Urban Maths: Pursuit Curves for Pirates

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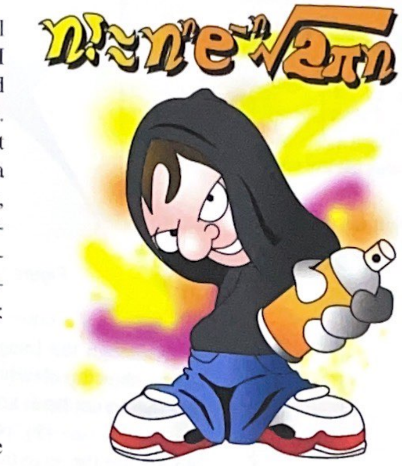
Introduction

While walking round a small lake at a local park recently I saw an excitable, unrestrained dog jump in the water to chase the ducks. Ignoring its owner's attempts to call it back, the dog kept swimming towards a duck, continually turning to face the duck, which swam rapidly away. I should, perhaps, have been indignant at the carelessness of the dog's owner, but, instead, looking at the wake left by the dog, I thought: 'Hmm, there's a real life pursuit curve!'



Pirates and merchants

In 1735 the French mathematician, Pierre Bouguer, published the now classic pursuit curve problem. He imagined a pirate ship chasing a merchant ship. The chase begins with the pirate ship at Cartesian coordinates $(0,0)$ and the merchant ship at $(x_0, 0)$. The merchant ship travels at constant speed, V_m, along the vertical line $x = x_0$, while the pirate ship chases it at constant speed, V_p, along a curved path such that the pirates are always pointed directly towards the merchant ship (see Figure 1). Bouguer wanted to know the equation of the line, or curve of pursuit of the pirate ship.

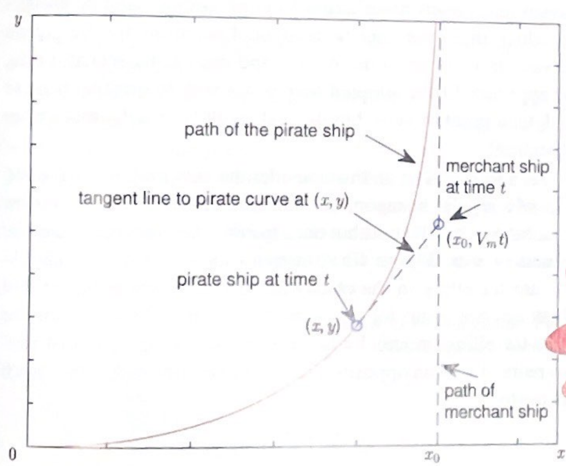


Figure 1: Paths of pirate and merchant ships

To determine this consider the positions of both ships at a time, t, with the pirates at coordinates (x, y) and the merchants at coordinates $(x_0, V_m t)$. We can see from Figure 1 that the gradient, g (which we will use as shorthand for $\mathrm{d}y/\mathrm{d}x$), of the tangent to the pursuit curve, is

$$g = \frac{V_m t - y}{x_0 - x},$$

which we can rearrange as:

$$V_m t = y + g.(x_0 - x). \tag{1}$$

In a time, t, travelling at constant speed, the pirate ship will travel a distance $V_p t$, and this must be equal to the distance along the pursuit curve; so integrating from $(0,0)$ to (x, y) we have:

$$V_p t = \int \sqrt{1+g^2}\,\mathrm{d}x. \tag{2}$$

Note that g is not a constant, it is a function of x. By eliminating time, t, between equations (1) and (2) we have:

$$\frac{V_m}{V_p}\int \sqrt{1+g^2}\,\mathrm{d}x = y + g.(x_0 - x). \tag{3}$$

Differentiating both sides of equation (3) with respect to x, and with some rearrangement, we get:

$$(x_0 - x)\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{V_m}{V_p}\sqrt{1+g^2}. \tag{4}$$

We can find g as a function of x from equation (4) by separating variables and integrating using the initial condition $g = 0$ when $y = 0$. After some algebraic manipulation we obtain:

$$g \equiv \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left[\left(1 - \frac{x}{x_0}\right)^{-V_m/V_p} - \left(1 - \frac{x}{x_0}\right)^{V_m/V_p}\right]. \tag{5}$$

Integrating this for the situation where $V_p > V_m$ and performing some straightforward but rather tedious manipulation, we find the

pursuit curve equation to be:

$$y = \frac{x_0}{2}\left(\frac{2r}{1-r^2} + \frac{(1-x/x_0)^{1+r}}{1+r} - \frac{(1-x/x_0)^{1-r}}{1-r}\right), \quad (6)$$

where $r = V_m/V_p$.

