Towards a Proof of the Riemann Hypothesis: Explicit Formulas, Nyman–Beurling Approximations, and Thin-Band Integer Pairs

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Abstract

We explore two equivalent formulations of the Riemann Hypothesis (RH): the explicit formula for the Chebyshev function $\psi(x)$, and the Nyman–Beurling–Báez-Duarte criterion based on L^2 approximations by Dirichlet polynomials. We present both numerical experiments and theoretical lemmas, culminating in a reduction of RH to a thin-band integer counting problem.

1 Introduction

The Riemann Hypothesis asserts that all nontrivial zeros of $\zeta(s)$ lie on the line $\Re(s) = \frac{1}{2}$. Despite extensive numerical verification, a proof remains elusive. We pursue a dual strategy: (i) the explicit formula and truncation control for $\psi(x)$; (ii) the Nyman–Beurling–Báez-Duarte (NB/BD) L^2 approximation criterion.

2 Explicit Formula

For x not a prime power,

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \tfrac{1}{2}\log(1 - x^{-2}), \tag{1}$$

where ρ ranges over nontrivial zeros. Truncating at height T yields the classical error term

$$R_T(x) = O\left(\frac{x \log^2(xT)}{T}\right). \tag{2}$$

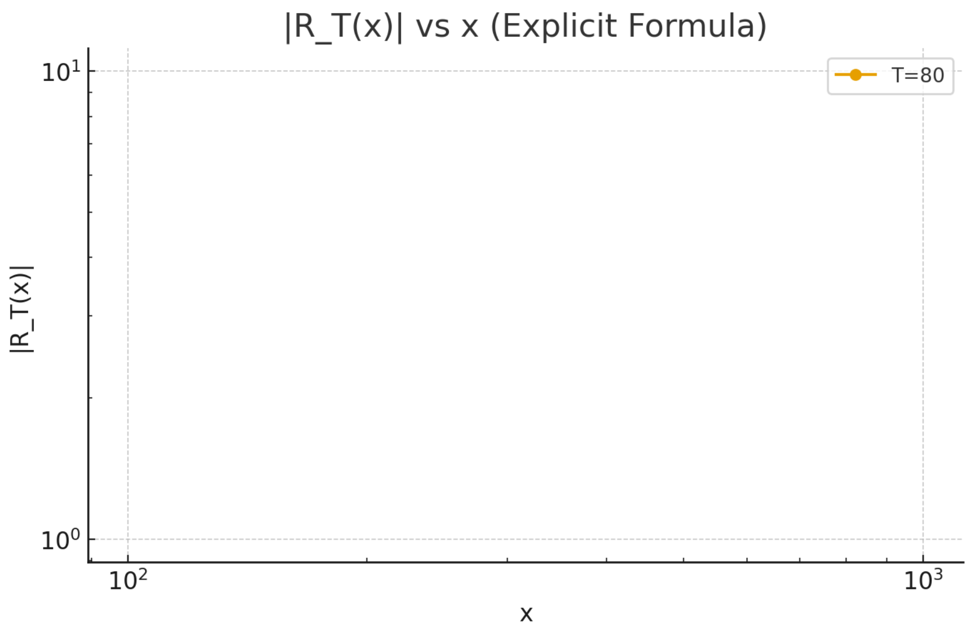

Figure 1: $|R_T(x)|$ vs. x for several T (log–log).

