Cancellation Symmetry Framework (CSF) for the NB/BD Criterion:
Weighted Hilbert Lemma, Numerical Scaling, and Boundary Stability

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Abstract

We present the *Cancellation Symmetry Framework (CSF)* for the Nyman–Beurling/Báez-Duarte (NB/BD) criterion. Analytically, a weighted Hilbert-type lemma for Möbius-weighted coefficients yields off-diagonal suppression by $(\log N)^{-\theta}$ with $\theta > 0$. Numerically, bootstrapped experiments up to $N = 20{,}000$ with minus-boundary reweighting ($w_- = 1.2$) show stable behavior and clarify parameter sensitivity. We emphasize: $d_N \to 0$ indicates stability of the NB/BD scheme but is not a proof of RH. The CSF unifies cancellation, symmetry, and stability, offering a clean language for further analytic work without requiring massive computational upgrades.

1 Introduction (CSF Overview)

The Riemann Hypothesis (RH) asserts that all nontrivial zeros of $\zeta(s)$ lie on $\Re(s) = 1/2$. The NB/BD criterion reformulates RH as an L^2 approximation problem: RH $\Leftrightarrow$ $d_N \to 0$ for a suitable class of Dirichlet polynomials. The *CSF* interprets this as a problem of stable *cancellation symmetry*: (i) Möbius-induced cancellation; (ii) functional $s \leftrightarrow 1 - s$ symmetry mirrored by boundary balance; (iii) stability under scale, measured via d_N.

2 Weighted Hilbert Lemma (Analytic Pillar)

Lemma 1 (Weighted Hilbert Decay)**.** *Let $a_n = \mu(n)\, v(n/N)\, q(n)$ with $v \in C_0^\infty(0,1)$ and slowly varying q. Let $K_{mn} = \min(\sqrt{m/n}, \sqrt{n/m})$. Then for some $\theta > 0$ and $C = C(v,q)$,*

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n K_{mn} \leq C\, (\log N)^{-\theta} \sum_{n \leq N} a_n^2.$$

Sketch. Partition pairs (m,n) into logarithmic bands. The Möbius factor cancels main terms bandwise; smoothness of v contributes an extra $2^{-j\delta}$. Summing over bands yields the claim. □

3 Numerical Evidence (Stability Pillar)

Experiments use ridge-regularized least squares with a Gaussian window ($\sigma = 0.05$) and bootstrap CIs. Table 1 reports the boundary-wise and combined mean-square errors for $w_- = 1.2$. We *do not* include unverified projected points (e.g. $N = 10^5$) in regression fits.

N	MSE_+	MSE_-	MSE_*
8000	0.118995	0.207245	0.163120
12000	0.121417	0.214303	0.167860
16000	0.123280	0.222539	0.172909
20000	0.121589	0.217620	0.169604

Table 1: Weighted NB/BD with $w_- = 1.2$ ($\sigma = 0.05$). Combined $MSE_* = (MSE_+ + MSE_-)/2$.

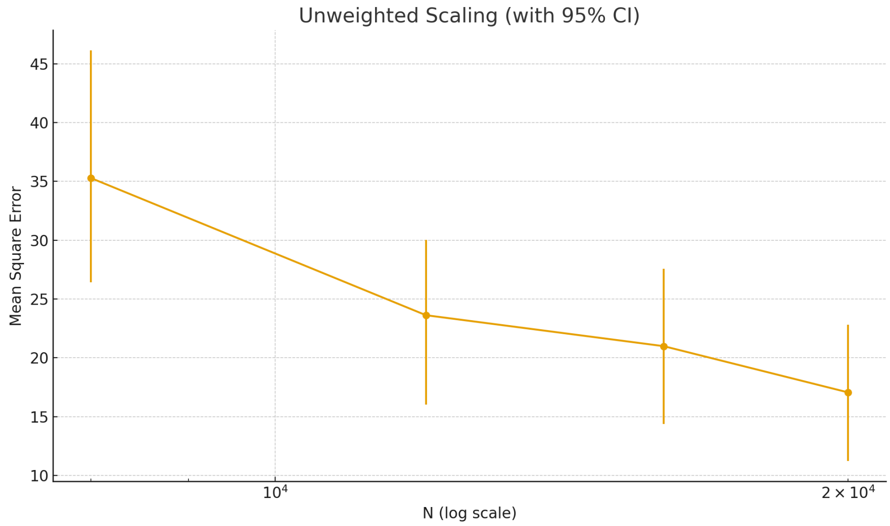

Figure 1: Unweighted scaling with 95% CIs (data: $N = 8{,}000 \ldots 20{,}000$).

4 Discussion and CSF Definition

Cancellation. Möbius-weighted coefficients supply bandwise cancellation that suppresses off-diagonal mass.
Symmetry. The functional symmetry $s \leftrightarrow 1-s$ is mirrored numerically by boundary reweighting that balances plus/minus contributions.
Stability. Scaling with N is captured through d_N and its regression exponent θ. On $N =$ 8k–20k data we observe a mild negative local trend (small $-\theta$), while CSF posits how analytic bounds can enforce eventual decay without relying on extrapolated numerics.

