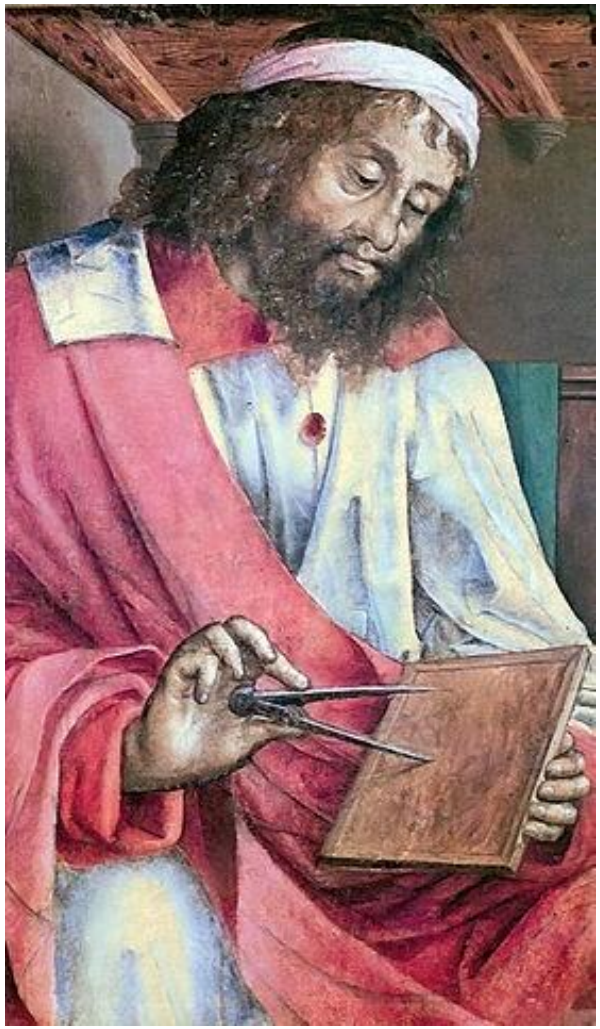
Video 1.1
Sampath Kannan

What is an algorithm?

Muhammad ibn Musa al-Khwarizmi: gave rise to the word "algorithm"



Euclid: Inventor of an algorithm for computing greatest common divisors

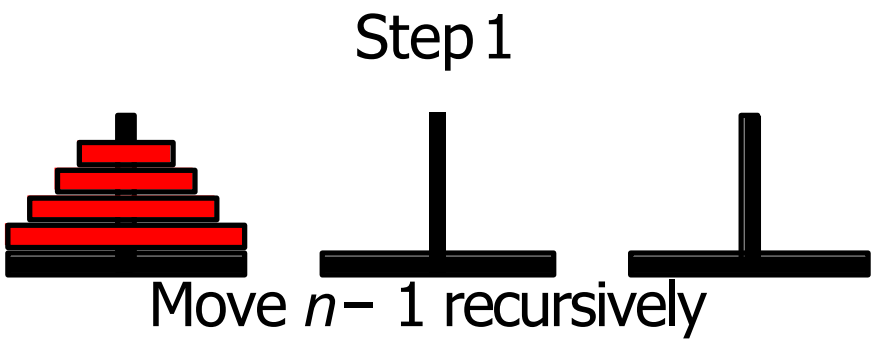
Why study algorithms?

As programs get complicated, thinking algorithmically allows us to:

- reason about their correctness and efficiency before implementing them
- focus on techniques for solving problems
- understand relationship between different computational problems

Induction + Algorithm Design

- A fundamental idea in algorithm design – solve a problem on bigger data sets using your knowledge of how to solve it on smaller ones.
- This idea embodies the proof technique of Mathematical Induction.
- Example: Towers of Hanoi



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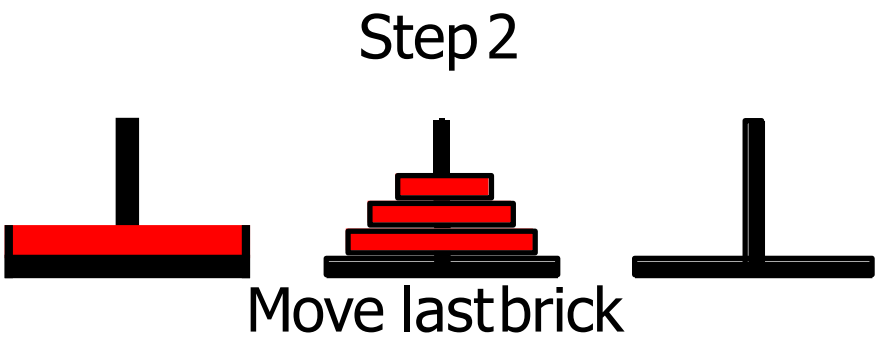

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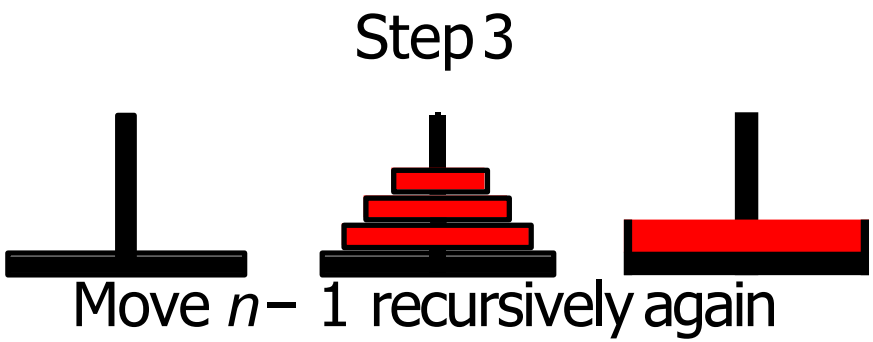

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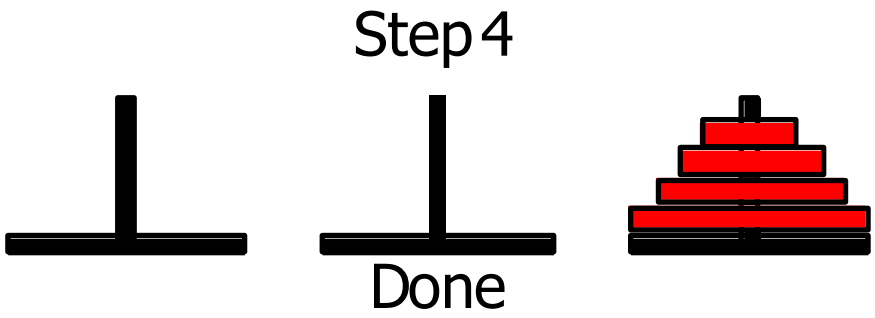

- Move top n-1 disks from rod A to rod B
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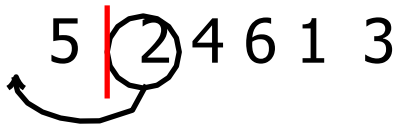
- Move top n-1 disks from rod A to rod B
- Move disk 1 from rod A to rod C
- Move the n-1 disks from rod B to rod C
- How long does this take? How can this be analyzed with **induction**?

Another Example: Insertion Sort



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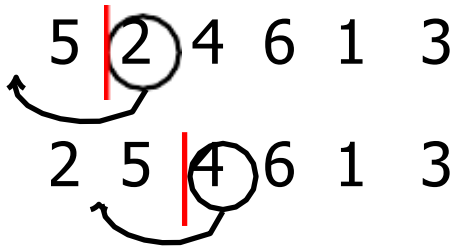

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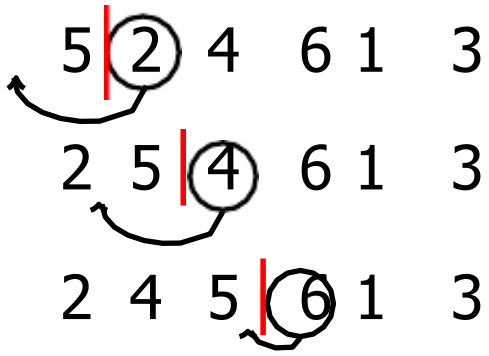

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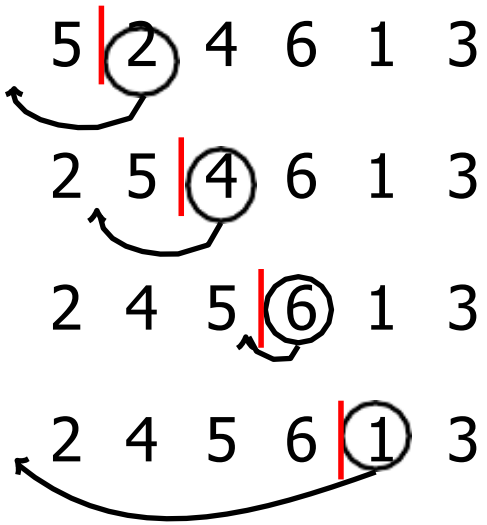

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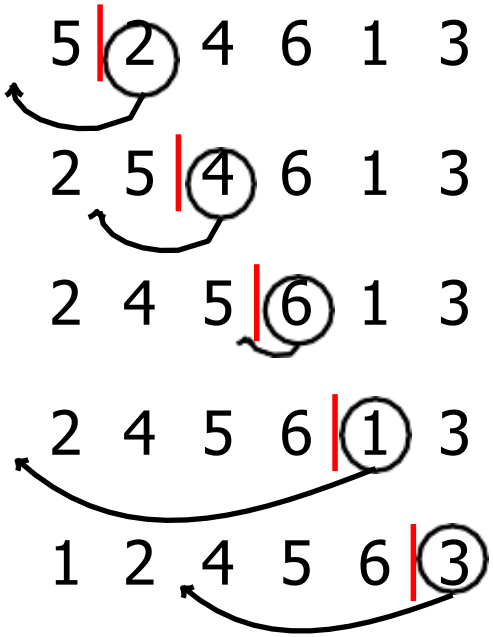

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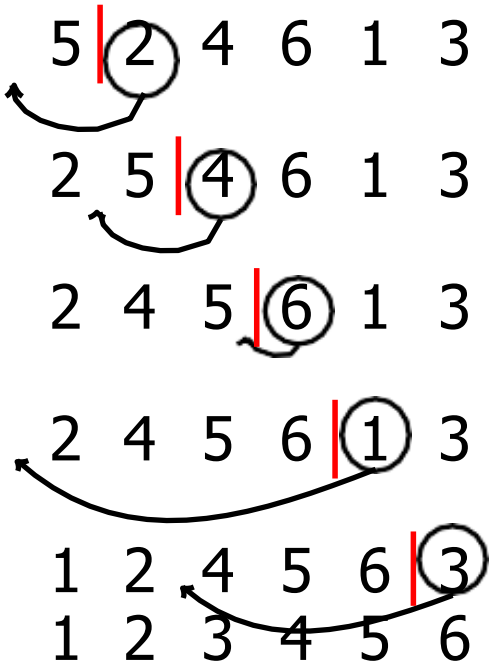

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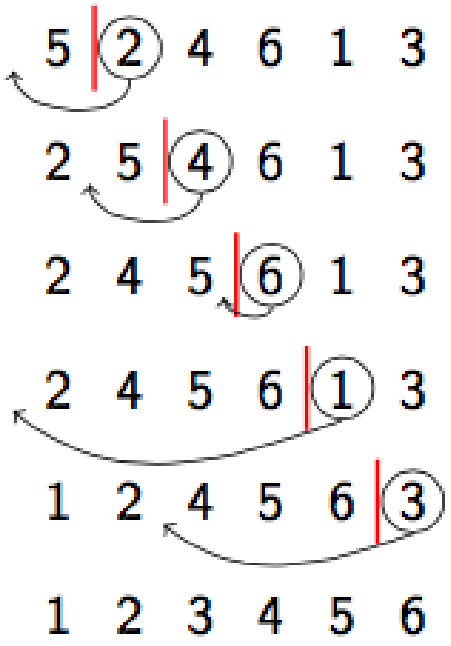

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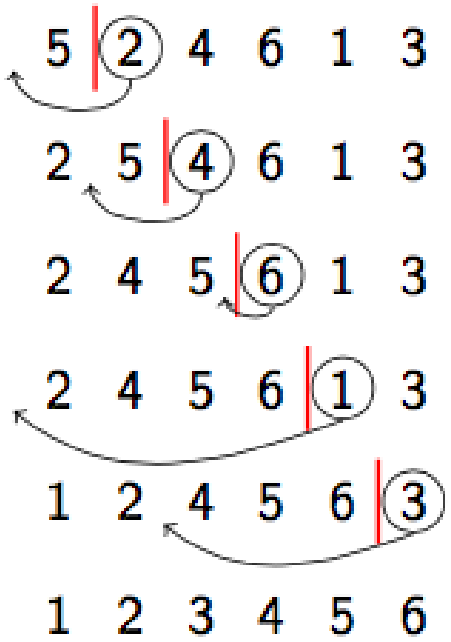

If we've already sorted the first **k elements** of the array, how long does it take to place the next element?

Recurrence Relations

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- How can we write the runtime of Towers of Hanoi using a recurrence?

Recurrence Relations

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- $T(n-1) =$ # operations required to solve a tower with $n-1$ disks
- Can we write $T(n)$ using $T(n-1)$?

Recurrence Relations

Towers of Hanoi recurrence: $T(n) = 2T(n-1) + 1$

Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

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Towers of Hanoi: Runtime

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- substituting in for $T(n-1)$:

Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping** substituting in for $T(n-1)$:

- $T(n) = 2(2T(n-2)+1)+1$

Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- substituting in for $T(n-1)$:
- $T(n) = 2(2T(n-2)+1) + 1$
- $T(n) = 4T(n-2)+2+1$

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- Can we generalize this to k?

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- $T(n) = 2^k T(n-k) + (\sum_{i=0}^{k-1} 2^i)$

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Result: Solving Towers of Hanoi requires $2^n - 1$ operations!

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- In general, kth iteration of the loop: at most $k - 1$ swaps required

Recurrence Relations: Back to Insertion Sort

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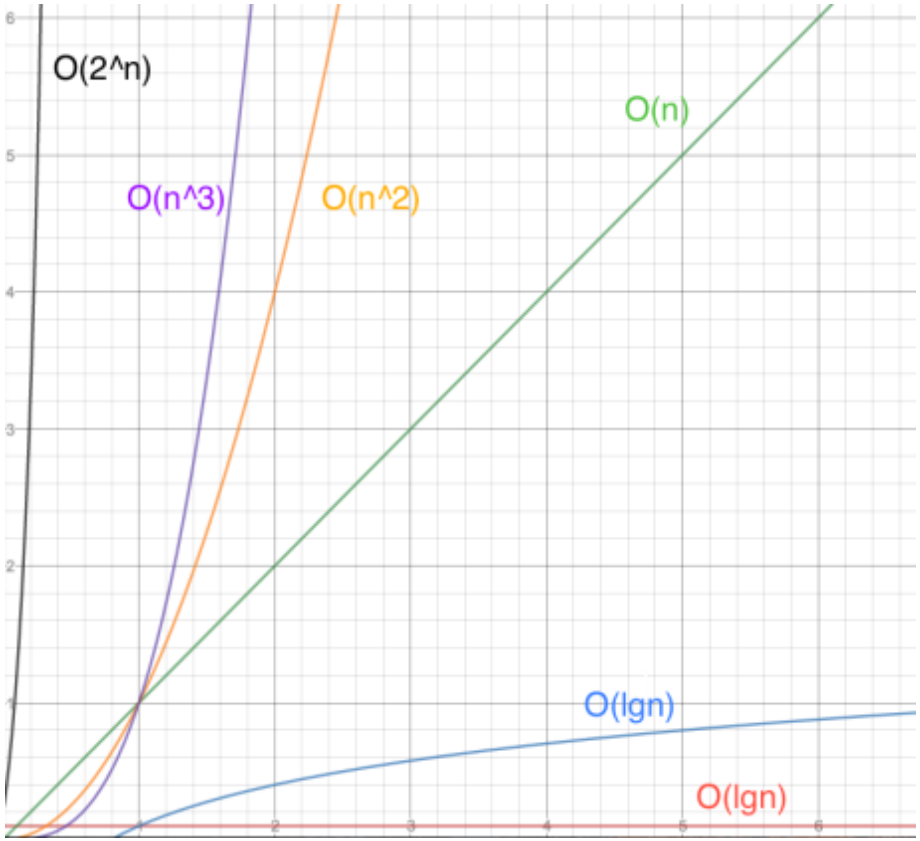
Number of swaps required for insertion sort: $\frac{n(n-1)}{2}$

Video 1.2
Sampath Kannan

Asymptotic Bounds

Motivation:



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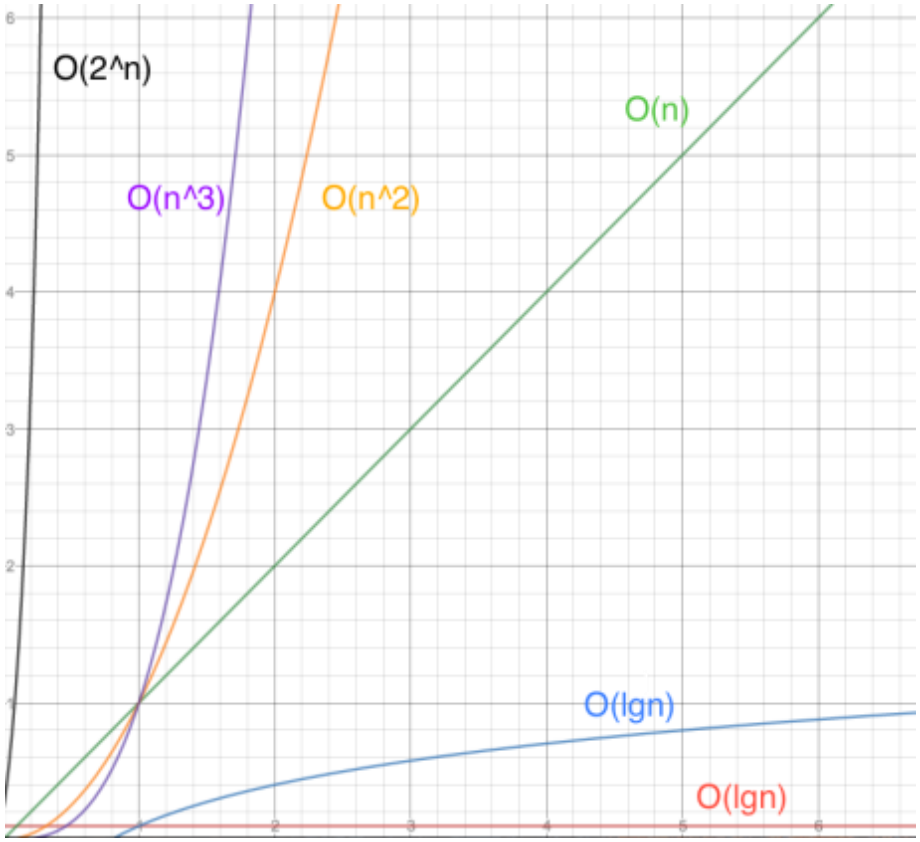

- Essentially a way to compare functions without worrying about their behavior on small n.

