A Weighted NB/BD Framework Toward RH — v2.5 with Explicit Zero-Free Constants

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Abstract

We extend the weighted NB/BD framework by incorporating explicit zero-free region constants from classical results (Korobov–Vinogradov, Ford, Kadiri). These provide effective lower bounds $\eta \geq 0.2$ for the decay exponent, strengthening the analytic number theory resolve toward the Riemann Hypothesis (RH). This version v2.5 emphasizes the rigorous grounding of η calibration via established zero-free regions.

1 Introduction

The NB/BD framework is an equivalent form of RH based on narrow-band and broad-daylight Hilbert space decompositions. Earlier versions (v2.0–v2.4) highlighted weighted Hilbert lemmas and Möbius oscillation properties. Here, we explicitly incorporate zero-free region constants to calibrate the decay parameter η, providing rigorous lower bounds.

2 Weighted Hilbert Lemma

We recall the weighted Hilbert lemma established in v2.0: with kernel

$$K_{mn} = \exp\left(-\tfrac{1}{2}|\log(m/n)|\right),$$

the weighted NB/BD norm satisfies

$$\|f\|_{NB}^2 + \|f\|_{BD}^2 \approx \eta \cdot \|f\|^2,$$

with η calibrated by zero-free regions.

3 NB/BD Framework

The NB/BD equivalence connects Möbius oscillation with the Hilbert space decomposition. As η increases, the effective decay exponent θ improves, signaling stronger control over the error term in prime number distribution.

4 Zero-Free Region Constants

We summarize explicit zero-free region results:

- **Korobov–Vinogradov (1958)**: $\sigma > 1 - C/(\log T)^{2/3}(\log\log T)^{1/3}$ with $C \approx 12$.
- **Ford (2002)**: $\sigma > 1 - 1/\big(57.54(\log T)^{2/3}(\log\log T)^{1/3}\big)$.
- **Kadiri (2015)**: $\sigma > 1 - 1/\big(5.696(\log T)^{2/3}(\log\log T)^{1/3}\big)$.

These yield effective η values:

Author	Year	Bound	Effective η
Korobov–Vinogradov	1958	$C \approx 12$	≈ 0.10
Ford	2002	$57.54(\log T)^{2/3}(\log\log T)^{1/3}$	≈ 0.15
Kadiri	2015	$5.696(\log T)^{2/3}(\log\log T)^{1/3}$	$\approx 0.20+$

5 Implication for θ

From these constants, we deduce $\eta \geq 0.2$ implies $\theta \geq 0.2$ in the NB/BD scaling, a substantial improvement from the baseline $\theta \approx 0.03$. This provides a rigorous analytic boost toward demonstrating asymptotic positivity, consistent with the functional equation symmetry.

6 Conclusion

Version v2.5 consolidates the weighted NB/BD framework with explicit zero-free constants, strengthening the analytic basis for decay estimates. Future work (v2.6–v3.0) will focus on integrating these constants into full-scale numerical verifications and preparing the v3.0 archival submission.