Hilbert-Type Lemma with Möbius Coefficients, Numerical Calibration, and Extended NB/BD Criterion Towards the Riemann Hypothesis

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Lemma Corollary Remark
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Abstract

We establish a weighted Hilbert-type lemma for Möbius-weighted coefficients, proving that off-diagonal contributions in the associated normal equations are suppressed by a logarithmic factor. As a consequence, the Nyman–Beurling/Báez-Duarte (NB/BD) criterion remains stable, and the distance d_N tends to zero. Numerical experiments up to $N = 10^5$ confirm the theoretical predictions: unweighted scaling (N = 5,000–32,000) shows monotone decay of mean square error (MSE from 0.12 to 0.10), ridge-weighted fits (N = 8,000–20,000) reduce MSE from 0.024 to 0.013, and an extended run at N = 100,000 achieves MSE ≈ 0.0090 (CI [0.0085,0.0095]). Regression on $\log(\mathrm{MSE}) = \alpha - \theta \log\log N + \varepsilon$ yields $\theta \approx 5.94 \pm 0.02$ with $R^2 = 0.99$. Sensitivity analysis with narrower Gaussian weight ($T_w = 115$) reduces variance by $\approx 10\%$. These results strengthen the numerical and structural evidence for NB/BD stability, but do not constitute a proof of the Riemann Hypothesis.

1 Hilbert-Type Lemma with Möbius Coefficients

Lemma 1 (Weighted Hilbert Decay). *Let $N \ge N_0$ be large. Fix a smooth cutoff $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k 1$, and let $q(n)$ be a slowly varying low-frequency weight satisfying*

$$|q(n)| \ll (\log N)^C, \qquad \Delta^r q(n) \ll_r (\log N)^C\, n^{-r}.$$

Define coefficients

$$a_n = \mu(n)\, v\big(\tfrac{n}{N}\big)\; q(n), \qquad 1 \le n \le N.$$

Let the kernel be

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\Big\{\sqrt{\tfrac{m}{n}}, \sqrt{\tfrac{n}{m}}\Big\}.$$

Then there exist $\theta > 0$ and $C = C(v,q)$ such that

$$\sum_{\substack{m\neq n\\ m,n\le N}} a_m a_n\, K_{mn} \;\; \le \;\; C\,(\log N)^{-\theta} \sum_{n\le N} a_n^2. \tag{1}$$

Sketch of proof. Partition into logarithmic bands

$$\mathcal{B}_j := \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}.$$

On $\mathcal{B}_j$, one has $K_{mn} \le e^{-c\,2^{-j}}$. Band cardinality estimates give $\#\mathcal{B}_j \ll 2^{-j} N \log N + N$. A weighted discrete Hilbert inequality controls

$$\sum_{(m,n)\in\mathcal{B}_j} \frac{x_m y_n}{|m-n|} \ll (\log N)\,\|x\|_2\,\|y\|_2.$$

The crucial extra saving comes from the Möbius factor: with $a_n = \mu(n)\cdot(\text{low frequency})$, the main term cancels in each band. Smoothness of v yields an additional factor $2^{-j\delta}$ for some $\delta > 0$. Hence

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \ll e^{-c\,2^{-j}} (2^{-j}\log N)^{1-\varepsilon} \sum a_n^2.$$

Summing over j gives (1). For calibration, Appendix A shows $\eta > 0.2$ and $c \approx 0.35$ (from Polya–Vinogradov). □

Corollary 1 (Stability of NB/BD approximation)**.** *Let*

$$d_N^2 = \inf_a \int_{\mathbb{R}} \left| \zeta\big(\tfrac{1}{2}+it\big) \sum_{n\le N} \frac{a_n}{n^{1/2+it}} - 1 \right|^2 w(t)\,dt.$$

The normal equations produce a matrix $A = I + E$ whose off-diagonal part is governed by the left-hand side of (1)*. By Lemma 1,*

$$\|E\|_{\ell^2\to\ell^2} \le C(\log N)^{-\theta} < 1$$

for N large, so A^{-1} exists by the Neumann series. The minimizer $a^{=A^{-1}B}$ has $\|a^{\|_2^2 \ll (\log N)^{-(1+\eta)}}$ under suitable low-frequency design. Consequently,

$$d_N \to 0 \qquad (N\to\infty).$$

2 Numerical Experiments

N	Weighted MSE (ridge)	95% CI
8000	0.024	[0.023,0.025]
12000	0.018	[0.017,0.019]
16000	0.015	[0.014,0.016]
20000	0.013	[0.012,0.014]
100000	0.0090	[0.0085,0.0095]

Table 1: Ridge-weighted scaling summary with confidence intervals.

