Operator–Spectral Roadmap toward RH: A Weighted Hilbert Embedding of the NB/BD Framework (v3.4)

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Abstract

We propose an operator–spectral framework that integrates the Nyman–Beurling/Báez–Duarte (NB/BD) L^2 approximation with a weighted Hilbert embedding and the functional equation of the completed zeta. The central object is a quasi self-adjoint operator T_ξ acting on a weighted L^2 space such that finite sections of the NB/BD kernel approximate T_ξ compactly. This clarifies how Möbius oscillation dampens off-diagonal interactions (a "weighted Hilbert lemma") and how the functional symmetry $\xi(s) = \xi(1-s)$ manifests as an operator conjugation. We outline a concrete, staged roadmap from this surrogate model to a self-adjoint realization whose real spectrum would imply the Riemann Hypothesis. This is a design note (v3.4): proofs are sketched, with complete proofs deferred to the v3.5–v3.9 series.

1 Introduction

The Riemann Hypothesis (RH) is equivalent to the statement that the nontrivial zeros of $\zeta(s)$ lie on $\Re s = \frac{1}{2}$. In the Nyman–Beurling/Báez–Duarte (NB/BD) program, RH is rephrased as the decay of an L^2-distance $d_N \to 0$. Previous versions established a *weighted Hilbert lemma* for Möbius-weighted coefficients, suppressing off-diagonal terms by $(\log N)^{-\theta}$ with $\theta > 0$ in a model setting. Here we assemble these pieces into an *operator–spectral* structure whose symmetry reflects the functional equation of the completed zeta $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Guiding principle. If an operator T_ξ encoding ξ admits a self-adjoint realization on a suitable Hilbert space, then its spectrum is real; under the canonical spectral map this would pin zeros to the critical line. Our framework builds a quasi self-adjoint surrogate and shows how finite NB/BD sections approximate it.

2 Weighted Hilbert Embedding and Kernel

Work on the space $H = L^2((0,\infty), w(x)\,dx)$ with

$$w(x) = x^{-1/2}e^{-x}.$$

For $m, n \in \mathbb{N}$ let

$$K(m,n) = \exp\Big(-\tfrac{1}{2}\Big|\log\frac{m}{n}\Big|\Big) = \min\Big\{\sqrt{\frac{m}{n}}, \sqrt{\frac{n}{m}}\Big\}.$$

Let K_N be the $N \times N$ matrix $(K(m,n))_{m,n\le N}$ and consider Möbius-weighted coefficients

$$a_n = \mu(n)\,v(n/N)\,q(n),$$

with $v \in C_0^\infty(0,1)$ and q slowly varying (finite differences $\Delta^r q(n) \ll_r (\log N)^C n^{-r}$).

Lemma 1 (Weighted Hilbert decay, model form). *There exist $\theta > 0$ and $C = C(v, q)$ such that*

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n \, K(m, n) \; \leq \; C(\log N)^{-\theta} \sum_{n \leq N} a_n^2.$$

Sketch. Partition into dyadic bands in $|\log(m/n)|$. On each band $K(m, n) \leq e^{-c2^{-j}}$. The Möbius factor cancels the main contribution within a band; smoothness of v yields an extra $2^{-j\delta}$. Summing j gives the $(\log N)^{-\theta}$ savings. □

Remark 1. Lemma 1 is a surrogate (model) statement: it isolates the mechanism (Möbius cancellation + smoothing) responsible for off-diagonal damping in NB/BD normal equations. In later versions we will prove a precise variant adapted to the operator below.

3 An Operator Surrogate for the Completed Zeta

Let $\mathcal{F}$ be the Mellin transform $(\mathcal{F}f)(s) = \int_0^\infty f(x)\, x^{s-1}\, dx$. Define the operator

$$T_\xi \; := \; \mathcal{M}^{-1} \circ \Xi(s) \circ \mathcal{M}, \qquad \Xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

On a dense subspace $\mathcal{S} \subset H$ (Schwartz functions against w), T_ξ maps $\mathcal{S} \to \mathcal{S}'$ and extends to a closed operator on H.

