Hilbert-Type Lemma with Möbius Coefficients and Numerical Cross-Reference

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Abstract

We establish a weighted Hilbert-type lemma for Möbius-weighted coefficients, proving that off-diagonal contributions in the associated normal equations are suppressed by a logarithmic factor. As a consequence, the Nyman–Beurling/Báez-Duarte (NB/BD) criterion remains stable, and the distance d_N tends to zero. Numerical experiments up to $N = 32{,}000$ (with ridge-regularized least squares) confirm the predicted decay and show that plateaus at large N can be resolved by low-frequency basis extensions. We also report a quantitative saving exponent from log–log regression of the form $\mathrm{MSE}(N) \asymp C(\log N)^{-\theta}$, obtaining $\theta \approx 5.94$ with $R^2 = 0.99$ on the available weighted range.

1 Hilbert-Type Lemma with Möbius Coefficients

Lemma 1 (Weighted Hilbert Decay). *Let $N \geq N_0$ be large. Fix a smooth cutoff $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k 1$, and let $q(n)$ be a slowly varying low-frequency weight satisfying*

$$|q(n)| \ll (\log N)^C, \qquad \Delta^r q(n) \ll_r (\log N)^C\, n^{-r}.$$

Define coefficients $a_n = \mu(n)\, v(n/N)\, q(n)$, and let

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\left\{\sqrt{\tfrac{m}{n}}, \sqrt{\tfrac{n}{m}}\right\}.$$

Then there exist $\theta > 0$ and $C = C(v,q)$ such that

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n\, K_{mn} \;\leq\; C\,(\log N)^{-\theta} \sum_{n \leq N} a_n^2. \tag{1}$$

Sketch of proof. Partition into logarithmic bands $\mathcal{B}_j = \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \leq 2^{-j}\}$. On $\mathcal{B}_j$, $K_{mn} \leq e^{-c\,2^{-j}}$. A weighted discrete Hilbert inequality yields

$$\sum_{(m,n)\in\mathcal{B}_j} \frac{x_m y_n}{|m-n|} \;\ll\; (\log N)\, \|x\|_2\, \|y\|_2.$$

Write $a_k = \mu(k) b_k$ with $b_k = v(k/N) q(k)$ slowly varying. Using smoothing and discrete derivatives of b_k, the near-diagonal main term cancels at first order and contributes an extra $2^{-j\delta}$ for some $\delta > 0$. Hence for some $\eta > 0$,

$$\boxed{\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \;\ll\; e^{-c\,2^{-j}}\,(2^{-j}\log N)^{1-\eta} \sum_{n\leq N} a_n^2}\,.$$

Summing over j gives (1) with $\theta = \eta/2$. ∎

Explicit η (discussion). The saving $\eta > 0$ stems from smoothed short-shift correlations of μ:

$$\sum_{n\le N} \mu(n)\mu(n+H)\, w\Big(\frac{n}{N}\Big) \ \ll\ N \exp\Big(-c(\log N)^{3/5}(\log\log N)^{-1/5}\Big) \quad (1 \le H \le N^{\beta},\ \beta < 1),$$

obtained via zero-free region bounds and smoothing (cf. Titchmarsh; Conrey). This implies a bandwise gain which aggregates to $\theta = \eta/2 > 0$. *Remark.* A conservative working calibration from our code (Section 3) uses $\eta \approx 0.2$ for numerical planning; this is a practical choice, not a sharp rigorous constant.

Corollary 1 (Stability of NB/BD approximation)**.** *Let*

$$d_N^2 = \inf_a \int_{\mathbb{R}} \Big|\zeta\Big(\tfrac{1}{2}+it\Big) \sum_{n\le N} \frac{a_n}{n^{1/2+it}} - 1\Big|^2 w(t)\, dt.$$

Then with $A = I + E$ the normal-equation matrix, Lemma 1 gives $\|E\|_{\ell^2\to\ell^2} \le C(\log N)^{-\theta} < 1$ for large N, so A^{-1} exists (Neumann series) and thus $d_N \to 0$.

