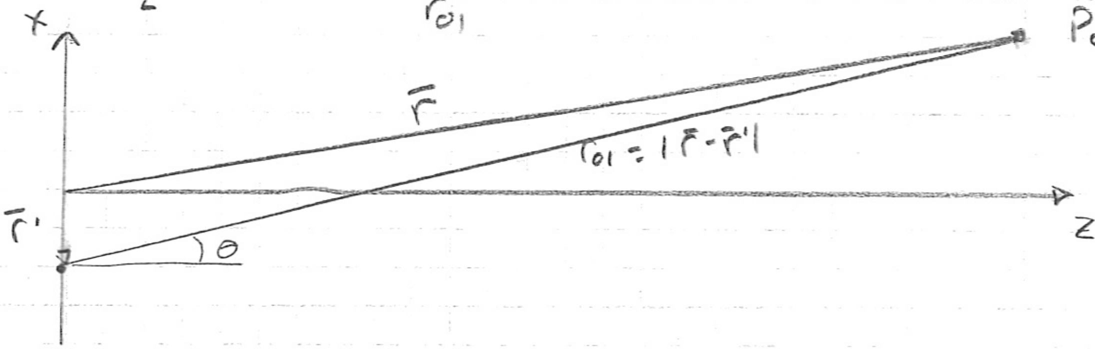
$$E(P_0) = -\frac{1}{2\pi}\int_{\Sigma} E \frac{\partial G}{\partial n}\, ds \qquad r_{01} = |\bar{r}-\bar{r}'| \gg \lambda$$

$$\simeq \frac{1}{j\lambda}\int_{\Sigma} E(x',y')\,\frac{e^{jkr_{01}}}{r_{01}}\cos\theta \, dx'\,dy'$$



$$\cos\theta = \frac{z}{r_{01}}$$

$$E(P_0) = \frac{1}{j\lambda}\int_{\Sigma}\left(\frac{z}{r_{01}}\right)\frac{e^{jkr_{01}}}{r_{01}}\, dx'\,dy'$$

$$= \frac{z}{j\lambda}\int_{\Sigma}\frac{e^{jkr_{01}}}{r_{01}^2}\, dx'\,dy' = \frac{z}{j\lambda}\int_{\Sigma}\frac{e^{jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|^2}\, dx'\,dy'$$

$$r_{01} = |\bar{r}-\bar{r}'| = \sqrt{(x-x')^2+(y-y')^2+z^2}$$

$$= z\sqrt{\left(\frac{x-x'}{z}\right)^2+\left(\frac{y-y'}{z}\right)^2+1}$$

we want to eliminate the square root using

$$(1+\Delta)^{1/2} = 1+\frac{\Delta}{2}-\frac{\Delta^2}{8}+\dots$$

$$r_{01} = z\left[1+\frac{1}{2}\left(\frac{x-x'}{z}\right)^2+\frac{1}{2}\left(\frac{y-y'}{z}\right)^2-\frac{1}{8}\left(\frac{x-x'}{z}\right)^4-\frac{1}{8}\left(\frac{y-y'}{z}\right)^4+\dots\right]$$

The first approximation is for r_{01}^2
we choose $r_{01} = z$.
The error is the next term so $\frac{1}{2}\left(\frac{x-x'}{z}\right)^2+\frac{1}{2}\left(\frac{y-y'}{z}\right)^2 \ll 1$

Inside the exponential it is more sensitive so we use the next term.

$$e^{jkr_{01}} \simeq \exp\left[jkz\left(1+\frac{1}{2}\left(\frac{x-x'}{z}\right)^2+\frac{1}{2}\left(\frac{y-y'}{z}\right)^2\right)\right]$$

The error is the next term. The exponential wraps around with a period of π so

$$kz\left(\frac{1}{8}\right)\left(\left(\frac{x-x'}{z}\right)^2+\left(\frac{y-y'}{z}\right)^2\right)^2 \ll \pi$$

However, this condition is way more stringent than it needs to be.

If the field in the aperture is slowly varying then the phase function $\exp(jkr)$ has it major contribution for small values of r. With large r the phase is changing so fast that the integral is close to zero.

This is called the principle of stationary phase.

This allows the Fresnel approximation for a uniformly illuminated aperture to be accurate. to very small values of z.

