A grating is a periodic structure.
There are a variety of different types.

Amplitude :

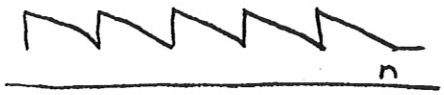
Binary amplitude

sinusoidal amplitude

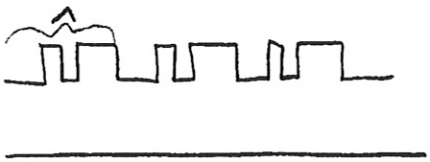
Phase surface relief

binary surface relief

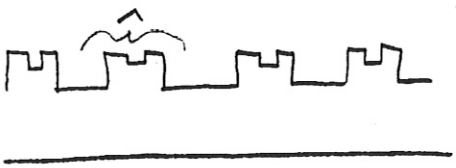
sinusoidal surface relief



Blazed surface relief

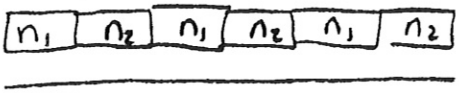


Damann surface relief

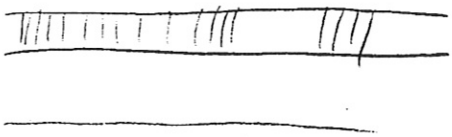


arbitrary profile surface relief

Phase index

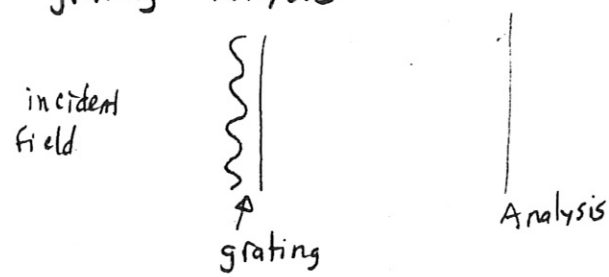


Binary index



Sinusoidal index

General grating analysis



Assume incident field is a plane wave $e_i(x) = e^{-j2\pi \frac{\sin\theta}{\lambda} x}$

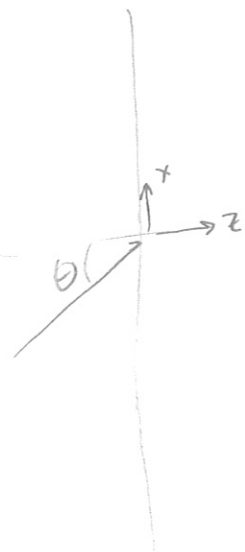
grating : $g(x) \circledast \sum_{m=-\infty}^{\infty} \delta(x - m\Lambda)$

$g(x)$ is the transmittance for a single period

Aperture $a(x)$

$t(x) = [\text{ grating }]$ Aperture

$$t(x) = \left[g(x) \circledast \sum_{m=-\infty}^{\infty} \delta(x - m\Lambda) \right] a(x)$$



$$\vec{E}_i = E_0 \, e^{+jk(\sin\theta \, x + \cos\theta \, z)}$$

$$t(x) = \left[g(x)\, e^{+j\, 2\pi \frac{\sin\theta}{\lambda} x} \otimes \sum_{m=-\infty}^{\infty} \delta(x - m\Lambda) \right] a(x)$$

$$T = \mathcal{F}\{t\} = \left[\sum_{m=-\infty}^{\infty} G\left(f_x + \frac{\sin\theta}{\lambda}\right) \delta\left(f_x + \frac{\sin\theta}{\lambda} - \frac{m}{\Lambda}\right) \right] \otimes A(f_x)$$

$$= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} A(\tau)\, G\left(\tau - f_x + \frac{\sin\theta}{\lambda}\right) \delta\left(\tau - f_x + \frac{\sin\theta}{\lambda} - \frac{m}{\Lambda}\right) d\tau$$