  Big-Oh is like $\leq$ (ignoring constant factors), and Big-Omega is like $\geq$

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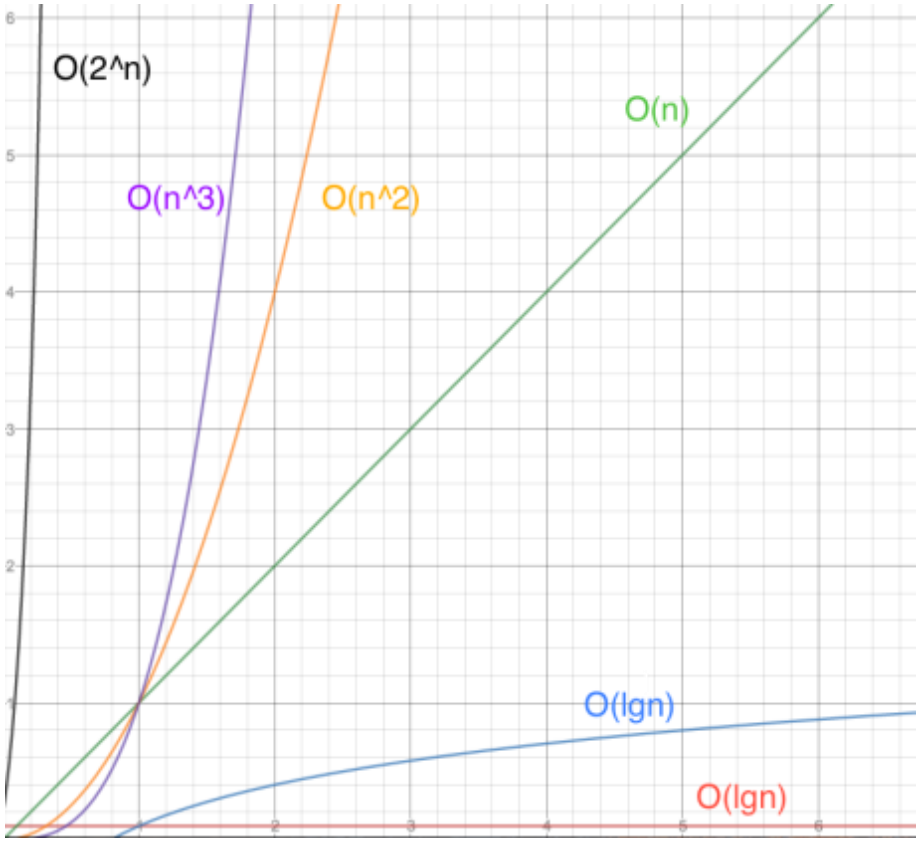

- Essentially a way to compare functions without worrying about their behavior on small n. In this sense Big-Oh is like $\leq$ (ignoring constant factors), and Big-Omega is like $\geq$
- Gives us an idea of how fast a function grows

- Note: $O(f(n))$ is a **set**.
  - $O(n^2)$: the set of all function that do not grow faster than n^2

Asymptotic Bounds: Examples

Some elements of $O(n^2)$:

- $2n^2 \in O(n^2)$
- $100n^2 + n + 1 \in O(n^2)$
- $n \in O(n^2)$

Some elements of $\Omega(n^2)$:

- $2n^2 \in \Omega(n^2)$
- $\frac{n^2}{1000} n \in \Omega(n^2)$
- $n^3 \in \Omega(n^2)$

What is the complexity of insertion sort?

- $T(n) = T(n-1) + n$
- $T(n) = \frac{n^2+n}{2}$
- $T(n) \in O(n^2)$

Insertion sort has a runtime of $O(n^2)$

Complexity

More on Insertion Sort...

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    for i <- 1 to length(A)
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- Each iteration requires no swaps! (constant time to check the first element)
- $\sum_{i=1}^{n} 1 = n \in O(n)$

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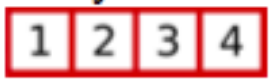

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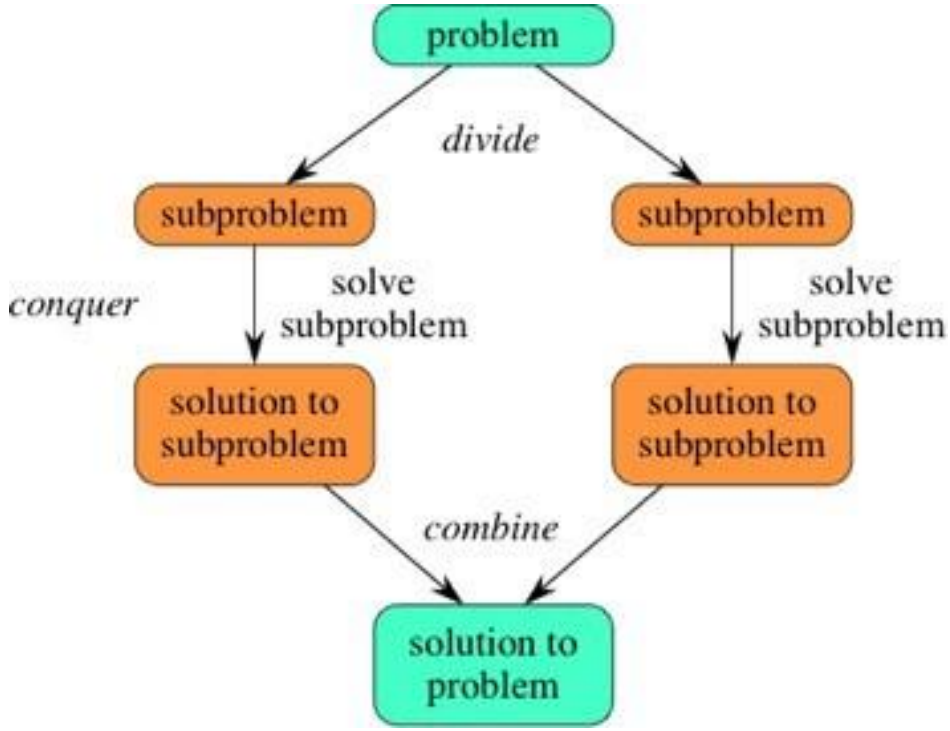
Takeaway: Can't assume anything about the input. Always assume the worst case!

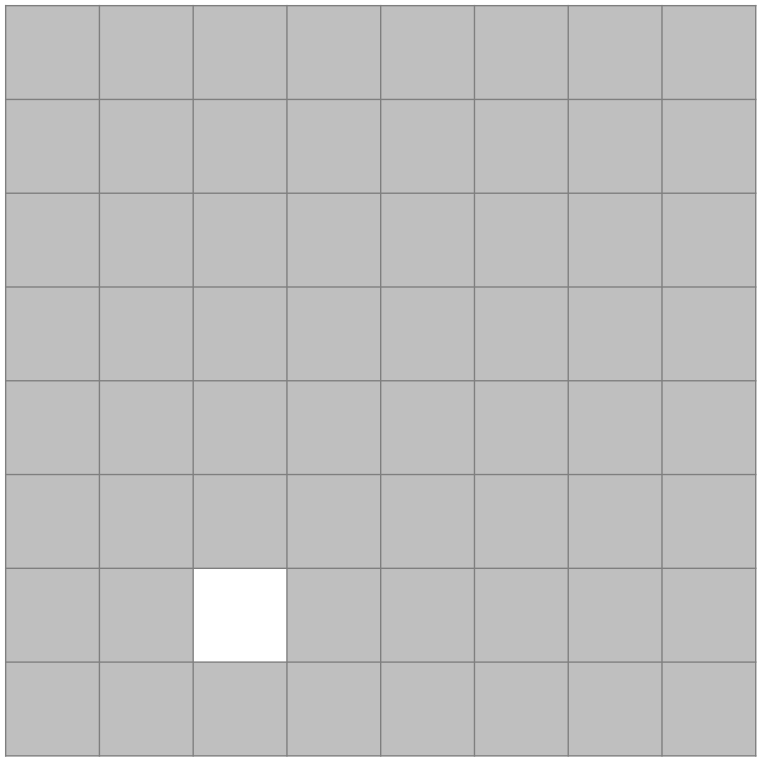
Algorithm Design: Divide and Conquer Paradigm

Idea: Solve a problem by splitting it into pieces, solving those pieces recursively, and merging them to solve the larger problem



Divide and Conquer Example: Triominos

- Input: *NxN* grid (assume N is a power of 2) with a single square removed, and a supply of corner shaped triomino tiles
- Goal: Fill the grid without any overlapping tiles



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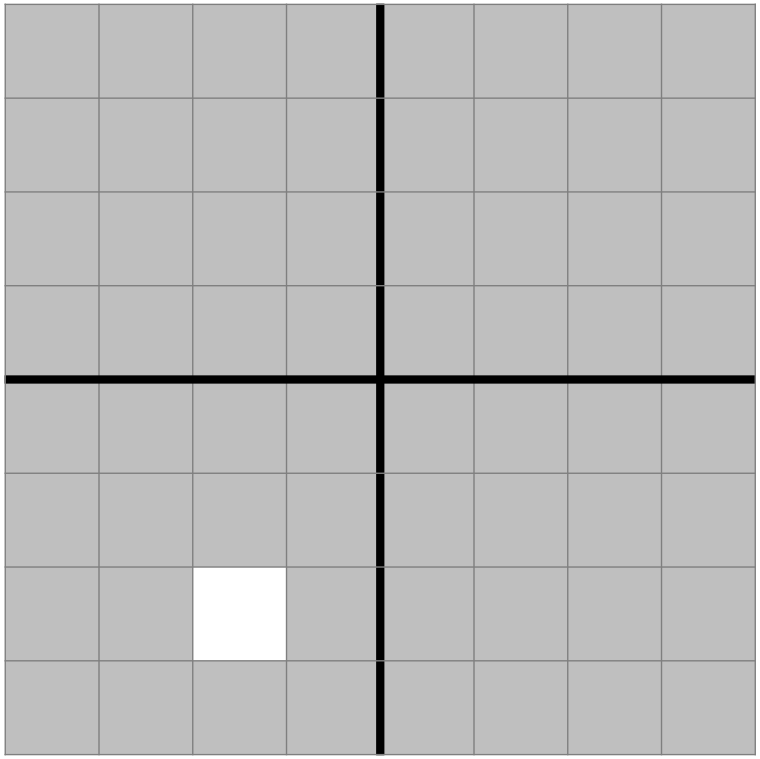

Algorithm:

- **Divide** the grid into 4 squares (size $2^{n-1} \times 2^{n-1}$).
- note: 1 of these 4 squares contains the missing piece

Video 1.3
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Binary Search

- How long does it take to search for an element in an array? O(n)
- Idea: Can we do better if we know that the array is sorted?

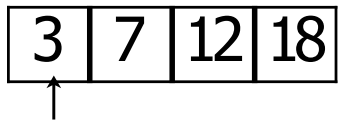
```
Binary-search(A, val):
    return Binary-search(A, val, 0, (length(A)-1))
```

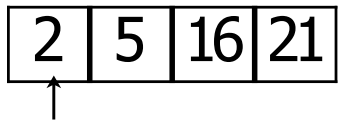
```
Binary-search(A, val, low, high): if
    high < low
        return -1 (not found)
    mid <- (low + high) / 2  if
    A[mid] > val
        return Binary-search(A, val, low, mid-1)
    else if A[mid] < val
        return Binary-search(A, val, mid+1, hi)
    else return mid
```

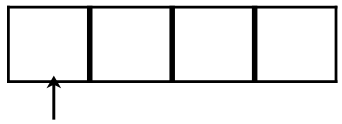
- Each step of the algorithm, the size of the input halves.
- $T(n) = T(\frac{n}{2}) + 1$
- How to solve this recurrence: How many times can we halve N before reaching 1? $\frac{N}{2}$, $\frac{N}{4}$, ...
- $\frac{N}{2^k} = 1, k = lg_2 N$
- binary search runs in $O(lgN)$

Merging two sorted lists

Input: two sorted arrays of size n and m Output: a single sorted array of size n+m







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merge(A, B):
    C = new array[len(A) + len(B)]
    i, j, k <- 0
    while i < len(A) and j < len(B):
        if A[i] < B[j]:
            C[k] <- A[i]
            i++, k++
        else:
            C[k] <- B[j]
            j++, k++
    while i < len(A):
        C[k++] <- A[i++]
    while j < len(B):
        C[k++] <- B[j++]
    return C
```

- How long does this take?
- Every time a comparison is made, either i or j is incremented
- Total number of comparisons is n + m
- merging runs in $O(n + m)$ time

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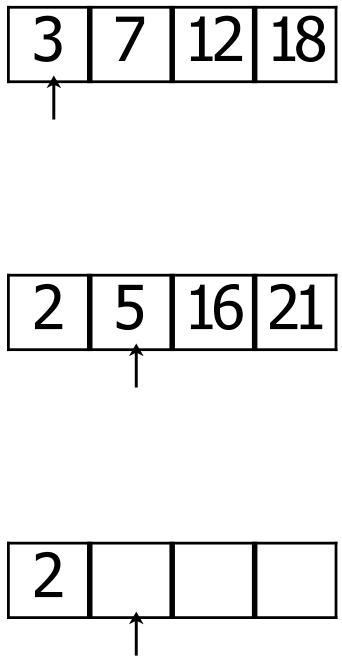


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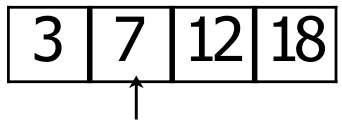

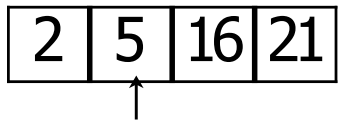

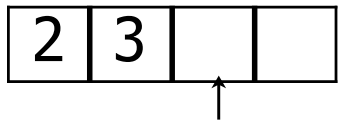


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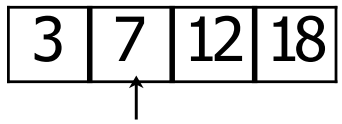

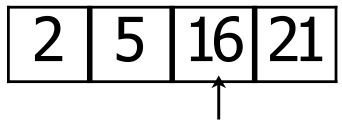

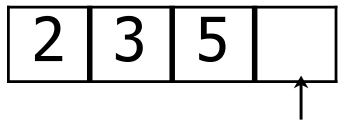


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Input: An array of size n, Output: A sorted array of size n

Can we apply the Divide and Conquer paradigm to sorting? Idea: Split the array, sort halves recursively, **merge** the result


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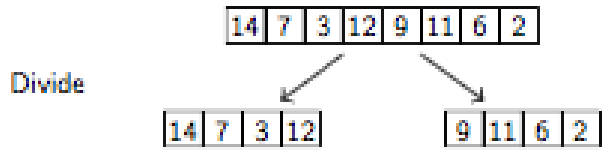


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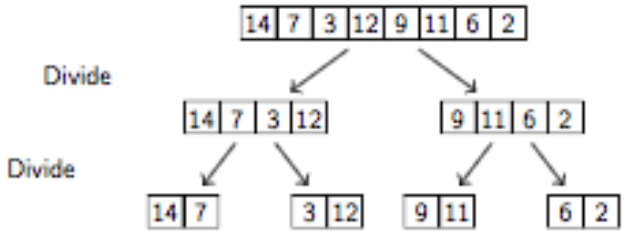


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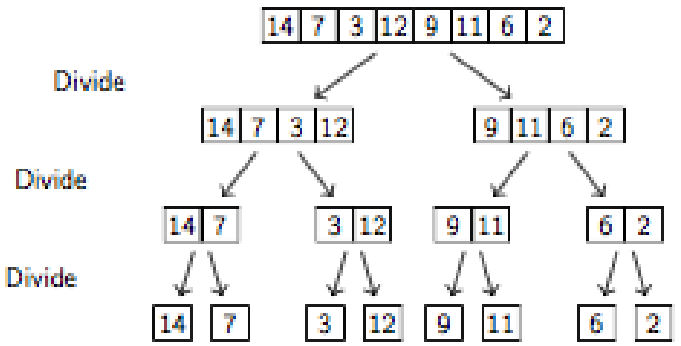

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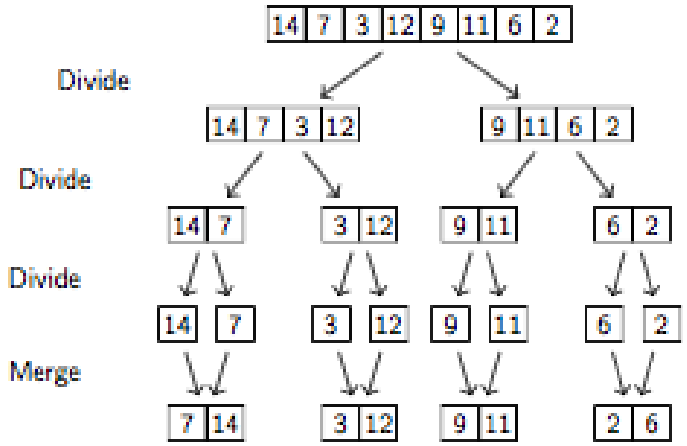

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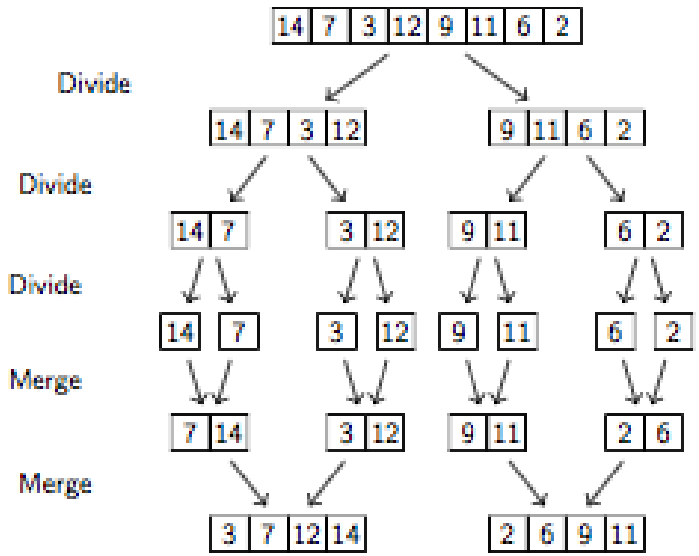

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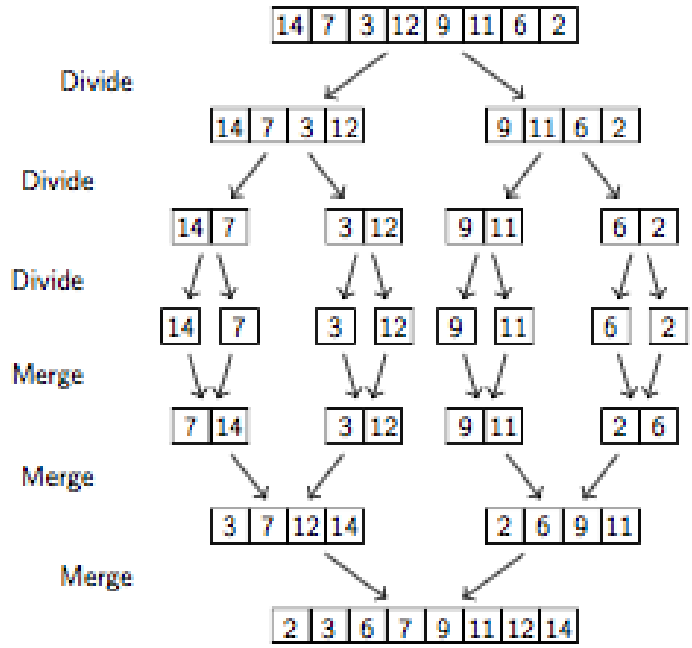


Video 1.4
Sampath Kannan

Algorithm Design: Using Randomness

- Remember from InsertionSort: Algorithm performance can depend on the input:
  - on a sorted list: $O(n)$ comparisons
  - on a reversed list: $O(n^2)$ comparisons
  - In general: somewhere between n and $\frac{n(n+1)}{2}$ comparisons
  - However, the **worst-case** is still $O(n^2)$
- An "adversary" can repeatedly construct an input to our algorithm that causes it to perform as poorly as possible
- Can we prevent our algorithm performance from depending on the input?
  - Shift the dependency: from **input** to **randomization**
- Idea: Write algorithms that toss a coin!

First: An Introduction to Probability

- For a stronger introduction, see:
  https://www.coursera.org/learn/probability- intro

- *Random Variable*: A function X from the results of an experiment to numbers
- $E[X]$: the expected value of the random variable X (a "weighted average
  - Formula: $E[X] = \Sigma\, i * P(X = i)$ (for all values i that X can take on)
  - Example:
    - Roll a 6-sided die. Let X = the value that the die lands on. What is $E[X]$?
    - X can take on each of the values 1 through 6, each with probability $\frac{1}{6}$
    - $E[X] = 1\frac{1}{6} + 2\frac{1}{6} + .. + 6\frac{1}{6} = \frac{21}{6} = 3.5$

Intro to Probability: Continued

- What is the expected sum of two dice?
- X = the sum of two dice. Want to find $E[X]$.
- X can take on values from 2…12
- E.x. $P(X = 5)$. Can result from two die rolls of:
  - (1, 4)
  - (4, 1)
  - (2, 3)
  - (3, 2)
- $P(X = 5) = 4\frac{1}{36} = \frac{1}{9}$
- $E[X] = \sum_{i=2}^{12} i * P(X = i) = 2\frac{1}{36} + 3\frac{2}{36} + \ldots + 12\frac{1}{36}$

sum	2	3	4	5	6	7	8	9	10	11	12
probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Calculation is not trivial. Solution: Linearity of Expectation!

Intro to Probability: Continued

- *Linearity of Expectation*: For n random variables, $X_1, .., X_n$, $E[X_1 + .. + X_n] = E[X_1] + .. + E[X_n]$
- Example:
  - What is the expected sum of rolling 2 dice?
  - let X_i be the random variable denoting the value of the i'th die rolled
  - let X be the r.v. denoting the sum of all 2 dice
  - then $X = X_1 + X_2$
  - $E[X] = E[X_1 + X_2]$
  - $E[X] = E[X_1] + E[X_2]$ (by lin. of exp.)
  - as shown above, for each i, $E[X_i] = 3.5$
  - $E[X] = 3.5 + 3.5 = 7$

Expectation Example: Hat Checking

n people go to a restaurant, take off their hats and throw them in a pile. Afterwards, they each take a hat from the pile at random. What is the expected number of people who get their hat back?

- We can analyze using random variables!
- Let:
  - X = the number of people who get their hats back

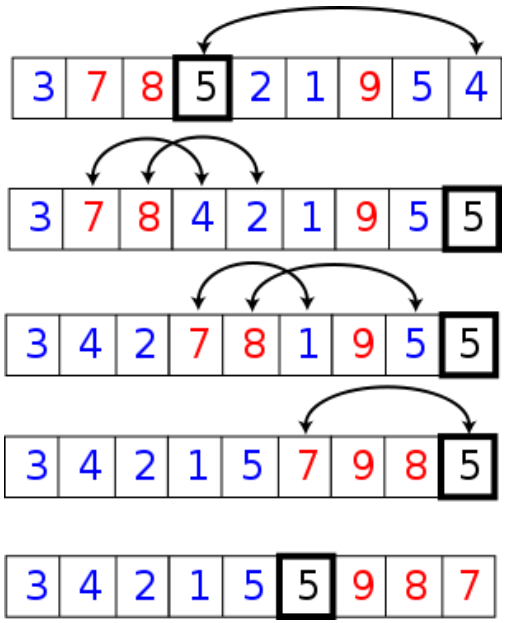
$$X_i = \begin{cases} 1 & \text{person } i \text{ chooses their own hat} \\ 0 & \text{person } i \text{ doesn't choose their own hat} \end{cases}$$

- What is $E[X_i]$?
  - From the definition:
  - $E[X_i] = 1 * P(\text{choose their hat}) + 0P(\text{don't choose their hat})$
  - $E[X_i] = P(\text{choose hat}) = \frac{1}{n}$
- Again, $X = X_1 + X_2 + .. + X_n$
- $E[X] = E[X_1 + .. + X_n] = E[X_1] + .. + E[X_n]$ by lin. of exp.
- $E[X] = n\frac{1}{n} = 1$

In expectation, one person will correctly take their own hat!

Quicksort: An Introduction

Goal: Another sorting algorithm that uses divide-and-conquer

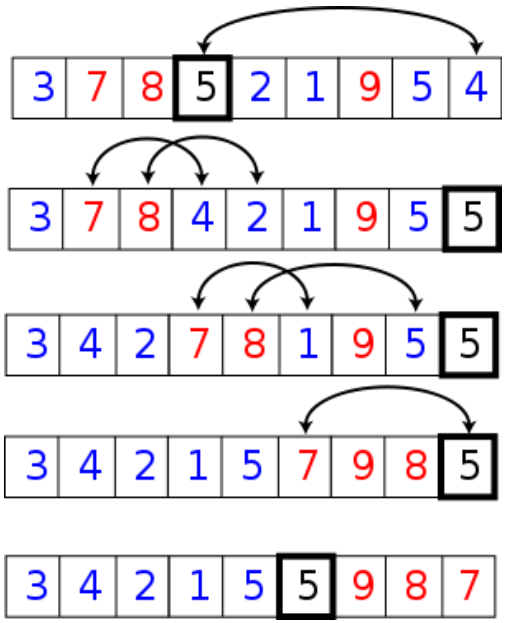


- Idea:
  - Select an element in the array
  - Partition the other elements of the array around it
- Is the array more sorted than it was before?
- Answer: yes!
- Next step: recursively sort the left and right sides of the array as well.

Problem: What about "adversarial inputs"? This algorithm will perform better on some inputs than others.

Quicksort: Randomized

Can we write an algorithm for sorting that uses coin tossing (randomness)?



- Idea:
  - *Randomly* select an element in the array
  - Partition the other elements of the array around it
  - Recursively sort the left and right sides of the array

Result: Another divide and conquer algorithm for sorting, that uses **randomness.**

Video 1.5
Sampath Kannan

Quicksort

Idea: Choose an element at random. Partition the array around this element. Recursively sort the left and right side.

Quicksort

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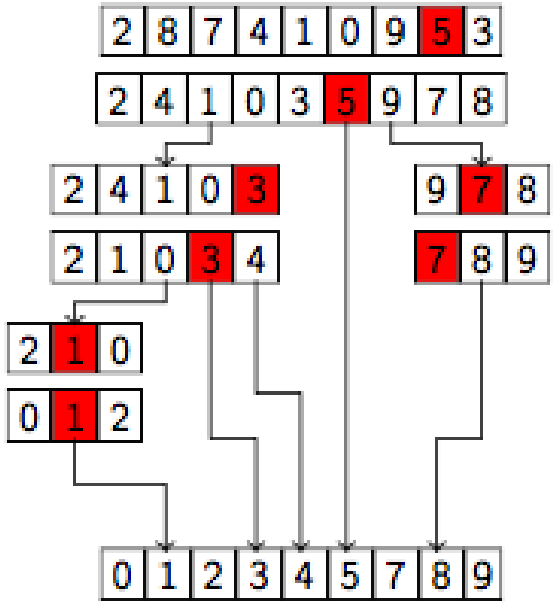
```
quicksort(A):
    quicksort(A, 0, len(A)-1)

