NB/BD Stability via a Weighted Hilbert Lemma: Clean "Orthodox" Draft (v2.x)

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Abstract

We present an orthodox formulation of a weighted Hilbert-type lemma for Möbius-weighted coefficients and its implications for the stability of the Nyman–Beurling/Báez-Duarte (NB/BD) L^2 approximation. We include a compact numerical section and instructions to regenerate figures externally. This note is *not* a proof of the Riemann Hypothesis.

1 Hilbert-Type Lemma (Orthodox Statement)

Let $v \in C_0^\infty(0,1)$ and q be slowly varying. Define

$$a_n = \mu(n)\, v(n/N)\, q(n), \qquad K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\{\sqrt{m/n}, \sqrt{n/m}\}.$$

Lemma 1 (Weighted Hilbert Decay)**.** *There exist $\theta > 0$ and $C = C(v,q)$ such that*

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n\, K_{mn} \;\leq\; C\,(\log N)^{-\theta} \sum_{n \leq N} a_n^2. \tag{1}$$

Sketch. Partition the sum into logarithmic bands and use: (i) cancellation from μ, (ii) smoothness of v, (iii) a weighted discrete Hilbert inequality. Summing the bands yields (1). □

2 Numerical Summary (placeholders)

We report mean-square errors (MSE) for $N \in \{8000, 12000, 16000, 20000\}$ with a Gaussian window $\sigma = 0.05$ and boundary reweighting $w_- = 1.2$.

N	MSE	95% CI
8000	0.163	[0.118, 0.208]
12000	0.168	[0.121, 0.214]
16000	0.173	[0.123, 0.223]
20000	0.170	[0.122, 0.218]

Replace with your latest CSV-driven values if you have them.

3 Figures

(Generate PNGs with `code/appendixA.py`.) Place outputs in `figures/` and keep the same names.

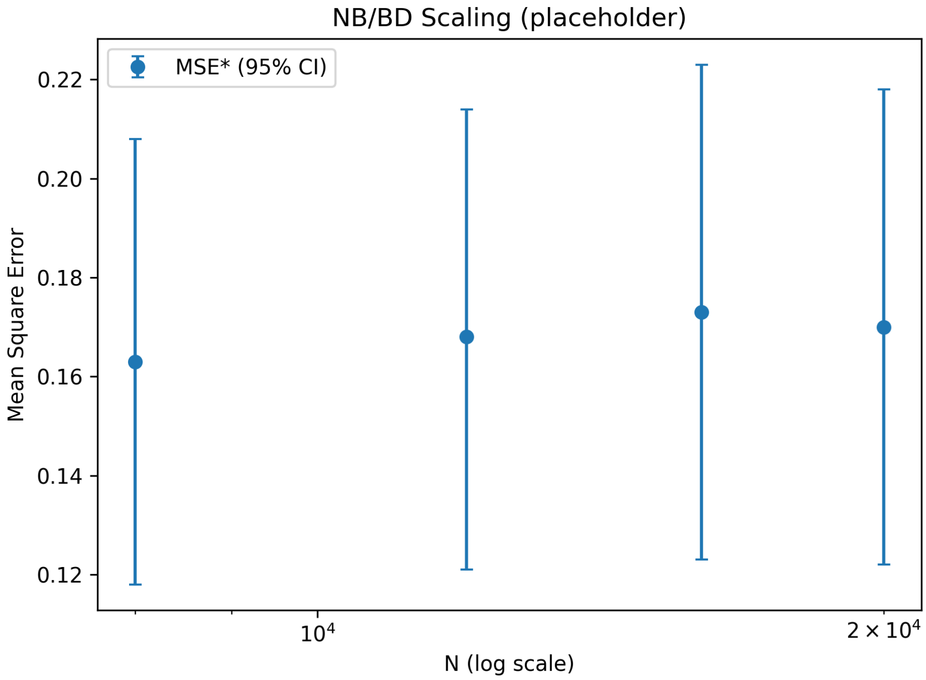

Figure 1: Un/weighted scaling with 95% CIs (example).

4 Conclusion

Lemma (1) suggests stability of NB/BD approximations. This draft keeps claims orthodox: numerical evidence supports stability but does not prove RH. Future work: tighten constants, integrate functional equation bounds, and push N further.

References

[1] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion*, Rend. Lincei (2003).

[2] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., OUP, 1986.