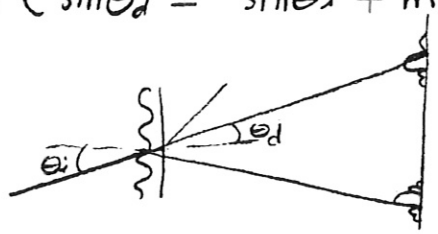
$\tau = f_x - \frac{\sin\theta_i}{\lambda} + \frac{m}{\Lambda}$

$$= \sum_{m=-\infty}^{\infty} A\left(f_x - \frac{\sin\theta}{\lambda} + \frac{m}{\Lambda}\right) G\left(f_x + \frac{\sin\theta}{\lambda} + \frac{m}{\Lambda} - f_x - \frac{\sin\theta}{\lambda}\right)$$

$$= \sum_{m=-\infty}^{\infty} A\left(f_x - \frac{\sin\theta}{\lambda} + \frac{m}{\Lambda}\right) G\left(\frac{m}{\Lambda}\right) \Big|_{f_x = \frac{x}{\lambda d} = \frac{\sin\theta_d}{\lambda}}$$

$$= \sum_{m=-\infty}^{\infty} \eta_m\, A\left(\frac{\sin\theta_d}{\lambda} - \frac{\sin\theta_i}{\lambda} + \frac{m}{\Lambda}\right)$$

$$= \sum_{m=-\infty}^{\infty} \eta_m\, A\left(\sin\theta_d - \sin\theta_i + m\frac{\lambda}{\Lambda}\right)$$



An arbitrary grating can be analyzed as:

(1) A direction for a given order $\sin\theta_d = \sin\theta_i - m\frac{\lambda}{\Lambda}$

(2) An efficiency of that order η_m

(3) A shape characterized by the incident beam and the aperture

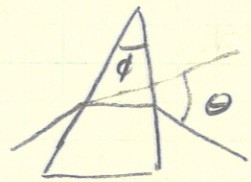
The direction of the order is independent of the shape of the grating period.

The efficiency of each order depends on the grating shape.

First of all, we are going to look at the direction of the grating order.

Dispersing Element

Compare grating with a dispersing prism
use a thin prism



$$\theta = (n-1)\,\phi$$

$$\frac{d\theta}{d\lambda} = \frac{dn}{d\lambda}\,\phi$$

$$n = A + B\lambda^{-2}$$

$$\frac{dn}{d\lambda} = -2\,B\lambda^{-3}$$

$$B = \frac{(n_d-1)}{V}\,\frac{\lambda_c^2\lambda_F^2}{(\lambda_c^2-\lambda_F^2)}$$

$$\frac{dn}{d\lambda} = \frac{(n_d-1)}{V}\,(-2)\,\frac{\lambda_c^2\,\lambda_F^2}{(\lambda_c^2-\lambda_F^2)}\left(\frac{1}{\lambda^3}\right)$$

use $\lambda_c = 656\,nm$
$\lambda_F = 486\,nm$
$\lambda = 588\,nm$

$$\frac{dn}{d\lambda} = -0.0052\left(\frac{n_d-1}{V}\right)$$

want a material with high n_d and low V
use SF2 $n_d = 1.648$ $V = 33.85$

$$\frac{dn}{d\lambda} = (-0.0052)\left(\frac{0.648}{33.85}\right) = -9.955\times10^{-5}\ nm^{-1}$$

$$\frac{d\theta}{d\lambda} = \frac{dn}{d\lambda}\,\phi = (-9.955\times10^{-5})\left(\frac{\pi}{6}\right)$$

$$\frac{d\theta}{d\lambda} = -5.2\times10^{-5}\ rad/nm$$

Now a grating

$$\sin\theta_d = \sin\theta_i - m\frac{\lambda}{\Lambda}$$

$$\theta_d = \sin^{-1}\left[\sin\theta_i - m\frac{\lambda}{\Lambda}\right]$$

$$\frac{d\theta_d}{d\lambda} = \frac{1}{\sqrt{1-\left(\sin\theta_i - m\frac{\lambda}{\Lambda}\right)^2}}\left(-\frac{m}{\Lambda}\right)$$

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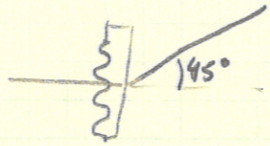
use a normal incident beam and 45° diffracted beam



$$\sin 45° = 0 - m\left(\frac{588}{\Lambda}\right)$$

$$m = -1$$

$$\Lambda = \frac{588}{\sin 45°} = 832 \text{ nm}$$

$$\frac{d\theta}{d\lambda} = \frac{1}{\sqrt{1-(\sin 45°)^2}}\left(\frac{1}{832 \text{ nm}}\right)$$

$$\frac{d\theta}{d\lambda} = 0.0017 \ \frac{\text{rad}}{\text{nm}}$$ compared to $$\frac{d\theta}{d\lambda} = -5.2\times10^{-5} \ \frac{\text{rad}}{\text{nm}}$$

