Urban Maths: Yo-yo

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I was watching a couple of youngsters doing tricks with yo-yos. They had them moving continuously in all sorts of directions, not just up and down vertically; they were swinging them pendulum style, wrapping and unwrapping them around their arms, etc., all without getting tangled. I was full of admiration for their skill. The best I can do is a single vertical down and up – if I'm lucky! However, since I'm not much good at the real thing, I thought I would have a go at producing a mathematical model of the activity. Here is what I came up with.

Referring to Figure 1, and using Newton's second law of motion we have:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = T - mg \qquad (1)$$

and

$$I\frac{\mathrm{d}\omega}{\mathrm{d}t} = Tr, \qquad (2)$$

where m is the mass, I is the moment of inertia, v is the velocity of the centre of mass (positive upwards), ω is the angular rotational speed, t is time, T is the tension in the string, r is the radius of the axle and g is gravitational acceleration.

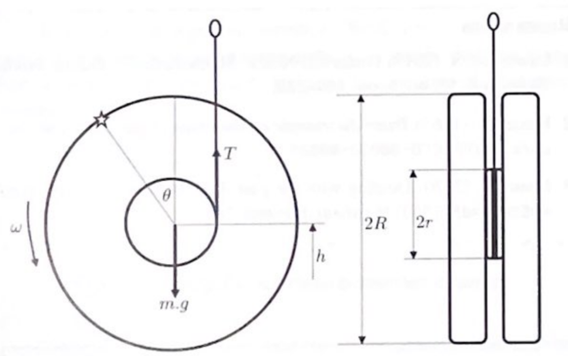


Figure 1: Two views of the yo-yo model.

We can also connect the rotational speed, ω, with the linear velocity of the centre of mass, v, through:

$$v = \omega r. \qquad (3)$$

(In one rotation, the centre of mass travels through a distance of $2\pi r$, because of the string ravelling or unravelling, in a time of $2\pi/\omega$.) Note that v is not the tangential velocity at the axle, though it has the same absolute magnitude. We also need to take into account that, no matter the direction of spin, the linear velocity will be negative during a descending phase and positive during an ascending phase (note that the yo-yo continues to spin in the same sense as it switches from descending to ascending).

We will simply take the moment of inertia as $I = mR^2/2$, assuming the gap between the two main sections of the yo-yo is small enough that we need not bother to correct for it. We can eliminate the string tension, T, between equations (1) and (2). Then, by differentiating equation (3) with respect to time (but noting that $\mathrm{d}\omega/\mathrm{d}t$ will be positive during a descent and negative during an ascent, whereas $\mathrm{d}v/\mathrm{d}t$ will be negative throughout) and with a little algebraic manipulation, we get the following expression for the rate of change of velocity of the centre of mass:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{g}{(1+(R/r)^2/2)}. \qquad (4)$$

We can integrate equation (4) immediately for the downward motion; so, with the assumption that there is zero initial velocity, i.e. we just let it unwind under the influence of gravity alone, we have:

$$v = -\frac{gt}{(1+(R/r)^2/2)} \qquad (5)$$

(with a corresponding expression for rotational speed, ω, obtained by using equation (3)).

We can integrate this again to get the height of the centre of mass as a function of time:

$$h = h_0 - \frac{gt^2}{2\,(1+(R/r)^2/2)}, \qquad (6)$$

where h_0 is the initial height.

For the upward motion, we again integrate equation (4), this time getting:

$$v = v_{\mathrm{lo}} - \frac{g(t-t_{\mathrm{lo}})}{(1+(R/r)^2/2)}, \qquad (7)$$

where t_{lo} is the time at which the yo-yo reaches its lowest height, h_{lo}, and v_{lo} is the magnitude of the linear velocity at that height (I am assuming the yo-yo changes direction instantly at its lowest height).

