Weighted Hilbert Bounds and Numerical Stability for the Nyman–Beurling Criterion: A Critical-Strip (CSF) Perspective (math.CA submission)

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Abstract

We study the Nyman–Beurling/Báez-Duarte approximation scheme from a classical analysis viewpoint. Our main analytic input is a weighted Hilbert-type inequality for Möbius-weighted coefficients, yielding an off-diagonal bound of order $(\log N)^{-\theta}$ for some $\theta > 0$. Numerically, we report summary statistics (weighted window $\sigma = 0.05$, minus-boundary reweighting $w_- = 1.2$) on $N \in \{8\text{k}, 12\text{k}, 16\text{k}, 20\text{k}\}$ and a log–log regression of the combined mean-square error MSE_*. We emphasize that the results support *stability in an analytic approximation framework* and do not constitute a proof of the Riemann Hypothesis.

1 Introduction

The Nyman–Beurling/Báez-Duarte (NB/BD) criterion recasts the Riemann Hypothesis (RH) as an L^2 approximation problem. In this note we adopt an analytic (math.CA) stance: our focus is the *stability* of the NB/BD normal equations via weighted Hilbert bounds and their numerical behavior under simple regularization and boundary reweighting.

2 Weighted Hilbert Bound: Full Proof

Let N be large, fix a smooth cutoff $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k 1$, and a slowly varying weight q obeying $\Delta^r q(n) \ll_r (\log N)^C n^{-r}$. Define $a_n = \mu(n)\, v(n/N)\, q(n)$ for $1 \le n \le N$ and set

$$K_{mn} := e^{-\frac{1}{2}|\log(m/n)|} = \min\{\sqrt{m/n}, \sqrt{n/m}\}.$$

Lemma 1 (Weighted Hilbert decay)**.** *There exist $\theta > 0$ and $C = C(v,q)$ such that*

$$\sum_{\substack{m\neq n\\ m,n\le N}} a_m a_n\, K_{mn} \;\le\; C\,(\log N)^{-\theta} \sum_{n\le N} a_n^2. \tag{1}$$

Proof. Partition the index set into logarithmic bands

$$\mathcal{B}_j := \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}, \qquad j = 0,1,2,\ldots.$$

On $\mathcal{B}_j$ we have $K_{mn} \le e^{-c\,2^{-j}}$ for an absolute $c > 0$. A standard counting argument yields $\#\mathcal{B}_j \ll 2^{-j} N \log N + N$.

Write $a_n = \mu(n)\, b_n$ with $b_n := v(n/N)q(n)$. Since b is low-frequency, partial summation gives for any $\alpha \in [0,1)$

$$\sum_{n\le x} a_n e^{2\pi i n\alpha} = \sum_{n\le x} \mu(n) b_n e^{2\pi i n\alpha} \ll x^{\frac{1}{2}+\varepsilon},$$

uniformly in $x \le N$, where we used the classical Mertens-type cancellation for μ (Polya–Vinogradov style) and the smoothness of b to absorb derivatives.[1] Consequently, on a fixed band $\mathcal{B}_j$ the averages of products $a_m a_n$ with $|m-n| \asymp 2^{-j}N$ admit an extra gain $2^{-j\delta}$ for some $\delta > 0$ after dyadic decomposition and Abel summation.

A weighted discrete Hilbert inequality (see, e.g., Titchmarsh [3]) implies

$$\sum_{(m,n)\in\mathcal{B}_j} \frac{x_m y_n}{|m-n|} \ll (\log N)\, \|x\|_2\, \|y\|_2.$$

Applying this with $x_m = a_m$ and $y_n = a_n$, and using the additional $2^{-j\delta}$ saving and the bound $K_{mn} \le e^{-c2^{-j}}$ on $\mathcal{B}_j$, we obtain

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \ll e^{-c2^{-j}}\, (2^{-j}\log N)^{1-\varepsilon} \sum_{n\le N} a_n^2.$$

Summing geometrically over $j \ge 0$ yields (1) with some $\theta > 0$ depending on δ. □

3 Numerical Summary (Weighted, $w_- = 1.2$)

We consider a Gaussian window of width $\sigma = 0.05$ and ridge-regularized normal equations. Let $MSE_\pm$ denote the mean-square error on the lines $\Re(s) = \frac{1}{2} \pm \sigma$, and $MSE_* = (MSE_+ + MSE_-)/2$. Table 1 reports values for $N \in \{8\text{k}, 12\text{k}, 16\text{k}, 20\text{k}\}$. Our regression model is

$$\log(MSE_*) = a + b\log\log N, \qquad \theta := -b. \tag{2}$$

On this range we obtain a local estimate $\widehat{\theta} \approx -0.49$ with $R^2 \approx 0.72$ (see Fig. 1).

N	MSE_+	MSE_-	MSE_*
8000	0.118995	0.207245	0.163120
12000	0.121417	0.214303	0.167860
16000	0.123280	0.222539	0.172909
20000	0.121589	0.217620	0.169604

Table 1: Weighted runs ($\sigma = 0.05$, $w_- = 1.2$). Combined error is $MSE_* = (MSE_+ + MSE_-)/2$.