The resulting Fresnel Diffraction integral is

$$\boxed{E(P_0)=\frac{e^{jkz}}{j\lambda z}\iint_{\Sigma} E(x',y')\exp\left[\frac{jk}{2z}\left((x-x')^2+(y-y')^2\right)\right]dx'dy'}$$

The next approximation is more harsh. It is called the Fraunhofer approximation.

Look at the exponential term of the Fresnel approximation

$$\exp\left[\frac{jk}{2z}\left(x^2 - 2xx' + x'^2 + y^2 - 2yy' + y'^2\right)\right]$$

$$= \underbrace{\exp\left[\frac{jk}{2z}(x^2+y^2)\right]}_{\text{independent of } x'} \exp\left[\frac{jk}{2z}(-2xx' - 2yy')\right] \underbrace{\exp\left[\frac{jk}{2z}(x'^2+y'^2)\right]}_{\text{term we want to neglect}}$$

So we need $\frac{k}{2z}\left(x'^2 + y'^2\right) \ll \pi$

$$\boxed{z \gg \frac{x'^2 + y'^2}{2\lambda}}$$

The Fraunhofer diffraction integral becomes

$$\boxed{E(P_0) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \iint_{\Sigma} E(x', y') \exp\left[-\frac{jk}{z}(xx' + yy')\right] dx'dy'}$$

The Fraunhofer diffraction integral is similar to the Fourier transform integral.

$$\int_{-\infty}^{\infty} E(x') \exp\left[-j\frac{k}{z}xx'\right] dx' \qquad \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi ft}\, dt$$

use the relationship:

$$x' = t$$

$$2\pi f = \frac{k}{z}x = \frac{2\pi}{\lambda z}x$$

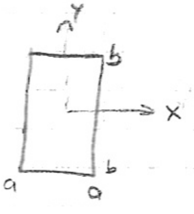
$$f = \frac{x}{\lambda z}$$

It is also a 2-dimension transform

$$f_x = \frac{x}{\lambda z}, \quad f_y = \frac{y}{\lambda z}$$

Fresnel slit diffraction

$$T(x) = \begin{cases} 1 & |x| < a \\ 0 & \text{else} \end{cases}$$



$$E(x,y) = \frac{e^{jkz}}{j\lambda z} \underbrace{\int_{-a}^{a} \exp\left[-\frac{jk}{2z}(x-x')^2\right]dx'}_{A(x)} \underbrace{\int_{-b}^{b} \exp\left[-j\frac{k}{2z}(y-y')^2\right]dy'}_{B(y)}$$

$$A(x) = \int_{-a}^{a} \exp\left[-j\frac{k}{2z}(x-x')^2\right]dx'$$

$$u = x'-x$$
$$du = dx'$$

$$A(x) = \int_{-a-x}^{a-x} \exp\left(-j\frac{k}{2z}u^2\right) du$$

Use Fresnel integrals $\quad C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt$

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt$$

$$\int_{-a-x}^{a-x} \cos\left(\frac{k}{2z}u^2\right)du - j\int_{-a-x}^{a-x} \sin\left(\frac{k}{2z}u^2\right)du$$

$$= -\int_0^{-a-x} \cos\left(\frac{k}{2z}u^2\right) du + \int_0^{a-x} \cos\left(\frac{k}{2z}u^2\right)du + j\int_0^{-a-x} \sin\left(\frac{k}{2z}u^2\right)du - j\int_0^{a-x} \sin(\,)\,du$$

$$\frac{\pi}{2}t^2 = \frac{k}{2z}u^2 = \frac{2\pi}{2\lambda z}u^2 = \frac{\pi}{\lambda z}u^2$$

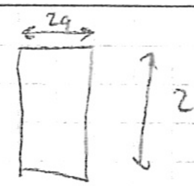
$$u^2 = \frac{\lambda z}{2}t^2$$
$$u = \sqrt{\frac{\lambda z}{2}}\,t \Rightarrow du = \sqrt{\frac{\lambda z}{2}}\,dt$$
$$u = -a-x \Rightarrow -a-x = \sqrt{\frac{\lambda z}{2}}\,t$$
$$t = -\sqrt{\frac{2}{\lambda z}}(a+x)$$