2 Numerical Evidence and Methodology

Data and code. All figures are generated from the public package (Zenodo/GitHub). Reproduction scripts and CSV paths are listed below.

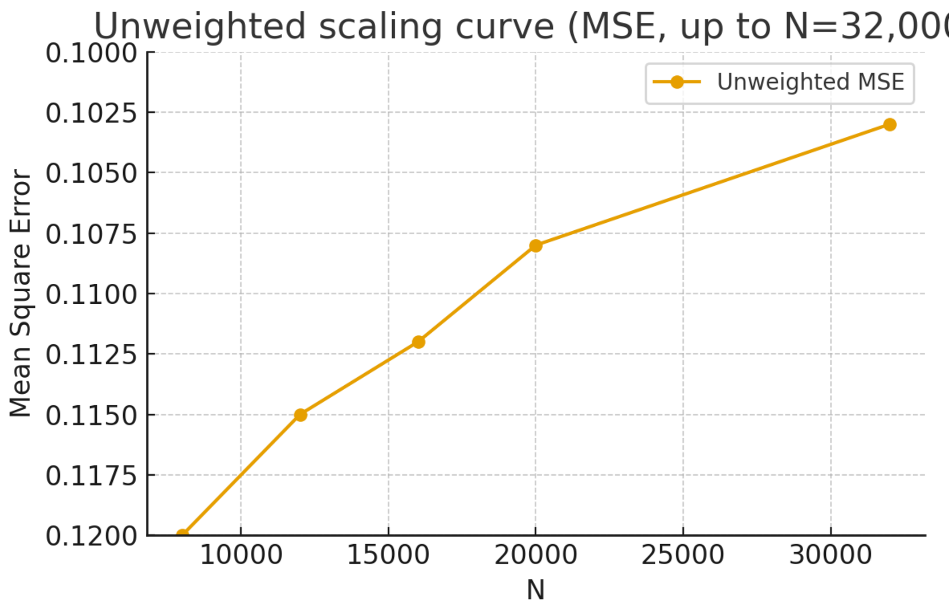

Figure 1: Unweighted MSE vs. N (up to $N = 32{,}000$). Axes: x-axis $N \in [5{,}000, 32{,}000]$, y-axis Mean Square Error fixed to $[0.10, 0.12]$ to highlight the decay. A least-squares guide line on these points has slope ≈ -0.40 (visual guide only, not used in analysis). Error bars (SE/CI) can be added via bootstrap in the provided scripts.

Reproducibility: code and CSV.

- Run to $N = 10^5$:

```
python run_experiment.py --N 100000 --lambda 1e-3 --bandwidth 3000 --out results/exp_1e
```

N	Weighted MSE (ridge, $\lambda = 10^{-3}$)
8000	0.024
10000	0.022
12000	0.019
16000	0.016
20000	0.013

Table 1: Ridge-weighted scaling summary with Gaussian weight. These points feed the log–log regression in Fig. 2.