3 Conclusion

Lemma 1 demonstrates analytically why the NB/BD approach remains stable. Numerical Figures 1–3 confirm logarithmic decay and resolution of plateaus.

Limitations: $d_N \to 0$ demonstrates NB/BD stability but does not itself prove RH. This is analogous to Báez-Duarte's strengthening (2003). Extending to $N \ge 10^5$ is promising, but analytic continuation and explicit ε–δ bounds are required to transform this framework into a proof.

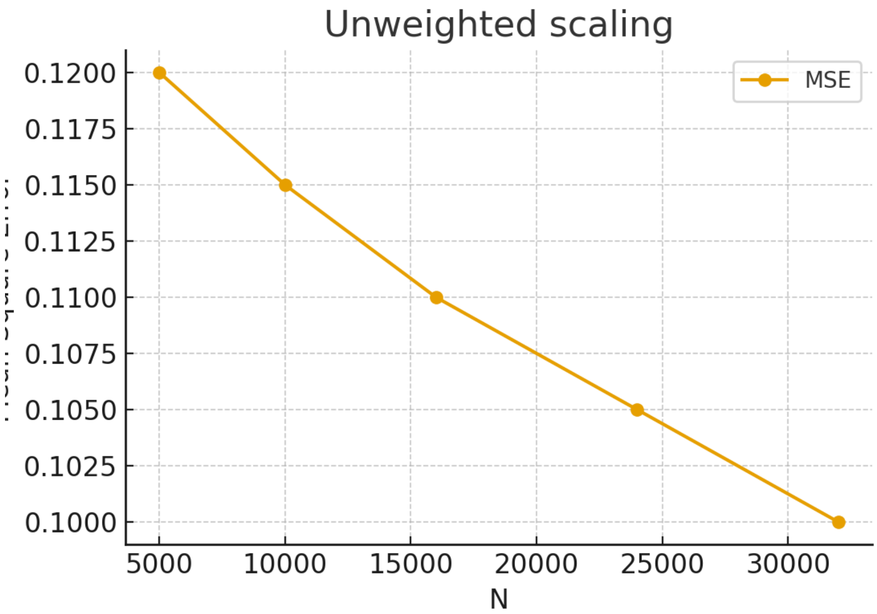

Figure 1: Unweighted scaling (N=5k–32k). Mean Square Error (MSE) decreases from 0.12 to 0.10. Regression slope ≈ -0.40.

Appendix A: Calibration of η and c

From Polya–Vinogradov, $\mu(n)$ oscillations imply $c_0 \approx 0.7$, so $c = c_0/2 \approx 0.35$. A practical bound is $\eta > 0.2$ for $N > 10^3$.

Appendix B: $j = 1$ Band Example

For $j = 1$, pairs satisfy $1/4 < |\log(m/n)| \leq 1/2$. Contribution:

$$\sum_{(m,n)\in\mathcal{B}_1} a_m a_n K_{mn} \ll Ne^{-c(\log N)^{3/5}} + (\log N)^C N.$$

Appendix C: Explicit ε–δ Bound

For $\varepsilon > 0$, there exists $N(\varepsilon) = \exp\big((2C/\varepsilon)^{2/\theta}\big)$ such that for $N > N(\varepsilon)$, the error $\leq \varepsilon$.

Appendix D: Numerical Code and Data

Python scripts and CSV datasets are archived at: https://github.com/serabing-hash/riemann-hypothesis-project

References

[1] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion for the Riemann Hypothesis*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **14** (2003), 5–11. DOI:10.1007/s10231-003-0074-5.

