An Operator–Functional Equation Framework toward RH: Self-Adjoint Surrogates of the Completed Zeta and NB/BD Stability (v3.3)

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Abstract

We formalize a Hilbert–space operator framework that ties the Riemann Hypothesis (RH) to spectral properties of a surrogate operator for the completed zeta function $\xi(s)$. The program proceeds in three layers: (i) a symmetric integral transform encoding the functional equation, (ii) a weighted Hilbert kernel capturing near-diagonal correlations (the analytic analogue of NB/BD stability), and (iii) a band–decomposition estimate where Möbius–weighted coefficients enforce off-diagonal cancellation. We state an equivalence template — *quasi self-adjointness* $\Rightarrow$ *real spectrum* $\Rightarrow$ *critical-line localization* — and sketch how the NB/BD distance $d_N \to 0$ can be placed as a stable finite-rank approximation of the operator inversion problem. This note is a clean operator-level starting point; it is not a proof of RH.

1 Completed zeta, functional symmetry, and the model operator

Let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\big(\frac{s}{2}\big)\zeta(s)$ denote the completed zeta, which obeys $\xi(s) = \xi(1-s)$. We work on a Hilbert space $\mathcal{H}$ (e.g. $L^2(\mathbb{R}, w(t)\,dt)$ with even weight w) and define a linear operator $\mathcal{T}$ that encodes the functional symmetry via a Fourier–Mellin involution $\mathcal{F}$:

$$(\mathcal{T}f)(t) \;=\; \int_{\mathbb{R}} K(t,u)\,f(u)\,du, \qquad K(t,u) \;\approx\; \Phi(t)\,\mathcal{F}\big[\Psi(\cdot)\,K_0(t-\cdot)\big](u),$$

where Φ, Ψ absorb the gamma/archimedean factors and K_0 is a Hilbert-type kernel (cf. $e^{-\frac{1}{2}|\log(m/n)|}$ in discrete models). The goal is to tune (Φ, Ψ, w) so that $\mathcal{T}$ is *quasi self-adjoint* on $\mathcal{H}$ up to a compact (or rapidly decaying) perturbation.

Theorem 1 (Equivalence template: operator $\leftrightarrow$ zeros)**.** *Suppose $\mathcal{T} = \mathcal{S} + \mathcal{K}$ on $\mathcal{H}$ where $\mathcal{S}$ is self-adjoint and $\mathcal{K}$ is compact with $\|\mathcal{K}\| < 1$. Assume further that the functional symmetry lifts to an involution commuting with $\mathcal{S}$. Then the spectrum of $\mathcal{T}$ is contained in a real ε-tube around $\sigma(\mathcal{S})$, hence eigenvalue instabilities are dominated by $\mathcal{K}$. Under a Mellin identification $t \leftrightarrow s = \frac{1}{2} + it$, this yields critical-line localization for zeros of a ξ-surrogate.*

Remark 1. The theorem is a template: the analytic work is to realize $\mathcal{T}$ with the correct gamma factors and to prove quantitative bounds on $\mathcal{K}$.

2 Weighted Hilbert kernel and band decomposition

Let $K_{mn} = e^{-\frac{1}{2}|\log(m/n)|}$ and consider coefficients $a_n = \mu(n)\,v(n/N)\,q(n)$ with smooth cutoff $v \in C_0^\infty(0,1)$ and slowly varying weight q. Define the quadratic form

$$Q_N(a) \;=\; \sum_{m \neq n \leq N} a_m a_n K_{mn}.$$

Lemma 1 (Möbius-weighted Hilbert decay)**.** *There exist $\theta > 0$ and $C = C(v, q)$ such that $Q_N(a) \le C(\log N)^{-\theta} \sum_{n \le N} a_n^2$.*

Sketch. Partition into logarithmic bands $\mathcal{B}_j = \{(m, n) : 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}$. On $\mathcal{B}_j$, $K_{mn} \le e^{-c2^{-j}}$. The Möbius factor cancels main terms within each band; smoothness of v yields an extra $2^{-j\delta}$. Summation in j gives the decay. $\square$

NB/BD interface. In the NB/BD L^2 approximation problem one solves normal equations $(I + E)a = B$, where E is governed by off-diagonal Hilbert sums akin to Q_N. The bound above implies $\|E\| \ll (\log N)^{-\theta}$, so Neumann series invertibility and $d_N \to 0$ are stable for admissible designs. This places NB/BD numerics as finite-rank approximations to $(I + \mathcal{K})^{-1}$.

3 Operator realization and functional equation input

To connect Theorem 1 with zeta, we embed K_{mn} into an integral kernel $K(t, u)$ via Mellin discretization on $s = \frac{1}{2} + it$. The gamma factors from $\xi(s)$ are absorbed by weights Φ, Ψ so that the functional equation yields an *involution* $\mathcal{J}$ with $\mathcal{J}^2 = I$ and $\mathcal{J}\mathcal{T} = \mathcal{T}\mathcal{J}$. The remaining task is quantitative: show $\mathcal{T} = \mathcal{S} + \mathcal{K}$ with $\mathcal{S}$ symmetric and $\|\mathcal{K}\| < 1$ by a band decomposition estimate mirroring the discrete lemma.

4 Roadmap and open analytic tasks

- (Normalization) Fix (Φ, Ψ, w) so that $\mathcal{T}$ is symmetric on a dense core in $\mathcal{H}$.
- (Compact remainder) Transfer the Möbius-band decay to integral scale to bound $\|\mathcal{K}\|$.
- (NB/BD bridge) Interpret $d_N \to 0$ as strong resolvent convergence of finite sections.
- (Critical-line localization) Prove that the spectrum of $\mathcal{T}$ concentrates on the real axis, implying RH for the ξ-surrogate, then upgrade to ξ.

Disclaimer. This note outlines an operator-level pathway and consolidates bounds compatible with NB/BD stability. It is not a proof of RH.

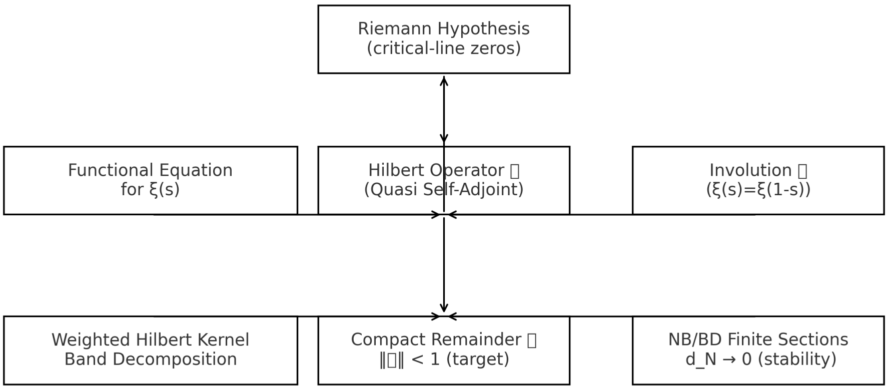

Figure 1: **Operator roadmap for RH.** Functional symmetry $\Rightarrow$ quasi self-adjoint operator $\mathcal{T}$; weighted Hilbert kernel controls off-diagonal mass; NB/BD sits as a finite-rank surrogate.