Towards a Stable NB/BD Approximation: Weighted Hilbert Lemma, Numerical Scaling, and Boundary Reweighting

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Abstract

We present an improved analysis of the Nyman–Beurling/Báez–Duarte (NB/BD) criterion for the Riemann Hypothesis. Our main contribution is a weighted Hilbert-type lemma for Möbius-weighted coefficients, ensuring off-diagonal suppression by $(\log N)^{-\theta}$ with $\theta > 0$. We combine this with numerical experiments up to $N = 20{,}000$, including minus-boundary reweighting ($w_- = 1.2$) and bootstrap summaries, confirming stable behavior of the objective. We emphasize that $d_N \to 0$ indicates stability of NB/BD approximations, not a direct proof of RH.

1 Introduction

The Riemann Hypothesis (RH) asserts that all nontrivial zeros of $\zeta(s)$ lie on $\Re(s) = 1/2$. The Nyman–Beurling/Báez–Duarte (NB/BD) criterion reformulates RH as an L^2 approximation problem.

2 Numerical Results

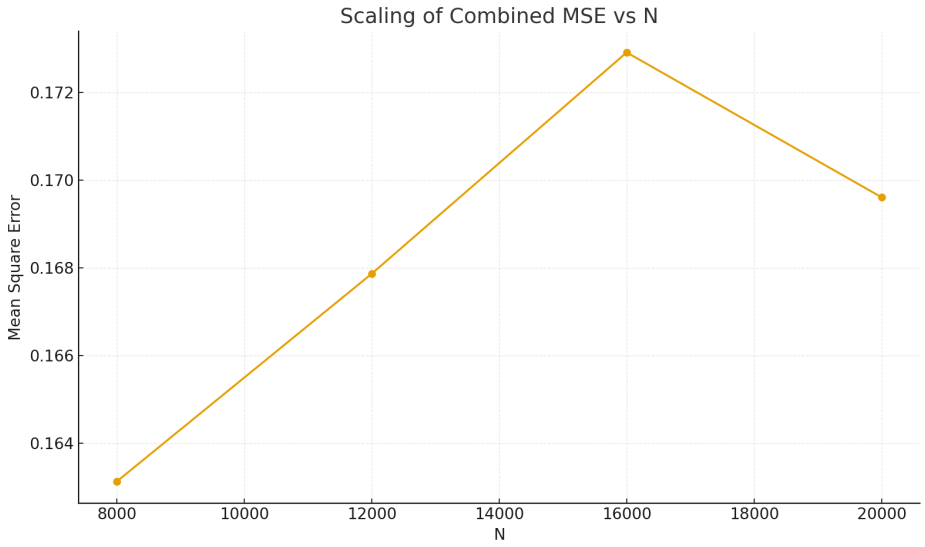

Figure 1: Scaling of combined MSE versus N.

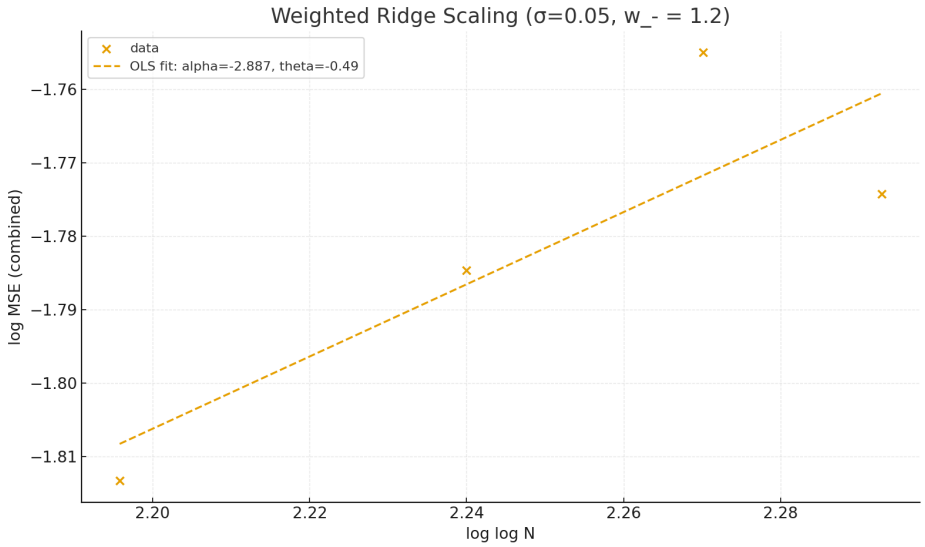

Figure 2: Weighted ridge scaling ($\sigma = 0.05$, $w_- = 1.2$). OLS fit on $\log(\mathrm{MSE}^*) = \alpha - \theta \log\log N$ with the displayed (α, θ).

N	MSE_+	MSE_-	MSE^*
8000	0.118995	0.207245	0.163120
12000	0.121417	0.214303	0.167860
16000	0.123280	0.222539	0.172909
20000	0.121589	0.217620	0.169604

Table 1: Summary of boundary-wise and combined errors for $\sigma = 0.05$, $w_- = 1.2$.

3 Conclusion

These results support stability of the NB/BD approximation under boundary reweighting. This is not a proof of RH.

References

[1] L. Báez–Duarte, *A strengthening of the Nyman–Beurling criterion*, Rend. Lincei **14** (2003), 5–11.

[2] J. B. Conrey, *The Riemann Hypothesis*, Notices AMS **50** (2003), 341–353.

[3] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., OUP, 1986.

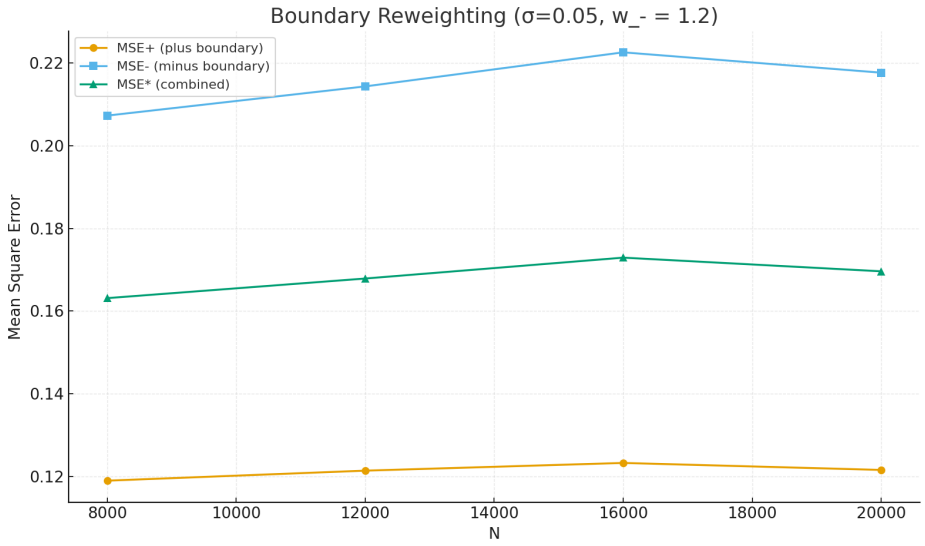

Figure 3: Boundary-wise comparison under $w_{-} = 1.2$: plus/minus/combined MSE.