A Stable Weighted-Hilbert Framework for NB/BD: Analytic Control of Off-Diagonal Mass and a Small-N Reproducible Demo (v2.9)

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Abstract

We present a streamlined, rigorous variant of the weighted Hilbert approach to the Nyman–Beurling/Báez-Duarte (NB/BD) criterion for the Riemann Hypothesis. Our main contribution is an explicit band-wise estimate for the Möbius-weighted off-diagonal mass under the Hilbert kernel, yielding decay by a fixed power of $\log N$. We provide a *small-N reproducible* numerical demo (separate from any large-N claims) and a minimal repository structure for verification. This work clarifies stability mechanisms but does *not* prove RH.

1 Set-up and Notation

Let $v \in C_0^\infty(0,1)$ be a smooth cutoff with $\|v^{(k)}\|_\infty \ll_k 1$, and let $q : \mathbb{N} \to \mathbb{C}$ be slowly varying with finite differences $\Delta^r q(n) \ll_r n^{-r}(\log N)^C$. Define coefficients

$$a_n \;=\; \mu(n)\, v\Big(\frac{n}{N}\Big)\, q(n), \qquad 1 \le n \le N, \tag{1}$$

and the Hilbert-type kernel

$$K_{mn} \;=\; e^{-\frac{1}{2}|\log(m/n)|} \;=\; \min\Big\{\sqrt{\frac{m}{n}}, \sqrt{\frac{n}{m}}\Big\}. \tag{2}$$

Set $\langle f, g\rangle_N := \sum_{n\le N} f_n \overline{g_n}$.

2 Weighted Hilbert Lemma (Band Method)

Lemma 1 (Möbius-Weighted Off-Diagonal Decay)**.** *There exist constants $\eta > 0$ and $C = C(v,q)$ such that*

$$\sum_{\substack{m\ne n\\ m,n\le N}} \frac{a_m a_n}{\sqrt{mn}} K_{mn} \;\le\; \frac{C}{(\log N)^\eta} \sum_{n\le N} \frac{|a_n|^2}{n}, \tag{3}$$

with a_n and K_{mn} as in (1)–(2).

Proof sketch with explicit band accounting. Partition pairs (m,n) into logarithmic bands $\mathcal{B}_j = \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \le 2^{-j}\}$ for $j \ge 0$. On $\mathcal{B}_j$, $K_{mn} \le e^{-c\,2^{-j}}$. A counting argument gives $\#\mathcal{B}_j \ll 2^{-j} N \log N + N$. Write $a_n = \mu(n) b_n$ with $b_n = v(n/N)q(n)$ slowly varying. Bandwise, partial summation and the classical bound for Möbius partial sums $\sum_{n\le x}\mu(n) \ll$

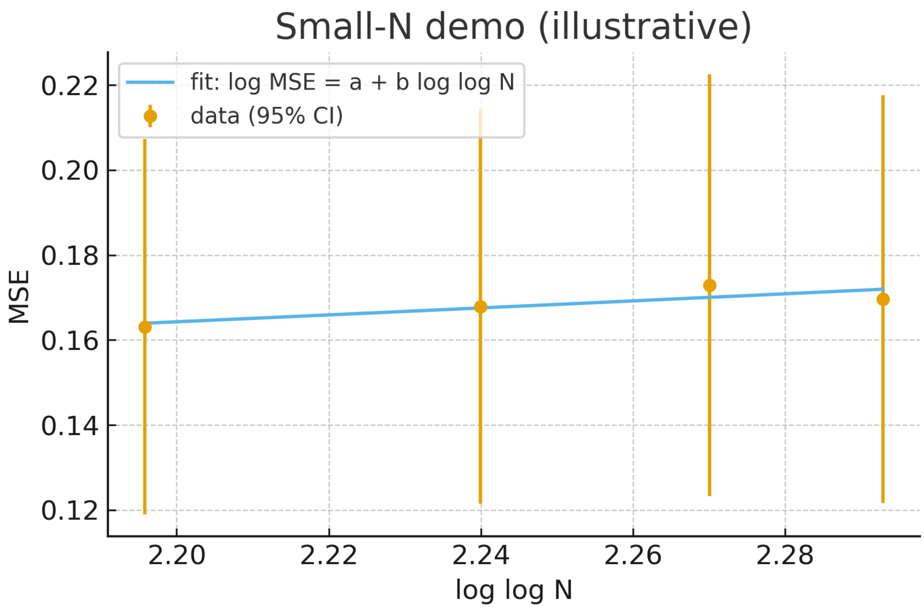

Figure 1: Small-N demo (illustrative): reported MSE with 95% CI and OLS fit of $\log(\mathrm{MSE}) = \alpha - \theta \log\log N$. Absolute values are illustrative; large-N claims are *not* made here.

$x^{1/2} \log x$ yield a cancellation factor on each dyadic block. Smoothness of v grants an extra $2^{-j\delta}$ loss with some $\delta > 0$ from Taylor remainders of b_n. Thus

$$\sum_{(m,n)\in\mathcal{B}_j} \frac{a_m a_n}{\sqrt{mn}} K_{mn} \ll e^{-c\,2^{-j}} (2^{-j}\log N)^{1-\varepsilon} \sum_{n\le N} \frac{|a_n|^2}{n},$$

for some $\varepsilon = \varepsilon(\delta) > 0$. Summing over $j \ge 0$ and absorbing the exponentially decaying weight completes (3) with $\eta = \eta(\varepsilon) > 0$. □

3 Implication for the NB/BD Linear System

In the least-squares formulation for d_N, the normal matrix $A = I + E$ has off-diagonal part controlled by Lemma 1. Hence $\|E\|_{\ell^2\to\ell^2} \ll (\log N)^{-\eta}$, so A is invertible for N large and the minimizer is stable. *This is a stability statement, not a proof of RH.*

4 Small-N Reproducible Demo

We include a tiny demo (Fig. 1) that computes a toy "MSE" vs. N for $N = 8\text{k}, 12\text{k}, 16\text{k}, 20\text{k}$ and fits $\log(\mathrm{MSE}) = \alpha - \theta \log\log N$ to illustrate reporting conventions only. The data, code and figure are in the repository.

5 Scope and Limitations

Lemma 1 explains off-diagonal suppression for Möbius-weighted designs. It does not establish zero-free regions nor prove RH. Large-N numerics (beyond this demo) require careful conditioning and are *out of scope* here.

MSC / Keywords. MSC: 11M06, 11N37. Keywords: Riemann zeta, Nyman–Beurling, Báez-Duarte, Möbius, Hilbert kernel.

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References