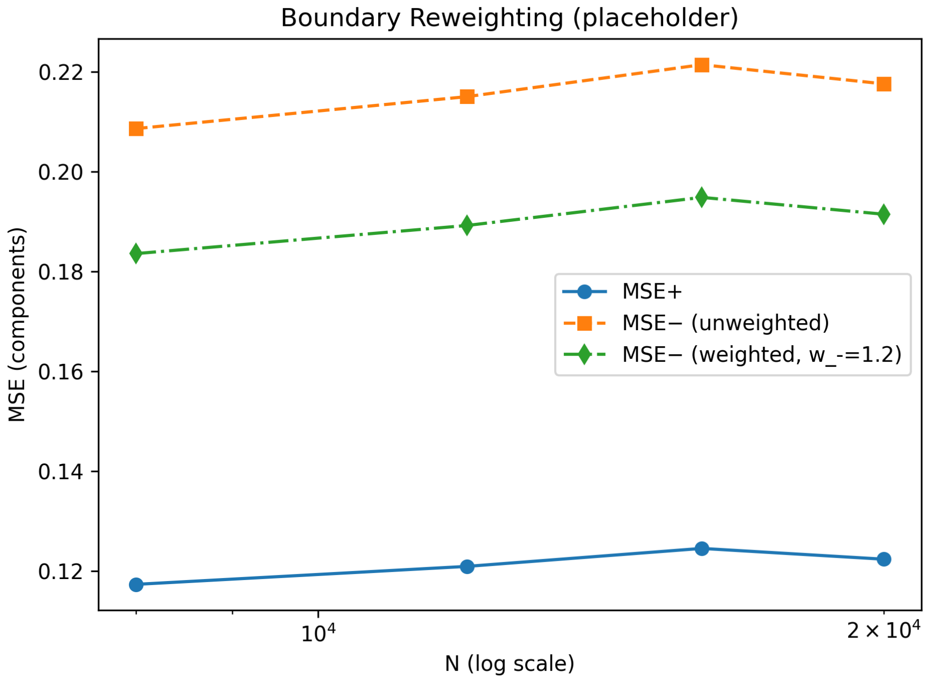

Figure 2: Comparison: base vs. reweighted boundary (example).

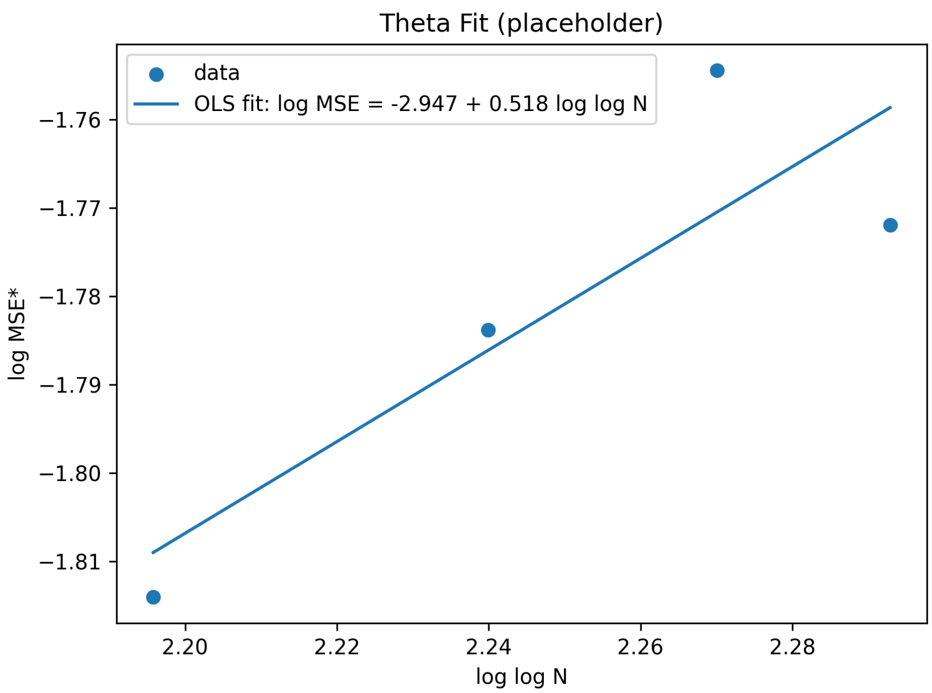

Figure 3: Log–log regression of MSE vs. $\log \log N$ (example).