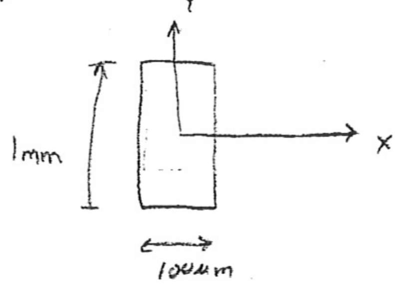
A $100\mu m \times 1mm$ aperture illuminated by a laser $\lambda = 500nm$.
(a) Find the diffraction pattern
(b) Find the necessary distance away from the aperture

Assume that the incident field is a plane wave

$$E(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \mathcal{F}\{P(x,y)\}\Big|_{f_X = \frac{x}{\lambda z},\ f_Y = \frac{y}{\lambda z}}$$



$$p(x) = \begin{cases} 1 & |x| \le 50\mu m \\ 0 & \text{else} \end{cases}$$

this is similar to

$$\text{rect}(x) = \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

$$\text{rect}(ax) = \begin{cases} 1 & |ax| \le \frac{1}{2} \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & |x| \le \frac{1}{2a} \\ 0 & \text{else} \end{cases}$$

let $a = \frac{1}{100\mu m}$

$$p(x) = \text{rect}\left(\frac{x}{100\mu m}\right) = \begin{cases} 1 & |x| \le 50\mu m \\ 0 & \text{else} \end{cases}$$

Now take the Fourier transform

$$\mathcal{F}\{\text{rect}(ax)\} = \frac{1}{a} \text{Sinc}\left(\frac{F_x}{a}\right)$$

$$P(F_x) = (100 \times 10^{-6}) \text{Sinc}(100 \times 10^{-6} F_x)$$

similar for the y-direction

$$P(F_y) = (10^{-3}) \text{Sinc}(10^{-3} F_y)$$

$$F_x = \frac{x}{\lambda z} \qquad F_y = \frac{y}{\lambda z}$$

$$E(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} (100 \times 10^{-6})(10^{-3}) \text{Sinc}\left(\frac{100 \times 10^{-6} x}{\lambda z}\right) \text{Sinc}\left(\frac{10^{-3} y}{\lambda z}\right)$$

Zeros at $\frac{100\times10^{-6}x}{\lambda z} = \pm 1$

$$\theta_x = \frac{x}{z} = \frac{\lambda}{100\times10^{-6}} = \frac{0.5\times10^{-6}}{100\times10^{-6}}$$

$$\theta_x = 5 \text{ mrad} = 0.29°$$

$$\frac{10^3 y}{\lambda z} = \pm 1$$

$$\theta_y = \frac{y}{z} = \frac{\lambda}{10^{-3}} = \frac{0.5}{1000} = 0.5 \text{mrad} = 0.029°$$

Valid distance $z \gg \frac{(x')^2 + (y')^2}{2\lambda}$

$$z \gg \frac{(50\times10^{-6})^2 + (500\times10^{-6})^2}{(2)(0.5\times10^{-6})}$$

$$z \gg 0.25 \text{ m}$$

Circular aperture

$$G(f_x, f_y) = \iint_{-\infty}^{\infty} g(x,y) \exp\left[-j2\pi (f_x x + f_y y)\right] dx\,dy$$

transform into polar coordinates

$$r = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\rho = \sqrt{f_x^2 + f_y^2}$$
$$\phi = \tan^{-1}\left(\frac{f_y}{f_x}\right)$$

For a rotationally symetric incident field the Fourier transform becomes

$$G(\rho) = \int_0^{2\pi} \int_0^{\infty} g(r) \exp\left[-j\,2\pi r\rho\,(\cos\theta \cos\phi + \sin\theta \sin\phi)\right] \rho\, d\theta\, dr$$

$$= \int_0^{2\pi} \int_0^{\infty} g(r) \exp\left[-j\,2\pi r\rho \cos(\theta-\phi)\right] \rho\, d\theta\, dr$$

