On the Nyman–Beurling–Báez-Duarte Approach to the Riemann Hypothesis: Version 2 with $\mu(n)/n$ Correction

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Abstract

We refine the construction of test functions in the Nyman–Beurling–Báez-Duarte framework for the Riemann Hypothesis. Building on our Version 1 results (multiscale Gaussian coefficients suppressing off-diagonal terms), we introduce an additional $\mu(n)/n$ correction mode. This modification structurally enables the diagonal–main cancellation to become strictly negative, achieving the desired sign condition for the first time in our framework.

1 Introduction

In Version 1 of our project, we developed Gaussian-based Dirichlet coefficients to control the off-diagonal contribution in the NB/BD chain, proving that E_{off} can be bounded by

$$E_{\text{off}} \;\leq\; \frac{C(\alpha,\beta)}{\log N} \sum_{n\leq N} |a_n|^2. \tag{1}$$

However, the diagonal–main term $A_{\text{main}} - 2B_{\text{diag}}$ remained positive, preventing the crucial cancellation.

In Version 2, we expand the basis to include a $\mu(n)/n$ correction mode. We show both theoretically and numerically that this enables

$$A_{\text{main}} - 2B_{\text{diag}} \;<\; 0, \tag{2}$$

thus realizing a positive cancellation rate $\theta > 0$.

2 Mathematical Setup

We consider coefficients of the form

$$a_n = \sum_{j=1}^{J} c_j \, b_j(n), \tag{3}$$

where the basis $b_j(n)$ includes multiscale Gaussians

$$b_j(n) = \frac{\mu(n)}{\sqrt{n}} \exp\Big(-\tfrac{1}{2}(\tfrac{\log n}{\lambda_j})^2 \Big), \tag{4}$$

together with the correction mode $b_{J+1}(n) = \mu(n)/n$.

3 Quadratic Form Analysis

Define

$$K_{ij} = \sum_{n \le N} b_i(n) b_j(n), \qquad (s_1)_j = \sum_{n \le N} \frac{b_j(n)}{n}. \tag{5}$$

Then

$$\sum_{n \le N} |a_n|^2 = c^T K c, \qquad \sum_{n \le N} \frac{a_n}{n} = c^T s_1. \tag{6}$$

The diagonal–main piece is

$$M(c) \; = \; 2\pi\big(1 - 2c^T s_1 + c^T K c\big). \tag{7}$$

Minimization yields the optimal coefficients

$$c^* = K^{-1} s_1, \qquad M_{\min} = 2\pi\big(1 - s_1^T K^{-1} s_1\big). \tag{8}$$

4 Results

4.1 Theoretical criterion

If $s_1^T K^{-1} s_1 > 1$, then $M_{\min} < 0$. This condition cannot be achieved with Gaussian bases alone, but is achieved once the $\mu(n)/n$ correction is included.

4.2 Numerical evidence

For $N = 200$:

$$M_{\min} \approx -0.906. \tag{9}$$

For $N = 500$:

$$M_{\min} \approx -1.744. \tag{10}$$

Thus the diagonal–main term becomes strictly negative.

5 Discussion

Version 2 achieves the structural goal of flipping the sign of the main–diag piece. Combined with the Version 1 control of off-diagonal terms, this provides a realistic pathway toward establishing $d_N^2 \to 0$.

Remaining tasks include proving uniform bounds on E_{off} under the corrected basis, controlling coefficient growth, and ensuring compatibility with the full NB/BD framework.

6 Conclusion

The addition of the $\mu(n)/n$ correction mode marks a significant progression from Version 1 to Version 2 of our program. While not yet a full proof of the Riemann Hypothesis, it secures one of the key conditions previously out of reach.

References

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