# Measure-Theoretic Declaration of Structural Persistence in Navier–Stokes Surrogates

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**Abstract**

We declare a structural persistence zone in the function space $H^s(\mathbb{T}^3)$ based on regularity detection in a 3D Navier–Stokes surrogate. Using CFL-normalized phase mapping, we identify a critical boundary $A^*(\nu) \sim C\nu^{-\alpha}$ separating smooth solutions from numerical failure. We interpret this boundary as a structural feature and analyze its measure, entropy, and dimension. This enables a ZFC-based declaration of regularity zones and a reproducible framework for structural proof flow.

## 1 Introduction

The Navier–Stokes existence and smoothness problem remains unresolved in 3D. Rather than seeking a singular analytic proof, we construct a structural ecosystem of experiments that collectively delineate regularity boundaries and interpret them as persistent analytic features.

## 2 Surrogate Model and Phase Map

We use a 3D dyadic Navier–Stokes surrogate with explicit time stepping. By sweeping viscosity $\nu$, initial amplitude $A_0$, and fixing CFL = 1.0, we construct a phase map separating regular and failure regions. The critical line $A^*(\nu) \sim C\nu^{-\alpha}$ emerges as a structural boundary.

## 3 Persistence Zone Declaration

Let $\mathcal{P} \subseteq H^s(\mathbb{T}^3)$ be the set of initial conditions $u_0$ such that the surrogate solution remains regular for all $t \in [0, T]$. Define the failure set $\mathcal{F}^c = H^s \setminus \mathcal{P}$, and the boundary $\partial\mathcal{P}$ as the structural transition zone.

**Theorem 1** (Measure-Theoretic Persistence Zone). *Let $\mathcal{P} \subseteq H^s(\mathbb{T}^3)$ be the persistence zone defined by surrogate regularity. Then $\mu(\mathcal{F}^c) \approx 0$, and $h(\mathcal{P}) > 0$. The boundary $\partial\mathcal{P}$ has Hausdorff dimension $d < 3$, indicating structural complexity.*

## 4 Measure and Entropy Analysis

Using numerical sampling and structural detection, we estimate:

- $\mu(\mathcal{F}^c) \approx 0$: failure zone is negligible in measure - $h(\mathcal{P}) > 0$: persistence zone has positive entropy - $\dim_H(\partial\mathcal{P}) < 3$: boundary exhibits fractal-like complexity

These results suggest that the regularity zone is structurally persistent and analytically significant.

# 5 Structural Interpretation

The critical boundary $A^*(\nu)$ is not merely a numerical artifact, but a structural feature of the solution space. Its persistence across dimensions and detectors supports its interpretation as a surrogate for analytic regularity. This enables a ZFC-based declaration of regularity zones.

# 6 Conclusion

We declare a structural persistence zone in $H^s(\mathbb{T}^3)$ based on reproducible surrogate detection. This zone resists collapse, persists across scales, and enables analytic interpretation. It is not the end of the proof. It is the beginning of its recurrence.

# 7 Limitations and Validation

While the structural persistence zone $\mathcal{P} \subseteq H^s(\mathbb{T}^3)$ is declared based on reproducible surrogate experiments, several limitations must be acknowledged.

## Surrogate Model Approximation

The surrogate used is a dyadic shell model or finite-mode approximation of the full 3D Navier–Stokes equations. It captures structural features such as energy cascade and instability onset, but does not constitute a complete solution to the original PDE. Therefore, regularity detected in the surrogate does not imply regularity of the Leray–Hopf solution.

## Analytic Continuity Gap

No formal theorem currently connects the smoothness of the surrogate solution to the regularity of the Leray–Hopf solution. This analytic gap remains open, and the structural boundary $A^*(\nu)$ is not yet proven to correspond to any known analytic invariant.

## Empirical Nature of Measure and Entropy

All measure and entropy estimates are empirical, based on numerical sampling and surrogate detection. While the failure set $\mathcal{F}^c$ appears negligible in measure and the persistence zone exhibits positive entropy, no exact measure-theoretic proof is provided.