A factor of 32 more dispersive and opposite direction

Gratings

The grating equation $\sin\theta_d = \sin\theta_i + m\frac{\lambda}{\Lambda}$

can be cast into a K-space formulation

$$\vec{K}_d = \vec{K}_i + m\vec{K}$$

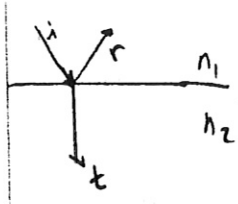
$\vec{K}_d$ is the wavevector of the diffracted wave inside of the grating
$\vec{K}_i$ is the incident wavevector
$\vec{K}$ is the grating vector $\vec{K} = \frac{2\pi}{\Lambda}$

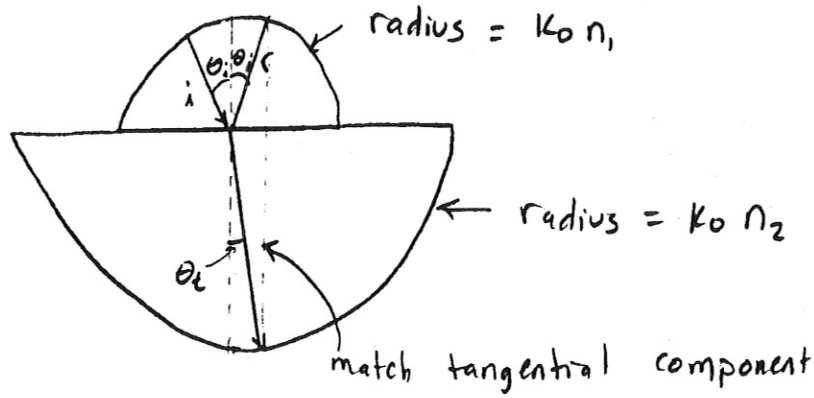
The wavevector in a particular media is determined by matching the tangential components.

We do this matching using K-space diagrams

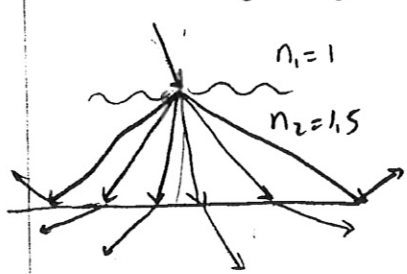
Let's first look at Snell's Law $n_1 \sin\theta_1 = n_2 \sin\theta_2$

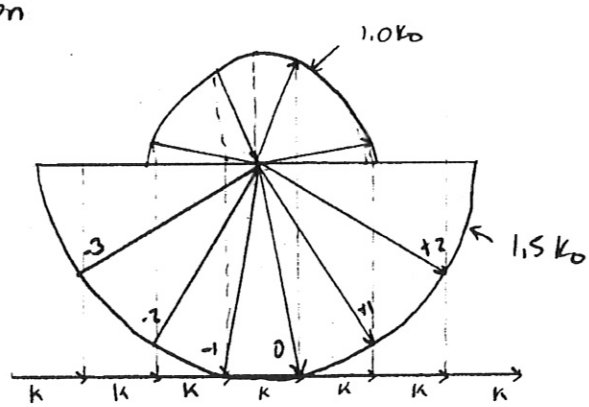
Draw 2 semi-circles for the incident and transmitted regions





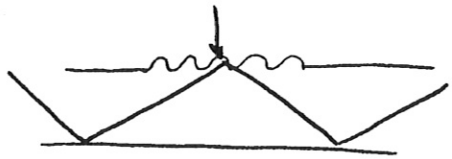
Now the grating equation

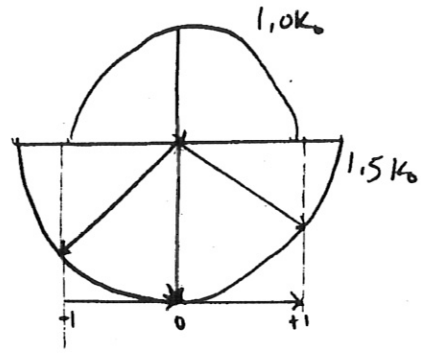




We want to design a grating to couple light into a glass substrate.