The definition of the Bessel function is

$$J_0(a) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left[-j\,a \cos(\theta-\phi)\right] d\theta$$

let $a = 2\pi r\rho$

$$G(\rho) = 2\pi \int_0^{\infty} r\, g(r)\, J_0(2\pi r\rho)\, dr$$

Since the Bessel function is even

$$g(r) = 2\pi \int_0^{\infty} \rho\, G(\rho)\, J_0(2\pi r\rho)\, d\rho$$

So the Fourier transform and inverse Fourier transform are the same

Now take the Fourier transform of a circular aperture

circle function $\quad \mathrm{circ}(r) = \mathrm{circ}\left(\sqrt{x^2+y^2}\right) = \begin{cases} 1 & \sqrt{x^2+y^2} < 1 \\ \frac{1}{2} & \sqrt{x^2+y^2} = 1/2 \\ 0 & \text{else} \end{cases}$

$$\mathcal{F}\{\mathrm{circ}(r)\} = 2\pi \int_0^1 r\, J_0(2\pi r\rho)\, dr$$

let $u = 2\pi r\rho \qquad du = 2\pi\rho\, dr \qquad r = \frac{u}{2\pi\rho}$

$r = 1 \rightarrow u = 2\pi\rho$

$$\mathcal{F}\{\mathrm{circ}(r)\} = 2\pi \int_0^{2\pi\rho} \frac{u}{2\pi\rho}\, J_0(u)\, \frac{du}{2\pi\rho}$$

$$= \frac{1}{2\pi\rho^2} \int_0^{2\pi\rho} u\, J_0(u)$$

Now we use the Bessel identity

$$\int_0^x \tau J_0(\tau)\, d\tau = x J_1(x)$$

$$\mathcal{F}\{circ(r)\} = \frac{1}{2\pi\rho^2}(2\pi\rho)\, J_1(2\pi\rho)$$

$$= \frac{J_1(2\pi\rho)}{\rho}$$

For a circular aperture with radius R

$$\mathcal{F}\{circ(\tfrac{r}{R})\} = 2\pi R^2 (2) \frac{J_1(2\pi R f_R)}{2\pi R f_R}$$

with a lens $E(r) = \dfrac{e^{jkz}\, e^{j\frac{kr^2}{2}}}{j\lambda f}\, 2\pi R^2 \left[2 \dfrac{J_1(2\pi R \frac{r}{\lambda f})}{2\pi R \frac{r}{\lambda f}}\right]$

$$I(r) = |E(r)|^2 = \left(\frac{2\pi R^2}{\lambda f}\right)^2 \left[2 \frac{J_1(2\pi R \frac{r}{\lambda f})}{2\pi R \frac{r}{\lambda f}}\right]^2$$

The location of the 1st null is $\left[2 \dfrac{J_1(\pi x)}{\pi x}\right]^2 = 0$ at $x = 1.22$

$$\frac{2R}{\lambda f} r = 1.22$$

$$r = \frac{1.22}{2} \lambda \frac{f}{R}$$

$$r = 1.22 \lambda \frac{f}{D} \quad \text{or} \quad d = 2.44 \lambda f\#$$

This is called the Airy Disk

A lens with a focal length of $f = 10mm$ has a limiting aperture of $D = 5mm$. The lens also has a square obscuration in the middle of the lens. The lens is illuminate with a uniform beam of illuminance $10 \frac{mW}{m^2}$. The obscaration has a size of $2mm \times 2mm$. Use $\lambda = 500nm$

Start with the circular aperture.

$$\text{circ}\left(\frac{r}{2.5mm}\right) = \begin{cases} 1 & r < 2.5mm \\ 0 & \text{else} \end{cases}$$

$$\mathcal{F}\left\{\text{circ}\left(\frac{r}{2.5mm}\right)\right\} = A_c\left[2\,\frac{J_1\left(2\pi(2.5\times10^{-3})f_x\right)}{(2\pi)(2.5\times10^{-3})f_x}\right]$$

Not keeping Constant amplitude terms

$$E_1(x,y) = A_c\left[2\,\frac{J_1\left((2\pi)(2.5\times10^{-3})\frac{r}{\lambda f}\right)}{(2\pi)(2.5\times10^{-3})\left(\frac{r}{\lambda f}\right)}\right]$$

$$E_1(x,y) = 6.25\times10^{-6}\,\pi\left[2\,\frac{J_1(10^6\pi r)}{10^6\pi r}\right]$$

Now the square obscuration

$$\mathcal{F}\left\{\text{rect}\left(\frac{x}{2\times10^{-3}}\right)\text{rect}\left(\frac{y}{2\times10^{-3}}\right)\right\} = w^2\,\text{sinc}(2\times10^{-3}f_x)\,\text{sinc}(2\times10^{-3}f_y)$$

