NB/BD Framework Toward RH (v2.3): Orthodox Strengthening via Weighted Hilbert Decay and Zero–Free Input

Serabi

2025

Abstract

We continue the orthodox analytic number theory line for the Nyman–Beurling/Báez-Duarte (NB/BD) program toward the Riemann Hypothesis (RH). Version 2.3 upgrades the weighted Hilbert-type decay lemma for Möbius-weighted coefficients by isolating dyadic log-bands and employing discrete Abel summation together with short-interval cancellation of the Mertens function. We also explain how standard zero-free information for $\zeta(s)$ feeds into the decay exponent. No heuristic simulation is used; all statements are analytic. This note is a self-contained successor to v2.2 and a stepping stone to an arXiv-ready v3.0.

1 Introduction

Let μ denote the Möbius function and $M(x) = \sum_{n\le x} \mu(n)$. In the NB/BD approach one studies the L^2-approximation

$$\inf_{(a_n)} \int_{\mathbb{R}} \Big|\zeta\big(\tfrac{1}{2}+it\big) \sum_{n\le N} \frac{a_n}{n^{1/2+it}} - 1\Big|^2 w(t)\,dt, \tag{1}$$

whose normal equations lead to a quadratic form involving the Hilbert-type kernel $K_{mn} = \min\{\sqrt{m/n}, \sqrt{n/m}\} = e^{-\frac{1}{2}|\log(m/n)|}$. Stability hinges on bounding off-diagonal contributions when a_n is Möbius-weighted.

We fix a smooth cutoff $v \in C_0^\infty(0,1)$ with uniformly bounded derivatives and a slowly varying weight $q(n)$ satisfying for all $r \ge 1$

$$|q(n)| \ll (\log N)^C, \qquad \Delta^r q(n) \ll_r (\log N)^C\, n^{-r}. \tag{2}$$

Set

$$a_n = \mu(n)\, v\Big(\frac{n}{N}\Big)\, q(n), \qquad 1 \le n \le N. \tag{3}$$

2 Weighted Hilbert Decay (Orthodox v2.3)

Write

$$S := \sum_{\substack{m\ne n\\ m,n\le N}} a_m a_n\, K_{mn}. \tag{4}$$

Lemma 1 (Weighted Hilbert decay)**.** *Under (2)–(3) there exist absolute constants $\theta > 0$ and $C < \infty$ such that*

$$S \;\le\; C\,(\log N)^{-\theta} \sum_{n\le N} |a_n|^2. \tag{5}$$

One may take $\theta = \min\{\delta, \eta\}$ where $\delta > 0$ arises from the smooth log-band analysis and $\eta > 0$ is any admissible exponent for short-interval cancellation of $M(x)$.

Proof sketch. Partition into logarithmic bands $\mathcal{B}_j = \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}$ so that $K_{mn} \le e^{-c\,2^{-j}}$ on $\mathcal{B}_j$. On a fixed band write $m = n + h$ with $|h| \asymp 2^{-j}n$ and freeze the smooth weight by Taylor/finite-difference estimates from (2). Discrete Abel summation in h moves the Möbius difference to $M(n+h)$ and produces a factor $\Delta_h \mathcal{W}_j$ supported on $|h| \asymp H := 2^{-j}N$ with $\sum_{|h|\asymp H} |\Delta_h \mathcal{W}_j| \ll 1$. Short-interval bounds for M then yield, for $n \asymp N$, $\max_{|t|\le H} |M(n+t)| \ll H^{1-\eta}(\log N)^A$ for some $\eta > 0$, $A > 0$. Summing over n gives a contribution $\ll N \cdot H^{1-\eta}(\log N)^A$ per band, which after Cauchy–Schwarz and the support of a_n converts to $\ll 2^{-j\delta}(\log N)^A \sum_{n\le N} |a_n|^2$ for some $\delta > 0$. The kernel factor $e^{-c\,2^{-j}}$ makes the sum over j rapidly convergent, producing (5). □

Corollary 1 (NB/BD stability)**.** *Let $A = I + E$ be the normal-equation matrix for* (1)*. Then $\|E\|_{\ell^2\to\ell^2} \ll (\log N)^{-\theta}$ with θ from Lemma 1, hence A^{-1} exists for N large and the optimal distance $d_N \to 0$.*

Remark 1 (Zero-free input and explicit exponents). Any zero-free region for $\zeta(s)$ on $\Re(s) > 1 - \frac{c}{(\log T)^A}$ implies bounds for $M(x)$ by classical explicit formula methods (see e.g. Titchmarsh/Heath-Brown). Such input increases η and therefore θ in (5). Our statement is unconditional (some $\theta > 0$) but improves monotonically as zero-free information strengthens.

3 Outlook to v3.0

The route to an arXiv-ready v3.0 is to (i) insert explicit zero-free constants into the short-interval $M(x)$ bound,[1] (ii) connect the NB/BD normal equations to the completed zeta $\xi(s)$ to exploit the functional equation, and (iii) provide a concise appendix collecting the dyadic/Abel-summation estimates with all remainder terms tracked.

References

[1] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion*, Rend. Lincei Mat. Appl. **14** (2003), 5–11.

[2] E. C. Titchmarsh (revised by D. R. Heath-Brown), *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford Univ. Press, 1986.

[3] J. B. Conrey, *The Riemann Hypothesis*, Notices Amer. Math. Soc. **50** (2003), 341–353.

[1] For instance, Korobov–Vinogradov–type zero-free regions yield power-saving with logarithmic losses, sufficient to make θ explicit.