The corresponding equation for the situation where $V_p = V_m$ is

$$y = \frac{x_0}{2}\left[\frac{(1-x/x_0)^2}{2} - \ln\left(1 - \frac{x}{x_0}\right) - \frac{1}{2}\right], \quad (7)$$

though there is probably little point in the pirates giving chase in this case (and even less if $V_p < V_m$)!

Figure 2 illustrates the shapes of the pursuit curves for three different relative values of and V_p and V_m.

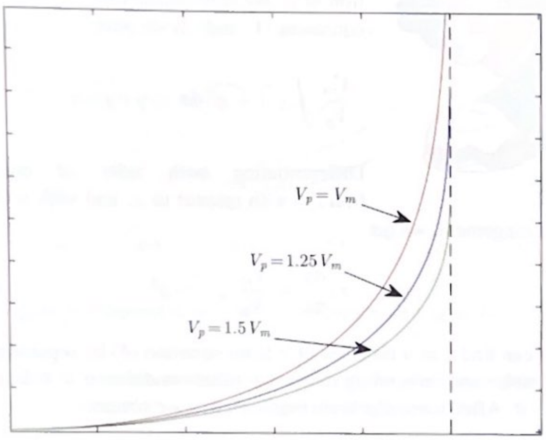


Figure 2: Bouguer's pursuit curves

We can find how long it takes the pirates to reach the merchant ship by taking the limit of equation (6) as x tends to x_0 (which gives us the appropriate value of y) and equating this to $V_m t_{\text{capture}}$. The result is

$$t_{\text{capture}} = \frac{r \times x_0}{V_m(1-r^2)}.$$

Of course, when $V_p = V_m$ $(r = 1)$ the pirates never catch the merchant ship.

Now, the analysis to determine the pursuit curves above is quite interesting, but visually the resulting curves are, frankly, rather boring! Let us turn to a different sort of pursuit problem to see if we can generate some more appealing curves.

Bugs

The n-Bug, or cyclic pursuit, problem places a number of ants (or bees, bugs, ...) at the vertices of a regular polygon with n sides. At some initial time they each start moving at the same speed towards an adjacent ant, tracing out n pursuit curves in so doing. What is the curve described by each of them?

Let us do the calculations for the simplest regular polygon, namely an equilateral triangle. Instead of just plotting the pursuit curves directly, we will plot the tangent line from each ant to the next at equal intervals of time throughout the chase. An example of what results is seen in Figure 3.

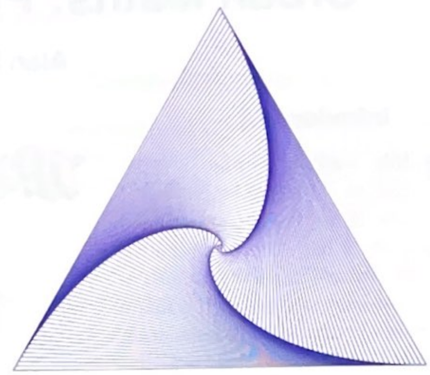


Figure 3: Triangular pursuit curves

Although the image is constructed of triangles, the pursuit curves show up clearly. It can be shown without much difficulty that the curves here, and for other regular polygons, are logarithmic spirals (see [1], pages 110–111, for example). However, I did not use the analytical result in generating Figure 3. Instead I numerically integrated the following time derivatives for changes in the x and y directions (simple Euler integration is sufficiently accurate here):

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = v\frac{(x_{i+1} - x_i)}{S_{i,i+1}}, \qquad \frac{\mathrm{d}y_i}{\mathrm{d}t} = v\frac{(y_{i+1} - y_i)}{S_{i,i+1}},$$

where $S_{i,i+1}$ is the direct distance between ants i and $i+1$ at time t, and v is the speed at which they are moving. This approach lends itself more easily to more complicated geometries for which there may not be analytical solutions for the pursuit curves. In contrast to the pirates and merchants scenario then, the approach I have adopted here is not very interesting from an analytical point of view, but the end result is visually much more appealing!

As a minor twist on this consider the ants initially on the vertices of a regular hexagon, but this time they each pursue, not the adjacent ant, but the next but one. In effect we have two triangular subsets of ants. Figure 4(a) illustrates the result when both subsets are travelling in the clockwise direction, while Figure 4(b) illustrates the result for one subset travelling clockwise and the other travelling anticlockwise (we will ignore the fact that various pairs of ants on opposite subsets will collide before they reach the centre!).

