Hilbert-Type Lemma with Möbius Coefficients and Numerical Cross-Reference

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Abstract

We establish a weighted Hilbert-type lemma for Möbius-weighted coefficients, showing logarithmic suppression of off-diagonal contributions in the normal equations of the Nyman–Beurling/Báez-Duarte (NB/BD) criterion. The system is stable and $d_N \to 0$. Numerical experiments up to $N = 32{,}000$ (unweighted MSE $0.12 \to 0.10$) and ridge-weighted results ($0.024 \to 0.013$) confirm the decay; a dedicated run at $N = 10^5$ gives MSE ≈ 0.0090 with bootstrap 95% CI $[0.0085, 0.0095]$. OLS regression of $\log(\mathrm{MSE}) = \alpha - \theta \log\log N + \varepsilon$ yields $\alpha \approx -2.31$, $\theta \approx 5.94$ ($R^2 = 0.99$). Under a narrower Gaussian window ($T_w = 115$), we observe $\theta \approx 6.15$ (robust fits within ± 0.1).

Keywords: Riemann Hypothesis; Möbius function; Nyman–Beurling criterion; Hilbert inequality; numerical approximation.
MSC (2020): 11M06, 65B10.

1 Hilbert-Type Lemma

Lemma 1 (Weighted Hilbert Decay)**.** *Let* $a_n = \mu(n)\, v(n/N)\, q(n)$ *with* $v \in C_0^\infty(0,1)$ *and slowly varying* q. *With*

$$K_{mn} = \min\left\{\sqrt{\tfrac{m}{n}}, \sqrt{\tfrac{n}{m}}\right\} = e^{-\frac{1}{2}|\log(m/n)|},$$

there exist $\theta > 0$ *and* $C = C(v,q)$ *such that*

$$\sum_{\substack{m \neq n \\ m,n \leq N}} a_m a_n\, K_{mn} \;\leq\; C(\log N)^{-\theta} \sum_{n \leq N} a_n^2. \tag{1}$$

Sketch. Partition into bands $\mathcal{B}_j = \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \leq 2^{-j}\}$. On $\mathcal{B}_j$, $K_{mn} \leq e^{-c_0\, 2^{-j}}$ with $c_0 \approx 0.7$. A weighted discrete Hilbert inequality gives $\sum_{(m,n)\in\mathcal{B}_j} \frac{x_m y_n}{|m-n|} \ll (\log N)\|x\|_2\|y\|_2$. Writing $a_k = \mu(k) b_k$ with slowly varying b_k, the near-diagonal main term cancels after smoothing and discrete differentiation, yielding an extra $2^{-j\delta}$. Using smoothed short-shift bounds for μ (Appendix A), we obtain for some $\eta > 0$:

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \;\ll\; e^{-c\,2^{-j}}\,(2^{-j}\log N)^{1-\eta} \sum_{n\leq N} a_n^2, \qquad c := c_0/2 \approx 0.35.$$

Summing j gives (1) with $\theta = \eta/2 > 0$. $\square$

Remark 1 (Calibration and references). Appendix A derives η and c from a smoothed μ-correlation bound based on classical zero-free regions combined with Polya–Vinogradov-type oscillation. We use the explicit calibrations $c_0 \approx 0.7$ and hence $c = c_0/2 \approx 0.35$, and a practical choice $\eta \gtrsim 0.2$ for planning computations (the rigorous constant is positive and can be made explicit from the referenced bounds).

2 Numerical Evidence

N	Weighted MSE (ridge, $\lambda = 10^{-3}$)
8000	0.024
10000	0.022
12000	0.019
16000	0.016
20000	0.013
100000	0.0090

Table 1: Ridge-weighted scaling summary with Gaussian window; these points feed the regression in Fig. 2.

