NB/BD Framework Toward RH (v2.4): Orthodox Strengthening via Log–Band/Abel Analysis and Zero–Free Input

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Abstract

We strengthen the orthodox analytic line for the Nyman–Beurling/Báez–Duarte (NB/BD) program toward the Riemann Hypothesis. Compared with v2.3, we provide a more explicit log–band decomposition and discrete Abel summation scheme for Möbius–weighted coefficients in the Hilbert–type kernel. The core bound is proved under a short–interval cancellation hypothesis for the Mertens function $M(x)$, and we explain how classical zero–free regions for $\zeta(s)$ improve the exponent. All statements here are analytic; a small numerical appendix is included only for sanity checks.

1 Setup and Notation

Write $M(x) = \sum_{n \le x} \mu(n)$. Fix a smooth cutoff $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k 1$ and a slowly varying weight $q(n)$ obeying, for all $r \ge 1$,

$$|q(n)| \ll (\log N)^C, \qquad \Delta^r q(n) \ll_r (\log N)^C \, n^{-r}. \tag{1}$$

Define Möbius–weighted coefficients

$$a_n = \mu(n)\, v\Big(\frac{n}{N}\Big)\, q(n), \qquad 1 \le n \le N, \tag{2}$$

and the Hilbert–type kernel

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\Big\{\sqrt{\frac{m}{n}}, \sqrt{\frac{n}{m}}\Big\}. \tag{3}$$

The off–diagonal quadratic form is

$$S(N) := \sum_{\substack{m \ne n \\ m,n \le N}} a_m a_n \, K_{mn}. \tag{4}$$

2 Short–Interval Hypothesis and Main Lemma

We formulate a standard short–interval hypothesis for $M(x)$.

Hypothesis $H_\eta(\beta)$. For some $\eta \in (0,1)$ and $\beta \in (0,1]$ there exist $A, C \ge 0$ such that for all N large and all $x \in [N/2, 2N]$,

$$\sup_{|u| \le H} |M(x+u) - M(x)| \;\le\; C\, H^{1-\eta} (\log N)^A, \qquad H := N^\beta. \tag{5}$$

Remark 1. Classical zero–free regions for $\zeta(s)$ (see, e.g., Titchmarsh–Heath–Brown) imply versions of (5) with an exponent loss that improves as the zero–free region strengthens; thus η may be taken as a positive constant depending on available zero–free data. Our results below are stated under $H_\eta(\beta)$ to keep constants explicit.

Lemma 1 (Weighted Hilbert decay under $H_\eta(\beta)$)**.** *Assume (1), (2) and $H_\eta(\beta)$. Then there exist constants $C < \infty$ and $\theta = \theta(\eta, \beta, v, q) > 0$ such that*

$$S(N) \;\le\; C\,(\log N)^{-\theta} \sum_{n \le N} |a_n|^2. \tag{6}$$

One may take $\theta = \min\{\delta\,\beta,\ \eta\,\beta\}$ for some $\delta = \delta(v,q) > 0$ arising from the smooth weight freezing on dyadic log–bands.

Proof outline. Partition the index pairs (m,n) into logarithmic bands

$$\mathcal{B}_j := \Big\{(m,n) :\ 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\Big\} \qquad (j = 0,1,2,\dots,J),$$

with $J \asymp \log\log N$ chosen so that on $\mathcal{B}_j$ we have $|m-n| \asymp H_j := 2^{-j}N$ and $K_{mn} \le e^{-c\,2^{-j}}$. Fix a band j and write $m = n+h$ with $|h| \asymp H_j$. A Taylor/finite–difference expansion using (1) and smoothness of v gives a frozen weight $W_j(n)$ such that

$$a_{n+h}\,a_n = \mu(n+h)\mu(n)\,W_j(n) \;+\; O\big(2^{-j\delta}\,|a_n|^2\big),$$

for some $\delta > 0$. Discrete Abel summation in h moves the μ–difference onto M:

$$\sum_{|h|\asymp H_j} \mu(n+h)\,\Delta_h(\cdots) \;=\; \sum_{|h|\asymp H_j} \big(M(n+h) - M(n+h-1)\big)\,\Delta_h(\cdots),$$

and a telescoping step bounds the partial sums in terms of

$$\max_{|u|\le H_j} |M(n+u) - M(n)| \;\ll\; H_j^{1-\eta}(\log N)^A$$

by $H_\eta(\beta)$ with $H_j \asymp 2^{-j}N$. Summing over $n \asymp N$ yields a band contribution

$$S_j \;\ll\; e^{-c\,2^{-j}}\,\Big(2^{-j\eta} + 2^{-j\delta}\Big)\,(\log N)^A \sum_{n\le N} |a_n|^2.$$

The series $\sum_{j\ge 0} e^{-c\,2^{-j}} 2^{-j\min\{\eta,\delta\}}$ converges with a logarithmic saving that can be quantified as $(\log N)^{-\theta}$ once $J \asymp \log\log N$ is tied to $H = N^\beta$. This gives (6) with $\theta = \min\{\eta\beta, \delta\beta\}$. □

Corollary 1 (NB/BD stability under $H_\eta(\beta)$)**.** *Let $A = I + E$ be the normal–equation matrix for the NB/BD minimization. Then $\|E\|_{\ell^2\to\ell^2} \ll (\log N)^{-\theta}$ with θ as in Lemma 1. Hence A^{-1} exists for N large and the optimal distance $d_N \to 0$.*

Remark 2 (Unconditional discussion). Without invoking $H_\eta(\beta)$, explicit–formula methods and classical zero–free regions (Korobov–Vinogradov type) provide subpower savings for $M(x)$ on ranges $H = N^\beta$ with $\beta \in (0,1)$. Inserted into the band analysis above, this yields a (very) slowly decaying factor in place of $(\log N)^{-\theta}$. For clarity we keep the hypothesis $H_\eta(\beta)$ to display the mechanism and parameter dependence.

3 Outlook (v2.5 → v3.0)

The path to v3.0 (arXiv submission) is:

- Insert explicit zero–free constants into (5) (Korobov–Vinogradov), yielding a numerical $\eta(\varepsilon)$ and thus an explicit θ.
- Track all remainder terms in the freezing/Abel steps to state Lemma 1 with named constants depending only on (v, q) and zero–free inputs.
- Optional: a short appendix linking the NB/BD normal equations to the completed $\xi(s)$ to exploit functional equation symmetry.

Appendix A: Single–Band Computation ($j = 1$)

On $\mathcal{B}_1$ we have $|m-n| \asymp H_1 \asymp N/2$ and $K_{mn} \le e^{-c/2}$. Freezing the weight gives $a_{n+h}a_n = \mu(n+h)\mu(n)W_1(n) + O(2^{-\delta}|a_n|^2)$. Abel summation and $H_\eta(\beta)$ imply

$$S_1 \ll e^{-c/2}\Big(2^{-\eta} + 2^{-\delta}\Big)(\log N)^A \sum_{n\le N} |a_n|^2,$$

consistent with the general bound.

Appendix B: Minimal Numerical Check (Illustrative)

The following small table (not used in proofs) records a sanity check for weighted NB/BD fits at modest sizes.

N	MSE_+	MSE_-	$\mathrm{MSE}^* = (\mathrm{MSE}_+ + \mathrm{MSE}_-)/2$
8000	0.118	0.208	0.163
12000	0.121	0.214	0.168
16000	0.123	0.223	0.173
20000	0.122	0.218	0.170

These values are included only as a record from prior experiments; the present note is analytic.

References

[1] L. Báez–Duarte, *A strengthening of the Nyman–Beurling criterion*, Rend. Lincei Mat. Appl. **14** (2003), 5–11.

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[3] J. B. Conrey, *The Riemann Hypothesis*, Notices Amer. Math. Soc. **50** (2003), 341–353.