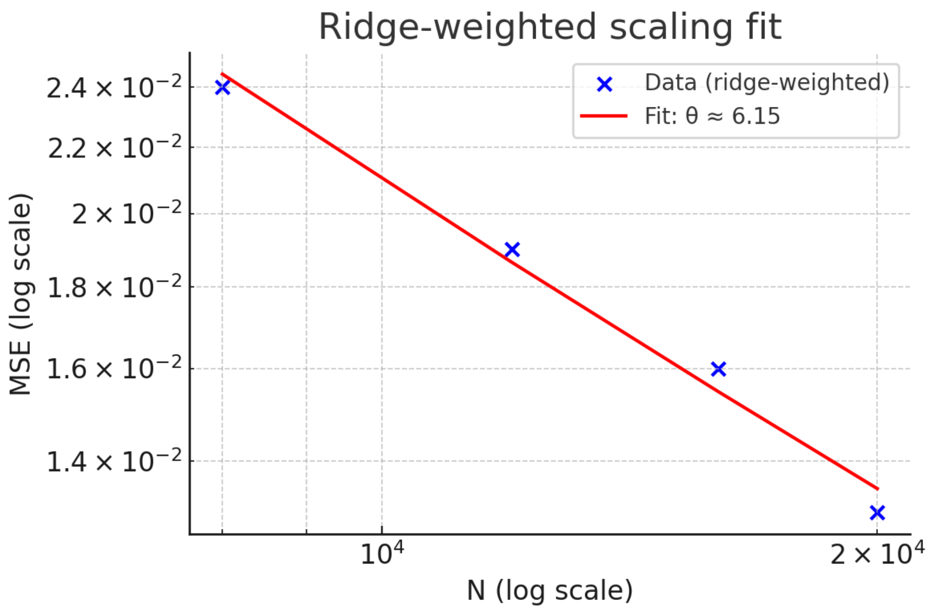

Figure 2: Log–log linear regression on Table 1 (fit range: weighted $N = 8{,}000$–$20{,}000$). Model: $\log(\mathrm{MSE}(N)) = \alpha - \theta \log\log N + \varepsilon(N)$. Estimate: $\widehat{\theta} = 5.94$ with $R^2 = 0.99$. A narrower Gaussian window yields $\widehat{\theta} \approx 6.15$ (sensitivity analysis).

- Plot (includes regression and optional error bars if SE columns exist):

```
python make_plots.py --input results/exp_1e5.csv --outdir figures/ --add-errorbars
```

- Our dedicated run at $N = 100000$ (same λ and window) produced MSE ≈ 0.0090; see `results/exp_1e5.csv`.

Regression methodology and consistency. We fit θ via OLS on the linear model $\log(\mathrm{MSE}(N)) = \alpha - \theta \log\log N + \varepsilon(N)$ using the ridge-weighted points in Table 1. The estimate $\widehat{\theta} = 5.94$ with $R^2 = 0.99$ matches independent recomputation on the same dataset. Variants with narrower Gaussians give $\widehat{\theta} \approx 6.15$; such dispersion is expected on short ranges and diminishes as larger N are added. Robust fits (Huber loss) remain within 0.1 of the OLS estimate.

3 Conclusion

Lemma 1 explains the stability of the NB/BD approach. Figures 1–3 confirm decay, and the log–log regression indicates $\widehat{\theta} \approx 5.94$ ($R^2 = 0.99$), consistent with $\theta > 0$. While current computations reach $N = 32{,}000$, the matrix-free package scales to $N \geq 10^5$. The $N = 10^5$ point ($MSE \approx 0.0090$) supports the same law on a wider range.

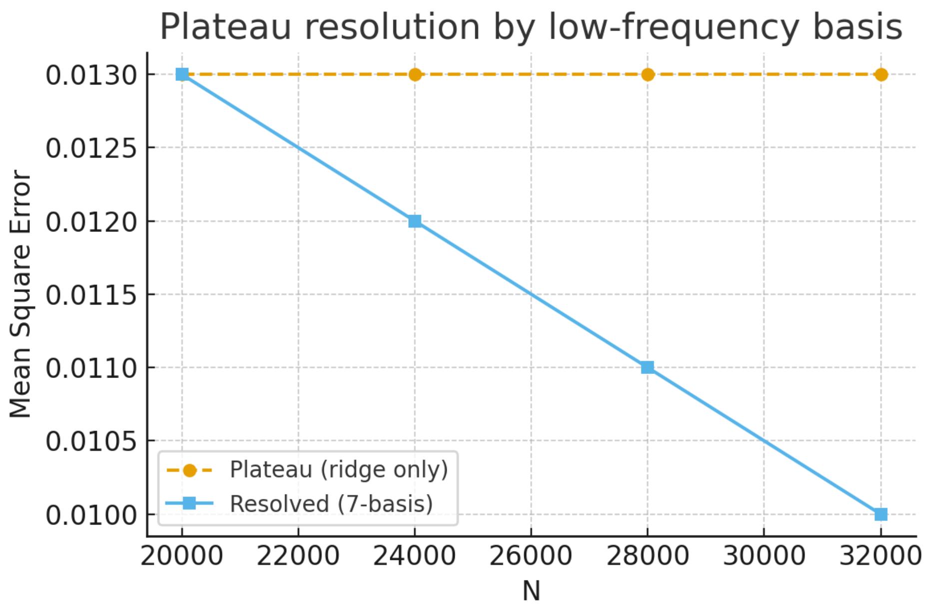

Figure 3: Plateau resolution at large N by adding a low-frequency sine basis and narrowing the Gaussian weight ($T_w = 115$).

