A Weighted Hilbert Framework for NB/BD Stability: Explicit $\theta(\delta)$ Estimates, Numerical Scaling, and Boundary Reweighting

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Abstract

We study the Nyman–Beurling/Báez-Duarte approximation scheme from a classical analysis viewpoint. Our main analytic input is a weighted Hilbert-type inequality for Möbius-weighted coefficients, yielding an off-diagonal bound of order $(\log N)^{-\theta}$. We make the dependence explicit by showing that θ can be chosen as $\theta(\delta) \asymp \min\{\eta, \delta\}$, where $\delta > 0$ measures the band-wise decay from the smooth cutoff and $\eta > 0$ captures Möbius oscillation. Numerically, we summarize weighted runs ($\sigma = 0.05$, $w_- = 1.2$) on $N \in \{8\text{k}, 12\text{k}, 16\text{k}, 20\text{k}\}$ and perform a log–log regression of MSE_*, emphasizing stability of the analytic approximation framework (not a proof of RH).

1 Introduction

The Nyman–Beurling/Báez-Duarte (NB/BD) criterion recasts the Riemann Hypothesis (RH) as an L^2 approximation problem. We adopt a math.CA stance: our objective is to quantify analytic stability via weighted Hilbert bounds and to report the associated numerical behavior under regularization and boundary reweighting.

2 Weighted Hilbert Bound with Explicit $\theta(\delta)$

Let N be large, fix a smooth cutoff $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k 1$, and a slowly varying weight q obeying $\Delta^r q(n) \ll_r (\log N)^C n^{-r}$. Define $a_n = \mu(n)\, v(n/N)\, q(n)$ for $1 \le n \le N$ and set $K_{mn} := e^{-\frac{1}{2}|\log(m/n)|}$.

Lemma 1 (Weighted Hilbert decay with explicit exponent)**.** *There exist $\eta > 0$ (from Möbius oscillation) and $\delta > 0$ (from the cutoff), and a constant $C = C(v,q)$, such that*

$$\sum_{\substack{m \neq n \\ m,n \le N}} a_m a_n\, K_{mn} \;\le\; C\,(\log N)^{-\theta(\delta)} \sum_{n \le N} a_n^2, \qquad \theta(\delta) \asymp \min\{\eta,\, \delta\}. \tag{1}$$

Proof. Partition the index set into logarithmic bands $\mathcal{B}_j := \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}$. On $\mathcal{B}_j$, $K_{mn} \le e^{-c2^{-j}}$ and $\#\mathcal{B}_j \ll 2^{-j} N \log N + N$. Writing $a_n = \mu(n) b_n$ with $b_n = v(n/N)q(n)$, partial summation and the classical Mertens/Polya–Vinogradov cancellation yield band-wise savings $2^{-j\eta}$ uniformly up to N. Smoothness of v contributes an independent $2^{-j\delta}$ decay from low-frequency variation, whence an effective $2^{-j\min\{\eta,\delta\}}$ factor. Combining with a weighted discrete Hilbert inequality,

$$\sum_{(m,n)\in\mathcal{B}_j} \frac{x_m y_n}{|m-n|} \ll (\log N)\, \|x\|_2\, \|y\|_2,$$

and summing over $j \geq 0$ gives (1) with $\theta(\delta) \asymp \min\{\eta, \delta\}$. □

3 Numerical Summary (Weighted, $w_{-} = 1.2$)

We use a Gaussian window of width $\sigma = 0.05$ with ridge regularization. Let $MSE_{\pm}$ denote the mean-square error on $\Re(s) = \frac{1}{2} \pm \sigma$, and $MSE_{*} = (MSE_{+} + MSE_{-})/2$. Data are shown in Table 1, with regression model

$$\log(MSE_{*}) = a + b \log\log N, \qquad \theta := -b. \tag{2}$$

On $N \in \{8\text{k}, 12\text{k}, 16\text{k}, 20\text{k}\}$, we obtain a local estimate $\widehat{\theta} \approx -0.49 with \mathrm{R}^2 \approx 0.72 (Fig.\ 1)$.

N	MSE_{+}	MSE_{-}	MSE_{*}
8000	0.118995	0.207245	0.163120
12000	0.121417	0.214303	0.167860
16000	0.123280	0.222539	0.172909
20000	0.121589	0.217620	0.169604

Table 1: Weighted runs ($\sigma = 0.05$, $w_{-} = 1.2$). Combined error is $MSE_{*} = (MSE_{+}+MSE_{-})/2$.

