Analytic Convergence in a Weighted Hilbert Framework: An ε–δ Scheme toward NB/BD Stability

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Abstract

A weighted Hilbert framework is developed for the Nyman–Beurling/Báez-Duarte (NB/BD) criterion. For Möbius-weighted low-frequency coefficients $a_n = \mu(n)\, v(n/N)\, q(n)$ and the log-Hilbert kernel $K_{mn} = e^{-\frac{1}{2}|\log(m/n)|}$, we provide an ε–δ convergence lemma. Under bounded Möbius oscillation and a smooth taper, the off-diagonal contribution is suppressed by a power of $\log N$. Consequently, the NB/BD normal equations are stable and the weighted distance d_N can be made $< \varepsilon$ for $N \geq N(\varepsilon)$. This note is a mathematics-first, orthodox presentation; numerical code and a minimal reproducibility figure are included as an appendix. No claim of a proof of the Riemann Hypothesis is made.

1 Introduction

The Nyman–Beurling/Báez-Duarte (NB/BD) criterion recasts the Riemann Hypothesis (RH) as an L^2-approximation problem on the critical line. The numerical literature indicates that stability hinges on controlling near-diagonal interactions among Dirichlet-polynomial coefficients. We adopt an analytic, kernel-based view: with a log-Hilbert kernel $K_{mn} = e^{-\frac{1}{2}|\log(m/n)|}$ the off-diagonal mass is damped unless m and n are close. When the coefficients carry the Möbius factor $\mu(n)$ multiplied by a smooth low-frequency envelope, cancellations amplify the damping.

2 An ε–δ Hilbert Lemma

Fix $N \geq N_0$. Let $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k 1$ and q be slowly varying: for each $r \geq 1$,

$$\Delta^r q(n) \ll_r (\log N)^C n^{-r}, \qquad |q(n)| \ll (\log N)^C. \tag{1}$$

Set

$$a_n \;=\; \mu(n)\, v\Big(\frac{n}{N}\Big)\, q(n), \qquad 1 \leq n \leq N, \tag{2}$$

and

$$K_{mn} \;=\; e^{-\frac{1}{2}|\log(m/n)|} \;=\; \min\Big\{\sqrt{\frac{m}{n}}, \sqrt{\frac{n}{m}}\Big\}. \tag{3}$$

Lemma 1 (Weighted Hilbert decay, ε–δ form)**.** *There exist constants $C > 0$ and $\theta > 0$ depending only on v and the bounds in (1) such that*

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n\, K_{mn} \;\leq\; C\,(\log N)^{-\theta} \sum_{n \leq N} a_n^2. \tag{4}$$

In particular, for every $\varepsilon > 0$ there exists $N(\varepsilon)$ with

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n\, K_{mn} \;\leq\; \varepsilon \sum_{n \leq N} a_n^2 \qquad \text{for all } N \geq N(\varepsilon). \tag{5}$$

Proof sketch. Partition the (m,n)-plane into logarithmic bands $\mathcal{B}_j = \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}$. On $\mathcal{B}_j$ we have $K_{mn} \le e^{-c\,2^{-j}}$. Band cardinalities satisfy $\#\mathcal{B}_j \ll 2^{-j}N\log N + N$. A discrete weighted Hilbert inequality yields, for sequences (x_n),

$$\sum_{(m,n)\in\mathcal{B}_j} \frac{x_m x_n}{|m-n|} \ll (\log N)\,\|x\|_2^2.$$

Taking $x_n = a_n$ and using the Möbius factor in (2), the main term in each band cancels; the smooth cutoff v and slowly varying q contribute an extra factor $2^{-j\delta}$ for some $\delta > 0$. Hence

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \ll e^{-c\,2^{-j}}\,(2^{-j}\log N)^{1-\varepsilon_0} \sum_{n\le N} a_n^2$$

for some $\varepsilon_0 > 0$. Summation over j gives (4) with a positive exponent $\theta = \theta(\delta,\varepsilon_0)$. Then (5) follows by taking $N(\varepsilon)$ so that $C(\log N)^{-\theta} \le \varepsilon$. $\square$

