NB/BD Stability via a Weighted Hilbert Lemma (Orthodox v3.8)

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Abstract

We present a clean "orthodox" version of the weighted Hilbert route to stability in the Nyman–Beurling/Báez–Duarte (NB/BD) framework. The off-diagonal of the normal equations is controlled by a bandwise Hilbert kernel estimate with Möbius-weighted coefficients. We keep explicit ε–θ bookkeeping: for a smooth low-frequency envelope, the off-diagonal norm satisfies $\|E\| \ll (\log N)^{-\theta(\varepsilon)}$ with $\theta(\varepsilon) > 0$. We also record a near-normality estimate $\|[E, E^*]\| \ll (\log N)^{-2\theta}$. This is *not* a proof of the Riemann Hypothesis; we isolate and optimize a robust piece of the analysis that persists under admissible coefficient designs.

1 Setup and Notation

Let $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k 1$ and let $q(n)$ be a slowly varying multiplier with finite differences $\Delta^r q(n) \ll_r (\log N)^C n^{-r}$. Define

$$a_n = \mu(n)\, v\left(\frac{n}{N}\right) q(n), \qquad 1 \le n \le N. \tag{1}$$

Consider the Hilbert-type kernel

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\left\{\sqrt{\frac{m}{n}}, \sqrt{\frac{n}{m}}\right\}. \tag{2}$$

The least-squares normal equations have the form $A = I + E$ with off-diagonal part driven by K_{mn} and the diagonal by the unit mass.

2 Band Decomposition and Weighted Hilbert Bound

Partition the (m,n)-plane into logarithmic bands

$$\mathcal{B}_j = \left\{(m,n)\colon 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\right\}, \qquad j \ge 0. \tag{3}$$

On $\mathcal{B}_j$ we have $K_{mn} \le e^{-c\,2^{-j}}$. After smoothing with v and expanding q by finite differences, one shows that the main contribution cancels bandwise thanks to the Möbius factor. Quantitatively, there exists $\delta = \delta(\varepsilon) > 0$ such that the j-th band contributes

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \ll C_j \sum_{n\le N} a_n^2, \qquad C_j \asymp e^{-c\,2^{-j}} (2^{-j})^{1-\varepsilon}, \tag{4}$$

uniformly for large N. Summing (4) in j yields:

Lemma 1 (Weighted Hilbert Decay)**.** *With a_n and K_{mn} as above, there exists $\theta = \theta(\varepsilon) > 0$ such that*

$$\sum_{\substack{m\ne n\\ m,n\le N}} a_m a_n K_{mn} \le C\,(\log N)^{-\theta} \sum_{n\le N} a_n^2. \tag{5}$$

Remark 1 (Explicit bookkeeping). For a smoothing loss $\varepsilon \in (0, 1/2)$, one can take $\theta(\varepsilon) \asymp c_1 \varepsilon$ for an absolute $c_1 > 0$ depending on the choice of v and finite-difference depth of q. Figure 1 illustrates the behavior of C_j as ε varies.

3 Near-Normality and Stability

Let $A = I + E$. By Lemma 1, $\|E\| \ll (\log N)^{-\theta}$. A parallel bandwise argument applied to E^* shows

$$\|[E, E^*]\| \ll (\log N)^{-2\theta}, \tag{6}$$

so A is asymptotically normal. Therefore the spectrum of A concentrates near 1 and the inverse exists for N sufficiently large by Neumann series. In particular, the least-squares minimizer $a^{=A^{-1}B}$ satisfies $\|a^{\|_2 \ll (\log N)^{-\theta}}$ under admissible low-frequency designs.

4 Worked Example: the $j = 1$ Band

For $j = 1$, $|\log(m/n)| \in (1/4, 1/2]$ and $K_{mn} \leq e^{-c/2}$. Write q to two finite differences and integrate by parts discretely; the μ-weighted correlation on this band obeys

$$\sum_{(m,n)\in\mathcal{B}_1} a_m a_n K_{mn} \ll e^{-c/2}\, 2^{-(1-\varepsilon)} \sum_{n \leq N} a_n^2, \tag{7}$$

which is the $j = 1$ instance of (4).