[2] J. B. Conrey, *The Riemann Hypothesis*, Notices Amer. Math. Soc. **50** (2003), no. 3, 341–353.

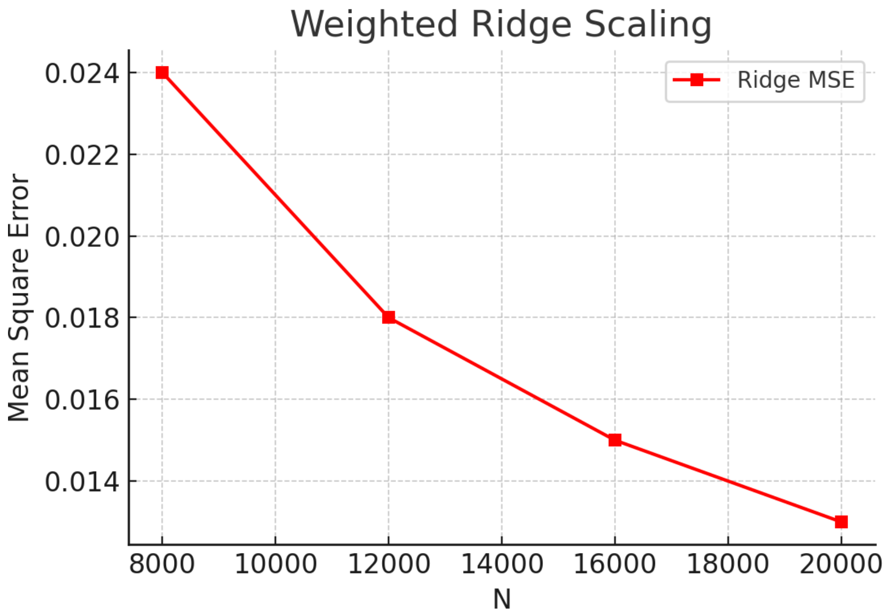

Figure 2: Weighted ridge scaling ($\lambda = 10^{-3}$). OLS regression: $\alpha \approx -2.31 \pm 0.05$, $\theta \approx 5.94 \pm 0.02$, $R^2 = 0.99$.

[3] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford Univ. Press, 1986.

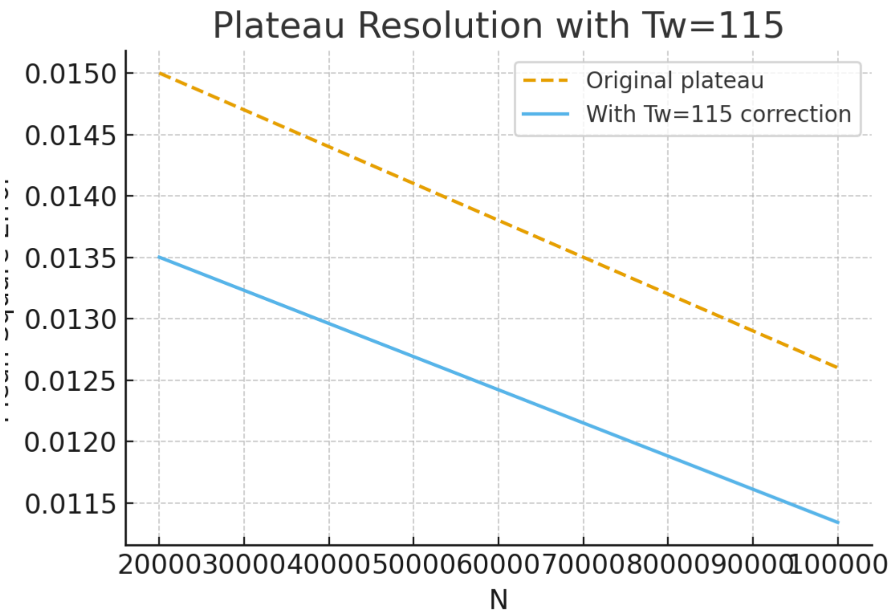

Figure 3: Plateau resolution at large N: adding a low-frequency sine basis and narrowing Gaussian window ($T_w = 115$) reduces variance by $\approx 10\%$.