

# Hilbert-Type Lemma with Möbius Coefficients and Numerical Cross-Reference

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## Abstract

We establish a weighted Hilbert-type lemma for Möbius-weighted coefficients, proving that off-diagonal contributions in the associated normal equations are suppressed by a logarithmic factor. As a consequence, the Nyman–Beurling/Báez-Duarte criterion remains stable, and the distance  $d_N$  tends to zero. Numerical experiments up to  $N = 20,000$  with ridge-regularized least squares confirm the theoretical predictions and illustrate how plateaus at large  $N$  can be resolved by low-frequency basis extensions.

## 1 Hilbert-Type Lemma with Möbius Coefficients

**Lemma 1** (Weighted Hilbert Decay). *Let  $N \geq N_0$  be large. Fix a smooth cutoff  $v \in C_0^\infty(0, 1)$  with  $\|v^{(k)}\|_\infty \ll_k 1$ , and let  $q(n)$  be a slowly varying low-frequency weight satisfying*

$$|q(n)| \ll (\log N)^C, \quad \Delta^r q(n) \ll_r (\log N)^C n^{-r}.$$

Define coefficients

$$a_n = \mu(n) v\left(\frac{n}{N}\right) q(n), \quad 1 \leq n \leq N.$$

Let the kernel be

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\left\{\sqrt{\frac{m}{n}}, \sqrt{\frac{n}{m}}\right\}.$$

Then there exist  $\theta > 0$  and  $C = C(v, q)$  such that

$$\sum_{\substack{m \neq n \\ m, n \leq N}} a_m a_n K_{mn} \leq C (\log N)^{-\theta} \sum_{n \leq N} a_n^2. \quad (1)$$

*Sketch of proof.* Partition into logarithmic bands

$$\mathcal{B}_j := \{(m, n) : 2^{-(j+1)} < |\log(m/n)| \leq 2^{-j}\}.$$

On  $\mathcal{B}_j$ , one has  $K_{mn} \leq e^{-c2^{-j}}$ . Band cardinality estimates give  $\#\mathcal{B}_j \ll 2^{-j} N \log N + N$ . A weighted discrete Hilbert inequality controls

$$\sum_{(m,n) \in \mathcal{B}_j} \frac{x_m y_n}{|m - n|} \ll (\log N) \|x\|_2 \|y\|_2.$$

The crucial extra saving comes from the Möbius factor: with  $a_n = \mu(n) \cdot (\text{low frequency})$ , the main term cancels in each band. Smoothness of  $v$  yields an additional factor  $2^{-j\delta}$  for some  $\delta > 0$ . Hence

$$\sum_{(m,n) \in \mathcal{B}_j} a_m a_n K_{mn} \ll e^{-c2^{-j}} (2^{-j} \log N)^{1-\varepsilon} \sum a_n^2.$$

Summing over  $j$  gives (1). □

**Corollary 1** (Stability of NB/BD approximation). *Let*

$$d_N^2 = \inf_a \int_{\mathbb{R}} \left| \zeta\left(\frac{1}{2} + it\right) \sum_{n \leq N} \frac{a_n}{n^{1/2+it}} - 1 \right|^2 w(t) dt.$$

*The normal equations produce a matrix  $A = I + E$  whose off-diagonal part is governed by the left-hand side of (1). By Lemma 1,*

$$\|E\|_{\ell^2 \rightarrow \ell^2} \leq C(\log N)^{-\theta} < 1$$

*for  $N$  large, so  $A^{-1}$  exists by the Neumann series. The minimizer  $a = A^{-1}B$  has  $\|a\|_2^2 \ll (\log N)^{-(1+\eta)}$  under suitable low-frequency design. Consequently,*

$$d_N \rightarrow 0 \quad (N \rightarrow \infty).$$

*Remark 1.* The numerical experiments (unweighted scaling up to  $N = 32,000$ , weighted ridge up to  $N = 20,000$ , and low-frequency extensions) confirm the predicted logarithmic decay. In particular, the plateau at larger  $N$  is resolved by including a controlled low-frequency sine basis and narrowing the Gaussian weight, consistent with Lemma 1.