$n = 1.5$
$\lambda = 0.633\ \mu m$
What is the maximum grating period to have no diffracted orders in air with normally incident light





$K_x d > 1.0 K_0$

$K_{xi} = 0$

$K > \frac{2\pi}{\lambda}$

$\frac{2\pi}{\Lambda} > \frac{2\pi}{\lambda}$

$\Lambda < \lambda$

we can relate the diffracted wave to an effective index

$K_x d = K_{xi} + m K \Rightarrow n_{eff} K_0 = n_1 \sin\theta_i K_0 + m \frac{2\pi}{\Lambda}$

$\frac{K_x d}{K_0} = \sin\theta_i + m \frac{\lambda}{\Lambda}$

$n_{eff} = n_1 \sin\theta_i + m \frac{\lambda}{\Lambda}$

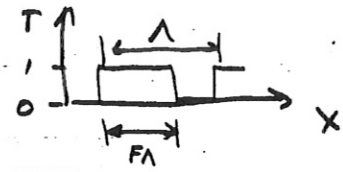
if $n_{eff} < n_1$ orders in region 1

if $n_2 < n_{eff} < n_1$ orders in substrate

$n_{eff} > n_2$ no orders

<u>Grating Applications</u>
Beam splitters
Array generators
Wavelength dispersers → Spectrometers

Binary Amplitude Grating



$$t(x) = \left(\mathrm{rect}\left(\frac{x}{F\Lambda}\right) e^{-j2\pi \frac{\sin\theta_i}{\lambda} x}\right) \circledast \sum_{m=-\infty}^{\infty} \delta(x - m\Lambda)$$

$q = \frac{1}{\Lambda}$

$$T(f_x) = \mathrm{Sinc}\left[F\Lambda\left(f_x - \frac{\sin\theta_i}{\lambda}\right)\right] \sum_{m=-\infty}^{\infty} \delta\left[\Lambda\left(f_x - \frac{\sin\theta_i}{\lambda}\right) - m\right]$$

$$= \mathrm{Sinc}\left[F\Lambda f_x - \frac{F\Lambda}{\lambda}\sin\theta_i\right] \sum_{m=-\infty}^{\infty} \delta\left[f_x - \frac{\sin\theta_i}{\lambda} - \frac{m}{\Lambda}\right]$$

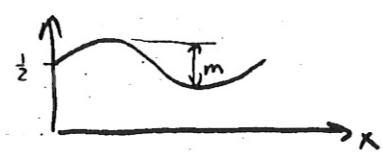
$$f_x = \frac{\sin\theta_i}{\lambda} + \frac{m}{\Lambda}$$

$$\eta_m = \mathrm{Sinc}\left[\cancel{\frac{F\Lambda}{\lambda}\sin\theta_i} + Fm - \cancel{\frac{F\Lambda}{\lambda}\sin\theta_i}\right]$$

$$\boxed{\eta_m = \mathrm{Sinc}(Fm)}$$

Sinusoidal Amplitude Grating

$$t(x) = \frac{1}{2} + \frac{m}{2}\cos\left(\frac{2\pi}{\Lambda}x\right)$$



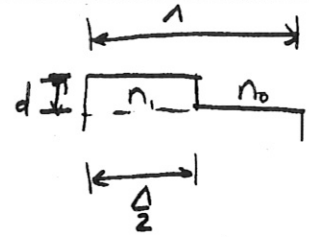
$$\mathcal{F}\left\{\frac{1}{2} + \frac{m}{2}\cos\left(\frac{2\pi}{\Lambda}x\right)\right\} = \frac{1}{2}\delta(f_x) + \frac{m}{4}\delta\left(f_x + \frac{2\pi}{\Lambda}\right) + \frac{m}{4}\delta\left(f_x - \frac{2\pi}{\Lambda}\right)$$

$$I^2 = \left(\frac{1}{4}\delta(f_x) + \frac{m^2}{16}\delta\left(f_x + \frac{2\pi}{\Lambda}\right) + \frac{m^2}{16}\delta\left(f_x - \frac{2\pi}{\Lambda}\right)\right) \circledast \text{Beam}$$

$\eta_0 = 1/4$

$\eta_1 = \eta_{-1} = m^2/16$

Binary Phase Grating



50% fill factor

Need to define both sections

Each section has
(1) Relative shift
(2) Relative Phase
(3) Defining rect (·)