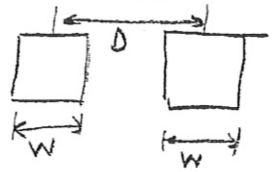
$$E_2(x,y) = w^2\,\text{sinc}\left(\frac{2\times10^{-3}}{(0.5\times10^{-6})(10\times10^{-3})}x\right)\text{sinc}\left(\frac{2\times10^{-3}}{(0.5\times10^{-6})(10\times10^{-3})}y\right)$$

$$(4\times10^{-6})\,\text{sinc}(4\times10^{5}x)\,\text{sinc}(4\times10^{5}y)$$

$$E_{tot} = E_1 - E_2$$

Example 3 | Separated squares

use $\lambda = 500\,nm$
$f = 10\,mm$



Aperture 1: $t_1(x,y) = \text{rect}\left(\frac{x - D/2}{W}\right) \text{rect}\left(\frac{y}{W}\right)$

$$T_1(f_x, f_y) = \mathcal{F}\{t\} = e^{-j2\pi D/2\, f_x}\ \text{sinc}(W f_x)\ \text{sinc}(W f_y)$$

$$E_1(x,y) = W^2 e^{-j\pi D x/\lambda f}\ \text{sinc}\left(W \tfrac{x}{\lambda f}\right)\ \text{sinc}\left(W \tfrac{y}{\lambda f}\right)$$

$$t_2(x,y) = \text{rect}\left(\frac{x + D/2}{W}\right) \text{rect}\left(\frac{y}{W}\right)$$

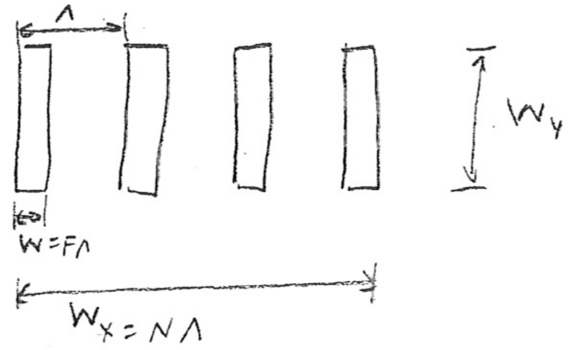
$$T_2(f_x, f_y) = W^2 e^{+j2\pi D/2\, f_x}\ \text{sinc}(W f_x)\ \text{sinc}(W f_y)$$

$$E_2(x,y) = W^2\ \text{sinc}\left(W \tfrac{x}{\lambda f}\right)\ \text{sinc}\left(W \tfrac{y}{\lambda f}\right)\ e^{+j\pi D x/\lambda f}$$

$$E_{tot}(x,y) = W^2\ \text{sinc}\left(\frac{Wx}{\lambda f}\right) \text{sinc}\left(\frac{Wy}{\lambda f}\right)(2)\ \frac{\left(e^{j\pi D x/\lambda f} + e^{-j\pi D \frac{x}{\lambda f}}\right)}{2}$$

$$E_{tot} = E_0\ \text{sinc}\left(\frac{Wx}{\lambda f}\right) \text{sinc}\left(\frac{Wy}{\lambda f}\right) \cos\left(\pi \frac{Dx}{\lambda f}\right)$$

$\lambda = 500nm$
$f = 10mm$



Model the aperture as 3 separate functions
(1) Individual infinite slit
(2) infinite sum of delta functions
(3) complete aperture size

(1) Individual slit

$$t_1(x,y) = rect\left(\frac{x}{W}\right)$$

$$T_1(f_x, f_y) = \mathcal{F}\{t_1\} = W \, sinc(Wf_x) \, \delta(f_y)$$

(2) Infinite sum of delta functions

First look at the perodicity in x-direction

$$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n) \qquad \text{we want} \quad \sum_{n=-\infty}^{\infty} \delta(x-n\Lambda)$$

$$comb\left(\frac{x}{\Lambda}\right) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{\Lambda}-n\right) = \sum_{n=-\infty}^{\infty} \delta\left[\frac{1}{\Lambda}(x-n\Lambda)\right]$$

from the definition of the delta function

$$1 = \int_{-\infty}^{\infty} \delta(x)\, dx \qquad \int_{-\infty}^{\infty} \delta\left(\frac{x}{\Lambda}\right) dx = \Lambda \int_{-\infty}^{\infty} \delta(u)\, du$$

$$\text{so} \quad \delta\left(\frac{1}{\Lambda}(x-n\Lambda)\right) = \Lambda\, \delta(x-n\Lambda)$$

$$comb\left(\frac{x}{\Lambda}\right) = \Lambda \sum_{n=-\infty}^{\infty} \delta(x-n\Lambda)$$

$$t_2(x,y) = \frac{1}{\Lambda}\, comb\left(\frac{x}{\Lambda}\right) \delta(y)$$

$$T_2(f_x, f_y) = \left(\frac{1}{\Lambda}\right)(\Lambda) \sum_{n=-\infty}^{\infty} \delta(\Lambda f_x - n)$$