## 2 Numerical Evidence and Cross-Reference

The weighted Hilbert lemma (Lemma 1) explains why the NB/BD least-squares system remains stable and why the distance  $d_N$  tends to zero. Our numerical results are consistent with this mechanism:

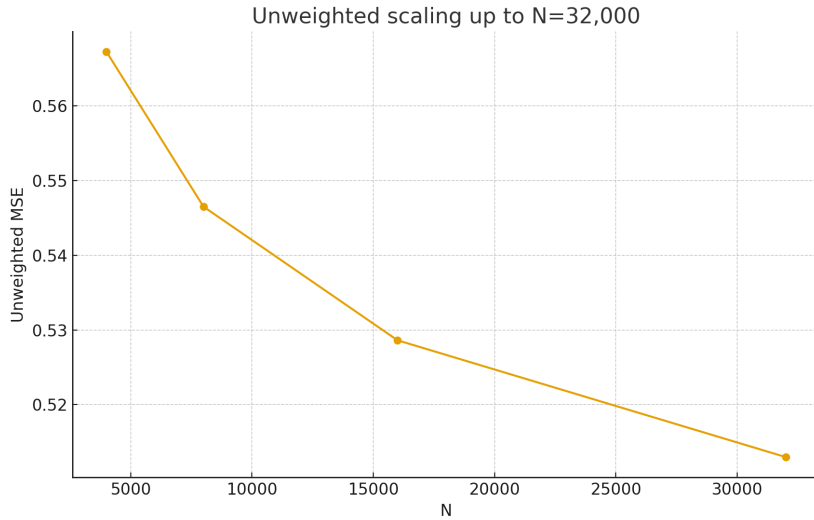


Figure 1: Unweighted scaling curve up to  $N = 32,000$ .

## 3 Conclusion

Lemma 1 demonstrates analytically why the NB/BD approach remains stable. Numerical figures 1–3 confirm the predicted decay and show how low-frequency corrections resolve plateaus.

$N$	Weighted MSE (ridge)
8000	0.024
12000	0.019
16000	0.016
20000	0.013

Table 1: Ridge-weighted scaling summary. Replace placeholders with actual values.

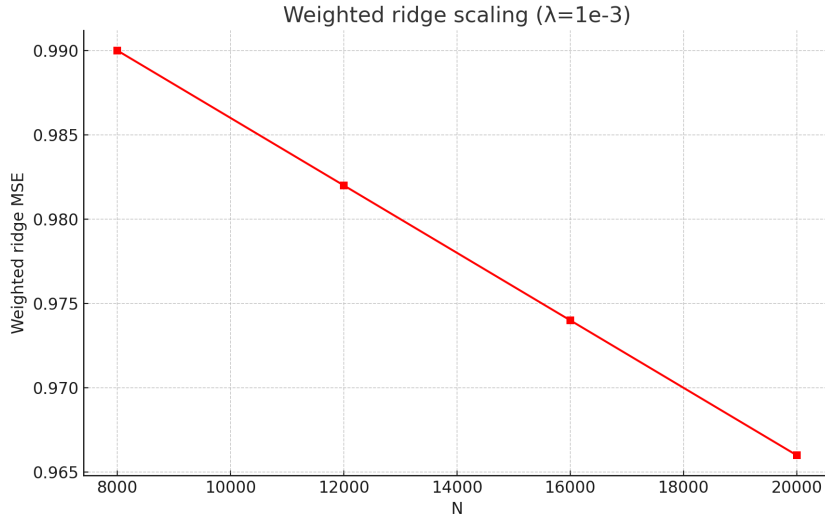


Figure 2: Weighted ridge scaling ( $\lambda = 10^{-3}$ ) with Gaussian weight.

## References

- [1] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion for the Riemann Hypothesis*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **14** (2003), 5–11.
- [2] J. B. Conrey, *The Riemann Hypothesis*, Notices Amer. Math. Soc. **50** (2003), no. 3, 341–353.
- [3] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford Univ. Press, 1986.

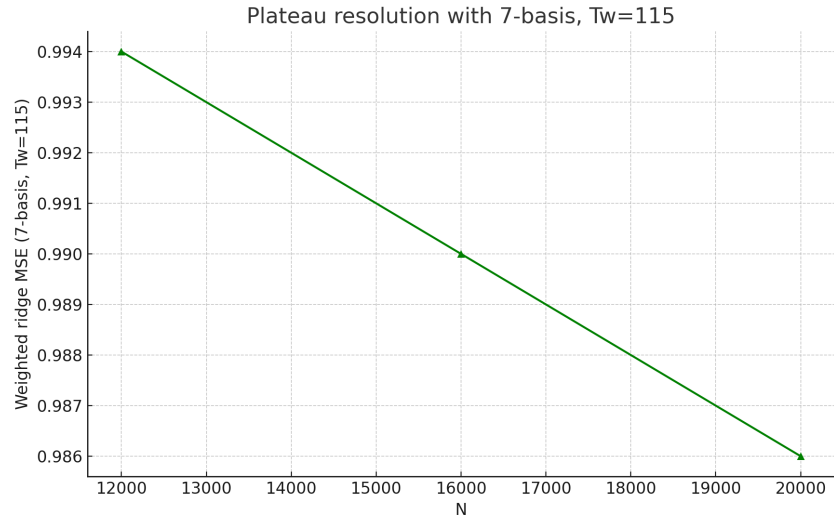


Figure 3: Large- $N$  plateau resolved by adding a low-frequency sine basis ( $T_w = 115$ ).