$$A(x) = \sqrt{\frac{\lambda z}{2}} \int_0^{\sqrt{\frac{2}{\lambda z}}(a-x)} \cos\left(\frac{\pi}{2}t^2\right)dt - \int_0^{-\sqrt{\frac{2}{\lambda z}}(a+x)} \cos\left(\frac{\pi}{2}t^2\right)dt + j\int_0^{-\sqrt{\frac{2}{\lambda z}}(a+x)} \sin\left(\frac{\pi}{2}t^2\right)dt - j\int_0^{\sqrt{\frac{2}{\lambda z}}(a-x)} \sin\left(\frac{\pi}{2}t^2\right)dt$$

$$E(x,y) = \left(\frac{e^{jkz}}{j\lambda z}\right)\left(\frac{\lambda z}{2}\right)\left[\left(C\left(\sqrt{\frac{2}{\lambda z}}(a-x)\right) - C\left(\sqrt{\frac{2}{\lambda z}}(-a-x)\right) - j\left(S\left(\frac{2}{\lambda z}(a-x)\right) - S\left(-\frac{2}{\lambda z}(a+x)\right)\right)\right)\right]$$

$$I(x,y) = \left(\frac{1}{4}\right)\left(\left[C\left(\sqrt{\frac{2}{\lambda z}}(a-x)\right) - C\left(-\sqrt{\frac{2}{\lambda z}}(a+x)\right)\right]^2 + \left[S\left(\sqrt{\frac{2}{\lambda z}}(a-x)\right) - S\left(\left(-\sqrt{\frac{2}{\lambda z}}(a+x)\right)\right]^2\right)$$

Fraunhofer slit diffraction



$$T(x) = \begin{cases} 1 & |x| < a \\ 0 & \text{else} \end{cases}$$

$$E(x,y) = \underbrace{\frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)}}_{E_0} \int_{-a}^{a} e^{-j\frac{k}{z}xx'} dx' \int_{-b}^{b} e^{-j\frac{k}{z}yy'} dy'$$

$$= E_0 \left.\frac{e^{-j\frac{k}{z}xx'}}{-j\frac{k}{z}x}\right|_{-a}^{a} \left.\frac{e^{-j\frac{k}{z}yy'}}{-j\frac{k}{z}y}\right|_{-b}^{b}$$

$$= E_0 \frac{e^{-j\frac{k}{z}xa} - e^{j\frac{k}{z}xa}}{-j\frac{k}{z}x} \frac{e^{-j\frac{k}{z}yb} - e^{j\frac{k}{z}yb}}{-j\frac{k}{z}x}$$

$$= E_0 \operatorname{Sinc}\left(\frac{k}{z}xa\right) \operatorname{Sinc}\left(\frac{k}{z}yb\right)$$

Now using Fourier transforms

$$\operatorname{rect}(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

$$\operatorname{rect}\left(\frac{x}{2a}\right) = \begin{cases} 1 & \left|\frac{x}{2a}\right| < \frac{1}{2} \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & |x| < \frac{2a}{2} = a \\ 0 & \text{else} \end{cases}$$

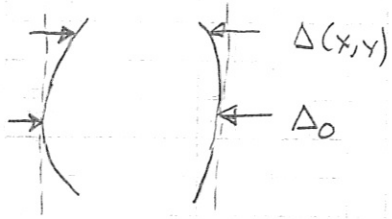
$$\mathcal{F}\left\{\operatorname{rect}\left(\frac{x}{2a}\right)\right\} = 2a \operatorname{Sinc}(2af_x)$$

$$f_x = \frac{x}{\lambda z}$$

$$E(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \mathcal{F}\left\{\operatorname{rect}\left(\frac{x}{2a}\right)\right\}\Big|_{f_x=\frac{x}{\lambda z}} \mathcal{F}\left\{\operatorname{rect}\left(\frac{y}{2b}\right)\right\}\Big|_{f_y=\frac{y}{\lambda z}}$$

$$E(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} (2a)(2b) \operatorname{Sinc}\left(2a\frac{x}{\lambda z}\right) \operatorname{Sinc}\left(\frac{2by}{\lambda z}\right)$$

