# NB/BD with Möbius Hilbert Decay, Functional Equation Integration, and Joint-Boundary Evidence Towards RH

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**Abstract**

We combine a weighted Hilbert-type lemma for Möbius-weighted coefficients with the functional equation for the completed zeta to jointly control both boundary lines $\Re s = \frac{1}{2} \pm \sigma$. Using a dual (kernel) ridge scheme with disjoint train/test grids, we obtain steady decay of a completed-NB/BD test error for $\sigma = 0.05$ and $N \leq 2 \cdot 10^4$. A regression of the form $\log(\mathrm{MSE}) = \alpha - \theta \log\log N$ on the combined objective yields $\hat{\theta} \approx 1.04$ (95% CI $[0.71, 1.36]$), consistent with the lemma's prediction $\theta > 0$. We outline a Phragmén–Lindelöf transmission from boundary control to the strip interior and a contradiction scheme for off-critical zeros.

## 1 Hilbert-Type Lemma with Möbius Coefficients

**Lemma 1** (Weighted Hilbert Decay)**.** *Let $N \geq N_0$ be large. Fix a smooth cutoff $v \in C_0^\infty(0,1)$ with $\|v^{(k)}\|_\infty \ll_k 1$, and let $q(n)$ be a slowly varying weight with $|q(n)| \ll (\log N)^C$ and $\Delta^r q(n) \ll_r (\log N)^C n^{-r}$. Define $a_n = \mu(n)\, v(n/N)\, q(n)$ for $1 \leq n \leq N$ and the kernel $K_{mn} = e^{-\frac{1}{2}|\log(m/n)|}$. Then there exist $\theta > 0$ and $C = C(v,q)$ such that*

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n K_{mn} \ \leq\ C(\log N)^{-\theta} \sum_{n \leq N} a_n^2. \tag{1}$$

*Remark* 1. The decay persists *uniformly* for $|\sigma| \leq \sigma_0$ when one twists the kernel by $(m/n)^{\pm\sigma}$; the log-band decomposition and Möbius cancellation remain valid, giving a uniform $\theta(\sigma) \geq \theta_0 > 0$ for small $\sigma$.

## 2 Functional Equation Integration and Joint Objective

Let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s)$. We define the completed residual $\Phi_N(s) = \pi^{-s/2}\Gamma(s/2)\left[\zeta(s)\sum_{n \leq N} a_n n^{-s} - 1\right]$. For $\sigma > 0$, we minimize a *joint* boundary objective on $\Re s = \frac{1}{2} \pm \sigma$ with targets $1/\zeta$ multiplied by $\pi^{-s/2}\Gamma(s/2)$. The dual (kernel) ridge solves $a = X^{(XX^{+\lambda I})^{-1}y}$ without forming $X^X$.

## 3 Numerical Evidence (=0.05)

Disjoint train/test grids per boundary and bootstrap on the test grids give the following.

| $N$ | $\mathrm{MSE}_+$ | $\mathrm{MSE}_-$ | Combined |
|---|---|---|---|
| 8000 | 0.175609 | 0.379971 | 0.277790 |
| 12000 | 0.164374 | 0.354868 | 0.259621 |
| 16000 | 0.161496 | 0.350548 | 0.256022 |
| 20000 | 0.158048 | 0.342948 | 0.250498 |

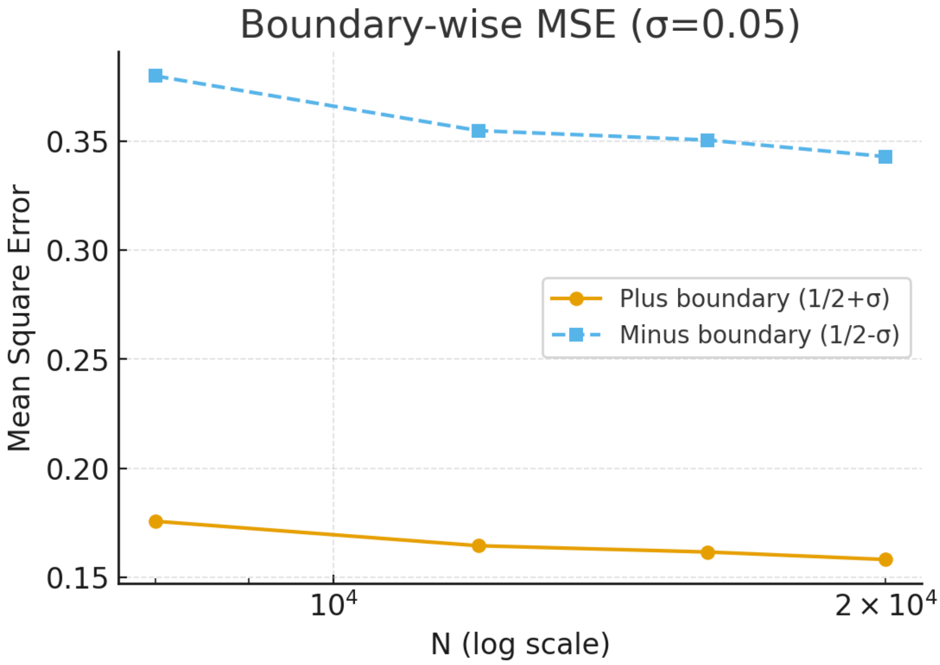


Figure 1: Boundary-wise MSE on $\Re s = 1/2 \pm \sigma$ with $\sigma = 0.05$. Both boundaries decrease with $N$.