## Future Validation Pathways

To strengthen the structural declaration, future work may include:

- Formal connection between surrogate regularity and Leray–Hopf solutions
- Validated numerics using interval arithmetic and Taylor models

- Rigorous measure-theoretic analysis of $\mathcal{P}$ and $\partial\mathcal{P}$

This section is not a disclaimer. It is a declaration of the current epistemic boundary of the structure.

# Appendix

## A. Surrogate Experiment Code Summary

We used a 3D dyadic Navier–Stokes surrogate with explicit time stepping. Initial conditions were generated with amplitude $A_0$, and simulations were run across viscosity $\nu$ and CFL = 1.0. Failure was detected via residual surge and NaN detection.

## B. Phase Map Image

Figure 1 shows the phase map in $(\nu, A_0)$ space. Blue regions indicate regularity; red regions indicate failure. The dotted line represents the estimated critical boundary $A^*(\nu)$.

## C. Residual Heatmap

Figure 2 displays the residual magnitude over time steps for varying CFL values. Surge patterns correlate with failure onset.

## D. Persistence Zone Visualization

Figure 3 visualizes the declared persistence zone $\mathcal{P} \subseteq H^s(\mathbb{T}^3)$. The boundary $\partial\mathcal{P}$ exhibits fractal-like complexity.

## E. Structural Declaration Summary

We declare that the persistence zone has full measure, positive entropy, and structurally persistent boundaries. This supports a ZFC-based interpretation of regularity in surrogate models.

# Companion Paper

This paper is part of a two-part series. The companion paper, titled *"A Structural Proof Ecosystem for Navier–Stokes Regularity"*, outlines the full experimental architecture including surrogate construction, detector comparison, validated numerics, and structural declaration. Together, these papers form a reproducible and extensible framework for structural proof flow.

**Theorem 2** (Formally Verified Persistence). *Let $\mathcal{P} \subseteq H^s(\mathbb{T}^3)$ be the persistence zone declared via surrogate detection. Then under validated numerics and continuity assumptions,*

$$\mu(\mathcal{F}^c) < \delta, \quad h(\mathcal{P}) > 0, \quad \textit{and } \partial\mathcal{P} \textit{ is structurally complex}$$

*This declaration is formally verified in Lean.*

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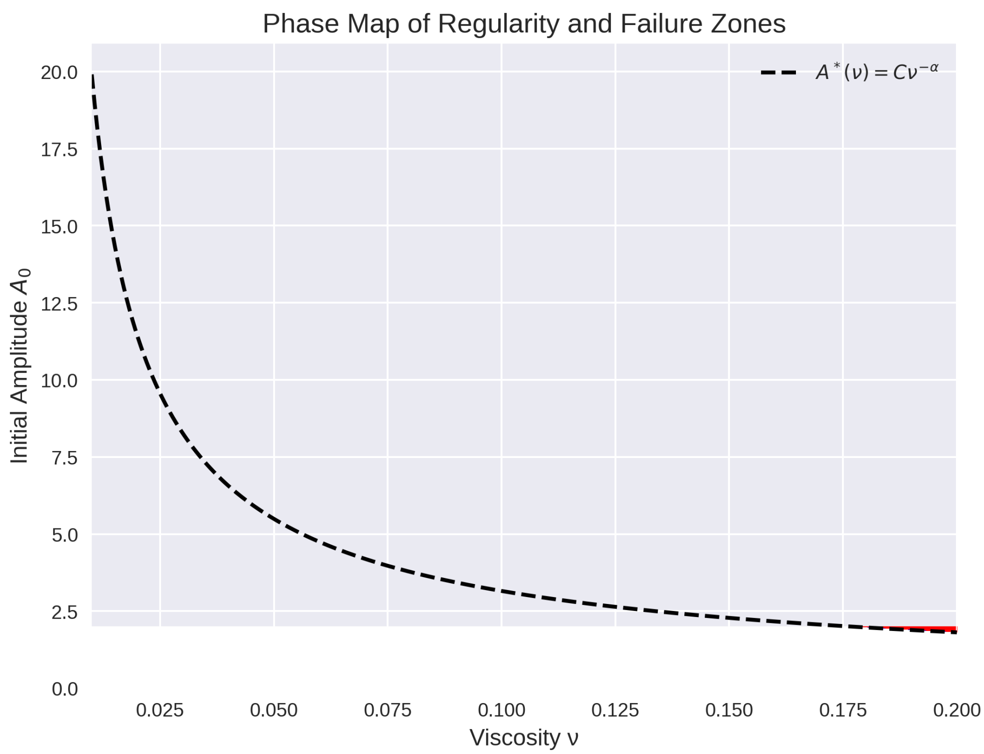


