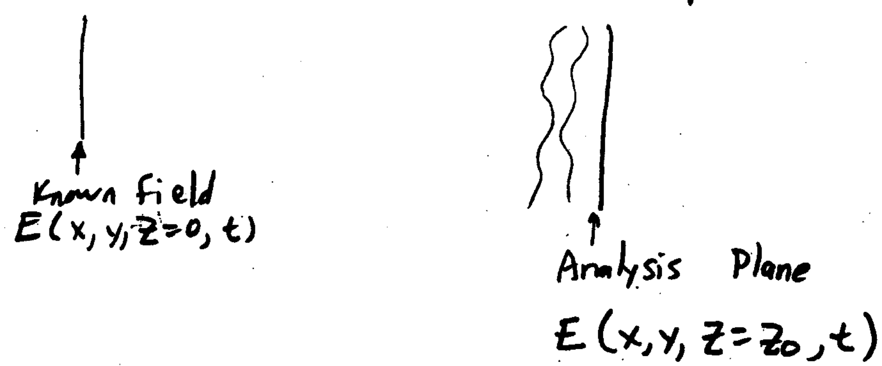
In RF electromagnetics the fields are often calculated from current and charge distributions on a source or antenna. This is not the case for most optical systems.

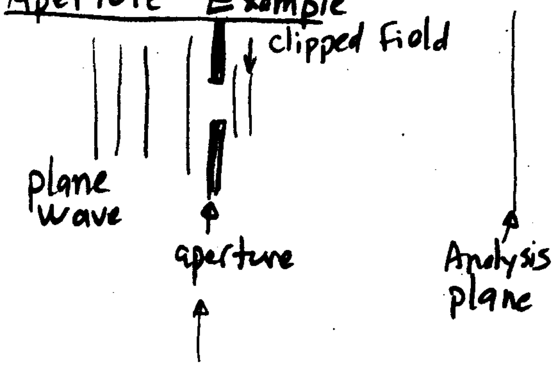
We usually analyze Maxwell's equations for a charge free environment.

Goal: Calculate the field at an arbitrary plane given a known field in a different plane.



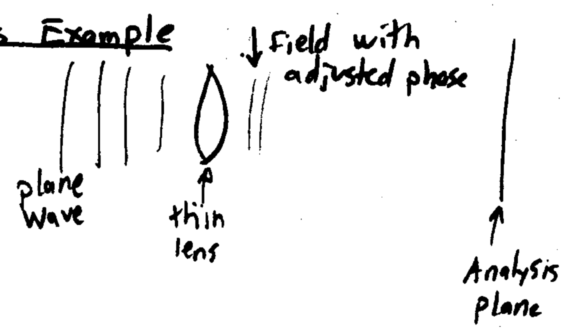
To analyze an optical system we determine the effect of the component on the field and then propagate the new field.

Aperture Example



Integrate over the aperture for each point in the analysis plane. Since no fields penetrates through the aperture, the aperture results in a change in the extent of the integration.

Lens Example



Start with Maxwell's equations in a Source free media

$$\nabla \times \vec{\mathcal{E}} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad \nabla \cdot \vec{D} = 0$$

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times (\nabla \times \vec{\mathcal{E}}) = \nabla \times -\mu \frac{\partial \vec{H}}{\partial t}$$

$$= -\mu \frac{\partial}{\partial t} \nabla \times \vec{H}$$

Assume homogeneous μ

$$= -\mu \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial \vec{\mathcal{E}}}{\partial t}\right)$$

$$\nabla \times \nabla \times \vec{\mathcal{E}} = -\mu \varepsilon \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2}$$

Using a vector identity

$$\nabla \times \nabla \times \vec{\mathcal{E}} = \nabla(\nabla \cdot \vec{\mathcal{E}}) - \nabla^2 \vec{\mathcal{E}}$$

for a linear homogeneous media $\nabla \cdot \vec{\mathcal{E}} = \nabla \cdot \varepsilon \vec{D} = \varepsilon \nabla \cdot \vec{D} = 0$

$$\nabla^2 \vec{\mathcal{E}} - \mu \varepsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = 0$$

