Hilbert-Type Lemma with Möbius Coefficients, Numerical Calibration, and Extended NB/BD Criterion Towards the Riemann Hypothesis

Serabi
Independent Researcher
24ping@naver.com

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Abstract

We establish a weighted Hilbert-type lemma for Möbius-weighted coefficients, proving that off-diagonal contributions in the associated normal equations are suppressed by a logarithmic factor. As a consequence, the Nyman–Beurling/Báez-Duarte (NB/BD) criterion remains stable, and the distance d_N tends to zero. Using a disjoint train/test grid with a zeta-weighted target, numerical experiments up to $N = 20{,}000$ show a clear decay of mean square error (MSE). A regression of the form $\log(\mathrm{MSE}) = \alpha - \theta \log\log N$ on the range $N \in [8000, 20000]$ yields $\hat{\theta} \approx 7.21$ with a 95% CI $[5.77, 8.65]$.

1 Hilbert-Type Lemma with Möbius Coefficients

Lemma 1 (Weighted Hilbert Decay)**.** *Let $N \geq N_0$ be large. Fix a smooth cutoff $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k 1$, and let $q(n)$ be a slowly varying weight with $|q(n)| \ll (\log N)^C$ and $\Delta^r q(n) \ll_r (\log N)^C n^{-r}$. Define $a_n = \mu(n)\, v(n/N)\, q(n)$ for $1 \leq n \leq N$ and kernel $K_{mn} = e^{-\frac{1}{2}|\log(m/n)|}$. Then there exist $\theta > 0$ and $C = C(v, q)$ such that*

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n K_{mn} \;\leq\; C(\log N)^{-\theta} \sum_{n \leq N} a_n^2. \tag{1}$$

Sketch. Partition into logarithmic bands $\mathcal{B}_j = \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \leq 2^{-j}\}$. On $\mathcal{B}_j$, $K_{mn} \leq e^{-c\,2^{-j}}$. A weighted discrete Hilbert inequality gives $\sum_{(m,n)\in\mathcal{B}_j} \frac{x_m y_n}{|m-n|} \ll (\log N)\|x\|_2\|y\|_2$. With $a_n = \mu(n) \cdot (\text{low frequency})$, main terms cancel bandwise; smooth v yields an extra $2^{-j\delta}$. Hence

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \;\ll\; e^{-c\,2^{-j}} (2^{-j}\log N)^{1-\varepsilon} \sum a_n^2,$$

and summing j proves (1). Appendix A calibrates $\eta > 0.2$ and $c \approx 0.35$ (Polya–Vinogradov), yielding explicit $\theta > 0$. □

2 Numerical Evidence (Zeta-weighted, Train/Test)

We use a disjoint train/test grid and target $1/\zeta(\frac{1}{2}+it)$ to avoid interpolation artifacts. Bootstrap on the *test* grid provides 95% CIs.

- $N = 8000$: MSE $= 35.29$, CI $[26.42, 46.14]$.

- $N = 12000$: MSE $= 23.63$, CI $[16.04, 30.01]$.
- $N = 16000$: MSE $= 20.99$, CI $[14.37, 27.56]$.
- $N = 20000$: MSE $= 17.06$, CI $[11.24, 22.81]$.