The corresponding expression for height is now:

$$h = h_{\mathrm{lo}} - \frac{g(t-t_{\mathrm{lo}})^2}{2\,(1+(R/r)^2/2)} + v_{\mathrm{lo}}(t-t_{\mathrm{lo}}). \qquad (8)$$

(Equation (8) is basically the standard kinetics equation $s = ut + at^2/2$ with suitably adjusted origin, t and a.) We set h_{lo} by specifying the number of turns, n, of the string that are on the yo-yo's axle (we will assume the string is thin enough that it does not significantly alter the effective radius of the axle). That is, we set: $h_{\mathrm{lo}} = h_0 - 2\pi rn$. Using this in equation (6), we find the time, t_{lo}, to be:

$$t_{\mathrm{lo}} = \sqrt{\frac{4\pi rn}{g}\left(1+\frac{1}{2}\left(\frac{R}{r}\right)^2\right)}.$$

Using equation (5), the maximum absolute velocity, v_{lo}, is

$$v_{\mathrm{lo}} = \sqrt{\frac{4\pi rng}{\left(1+(R/r)^2/2\right)}}.$$

We can also calculate the rotational speed, ω, and angle, θ, as functions of time (Figure 1), though I will not spell out the equations here. Figure 2 shows values of h, v, θ and ω for a single unravelling and re-ravelling of the yo-yo (assuming no energy losses). The data used to generate Figure 2 are: $R = 6\,\text{cm}$, $r = 1\,\text{cm}$, $n = 15$ and $h_0 = 1.5\,\text{m}$. The points are plotted at equal intervals of time.

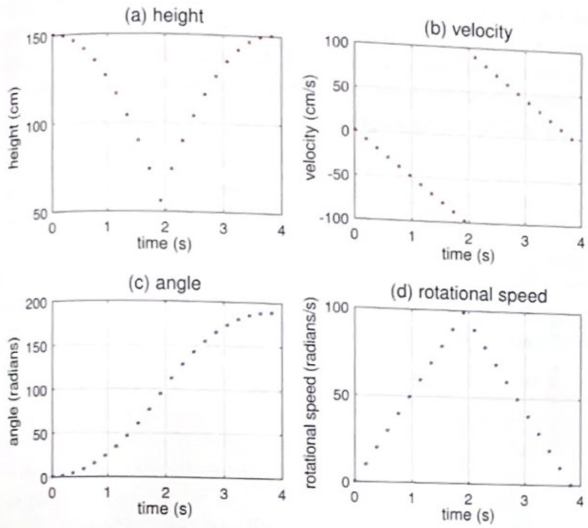


Figure 2: Yo-yo height, velocity, angle and rotational speed as functions of time.

Note that Figure 2(b) clearly shows the instantaneous change in sign of v when the string reaches its maximum length.

Parenthetically, it is interesting to note that the trajectories of height and velocity seen in (a) and (b) of Figure 2 are those of a perfectly elastic bouncing ball dropped onto a hard surface from a height equal to the length of string wound around the yo-yo's axle, and subject to a reduced gravitational acceleration. The effective reduced gravitational acceleration, g_{reduced}, is given by:

$$g_{\text{reduced}} = \frac{g}{\left(1 + (R/r)^2 / 2\right)}.$$

But back to the yo-yo. We can determine the path taken by a point on the rim of the yo-yo, such as that of the star shape indicated in Figure 1. Such a point has a vertical position relative to the ground at any time that is a combination of the vertical component of its position relative to the yo-yo's centre of mass and the height of the centre of mass. Using y for its height above the ground, we have:

$$y = h + R\cos\theta,$$

where θ is as indicated in Figure 1, and h is obtained from equations (6) or (8) as appropriate. Its horizontal position, x (positive to the right), is

$$x = -R\sin\theta.$$

Figure 3 shows an xy trace of the path of the point on the rim. However, the point is also traveling in time, so Figure 4 illustrates a 3D view of the movement. Again, the points in Figure 4 are plotted at equal intervals of time (more points are used than in Figure 2 to show more clearly the spiral nature of the path).

