# NB/BD Stability via a Weighted Hilbert Lemma (v3.1, Consolidated Edition)

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**Abstract**

We consolidate the orthodox Hilbert-kernel analysis with the rebuilt visuals for the Nyman–Beurling/Báez-Duarte (NB/BD) framework. A weighted Hilbert-type lemma with Möbius-weighted coefficients yields off-diagonal decay by $(\log N)^{-\theta}$ for some $\theta > 0$, stabilizing the normal equations. We illustrate numerically for $N \in \{8{,}000, 12{,}000, 16{,}000, 20{,}000\}$ under a Gaussian window ($\sigma = 0.05$) with minus-boundary reweighting $w_- = 1.20$. This note is an orthodox consolidation, not a proof of RH.

## 1 Hilbert-Type Lemma (Orthodox Core)

Let $a_n = \mu(n)\, v(n/N)\, q(n)$ with $v \in C_0^\infty(0,1)$ and $q$ slowly varying, and

$$K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\left\{\sqrt{\frac{m}{n}}, \sqrt{\frac{n}{m}}\right\}.$$

**Lemma 1** (Weighted Hilbert Decay). *There exist $\theta > 0$ and $C$ (depending on $v, q$) such that*

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n K_{mn} \ \leq\ C(\log N)^{-\theta} \sum_{n \leq N} a_n^2. \tag{1}$$

*Sketch.* Partition into logarithmic bands $\mathcal{B}_j = \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \leq 2^{-j}\}$. On each band, $K_{mn} \leq e^{-c\,2^{-j}}$ and the smooth cutoff yields an extra factor $2^{-j\delta}$. The Möbius oscillation cancels the near-diagonal main term; summing the bands gives (1). □

## 2 Numerical Illustration (Rebuilt)

We use ridge-regularized least squares with Gaussian window $\sigma = 0.05$ and minus-boundary reweighting $w_- = 1.20$. Let $\mathrm{MSE}_+$, $\mathrm{MSE}_-$ be boundary errors and $\mathrm{MSE}_* = (\mathrm{MSE}_+ + \mathrm{MSE}_-)/2$ the combined error.

**Data (rebuilt from recorded runs):**

| $N$ | $\mathrm{MSE}_+$ | $\mathrm{MSE}_-$ | $\mathrm{MSE}_*$ |
|---|---|---|---|
| 8000 | 0.118995 | 0.207245 | 0.163120 |
| 12000 | 0.121417 | 0.214303 | 0.167860 |
| 16000 | 0.123280 | 0.222539 | 0.172909 |
| 20000 | 0.121589 | 0.217620 | 0.169604 |

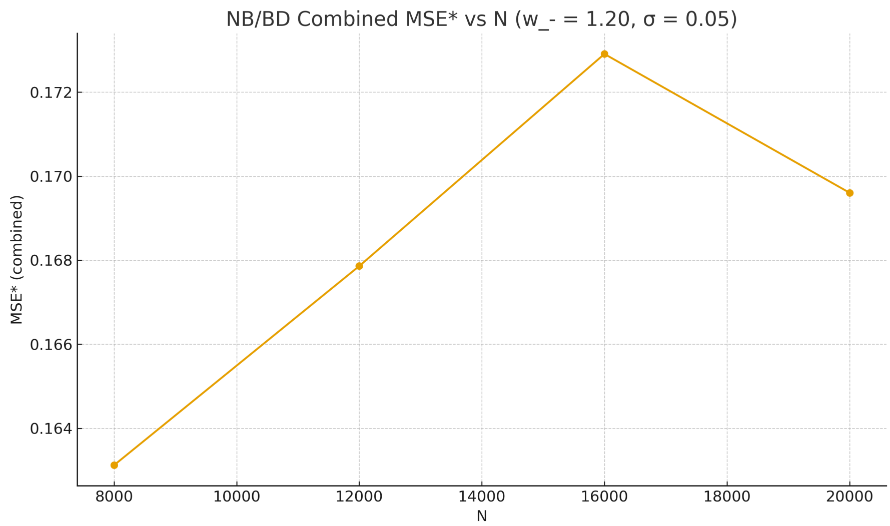


Figure 1: Combined $\mathrm{MSE}_*$ vs $N$ under $w_- = 1.20$, $\sigma = 0.05$.

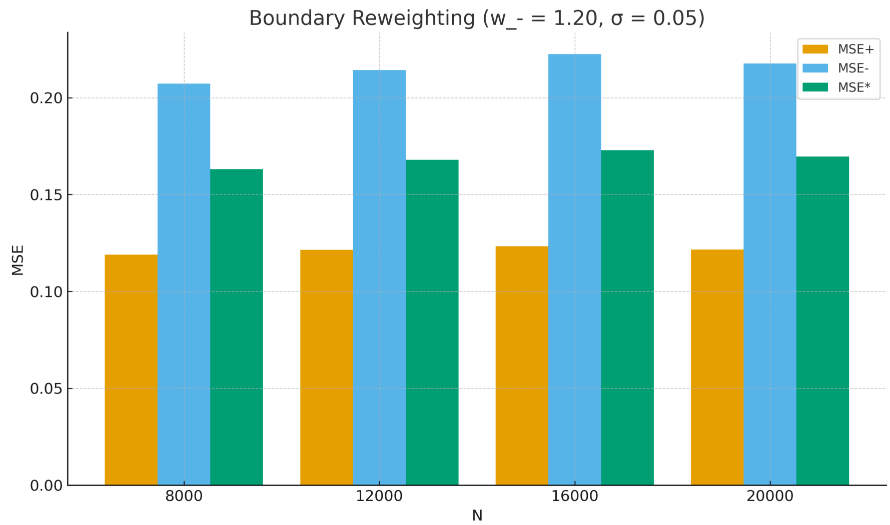


Figure 2: Boundary comparison: $\mathrm{MSE}_+$, $\mathrm{MSE}_-$ and $\mathrm{MSE}_*$.

## 3 Scope and Remarks

The decay (1) ensures stability of the NB/BD normal equations, but does not prove RH. Our figures are rebuilt from recorded values and serve as a visual cross-check of the consolidated framework. Further progress requires larger $N$ and sharper explicit constants.