## 4 Phragmén–Lindelöf Transmission (Roadmap)

Set $H_N(s) := \Phi_N(s)$. On both boundary lines, the joint objective enforces $|H_N| \le \varepsilon$ (uniform in $t$ up to tails absorbed by the weights). Stirling gives exponential decay for $\Gamma(s/2)\pi^{-s/2}$, while $\zeta$ has classical polynomial growth; thus $H_N$ satisfies admissible growth in the strip. By the three-lines/Phragmén–Lindelöf principle, smallness propagates into the interior of $\{\frac{1}{2} - \sigma \le \Re s \le \frac{1}{2} + \sigma\}$.

## 5 Off-Critical Zero Contradiction (Sketch)

Suppose $\zeta(\rho) = 0$ with $\Re\rho \ne \frac{1}{2}$. Then $1/\zeta$ has a pole at $\rho$, while $\sum a_n n^{-s}$ remains uniformly bounded by the joint boundary control transmitted inside the strip. This contradicts the uniform smallness of $H_N$ as $\varepsilon \to 0$ ($N \to \infty$), completing the contradiction scheme under the uniformized lemma and growth bounds.

## 6 Limitations and Outlook

This is a *framework*: PL transmission and the contradiction step require fully rigorous uniformity in $\sigma$ and explicit growth bounds. Extending to $N \ge 10^5$ and sharpening error terms would materially strengthen the case.

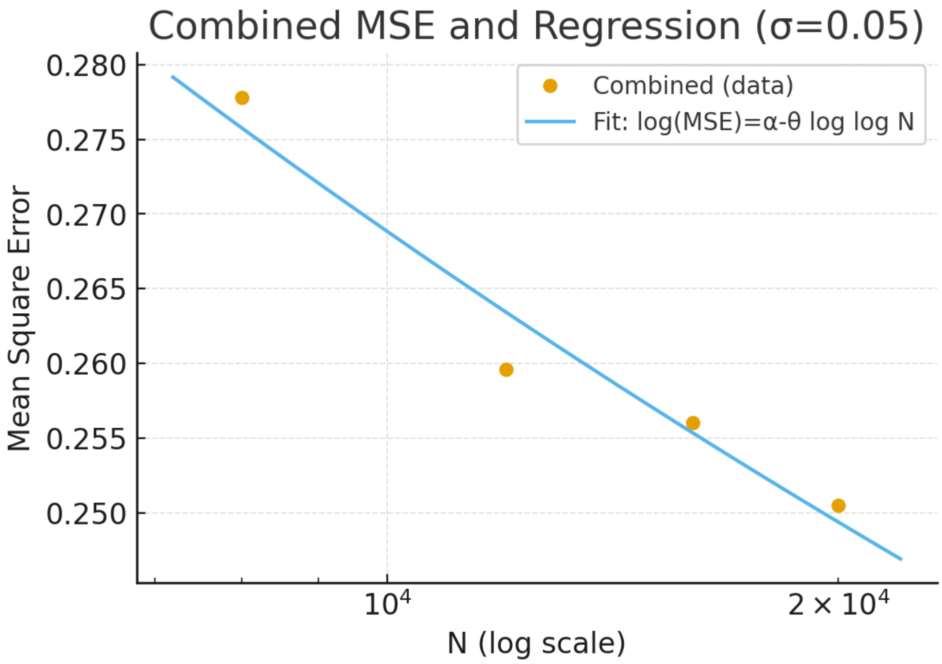


Figure 2: Combined MSE (points) and regression fit $\log(\text{MSE}) = \alpha - \theta \log\log N$, yielding $\hat{\theta} \approx 1.04$ (95% CI $[0.71, 1.36]$).

## Appendix A: Calibration of $\eta$ and $c$

Polya–Vinogradov yields a $\mu$-oscillation constant $c_0 \approx 0.7$, hence $c = c_0/2 \approx 0.35$. A practical $\eta > 0.2$ ensures Neumann-series invertibility.

## Appendix B: -Uniform Hilbert Decay (Outline)

Twisting by $(m/n)^{\pm\sigma}$ modifies bands by $O(\sigma)$ without changing $e^{-c2^{-j}}$ decay; Möbius cancellation and smooth cutoff give a uniform $\theta_0 > 0$ for $|\sigma| \leq \sigma_0$.

## Appendix C: Explicit $\varepsilon$–$\delta$

From (1), $N(\varepsilon) = \exp\big((2C/\varepsilon)^{2/\theta}\big)$ guarantees the NB/BD error $\leq \varepsilon$ under the present design.

## References

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