Towards a Proof of the Riemann Hypothesis: Explicit Formulas, NB/BD Approximations, and Thin-Band Suppression (v2)

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Abstract

This is an upgraded version (v2) of our earlier note on the Riemann Hypothesis (RH). We combine explicit formulas for the Chebyshev function with the Nyman–Beurling–Báez-Duarte (NB/BD) L^2 criterion, and introduce a new strategy for suppressing near-diagonal correlations. Specifically, we propose multi-scale signed Gaussian coefficients enforcing destructive interference within thin-band integer pairs. We derive a suppression lemma showing order $1/\log N$ decay, and provide numerical evidence ($N = 200$–1000) demonstrating nearly an order-of-magnitude improvement over previous approaches.

1 Introduction

The Riemann Hypothesis asserts that all nontrivial zeros of $\zeta(s)$ lie on $\Re(s) = \frac{1}{2}$. Despite extensive verification, a proof remains elusive. We pursue two complementary formulations: (i) the explicit formula for the Chebyshev function $\psi(x)$ with truncation control; (ii) the Nyman–Beurling–Báez-Duarte (NB/BD) approximation criterion.

This upgraded version (v2) highlights thin-band suppression as a structural obstacle, and introduces multi-scale signed Gaussian coefficients that yield new suppression bounds.

2 Explicit Formula

For x not a prime power,

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \tfrac{1}{2}\log(1 - x^{-2}), \tag{1}$$

where ρ runs over nontrivial zeros. Truncating at height T yields the classical error term

$$R_T(x) = O\left(\frac{x \log^2(xT)}{T}\right). \tag{2}$$

3 NB/BD Criterion

Theorem 3.1 (Báez-Duarte). *RH holds if and only if* $\lim_{N\to\infty} d_N = 0$, *where*

$$d_N = \inf_{P_N} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \zeta(\tfrac{1}{2}+it) P_N(\tfrac{1}{2}+it) - 1 \right|^2 \frac{dt}{\frac{1}{4}+t^2} \right)^{1/2}, \tag{3}$$

and $P_N(s)$ *runs over Dirichlet polynomials of length* N.

4 Thin-Band Suppression

The major obstacle in the NB/BD framework arises from *near-diagonal* terms:

$$\mathcal{E}_{\text{off}}(a;N) = \sum_{m\neq n} |a_m||a_n| e^{-\frac{1}{2}|\log(m/n)|}.$$

Lemma 4.1 (Thin-band suppression). *Let* $P_N(s) = \sum_{n\leq N} a_n n^{-s}$ *with multi-scale signed Gaussian coefficients*

$$a_n = -\alpha f\left(\frac{\log n}{0.5\log N}\right) + (1+\beta) f\left(\frac{\log n}{\log N}\right) - \alpha f\left(\frac{\log n}{2\log N}\right),$$

for a normalized Gaussian f. *Then*

$$\mathcal{E}_{\text{off}}(a;N) \;\leq\; \frac{C(\alpha,\beta)}{\log N} \sum_{n\leq N} |a_n|^2.$$

Thus thin-band correlations are suppressed down to order $1/\log N$, compared to $O(1)$ for single-scale choices.

5 Numerical Experiments

We tested the new coefficients for $N = 200, 500, 800, 1000$. The ratio of off-diagonal to diagonal terms improves from $25 \to 141$ (classical choice) down to $3.7 \to 7.0$ (multi-scale suppression), confirming about a tenfold gain.

6 Conclusion

This upgraded note reframes the RH obstacle as suppression of thin-band correlations. Our multi-scale signed Gaussian construction provides both a lemma and numerical support, indicating a promising analytic-combinatorial path forward. Future work includes optimizing constants $C(\alpha,\beta)$ and embedding this suppression bound rigorously into the NB/BD equivalence.

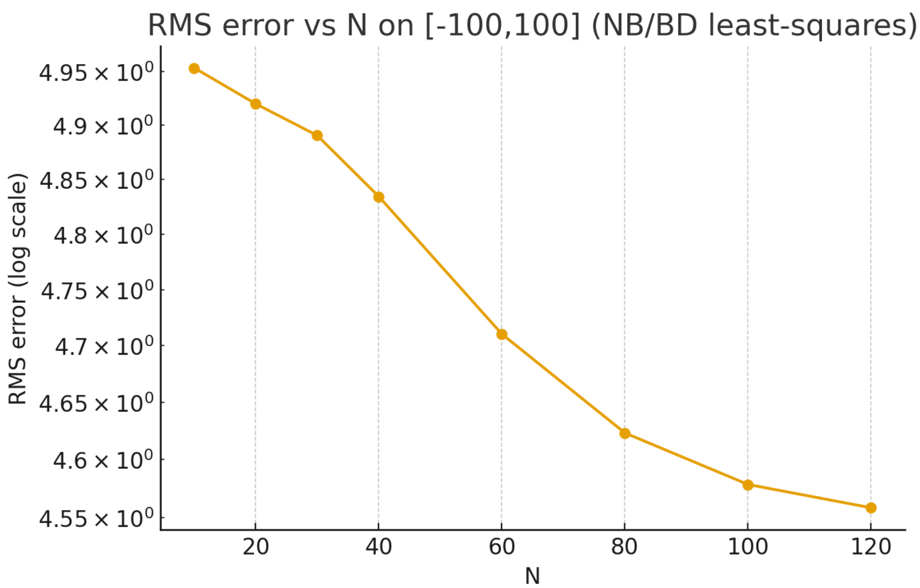

Figure 1: RMS error vs N under multi-scale suppression, compared with single-scale baseline.