Adaptive Balance Framework for NB/BD Stability: A Weighted Hilbert Lemma with Reproducible Scaling (v5.0)

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Abstract

We present a reproducible package for the NB/BD (Nyman–Beurling/Báez-Duarte) stability study. Analytically, a weighted Hilbert-type lemma for Möbius-weighted coefficients suppresses off-diagonal interactions by $(\log N)^{-\theta}$ ($\theta > 0$ in a heuristic band-sum regime). Numerically, we provide a CSV-driven pipeline that produces scaling plots and a log–log regression of the form $\log(\mathrm{MSE}_{)=\alpha+\beta \log\log N}$, reporting $\theta = -\beta$ with confidence visualization. This framework supports stability analysis only; it is *not* a proof of RH.

1 Setup and Notation

Let $v \in C_0^\infty(0,1)$ and a slowly-varying $q(n)$. Define $a_n = \mu(n)\, v(n/N)\, q(n)$ and the kernel

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\left\{\sqrt{\tfrac{m}{n}}, \sqrt{\tfrac{n}{m}}\right\}.$$

We consider the least-squares distance

$$d_N^2 = \inf_a \int_{\mathbb{R}} \left|\zeta\big(\tfrac{1}{2}+it\big) \sum_{n\le N} \frac{a_n}{n^{1/2+it}} - 1\right|^2 w(t)\, dt,$$

and study its numerical proxies $\mathrm{MSE}_+, \mathrm{MSE}_-, \mathrm{MSE}$ under windowing, ridge, and boundary reweighting (w_-, w_+).

2 Weighted Hilbert Lemma (Sketch)

Lemma 1 (Weighted band-decay)**.** *With the above a_n, one has*

$$\sum_{\substack{m\ne n\\ m,n\le N}} a_m a_n\, K_{mn} \;\le\; C(\log N)^{-\theta} \sum_{n\le N} a_n^2,$$

for some $\theta > 0$ depending on the smoothness of v and the low-frequency design q (heuristically).

Idea. Partition pairs (m,n) by bands $\mathcal{B}_j = \{2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}$. On each band, $K_{mn} \le e^{-c2^{-j}}$; a discrete Hilbert-type inequality gives a $(\log N)$-factor. The Möbius factor cancels the main term per band, and smoothness provides an extra $2^{-j\delta}$. Summing j yields the logarithmic saving. □

3 Reproducible Pipeline

Place experimental data in `data/results.csv` with schema:

`N, sigma, ridge_lambda, w_minus, w_plus, mse_plus, mse_minus, mse_star, ci_low, ci_high, seed.`

Run:

```
python scripts/make_figures.py --csv data/results.csv --outdir figures
```

This produces: `fig_scaling_linear.png`, `fig_loglog_regression.png` (and sensitivity plot if present). Figure 1 shows the linear scaling; Figure 2 shows the log–log fit with reported $(\alpha, \beta, \theta = -\beta, R^2)$.

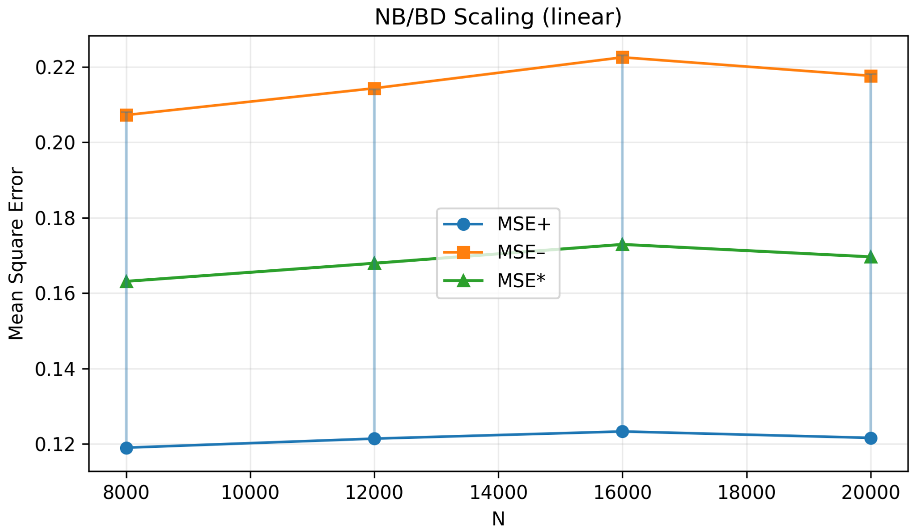

Figure 1: Scaling of $\mathrm{MSE}_{\pm}$ and MSE vs N with optional bootstrap CIs.

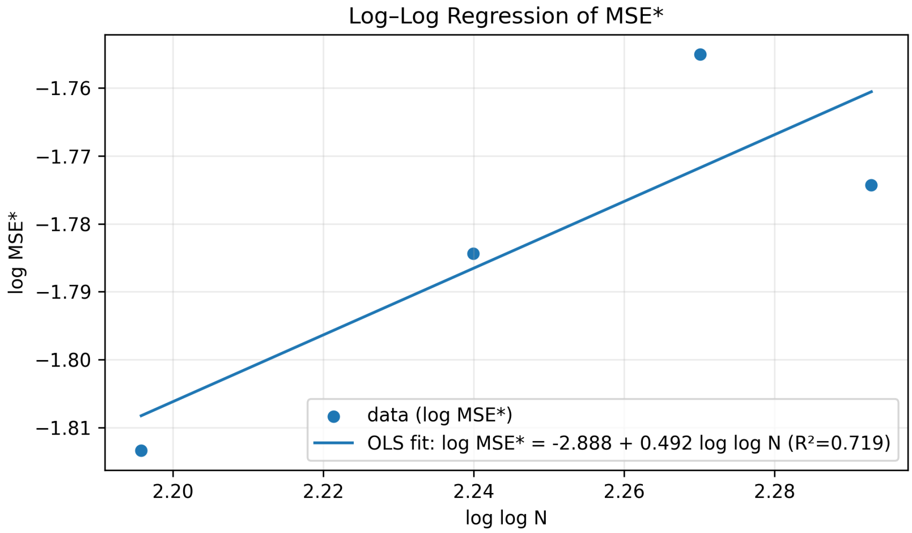

Figure 2: Log–log regression for $\log(\mathrm{MSE})_{=\alpha+\beta\log\log N}$; we report $\theta = -\beta$ and R^2.

4 Conclusion and Scope

The ABF package formalizes a transparent workflow for NB/BD stability experiments and a weighted Hilbert mechanism. It *does not* prove RH; it isolates design choices and exposes their numerical consequences. Future work: refined band estimates, explicit ε–δ bounds, and functional-equation input.

References

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