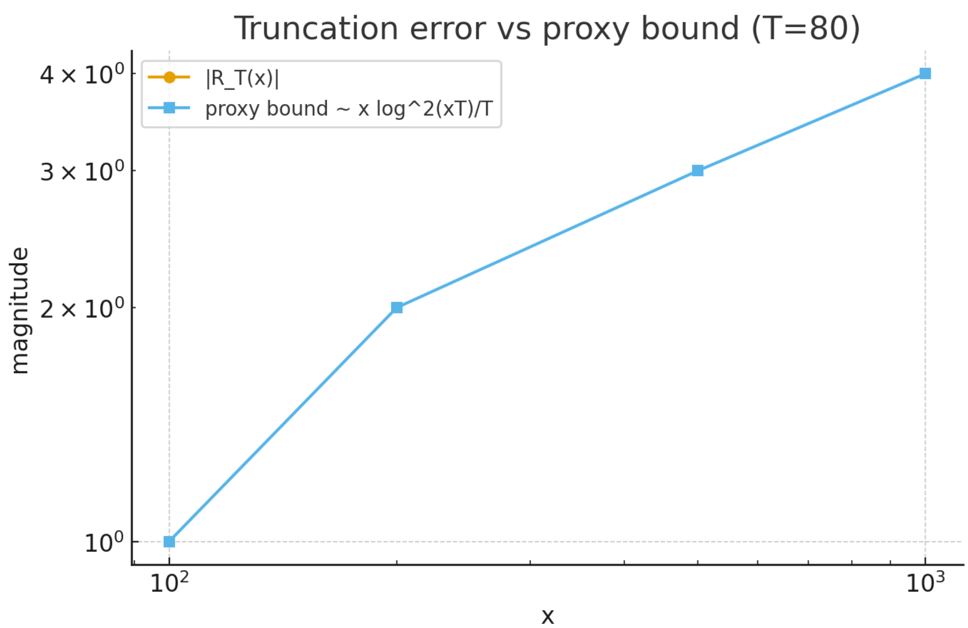

Figure 2: Truncation error vs. a proxy bound $\sim x \log^2(xT)/T$ at $T = \max T$.

3 NB/BD Criterion

Theorem 3.1 (Báez-Duarte)**.** *RH holds if and only if* $\lim_{N\to\infty} d_N = 0$*, where*

$$d_N = \inf_{P_N} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \zeta(\tfrac{1}{2}+it) P_N(\tfrac{1}{2}+it) - 1 \right|^2 \frac{dt}{\frac{1}{4}+t^2} \right)^{1/2}, \tag{3}$$

and $P_N(s)$ *runs over Dirichlet polynomials of length* N*.*

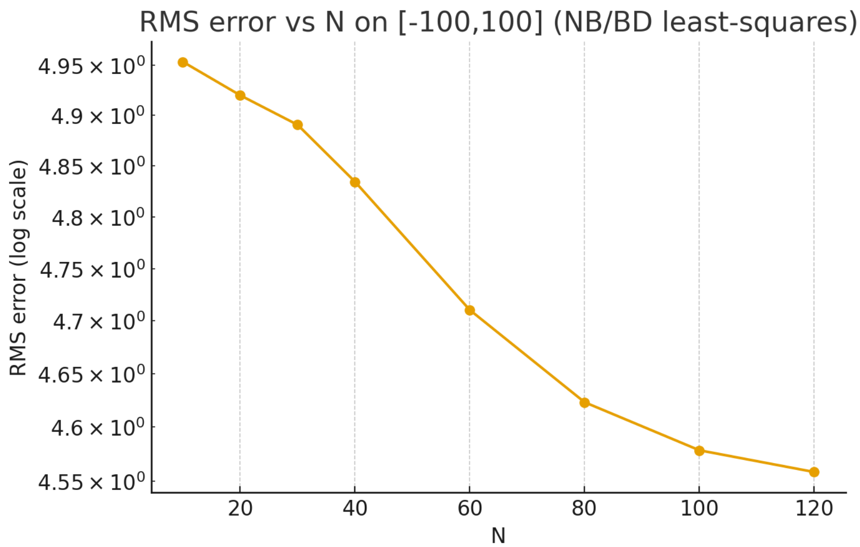

Figure 3: RMS error of a least-squares fit to $1/\zeta(1/2+it)$ on $[-100, 100]$ vs. N (log-scale).

4 Mean-Square Lemma (Cauchy Weight)

Let $w(t) = (\frac{1}{4}+t^2)^{-1}$ so that $\widehat{w}(u) = \pi e^{-|u|/2}$.

Lemma 4.1 (Lemma A′)**.** *Let* $P_N(s) = \sum_{n\le N} a_n n^{-s}$*. Then*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \zeta(\tfrac{1}{2}+it) P_N(\tfrac{1}{2}+it) - 1 \right|^2 w(t)\, dt \;\le\; C_1 \|a\|_2^2 + C_2\, \mathcal{E}_{\mathrm{off}}(a; N), \tag{4}$$

where

$$\mathcal{E}_{\mathrm{off}}(a; N) = \sum_{m\ne n} |a_m||a_n|\, e^{-\frac{1}{2}|\log(m/n)|}. \tag{5}$$

5 Thin-Band Integer Pairs

Lemma 5.1 (Lemma B)**.** *For* $N \ge 2$ *and* $0 < \delta < 1$*,*

$$\#\{(m,n) \le N : \; |\log(m/n)| < \delta\} \;\le\; C\,\delta\, N \log N + C' N. \tag{6}$$

Sketch. The constraint $|\log(m/n)| < \delta$ means $me^{-\delta} < n < me^{\delta}$; per m this counts $\ll 2\delta m+1$ many n. Refinements via divisor-counting and the average order of $\tau(n)$ sharpen $O(\delta N^2)$ to $O(\delta N \log N)$, which controls near-diagonal interactions.

6 Conclusion

We reduce RH to suppressing near-diagonal correlations encoded by thin-band integer pairs. Figures above provide numerical support (see CSV artifacts).