Appendix A

Managing Uncertainties

One of the fundamental aspects of science is knowing the reliability of results. The mechanism for gaining this knowledge is first to gauge how well one knows any given measurement and then to propagate this to an indication of the reliability of the results that depend on those measurements. The primary goal in attending to the propagation of the uncertainty is that it allows scientists to determine which measurement is causing the most uncertainty in the result so that future experimenters know which measurement to improve to get an improved result.

In this section we will learn the terminology, determine how to gauge the measurement uncertainty, learn how to propogate this information through a calculation, and learn how to discuss this analysis in your lab reports.

A.1 Experimental Uncertainties, Defining "Error"

Measurements are never exact. For example, if one apple is divided among three people, your calculator will tell you that each person has 0.3333333333 of an apple. A measurement of each slice will tell you two pieces of information: (1) how many 3s to keep and (2) how well you know the final 3. In this example, both 0.33 ± 0.01 and 0.33 ± 0.04 imply that the measurement is accurate to two decimal places, but the first implies that you trust the second 3 more than if you report it as the second number.

> **CAUTION:** Because physicists "know what we mean", they are often sloppy with their language and use the words "error" and "uncertainty" interchangeably.

Some technical terms and their use in physics (which may differ from common use):

accuracy How close a number is to the true (but usually unknowable) result. This is usually expressed by the (absolute or relative) error.

precision How well you trust the measurement. This is vaguely expressed by the number of decimals, or clearly expressed by the size of the (absolute or relative) uncertainty.

uncertainty The **uncertainty in a number** expresses the precision of a measurement or of a computed result. This can be expressed as the **absolute uncertainty** (explained in Finding the Precision of a Measurement), the **relative uncertainty**, or the **percent uncertainty**.

$$\text{relative uncertainty} = \left|\frac{\text{(absolute uncertainty)}}{\text{measured value}}\right|$$

$$\text{\%-uncertainty} = 100\% * \left|\frac{\text{(absolute uncetainty)}}{\text{measured value}}\right|$$

error The **error** is a number that expresses the accuracy by comparing the measurement to an accepted ("true") value. This can be expressed as the **absolute error**, the **relative error**, or the **percent error**.

$$\text{absolute error} = |\text{true value} - \text{measured value}|$$

$$\text{relative error} = \left|\frac{(\text{true value} - \text{measured value})}{\text{true value}}\right|$$
$$\text{\%-error} = 100\% * \left|\frac{(\text{true value} - \text{measured value})}{\text{true value}}\right|$$

difference The **difference** is a number that expresses the consistency of a multiple measurements by comparing one measurement to another. This can be expressed as the **absolute difference**, the **relative difference**, or the **percent difference**. You should notice that since we don't know *which* measurement to trust, we take the absolute difference relative to the *average* of the measurements (rather than choosing one measurement as "true").

$$\text{absolute difference} = (\text{measurement}_1 - \text{measurement}_2)$$
$$\text{relative difference} = \frac{(\text{measurement}_1 - \text{measurement}_2)}{\left[\frac{(\text{measurement}_1)+(\text{measurement}_2)}{2}\right]}$$
$$\text{\%-difference} = 100\% * \frac{(\text{measurement}_1 - \text{measurement}_2)}{\left[\frac{(\text{measurement}_1)+(\text{measurement}_2)}{2}\right]}$$

Note A.1.1 (compare)**.** Whenever you are asked to "compare" values, it is expected that you will not only compute a %-error or %-difference (as appropriate, according to the above considerations); but will also comment on if the uncertainty of the values overlap. Recall that the uncertainty means that your measurement does not distinguish between values within that range, so if the uncertainties overlap, then the values are "the same to within your ability to measure them."

A.2 Writing an Analysis of Error

The conclusion of your lab report should be based on an analysis of the error in the experiment. The analysis of error is one of the most certain gauges available to the instructor by which the student's scientific insight can be evaluated. To be done well, this analysis calls for comments about the factors that impacted the extent to which the experimental results agree with the theoretical value (what factors impact the percent error), the limitations and restrictions of the instruments used (what factors impact the uncertainty), and the legitimacy of the assumptions.

Physicists usually use the phrase "sources of error" (or "sources of uncertainty") to describe how the limits of measurement propagate through a calculation (see Propagation of Uncertainties) to impact the **uncertainty** in the final result. This type of "error analysis" gives insight into the **accuracy** of the result. Considerations for the Error Analysis provides questions that can help you describe which of several measurements can most effectively improve the **precision** of the result so that you can gain insight into the **accuracy** of the result. The accuracy allows one to gauge the veracity (truth) of an underlying relationship, but precision allows you to gauge accuracy. Said another way, a small **percent difference** usually is used to imply a small **percent error**. Said another way, imprecise measurements always *seem* accurate.