Choose relative shift from the center $x = \frac{\Lambda}{2}$

RIDGE

(1) relative shift of $-\frac{\Lambda}{4}$
(2) relative phase $e^{j\frac{2\pi}{\lambda}dn_1}$
(3) Defining rect $\mathrm{rect}\left(\frac{x+\frac{\Lambda}{4}}{\Lambda/2}\right)$

$$t_{ridge} = e^{j\frac{2\pi}{\lambda}dn_1}\,\mathrm{rect}\left(\frac{x+\frac{\Lambda}{4}}{\Lambda/2}\right)$$

GROOVE

(1) Relative shift of $\Lambda/4$
(2) relative phase of $e^{j\frac{2\pi}{\lambda}dn_0}$
(3) Defining rect $\mathrm{rect}\left(\frac{x-\Lambda/4}{\Lambda/2}\right)$

$$t_{groove} = \mathrm{rect}\left(\frac{x-\Lambda/4}{\Lambda/2}\right)e^{j\frac{2\pi}{\lambda}n_0}$$

Total transmission function is

$$t = e^{j\frac{2\pi}{\lambda}n_1 d}\,\mathrm{rect}\left(\frac{x+\frac{\Lambda}{4}}{\Lambda/2}\right) + e^{j\frac{2\pi}{\lambda}n_0 d}\,\mathrm{rect}\left(\frac{x-\frac{\Lambda}{4}}{\Lambda/2}\right)$$

factor out and eliminate common phase

$$t = e^{j\frac{2\pi}{\lambda}(n_1-n_0)d}\,\mathrm{rect}\left(\frac{x+\Lambda/4}{\Lambda/2}\right) + \mathrm{rect}\left(\frac{x-\Lambda/4}{\Lambda/2}\right)$$

$$T = \mathcal{F}\{t\}\Big|_{f_x = \frac{x}{\lambda d}}$$

$$T = e^{j\frac{2\pi}{\lambda}d(n_1-n_0)} \frac{\Delta}{2} e^{j2\pi f_x \Delta/4} \operatorname{Sinc}\left(\frac{\Delta}{2} f_x\right)$$

$$+ \frac{\Delta}{2} e^{-j2\pi f_x \Delta/4} \operatorname{sinc}\left(\frac{\Delta}{2} f_x\right)$$

$$= e^{+j\frac{\pi}{\lambda}d(n_1+n_0)} \frac{\Delta}{2} \operatorname{sinc}\left(\frac{\Delta}{2} f_x\right)\left[e^{-j\frac{\pi}{\lambda}d(n_1-n_0)} e^{j2\pi f_x \frac{\Delta}{4}}\right.$$

$$\left. + e^{-j\frac{\pi}{\lambda}d(n_1-n_0)} e^{-j2\pi f_x \Delta/4}\right]$$

Ignoring common factors

$$T = \operatorname{Sinc}\left(\frac{\Delta}{2} f_x\right) \cos\left[\frac{\pi}{\lambda}d(n_1-n_0) + 2\pi\left(\frac{\Delta}{4}\right) f_x\right]$$

Evaluate at the angle points of $f_x = \frac{m}{\Lambda}$

$$\eta_m = \operatorname{Sinc}\left(\frac{\Delta}{2}\frac{m}{\Lambda}\right) \cos\left[\frac{\pi}{\lambda}d(n_1-n_0) + \frac{\pi}{2}\lambda\frac{m}{\Lambda}\right]$$

$$\eta_m = \operatorname{Sinc}\left(\frac{m}{2}\right) \cos\left[\frac{\pi}{\lambda}d(n_1-n_0) + \frac{\pi}{2}m\right]$$

if we want $\eta_0 = 0$ choose $\frac{\pi}{\lambda}d(n_1-n_0) = \frac{\pi}{2}$

$$d = \frac{\lambda}{2(n_1-n_0)}$$

$$\eta_m = \operatorname{Sinc}\left(\frac{m}{2}\right) \cos\left[\frac{\pi}{2}(1+m)\right]$$

$\eta_0 = 0$

$\eta_1 = \operatorname{Sinc}^2\left(\frac{1}{2}\right)$

$\eta_{-1} = \operatorname{sinc}^2(1/2)$

$\eta_2 = \cancel{\operatorname{Sinc}^2(1)}^{0} \cancel{\cos(3\pi/2)}^{0}$

$\eta_{-2} = \operatorname{Sinc}^2(1) \cancel{\cos\left(-\frac{\pi}{2}\right)}^{0}$

$\eta_{\pm 3} = \operatorname{Sinc}^2\left(\frac{3}{2}\right) \cos\left[\frac{\pi}{2}(4)\right] = \operatorname{Sinc}^2\left(\frac{3}{2}\right)$