What happens in a non-homogeneous material?

$$\nabla \cdot \vec{\mathcal{E}} = -2\vec{\mathcal{E}} \cdot \nabla \ln n \qquad \text{[Do for homework]}$$

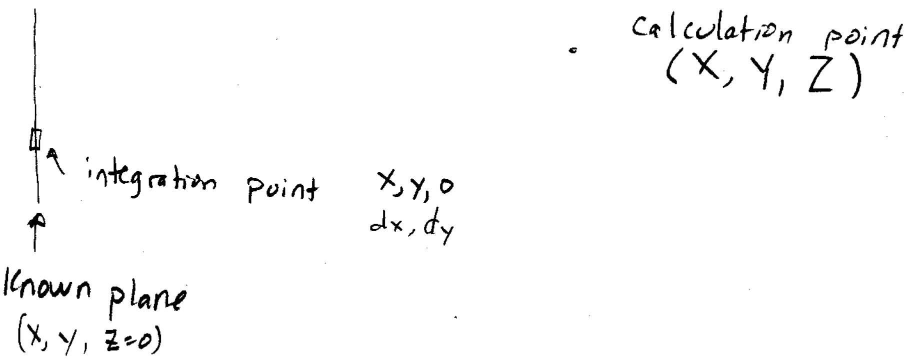
$$\nabla^2 \vec{\mathcal{E}} + 2\nabla(\vec{\mathcal{E}} \cdot \nabla \ln n) - \mu\varepsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = 0$$

For Scalar diffraction: $2\nabla(\vec{\mathcal{E}} \cdot \nabla \ln n) \approx 0$

This is valid if the diffracting structure is large compared to the wavelength.

Using Huygens principle, the know field is broken up into spherical point sources.

$$E_{diff}(R) \propto \int E_{known}(r) \frac{e^{ik|\bar{R}-\bar{r}|}}{|\bar{R}-\bar{r}|} ds$$



$$\vec{r} = x'\hat{x} + y'\hat{y}$$

$$\vec{R} = X\hat{x} + Y\hat{y} + Z\hat{z}$$

$$|\bar{R}-\bar{r}| = \sqrt{(X-x')^2 + (Y-y')^2 + Z^2}$$

$$E_{diff}(R) = \int E_{known}(x',y') \frac{e^{ik|\bar{R}-\bar{r}|}}{|\bar{R}-\bar{r}|} dx'\,dy'$$

$$E_{diff}(X,Y,Z) = \int E_{known}(x',y') \frac{e^{ik\sqrt{(X-x')^2+(Y-y')^2+Z^2}}}{\sqrt{(X-x')^2+(Y-y')^2+Z^2}} dx'\,dy'$$

Now we switch over to time harmonic fields

$$\mathcal{E}(r,t) = A(r)\cos[2\pi f t + \phi(r)]$$

using complex notation this becomes

$$\mathcal{E}(r,t) = \mathrm{Re}\{E(r)\exp(-j2\pi f t)\}$$

where $E(r) = A(r)\exp[-j\phi(r)]$

The wave equation: $\nabla^2\mathcal{E} - \mu\varepsilon\dfrac{\partial^2\mathcal{E}}{\partial t^2} = 0$

$$\nabla^2\mathcal{E} - \mu_0\varepsilon_0\varepsilon_r\frac{\partial^2\mathcal{E}}{\partial t^2} = 0$$

$$\nabla^2\mathcal{E} - \frac{n^2}{c^2}\frac{\partial^2\mathcal{E}}{\partial t^2} = 0$$

$$\nabla^2\mathcal{E} - \frac{n^2}{c^2}(j\omega)^2\,\mathcal{E} = 0$$

$$\nabla^2 E + k^2 E = 0$$

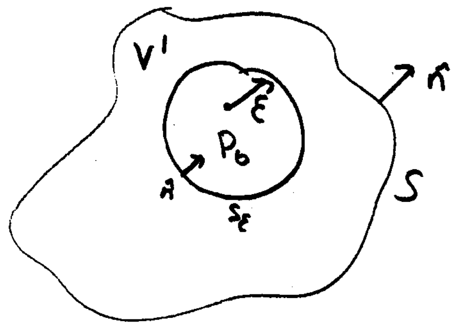
We want to calculate complex field E at an observation point in space.
We accomplish this using Green's theorem.