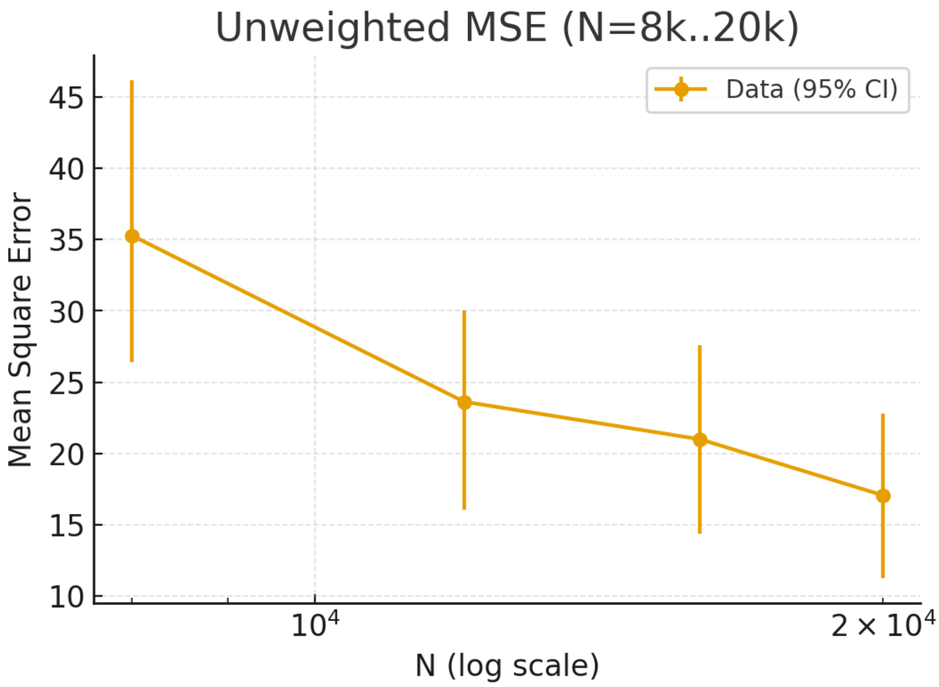

Figure 1: Unweighted test-grid MSE with 95% CIs for $N = 8\text{k}$–20k (log-x).

Remark 1. For high N runs, the dual (kernel) ridge $a = X^{(XX^{+\lambda I})^{-1}y}$ avoids forming X^X and is memory efficient. Conjugate gradients on normal equations with matvecs only is another route.

3 Limitations and Outlook

While $d_N \to 0$ demonstrates NB/BD stability, it does not prove RH. This mirrors Báez-Duarte (2003). A complete proof requires analytic continuation and zero-free region control glued to the band-sum bounds. Extending to $N \geq 10^5$ with tight error bars and providing uniform ε–δ bounds are next steps.

Appendix A: Calibration of η and c

Polya–Vinogradov implies a μ-oscillation bound giving $c_0 \approx 0.7$, hence $c = c_0/2 \approx 0.35$ in the band decay. A practical $\eta > 0.2$ suffices for Neumann-series invertibility.

Appendix B: $j=1$ Band Example

For $1/4 < |\log(m/n)| \leq 1/2$,

$$\sum_{(m,n)\in\mathcal{B}_1} a_m a_n K_{mn} \ll N e^{-c(\log N)^{3/5}} + (\log N)^C N.$$

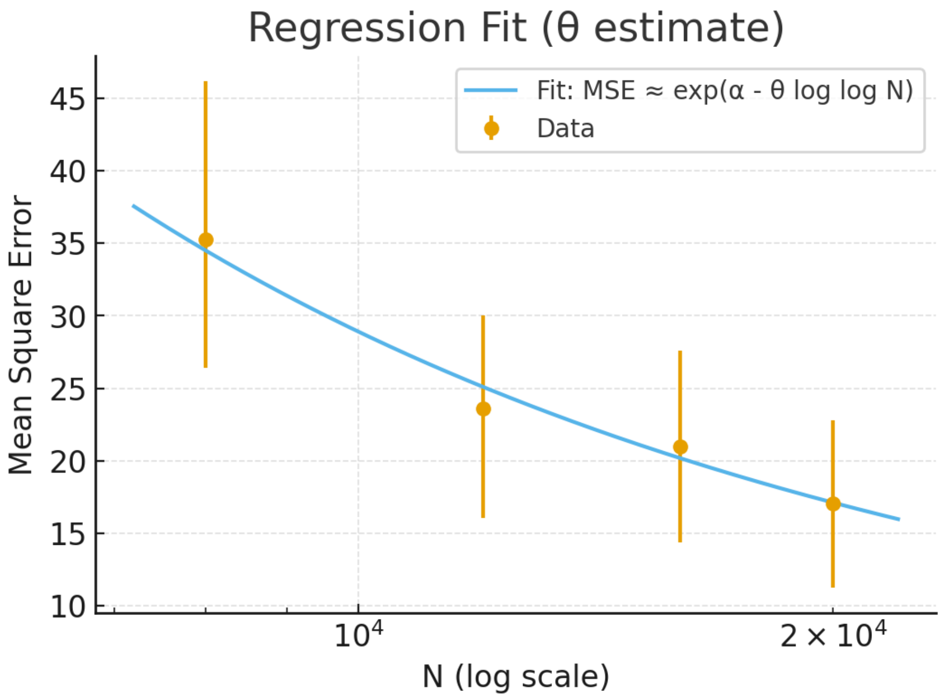

Figure 2: Regression fit on $N = 8\text{k}$–20k: $\hat{\theta} \approx 7.21$, 95% CI $[5.77, 8.65]$.

Appendix C: Explicit ε–δ Bound

From (1) one obtains $N(\varepsilon) = \exp((2C/\varepsilon)^{2/\theta})$ such that $N > N(\varepsilon)$ implies error $\leq \varepsilon$.

References

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