$\eta_{\pm 4} = 0$

$\eta_{\pm 5} = \operatorname{sinc}^2\left(\frac{5}{2}\right)$

Thin Sinusoidal Phase Grating

$$t_A(\xi, \eta) = \exp\left[j \frac{m}{2} \sin\left(\frac{2\pi}{L} \xi\right)\right] \operatorname{rect}\left(\frac{\xi}{2w}\right) \operatorname{rect}\left(\frac{\eta}{2w}\right)$$

the average phase has been dropped

The complex field after the grating with illumination by a plane wave is

$$U(\xi, \eta) = t_A(\xi, \eta)$$

This is simplified by using the identity

$$\exp\left[j \frac{m}{2} \sin\left(\frac{2\pi}{L} \xi\right)\right] = \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) \exp\left(j \frac{2\pi}{L} q \xi\right)$$

J_q is the Bessel function of the first kind and order q.

$$\mathcal{F}\left\{\exp\left(j \frac{m}{2} \sin \frac{2\pi}{L} \xi\right)\right\} = \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) \mathcal{F}\left\{\exp\left(j \frac{2\pi}{L} q\right)\right\}$$

$$= \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) \delta\left(f_x - \frac{q}{L}, f_y\right)$$

$$\mathcal{F}\{U(\xi, \eta)\} = w^2 \operatorname{sinc}(2wf_x) \operatorname{sinc}(2wf_y) \otimes \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) \delta\left(f_x - \frac{q}{L}, f_y\right)$$

$$\mathcal{F}\{U(\xi,\eta)\} = \sum_{q=-\infty}^{\infty} W^2 J_q\left(\frac{m}{2}\right) \operatorname{sinc}\left[2W\left(f_x - \frac{q}{L}\right)\right] \operatorname{sinc}(2Wf_y)$$

$$U(x,y) = \frac{e^{jkz}\, e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z} \mathcal{F}\{U(\xi,\eta)\}$$

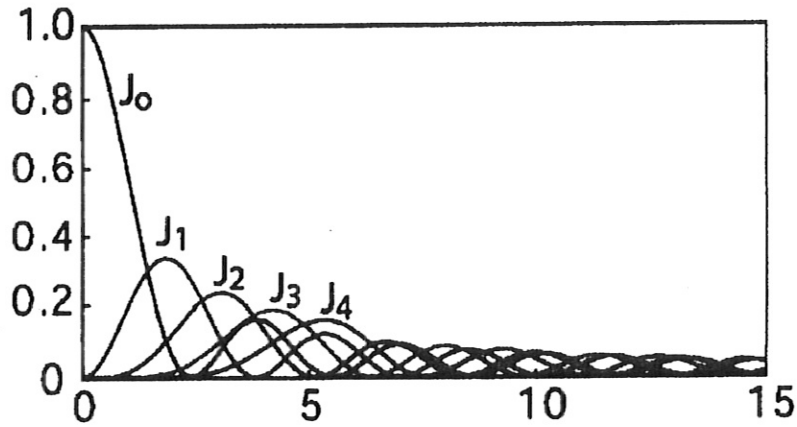
$$= \frac{e^{jkz}\, e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z} \sum_{q=-\infty}^{\infty} W^2 J_q\left(\frac{m}{2}\right) \operatorname{sinc}\left[2W\left(f_x - \frac{q}{L}\right)\right] \operatorname{sinc}(2Wf_y)$$

$$f_x = \frac{x}{\lambda z} \qquad f_y = \frac{y}{\lambda z}$$

$$I(x,y) = |U(x,y)|^2$$

$$I(x,y) = \left(\frac{W^2}{\lambda z}\right)^2 \sum_{q=-\infty}^{\infty} J_q^2\left(\frac{m}{2}\right) \operatorname{sinc}^2\left[\frac{2W}{\lambda z}\left(x - q\frac{\lambda z}{L}\right)\right] \operatorname{sinc}^2\left(\frac{2Wy}{\lambda z}\right)$$

Bessel Function



The zeros of each Bessel function are

J_o	J_1	J_2	J_3	J_4
2.4048	0	0	0	0
5.5201	3.8317	5.1356	6.3802	7.5883
8.6537	7.0156	8.4172	9.7610	11.0647
11.7991	10.1735	11.6198	13.0152	14.3725

