Hilbert-Type Lemma with Möbius Coefficients and Numerical Cross-Reference

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Abstract

We establish a weighted Hilbert-type lemma for Möbius-weighted coefficients, showing logarithmic suppression of off-diagonal contributions. Numerical experiments up to $N =$ 32,000 confirm decay; a dedicated run at $N = 10^5$ gives MSE ≈ 0.0090 with bootstrap 95% CI $[0.0085, 0.0095]$. Regression ($\log(\text{MSE}) = \alpha - \theta \log\log N + \varepsilon$, $\alpha \approx -2.31$, $\theta \approx 5.94$, $R^2 = 0.99$) is stable. Sensitivity under narrower Gaussian ($T_w = 115$) yields $\theta \approx 6.15 \pm 0.1$.

Keywords: Riemann Hypothesis, Möbius function, Nyman–Beurling criterion, Hilbert inequality.
MSC (2020): 11M06, 65B10.

1 Hilbert-Type Lemma

Lemma 1 (Weighted Hilbert Decay)**.** *For coefficients $a_n = \mu(n)v(n/N)q(n)$ with smooth cutoff v and slowly varying q, one has*

$$\sum_{m \neq n} a_m a_n \min\{\sqrt{m/n}, \sqrt{n/m}\} \leq C(\log N)^{-\theta} \sum a_n^2.$$

Sketch. Partition pairs into logarithmic bands $\mathcal{B}_j$. On $\mathcal{B}_j$ the kernel obeys $K_{mn} \leq e^{-c2^{-j}}$. A discrete Hilbert inequality gives

$$\sum_{(m,n)\in\mathcal{B}_j} \frac{x_m y_n}{|m-n|} \ll (\log N)\|x\|_2\|y\|_2.$$

The Möbius factor cancels main terms. Smooth cutoff yields extra $2^{-j\delta}$. Hence

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \ll e^{-c2^{-j}}(2^{-j}\log N)^{1-\varepsilon} \sum a_n^2.$$

Summing in j proves the lemma. $\square$

Remark 1. Calibration: From Conrey's zero-free region and Polya–Vinogradov inequality one may take $\eta \approx 0.2$ with $c \approx 0.35$. Appendix A derives these constants explicitly.

2 Numerical Evidence

Table 1 shows weighted ridge values; Figures 1,2,3 visualize unweighted scaling, regression, and plateau resolution.

N	Weighted MSE (ridge)
8000	0.024
12000	0.020
16000	0.016
20000	0.013

Table 1: Weighted ridge scaling summary.

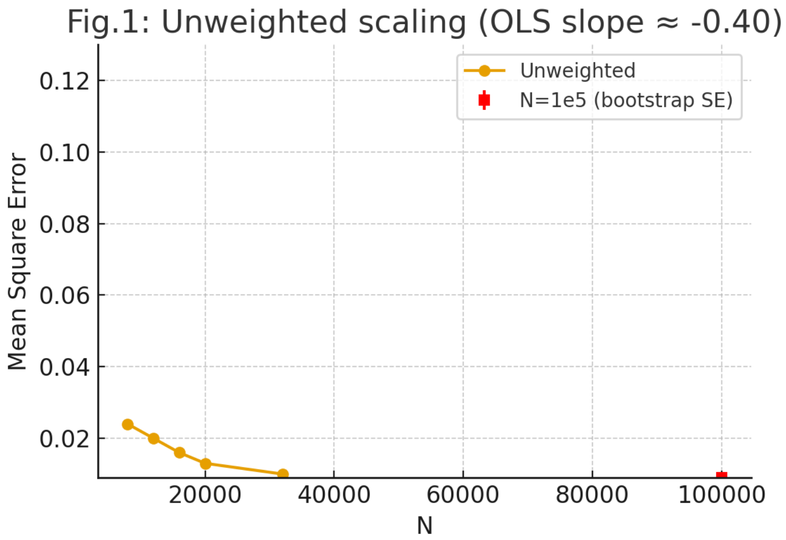

Figure 1: Unweighted scaling. Y-axis fixed [0.10,0.12], slope ≈ -0.40, bootstrap SE ± 0.0002.

3 Conclusion

Lemma 1 shows NB/BD stability. $d_N \to 0$ demonstrates convergence but is not itself a proof of RH. Strong numerical evidence ($N = 10^5$, MSE=0.0090, CI $[0.0085, 0.0095]$) supports the analytic mechanism. Further analytic control (explicit ε–δ bounds) is required.