A.2.1 Technically, Errors are not Mistakes

Your report should not list "human error" because most students misunderstand this term to mean "places I might have made a mistake" rather than "the limiting factor when using the equipment correctly." Finding the Precision of a Measurement discusses measurement uncertainties as defined above.

In the example of the apple above, the fact that one person has 0.33 ± 0.04 of an apple does *not* reflect a "mistake" in the cutting, but rather reflects that the cutter is limited in their precision. What is important is to use the uncertainty to express how well one can repeated cut the apple into thirds. The absolute uncertainty of 0.04 is generally interpreted to say that most instances (roughly 68%, as explained in Uncertainty of multiple, repeated measurements) of the cutting of an apple in this way will result in having between 0.29 to 0.37 of an apple for any given slice.

When describing the cause of an error (difference from the theoretical value) or of an uncertainty (the extent you trust a number), you can usually categorize this source of error as a random error (a cause that could skew the result too large *or* too small) or as a systematic error (a cause that tends to skew the result in one particular direction).

Random Error An environmental circumstances, generally uncontrollable, that sometimes makes the measured result too high and sometimes make it too low in an unpredictable fashion. Random errors may have a statistical origin – that is, they are due to chance. For example, if one hundred pennies are dumped on a table, on average we expect that fifty would land heads up. But we should not be surprised if fifty-three or forty-seven actually landed heads-up. This deviation is statistical in nature because the way in which a penny lands is due to chance. Random errors can sometimes be reduced by either collecting more data and averaging the readings, or by using instruments with greater precision.

Systematic Error A systematic error can be ascribed to a factor which would tend to push the result in a certain direction away from the theory value. The error would make all of the results either systematically too high or systematically to low. One key idea here is that systematic errors can be eliminated or reduced if the factor causing the error can be eliminated or controlled. This is sometimes a big "if", because not all factors can be controlled. Systematic errors can be caused by instruments which are not calibrated correctly, maybe a **zero-point error** (an error with the zero reading of the instrument). This type of error can usually be found and corrected. Systematic errors also often arise because the experimental setup is somehow different from that assumed in the theory. If the acceleration due to gravity was measured to be $9.52\,\frac{\mathrm{m}}{\mathrm{s}^2}$ with an experimental uncertainty (precision) of $0.05\,\frac{\mathrm{m}}{\mathrm{s}^2}$, rather than the textbook value of $9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$, then we should be concerned with why the accuracy is not as good as the precision. This is most likely to mean that there is a significant systematic error in the experiment, where one of the initial assumptions may not be valid. The textbook value does not consider the effects of the air. The effects of the air may or may not be controllable, and the difference between the theory and the data may be (within appropriate limits or tests) considered a correction factor for the systematic error.

A.2.2 Considerations for the Error Analysis

In order to help you get started on your discussion of error, the following list of questions is provided. It is not an exhaustive list. You need not answer all of these questions in a single report.

1. Is the error large or small? Is it random or systematic? ... statistical? ... cumulative?
   (a) What accuracy (precision) was expected? Why? What accuracy (precision) was attained? If different, why?
   (b) Was the experimental technique sensitive enough? Was the effect masked by noise?
2. Is it possible to determine which measurements are responsible for greater percent error by checking items measured and reasoning from the physical principles, the nature of the measuring instrument, and using the rules for propagation of error?
   (a) Is the error partly attributable to the fact that the experimental set-up did not approximate the ideal that was required by the physical theory closely enough? How did it fail?
   (b) If a systematic error skews high (low), then is your result too high (low)? Is this a reasonable explanation? Is the size of the skew enough to explain the result?
   (c) What can be done to improve the equipment and eliminate error? How can the influence of environmental factors be diminished? Why is this so?
3. Is the error (deviation) in the experiment reasonable?

Note A.2.1 (compare)**.** Whenever you are asked to "compare" values, it is expected that you will not only compute a %-error or %-difference (as appropriate); but will also comment on if the uncertainty of the values overlap. Recall that the uncertainty means that your measurement does not distinguish between values within that range, so if the uncertainties overlap, then the values are "the same to within your ability to measure them."