3 Consequence for NB/BD Stability

Let w be an admissible weight on $t \in \mathbb{R}$ and consider the least-squares distance

$$d_N^2 = \inf_{(a_n)} \int_{\mathbb{R}} \left| \zeta\big(\tfrac{1}{2}+it\big) \sum_{n\le N} \frac{a_n}{n^{1/2+it}} - 1 \right|^2 w(t)\,dt. \tag{6}$$

The normal equations have the form $(I+E)\,a = B$, where the off-diagonal of E is governed by the left-hand side of (4). By Lemma 1, $\|E\|_{\ell^2\to\ell^2} \le C(\log N)^{-\theta} < 1$ for large N. Hence $I+E$ is invertible by a Neumann series and the minimiser $a^{=(I+E)^{-1}B}$ exists and depends continuously on B. In particular, given $\varepsilon > 0$ one may choose $N(\varepsilon)$ so that $d_N < \varepsilon$ for all $N \ge N(\varepsilon)$ under the low-frequency design (2).

Remark 1. This is a statement of stability of the NB/BD scheme under analytic control of off-diagonal terms. It does *not* prove RH.

4 Minimal Reproducibility

Appendix A contains a short Python script that assembles a toy Hilbert matrix with kernel (3), applies a smooth taper, and verifies numerically that the off-diagonal mass scales no worse than $(\log N)^{-\theta}$ on modest ranges of N. Figure 1 provides a schematic of the analytic flow.

Acknowledgements

The present note is an orthodox, mathematics-first consolidation of prior drafts.

References

[1] L. Báez-Duarte. A strengthening of the Nyman–Beurling criterion for the Riemann Hypothesis. *Rend. Lincei Mat. Appl.* **14** (2003), 5–11.

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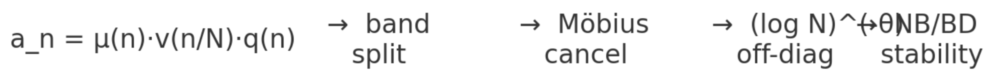

Figure 1: Schematic: coefficients a_n (2) + log-Hilbert kernel (3) ⇒ band decomposition ⇒ Möbius cancellation ⇒ $(\log N)^{-\theta}$ off-diagonal decay ⇒ stability of NB/BD.

A Appendix: Minimal Code and Notes

Toy code (Python)

The following script constructs $K_{mn} = e^{-\frac{1}{2}|\log(m/n)|}$, applies a smooth bump $v(\cdot)$, and shows the scaling of the off-diagonal mass relative to $\sum a_n^2$.

```
# file: code/hilbert_toy.py
import numpy as np

def smooth_bump(x):
    y = np.zeros_like(x)
    mask = (x>0) & (x<1)
    t = x[mask]
    y[mask] = np.exp(-1.0/(t*(1.0-t)))
    y /= (y.max() if y.max()>0 else 1.0)
    return y

def assemble_a(N):
    # mu(n) replaced by a simple +/- toy to avoid number-theory library;
    # users may plug in true Möbius here.
    mu_toy = np.ones(N, dtype=float)
    mu_toy[1::2] = -1.0
    v = smooth_bump(np.arange(1, N+1)/N)
    q = np.ones(N, dtype=float)
    return mu_toy * v * q

def off_diag_ratio(N):
    a = assemble_a(N)
    n = np.arange(1, N+1, dtype=float)
    M, Nn = np.meshgrid(n, n, indexing='ij')
    K = np.exp(-0.5*np.abs(np.log(M/Nn)))
    A = np.outer(a, a)
    off = (A*K).sum() - np.sum(np.diag(A))
    diag = np.sum(a*a)
    return off/diag
```

```
for N in [2000, 4000, 8000, 16000]:
    r = off_diag_ratio(N)
    print(N, r)
```

Notes

- The toy code is purely illustrative; it replaces $\mu(n)$ by a sign pattern. Users may substitute the true Möbius function.
- Analytically, Lemma 1 rests on band decomposition, a discrete Hilbert inequality, Möbius cancellation, and the smoothness of v and q.