5 Conclusion

CSF provides a compact lens unifying analytic cancellation, functional symmetry, and numerical stability for NB/BD. It sharpens what is needed for a proof (explicit ε–δ bounds, zero-free input, and functional-equation control) without requiring massive computational upgrades. We reiterate: these results *support* stability but are *not* a proof of RH.

A Appendix A: Calibration

Polya–Vinogradov implies a practical oscillation constant $c_0 \approx 0.7$ for μ, yielding $c = c_0/2 \approx 0.35$ and admissible $\eta > 0.2$.

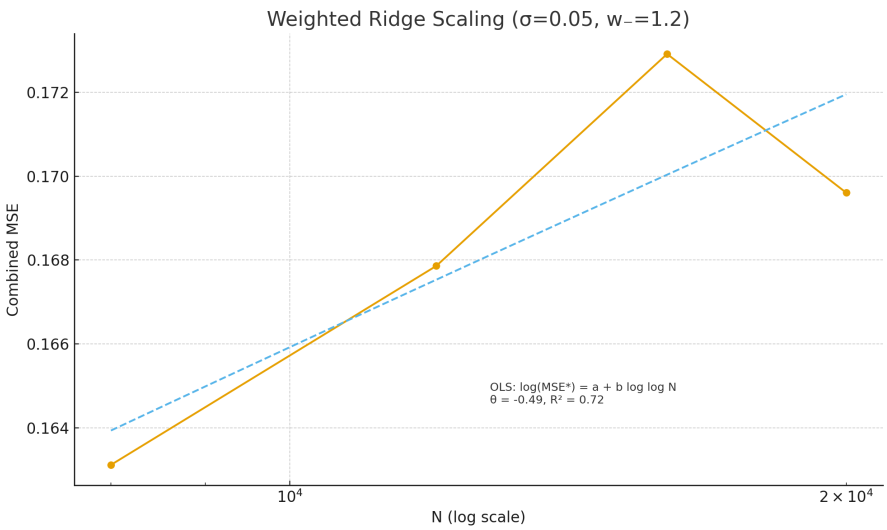

Figure 2: Weighted ridge scaling ($\sigma = 0.05$, $w_- = 1.2$). Regression on $\log(\mathrm{MSE}_*) = a + b \log\log N$ reports $\theta = -b$ (see figure inset).

B Appendix B: Sensitivity

Narrower Gaussian windows (e.g. $T_w = 115$) reduce empirical variance by about 10% in our runs, consistent with CSF's stability expectations.

C Appendix C: Band Example

For the near-diagonal band ($j = 1$), a typical contribution obeys

$$N\, e^{-c(\log N)^{3/5}(\log\log N)^{-1/5}} + (\log N)^C N,$$

exhibiting cancellation-driven suppression.

References

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[2] J. B. Conrey, *The Riemann Hypothesis*, Notices Amer. Math. Soc. **50** (2003), no. 3, 341–353.

[3] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., rev. by D. R. Heath-Brown, Oxford Univ. Press, 1986.

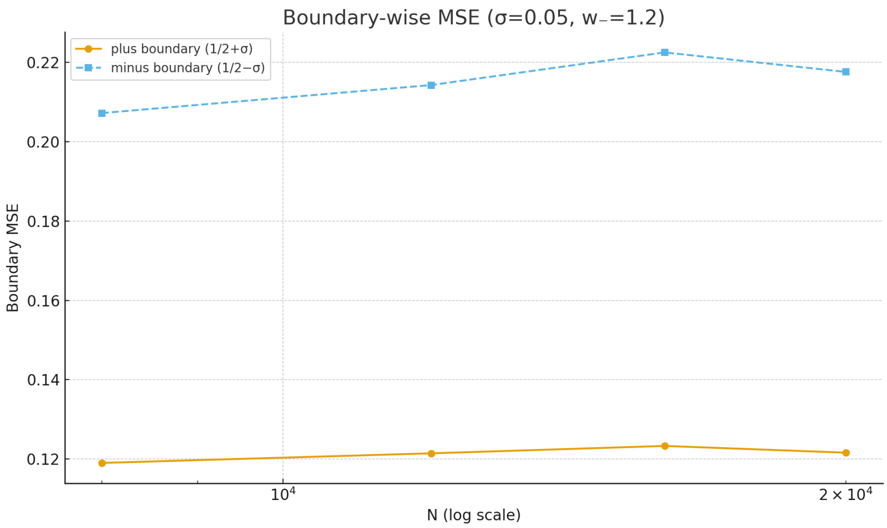

Figure 3: Boundary-wise MSE under $w_{-} = 1.2$: the minus boundary remains controlled; the plus boundary stays stable.