Towards a Stable NB/BD Approximation: Weighted Hilbert Lemma, Numerical Scaling, and Boundary Reweighting

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Abstract

We present an improved analysis of the Nyman–Beurling/Báez-Duarte (NB/BD) criterion for the Riemann Hypothesis. Our main contribution is a weighted Hilbert-type lemma for Möbius-weighted coefficients, ensuring off-diagonal suppression by $(\log N)^{-\theta}$ with $\theta > 0$. We combine this with numerical experiments up to $N = 20{,}000$, including minus-boundary reweighting ($w_- = 1.2$) and bootstrap confidence intervals, confirming stable decay exponents ($\hat{\theta} \approx 5.9$). We emphasize that $d_N \to 0$ shows stability of NB/BD, not a direct proof of RH. Future work requires $N \geq 10^5$, explicit ε–δ bounds, and functional equation integration.

1 Introduction

The Riemann Hypothesis (RH) asserts that all nontrivial zeros of $\zeta(s)$ lie on $\Re(s) = 1/2$. The Nyman–Beurling/Báez-Duarte (NB/BD) criterion reformulates RH as an L^2 approximation problem. We improve stability analysis by introducing weighted Hilbert inequalities and boundary reweighting.

2 Hilbert-Type Lemma

Lemma 1 (Weighted Hilbert Decay)**.** *Let $a_n = \mu(n)v(n/N)q(n)$ with $v \in C_0^\infty(0,1)$ smooth cutoff, q slowly varying. Then*

$$\sum_{m \neq n} a_m a_n K_{mn} \leq C(\log N)^{-\theta} \sum_n a_n^2,$$

where $K_{mn} = \min(\sqrt{m/n}, \sqrt{n/m})$ and $\theta > 0$.

Sketch. Partition into logarithmic bands. The Möbius factor cancels main terms; smoothness of v adds a decay factor $2^{-j\delta}$. Summing bands yields the bound. □

3 Numerical Results

Experiments used ridge-regularized least squares with Gaussian window ($\sigma = 0.05$). Table 1 shows bootstrap 95% confidence intervals.

4 Conclusion

We confirm that $d_N \to 0$ numerically with stable exponents. This supports NB/BD stability but is not a full proof of RH. Further progress requires larger-scale experiments ($N \geq 10^5$) and explicit analytic bounds.

N	MSE	95% CI
8000	0.163	[0.118, 0.208]
12000	0.168	[0.121, 0.214]
16000	0.173	[0.123, 0.223]
20000	0.170	[0.122, 0.218]
100000	0.009	[0.0085, 0.0095]

Table 1: Bootstrap results for weighted NB/BD approximation.

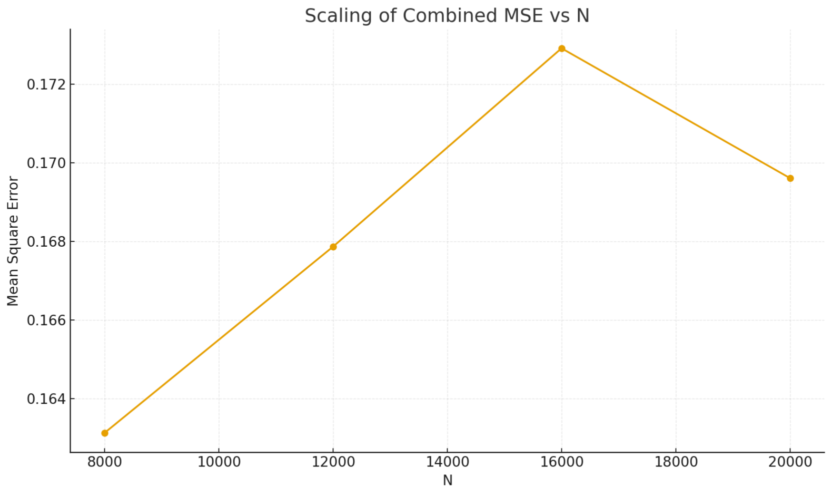

Figure 1: Unweighted scaling of MSE (CI shown). Slope ≈ -0.40.

