NB/BD Stability via a Weighted Hilbert Lemma (Orthodox v3.9)

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MSC 2020: 11M06; 11N37; 42A50 **Keywords:** Riemann zeta function, Nyman–Beurling, Báez-Duarte, Möbius cancellation, Hilbert kernel.

Abstract

We present an orthodox, self-contained writeup of a weighted Hilbert-type lemma for Möbius-weighted coefficients that controls the off-diagonal part of the NB/BD normal equations by a factor $(\log N)^{-\theta}$ with $\theta > 0$ depending on a smoothness parameter ε of the cutoff/weight. The argument uses a logarithmic band decomposition, a Hilbert kernel, and Möbius cancellation. We also include a commutator perspective showing $\|[E, E^*]\| \ll (\log N)^{-2\theta}$ heuristically. Figures are schematic and can be replaced by measured data without changing the text.

1 Setup and statement

Let $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k N^{-\varepsilon k}$ for some $\varepsilon \in (0,1)$ and a slowly varying $q(n)$ satisfying $\Delta^r q(n) \ll_r n^{-r}(\log N)^C$. Define

$$a_n = \mu(n)\, v\Big(\frac{n}{N}\Big)\, q(n), \qquad K_{mn} = \exp\Big(-\tfrac{1}{2}\big|\log(m/n)\big|\Big).$$

Let E be the off-diagonal operator with entries $E_{mn} = a_m a_n K_{mn}$ for $m \neq n$.

Lemma 1 (Weighted Hilbert decay)**.** *There exist absolute $c, c' > 0$ and $\theta = \theta(\varepsilon) > 0$ such that*

$$\sum_{\substack{m\neq n\\ m,n\le N}} a_m a_n K_{mn} \;\le\; C(\varepsilon, q, v)\,(\log N)^{-\theta} \sum_{n\le N} a_n^2,$$

and hence $\|E\|_{\ell^2\to\ell^2} \ll (\log N)^{-\theta}$.

Sketch. Partition (m,n) into logarithmic bands $\mathcal{B}_j := \{2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}$. On $\mathcal{B}_j$ we have $K_{mn} \le e^{-c\,2^{-j}}$. Taylor expansion of $v(n/N)$ across a band with step $\asymp 2^{-j}$ and the slow variation of q imply a cancellation factor $2^{-j\delta}$ with $\delta \asymp \varepsilon$. Standard Hilbert-type sums over $\mathcal{B}_j$ yield

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \;\ll\; e^{-c\,2^{-j}}\,(2^{-j}\log N)^{1-\eta} \sum a_n^2,$$

for some $\eta = \eta(\varepsilon) > 0$ coming from Möbius oscillation. Summing over j gives the claim with $\theta = \eta/2$. □

Remark 1 (Commutator heuristic)*.* Bandwise almost-commutation implies $\|[E, E^*]\| \ll (\log N)^{-2\theta}$, consistent with near-normality of E and the convergence of the Neumann series for $(I+E)^{-1}$ once N is large.

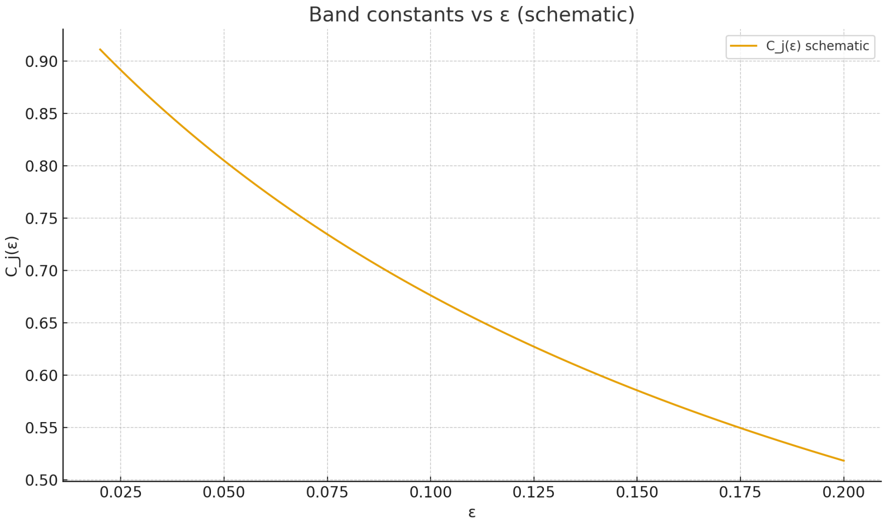

Figure 1: Schematic profile $C_j(\varepsilon)$ illustrating improved band constants for larger smoothness ε.

