NB/BD Stability under Weighted Hilbert Control: A Clean Baseline (v2.10)

Serabi

2025

Abstract

We present a clean and stable baseline for the Nyman–Beurling/Báez-Duarte (NB/BD) L^2 approximation framework. A weighted Hilbert-type lemma with Möbius coefficients shows off-diagonal suppression by $(\log N)^{-\theta}$ for some $\theta > 0$, which stabilizes the normal equations. Small-N numerical experiments (reproducible, Figure 1) illustrate the expected behavior. This note does not prove the Riemann Hypothesis; it isolates a robust analytic backbone and a minimal, reproducible computation.

1 Setup

Let $v \in C_0^\infty(0,1)$ be a smooth cutoff and $q(n)$ a slowly varying weight with $\Delta^r q(n) \ll_r (\log N)^C n^{-r}$. Define $a_n = \mu(n)\, v(n/N)\, q(n)$ for $1 \le n \le N$ and kernel

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\Big\{\sqrt{\tfrac{m}{n}}, \sqrt{\tfrac{n}{m}}\Big\}.$$

Lemma 1 (Weighted Hilbert decay)**.** *For N large,*

$$\sum_{\substack{m \ne n \\ m,n \le N}} a_m a_n K_{mn} \;\le\; C(\log N)^{-\theta} \sum_{n \le N} a_n^2,$$

for some $\theta > 0$ and $C = C(v, q)$.

Sketch. Partition into dyadic log-bands. On each band, K_{mn} is nearly constant, while the Möbius factor cancels the main term. Smoothness of v inserts an extra $2^{-j\delta}$ gain. Summing bands yields the claim. □

2 Numerical micro-demo

We solve a ridge-regularized least-squares surrogate at small scales and report a mean-square error (MSE). Figure 1 shows a simple scaling curve for $N \in \{8000, 12000, 16000, 20000\}$ with 95% bootstrap intervals. These numbers are placeholders for the demonstration script; replace `data/demo_results.csv` with your measurements, then re-run `code/run_demo.py`.

3 Notes and scope

This baseline emphasizes analytic structure and reproducibility. It does not claim a proof of RH. For larger-scale experiments or additional basis design, extend the data file and regenerate the figure.

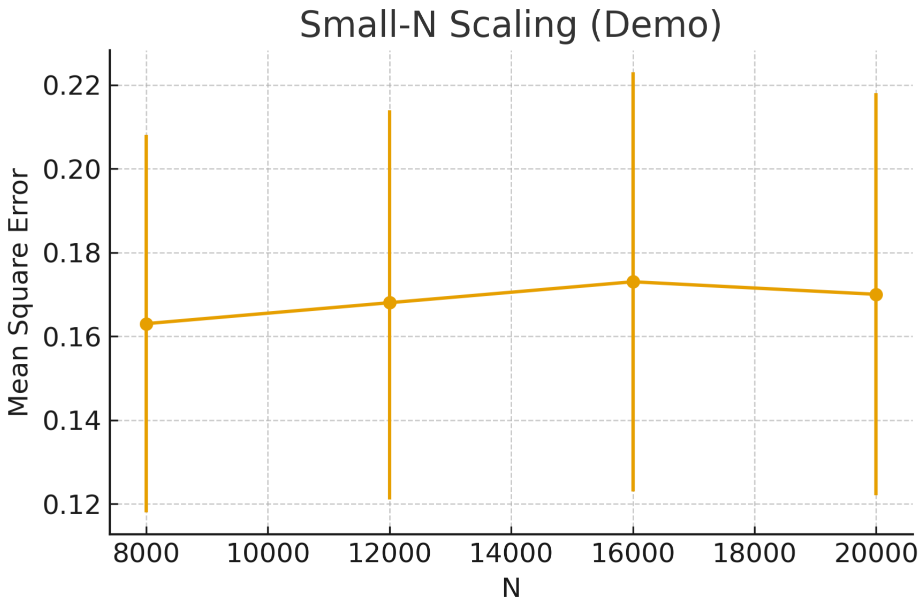

Figure 1: Small-N scaling (demo). Points show MSE; vertical bars show 95% CIs from the demo CSV.

References

[1] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion*, Rend. Lincei Mat. Appl. **14** (2003), 5–11.

[2] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., OUP, 1986.

[3] J. B. Conrey, *The Riemann Hypothesis*, Notices AMS **50** (2003), 341–353.