quicksort(A, lo, hi):
    if (hi >= lo) return
    pivot_location <- partition(A, lo, hi)
    quicksort(A, lo, pivot_location - 1)
    quicksort(A, pivot_location + 1, hi)

partition(A, lo, hi):
    pivot_index <- random(lo, hi)
    swap(A, pivot_index, hi)
    pivot <- A[hi]
    i <- lo, j <- hi, C <- new array
    for k = lo to hi-1:
        if A[k] <= pivot:
            C[i++] <- A[k]
        else:
            C[j--] <- A[k]
        k++
    C[i] <- A[hi] (copy the pivot in)
    copy C[lo:hi] back into A
    return i
```



Quicksort

Quicksort (compare to Mergesort)

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- divide-and-conquer algorithm
- First partition, then sort recursively

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Quicksort (compare to Mergesort)

- divide-and-conquer algorithm
- First partition, then sort recursively
- Can be done with no extra space
- runtime: See next slide

Quicksort: Analysis

- First: the recurrence for quicksort
- Step 1: Partition requires $O(n)$
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- $T(n) = T(k) + T(n - k - 1) + O(n)$

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Worst case (bad partition):

- partition does not split array at all
  at every step ($k = 1$ or $n - 1$)
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Best case (good partition):

- partition splits array evenly at ever step ($k = \frac{n}{2}$)
- $T(n) = T(\frac{n}{2}) + T(\frac{n}{2}) + O(n)$
- $T(n) = O(n \lg n)$ (recall from mergesort)

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- Step 1: Partition requires $O(n)$
- Step 2: Recursively sort left and right sides of the array
  - What are the sizes of these two arrays?
  - k and $n - k - 1$, for some k
- $T(n) = T(k) + T(n - k - 1) + O(n)$

Worst case (bad partition):

- partition does not split array at all at every step ($k = 1$ or $n - 1$)
- $T(n) = T(1) + T(n - 1) + n$
- $T(n) = O(n^2)$ (similar to insertion sort)

Best case (good partition):

- partition splits array evenly at ever step ($k = \frac{n}{2}$)
- $T(n) = T(\frac{n}{2}) + T(\frac{n}{2}) + O(n)$
- $T(n) = O(n \lg n)$ (recall from mergesort)

How does the algorithm perform on average?
We can analyze with **expectation**

Quicksort: Analysis

Recurrence for quicksort:

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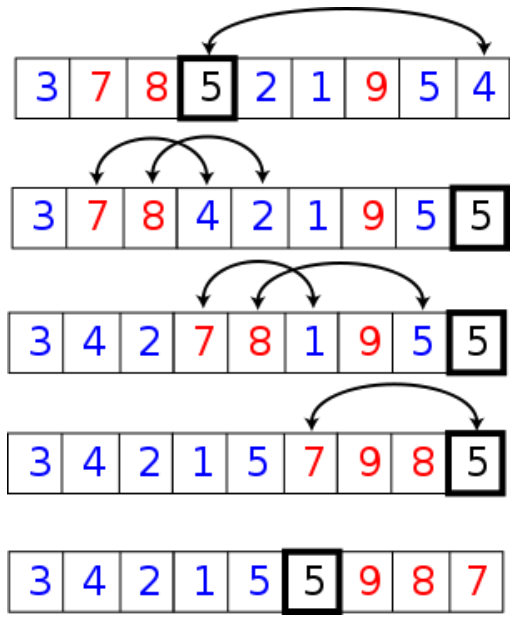
Partition step



Idea: Any two elements are never compared more than once

- What happens after an element is compared to the partitioning element?
  - these two elements won't be compared again

Quicksort: Analysis

Analyze with random variables:

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- Then $X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$
- $E[X] = E[\Sigma \Sigma X_{ij}] = \Sigma \Sigma E[X_{ij}]$ by lin. of exp.
- Recall: $E[X_{ij}] = 1 * P(X_{ij} = 1) + 0 * P(X_{ij} = 0)$
- $E[X_{ij}] = P(X_{ij} = 1)$

Quicksort: Analysis

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Analyze with random variables:

What is the probability that e_i and e_j are compared?

- e_i and e_j will be compared if either is selected as a pivot
- e_i and e_j will not be compared if some e_k, $i < k < j$ is selected as a pivot **first**
  - e_i will be to the left of e_k, and e_j will be to the right.

Quicksort: Analysis

- Which pivots must be chosen for e_i and e_j to be compared?
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- $E[X_{ij}] = \frac{2}{(j-i-1)+2} = \frac{2}{j-i+1}$
- $E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$
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Result: Randomized Quicksort makes an expected $O(nlgn)$ comparisons!

Quick Select

Goal: select the kth smallest element of an array

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Option 1:

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Key Idea:

- When we partition the array, the kth smallest element will only be on one side of this partition
- No need to recursively sort both sides of the array: Only the side containing the element we want

Quickselect

```
quickselect(A, k):
    quickselect(A, 0, len(A)-1, k)

quickselect(A, lo, hi, k):
    if (hi == lo) return A[lo]
    pivot_location <- partition(A, lo, hi)
    if pivot_location == k:
        return A[k]
    else if pivot_location < k:
        return quickselect(A, lo, pivot_location - 1, k)
    else :
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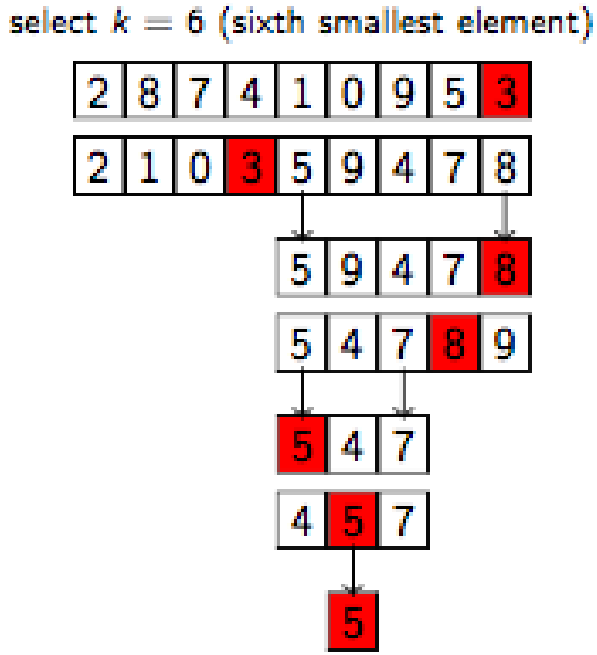

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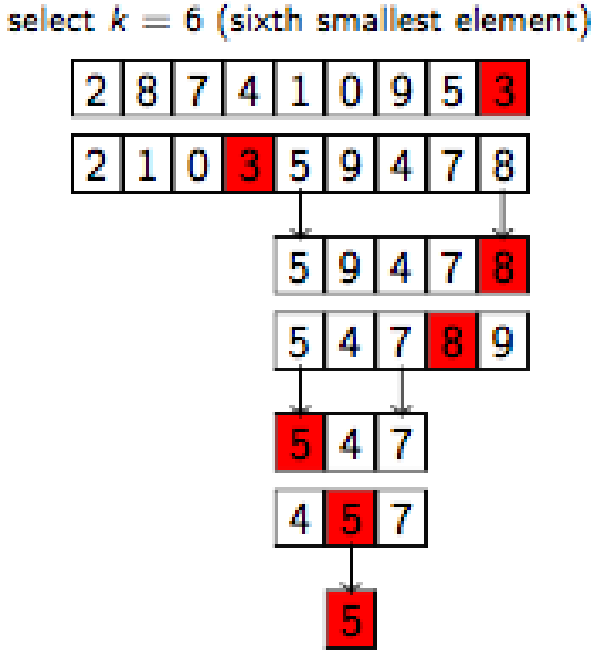
- Analysis?



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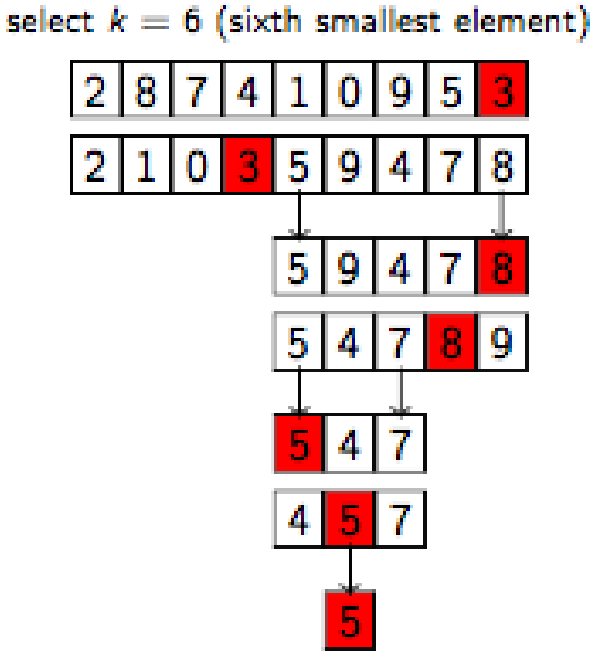

- Analysis?
  - We will use a similar analysis to Quicksort
  - What will change? Are certain elements less likely to be compared?

Quickselect: Analysis

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- case 1: $k < i < j$.
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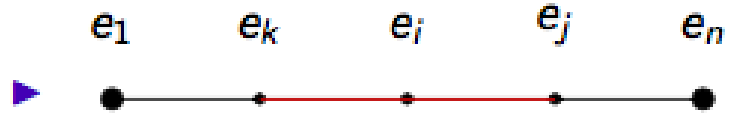

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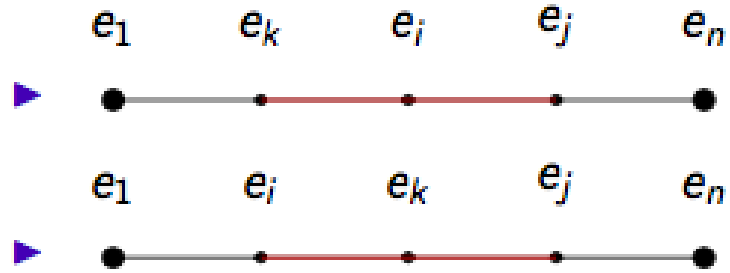

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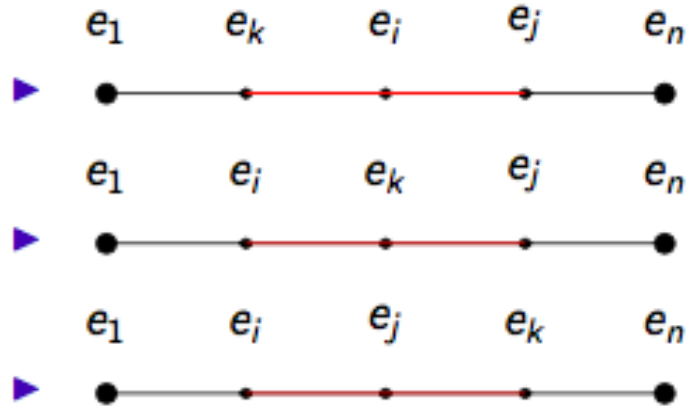

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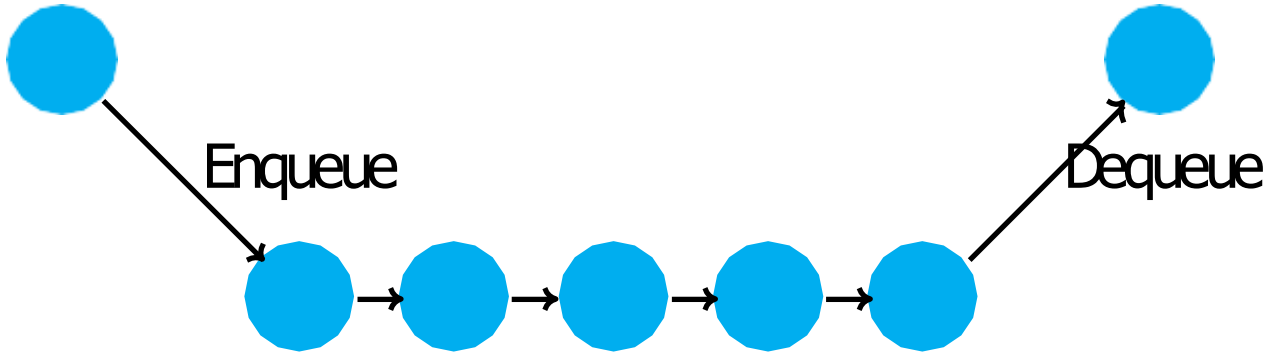
Outcome:

- Quickselect is faster than quicksort!
- Note: quickselect is randomized
- Can we make it deterministic, and still keep the worst case $O(n)$?
- Yes, with some extra work

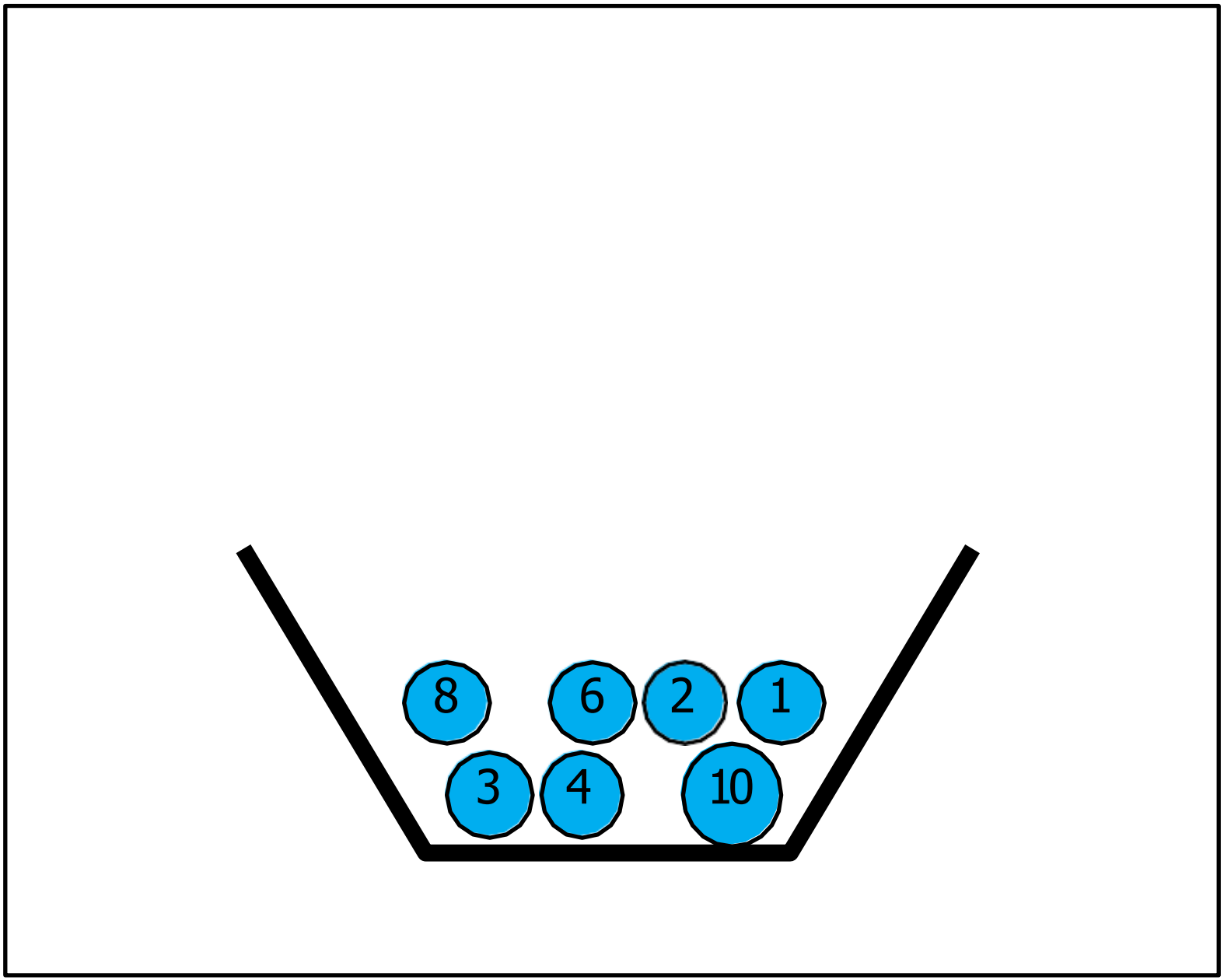
Video 1.6
Sampath Kannan

Queues



- Sometimes we want to extract elements not in the order we insert them but instead in the order of some given keys. We call this a *priority queue*
- For example your operating system is constantly getting jobs to complete, it needs a fast way of getting the highest priority job to schedule next