Let $U(r)$ and $G(r)$ be two complex-valued functions of position, and let S be a closed surface surrounding a volume V. If U, G, and their first and second partial derivatives are single-valued and continuous within and on S, then we have

$$\iiint_V \left(U\nabla^2 G - G\nabla^2 U\right)dv = \iint_S \left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right)ds$$

where $\frac{\partial}{\partial n}$ signifies a partial derivative in the outward normal direction at each point on S.

Let the point of observation be denoted P_0, and let S denote an arbitrary closed surface surrounding P_0



The problem is to express the optical disturbance at P_0 in terms of its values on the surface S.

We choose as an auxillary function a unit-amplitude spherical wave expanding about the point P_0.

$$G(P_1) = \frac{\exp(jkr_{01})}{r_{01}}$$

To be used in Green's Theorem, the function G, its first and second derivatives must be continuous in V.

Choose the volume V' between S and S_ε $\quad S' = S + S_\varepsilon$

Within V', G satisfies the Helmholtz equation

$$(\nabla^2 + k^2)\,G = 0 \quad \Rightarrow \nabla^2 G = -k^2 G$$

Substitute into Green's Theorem

$$\iiint_{V'} (U\nabla^2 G - G\nabla^2 U)\,dv = \iint_{S'} \left(U \frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right) ds$$

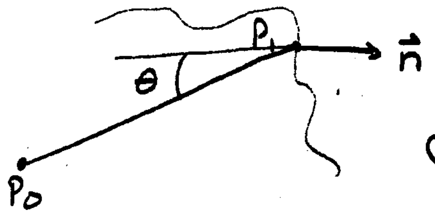
$$-\iiint_{V'} (Uk^2 G - Gk^2 U)\,dv$$

$$= -\iiint_{V'} (k^2 UG - k^2 UG)\,dV$$

$$0 = \iint_{S'} \left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right) ds \qquad S' = S + S_\varepsilon$$

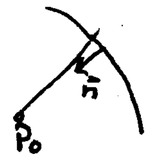
$$-\iint_{S_\varepsilon} \left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right) ds = \iint_{S} \left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right) ds$$

$$\frac{\partial G}{\partial n} = \cos(\vec{n}, \vec{r}_{01}) \left(jk - \frac{1}{r_{01}} \right) \frac{\exp(jk r_{01})}{r_{01}}$$



$$\cos(\vec{n}, \vec{r}_{01}) = \cos\theta$$

For the inner sphere



$$\cos(\vec{n}, \vec{r}_{01}) = -1$$

for P_1 on S_ε : $G(P_1) = \dfrac{e^{jk\varepsilon}}{\varepsilon}$ and $\dfrac{\partial G}{\partial n} = \dfrac{e^{jk\varepsilon}}{\varepsilon}\left(\dfrac{1}{\varepsilon} - jk\right)$

letting $\varepsilon \to 0$ $\displaystyle\iint_{S_\varepsilon} \left(U \frac{\partial G}{\partial n} - G \frac{\partial U}{\partial n} \right) ds$

$$= \iint_{S_\varepsilon} \left(U(P_1) \frac{e^{jk\varepsilon}}{\varepsilon} \left(\frac{1}{\varepsilon} - jk \right) - \frac{e^{jk\varepsilon}}{\varepsilon} \frac{\partial U(P_1)}{\partial n} \right) ds$$