(3) Complete aperture size

$$t_3(x,y) = \text{rect}\left(\frac{x}{N\Lambda}\right) \text{rect}\left(\frac{y}{W_y}\right)$$

$$T_3(f_x, f_y) = N\Lambda W_y \; \text{sinc}(N\Lambda f_x) \; \text{sinc}(W_y f_y)$$

Now put them together

$$t(x,y) = \left[t_3(x,y)\right]\left[t_1(x,y) \otimes t_2(x,y)\right]$$

$$T(f_x, f_y) = \left[T_3(f_x, f_y)\right] \otimes \left[T_1(f_x, f_y) \; T_2(f_x, f_y)\right]$$

$$T = \iint_{-\infty}^{\infty} \text{sinc}[N\Lambda(f_x - \alpha)] \; \text{sinc}[W_y(f_y - \beta)] \; \text{sinc}(W\alpha) \; \delta(\beta) \sum_{n=-\infty}^{\infty} \delta(\Lambda\alpha - n) \; d\alpha \, d\beta$$

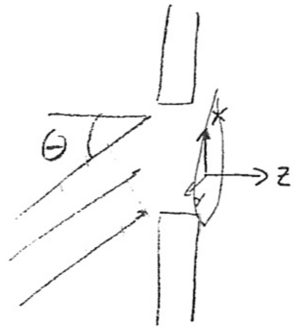
$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \text{sinc}[N\Lambda(f_x - \alpha)] \; \text{sinc}(W\alpha) \; \delta(\Lambda\alpha - n) \, d\alpha \int_{-\infty}^{\infty} \text{sinc}[W_y(f_y - \beta)] \; \delta(\beta) \, d\beta$$

$$= \sum_{n=-\infty}^{\infty} \text{sinc}\left[N\Lambda\left(f_x - \frac{n}{\Lambda}\right)\right] \text{sinc}\left(W\frac{n}{\Lambda}\right) \text{sinc}(W_y f_y)$$

$$T(f_x, f_y) = \left[\text{sinc}(W_y f_y)\right] \left[\sum_{n=-\infty}^{\infty} \text{sinc}\left[N\Lambda\left(f_x - \frac{n}{\Lambda}\right)\right] \text{sinc}\left(W\frac{n}{\Lambda}\right)\right]$$

$$T(x,y) = \left[\text{sinc}\left(W_y \frac{y}{\lambda f}\right)\right] \left[\sum_{n=-\infty}^{\infty} \text{sinc}\left(W\frac{n}{\Lambda}\right) \; \text{sinc}\left[\frac{N\Lambda}{\lambda f}\left(x - \frac{n\lambda f}{\Lambda}\right)\right]\right]$$

sinc functions centered at $x = \frac{n\lambda f}{\Lambda}$ and $y = 0$
width of orders defined by whole aperture size W_y and $N\Lambda$
Amplitude of orders defined by width of slit $\text{sinc}\left(W\frac{n}{\Lambda}\right)$



incident field is a plane wave

$E_i = E_0 e^{-j\vec{k}\cdot\vec{r}}$ $\quad \vec{k} = k_0 \sin\theta\, \hat{x} + k_0 \cos\theta\, \hat{z}$

$E_i = E_0 \exp(-jk_0(\sin\theta\, x + \cos\theta\, z))$

at $z=0$ plane

$E_i = E_0 e^{-jk\sin\theta x}$

Multiply by the aperture

$E_i = E_0 e^{-jk\sin\theta x} \operatorname{rect}\left(\frac{x}{W_x}\right) \operatorname{rect}\left(\frac{y}{W_y}\right)$

$\mathcal{F}\{E_i\}\Big|_{f_x = \frac{x}{\lambda f},\ f_y = \frac{y}{\lambda f}}$

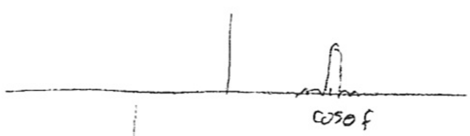
$E_{out}(x,y) = W_y \operatorname{sinc}(W_y f_y)\ \mathcal{F}\left\{\operatorname{rect}\left(\frac{x}{W_x}\right) e^{-jk\sin\theta x}\right.$

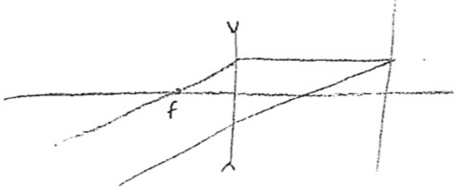
$= W_y \operatorname{sinc}(W_y f_y)\ \mathcal{F}\left\{\operatorname{rect}\left(\frac{x}{W_x}\right) e^{-j2\pi \frac{\sin\theta}{\lambda} x}\right.$

