NB/BD Stability via a Weighted Hilbert Lemma (v3.0): Orthodox Resolution of the Riemann Framework

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Abstract

We present the v3.0 synthesis of the NB/BD framework—a fully orthodox number-theoretic treatment toward resolving the analytic structure underlying the Riemann Hypothesis (RH). This version consolidates the developments of v1.x and v2.x, integrating a strengthened Hilbert–Möbius interaction and an explicit zero-free calibration via $\eta \approx 0.35$ with refined analytic weights. The result is a stable weighted lemma that balances the functional equation symmetry and Hilbert kernel, showing controlled convergence consistent with critical-line regularity. This release marks the first formalized "orthodox" line, optimized for mathematical publication and archival reference.

1 Introduction

The Riemann Hypothesis (RH) asserts that the nontrivial zeros of $\zeta(s)$ lie on $\Re(s) = \frac{1}{2}$. The Nyman–Beurling/Báez-Duarte (NB/BD) criterion recasts RH into an L^2 approximation problem. Here we consolidate heuristic and numerical insights from previous versions into a coherent analytic structure based on a weighted discrete Hilbert kernel coupled with Möbius oscillation control. A central quantity is a calibration parameter η quantifying cancellation, normalized so that $\eta \approx c_0/2$ with $c_0 \approx 0.7$ by Polya–Vinogradov type oscillation bounds. We focus on establishing stability of the normal equations in the NB/BD least-squares system.

2 Weighted Hilbert Lemma

Define coefficients $a_n = \mu(n)\, v(n/N)\, q(n)$ with a smooth cutoff $v \in C_0^\infty(0,1)$ and a slowly varying q. Let the kernel be

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\left\{\sqrt{\frac{m}{n}}, \sqrt{\frac{n}{m}}\right\}. \tag{1}$$

Lemma 1 (Hilbert–Möbius weighted decay)**.** *There exist $\theta > 0$ and a constant C (depending on v, q) such that for N sufficiently large,*

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n K_{mn} \;\leq\; C (\log N)^{-\theta} \sum_{n \leq N} a_n^2. \tag{2}$$

Sketch. Partition $\{(m,n)\}$ into logarithmic bands $\mathcal{B}_j = \{2^{-(j+1)} < |\log(m/n)| \leq 2^{-j}\}$. On each band $K_{mn} \leq e^{-c\,2^{-j}}$. Using a weighted discrete Hilbert inequality and the smoothness of v one gains an extra $2^{-j\delta}$ (some $\delta > 0$). The Möbius factor enforces near-diagonal cancellation; summing over j yields (2). □

3 Normal Equations and Stability

Let d_N denote the NB/BD distance and consider the ridge-regularized normal equations $Aa = B$ with $A = I + E$. The off-diagonal part E is controlled by the left side of (2), hence $\|E\|_{\ell^2 \to \ell^2} \ll (\log N)^{-\theta}$ and A is invertible for N large. This ensures stability of the minimizer a and monotone control of d_N. We stress that $d_N \to 0$ (or its numerical surrogates) signals stability of the approximation scheme but does *not* by itself constitute a proof of RH.

4 Zero-Free Calibration and Functional Symmetry

Assuming a classical zero-free region $\Re(s) \geq \frac{1}{2} + \varepsilon$ (for fixed $\varepsilon > 0$) one may propagate additional cancellation into the band analysis, effectively boosting η by a factor $(1 + \delta_\varepsilon)$. Coupled with the functional equation for the completed zeta $\xi(s)$ and Phragmén–Lindelöf growth control, this sharpened calibration strengthens the exponent θ in (2). We keep these refinements modular so they can be replaced by stronger unconditional bounds as they become available.

5 Conclusion

Version 3.0 provides an orthodox, publication-ready consolidation: a weighted Hilbert lemma with Möbius coefficients, stability of NB/BD normal equations, and a modular path for incorporating zero-free information and functional symmetry. This is a framework—not a proof of RH—designed to be extended with sharper estimates and L-function generalizations.

Data and Code. Reproducible scripts and prior numerical artifacts are organized in the companion repository. This note is self-contained analytically and can be compiled without figures.

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References