U is constant over the small area S_0

$$= \left[U(P_0) \frac{e^{jk\varepsilon}}{\varepsilon} \left(\frac{1}{\varepsilon} - jk \right) - \frac{e^{jk\varepsilon}}{\varepsilon} \frac{\partial U(P_0)}{\partial n} \right] \iint_{S_\varepsilon} ds$$

$$= \lim_{\varepsilon \to 0} 4\pi \varepsilon^2 \left[U(P_0) \frac{e^{jk\varepsilon}}{\varepsilon} \left(\frac{1}{\varepsilon} - \cancel{jk}^{\,0} \right) - \frac{e^{jk\varepsilon}}{\varepsilon} \cancel{\frac{\partial U(P_0)}{\partial n}}^{\,0} \right]$$

$$= 4\pi\, U(P_0)$$

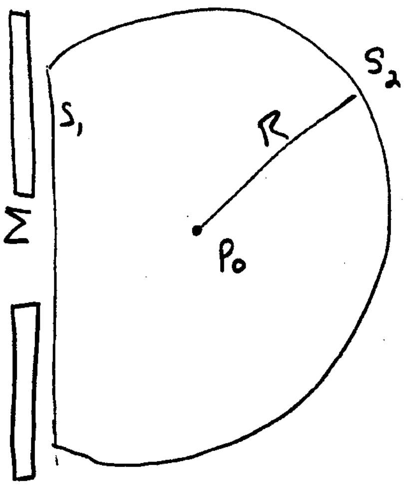
And this results in : $\displaystyle\iint_S \left(U \frac{\partial G}{\partial n} - G \frac{\partial U}{\partial n} \right) ds = -4\pi\, U(P_0)$

$$U(P_0) = \frac{1}{4\pi} \iint_S \left(U \frac{\partial G}{\partial n} + G \frac{\partial U}{\partial n} \right) ds$$

$$\boxed{U(P_0) = \frac{1}{4\pi} \iint_S \left[\frac{\partial U}{\partial n} \left[\frac{\exp(jk r_{01})}{r_{01}} \right] - U \frac{\partial}{\partial n} \left[\frac{\exp(jk r_{01})}{r_{01}} \right] \right] ds}$$

This relates the field at a point to the field on a surface.

Let's look at a uniform field over an aperture



The surface is an infinite sheet and a partial infinite sphere.

$$U(P_0) = \frac{1}{4\pi} \iint_S \left(\frac{\partial U}{\partial n} G - U \frac{\partial G}{\partial n} \right) ds$$

$$= \frac{1}{4\pi} \iint_{S_1} \left(\frac{\partial U}{\partial n} G - U \frac{\partial G}{\partial n} \right) ds + \frac{1}{4\pi} \iint_{S_2} \frac{\partial U}{\partial n} G - U \frac{\partial G}{\partial n} ds$$

Start by looking at the surface S_2 as $R \to \infty$

$$G = \frac{e^{jkR}}{R} \approx 0 \qquad \frac{\partial G}{\partial n} = \cos(\hat{n}, \vec{r}_{01}) \left(jk - \frac{1}{R} \right) \frac{e^{jkR}}{R} \approx 0$$

$$\iint_{S_2} (\,)\, ds = 0$$

The screen is opaque except over the aperture Σ so the following assumptions are made:

(1) Across Σ U and $\partial U/\partial n$ are exactly the same
(2) over the rest of S_1 $U = \partial U/\partial n = 0$

These are not exactly true because of fringing at the boundary of the aperture. These assumptions are valid as long as $\Sigma \gg \lambda$

Optical wavelength is fairly small so we use the approximation $k \gg 1/r_{01}$

$$\frac{\partial G}{\partial n} = \cos(\vec{n}, \vec{r}_{01}) \left(jk - \frac{1}{r_{01}} \right) \frac{e^{jkr_{01}}}{r_{01}} \approx \cos(\vec{n}, \vec{r}_{01})\, jk \frac{e^{jkr_{01}}}{r_{01}}$$

$$U(P_0) = \frac{1}{4\pi} \iint_{\Sigma} \frac{\partial U}{\partial n} \frac{e^{jkr_{01}}}{r_{01}} - U \cos(\vec{n}, \vec{r}_{01})\, jk \frac{e^{jkr_{01}}}{r_{01}}\, ds$$

$$U(P_0) = \frac{1}{4\pi} \iint_{\Sigma} \frac{e^{jkr_{01}}}{r_{01}} \left[\frac{\partial U}{\partial n} - jkU \cos(\vec{n}, \vec{r}_{01}) \right]$$