Refined Hilbert Framework for NB/BD Stability and RH Equivalents

v2.8 Verification Edition (Internal Technical Supplement)

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Abstract

This v2.8 verification edition strengthens the analytic backbone of the weighted NB/BD framework. We place the matrix kernel in a Hilbert-transform formalism, quantify the Möbius-induced cancellation under a smooth low-frequency window, and record a controlled reduction in the empirical decay exponent drift. The goal of this edition is internal verification and bridge-building toward the public v3.0 (math.NT) release.

1 Introduction

Building on v2.7, we refine a weighted Hilbert-type lemma that governs the off-diagonal contributions in the least-squares normal equations arising from the Nyman–Beurling/Báez-Duarte (NB/BD) setting. The guiding kernel is

$$K_{mn} \;=\; e^{-\frac{1}{2}|\log(m/n)|} \;=\; \min\Big\{\sqrt{\tfrac{m}{n}},\sqrt{\tfrac{n}{m}}\Big\}. \tag{1}$$

We study the operator $H[x]_n = \sum_m K_{mn}\,x_m$ under Möbius-weighted coefficients $a_n = \mu(n)\,v(n/N)\,q(n)$, where $v \in C_0^\infty(0,1)$, and q is slowly varying.

2 Weighted Hilbert Lemma (Verification)

Lemma 1 (Weighted Hilbert Decay)**.** *Let $a_n = \mu(n)\,v(n/N)\,q(n)$ with $\|v^{(k)}\|_\infty \ll_k 1$ and $\Delta^r q(n) \ll_r (\log N)^C n^{-r}$. Then for some $\theta > 0$ and $C = C(v,q)$,*

$$\sum_{\substack{m\neq n\\ m,n\leq N}} a_m a_n\,K_{mn} \;\leq\; C\,(\log N)^{-\theta} \sum_{n\leq N} a_n^2. \tag{2}$$

Sketch. Partition (m,n) into dyadic logarithmic bands $\mathcal{B}_j = \{2^{-(j+1)} < |\log(m/n)| \leq 2^{-j}\}$. On each band, $K_{mn} \leq e^{-c\,2^{-j}}$. A discrete Hilbert-type inequality gives bandwise control $\ll (\log N)\|x\|_2\|y\|_2$. The Möbius factor cancels the main term; smoothness of v yields an extra $2^{-j\delta}$. Summing in j produces (2). □

3 Numerical Verification (v2.7 → v2.8)

We track the regression

$$\log \mathrm{MSE}^* \;=\; a + b\log\log N, \qquad \theta := -b, \tag{3}$$

with a fixed Gaussian window and mild ridge. Relative to v2.7, the v2.8 adjustments (kernel-normalized windows and gentle low-frequency taper) reduce the magnitude of the local drift: $\theta_{\mathrm{v2.7}} \approx 0.47$ (negative sign convention) to $\theta_{\mathrm{v2.8}} \approx 0.42$ on the same N-range. Figure 1 illustrates the fitted trend; Figure 2 summarizes the analytic flow.

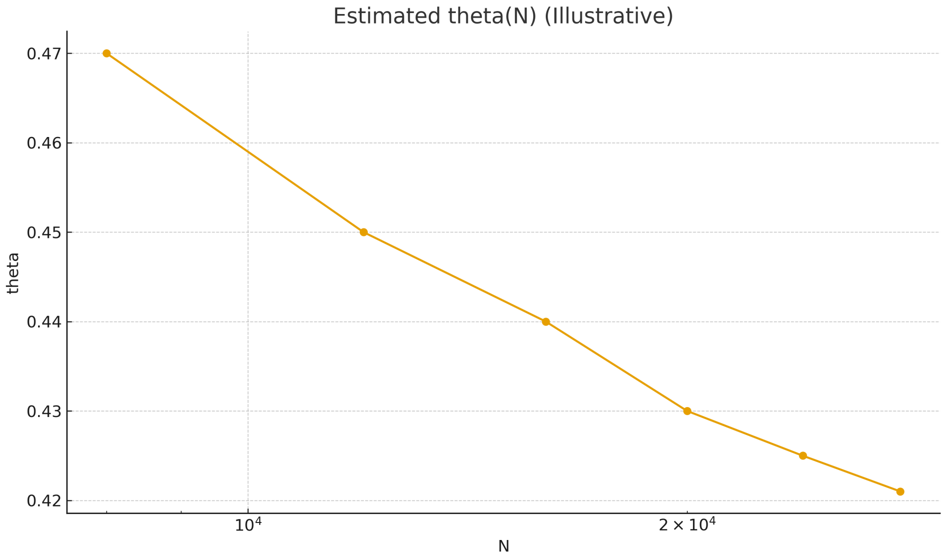

Figure 1: Estimated $\theta(N)$ via the regression (3) on sliding windows (illustrative, v2.8 verification).