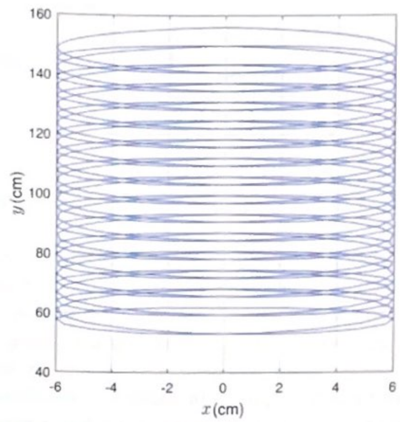


Figure 3: Spatial path traced by a point on the rim of the yo-yo.

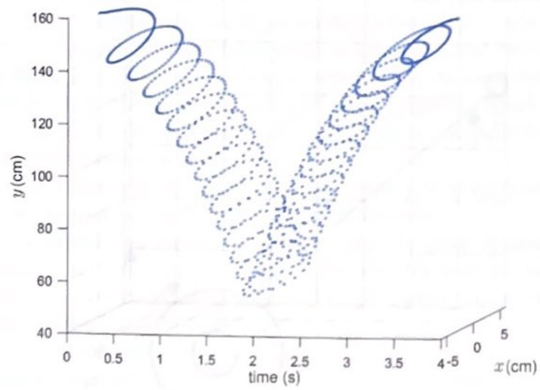


Figure 4: Space-time path of a point on the rim of the yo-yo.

The data used to generate Figures 2, 3 and 4 represent a reasonably realistic size for the yo-yo. What happens if we change its size? The advantage of a mathematical model is that we can do that quite easily. Let's make it 10 times larger; that is increase the radii R and r by a factor of 10. We will reduce the number of turns of the string around the axle by the same factor to keep the change in height the same. In Figure 5, we plot the space-time spirals of the point on the rim.

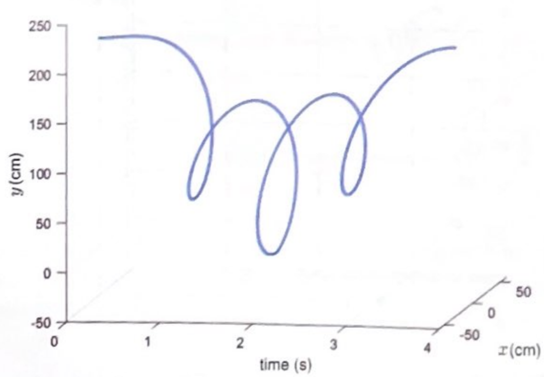


Figure 5: Space-time path of a point on the rim of a large yo-yo.

We can see that the number of turns of the spiral is also reduced by a factor of 10. Of course, we do not need the graph to tell us that – the number of rotations, hence spirals, is determined by n, the number of times we wind the string around the axle – but it's nice to have visual confirmation of the fact! The time from start to finish is the same as for the smaller yo-yo, but with each rotation taking much longer. I suspect that psychologically, if we were watching a real yo-yo of this size, the whole thing would feel slower.

We can develop the model further to incorporate more complicated manoeuvres than the simple gravitationally induced vertical motion considered above. For example, suppose we start by throwing the yo-yo at an angle to the vertical, letting it travel a short distance on loose, unwound string before the tension in the wound string acts to spin it. In addition to travelling up and down the string, the yo-yo will become a pendulum with a varying length (Figure 6). In Figure 6 the parameters v and ω are as above, while s is the string length, ϕ is the angle from vertical and Ω is $\mathrm{d}\phi/\,\mathrm{d}t$.