Operations of Priority Queues



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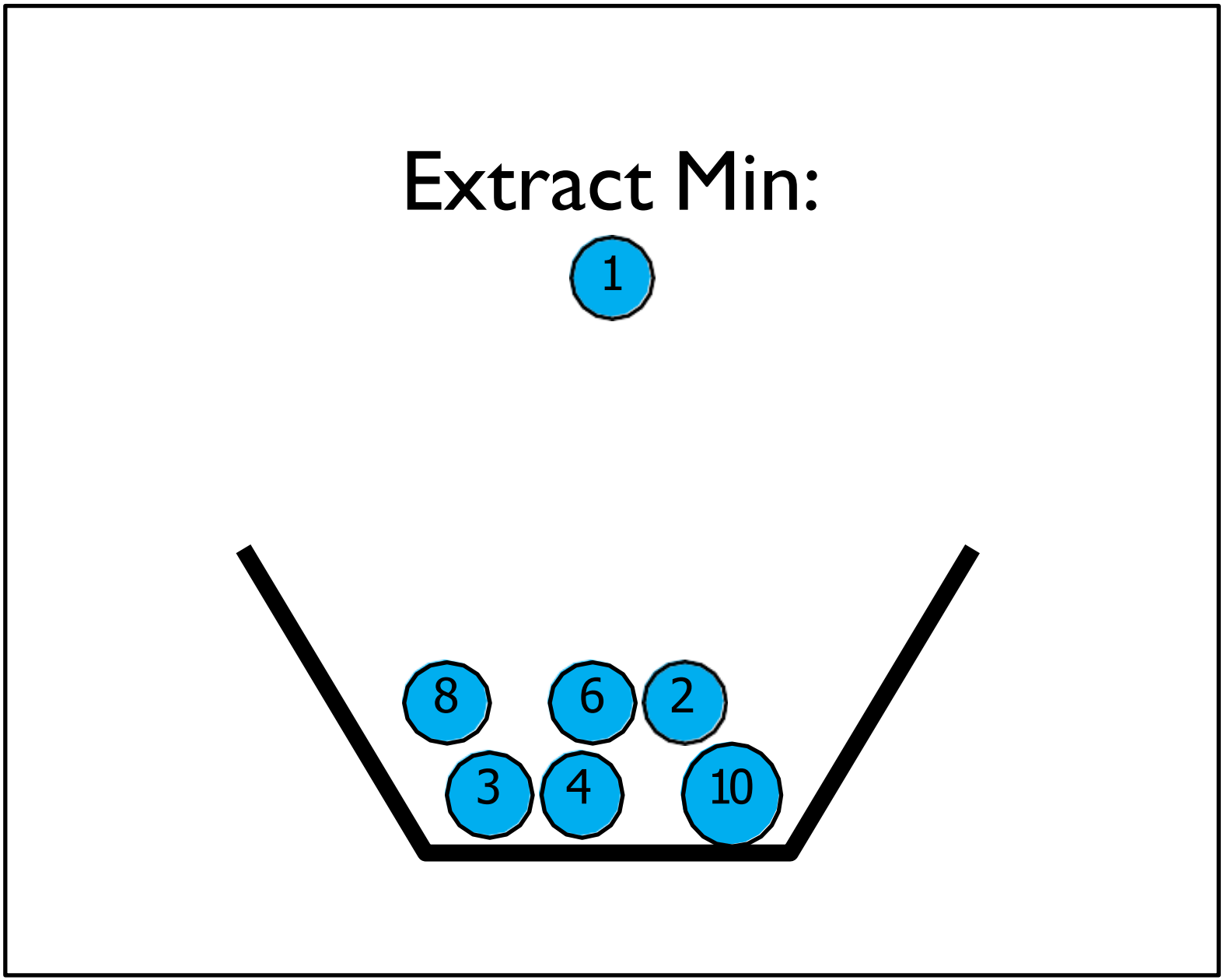

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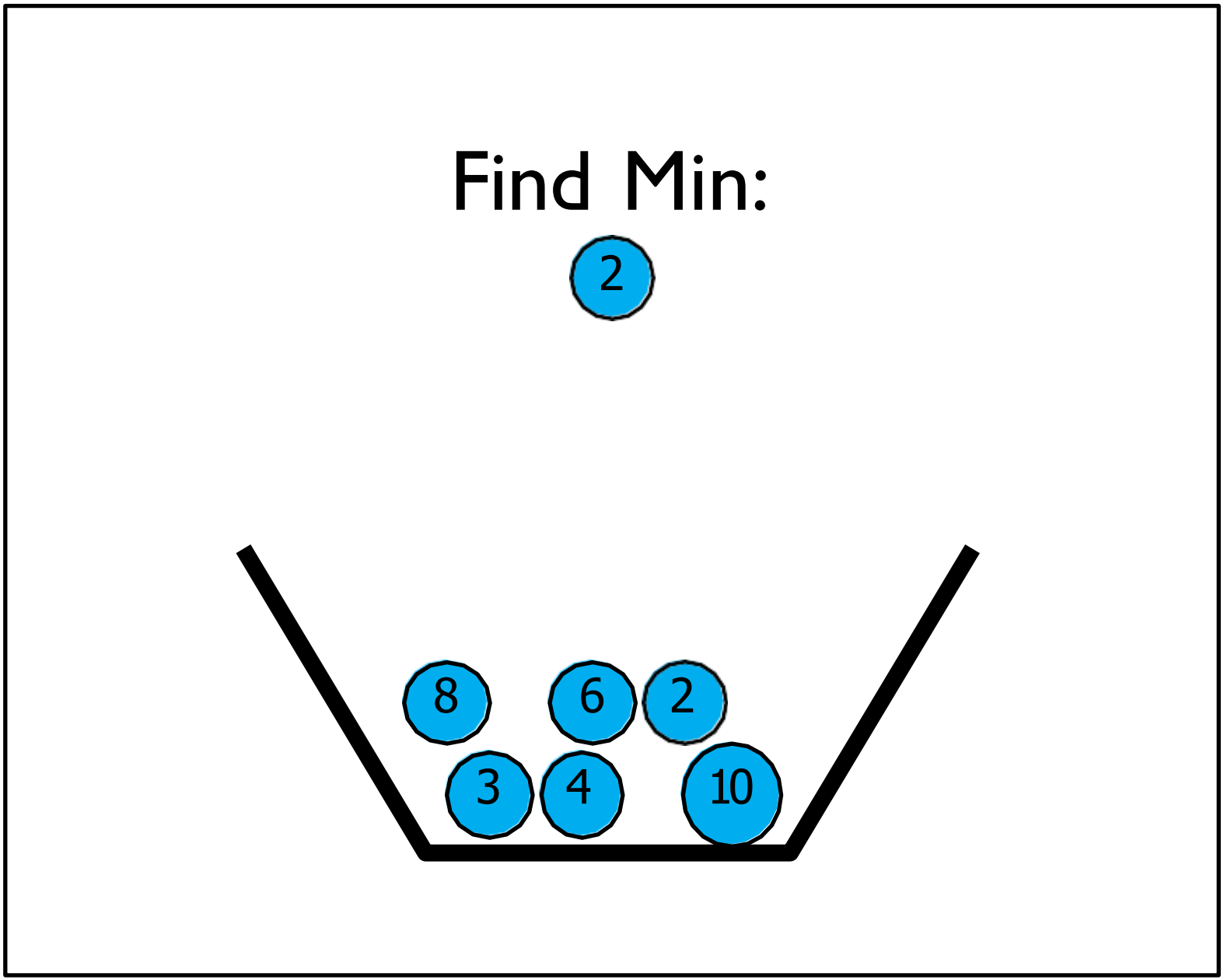
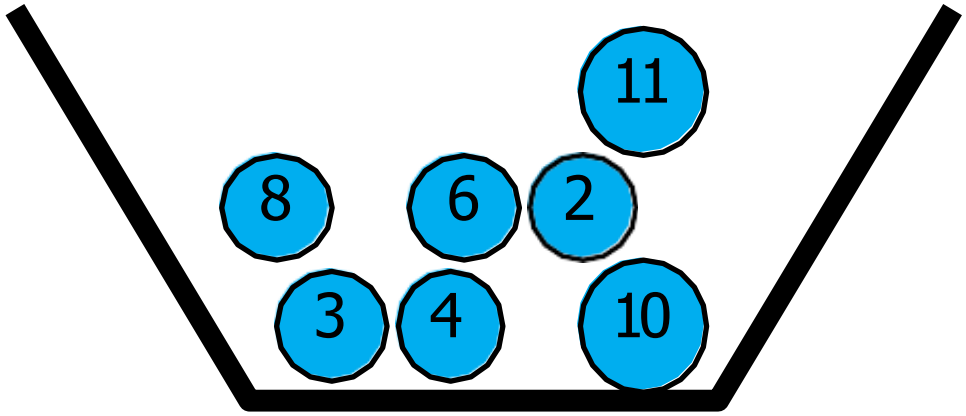


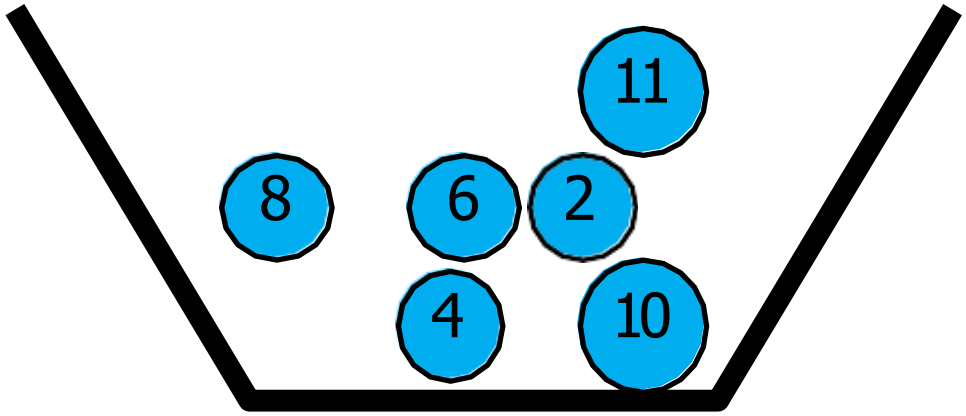
Operations of Priority Queues

Insert(11):



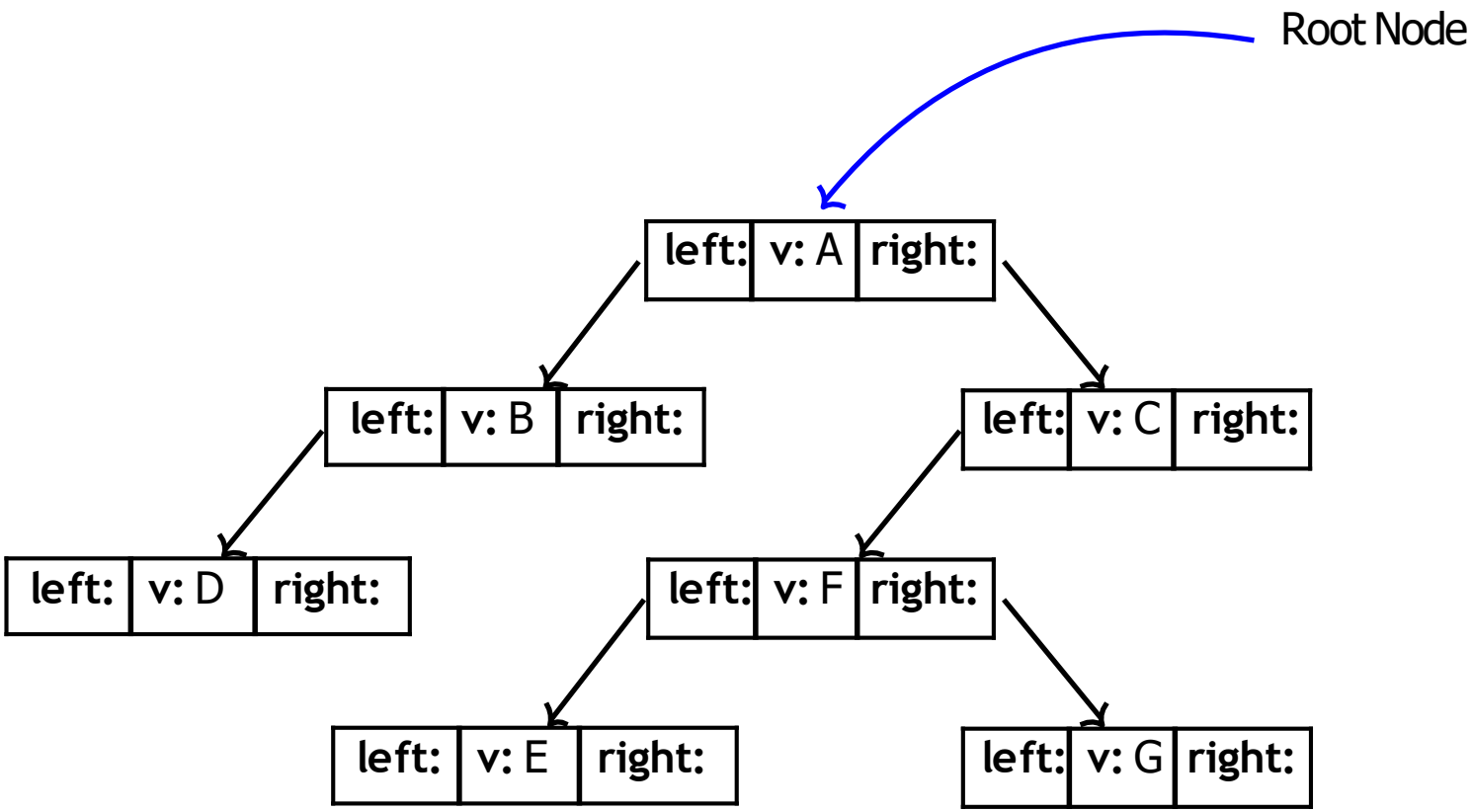
Operations of Priority Queues

Delete(3):



Trees

In order to make an efficient priority heap we will use a more general data structure called a tree.



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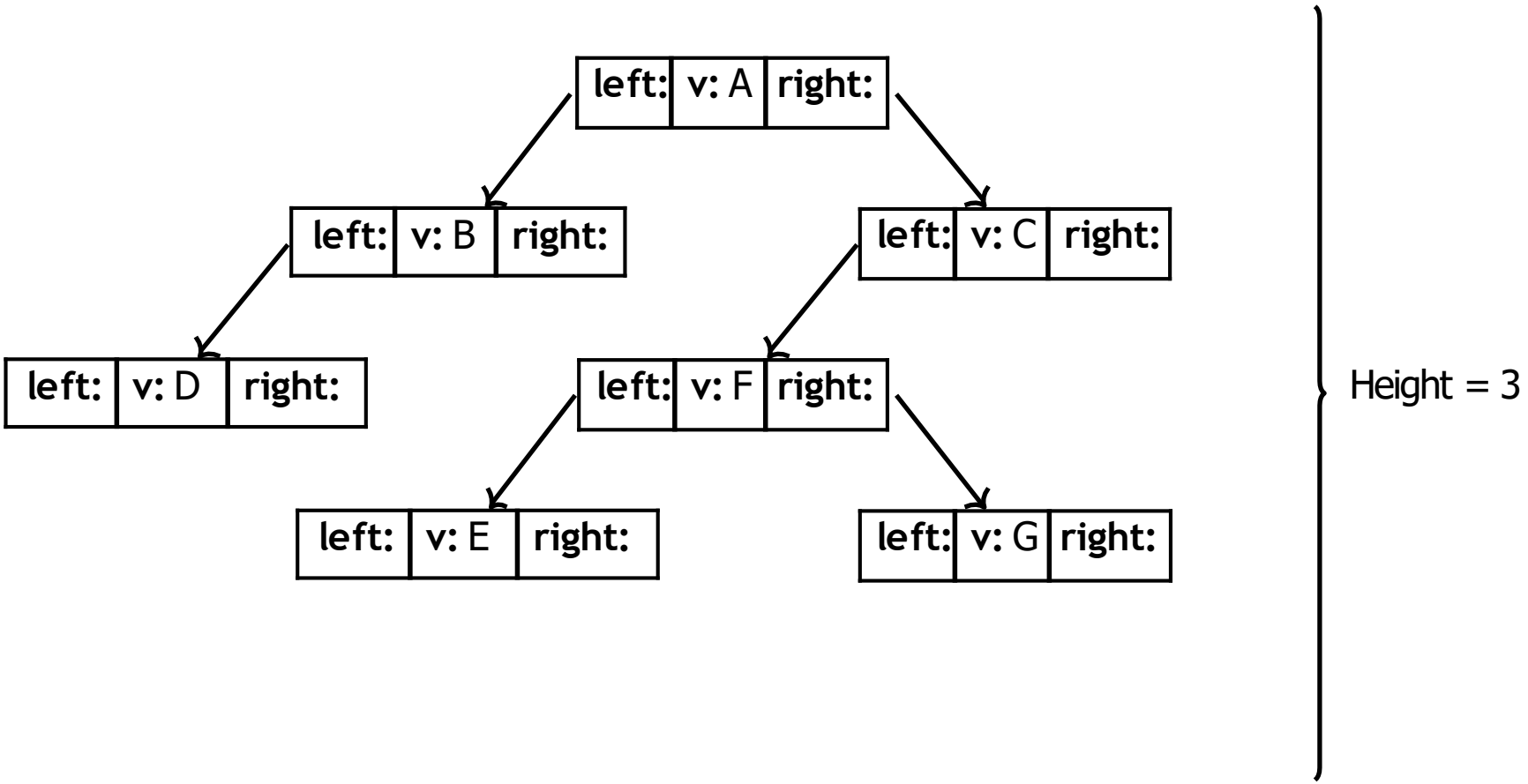
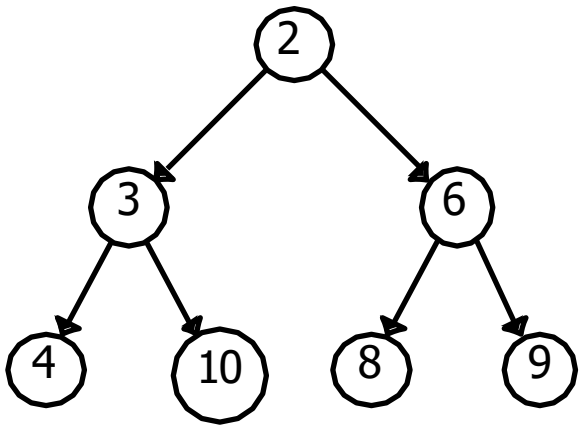