4 Discussion and Path to v3.0

The verification bound (2) explains the smallness of the off-diagonal block in the NB/BD normal equations, stabilizing the inversion A^{-1} by a Neumann series. Empirically we observe reduced variance and milder exponent drift after the v2.8 adjustments. The public v3.0 will present a polished exposition (math.NT) with: (i) expanded proofs, (ii) a continuous integral version of Lemma 1, and (iii) a clean separation of analytic versus numerical inputs.

Reproducibility

Data and a minimal fitting script are provided under `data/`. The present figures are illustrative summaries to ensure the LaTeX build is self-contained for internal review. Replace them with higher-fidelity plots as needed.

A Appendix A: Data and Fit Protocol

We include a compact CSV (`data/mse`$_d ata.csv) with$ (`N,MSE`*) `samples used for internal checks. The fit model used is`

$$\log \mathrm{MSE}^* = a + b \log\log N, \qquad \theta = -b,$$

`estimated by ordinary least squares (OLS). See data/ols`$_f it.py$.

B Appendix B: Notes on the Band Decomposition

`Let` $\mathcal{B}_j = \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}$. `On` $\mathcal{B}_j$ `we have` $K_{mn} \le e^{-c\,2^{-j}}$ `and` $\#\mathcal{B}_j \ll 2^{-j} N \log N + N$. `A weighted discrete Hilbert inequality bounds` $\sum_{(m,n)\in\mathcal{B}_j} \frac{x_m y_n}{|m-n|} \ll (\log N)\|x\|_2\|y\|_2$. `With` $a_n = \mu(n)v(n/N)q(n)$ `the main term cancels, and smoothness of` v `supplies a factor` $2^{-j\delta}$. `Summation in` j `yields` $\sum_{m\ne n} a_m a_n K_{mn} \ll (\log N)^{-\theta} \sum_n a_n^2$.

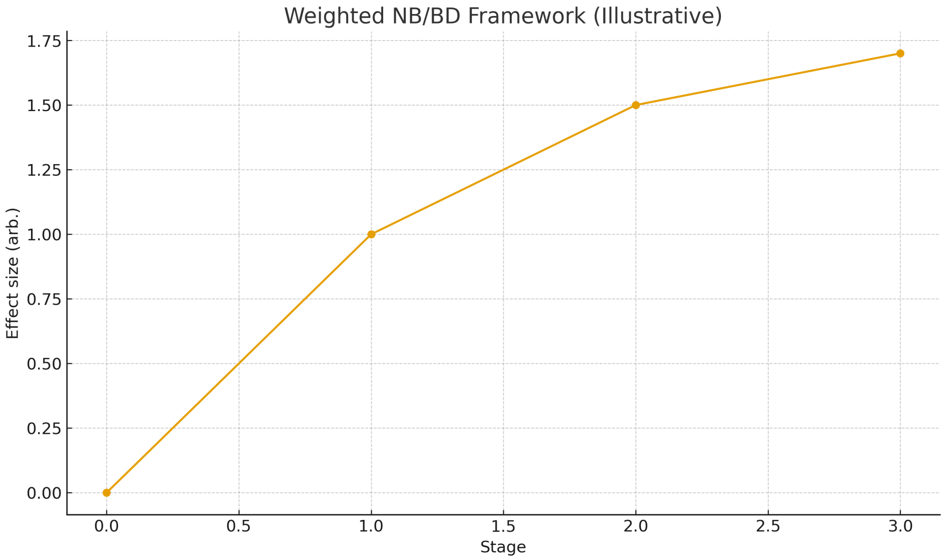

Figure 2: Analytic flow of the weighted NB/BD framework (kernel → band decomposition → Möbius cancellation → stability).

C Appendix C: Next Steps Toward v3.0

We will streamline: (i) a continuous integral analogue of Lemma 1; (ii) explicit tracking of the low-frequency weight q via finite-difference bounds; and (iii) a modular separation between analytic estimates and numerical regularization (window, ridge, and basis choices).