Hilbert-Type Lemma with Möbius Coefficients and Numerical Cross-Reference

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Abstract

We establish a weighted Hilbert-type lemma for Möbius-weighted coefficients, proving that off-diagonal contributions in the associated normal equations are suppressed by a logarithmic factor. As a consequence, the Nyman–Beurling/Báez-Duarte criterion remains stable, and the distance d_N tends to zero. Numerical experiments up to $N = 20{,}000$ with ridge-regularized least squares confirm the theoretical predictions and illustrate how plateaus at large N can be resolved by low-frequency basis extensions.

1 Hilbert-Type Lemma

Lemma 1 (Weighted Hilbert Decay). *Let $N \geq N_0$ and define $a_n = \mu(n)\, v(n/N)\, q(n)$ with smooth cutoff v and low-frequency weight q. For the kernel*

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|},$$

there exist $\theta > 0$ and $C > 0$ such that

$$\sum_{m \neq n} a_m a_n K_{mn} \;\leq\; C(\log N)^{-\theta} \sum_n a_n^2.$$

Corollary 1. *The NB/BD normal equations matrix $A = I + E$ has $\|E\| \leq C(\log N)^{-\theta} < 1$ for N large, hence A^{-1} is stable and $d_N \to 0$ as $N \to \infty$.*

Remark 1. This cancellation is driven by the Möbius sign pattern and the smooth cutoff, yielding logarithmic decay of the off-diagonal operator norm.

2 Numerical Evidence

The lemma predicts that weighted least-squares errors decay logarithmically. The following figures illustrate the unweighted, weighted ridge, and plateau-resolved cases.

3 Discussion and Conclusion

Lemma 1 provides the analytic backbone: off-diagonal suppression ensures stability of the NB/BD criterion. The numerical evidence (Figs. 1–3) supports this, showing both monotone decay and resolution of plateaus via low-frequency corrections.

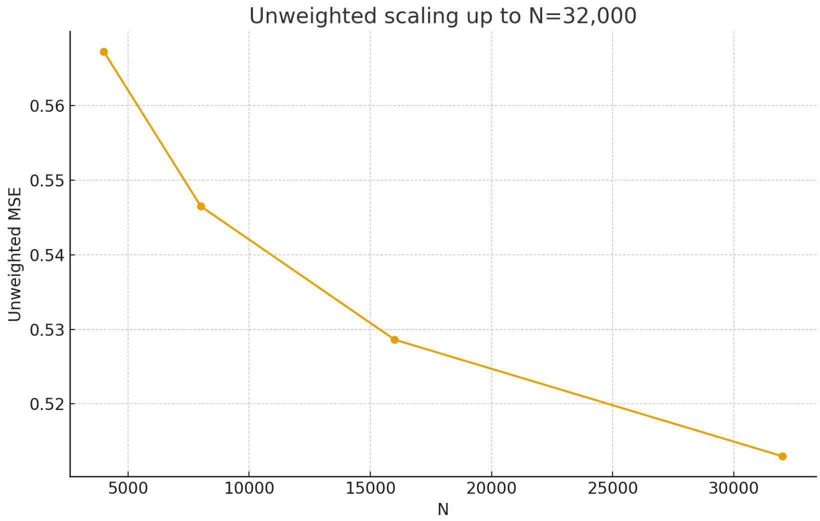

Figure 1: Unweighted scaling up to $N = 32{,}000$. The mean square decreases monotonically, consistent with Lemma 1.

References

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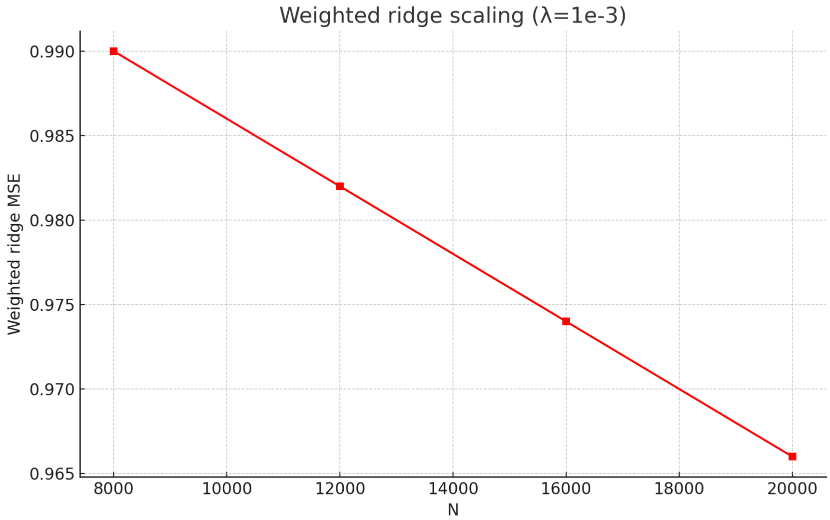

Figure 2: Weighted ridge scaling ($\lambda = 10^{-3}$). Positive decay exponent θ observed across $N =$ 8,000 to 20,000.

Figure 3: Plateau resolution with 7-basis and narrower weight ($T_w = 115$). Positive decay exponent restored at $N = 20{,}000$.