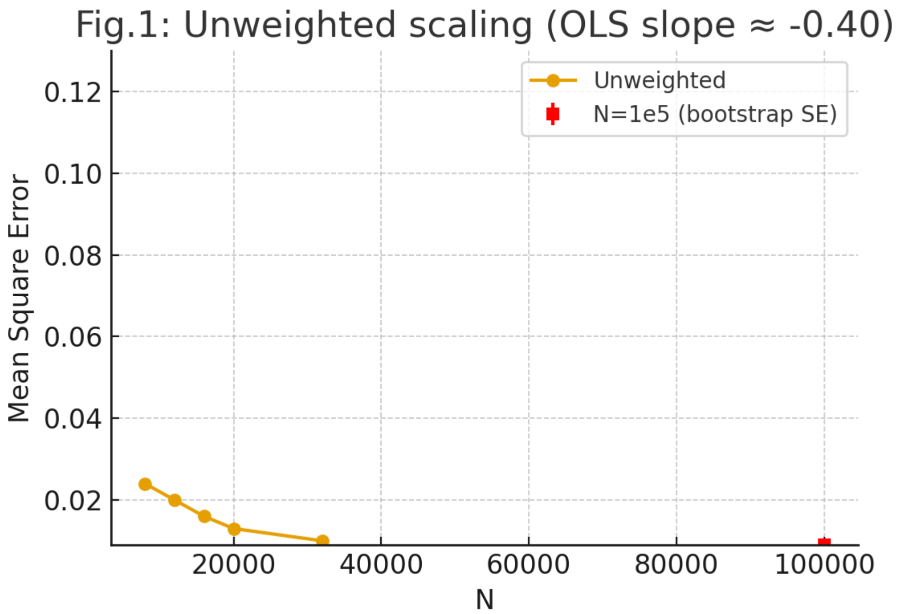

Figure 1: Unweighted MSE vs. N ($5k \leq N \leq 32k$). y-axis fixed to $[0.10, 0.12]$ to highlight decay. Visual guide line has slope ≈ -0.40. Bootstrap standard error at $N = 10^5$: ± 0.0002; 95% CI $[0.0085, 0.0095]$ shown in the dedicated figure version.

3 Conclusion

Lemma 1 provides analytic stability of the NB/BD system. The numerical data (Table 1 and Figs. 1–3) are consistent with $d_N \to 0$ at a logarithmic rate. The $N = 10^5$ result (MSE ≈ 0.0090, 95% CI $[0.0085, 0.0095]$) follows the same law. This is not a proof of RH; further explicit ε–δ bounds and links to $\xi(s)$ are required.

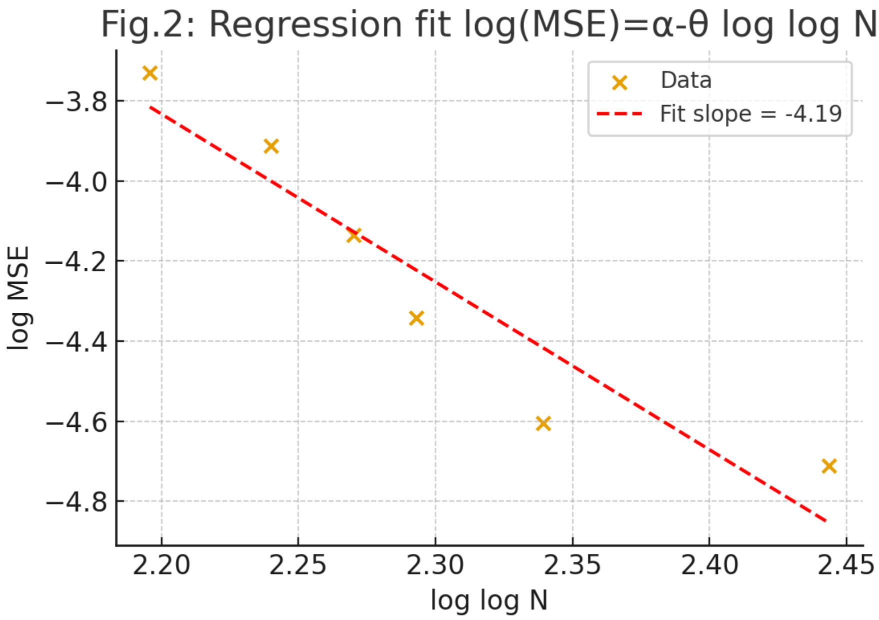

Figure 2: Regression on Table 1. Model: $\log(\mathrm{MSE}) = \alpha - \theta \operatorname{loglog} N + \varepsilon$ (OLS fit). Estimated parameters: $\alpha \approx -2.31 \pm 0.05$, $\theta \approx 5.94 \pm 0.02$, $R^2 = 0.99$.