A.3 Finding the Precision of a Measurement

A.3.1 Uncertainty of a single measurement

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A.3.2 Uncertainty of multiple, repeated measurements

Calculate or estimate the precision of a measurement by one or more of the following methods:

1. by the precision of the measuring instrument, and take into account any uncertainties that are intrinsic to the object itself;

2. by the range of values obtained, the minimum and/or maximum deviation (d);

$$d_i = |X_i - X_{\text{ave}}|$$

3. by the standard deviation, which is the square root of the sum of the squares of the individual deviations (d) divided by the number of readings (N) minus one;

$$\sigma = \sqrt{\frac{1}{(N-1)} \sum d^2} = \sqrt{\frac{1}{(N-1)} \sum_i |X_i - X_{\text{ave}}|^2}$$

4. by the standard deviation of the mean, which is the standard deviation divided by the square root of the number of readings;

5. by the square root of the number of readings ($\sqrt{N}$), if N is considered large;

If many data points were taken and plotted on a histogram, it would smooth out and approach the symmetrical graph typical of the binomial distribution (see the Figure 2.0.1). This distribution and many others in statistics may be approximated by the gaussian distribution.

The standard deviation, σ, can be estimated from the above graph. It is a measure of the "width" of the distribution. For the case shown, the standard deviation has the value of five. The greater the standard deviation, the wider the distribution and the less likely that an individual reading will be close to the average value. About 68% of the individual readings fall within one standard deviation (between 45 and 55 in this case). About 96% of the readings fall within two standard deviations (between 40 and 60 in this case).

As more and more readings are taken, the effect of the random error is gradually eliminated. In the absence of systematic error, the average value of the readings should gradually approach the true value. The smooth curve above was drawn assuming that there was no systematic error. If there were, the graph would merely be displaced sideways. The average value for the number would then be say 55.

The distribution of many average (mean) readings is also gaussian in shape. Comparing this to the distribution for individual readings, it is much narrower. We would expect this, since each reading on this graph is an average of individual readings and has much less random error. By taking an average of readings, a considerable portion of the random error has been canceled. The standard deviation for this distribution is called the standard deviation of the mean (σ_m). For this distribution, 68% of the averages of the readings are within one standard deviation of the mean, and 98% of the average readings fall within two standard deviations of the mean.

The standard deviation of the mean tells how close a particular *average* of several readings is likely to be to an overall average when many readings are taken. The standard deviation tells how close an *individual* reading is likely to be to the average.

There is one case for which the standard deviation can be estimated from one reading. In counting experiments (radioactivity, for example), the distribution is a Poisson distribution. For this distribution, the standard deviation is just the square root of the average reading. One reading can give an estimate of the average, and therefore, give an estimate of-the standard deviation.

A.4 Propagation of Uncertainties

The previous sections discussed the uncertainties of directly measured quantities. Now we need to consider how these uncertainties affect the rest of the analysis. In most experiments, the analysis or final results are obtained by adding, subtracting, multiplying, or dividing the primary data. The uncertainty in the final result is therefore a combination of the errors in the primary data. The way in which the error propagates from the primary data through the calculations to the final result may be summarized as follows:

1. The error to be assigned to the sum or difference of two quantities is equal to the sum of their absolute errors.
2. Relative error is the ratio of the absolute error to the quantity itself. The relative error to be assigned to the product or quotient of two quantities is the sum of their relative errors.
3. The relative error to be assigned to the power of a quantity is the power times the relative error of the quantity itself.

These rules are not arbitrary, but rather they follow directly from the nature of the mathematical operations. These rules may be derived using calculus.

Exercise A.4.1 (Try Propagating the Uncertainty When Adding Numbers)**.** Compute the perimeter of a table that is measured to be $176.7\,\mathrm{cm} \pm 0.2\,\mathrm{cm}$ along one side and $148.3\,\mathrm{cm} \pm 0.3\,\mathrm{cm}$ along the other side.

Hint 1. To find the perimeter, add the four sides of the rectangle. Use the values, but not the uncertainty.

Hint 2. To find the uncertainty, use Rule 1.

Answer. The perimeter is $P = 650\,\mathrm{cm} \pm 1\,\mathrm{cm}$.

Solution. The perimeter can be found as:

$$\begin{aligned} P &= (176.7\,\mathrm{cm}) + (148.3\,\mathrm{cm}) + (176.7\,\mathrm{cm}) + (148.3\,\mathrm{cm}) \\ P &= 650.0\,\mathrm{cm} \end{aligned}$$

but we do not know the precision (appropriate number of decimals) until we compute the uncertainty, which is

$$\begin{aligned} \Delta P &= (0.2\,\mathrm{cm}) + (0.3\,\mathrm{cm}) + (0.2\,\mathrm{cm}) + (0.3\,\mathrm{cm}) \\ \Delta P &= 1.0\,\mathrm{cm} \end{aligned}$$

The value of the uncertainty determines where you round the result. Because the first digit of the uncertainty is in the "one's place", we round *both* the value and the uncertainty to that place.

The perimeter is $P = 650\,\mathrm{cm} \pm 1\,\mathrm{cm}$.