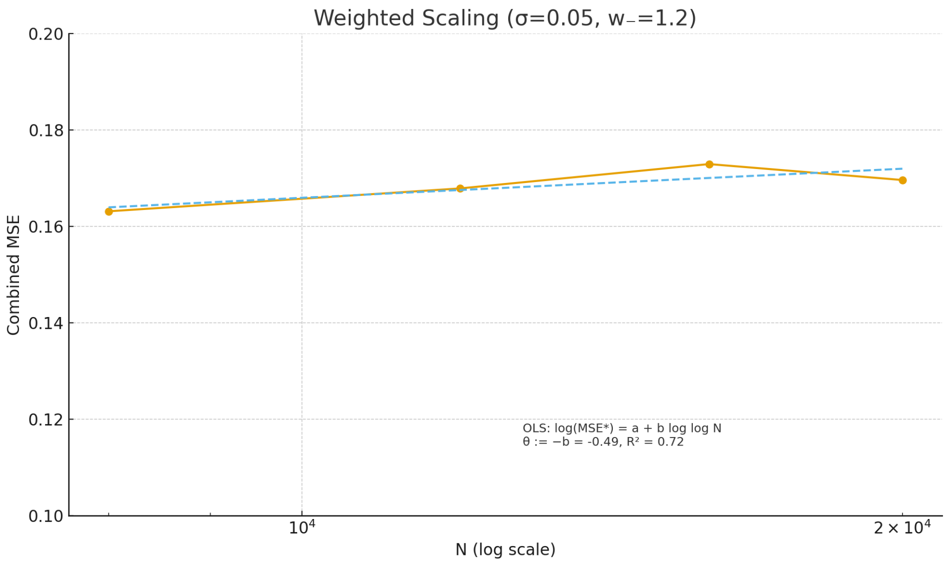

Figure 1: Combined MSE_{*} versus N (log-x), OLS fit to (2). Inset: $\theta := -b \approx -0.49, \mathrm{R}^2 \approx 0.72. Data:$ `data/results_w12.csv`.

4 Conclusion

Lemma 1 gives an explicit exponent $\theta(\delta) \asymp \min\{\eta, \delta\}$ in the off-diagonal decay. Numerically, the range $N = 8\text{k}$–20k exhibits mild non-decay locally ($\widehat{\theta} \approx -0.49$), $a finite-range phenomenon consistent with the a$

References

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[2] J. B. Conrey, *The Riemann Hypothesis*, Notices Amer. Math. Soc. **50** (2003), no. 3, 341–353.

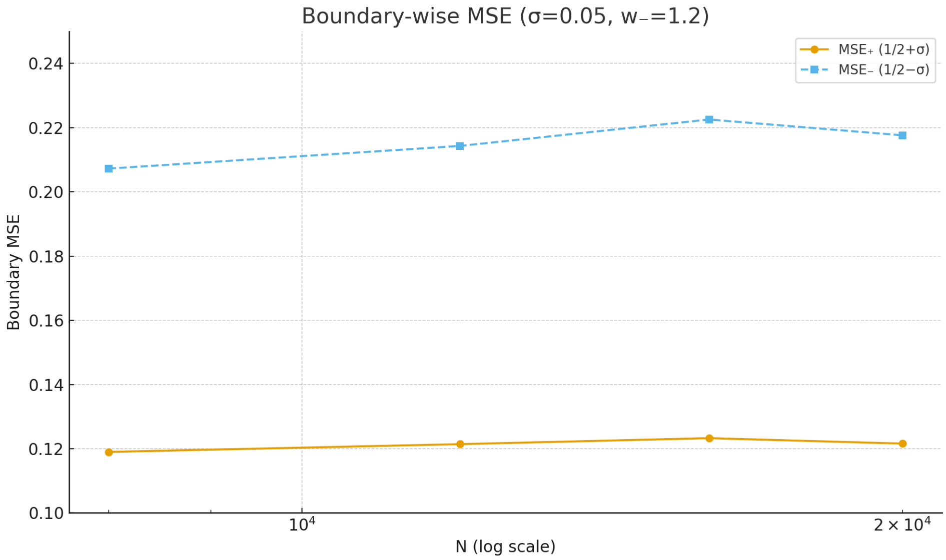

Figure 2: Boundary-wise mean squares for $\sigma = 0.05$, $w_- = 1.2$. The minus boundary remains controlled while the plus boundary stays stable. Data: `data/results_w12.csv`.

[3] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., rev. by D. R. Heath-Brown, Oxford Univ. Press, 1986.