi_1, \psi_2\rangle$$

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Theorem

$$P_2$$
  $\Delta \varphi = \pi$   $P_2$   $Y$ 

$$\Rightarrow |\psi_2, \psi_1\rangle = e^{i\pi\nu} |\psi_1, \psi_2\rangle$$

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Applications

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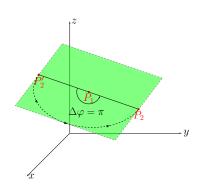
What is Spin?

Representations of Spin

## Topology of

What is Topology Spin-Statistics Theorem

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$$\Rightarrow |\psi_2, \psi_1\rangle = e^{i\pi\nu} |\psi_1, \psi_2\rangle$$

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Applications

# Spin and Representation

Theory

Deriving Spin
Representations

## Topology of

What is Topology Spin-Statistics

> Fractional Spin in 2D Topology

 $\varphi = -\pi$   $\varphi = -\pi$   $\varphi$ 

$$\Rightarrow |\psi_2, \psi_1\rangle = e^{i\pi\nu}|\psi_1, \psi_2\rangle = e^{-i\pi\nu}|\psi_1, \psi_2\rangle \Rightarrow e^{2\pi i\nu} = 1$$

### Quantum **Topology**

Theorem

## **Statistics**

$$\Rightarrow |\psi_2, \psi_1\rangle = e^{i\pi\nu} |\psi_1, \psi_2\rangle = e^{-i\pi\nu} |\psi_1, \psi_2\rangle \Rightarrow e^{2\pi i\nu} = 1$$

$$\Rightarrow \nu \in \{0, 1\} \mod 2$$

$$\Rightarrow e^{i\theta} \in \{-1, 1\}$$

 Hence we know our spins being half-integer/integer is a topological property for n > 3 dimensions.

### Quantum Topology

Spin-Statistics

$$\Rightarrow |\psi_2, \psi_1\rangle = e^{i\pi\nu} |\psi_1, \psi_2\rangle = e^{-i\pi\nu} |\psi_1, \psi_2\rangle \Rightarrow e^{2\pi i\nu} = 1$$

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- Hence we know our spins being half-integer/integer is a topological property for  $n \ge 3$  dimensions.
- If we tried to do this in a 2D topology, then the rotation argument wouldn't work.

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Physical Applications Physics Origins

#### Spin and Rep resentation Theory

What is Spin?
Deriving Spin
Representations of Spin

Spin
What is Topology
Spin-Statistics
Theorem
Fractional Spin ir

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$$\Rightarrow e^{i\theta} \in \{-1, 1\}$$

- Hence we know our spins being half-integer/integer is a topological property for  $n \ge 3$  dimensions.
- If we tried to do this in a 2D topology, then the rotation argument wouldn't work.
- Indeed, the Spin-Statistics theorem breaks down in a 2D topological space.

# Fractional Spin in 2D Topology

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Motivation
Physical
Applications
Physics Origins

resentation Theory What is Spin? Deriving Spin Representations of Spin

Topology of Spin What is Topology? Spin-Statistics Theorem Fractional Spin in 2D Topology

- The spin representations in 2D come from  $SO(2,\mathbb{R})$ .
- This group is abelian, so its Lie Algebra is abelian and it has a trivial Lie Bracket.

## 2D Representations

$$A, B \in \mathfrak{so}(2, \mathbb{R}) \Rightarrow AB = BA$$
  
  $\Rightarrow [A, B] = AB - BA = 0$ 

• This means any  $r \in \mathbb{Q}$  is a possible eigenvalue, giving a fractional spectrum for spin states.

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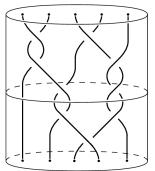
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Theory
What is Spin?

What is Spin? Deriving Spin Representations o Spin

Spin
What is Topology:
Spin-Statistics
Theorem
Fractional Spin in

2D Topology

 This gives rise to Anyons, which are quasi-particles of this type.



 Studying these objects is possible through braid groups and HOMFLY polynomials.