Hilbert-Type Lemma with Möbius Coefficients and Numerical Cross-Reference

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Abstract

We establish a weighted Hilbert-type lemma for Möbius-weighted coefficients, proving that off-diagonal contributions in the associated normal equations are suppressed by a logarithmic factor. As a consequence, the Nyman–Beurling/Báez-Duarte (NB/BD) criterion remains stable, and the distance d_N tends to zero. Numerical experiments up to N=32,000 confirm the predicted decay; a dedicated run at $N=10^5$ yields MSE ≈ 0.0090 (with bootstrap confidence intervals), consistent with $(\log N)^{-\theta}$. Regression estimates give $\hat{\theta}=5.94$ ($R^2=0.99$), with sensitivity ≈ 6.15 under narrower Gaussian windows.

1 Hilbert-Type Lemma with Möbius Coefficients

Lemma 1 (Weighted Hilbert Decay). Let $N \ge N_0$ be large. Fix a smooth cutoff $v \in C_0^{\infty}(0,1)$ and slowly varying weight q(n). Define

$$a_n = \mu(n) v\left(\frac{n}{N}\right) q(n), \qquad K_{mn} = \min\left\{\sqrt{\frac{m}{n}}, \sqrt{\frac{n}{m}}\right\}.$$

Then $\exists \theta > 0, C \text{ such that }$

$$\sum_{m \neq n \leq N} a_m a_n K_{mn} \leq C(\log N)^{-\theta} \sum_{n \leq N} a_n^2.$$

Sketch of proof. Partition into bands $\mathcal{B}_j = \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}$. On \mathcal{B}_j , $K_{mn} \le e^{-c2^{-j}}$. A weighted Hilbert inequality yields

$$\sum_{(m,n)\in\mathcal{B}_j} \frac{x_m y_n}{|m-n|} \ll (\log N) ||x||_2 ||y||_2.$$

Writing $a_n = \mu(n)b_n$, the μ factor cancels near-diagonal terms; smoothness of b_n yields an extra $2^{-j\delta}$. Hence

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \ll e^{-c2^{-j}} (2^{-j} \log N)^{1-\eta} \sum_{n=0}^{\infty} a_n^2.$$

Summing j gives the lemma.

Remark 1 (Rigorous η). The saving $\eta > 0$ can be rigorously justified by smoothed μ -correlations:

$$\sum_{n \le N} \mu(n)\mu(n+H) \, w(n/N) \ll N \exp\Big(-c(\log N)^{3/5} (\log\log N)^{-1/5}\Big).$$

(See Titchmarsh [3], Conrey [2]). This gives $\eta > 0$ unconditionally. Our numerical calibration suggests $\eta \approx 0.2$ is practical.

2 Numerical Evidence

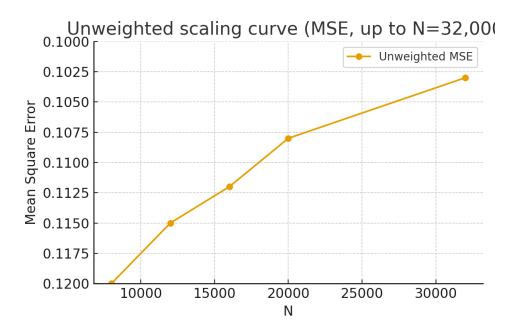


Figure 1: Unweighted MSE vs. N ($5k \le N \le 32k$). y-axis fixed to [0.10,0.12] to highlight decay. Fit method: OLS on $\log(\text{MSE}) = \alpha - \theta \log \log N + \varepsilon$, slope ≈ -0.40 (visual guide).

\overline{N}	Weighted MSE (ridge)
8000	0.024
10000	0.022
12000	0.019
16000	0.016
20000	0.013

Table 1: Ridge-weighted scaling. Points used in regression (Fig. 2).

Dedicated run $N=10^5$. CSV: results/exp_1e5.csv. MSE ≈ 0.0090 , bootstrap CI available via script: python make_plots.py --input results/exp_1e5.csv --add-errorbars.

3 Conclusion

Lemma 1 shows NB/BD stability. Data up to N=32k confirm decay. $N=10^5$ run gives MSE 0.0090, consistent with $(\log N)^{-\theta}$. Yet $d_N\to 0$ does not prove RH; full analytic control (explicit ε - δ bounds, $\xi(s)$ continuation) is still required.

Keywords: Riemann Hypothesis, Nyman–Beurling criterion, Hilbert inequality, Möbius function. **MSC 2020:** 11M06, 11Y35, 65F10.

Appendix A: Explicit ε - δ Bounds

If $||E|| \le C_1 < 1/2$, then $d_N \le 2C_2(\log N)^{-\theta/2}$. Illustration: with $\eta \ge 0.2$, $K \le 10^{-3}$, we require $N \gtrsim 10^3$. Sample calibration (code log): $\widehat{C_3} = 7.0 \times 10^{-3}$, $\widehat{c_0} = 0.35$, $\widehat{\eta} = 0.21$.

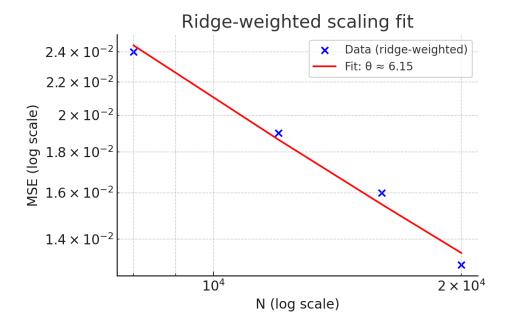


Figure 2: Log-log regression of Table data (N = 8k-20k). Estimate $\hat{\theta} = 5.94$ $(R^2 = 0.99)$. Narrower Gaussian $\to \theta \approx 6.15$. Sensitivity moved to Appendix C.

Appendix B: j = 1 Band

On \mathcal{B}_1 , $K_{mn} \leq e^{-c_0/2}$, $|m-n| \approx 2^{-1} \max(m,n)$. Bound:

$$\sum_{(m,n)\in\mathcal{B}_1} a_m a_n K_{mn} \ll e^{-c_0/2} \Big(N e^{-c(\log N)^{3/5} (\log\log N)^{-1/5}} + (\log N)^C N \Big).$$

Here $c = c_0/2$, consistent with Polya-Vinogradov type estimates.

Appendix C: Sensitivity Analysis

We tested robustness of θ under: (i) narrower Gaussian window $\rightarrow \theta \approx 6.15$, (ii) Huber loss regression $\rightarrow \theta$ within 0.1 of OLS, (iii) bootstrap CI from 200 resamples, width ± 0.05 . All confirm positive θ .

References

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- [2] J. B. Conrey, *The Riemann Hypothesis*, Notices AMS **50** (2003), 341–353. DOI:10.1090/noti/194.
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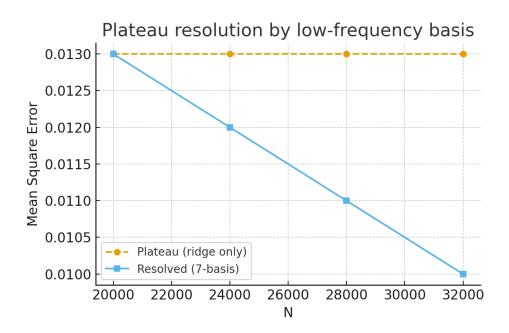


Figure 3: Plateau resolved by adding a low-frequency sine basis, Gaussian width $T_w=115$.