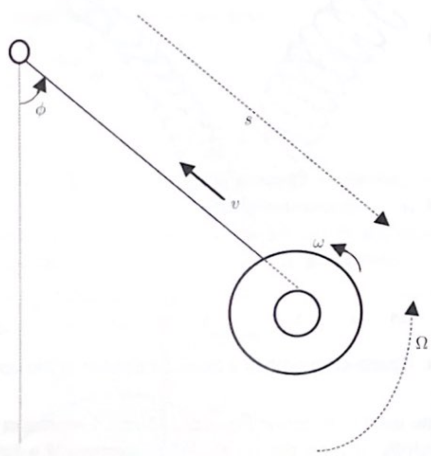


Figure 6: Yo-yo pendulum

We continue to use equations (1) through (3), though we must multiply g by $\cos\phi$. We also need:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \Omega \quad \text{and} \quad s\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -g.\sin\phi.$$

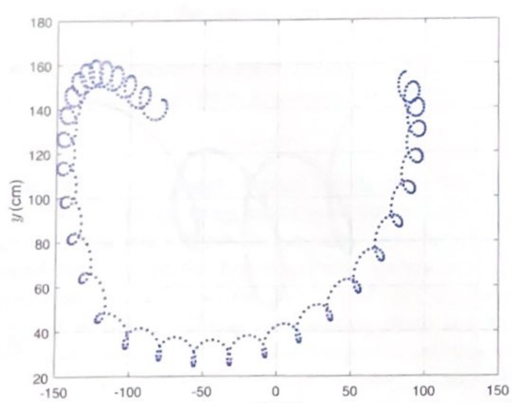


Figure 7: Spatial path traced by a point on the rim of the pendulum yo-yo.

The linear velocity, v, is now the rate of change of the string length, s. We now have several non-linear simultaneous differential equations to solve, so there is no nice set of analytical solutions as there was above. However, the equations are not difficult to solve numerically.

We will assume the length of unwound string is 1 m and that we throw the pendulum in such a way that it is travelling radially at $1\,\mathrm{m\,s^{-1}}$ at an initial angle of 60° ($\pi/3$ radians) from the vertical when the string becomes taut. Figure 7 shows the resulting xy trace of the point on the rim of the (original size) yo-yo, while Figure 8 shows the interesting relationship between the pendulum angle and the string length.

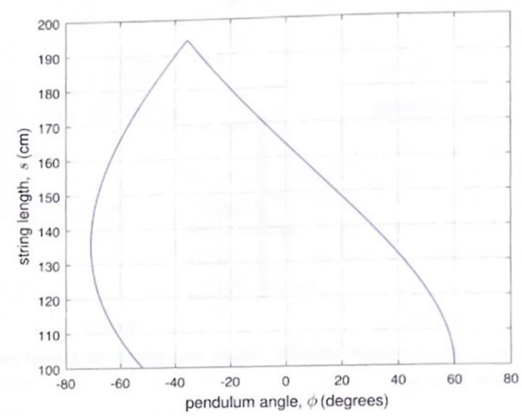
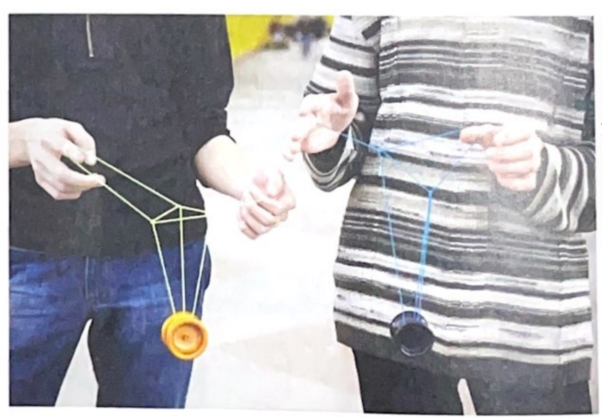


Figure 8: Pendulum yo-yo string length versus pendulum angle.

In contrast to the simple vertical yo-yo, there is a distinct asymmetry exhibited here. This reflects that the initial conditions were chosen such that the pendulum length (i.e. the string length) is not at its maximum when the pendulum reaches the vertical position.

We could complicate matters further by allowing the yo-yo to swing a full 360° and wrapping around a wrist, for example. However, I think I will leave anything more complicated to the experts with their real, physical yo-yos!



Acknowledgement

'Urban Maths' cartoonist: Adrian Metcalfe – `www.thisisfruittree.com`