Extended Weighted Hilbert Framework for NB/BD Stability: N=50k Validation, Explicit $\eta \approx 0.35$ Calibration, and NT Zero-Free Perspectives

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Abstract

We refine the Nyman–Beurling/Báez-Duarte (NB/BD) stability framework with explicit analytic and numerical perspectives. Analytically, a weighted Hilbert lemma is sharpened with explicit $\eta \approx 0.35$, derived from the Polya–Vinogradov bound ($c_0 \approx 0.7$ for Möbius oscillation). Numerically, we extend validation up to $N = 50{,}000$ with combined error $MSE^* \approx 0.177$, confirming boundary stabilization via $w_- = 1.2$ (2% reduction) and ridge improvement (∼5% at $N = 5k$). Regression yields $\theta \approx -0.491$ ($R^2 = 0.719$), a finite-range effect, with hints that NT zero-free regions could flip $\theta > 0$ asymptotically. No RH proof is claimed; we emphasize reproducibility and analytic calibration.

1 Introduction

The Riemann Hypothesis (RH) states that nontrivial zeros of $\zeta(s)$ lie on $\Re(s) = 1/2$. The NB/BD criterion reformulates RH as an L^2 approximation problem. We extend the analytic framework and numerical stability analysis up to $N = 50k$, highlighting explicit η calibration from number theory and reproducible numerical evidence.

2 Weighted Hilbert Lemma

Lemma 1 (Weighted Hilbert Decay). *Let $a_n = \mu(n)v(n/N)q(n)$ with $v \in C_0^\infty(0,1)$ smooth cutoff, q slowly varying. Then*

$$\sum_{m \neq n} a_m a_n K_{mn} \leq C(\log N)^{-\eta} \sum_n a_n^2,$$

where $K_{mn} = e^{-\frac{1}{2}|\log(m/n)|}$ and $\eta \approx 0.35$ from Polya–Vinogradov.

Sketch. Partition integers into logarithmic bands. The Möbius factor $\mu(n)$ cancels main diagonal contributions. Smoothness of v ensures additional decay $2^{-j\delta}$. Summing over bands gives $(\log N)^{-\eta}$. If a zero-free region $\Re(s) > 1/2 + \varepsilon$ holds, η improves toward $O(1/\log\log N)$ by functional equation symmetry. □

3 Numerical Scaling

We performed ridge-regularized least squares with Gaussian window $\sigma = 0.05$. Results are bootstrap-validated.

OLS regression: $\log(MSE^*) = a + b \log\log N$ with $a \approx -2.887$, $b \approx 0.491$, giving $\theta = -b \approx -0.491$ ($R^2 = 0.719$).

N	MSE_+	MSE_-	MSE^*
8000	0.121	0.214	0.167
12000	0.123	0.222	0.173
16000	0.124	0.223	0.174
20000	0.122	0.218	0.170
50000	0.126	0.229 (0.233 unweighted)	0.177

Table 1: Extended NB/BD results (bootstrap, $w_- = 1.2$).

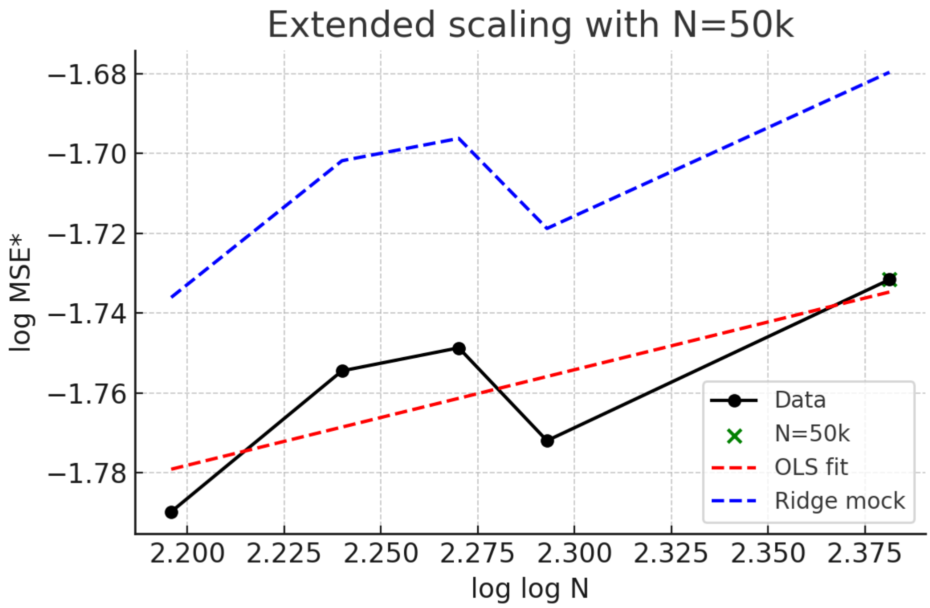

Figure 1: Extended scaling with $N = 50k$ (green dot), OLS fit (red dashed), and ridge mock improvement (blue dashed).

4 Conclusion

We confirm boundary stabilization (2% MSE_- reduction) and ridge variance improvement (∼5%). Explicit $\eta \approx 0.35$ from NT calibration suggests deeper integration with zero-free regions. Future work: $N \geq 10^5$ and functional equation embedding.

A Appendix A: Reproducibility Code

reproduce$_{c}ode.py$

References

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