NB/BD Stability via a Weighted Hilbert Operator: Operator–Spectral Roadmap (v3.5)

Serabi
Independent Researcher
24ping@naver.com

2025

Abstract

We refine a weighted Hilbert-operator approach to the Nyman–Beurling/Báez-Duarte (NB/BD) criterion. Our operator formulation isolates near-diagonal interactions while damping off-diagonal terms through a log-banded partition and a low-frequency cutoff. We provide a self-contained lemma (Hilbert-type decay with Möbius-weighted coefficients) with explicit assumptions, clarify where numerical heuristics enter, and outline a spectral roadmap (near-normality, compact perturbations, and stability of normal equations). This note is a *mathematical framework*, not a proof of the Riemann Hypothesis (RH).

1 Setup and Operator Form

Let N be large. Fix a smooth cutoff $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k 1$ and a slowly-varying weight q with

$$|q(n)| \ll (\log N)^C, \qquad \Delta^r q(n) \ll_r (\log N)^C n^{-r}.$$

Define coefficients $a_n = \mu(n)\, v(n/N)\, q(n)$ and the kernel

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\left\{\sqrt{\tfrac{m}{n}}, \sqrt{\tfrac{n}{m}}\right\}.$$

For vectors $x = (x_n)_{n \le N}$ set the discrete operator

$$(\mathcal{H}x)_m = \sum_{n \le N} K_{mn}\, x_n, \qquad m \le N.$$

NB/BD normal equations produce $A = I + E$ where E collects off-diagonal interactions governed by $\mathcal{H}$.

2 Hilbert-Type Decay Lemma

Lemma 1 (Weighted Hilbert Decay)**.** *With $a_n = \mu(n)v(n/N)q(n)$ as above, there exist constants $\theta > 0$ and $C = C(v,q)$ such that*

$$\sum_{\substack{m \ne n \\ m,n \le N}} a_m a_n\, K_{mn} \;\le\; C\,(\log N)^{-\theta} \sum_{n \le N} a_n^2.$$

Sketch. Partition pairs (m,n) into dyadic log-bands $\mathcal{B}_j := \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}$. On $\mathcal{B}_j$ we have $K_{mn} \le e^{-c\,2^{-j}}$. A discrete Hilbert-type bound with smooth cutoffs controls the raw band sum by $(\log N)\|a\|_2^2$. With $a_n = \mu(n)\cdot$(low-frequency), summation by parts across each band gains an extra $2^{-j\delta}$ via smoothness of v and the oscillation of $\mu(n)$, yielding

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \ll e^{-c\,2^{-j}}(2^{-j}\log N)^{1-\varepsilon}\sum a_n^2.$$

Summing j gives the stated $(\log N)^{-\theta}$ suppression for some $\theta > 0$. $\square$

Remark 1 (Scope). Lemma 1 is qualitative: it ensures *some* $\theta > 0$ under the stated smooth and low-frequency assumptions. Sharper, explicit θ requires zero-free input or stronger mean value bounds for Möbius correlations; this is outside the present note.

3 Spectral Roadmap

Write $A = I + E$ for the NB/BD normal matrix. Lemma 1 gives $\|E\|_{\ell^2\to\ell^2} \le C(\log N)^{-\theta} < 1$ for N large, so A is invertible via a Neumann series. This supports stability of the least-squares coefficients and the NB/BD distance d_N in the weighted, low-frequency regime. A rigorous spectral path forward is:

(i) **Near-normality:** show $[\mathcal{H}, \mathcal{H}^*]$ is compact/small on the weighted subspace;

(ii) **Band-limited compactness:** the off-diagonal tail is compact under log-banding and smoothing;

(iii) **Fredholm alternative:** stability of $A = I + E$ under compact perturbations yields control of minimizers.

4 Limitations and Outlook

Our analysis does not prove RH. It isolates why the NB/BD linear system is *stable* under weighted, low-frequency designs. Future steps (all purely NT):

- explicit band-by-band constants and an effective $\theta(\delta)$;
- zero-free region input to trade oscillation for quantitative decay;
- functional equation/explicit formula integration to close remaining gaps.

References

[1] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion for the Riemann Hypothesis*, Rend. Lincei Mat. Appl. **14** (2003), 5–11.

[2] E. C. Titchmarsh (rev. D. R. Heath-Brown), *The Theory of the Riemann Zeta-Function*, 2nd ed., OUP, 1986.

[3] J. B. Conrey, *The Riemann Hypothesis*, Notices AMS **50** (2003), 341–353.