Phase of a sinusoidal phase grating

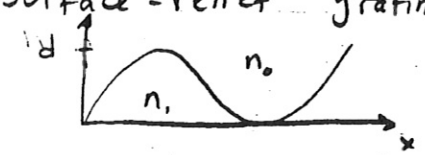
Index grating $n = n_0 + n_1 \sin(Kx)$

$$\phi = k_0\, dn = k_0\, d\,(n_0 + n_1 \sin Kx)$$

Ignore common phase

$$\phi = k_0\, dn_1 \sin(Kx)$$

Surface-relief grating



for propagating only through n_1

$$h = \frac{d}{2}(1 + \sin Kx)$$

$$\phi_1 = \frac{d}{2} k_0 n_1 (1 + \sin Kx)$$

for propagating through n_0

$$h = d - \frac{d}{2}(1 + \sin Kx)$$

$$= d - \frac{d}{2} - \frac{d}{2}\sin Kx$$

$$h = \frac{d}{2}(1 - \sin Kx)$$

$$\phi_0 = \frac{d}{2} k_0 n_0 (1 - \sin Kx)$$

The total phase is

$$\phi = \phi_0 + \phi_1 = \frac{d}{2} k_0 (n_1 + n_1 \sin Kx + n_0 - n_0 \sin Kx)$$

$$= \frac{d}{2}(k_0)(n_1 + n_0) + \frac{d}{2}(k_0)(n_1 - n_0)\sin Kx$$

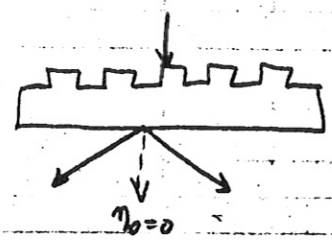
ignore non-spatially varying phase term

$$\phi = \frac{d}{2}(k_0)(n_1 - n_0)\sin Kx$$

comparing to $\phi = \frac{m}{2}\sin\left(\frac{2\pi}{\Lambda}x\right)$

$$m = d\, k_0 (n_1 - n_0)$$

A phase mask is designed to have no power in the m=0 order



Compare the thickness of a binary and a sinusoidal

<u>Binary</u>

$$\eta_i = \mathrm{Sinc}^2\left(\frac{i}{2}\right) \cos\left[\frac{\pi}{\lambda} d\,(n_1 - n_0) + \frac{\pi}{2} m\right]$$

We need $\cos\left[\frac{\pi}{\lambda} d\,(n_1 - n_0) + \frac{\pi}{2}(0)\right] = 0$

$$\frac{\pi}{\lambda}(d)(n_1 - n_0) = \frac{\pi}{2}$$

$$d = \frac{\lambda}{2(n_1 - n_0)} = 0.5 \frac{\lambda}{(n_1 - n_0)}$$

<u>Sinusoidal</u>

$$\eta_i = J_i^2\left(\frac{m}{2}\right)$$

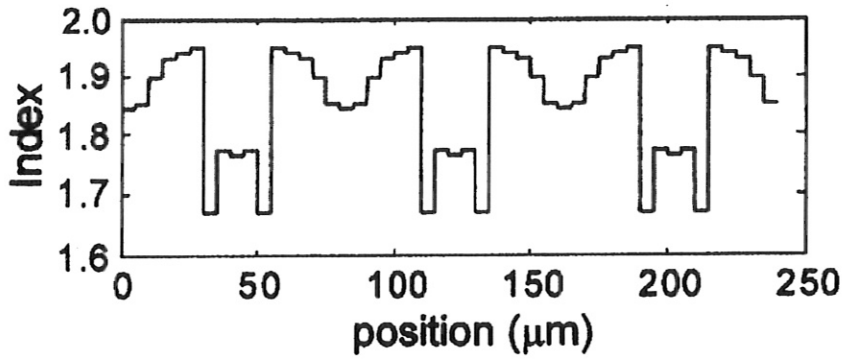
$$\eta_0 = 0 = J_0^2\left(\frac{d}{2} K_0 (n_1 - n_0)\right)$$

$$\frac{d}{2}\left(\frac{2\pi}{\lambda}\right)(n_1 - n_0) = 2.404$$

$$d = \frac{2.4048}{\pi} \frac{\lambda}{(n_1 - n_0)} = 0.77 \frac{\lambda}{(n_1 - n_0)}$$

for $\lambda = 244\,nm$
$n_0 = 1.0$
$n_1 = 1.5$

$d_{Bin} = 244\,nm$ $\quad$ $d_{sin} = 376\,nm$



This is a periodic structure with:

Period: $\Lambda = q\,x_c = 80\,\mu m$

Segments: $q=16$

Segment length: $x_c=5$

The complex transmission is given by

$$t(x) = \sum_{m=0}^{q-1} \exp(j\Phi_m)\,\mathrm{rect}\left(\frac{x - m\,x_c}{x_c}\right) \otimes \sum_{k=-\infty}^{\infty} \delta(x - k\,q\,x_c)$$

The diffracted field is given by

$$E(x) \propto \Im\{t_A\}\Big|_{x/\lambda d} = \Im\left\{\sum_{m=0}^{q-1} \exp(j\Phi_m)\ \mathrm{rect}\left(\frac{\xi - m\,x_c}{x_c}\right)\right\}\ \ \Im\left\{\sum_{k=-\infty}^{\infty} \delta(\xi - k\,q\,x_c)\right\}$$

$$\Im\{\mathrm{rect}(\xi - m\,x_c)\} = \exp(-\,j2\pi\,m\,f_{xo})\,\mathrm{sinc}(\mathrm{f_{xo}})$$

$$\Im\left\{f\left(\frac{x}{a}\right)\right\} = a\,F(a\,f_{xo})$$

$$\Im\left\{\mathrm{rect}\left(\frac{\xi - m\,x_c}{x_c}\right)\right\} = x_c\,\exp(-\,j2\pi\,m\,x_c f_{xo})\ \mathrm{sinc}(\mathrm{x_c\ f_{xo}})$$

$$\Im\left\{\sum_{k=-\infty}^{\infty}\delta(\xi-k)\right\}=\sum_{k=-\infty}^{\infty}\delta(f_{xo}-k)$$

Use the substitution $\xi=\xi/(qx_c)$

$$\sum_{k=-\infty}^{\infty}\delta\left(\frac{\xi}{q\,x_c}-k\right)=\sum_{k=-\infty}^{\infty}\delta(\xi-k\,q\,x_c)$$

$$\Im\left\{f\left(\frac{x}{a}\right)\right\}=a\,F(a\,f_{xo})$$

with $a=qx_c$

$$\Im\left\{\sum_{k=-\infty}^{\infty}\delta\left(\frac{\xi}{q\,x_c}-k\right)\right\}=q\,x_c\sum_{k=-\infty}^{\infty}\delta(q\,x_c\,f_{xo}-k)=q\,x_c\sum_{k=-\infty}^{\infty}\delta\left(f_{xo}-\frac{k}{q\,x_c}\right)$$

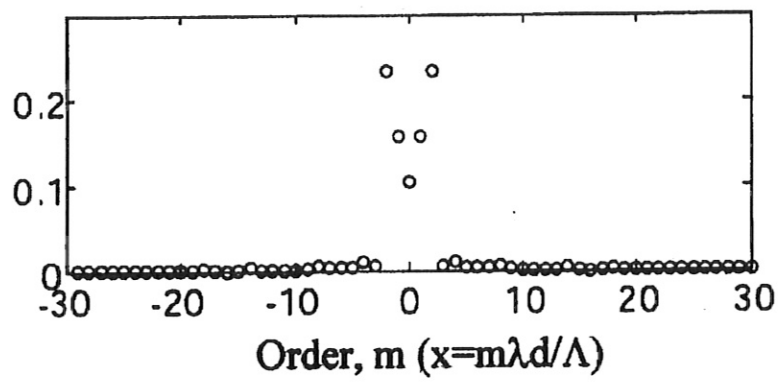
$$f_{xo}=\frac{x}{\lambda\,d}$$

$$\Im\left\{\sum_{k=-\infty}^{\infty}\delta\left(\frac{\xi}{q\,x_c}-k\right)\right\}=q\,x_c\sum_{k=-\infty}^{\infty}\delta\left(x-\frac{k\lambda d}{q\,x_c}\right)$$

$$E(x)\propto\sum_{m=0}^{q-1}\exp(j\Phi_m)\exp\left(-j2\pi m\,x_c\frac{x}{\lambda f}\right)$$

$$*\quad\sum_{k=-\infty}^{\infty}\delta\left(x-\frac{k\lambda d}{q\,x_c}\right)$$

$$*\quad\sin c\left(\frac{x_c\,x}{\lambda\,f}\right)$$



89% of the power is in the center 5 orders