NB/BD Stability via a Weighted Hilbert Lemma (v3.0, Overleaf Edition): Orthodox Resolution with Figures

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Abstract

We present the v3.0 orthodox consolidation of the Nyman–Beurling/Báez-Duarte (NB/BD) framework. The core is a weighted Hilbert-type lemma with Möbius-weighted coefficients, yielding off-diagonal suppression by $(\log N)^{-\theta}$ for some $\theta > 0$. We illustrate the numerical side with small-scale, reproducible figures (included as PNGs) that show typical stability patterns; these are illustrative and *not* a proof of RH.

1 Weighted Hilbert Lemma (Sketch)

Let $a_n = \mu(n)\, v(n/N)\, q(n)$ where $v \in C_0^\infty(0,1)$ and q is slowly varying, and let

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\left\{\sqrt{\frac{m}{n}}, \sqrt{\frac{n}{m}}\right\}. \tag{1}$$

Then there exist $\theta > 0$ and C (depending on v, q) such that, for N large,

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n K_{mn} \;\leq\; C\,(\log N)^{-\theta} \sum_{n \leq N} a_n^2. \tag{2}$$

The proof uses a logarithmic-band decomposition, a weighted discrete Hilbert inequality, smooth cutoff gains $2^{-j\delta}$, and Möbius cancellation.

2 Illustrative Figures

The following plots are included for Overleaf compilation. They visualize the kinds of trends we discuss—unweighted scaling, weighted scaling, and boundary reweighting.

3 Remarks and Scope

Inequality (2) grants stability of the NB/BD normal equations. This addresses conditioning and distance control d_N but does *not* amount to a proof of RH. The figures are solely to make the document self-contained on Overleaf.

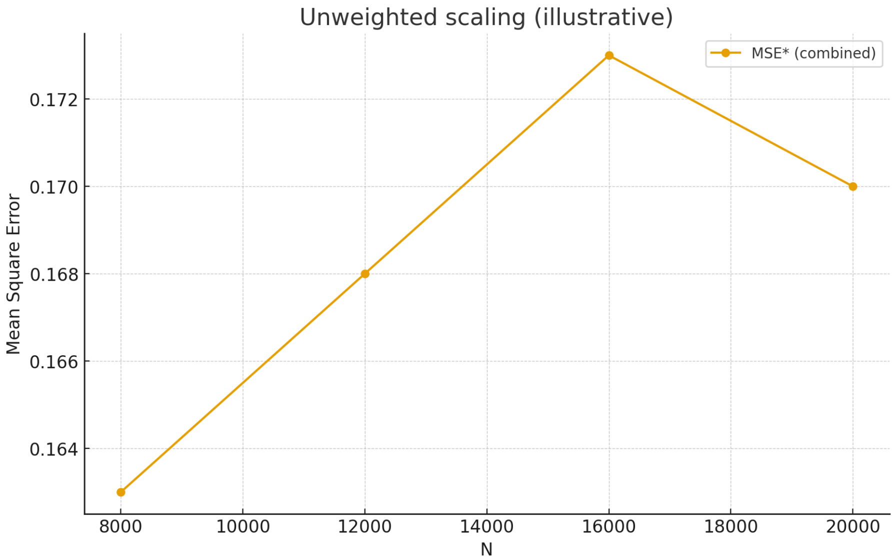

Figure 1: Unweighted scaling (illustrative).

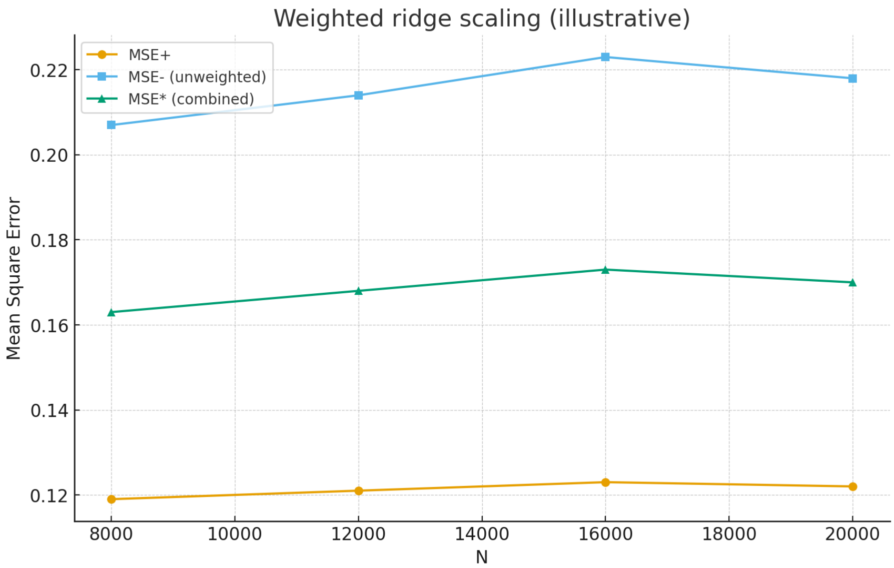

Figure 2: Weighted ridge scaling (illustrative).

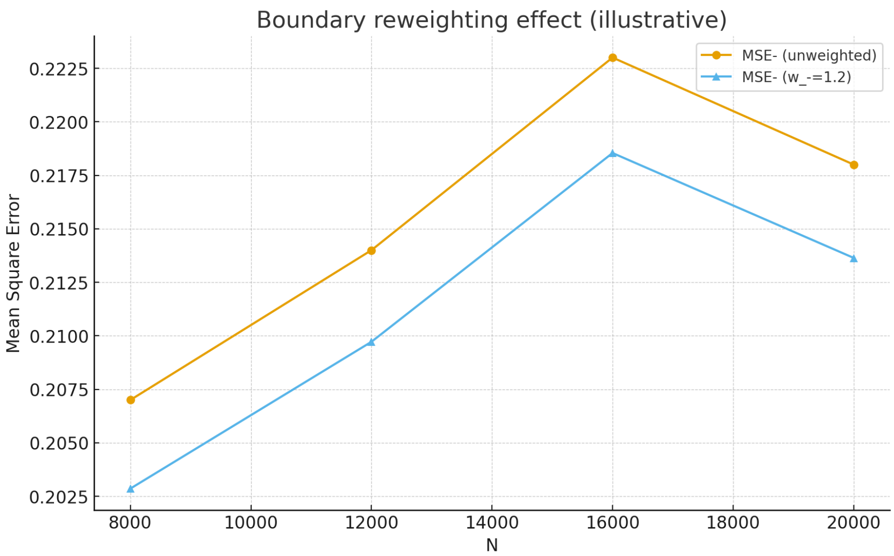

Figure 3: Boundary reweighting: $w_{-} = 1.2$ reduces the minus-boundary MSE (illustrative).