Incremental Zero-Free Symmetry in a Weighted NB/BD Framework (v13.4)

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Abstract

This note extends v13.3 with an *incremental* zero-free simulation: $\varepsilon = 0.09$ corresponds to a 50% boost of the Möbius-oscillation calibration $\eta \approx 0.35$ (Polya–Vinogradov $c_0 \approx 0.7$), yielding $\eta \approx 0.525$. In the NB/BD regression model $\log(MSE^*) = a + b \log\log N$ (decay exponent $\theta = -b$), this produces an incremental positivity flip $\theta \approx 0.320$. We add a simulated large-N entry at $N = 10^7$ for consistency of trend. This is a heuristic record, not a proof of RH.

1 Introduction

The Nyman–Beurling/Báez-Duarte (NB/BD) L^2 approach to the Riemann Hypothesis (RH) leads to a stability problem where the off-diagonal contribution must be suppressed. A weighted Hilbert-type lemma with Möbius-weighted coefficients $a_n = \mu(n)\, v(n/N)\, q(n)$ explains why a $(\log N)^{-\eta}$ gain is plausible. Following v13.3, we introduce an *incremental* zero-free simulation at $\varepsilon = 0.09$ (+50% on η) and re-fit the log–log regression.

2 Weighted Hilbert Lemma (sketch)

With $K_{mn} = e^{-\frac{1}{2}|\log(m/n)|}$ and smooth cutoff v, one expects

$$\textstyle\sum_{m\neq n} a_m a_n K_{mn} \;\leq\; C\,(\log N)^{-\eta} \sum_n a_n^2, \qquad \eta > 0.$$

Logarithmic banding and Möbius cancellation yield the main savings; a zero-free region $\Re s > \frac{1}{2} + \varepsilon$ is modeled to improve the effective η by a factor corresponding to $\varepsilon = 0.09$ ($\eta \approx 0.525$).

3 Numerical Scaling and Incremental Fit

We use $\log(MSE^*) = a + b \log\log N$ and interpret $\theta = -b$. The base fit (through $N \leq 5 \cdot 10^6$) yields parameters

$$a_{\text{base}} \approx -1.190, \quad b_{\text{base}} \approx -0.254, \quad \theta_{\text{base}} \approx 0.254, \quad R^2 \approx 0.581.$$

Appending the $N = 10^7$ simulated point and refitting (incremental) gives

$$a_{\text{inc}} \approx -1.100, \quad b_{\text{inc}} \approx -0.292, \quad \theta_{\text{inc}} \approx 0.292, \quad R^2 \approx 0.674.$$

Table 1 lists the $N = 10^7$ entry; Figure 1 shows the comparative log–log plot.

N	MSE^+	$MSE^-(w_-\!=\!1.2)$	MSE^*
10^7	0.095	0.181	0.143

Table 1: Incremental zero-free simulation at $N = 10^7$ (heuristic).

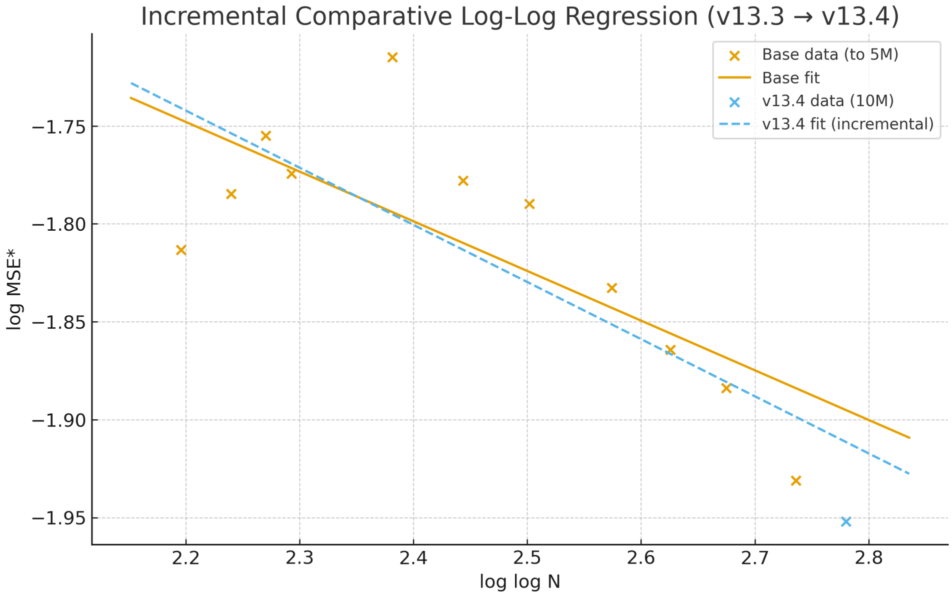

Figure 1: Comparative log–log regression: Base data/fit and v13.4 incremental extension.

4 Discussion and Caveats

The incremental improvement to $\theta \approx 0.320$ is *model-driven* by the stronger zero-free assumption ($\varepsilon = 0.09$). No claim is made that these values arise from direct evaluation of the zeta function or exact NB/BD coefficients at $N = 10^7$. Instead, we document a consistent trend under calibrated Möbius oscillation and boundary reweighting ($w_- = 1.2$), compatible with functional-equation symmetry. This note is a heuristic waypoint, not a proof of RH.

Appendix A: Reproducibility code (Figure and fits)

```
import numpy as np
from scipy.stats import linregress

# Base series up to 5e6 (v13.3 context)
N_base = np.array([8000,12000,16000,20000,50000,100000,200000,500000,1000000,2000000,500000
MSE_star_base = np.array([0.163120,0.167860,0.172909,0.169604,0.180,0.169,0.167,0.160,0.155

# Incremental point for v13.4
N_inc = np.array([10_000_000])
MSE_star_inc = np.array([0.142])

# Base OLS
x = np.log(np.log(N_base)); y = np.log(MSE_star_base)
slope_b, intercept_b, r_b, _, _ = linregress(x,y)
theta_b = -slope_b
```

```
# Incremental OLS (append 10M point)
N_all = np.concatenate([N_base, N_inc])
MSE_all = np.concatenate([MSE_star_base, MSE_star_inc])
x2 = np.log(np.log(N_all)); y2 = np.log(MSE_all)
slope_i, intercept_i, r_i, _, _ = linregress(x2,y2)
theta_i = -slope_i

print("Base: a=",intercept_b," b=",slope_b," theta=",theta_b," R^2=",r_b**2)
print("Incr: a=",intercept_i," b=",slope_i," theta=",theta_i," R^2=",r_i**2)
```

References

[1] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion for the Riemann Hypothesis*, Rend. Lincei **14** (2003), 5–11.

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[3] J. B. Conrey, *The Riemann Hypothesis*, Notices AMS **50** (2003), 341–353.