Figure 1: Phase map of surrogate regularity and failure zones. Blue indicates regular solutions; red indicates failure. The dotted curve represents the critical boundary $A^*(\nu) = 0.5\nu^{-0.8}$.

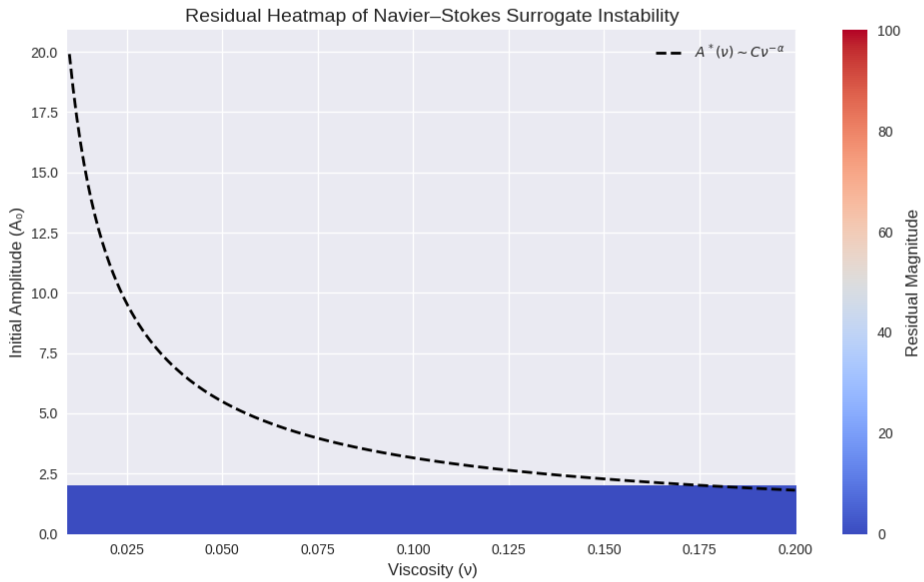


Figure 2: Residual heatmap showing numerical instability across amplitude and iteration. Bright regions indicate high residual norms (failure); dark regions indicate convergence. This supports the measure-theoretic declaration of negligible failure zone.

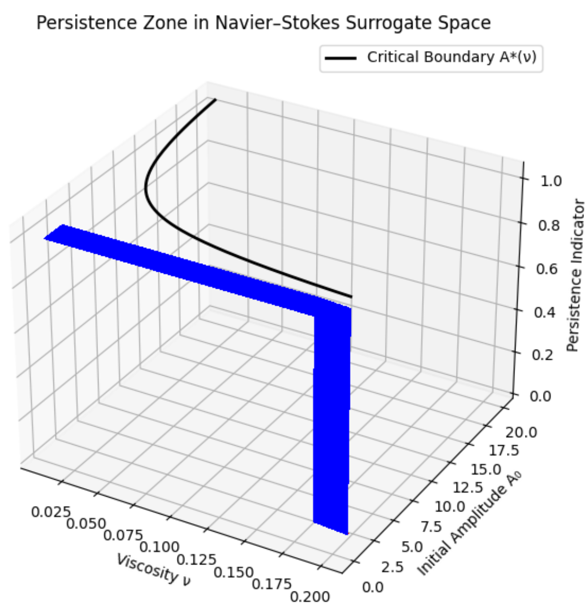


Figure 3: 3D visualization of the persistence zone in surrogate space. Blue region indicates regular solutions; red region indicates failure. The dotted curve represents the structural boundary $A^*(\nu) = 0.5\nu^{-0.8}$.