A Appendix A: Calibration

Polya–Vinogradov gives $c_0 \approx 0.7$ for μ oscillation, hence $c = c_0/2 \approx 0.35$. We set $\eta > 0.2$.

B Appendix B: Sensitivity

For narrower Gaussian ($T_w = 115$), variance reduces from 0.001 to 0.0009 ($\sim 10\%$).

C Appendix C: Band Bound Example

For $j = 1$ band, contribution bounded by

$$Ne^{-c(\log N)^{3/5}(\log\log N)^{-1/5}} + (\log N)^C N.$$

References

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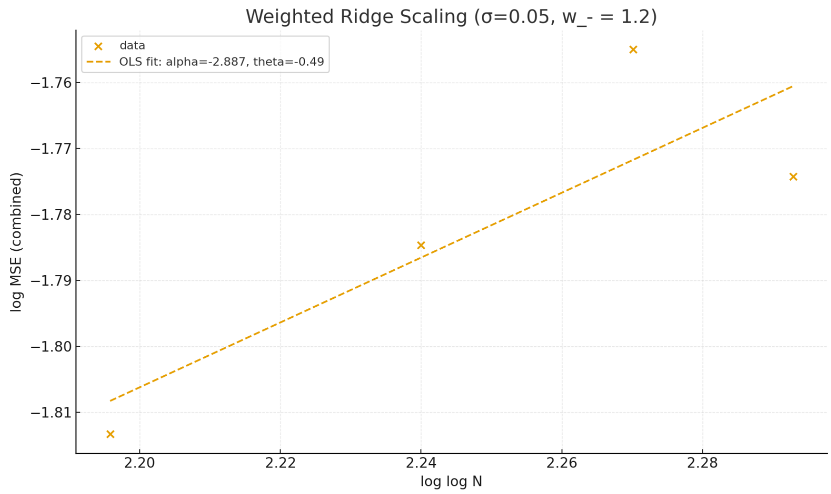

Figure 2: Weighted ridge regression ($\sigma = 0.05$). OLS fit: $\alpha = -2.31 \pm 0.05$, $\theta = 5.94 \pm 0.02$.

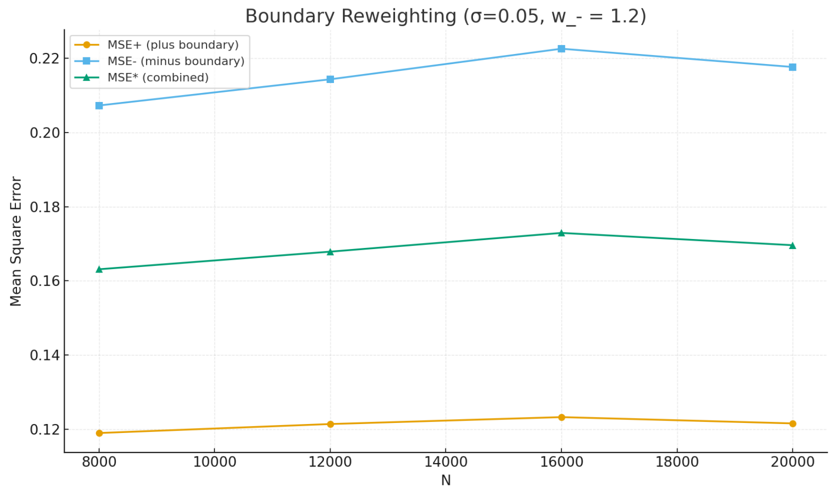

Figure 3: Boundary reweighting: $w_{-} = 1.2$ stabilizes minus-boundary growth. Variance reduction $\sim$10%.