A lens adds a phase transformation



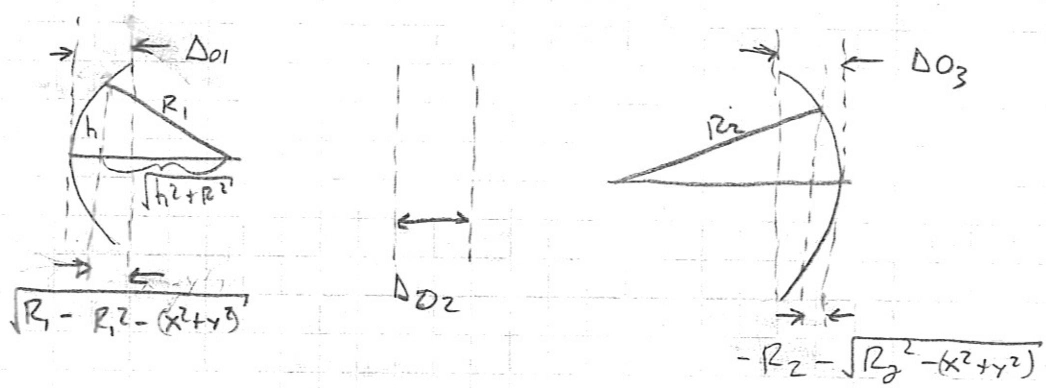
The total phase delay caused by the lens is

$$\phi(x,y) = kn\,\Delta(x,y) + k\left(\Delta_0 - \Delta(x,y)\right)$$

The lens transmission becomes

$$T_{lens} = \exp(jk\Delta_0)\,\exp\left[jk(n-1)\,\Delta(x,y)\right]$$

To find the thickness function $\Delta(x,y)$ separate the lens into 3 parts



$$\Delta_1(x,y) = \Delta_{01} - \left(R_1 - \sqrt{R_1^2 - (x^2+y^2)}\right)$$

$$= \Delta_{01} - R_1\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_1^2}}\right)$$

$$\Delta_3(x,y) = \Delta_{03} - \left(-R_2 - \sqrt{R_2^2 - (x^2+y^2)}\right)$$

$$= \Delta_{03} + R_2\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_2^2}}\right)$$

$$\Delta(x,y) = \Delta_{01} + \Delta_{02} + \Delta_{03} - R_1\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_1^2}}\right) + R_2\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_2^2}}\right)$$

In the paraxial approximation

$$\sqrt{1 - \frac{x^2+y^2}{R_1^2}} \approx 1 - \frac{x^2+y^2}{2R_1^2}$$

$$\sqrt{1 - \frac{x^2+y^2}{R_2^2}} \approx 1 - \frac{x^2+y^2}{2R_2^2}$$

$$\Delta(x,y) = \Delta_{01} + \Delta_{02} + \Delta_{03} - R_1\left(\cancel{R_1} - \cancel{R_1} + \frac{x^2+y^2}{2R_1^2}\right) + R_2\left(\cancel{R_2} - \cancel{R_2} + \frac{x^2+y^2}{2R_2^2}\right)$$

$$= \Delta_{01} + \Delta_{02} + \Delta_{03} - \left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$T_{lens} = \exp(jk\Delta_0)\ \exp\left(-jk(n-1)\left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right)$$

$$\frac{1}{f} \equiv (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$T_{lens} = \exp(jk\Delta_0)\ \exp\left(-j\frac{k}{2f}(x^2+y^2)\right)$$

So the incident field is multiplied by the lens transmission function

$$E(x,y) = \frac{e^{jkz}}{j\lambda z}\, e^{jk\Delta_0} \iint E(x',y')\, e^{-j\frac{k}{2f}(x'^2+y'^2)}\, e^{j\frac{k}{2z}(x'^2+y'^2)}\, e^{-j\frac{k}{z}(xx'+yy')}\, dx'dy'$$

$$= \frac{e^{jkz}}{j\lambda z}\, e^{jk\Delta_0} \iint E(x',y')\, e^{-j\frac{k}{2}(x'^2+y'^2)\left(\frac{1}{f}-\frac{1}{z}\right)}\, e^{-j\frac{k}{z}(xx'+yy')}\, dx'\,dy'$$

if the analysis plane is at the lens focus $f = z$

$$E(x,y) = \frac{e^{jkz}}{j\lambda z}\, e^{jk\Delta_0} \iint E(x',y')\, e^{-j\frac{2\pi}{\lambda f}(xx'+yy')}\, dx'\,dy'$$

$$E(x,y) = \frac{e^{jkz}}{j\lambda z}\, e^{jk\Delta_0}\, \mathcal{F}\{E(x,y)\}\Big|_{f_x = \frac{x}{\lambda f},\ f_y = \frac{y}{\lambda f}}$$