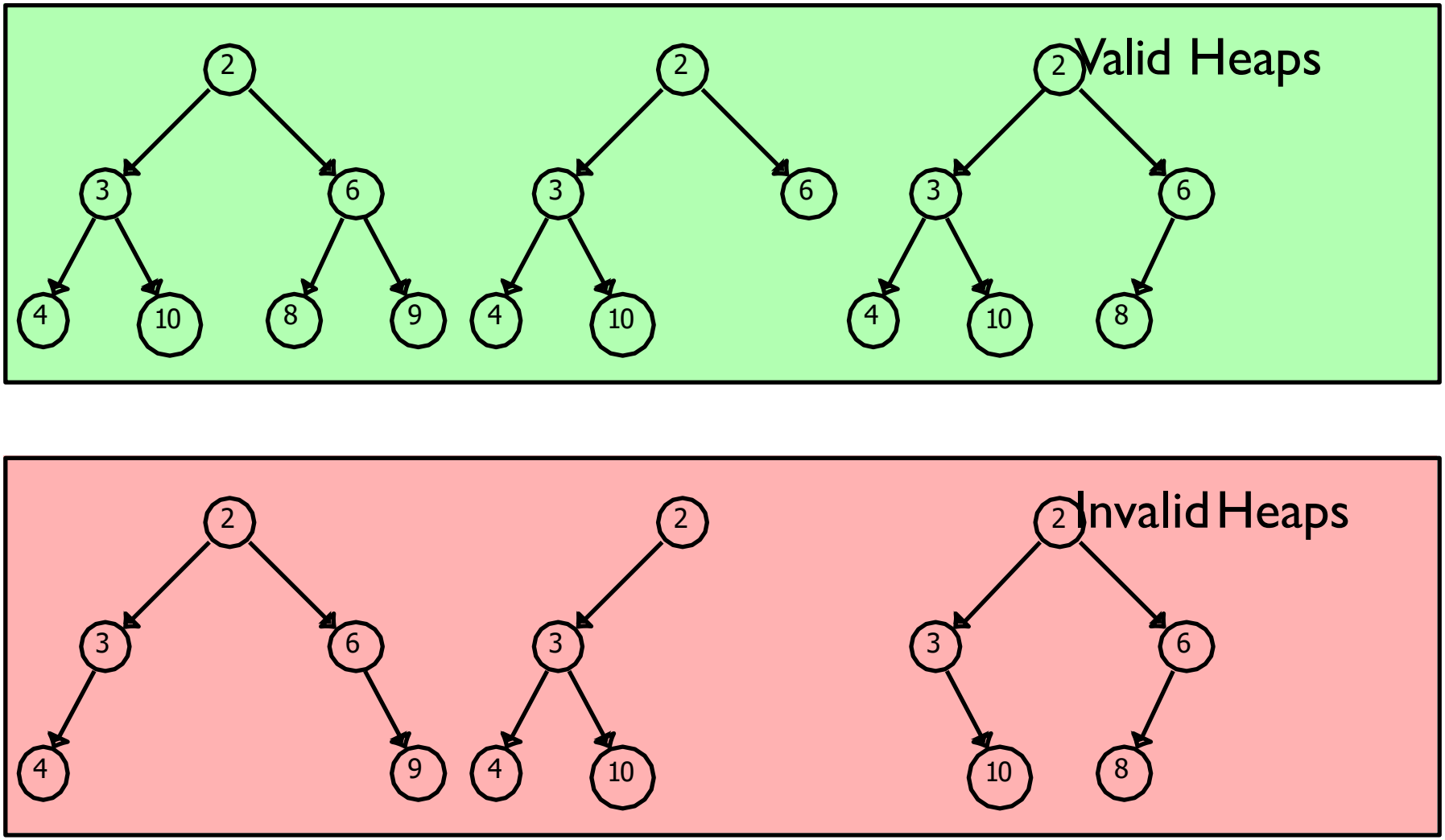
Heaps as trees

We can use a tree to make a heap by enforcing the properties that node will have a key value that is less than both of it's children, and that the tree will always be complete except for the last layer.



- This makes finding the minimum very easy. It's always on top!
- We will see that removing the root (minimum) element can be done in a number of operations proportional to the height.
- However if we want to find an arbitrary element we will have to search the whole tree.

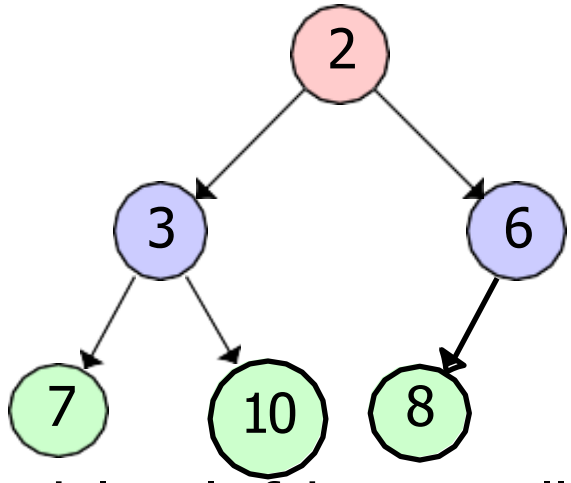
Heaps Shapes



Video 1.7
Sampath Kannan

Heap Representation

Since the tree for a heap will always be contiguous we can represent them implicitly with an array



2	3	6	7	10	8

So the i th level of the tree will occupy spots 2^{i-1} to $2^i - 1$ (we are using 1 based indexing for convenience)

Heap Representation

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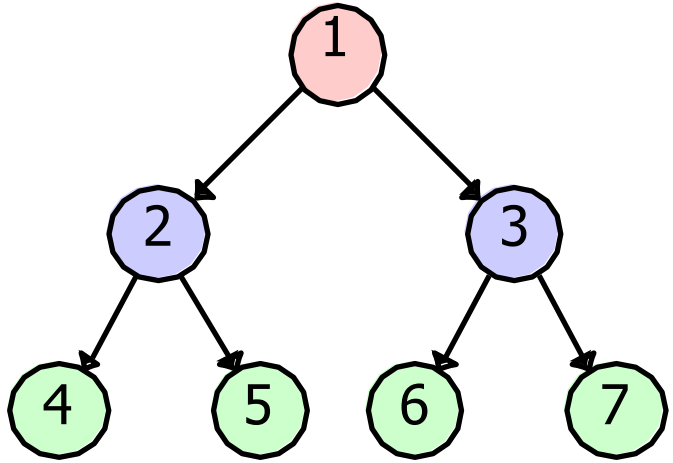

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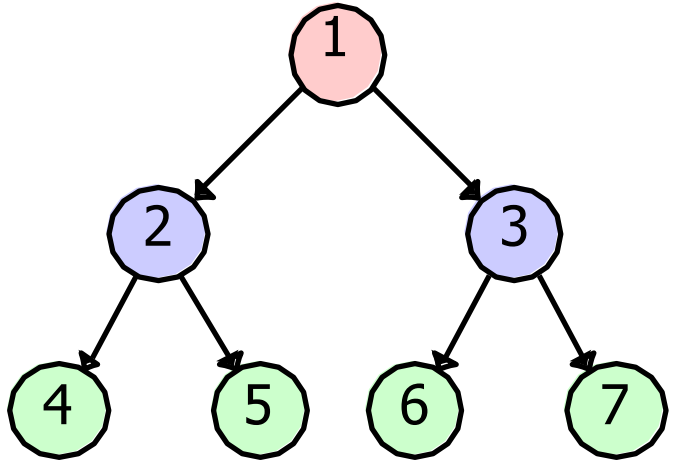
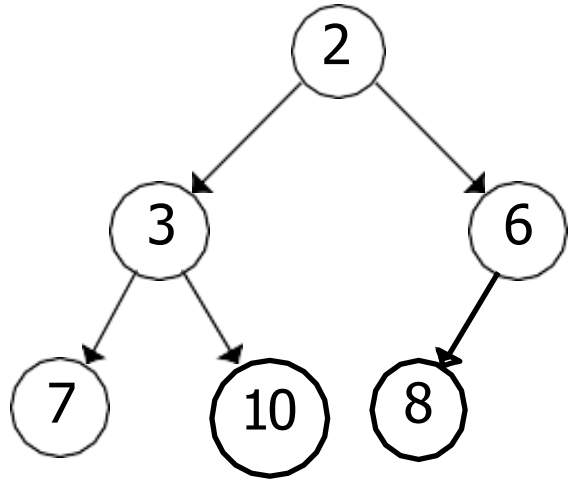


- Left child of 1 is 2, left child of 2 is 4, left child of 3 is 6, etc...
- In general the left child of node k is at position $2k$. So the right child is at $2k + 1$

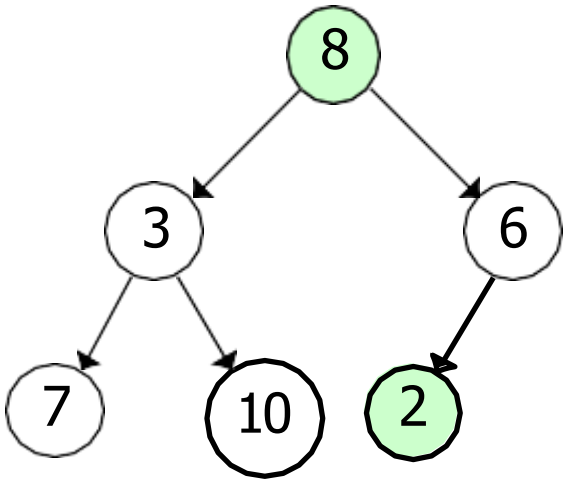
Operations on Heaps: Extract Min

We want to remove the minimum element (root) while maintaining the two heap properties: order and shape



Operations on Heaps: Extract Min

Step 1: Swap the root node with the node in the bottom right

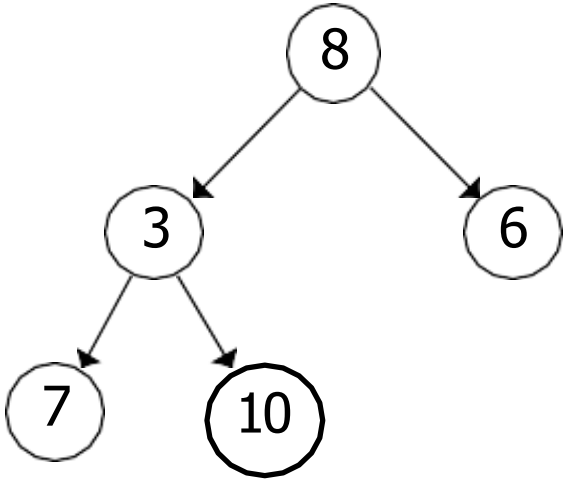



```
sink(A, k):
  N = length(A)
  while 2*k <= N
    smallest = 2*k
    if A[2*k] < A[2*k+1]
       smallest = 2*k+1
    if A[k] < smallest: break
    swap(A[k], A[smallest])
    k = smallest

extract-min(A, k):
  N = length(A)
  min = A[1]
  A[1] = A[N]
  sink(A, 1)
  return min
```

Operations on Heaps: Extract Min

Step 2: Now we can remove (2) while maintaining the shape property

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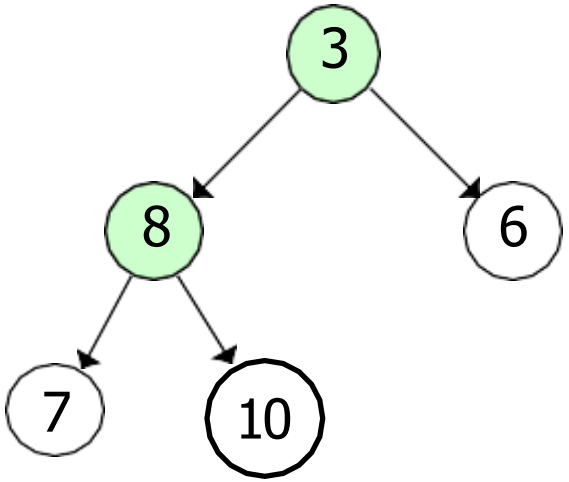
    if A[2*k] < A[2*k+1]
       smallest = 2*k+1
    if A[k] < smallest: break
    swap(A[k], A[smallest])
    k = smallest

extract-min(A, k):
  N = length(A)
  min = A[1]
  A[1] = A[N]
  sink(A, 1)
  return min
```

Operations on Heaps: Extract Min

Step 3: We will fix the order property by swapping (8) with it's smallest child




```
sink(A, k):
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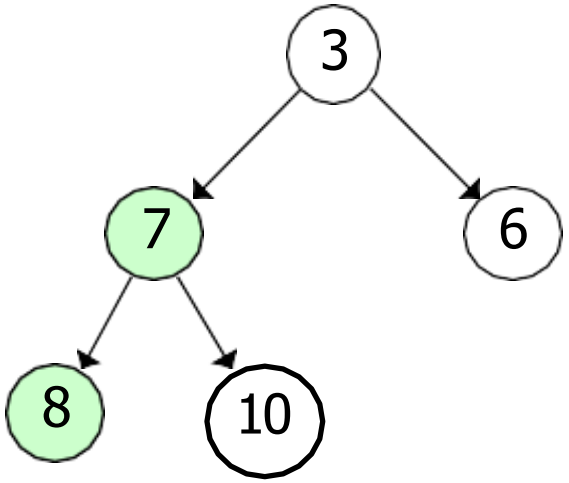
  A[1] = A[N]
  sink(A, 1)
  return min
```

Operations on Heaps: Extract Min

Step 4: Keep fixing the order property by swapping (8) with it's smallest child again

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  N = length(A)
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extract-min(A, k):
  N = length(A)
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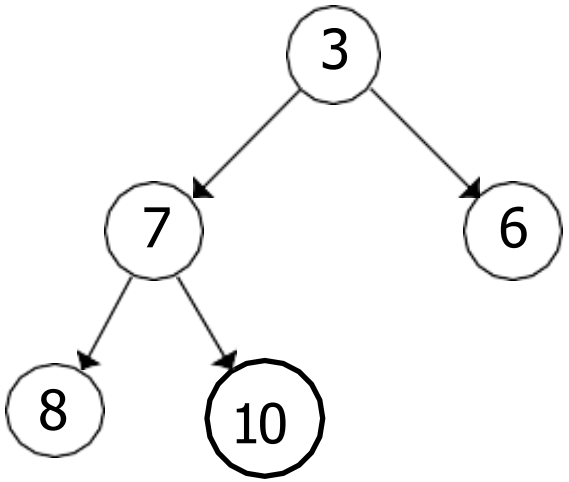
  A[1] = A[N]
  sink(A, 1)
  return min
```

Operations on Heaps: Extract Min

Step 5: The heap properties have been preserved so we're done!

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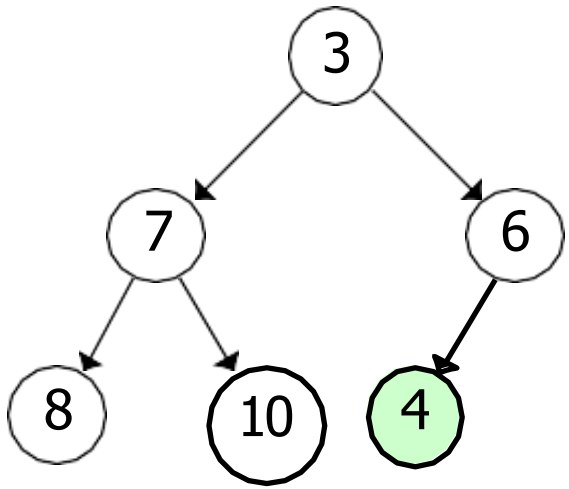
```

Operations on Heaps: Insert

Step 1: Preserve the shape property by inserting the new element at the bottom right

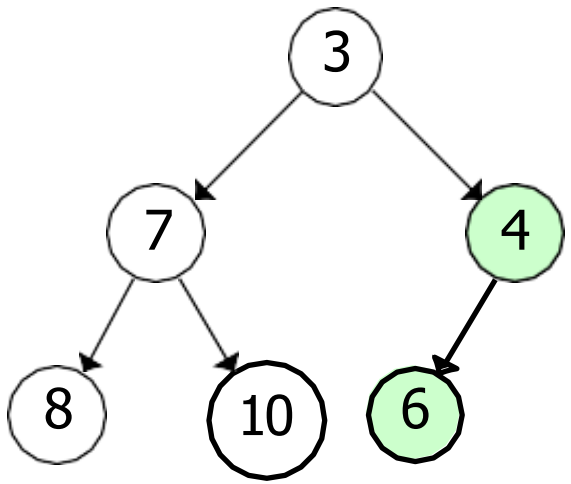



```
swim(A, k):
    while k > 1 and A[k/2] < A[k]:
        swap(A[k], A[k/2])
        k = k/2

insert(A, k, val):
    N = length(A)
    A[N+1] = val
    swim(A, N+1)
```

Operations on Heaps: Insert

Step 2: Fix the order property by swapping (4) with its parent since it's smaller

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swim(A, k):
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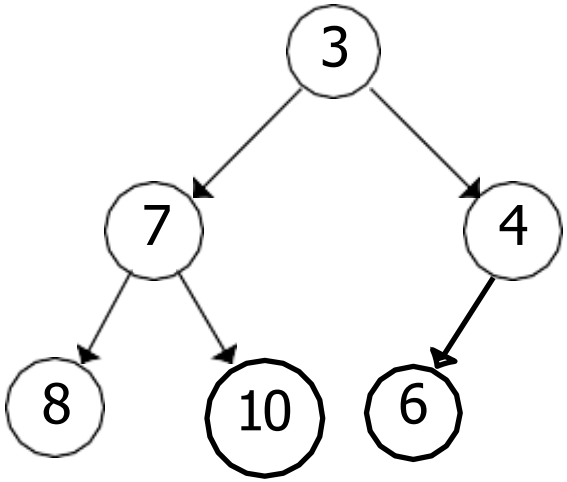
insert(A, k, val):
    N = length(A)
    A[N+1] = val
    swim(A, N+1)
```

Operations on Heaps: Insert

Step 3: (4) is bigger than its parent now so we're done!