$= W_y W_x \operatorname{sinc}(W_y f_y) \operatorname{sinc}\left(W_x\left(f_x - \frac{\sin\theta}{\lambda}\right)\right)\Big]$

sinc is centered at $\frac{x}{\lambda f} = \frac{\sin\theta}{\lambda}$

$x = \sin\theta\, f$ $\quad$ in paraxial approximation $x = \theta f$





$\frac{x}{f} = \theta \quad x = f\theta$

$E_i(x,y) = E_0 \exp\left(- \frac{x^2+y^2}{W^2}\right)$

Pass through a lens

$E_{out}(x,y) = \mathcal{F}\{E_i\}\Big|_{f_x = \frac{x}{\lambda f},\ f_y = \frac{y}{\lambda f}}$

From the table:

$f(x) = \exp(-\pi x^2) \Rightarrow \exp(-\pi f_x^2)$

$f(ax) = e^{-\pi (ax)^2} = e^{-\left(\frac{x}{W}\right)^2}$

$\pi (ax)^2 = \left(\frac{x}{W}\right)^2$

$a^2 = \frac{1}{\pi W^2} \qquad a = \frac{1}{\sqrt{\pi} W}$

$F(f_x) = \sqrt{\pi}\, W \exp\left(-\pi \left(\sqrt{\pi} W f_x\right)^2\right)$

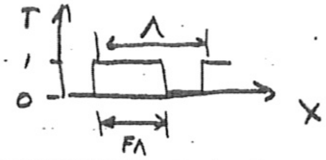
$= \sqrt{\pi}\, W \exp\left(-\left(\pi W f_x\right)^2\right)$

$E_{out}(x,y) = \exp\left(-\left(\frac{\pi W x}{\lambda f}\right)^2\right)$

Gaussian with new waist radius of $\boxed{\frac{\lambda f}{\pi W} = W_0}$

Binary Amplitude Grating

$$t(x) = \left(\mathrm{rect}\left(\frac{x}{F\Lambda}\right) e^{-j2\pi \frac{\sin\theta_i}{\lambda} x} \right) \otimes \sum_{m=-\infty}^{\infty} \delta(x - m\Lambda)$$



$a = \frac{1}{\Lambda}$

$$T(f_x) = \mathrm{Sinc}\left[F\Lambda\left(f_x - \frac{\sin\theta_i}{\lambda}\right)\right] \sum_{m=-\infty}^{\infty} \delta\left[\Lambda\left(f_x - \frac{\sin\theta_i}{\lambda}\right) - m\right]$$

$$= \mathrm{Sinc}\left[F\Lambda f_x - \frac{F\Lambda}{\lambda}\sin\theta_i\right] \sum_{m=-\infty}^{\infty} \delta\left[f_x - \frac{\sin\theta_i}{\lambda} - \frac{m}{\Lambda}\right]$$

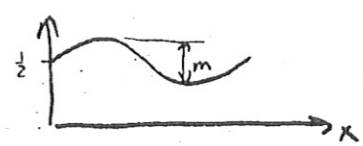
$$f_x = \frac{\sin\theta_i}{\lambda} + \frac{m}{\Lambda}$$

$$\eta_m = \mathrm{Sinc}\left[\cancel{\frac{F\Lambda}{\lambda}\sin\theta_i} + Fm - \cancel{\frac{F\Lambda}{\lambda}\sin\theta_i}\right]$$

$$\boxed{\eta_m = \mathrm{sinc}(Fm)}$$

Sinusoidal Amplitude Grating

$$t(x) = \frac{1}{2} + \frac{m}{2}\cos\left(\frac{2\pi}{\Lambda}x\right)$$



$$\mathcal{F}\left\{\frac{1}{2} + \frac{m}{2}\cos\left(\frac{2\pi}{\Lambda}x\right)\right\} = \frac{1}{2}\delta(f_x) + \frac{m}{4}\delta\left(f_x + \frac{2\pi}{\Lambda}\right) + \frac{m}{4}\delta\left(f_x - \frac{2\pi}{\Lambda}\right)$$

$$I^2 = \left(\frac{1}{4}\delta(f_x) + \frac{m^2}{16}\delta\left(f_x + \frac{2\pi}{\Lambda}\right) + \frac{m^2}{16}\delta\left(f_x - \frac{2\pi}{\Lambda}\right)\right) \otimes \text{Beam}$$

$$\eta_0 = 1/4$$

$$\eta_1 = \eta_{-1} = m^2/16$$