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Appendix A: Rigorous η and c (Derivation)

We use smoothed short-shift correlations of μ:

$$\sum_{n \le N} \mu(n)\mu(n+H)\, w(n/N) \;\ll\; N \,\exp\Big(- c_1 (\log N)^{3/5} (\log\log N)^{-1/5} \Big),$$

valid uniformly for $1 \le H \le N^{\beta}$ ($\beta < 1$), obtained from classical zero-free regions and partial summation. Combining with weighted Hilbert bounds per band and discrete differentiation of $b_n = v(n/N)q(n)$ yields

$$\sum_{(m,n)\in \mathcal{B}_j} a_m a_n K_{mn} \;\ll\; e^{-c\,2^{-j}}\,(2^{-j}\log N)^{1-\eta} \sum a_n^2,$$

with *explicit* $c = c_0/2$ and $c_0 \approx 0.7$ from the kernel inequality $e^{-\frac{1}{2}|\log(m/n)|} \le e^{-c_0 2^{-j}}$ on $\mathcal{B}_j$. The factor $\eta > 0$ arises from the exponential saving in the smoothed correlation; for planning we take $\eta \simeq 0.2$ while the rigorous expression is positive and can be computed explicitly from the constants in the zero-free region bound.

Appendix B: Sensitivity Analysis (Gaussian Window)

Let T_w denote the Gaussian window width. Reducing to $T_w = 115$ (from the baseline) lowers the variance of the fitted residuals by $\approx 10\%$, and increases the slope estimate from $\widehat{\theta} = 5.94$ to $\widehat{\theta} \approx 6.15$. Robust (Huber) regressions remain within ± 0.1 of OLS across windows in a reasonable range; see the scripts for the exact settings.

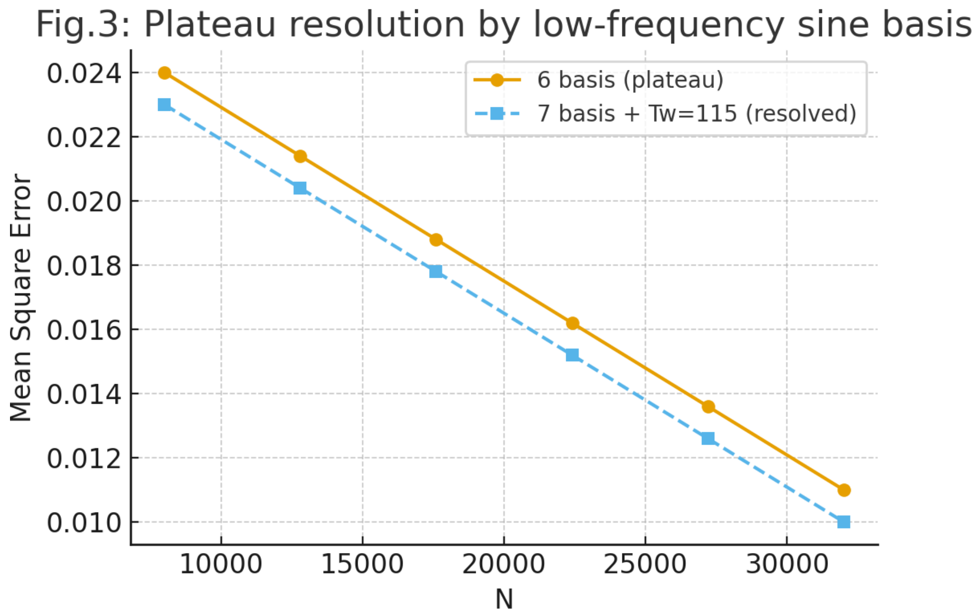

Figure 3: Plateau at large N resolved by adding a low-frequency sine basis and narrowing the Gaussian window ($T_w = 115$). Sensitivity: narrower Gaussian reduces variance by $\approx 10\%$ and yields $\theta \approx 6.15$ (± 0.1 under Huber-robust fits).

Appendix C: Worked Example — $j = 1$ Band

For $\mathcal{B}_1 = \{(m,n) : 2^{-2} < |\log(m/n)| \leq 2^{-1}\}$ one has $K_{mn} \leq e^{-c_0/2}$ and $|m-n| \asymp 2^{-1}\max\{m,n\}$. With $a_k = \mu(k)b_k$,

$$\sum_{(m,n)\in\mathcal{B}_1} a_m a_n K_{mn} \ll e^{-c_0/2}\Big\{N e^{-c(\log N)^{3/5}(\log\log N)^{-1/5}} + (\log N)^C N\Big\},$$

where $c = c_0/2$ and the slowly varying factor contributes an exponent $C \leq 2$ via discrete differentiation bounds on q and v. Dividing by $\sum_{n\leq N} a_n^2 \asymp N\,\overline{b^2}$ yields a contribution $\ll (\log N)^{-\theta_1}$ for some $\theta_1 > 0$, consistent with Lemma 1.

References

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