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insert(A, k, val):
    N = length(A)
    A[N+1] = val
    swim(A, N+1)
```

Heap efficiency

- All operations on the heap are a combination of a constant number of operations and sink or swim operation.
- Swim operation executes as long as $k > 1$ and divides it by 2 on every iteration
- Can execute at most $\log_2 k$ times. Since k is initially at most n, the number of elements, swim has a run time that is $O(\log n)$
- By the same logic sink has run time that is $O(\log n)$ as well.
- So all the operations are $O(\log n)$. Except for delete which must first take potentially $O(n)$ steps to locate the given element in the array.

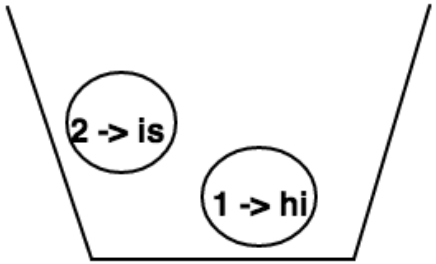
Video 1.8
Sampath Kannan

Dynamic Dictionaries

Dynamic Dictionaries support three main operations:

- **insert** into a dictionary
- **delete** from a dictionary
- **search** for an element in a dictionary

Abstract representation:



Dynamic dictionaries are used in applications everywhere:

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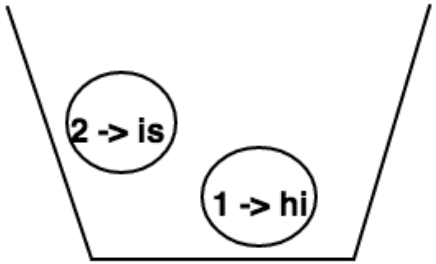

next: insert the pair (3, "the")

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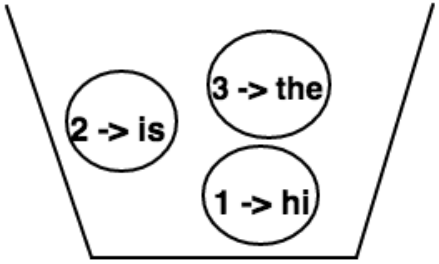

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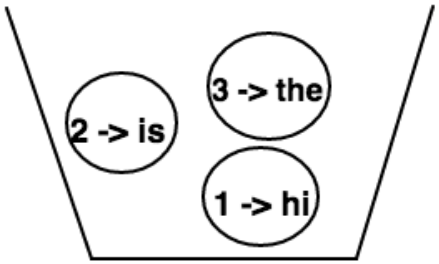

next: lookup 1

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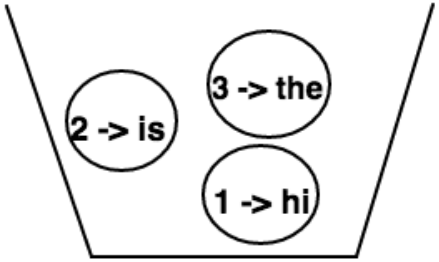

next: lookup 1 returns "hi"

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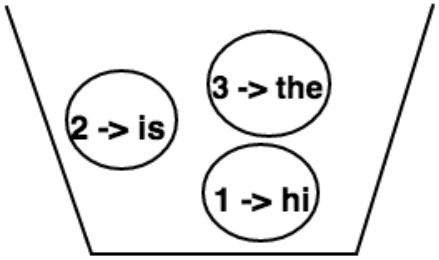

next: lookup 1
returns "hi"
next: delete 3 from dictionary

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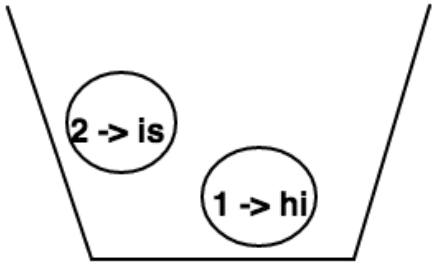

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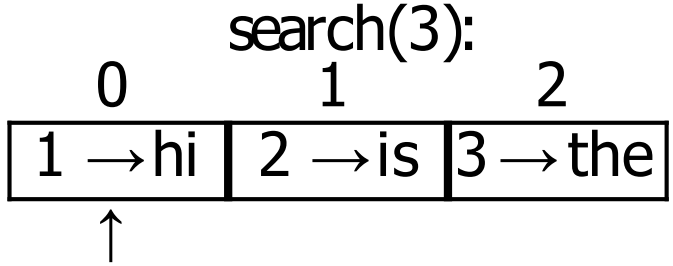
Implementations of Dictionaries

Can we find an efficient implementation for dictionaries?

Attempt 1: Arrays

- search: $O(n)$
  - entire array must be traversed
- insertion, deletion: $O(n)$
  - Array may need to be resized (requires copying all elements to a new array



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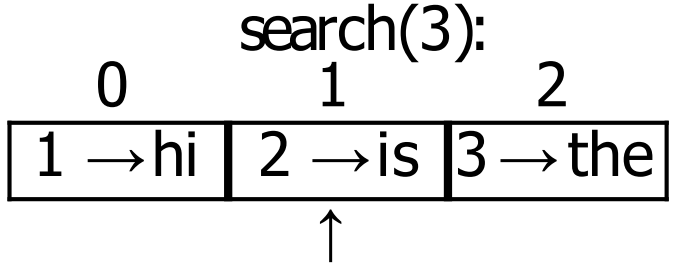

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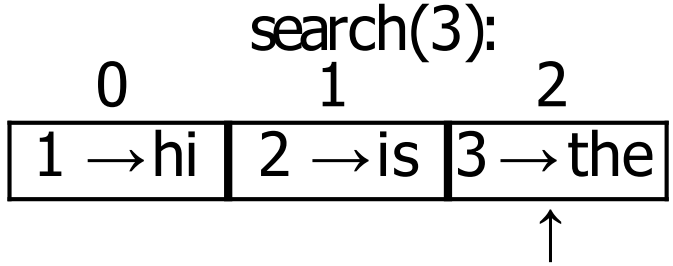

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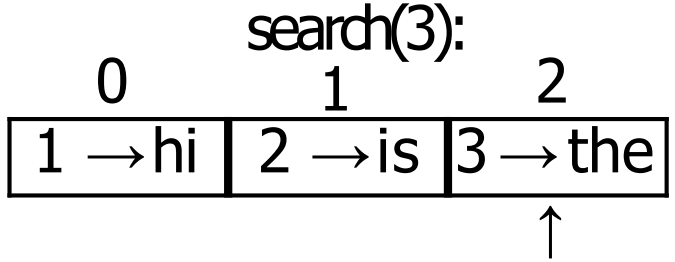

Attempt 2: Linked Lists

- search, deletion: $O(n)$
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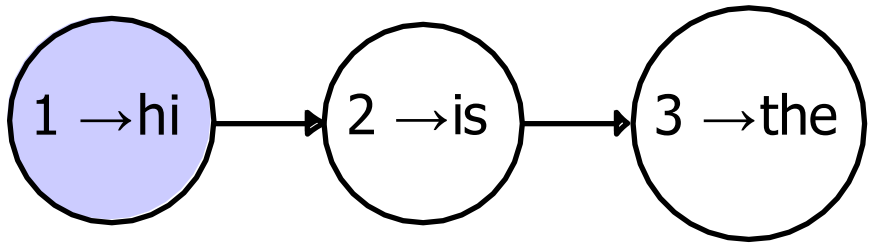
- insertion: $O(1)$
  - Can easily insert at the front of the list



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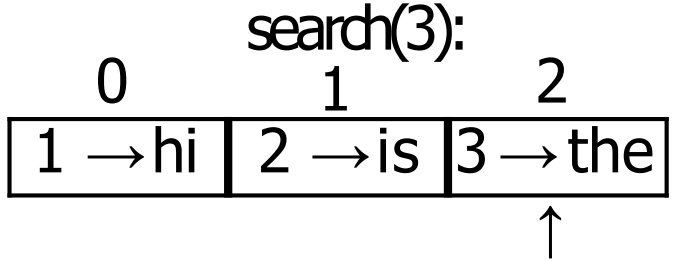

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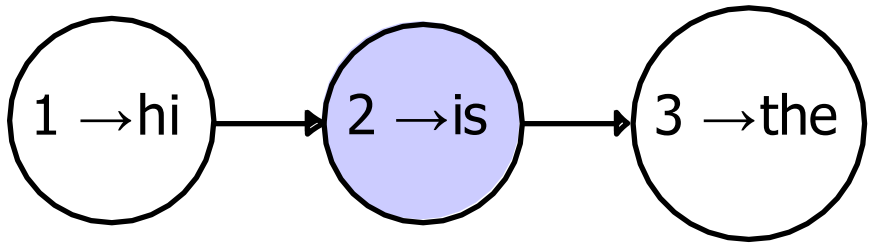