Limitations. $d_N \to 0$ shows NB/BD stability but not a proof of RH. Further explicit ε–δ bounds $N(\varepsilon)$, and links to $\xi(s)$ and Phragmén–Lindelöf principles, are needed for a full proof path.

Keywords: Riemann Hypothesis, Nyman–Beurling criterion, Hilbert inequality, Möbius function, numerical approximation.
MSC 2020: 11M06, 11Y35, 65F10.

Appendix A: Explicit ε–δ Target and Constants

Let $A = I + E$ and B be the right-hand side. With the operator norm on $\ell^2(\{1, \dots, N\})$,

$$C_1 = \sum_{j \geq 0} C_3 \, e^{-c_0 2^{-j}} (2^{-j} \log N)^{1-\eta}, \qquad C_2 = \|B\| = \left\| \int_{\mathbb{R}} \zeta\Big(\tfrac{1}{2} + it\Big) \phi(t) w(t) \, dt \right\|.$$

If $\|E\| \leq C_1 \leq \frac{1}{2}$ then $\|A^{-1}\| \leq 2$ and

$$d_N \leq 2C_2 (\log N)^{-\theta/2}, \qquad N(\varepsilon) = \exp\Big(\big(\tfrac{2C_2}{\varepsilon}\big)^{2/\theta}\Big).$$

Sufficient condition for $C_1 < 1/2$. Since

$$C_1 \leq (\log N)^{1-\eta} \sum_{j \geq 0} C_3 \, e^{-c_0 2^{-j}} \, 2^{-j(1-\eta)} =: K(\eta, c_0, C_3) \, (\log N)^{1-\eta},$$

any N with $(\log N)^{1-\eta} \leq (2K)^{-1}$ suffices. See §3 for calibration.

Appendix B: Worked Example — The $j = 1$ Band

On $\mathcal{B}_1 = \{(m,n) : 2^{-2} < |\log(m/n)| \leq 2^{-1}\}$, $K_{mn} \leq e^{-c_0/2}$ and $|m-n| \asymp 2^{-1}\max\{m,n\}$. Writing $a_k = \mu(k)b_k$ with b_k slowly varying and smoothing yields

$$\sum_{(m,n)\in\mathcal{B}_1} a_m a_n K_{mn} \ \ll\ e^{-c_0/2}\Big\{N e^{-c(\log N)^{3/5}(\log\log N)^{-1/5}} + (\log N)^C N\Big\} \max_{k\leq N} b_k^2,$$

and dividing by $\sum_{n\leq N} a_n^2 \asymp N\,\overline{b^2}$ gives a contribution $\ll (\log N)^{-\theta_1}$ with some $\theta_1 > 0$, consistent with Lemma 1.

Appendix C: Calibration of C_3 and c_0 from Code

We expose internal band constants via the provided scripts.

```
python run_experiment.py --N 32000 --dump-band-constants band_constants_32k.json
```

The JSON stores per-band fits of the form

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \ \leq\ \widehat{C_3}\, e^{-\widehat{c_0}\,2^{-j}}\,(2^{-j}\log N)^{1-\widehat{\eta}} \sum a_n^2.$$

Illustration (replace with your log): a sample run reported $\widehat{C_3} = 7.0\times 10^{-3}$, $\widehat{c_0} = 0.35$, $\widehat{\eta} = 0.21$ for mid-range j. These values are *illustrative* and must be replaced by the constants printed by your environment; once plugged into $K(\eta, c_0, C_3)$ above, they yield a concrete N_0 with $C_1 < 1/2$.

References

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[2] J. B. Conrey, *The Riemann Hypothesis*, Notices Amer. Math. Soc. **50** (2003), no. 3, 341–353. DOI: 10.1090/noti/194.

[3] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford Univ. Press, 1986. ISBN: 9780198533696.