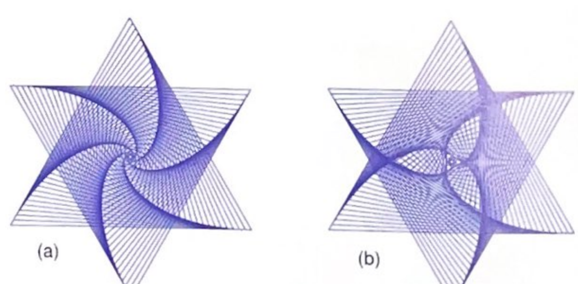


Figure 4: (a) Clockwise pursuit and (b) Clockwise and anticlockwise pursuit

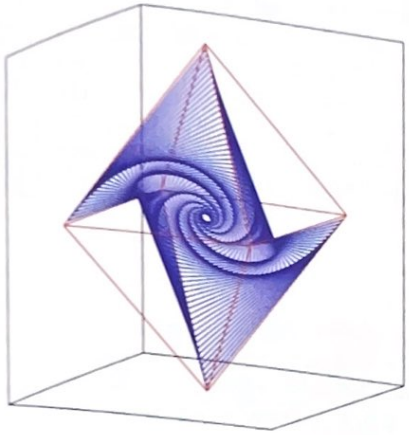


Figure 5: Pursuit in a diamond

We do not need to restrict ourselves to two dimensions of course. Figure 5 shows a three-dimensional example in which the ants start on the vertices of a diamond shape.

Finally

There are many tunes one can play with these pursuit curves: different two- and three-dimensional shapes, different speeds, etc. I have presented them here simply with their intrinsic shapes and patterns in mind. However, they are not entirely without application in the real world. Think missiles chasing aeroplanes, hawks chasing sparrows, dogs chasing ducks; even modern pirates chasing ships!

Incidentally, the duck was much faster over the water than the dog, so there was no chance the dog was going to catch it!

Acknowledgement

'Urban Maths' cartoonist: Adrian Metcalfe – `www.thisisfruittree.com`

References

1 Nahin, P.J. (2007) *Chases and Escapes: The Mathematics of Pursuit and Evasion*, Princeton University Press.

letters

Poems and Names

When I was in my teens I remember being greatly amused by somebody on the radio who said in a discussion on Homer that it was now believed that the writer of the Odyssey was not the same person who wrote the Iliad but another writer of the same name!

I little thought that I would one day find myself in a similar position. The implication of February's editorial was that I am the author of the fine poem which accompanied the obituary to Mike Walker. I knew Mike quite well and liked him a lot but I am not the author. Many people may have been misled by this and so I must put the record straight. The other Charles may also be a mathematician but I am Charles W Evans and he is Charles E Evans; there was no misprint in the name!

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IMA Honorary Secretary

Strictly Birthdays

I read with interest Sir Frederick Crawford's fascinating article 'Fun with Birthdays' [1] which highlighted the issues of testing the classic birthday problem using the relatively low numbers present at a small informal social gathering. Initially I thought that perhaps the probabilities of 2 people having a common day of the week for their births might be more suitable for small groups. A quick look indicated that the probability of this for a group of 3 was about 0.39 and for a group of 4 was about 0.65, which obviously may be tested in a small group but is entirely consistent with an intuitive view and lacks the counter-intuitive novelty and surprise of those in the classic birthday problem, so I abandoned this line of thought.

However, it reminded me that the relationship of day of the week to births and birthdays is a suitable topic of interest and fun for a small social gathering. A while ago my wife mentioned that some of our family members' birthdays 'occasionally' occur on the same day of the week as their original births and that this appears to happen to several members in the same years. After some investigation I was surprised to find that we all belong to one of 21 unique birthday groups, which define when our birthdays occur on the same day of the week as our birth. At the time I produced a crib sheet to allow an easy look-up of these special events. I have reproduced (Table 1) a cut-down version of my original table as 'a ready reckoner for use in a social setting' [1, p. 252] (to quote Sir Frederick).

We call the anniversary of our birth our 'birthday' but when this coincides with the same day of the week as our birth surely this is *strictly* our *birth day*. Hence the contemporary title 'Strictly Birthdays' (in the spirit of 'Fun with Birthdays')!

Although quite rudimentary, equation (1) provides an indicative view of the periodic cycles of strictly birthdays and much of the format of Table 1 but does not directly reflect the behaviour of each part of the year:

$$D = \left(d + Y + \left\lfloor \frac{Y}{4} \right\rfloor\right) \mod 7. \qquad (1)$$

(In the absence of any leap year effect, any date during a year moves forward to the next day of the week from its value the previous year. During a leap year, the dates after leap day – 29 February – move forward by 2 days of the week and this continues into the following year up to 28 February. Thereafter all dates revert to the 1-day movement until the next leap year. This explains why the table addresses each part of the year separately and why all days within each part of the year behave identically.)