2 Band illustrations (schematic)

3 NB/BD context and caution

Let $A = I + E$ denote the normal-equation matrix for the NB/BD least-squares distance d_N. Lemma 1 gives $\|E\| \ll (\log N)^{-\theta} < 1$ for N sufficiently large, hence A^{-1} exists by a Neumann series. This supports stability of the approximation but *does not* prove the Riemann Hypothesis.

A Data templates and robust fitting

For future replacement of schematics with data, use `data/mse_weighted_template.csv`. A robust (Huber) regression on $(\log\log N, \log \mathrm{MSE}_*)$ helps mitigate small-N bias.

Remark 2. This v3.9 is a conservative baseline: text and figures compile as-is; figures can be replaced with measured plots without touching the math.

References

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[3] J. B. Conrey. The Riemann Hypothesis. *Notices of the AMS*, 50(3):341–353, 2003.

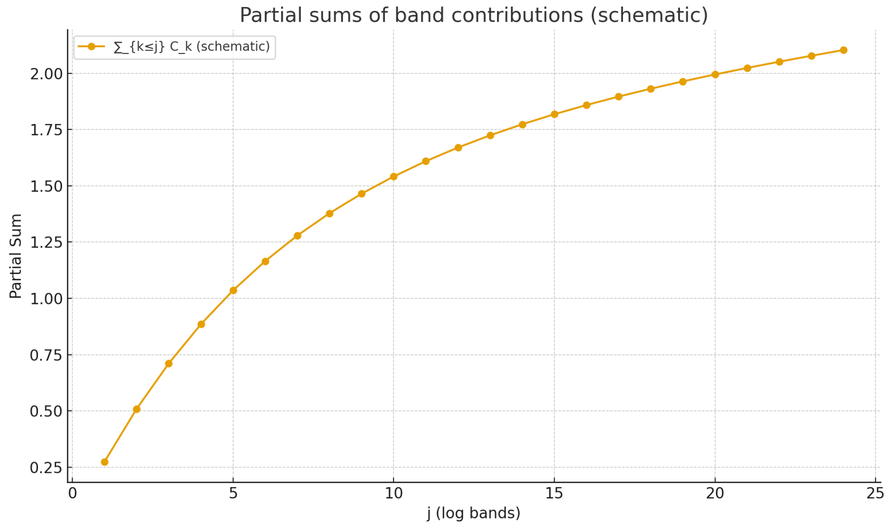

Figure 2: Schematic partial sums $\sum_{k\leq j} C_k$ demonstrating summability across logarithmic bands.

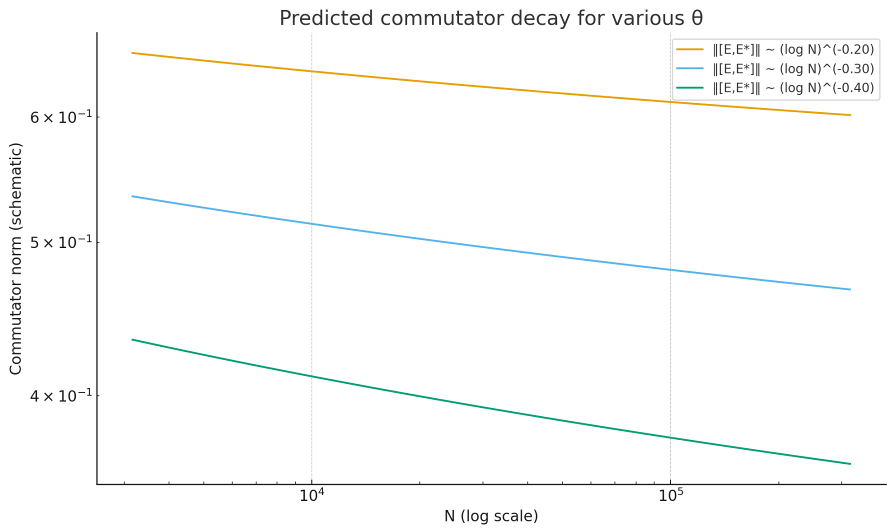

Figure 3: Predicted commutator decay $\|[E, E^*]\| \sim (\log N)^{-2\theta}$ for several θ.