Proposition 1 (Functional symmetry as conjugation). *Let J be complex conjugation on H. Then, formally,*

$$JT_\xi J \; = \; T_\xi^{;}$$

reflecting $\Xi(s) = \Xi(1 - s)$ on the Mellin line. In particular, if T_ξ admits a self-adjoint realization on H, then $\mathrm{Spec}(T_\xi) \subset \mathbb{R}$, which is compatible with RH.

Proof idea. On the Mellin side J corresponds to $\overline{(\cdot)}$ and inversion $s \mapsto 1 - \overline{s}$. The functional equation gives $\Xi(s) = \overline{\Xi(1 - \overline{s})}$ on the critical strip, which yields the stated conjugation identity at the operator level. □

4 Finite Sections as Compact Approximants

Let $P_N : H \to \mathbb{C}^N$ be the sampling map $(P_N f)_n = \langle f, \phi_n \rangle_w$ with $\phi_n(x) = n^{1/2}\, \mathbf{1}_{[n,n+1)}(x)$ transported to the $x \leftrightarrow \log n$ scale. Define the finite section operator $T_N := P_N T_\xi P_N$.

Proposition 2 (Compact approximation). *Under the weighted embedding above, T_N is unitarily equivalent, up to uniformly Hilbert–Schmidt error, to the NB/BD kernel matrix K_N perturbed by a banded term arising from the weight w and the cutoff v. Consequently, $T_N \to T_\xi$ in the strong resolvent sense along a subsequence if the off-diagonal part obeys Lemma 1.*

Sketch. Identify $P_N^{P_N}$ with a localized frame. The kernel $K(m, n)$ appears as the discrete Hilbert kernel for log-spacing. Weighted localization produces a compact (Hilbert–Schmidt) remainder. Decay of off-diagonal mass follows from Lemma 1. □

5 Roadmap to a Self-Adjoint Realization

We outline concrete, checkable targets.

(R1) Domain & closure. Construct a core $\mathcal{D} \subset H$ on which T_ξ is symmetric; show essential self-adjointness via a Hardy-type inequality (Carleman criterion on the Mellin axis).

(R2) Spectral calibration. Compare eigenvalues of T_N with spectral measures of T_ξ; prove tightness using Hilbert–Schmidt remainders, then identify limit points by Weyl's criterion.

(R3) NB/BD link. Show that the normal equations matrix $A_N = I + E_N$ satisfies $\|E_N\|_{\ell^2 \to \ell^2} \ll (\log N)^{-\theta}$, implying $d_N \to 0$ for the surrogate system, matching operator resolvent convergence.

(R4) Functional symmetry. Promote Proposition 1 to a unitary conjugation $UT_\xi U^{-1} = T_\xi$ with $U^2 = I$ by choosing a $\mathcal{F}$-compatible weight.

(R5) Self-adjoint model. Replace $\Xi(s)$ by a smoothed symbol $\Xi_\varepsilon(s)$ in a Beurling–Selberg window to obtain an explicitly self-adjoint $T_{\xi,\varepsilon}$; pass to $\varepsilon \downarrow 0$.

Outlook (v3.5–v4.0)

- **v3.5**: Explicit domain and closability of T_ξ; compactness of remainders.
- **v3.6**: Eigenvalue comparison between K_N and T_ξ (finite-section method).
- **v3.7**: Zero-free input $\Rightarrow$ quantitative band decay (refined θ).
- **v3.8**: Unitary symmetry implementing $\xi(s) = \xi(1-s)$ on H.
- **v3.9–4.0**: Integrated manuscript: *An Operator-Theoretic Path to RH.*

Disclaimer. This note is a roadmap. No claim of a proof of RH is made; all "$\Rightarrow$ RH" statements are conditional on establishing self-adjointness of the limiting operator.

References

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