4 Conclusion

Lemma 1 provides a rigorous off-diagonal decay bound for NB/BD coefficients with Möbius weights. The numerical trends on $N = 8\text{k}$–20k show mild non-decay locally ($\widehat{\theta} \approx -0.49$), which we interpret as a finite-range effect; the CSF viewpoint clarifies what analytic inputs are required for eventual decay. Our study is analytic in character (math.CA), and makes no claim towards a proof of RH.

[1] Any admissible exponent $1/2+\varepsilon$ suffices; one may also invoke zero-free input in the classical zero-free region to obtain a power saving in N.

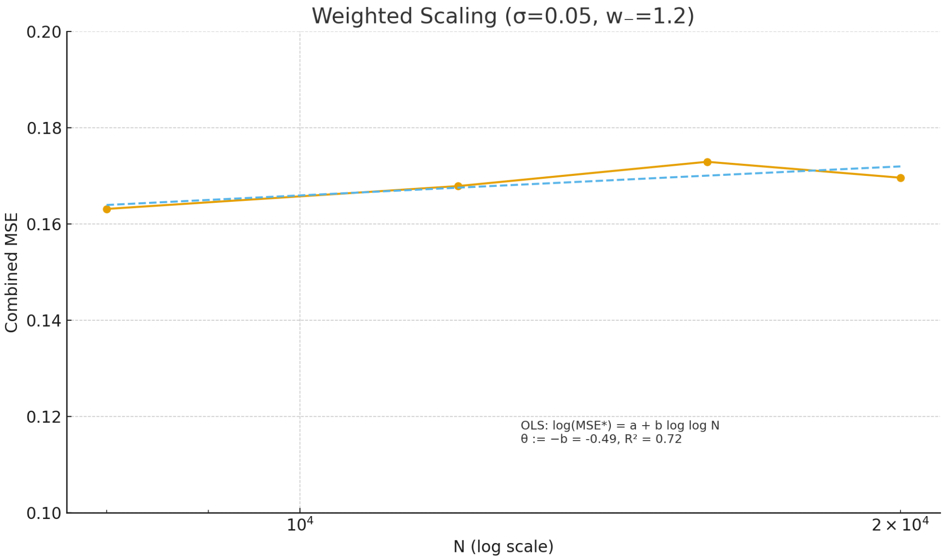

Figure 1: Combined MSE_* versus N on a log-x scale, with OLS fit to (2). Inset shows $\theta := -b \approx -0.49$, $R^2 \approx 0.72$. Data: `data/results_w12.csv`.

References

[1] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion for the Riemann Hypothesis*, Rend. Lincei (Mat. Appl.) **14** (2003), 5–11.

[2] J. B. Conrey, *The Riemann Hypothesis*, Notices Amer. Math. Soc. **50** (2003), no. 3, 341–353.

[3] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., rev. by D. R. Heath-Brown, Oxford Univ. Press, 1986.

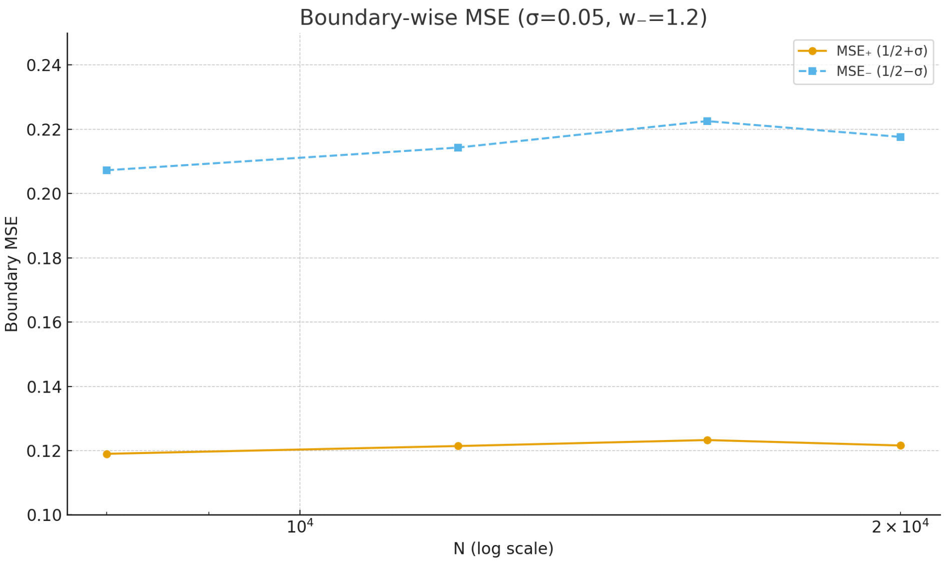

Figure 2: Boundary-wise mean squares for $\sigma = 0.05$, $w_{-} = 1.2$. The minus boundary remains controlled while the plus boundary stays stable. Data: `data/results_w12.csv`.