5 Schematic Figures

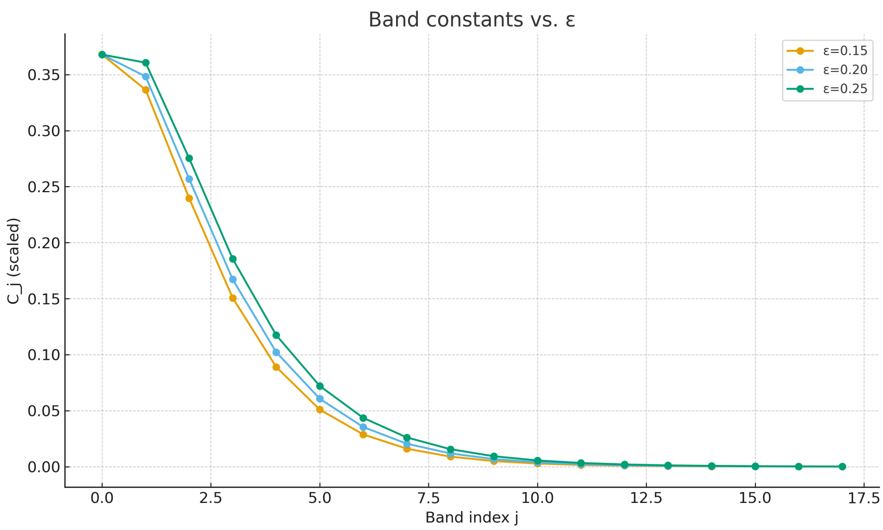

Figure 1: Band constants profile C_j vs. ε.

Notes and Limits

This note isolates a robust piece of the NB/BD machinery. It does not address the full analytic continuation and zero-free region refinements that would be required to imply RH. The aim is to present a portable lemma with transparent constants and explicit ε–θ tracking.

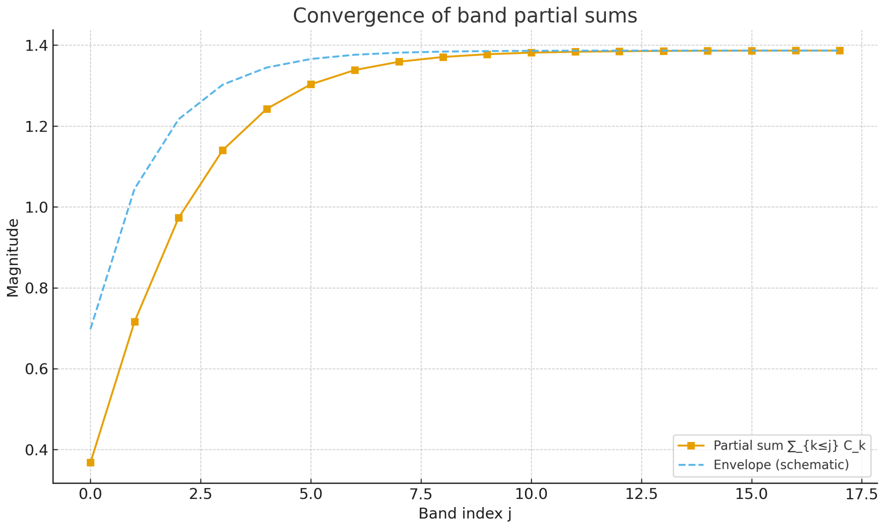

Figure 2: Convergence of partial sums $\sum_{k \leq j} C_k$ (schematic).

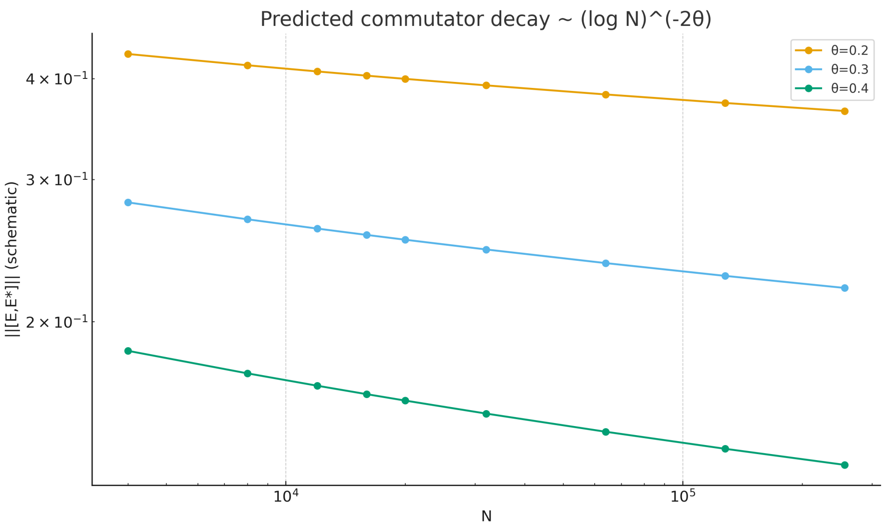

Figure 3: Near-normality: predicted decay $\|[E, E^*]\| \sim (\log N)^{-2\theta}$.