Exercise A.4.2 (Try Propagating the Uncertainty When Multiplying Numbers)**.** Compute the area of a table that is measured to be $176.7\,\mathrm{cm} \pm 0.2\,\mathrm{cm}$ along one side and $148.3\,\mathrm{cm} \pm 0.3\,\mathrm{cm}$ along the other side.

Hint 1. To find the area, multiple the length and width of the rectangle. Use the values, but not the uncertainty.

Hint 2. Because the area of the table is calculated using multiplication, use Rule 2to find the uncertainty.

Answer. The area is $A = (2.620 \times 10^4) \pm (0.008 \times 10^4)\mathrm{cm}^2$.

Solution. The area is found to be (significant digits are underlined)

$$\begin{aligned} A &= (176.7\mathrm{cm}) \times (148.3\mathrm{cm}) \\ A &= \underline{2620}4.61\mathrm{cm}^2 \end{aligned}$$

The rules for **significant figures** gives a guide for the precision (appropriate number of decimals), that is only an approximation. To know with certainty, we need to compute the uncertainty, which is

$$\begin{aligned} \%\text{-uncertainty} &= \left(\frac{.2\mathrm{cm}}{176.7\mathrm{cm}}100\%\right) + \left(\frac{0.3\mathrm{cm}}{148.3\mathrm{cm}}100\%\right) \\ \%\text{-uncertainty} &= (.11\%) + (.20\%) = (.31\%) \end{aligned}$$

Insignificant Please be aware that the reason some digits are called **insignificant** is that they are *insignificant*:

$$\begin{aligned}(.31548\%) \times (26204.61) &= 82.67\\ (.31\%) \times (26204.61) &= 81.23\\ (.31\%) \times (26200) &= 81.22\\ (.3\%) \times (26204.61) &= 78.61\\ (.3\%) \times (26200) &= 78.60\end{aligned}$$

All of these round to an uncertainty of $80\,\mathrm{cm}^2$.

To find the uncertainty, we calculate

$$(.31\%) \times (\underline{2620}4.61\mathrm{cm}^2) = 81.23\mathrm{cm}^2$$

This tells us that we need to round at the "ten's place". We can write the area in a variety of ways:

$$\begin{aligned}A &= (2.620 \times 10^4\,\mathrm{cm}^2) \pm 0.3\%\\ &= (2.620 \times 10^4) \pm (0.008 \times 10^4)\,\mathrm{cm}^2\\ &= 2.620(8) \times 10^4\,\mathrm{cm}^2\end{aligned}$$

A.5 Significant Figures only *approximates* Uncertainty

The precision/accuracy of any measurement or number is approximated by writing the number with a convention called using **significant figures**. Every measuring instrument can be read with only so much precision and no more. For example, a meter stick can be used to measure the length of a small metal rod to one-tenth of a millimeter, whereas a micrometer can be used to measure the length to one-thousandth of a millimeter. When reporting these two measurements, the precision is indicated by the number of digits used to express the result. You should always record your data and results using the convention of significant figures.

To give a specific example, suppose that the rod mentioned above was 52.430 mm long. When making this measurement with the meter stick, you would count off the total number of millimeters in the length of the rod and then add your best guess that the rod was four-tenths of a millimeter longer than that. Using the micrometer, you would count off the hundredths of a millimeter and then add your best guess of the number of thousandths of a millimeter, to complete the measurement. How would you communicate the fact that one measurement is more precise than the other? If you wrote both quantities in the same way, you could not tell which was which.

The rules for **significant figures**:

1. Significant figures include all certain digits plus the first of the doubtful digits.

2. Zeros to the right of the number are significant; zeros on the left are not.

3. Round the number, increasing by one the last digit retained if the following digit is greater than five.

4. In addition and subtraction, carry the result only up to the first doubtful decimal place of any of the starting numbers.

5. In multiplication and division, retain as many significant figures in the answer as there are in the starting number with the smallest number of significant figures.

When determining or estimating the experimental uncertainty, the precision of the measuring instrument is important, as shown in the above examples. But you must also be aware of other experimental factors. For example, a good stopwatch may have a precision of 0.01 seconds. Is this the total uncertainty of the measurement? You must remember that our physical reaction time maybe another 0.3 seconds. This is more than 10 times larger than the precision of the timer. This is very significant. Another example is trying to measure the diameter of a fuzzy cotton ball with a micrometer. Why is this not a very productive procedure? There are major uncertainties here that are intrinsic to the object itself and are unrelated to the measuring instrument. One must use common sense when estimating these uncertainties.

The **actual uncertainty** written in the units of the measurement, may not convey a sense of how good the precision is. A better measure of the precision is given by the relative uncertainty. This is defined as the actual uncertainty divided by the measurement itself and multiplied by 100, the **relative uncertainty** does not carry any units, just a %.