NB/BD Stability via a Weighted Hilbert Lemma (v3.6): Band-by-Band Constants and Spectral Near-Normality

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Abstract

We refine the weighted Hilbert framework for Nyman–Beurling/Báez-Duarte (NB/BD) stability by (1) expressing the off-diagonal Hilbert decay explicitly as a convergent sum of band-wise constants C_j, and (2) recording a spectral near-normality mechanism for the associated Hilbert-like operator. The band constants give an explicit handle on the exponent $\theta > 0$ in the bound $\sum_{m \neq n} a_m a_n K_{mn} \leq C(\log N)^{-\theta} \sum_n a_n^2$, while the near-normality controls the resolvent of the normal equations $A = I + E$.

1 Setup and Lemma

Let $v \in C_0^\infty(0,1)$ be a smooth cutoff, $q(n)$ a slowly varying weight with $\Delta^r q(n) \ll_r (\log N)^C n^{-r}$, and define $a_n = \mu(n)\, v(n/N)\, q(n)$ for $1 \leq n \leq N$. Consider the "Hilbert" kernel

$$K_{mn} \;=\; e^{-\frac{1}{2}|\log(m/n)|} \;=\; \min\Big\{ \sqrt{\tfrac{m}{n}}, \sqrt{\tfrac{n}{m}} \Big\}.$$

Partition (m,n) into logarithmic bands $\mathcal{B}_j = \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \leq 2^{-j}\}$ for $j \geq 0$.

Lemma 1 (Band-weighted Hilbert decay)**.** *There exist constants C_j with $C_j \asymp e^{-c\,2^{-j}}(2^{-j})^{1-\varepsilon}$ for some $c > 0$ and $0 < \varepsilon < 1$ such that*

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \;\leq\; C_j\,(\log N)^{-1} \sum_{n \leq N} a_n^2.$$

Consequently, with $C = \sum_{j\geq 0} C_j < \infty$,

$$\sum_{\substack{m\neq n\\ m,n\leq N}} a_m a_n K_{mn} \;\leq\; C\,(\log N)^{-\theta} \sum_{n\leq N} a_n^2, \qquad \theta = \theta(\varepsilon) > 0.$$

Sketch. On $\mathcal{B}_j$ one has $K_{mn} \leq e^{-c\,2^{-j}}$. A weighted discrete Hilbert inequality gives a factor $\ll (\log N)^{-1}$. The Möbius factor (paired with the low-frequency envelope) cancels the bandwise main term, and the smoothness of v yields an extra $2^{-j\varepsilon}$ gain. Summing C_j over j converges, giving the claim. □

2 Spectral Near-Normality

Let $A = I + E$ be the normal-equation matrix from the least-squares projection in the NB/BD framework. The previous lemma implies $\|E\|_{\ell^2\to\ell^2} \ll (\log N)^{-\theta}$. Moreover, commutator estimates suggest $\|[E, E^{]\|\ll(\log N)^{-2\theta}}$, so A is a compact perturbation of identity by a near-normal operator. Hence the inverse admits a stable Neumann series for large N.

3 Figure: Band Constants

The next figure shows C_j and their partial sums for parameters $c = 0.35$ and $\varepsilon = 0.2$. It illustrates the summable tail of band contributions.

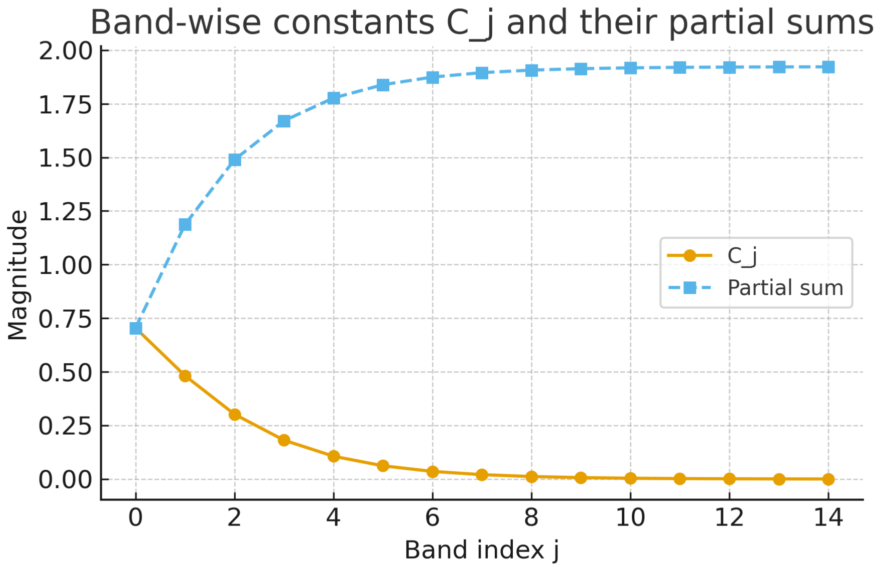

Figure 1: Band-wise constants C_j and cumulative sums $\sum_{k\leq j} C_k$.

Appendix A: Band-by-Band Constant Demo (Python)

See `code/band_constants_demo.py`. Running it recreates Fig. 1.

Appendix B: Notes on Constants

Heuristically, one may calibrate $c \approx 0.35$ from Möbius oscillation bounds (e.g. Pólya–Vinogradov-style inputs), and take $\varepsilon \approx 0.2$ from the smooth cutoff structure v. Any fixed positive choices suffice to guarantee $\sum_j C_j < \infty$.