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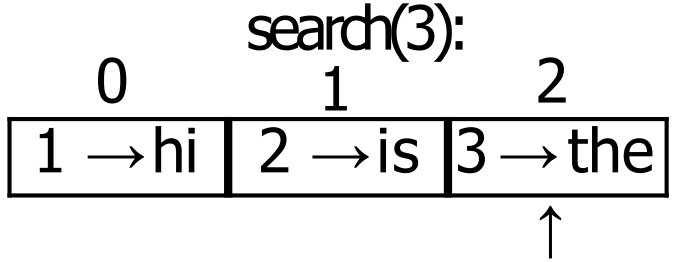

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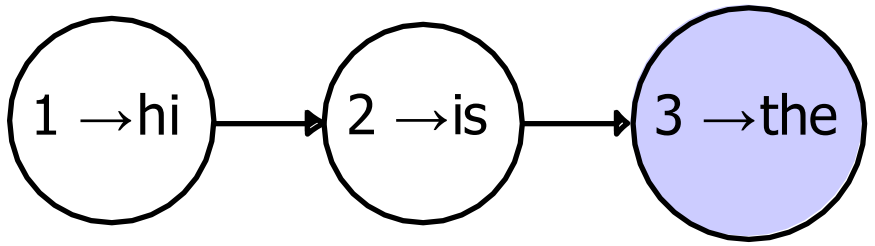

Dictionaries: Binary Search Trees

Attempt 3: Binary Search Tree

- Store items in nodes of a binary tree

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  search tree property:
  - All keys to the left of a node are < that node's key
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insert (4, "a")

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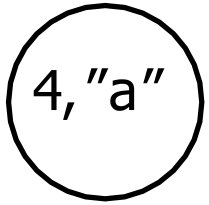
insert (2, "b")



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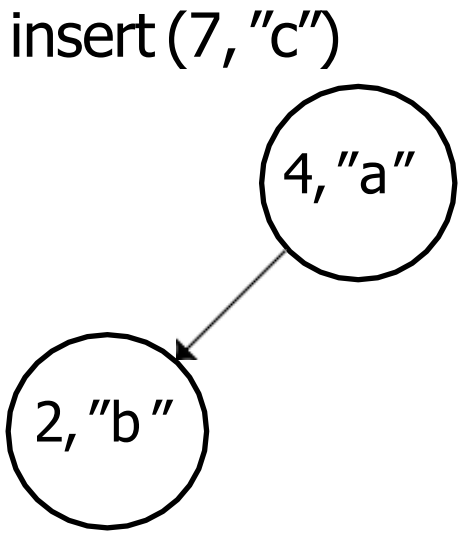

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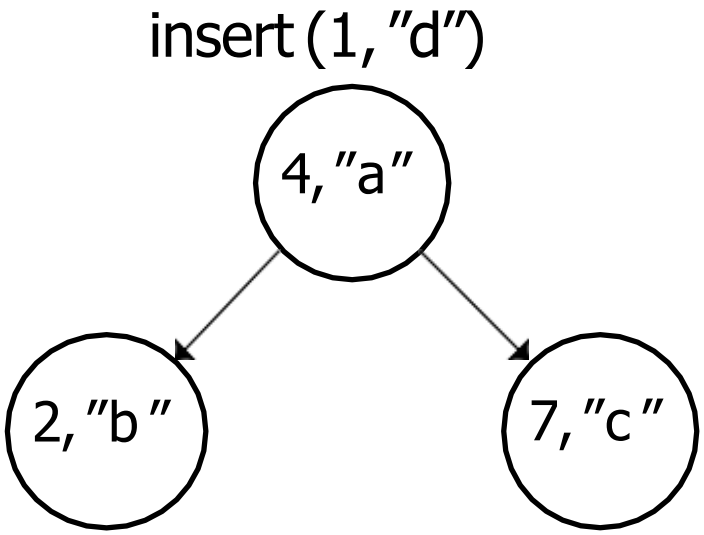

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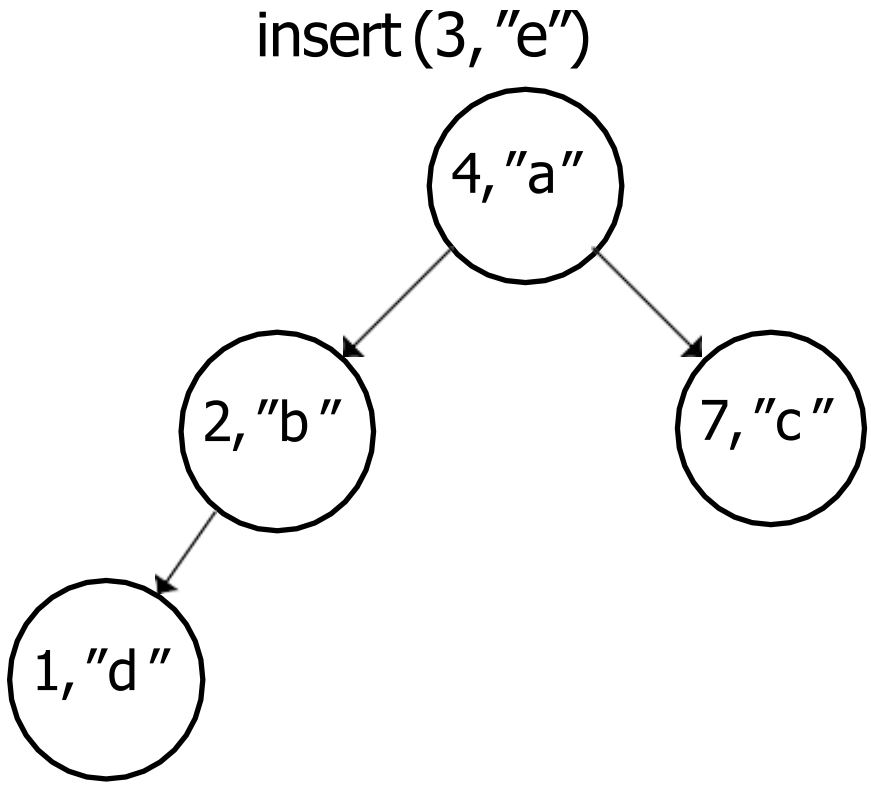

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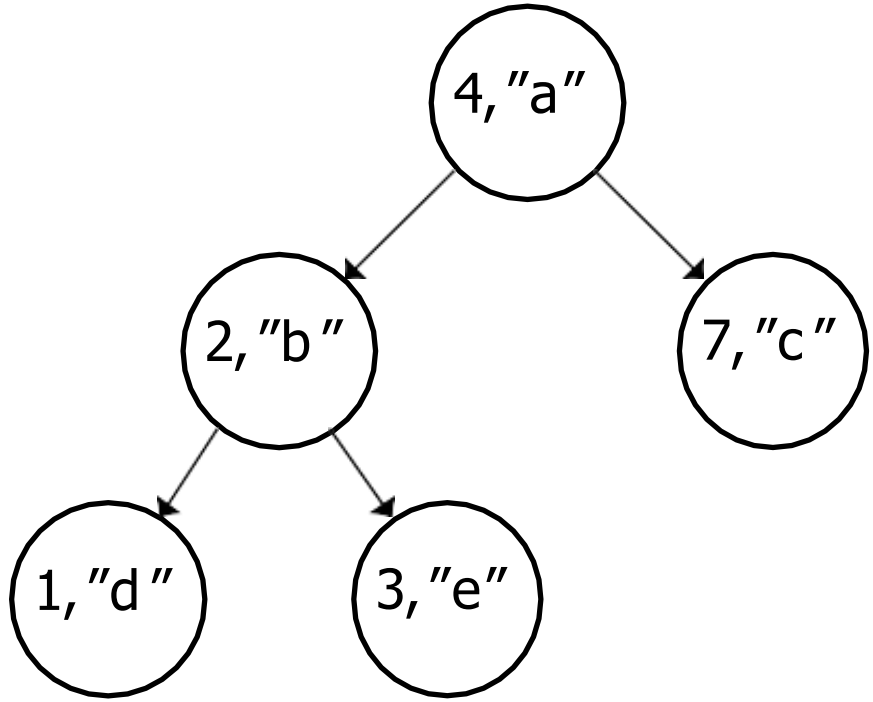

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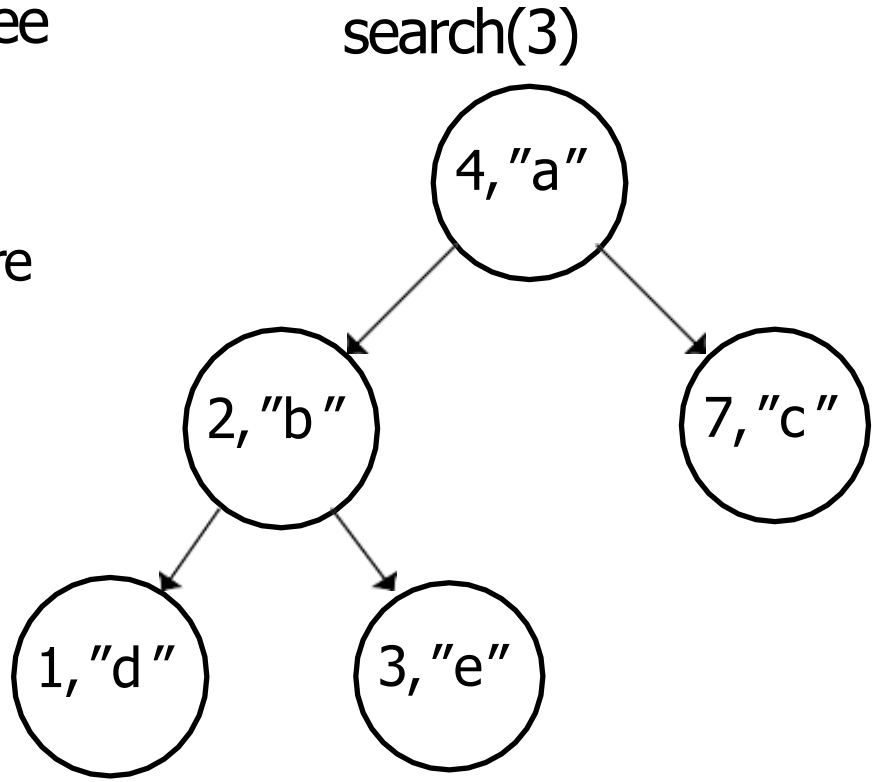

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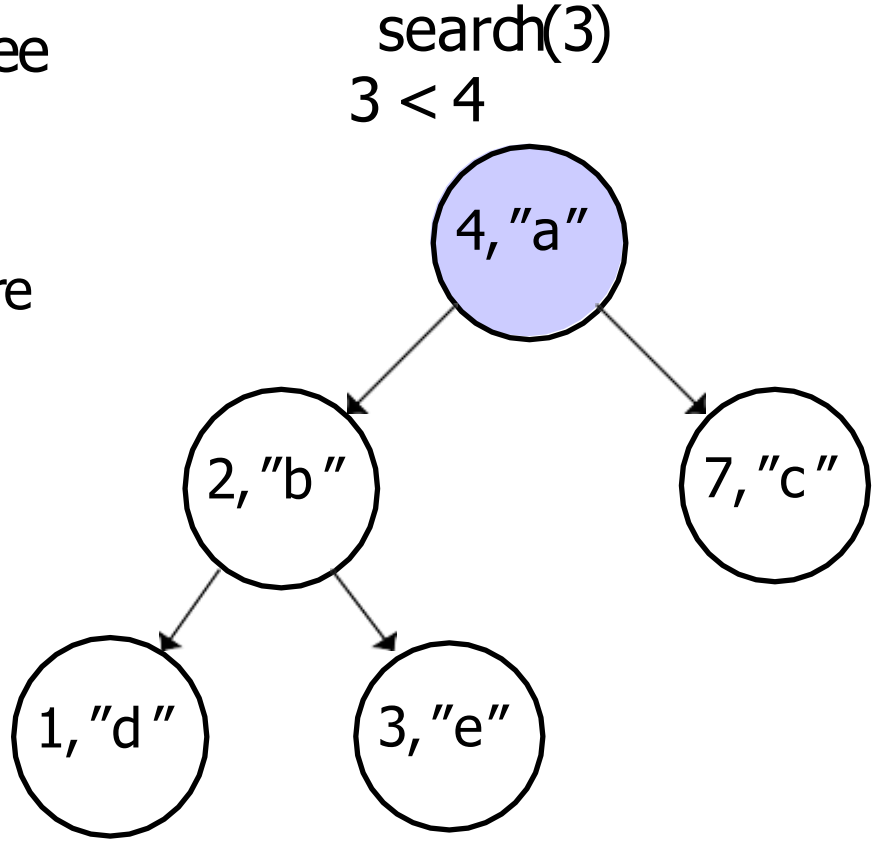

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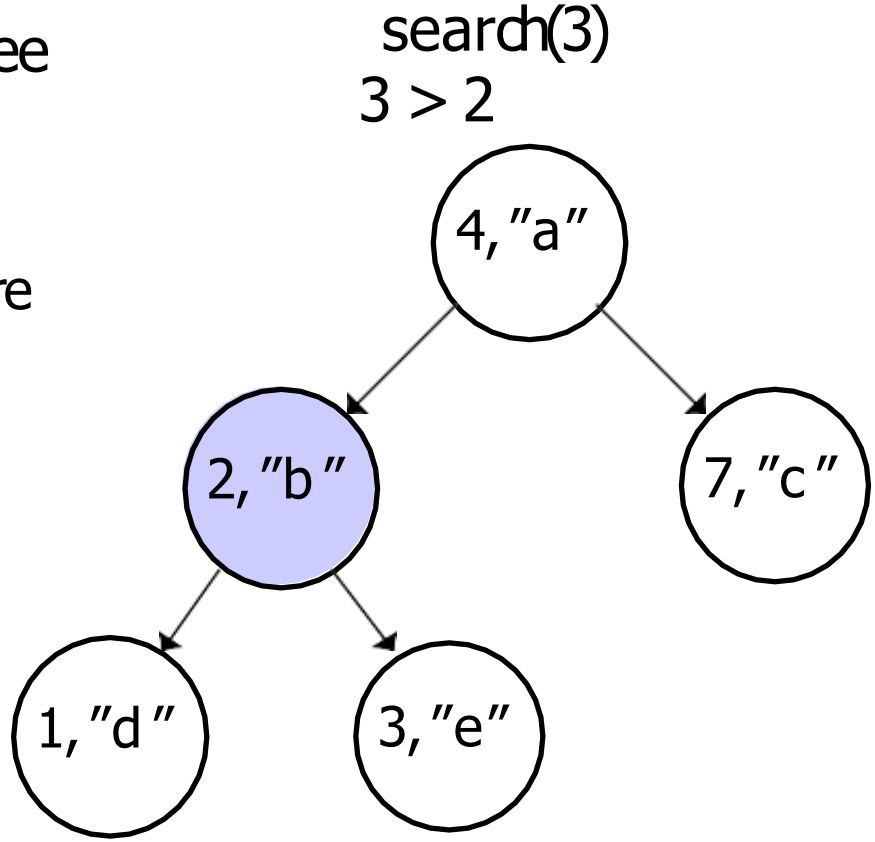

Dictionaries: Binary Search Trees

Attempt 3: Binary Search Tree

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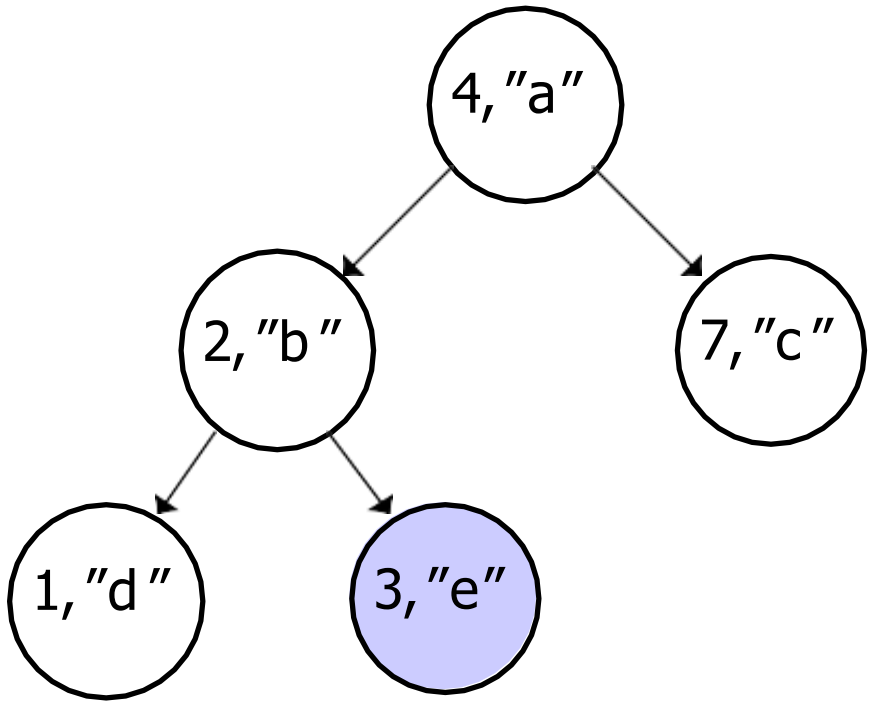
search(3)
return "e"



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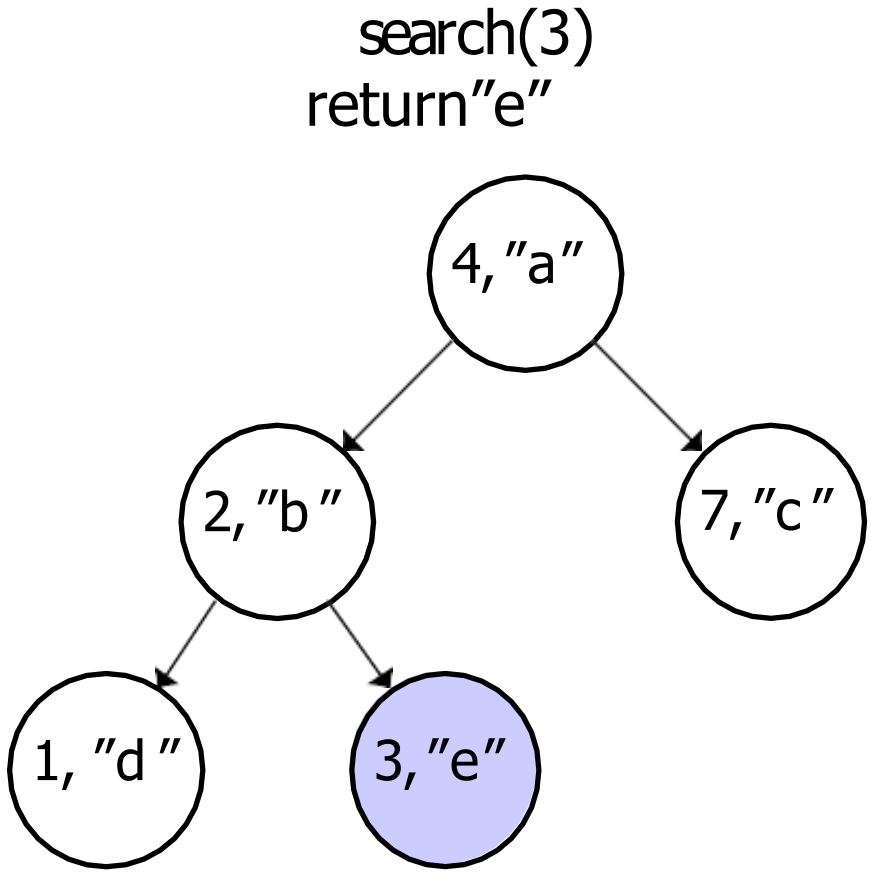

Time to insert, search and delete is proportional to the height of the tree!

Binary Search Trees: Runtime

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insert(1, "a")

Binary Search Trees: Runtime

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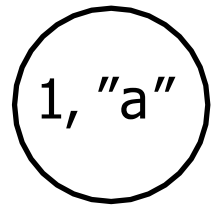
insert (2, "b")



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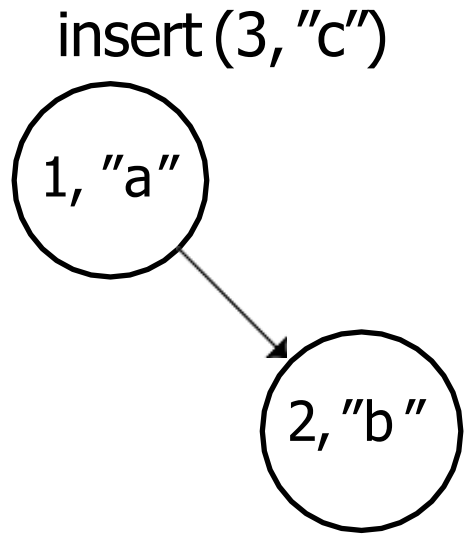

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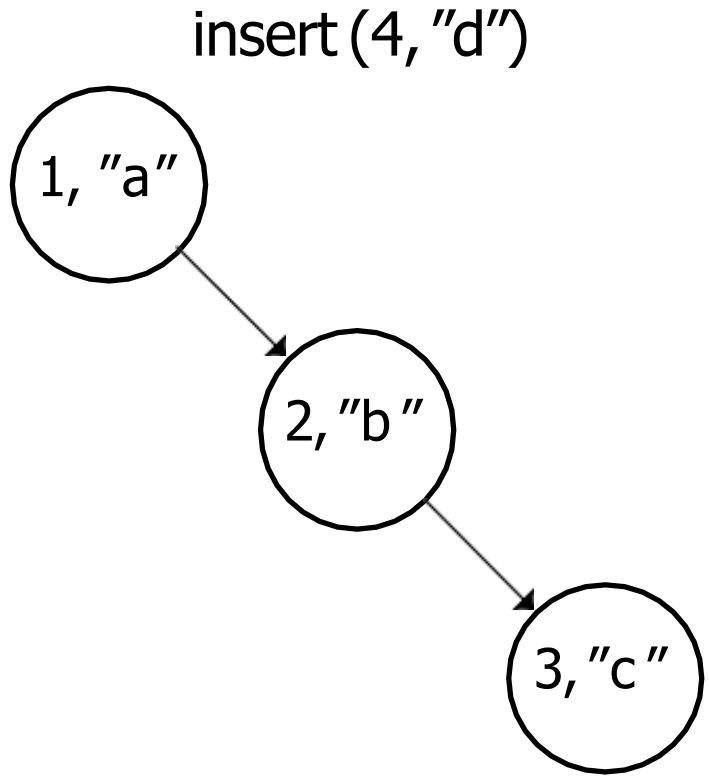

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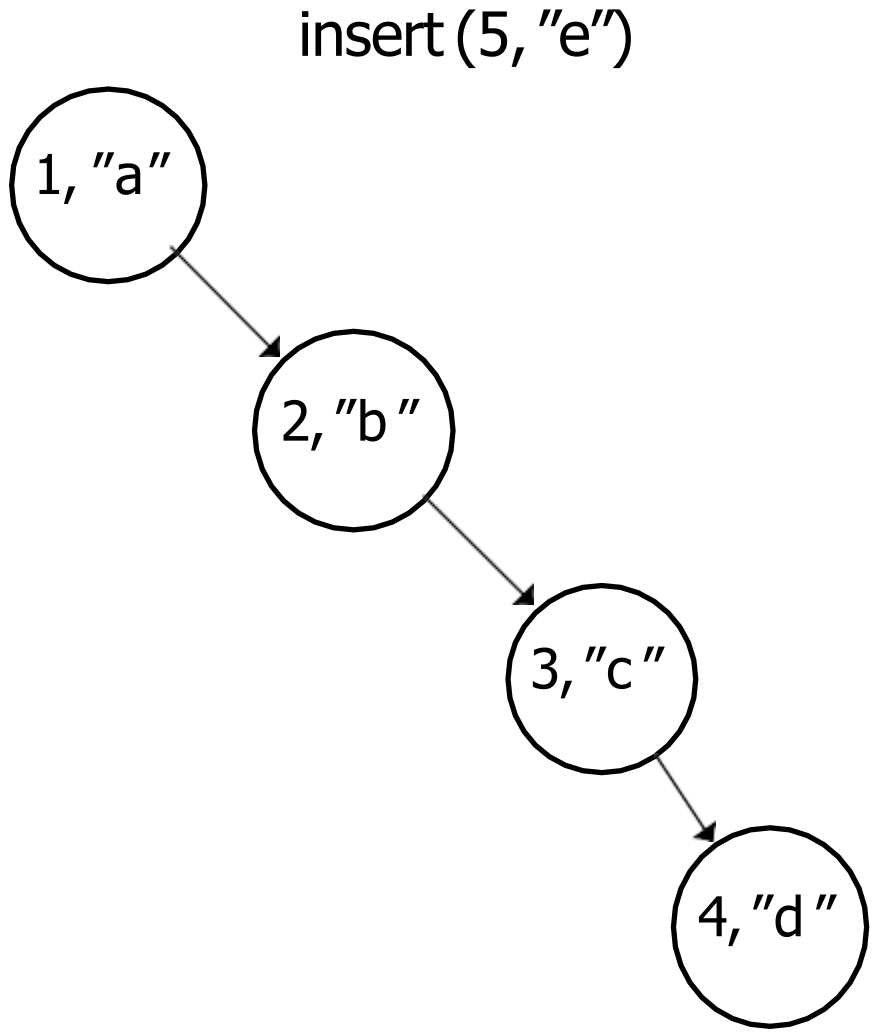

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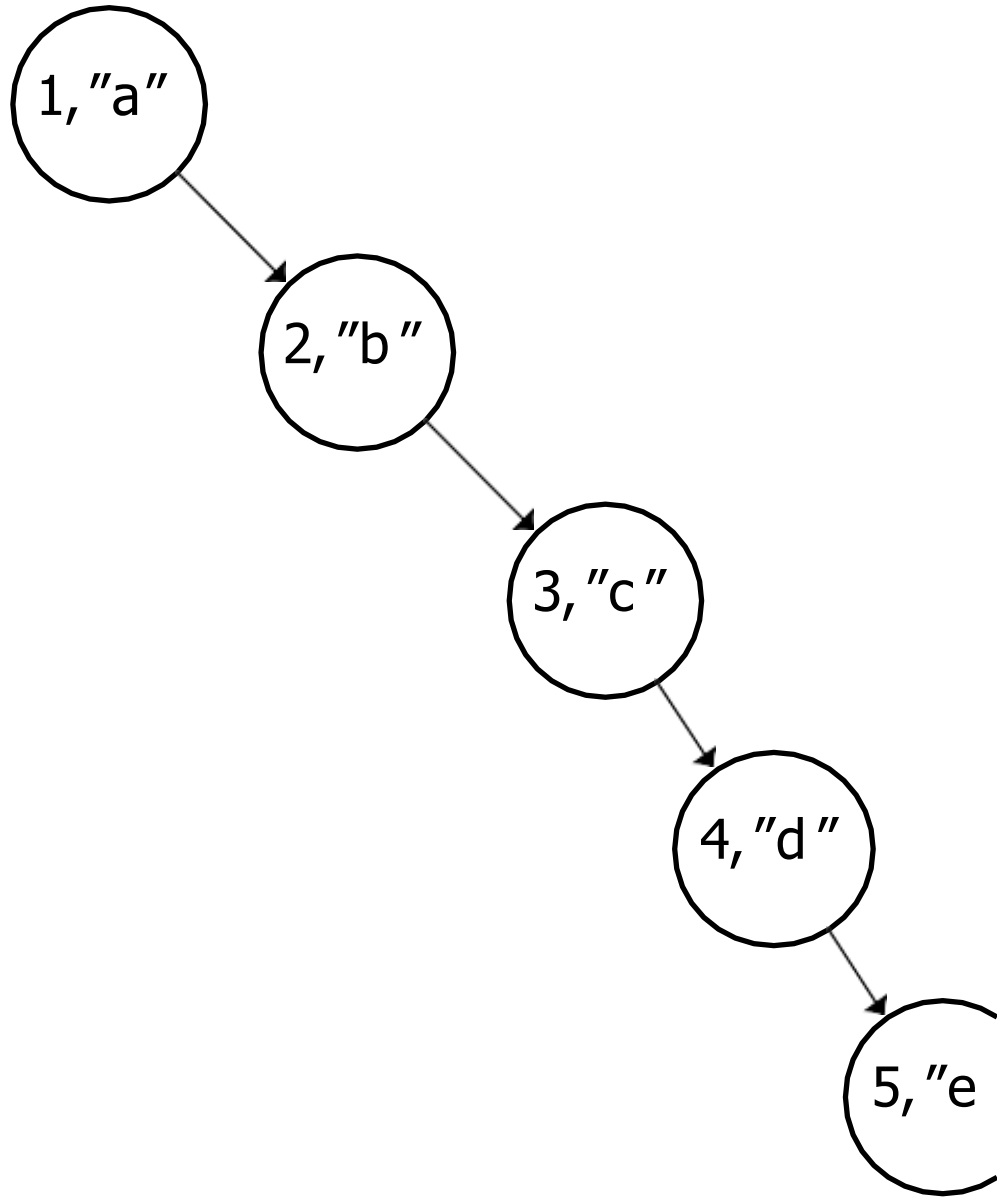

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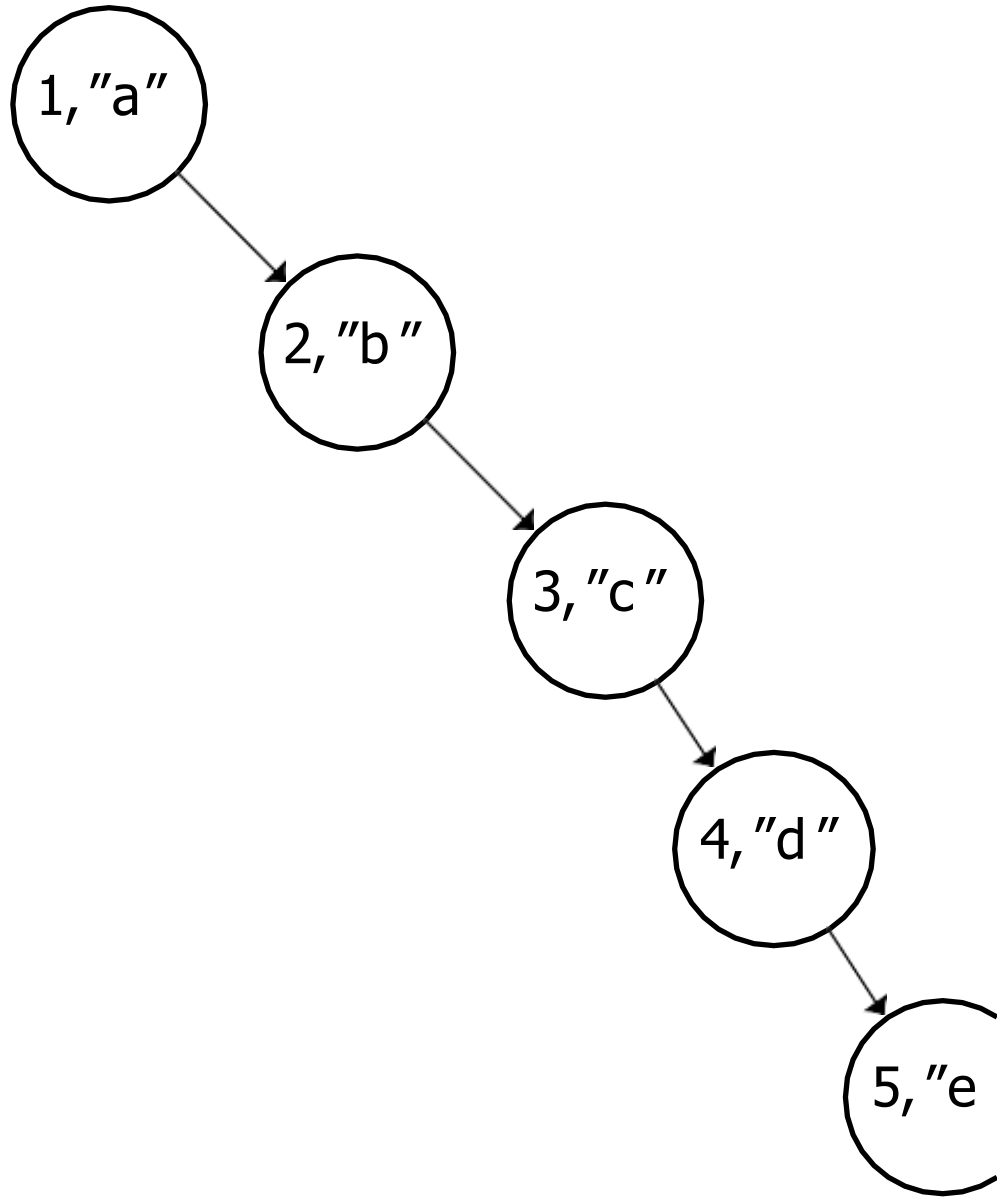

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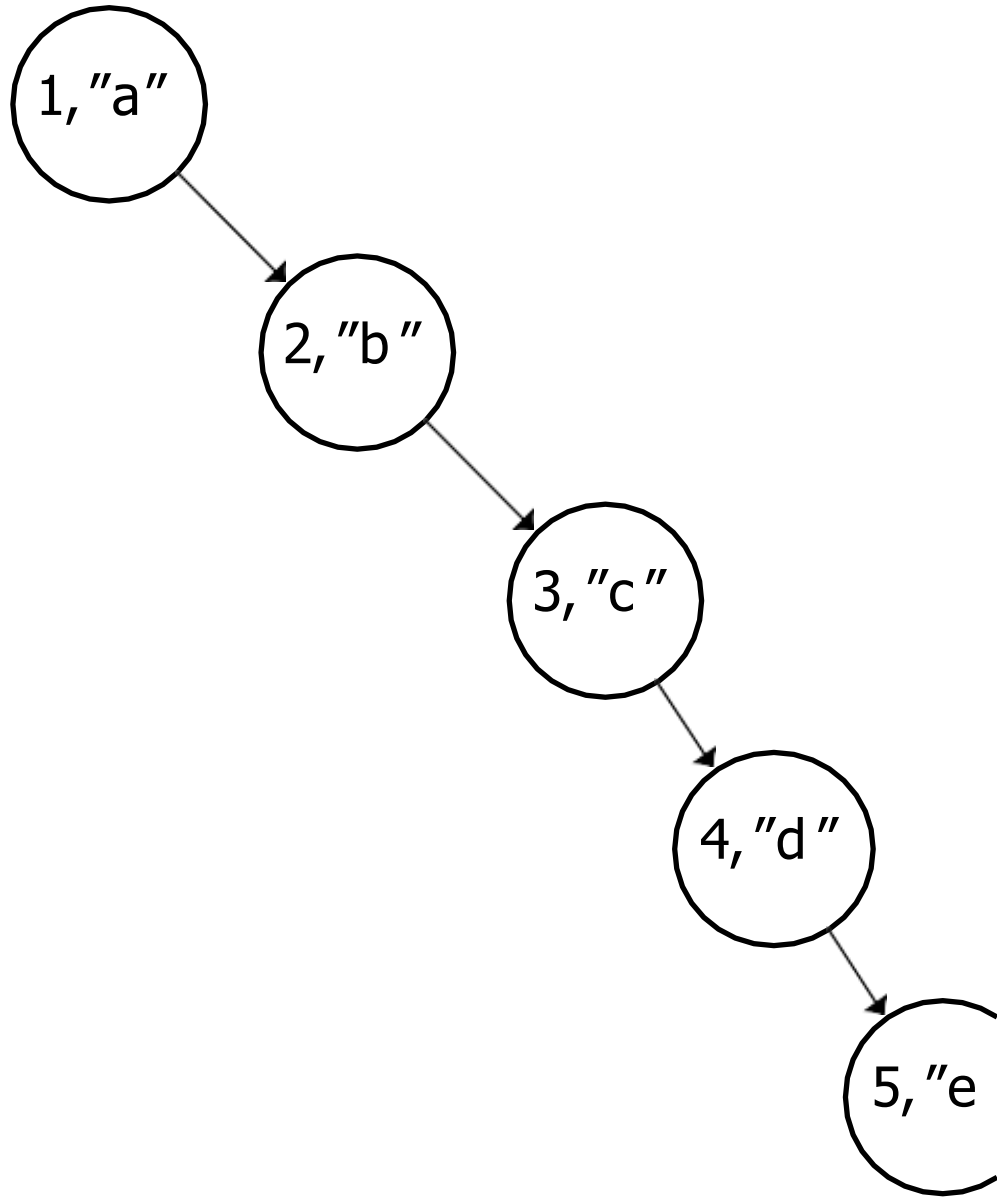

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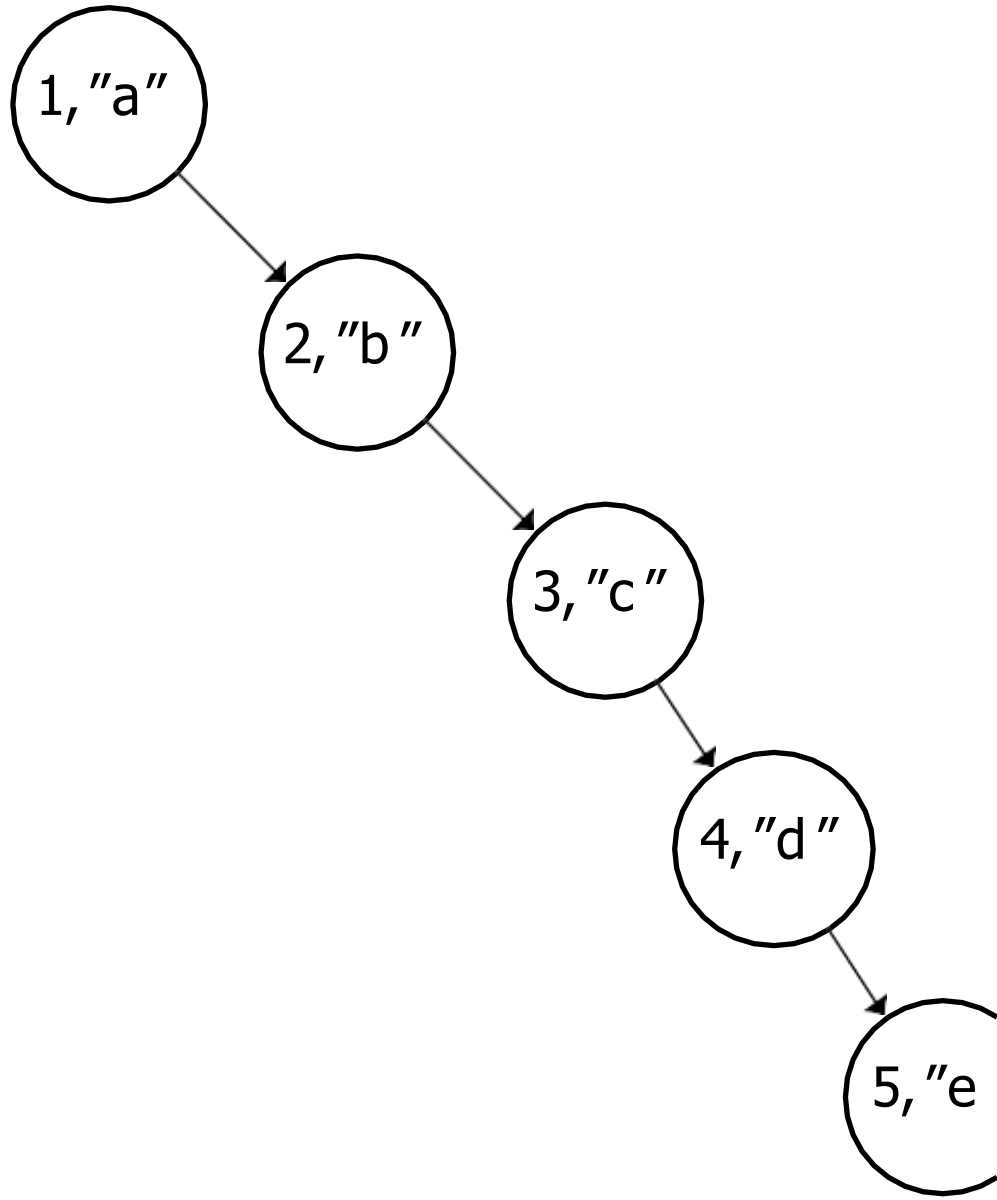
  - kth level: 2^k nodes
  - $n = 1 + 2 + 2^2 + \ldots + 2^k$
  - $2^{k+1} - 1 = n$, $k = O(lgn)$



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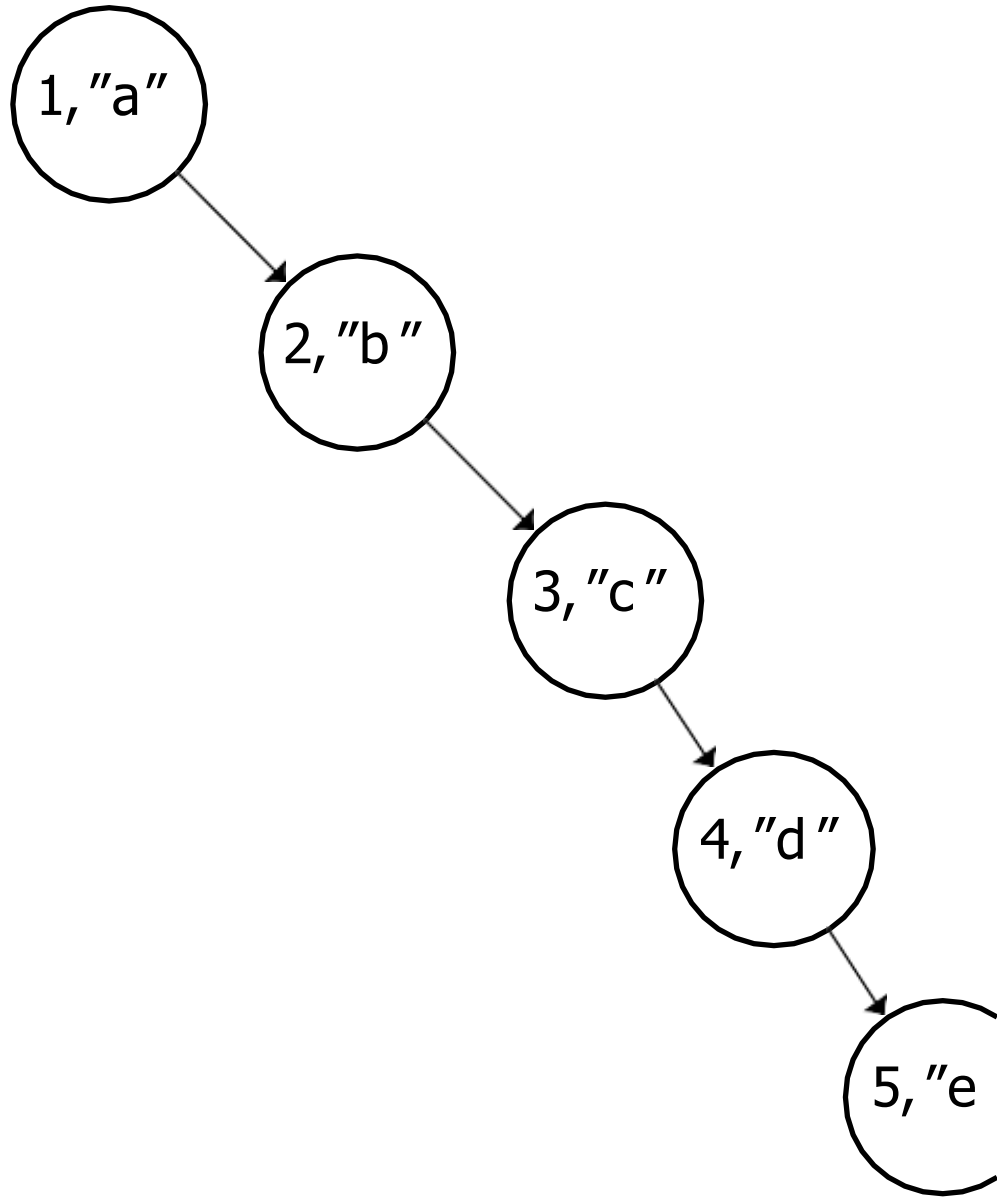

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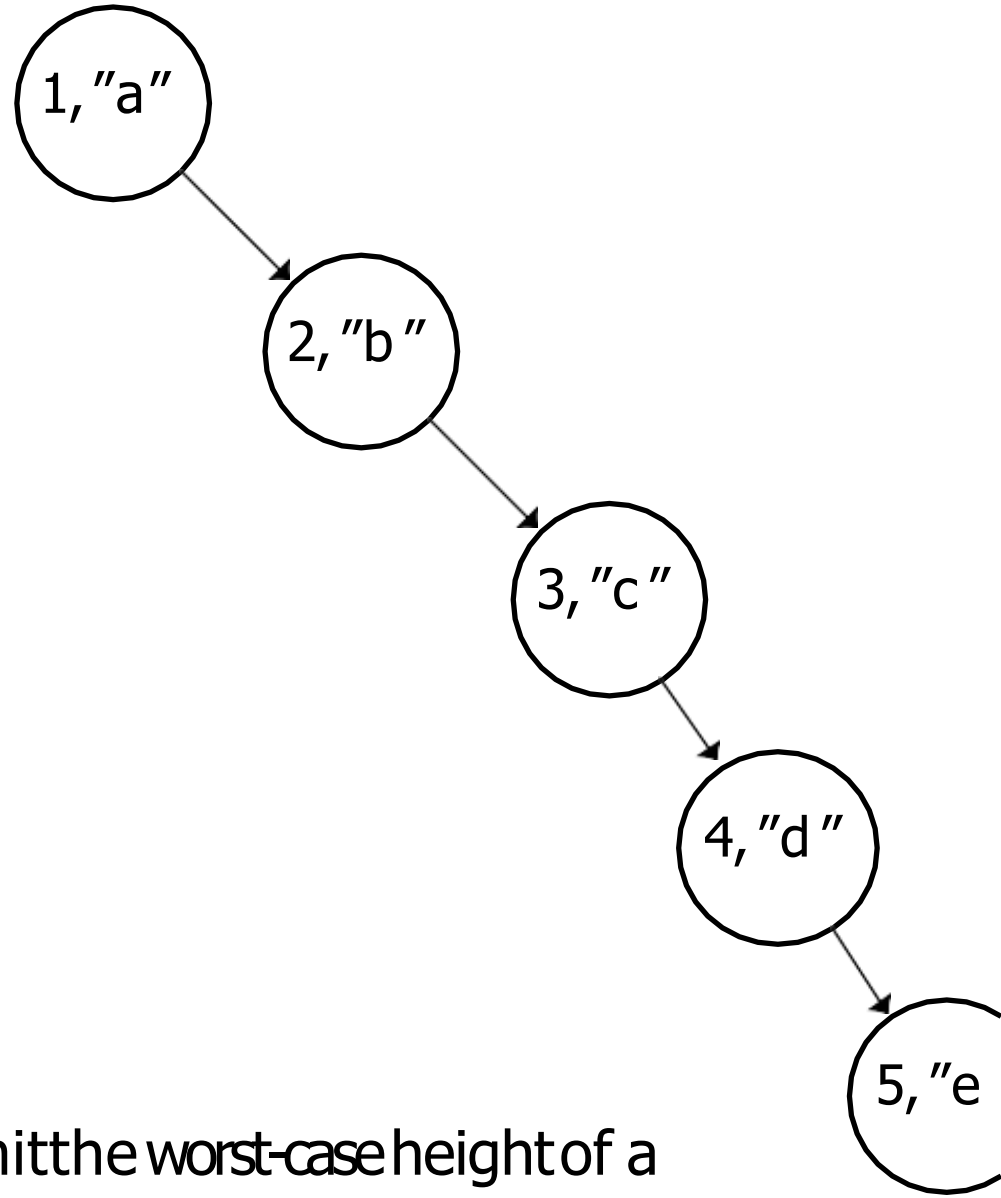


