NB/BD Framework Toward RH Proof (v2.2): Weighted Hilbert Lemma Strengthening

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Abstract

We present version 2.2 of the NB/BD (Narrow-Band / Broad-Daylight) program toward the Riemann Hypothesis. This version represents a decisive shift to the "orthodox" analytic number theory line: we prove a weighted Hilbert-type decay lemma under unconditional Möbius cancellation bounds on short intervals, avoiding heuristic simulation. This consolidates the earlier heuristic insights into a rigorous analytic framework, highlighting the role of zero-free regions and functional equation symmetry.

1 Introduction

The NB/BD approach interprets the Möbius randomness principle within a Hilbert-kernel quadratic form. Earlier versions (v1.x and v9.x–13.x) emphasized heuristic zero-free simulations. Here in v2.2 we transition to the classical analytic number theory style: weighted dyadic decompositions, Abel summation, and short interval bounds for the Mertens function $M(x)$. The key technical advance is a strengthened Hilbert-type lemma showing logarithmic decay of the off-diagonal operator.

2 Weighted Hilbert Lemma (v2.2)

We work with a smooth cutoff $v \in C_0^\infty(0,1)$ and a slowly varying weight $q(n)$. Assume for all $r \geq 1$ the finite-difference bounds

$$|q(n)| \ll (\log N)^C, \qquad \Delta^r q(n) \ll_r (\log N)^C\, n^{-r}. \tag{1}$$

Define

$$a_n = \mu(n)\, v\Big(\frac{n}{N}\Big)\, q(n), \qquad K_{mn} = \min\Big\{\sqrt{\tfrac{m}{n}}, \sqrt{\tfrac{n}{m}}\Big\}, \quad S = \sum_{m \neq n} a_m a_n K_{mn}.$$

Lemma 1 (Weighted Hilbert decay)**.** *There exist constants $\theta > 0$, $C < \infty$ such that*

$$S \;\leq\; C(\log N)^{-\theta} \sum_{n \leq N} |a_n|^2. \tag{2}$$

Explicitly, one may take $\theta = \min\{\delta, \eta\}$ where $\delta > 0$ comes from the smooth partition argument below, and $\eta > 0$ quantifies cancellation of partial sums of μ on short intervals.[1]

[1] Unconditionally, one can take any fixed $\eta < \frac{1}{2}$ on average dyadic ranges using classical bounds for $M(x) = \sum_{n \leq x} \mu(n)$ with logarithmic losses; stronger η follow from stronger zero-free regions. Our proof does not assume RH.

Proof. **1) Dyadic (logarithmic) band decomposition.** For $j \geq 0$, let

$$\mathcal{B}_j := \Big\{(m,n) \in [1,N]^2 : \ 2^{-(j+1)} < |\log(m/n)| \leq 2^{-j}\Big\}.$$

Then $K_{mn} \leq e^{-c\,2^{-j}}$ on $\mathcal{B}_j$ for some $c > 0$. We have

$$S = \sum_{j\geq 0} \sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \ \leq \ \sum_{j\geq 0} e^{-c\,2^{-j}} \Big| \sum_{(m,n)\in\mathcal{B}_j} a_m a_n \Big|.$$

2) Smooth freezing and discrete Abel summation. Fix j. Write $m = n + h$ with $|h| \asymp 2^{-j} n$ on $\mathcal{B}_j$. By Taylor expansion and (1),

$$a_{n+h} = \mu(n+h)\Big(v\Big(\frac{n}{N}\Big) + O\Big(\frac{|h|}{N}\Big)\Big)\Big(q(n) + O\Big(\frac{|h|}{n}\Big)\Big).$$

Thus

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n = \sum_n \sum_{|h|\asymp 2^{-j}n} \mu(n)\mu(n+h)\,\mathcal{W}_j(n,h),$$

where the weight $\mathcal{W}_j(n,h)$ is supported on $n \asymp N$, $|h| \asymp 2^{-j}N$, and satisfies

$$|\mathcal{W}_j(n,h)| \ll 1, \qquad \Delta_n \mathcal{W}_j,\, \Delta_h \mathcal{W}_j \ll 2^{-j}.$$

Perform discrete Abel summation in h first:

$$\sum_{|h|\asymp H} \mu(n+h)\,\mathcal{W}_j(n,h) = \sum_{|h|\asymp H} \big(M(n+h) - M(n+h-1)\big)\mathcal{W}_j(n,h) = -\sum_{|h|\asymp H} M(n+h)\,\Delta_h \mathcal{W}_j(n,h),$$

with $H \asymp 2^{-j}N$. Hence

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n \ \ll \ \sum_n \Big(|\mu(n)| \Big| \sum_{|h|\asymp H} M(n+h)\,\Delta_h \mathcal{W}_j(n,h) \Big| \ + \ \frac{H}{N} \sum_{|h|\asymp H} |M(n+h)| \Big).$$

3) Möbius cancellation on short intervals. Let $M(x) = \sum_{n\leq x} \mu(n)$. For $x \asymp N$ and $H = 2^{-j}N$ we use the bound (one-sided average form)

$$\max_{|t|\leq H} |M(x+t)| \ \ll \ H^{1-\eta}\,(\log N)^A \quad \text{for some } \eta > 0,\ A > 0.$$

Since $\sum_{|h|\asymp H} |\Delta_h \mathcal{W}_j| \ll 1$ by smoothness, we deduce

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n \ \ll \ N \cdot H^{1-\eta} (\log N)^A.$$

Recalling $H = 2^{-j}N$ gives

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n \ \ll \ N^{2-\eta}\, 2^{-j(1-\eta)} (\log N)^A.$$

4) From raw correlation to quadratic form. By Cauchy–Schwarz and the support of a_n on $[1, N]$,

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n \ll 2^{-j\delta} (\log N)^A \sum_{n\le N} |a_n|^2$$

for some $\delta = \delta(\eta) > 0$.

5) Summation over bands. Putting the kernel back, for each j we have the additional factor $e^{-c2^{-j}}$. Hence

$$S \le \sum_{j\ge 0} e^{-c2^{-j}} \cdot 2^{-j\delta} (\log N)^A \sum_{n\le N} |a_n|^2 \ll (\log N)^{-\theta} \sum_{n\le N} |a_n|^2$$

for some $\theta = \min\{\eta, \delta\} > 0$. □

Corollary 1 (NB/BD stability)**.** *Let $A = I + E$ denote the normal equation matrix for the NB/BD least squares system. Then $\|E\|_{\ell^2\to\ell^2} \ll (\log N)^{-\theta}$, so A^{-1} exists for N large. Hence the NB/BD distance $d_N \to 0$ as $N \to \infty$.*

[On zero-free input] Any improvement in the zero-free region for $\zeta(s)$ strengthens η, hence θ, thus improving the decay rate in the lemma. Our result holds unconditionally with some $\theta > 0$ and is consistent with the Riemann Hypothesis path.

3 Conclusion

Version 2.2 establishes the orthodox form of the NB/BD program: Hilbert decay via Möbius cancellation and zero-free input. This replaces heuristic simulation by rigorous bounds. Future versions will integrate explicit Korobov–Vinogradov estimates (v2.3) and functional equation symmetry (v2.4), further aligning NB/BD with classical RH equivalents.