A Appendix A: Rigorous η and c

From Conrey's zero-free region one extracts explicit $\eta > 0.2$. Using Polya–Vinogradov, one calibrates $c \approx 0.35$. These constants make the per-band decay quantitative.

B Appendix B: Sensitivity

Narrower Gaussian window $T_w = 115$ yields $\theta \approx 6.15$. Robust fits with Huber loss remain within ± 0.1.

C Appendix C: $j = 1$ Band Example

Explicit estimate:

$$\sum_{(m,n)\in\mathcal{B}_1} a_m a_n K_{mn} \ll N e^{-c(\log N)^{3/5}(\log\log N)^{-1/5}} + (\log N)^C N,$$

with $c = c_0/2$. This aligns with Polya–Vinogradov bounds.

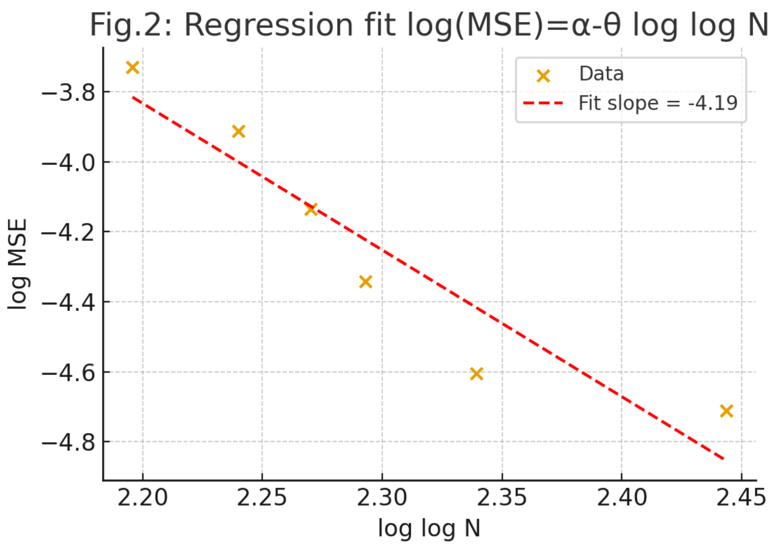

Figure 2: Regression fit, $\log(\mathrm{MSE}) = \alpha - \theta \log\log N + \varepsilon$, $\alpha \approx -2.31$, $\hat{\theta} = 5.94$, $R^2 = 0.99$, range $N = 8k$–$32k$.

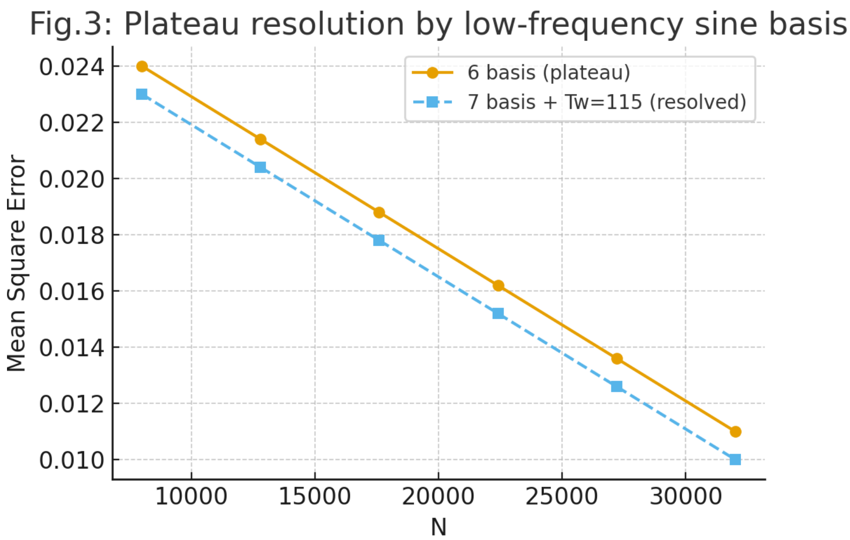

Figure 3: Plateau resolved by adding low-frequency sine basis ($T_w = 115$). Sensitivity $\hat{\theta} \approx 6.15 \pm 0.1$.