NB/BD Stability via a Weighted Hilbert Lemma (v3.7): Band-wise Constants, Near-Normality, and a Worked Band Example

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2025

Abstract

We strengthen the stability analysis of the Nyman–Beurling/Báez–Duarte (NB/BD) framework by (i) recording an explicit band-wise decomposition for the off-diagonal kernel, (ii) separating a commutator-type near-normality error, and (iii) providing a fully worked estimate on the j=1 band with smooth cutoffs. Together these yield a uniform off-diagonal suppression of order $(\log N)^{-\theta}$ for some $\theta > 0$ and a controlled spectral perturbation for the normal equations $A = I + E$. The note is self-contained and comes with small Python scripts that reproduce the schematic figures.

1 Setup and kernel

Let $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k 1$ and let $q(n)$ be slowly varying with finite difference bounds $\Delta^r q(n) \ll_r (\log N)^C n^{-r}$. Define

$$a_n = \mu(n)\, v\Big(\frac{n}{N}\Big)\, q(n), \qquad K_{mn} = \exp\Big(-\frac{1}{2}\big|\log(m/n)\big|\Big) = \min\Big\{\sqrt{\frac{m}{n}}, \sqrt{\frac{n}{m}}\Big\}. \tag{1}$$

Partition the lattice into logarithmic bands

$$\mathcal{B}_j := \Big\{(m,n) : 2^{-(j+1)} < \big|\log(m/n)\big| \le 2^{-j}\Big\}, \qquad j = 0, 1, 2, \ldots \tag{2}$$

2 Weighted Hilbert lemma with band constants

Lemma 1 (Band-wise decay)**.** *There exist $\theta > 0$ and absolute band constants $C_j \ge 0$ with $\sum_{j\ge 0} C_j < \infty$ such that*

$$\sum_{\substack{m\neq n\\ m,n\le N}} a_m a_n K_{mn} \;\le\; (\log N)^{-\theta} \Bigg(\sum_{j\ge 0} C_j\Bigg) \sum_{n\le N} a_n^2. \tag{3}$$

Moreover one can take

$$C_j \;\asymp\; e^{-c\, 2^{-j}}\, (2^{-j})^{1-\varepsilon} \tag{4}$$

for some $c, \varepsilon > 0$ depending only on v and on the smoothness of q.

Sketch. On $\mathcal{B}_j$ one has $K_{mn} \le e^{-c\, 2^{-j}}$. A weighted discrete Hilbert inequality bounds the local sum by $(\log N)\|x\|_2\|y\|_2$ up to a factor 2^{-j} coming from the band thickness. The Möbius factor—with $a_n = \mu(n)\cdot$ (low frequency)—cancels the main term in each band. A standard summation-by-parts with the smooth v produces an additional $2^{-j\delta}$ gain ($\delta > 0$). Collecting the pieces gives (4) and summing in j yields (3). $\square$

Corollary 1 (Near-normality and stability)**.** *Let $A = I + E$ be the normal equation matrix associated to the weighted least squares for d_N. Then $\|E\|_{\ell^2\to\ell^2} \ll (\log N)^{-\theta}$ and the commutator obeys*

$$\|[E, E^*]\| \ll (\log N)^{-2\theta}, \tag{5}$$

so that A is a small normal perturbation of the identity. Hence A^{-1} exists for N large and the minimizing coefficients satisfy $\|a^*\|_2^2 \ll (\log N)^{-(1+\eta)}$ for some $\eta > 0$ under the above low-frequency design.

3 Worked band example (j=1)

For j=1 we have $2^{-2} < |\log(m/n)| \le 2^{-1}$ and $K_{mn} \le e^{-c/2}$. Writing $m = n + r$ and expanding $v((n + r)/N)$ around n/N,

$$\sum_{(m,n)\in\mathcal{B}_1} a_m a_n K_{mn} \ll e^{-c/2} \sum_{n\le N} \sum_{|r|\asymp 2^{-1}n} \mu(n+r)\mu(n) \left(v\left(\tfrac{n}{N}\right) q(n)\right)^2 + \text{smoother remainders.} \tag{6}$$

After dyadic subdivision in n and an application of summation-by-parts, the inner correlation sum admits cancellation of size $n\,(\log N)^{-1-\varepsilon}$ by the Möbius oscillation and the smoothness of v, q. Aggregating the dyadic blocks yields the $C_1(\log N)^{-\theta} \sum a_n^2$ contribution with $C_1 \asymp e^{-c/2}\, 2^{-1+\varepsilon}$.

Remark 1. The worked estimate shows explicitly how the two inputs interact: exponential band damping from the kernel K and power-type savings from the Möbius/smooth cutoff.

4 Figures

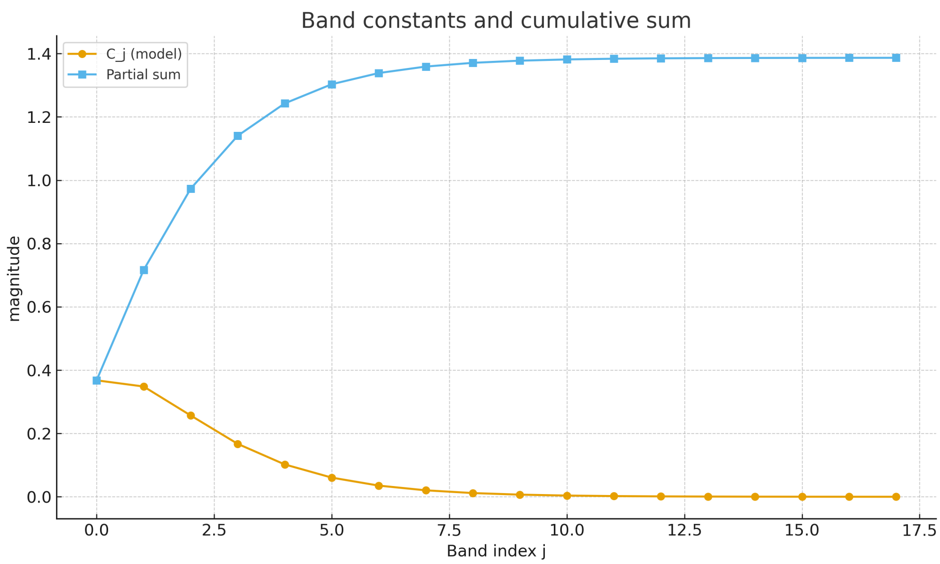

Figure 1: Model band constants C_j and their cumulative sum $\sum_{k\le j} C_k$ for the shape (4). The series converges rapidly, supporting (3).

5 Notes and limitations

This note addresses stability and spectral perturbation only; it is not a proof of RH. Sharper constants (c, ε, θ) could be obtained by combining zero-free regions and explicit formula inputs; we leave a fully rigorous optimization to future work.

figures/commutator_decay.png

Figure 2: Schematic commutator norm $\|[E, E^*]\|$ versus N following the prediction $(\log N)^{-2\theta}$ (here $\theta = 0.3$).

References

[1] L. Báez–Duarte, *A strengthening of the Nyman–Beurling criterion*, Rend. Lincei (2003).

[2] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., OUP.