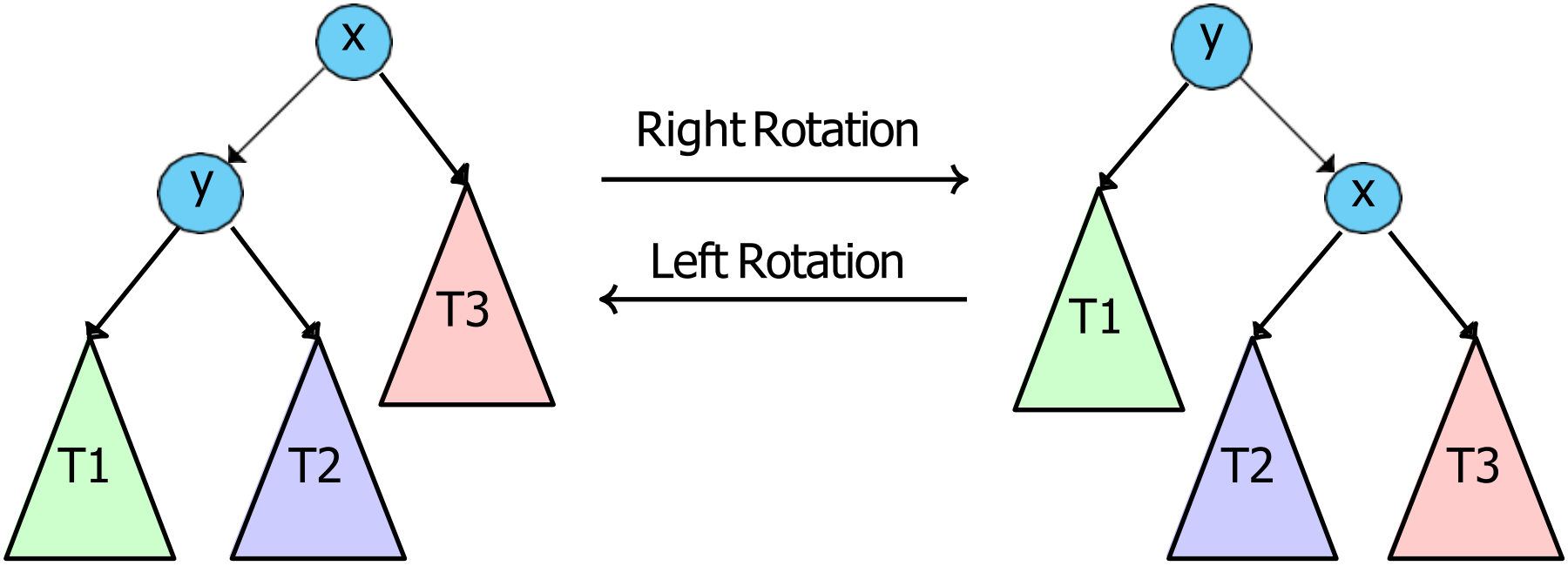
Is there anything we can do to limit the worst-case height of a binary search tree?

Video 1.9
Sampath Kannan

Balanced Binary Search Trees

- BSTs can become unbalanced leading to $O(n)$ run times for operations.
- We need a way to modify them so that their height is $O(\log n)$ instead of $O(n)$.
- Intuitively we can get this property if the left and right sub-trees always have similar heights
- Modifications must preserve search tree property

Rotations



We use rotations to keep left and right sub-trees balanced. In an AVL tree we maintain the invariant that all left and right sub-trees have a height difference of at most 1.

Hashing

- To use an array to implement a dictionary we need a way to map elements from our universe to indices. This mapping is called a **hash function** and the array is called a **hash table**
- Example: If our universe is all the integers and we have a hash table of size 37 we could use $h(x) = x \bmod 37$ as our hash function.
- If only one item gets mapped to each index then all operations are $O(1)$!

Load factor

- Suppose we have m different keys and a hash table of size n, and suppose that for each key we randomly choose an index to map it to.

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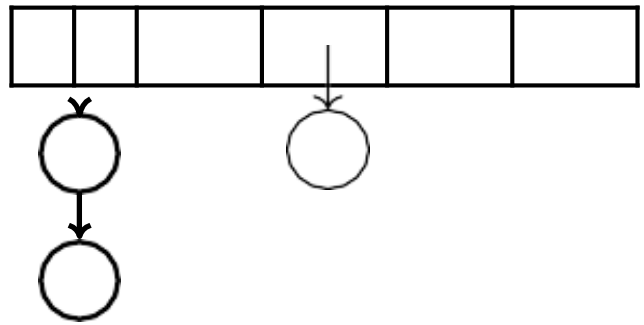
- $P(h(k) = i) = 1/n$.
- Let X_i be the number of keys mapped to index i and

$$E[X] = \sum_k P(h(k) = i) * (1) = \sum_k (1/n) * (1) = m/n$$

- **load factor** = α.

Handling Collisions

- Can't get rid of collisions so we need to store multiple items in a single bin
- One approach to this is **chaining**:



- Instead of storing each item directly in the array, we store a linked list of all the items that map to that index
- Run-time of all operations is now proportional to the length of the linked lists at the index we are operating on. We just saw that this gives *expected* $O(a)$ performance.
- Note that the worst case is still $O(m)$!

Handling Collisions 2

- Instead of chaining we can use **open addressing** where keys that map to the same index are stored in separate locations in the table.
- One approach to this is **double hashing**, where we use 2 hash functions $h(x)$ and $g(x)$.
- When there is a collision at $h(x)$ we try to insert at $h(x) + g(x)$, then $h(x) + 2g(x)$, ... etc

- Pros: No extra storage required, we don't have to deal with pointers
- Cons: Deletion is very tricky and easy to mess up

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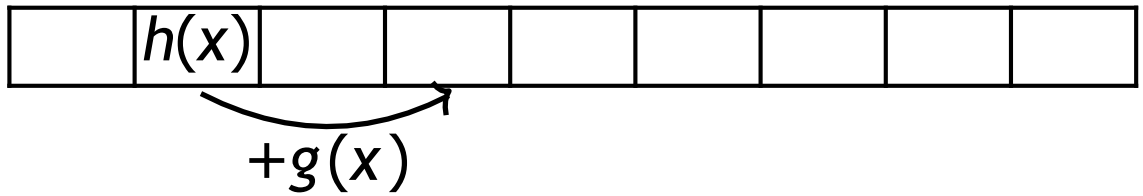

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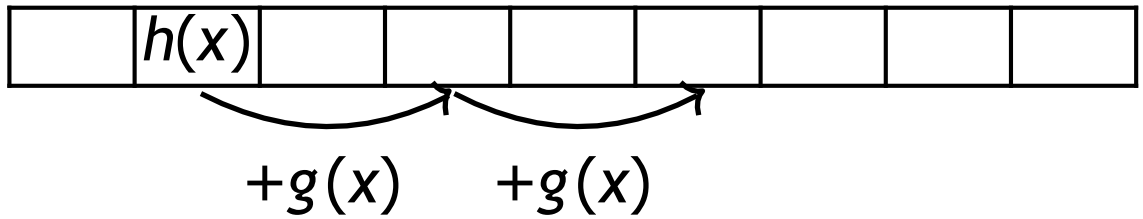

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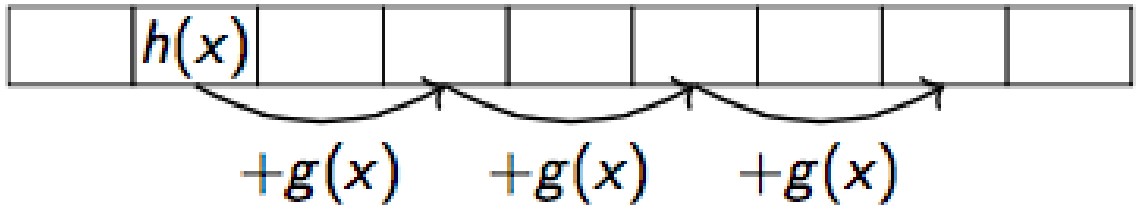

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