Hilbert-Type Lemma with Möbius Coefficients and Numerical Cross-Reference

Serabi

Independent Researcher

24ping@naver.com

2025

Abstract

We establish a weighted Hilbert-type lemma for Möbius-weighted coefficients, showing logarithmic suppression of off-diagonal contributions in the NB/BD normal equations. Numerically, unweighted MSE decays from 0.12 to 0.10 on 5k $\leq N \leq$ 32k; ridge-weighted MSE decreases from 0.024 to 0.013 ($N =$ 8k $\to$ 20k). A dedicated run at $N = 10^5$ yields MSE ≈ 0.0090 with bootstrap 95% CI $[0.0085, 0.0095]$. An OLS regression of $\log(\text{MSE}) = \alpha - \theta \log\log N + \varepsilon$ gives $\alpha \approx -2.31 \pm 0.05$ and $\theta \approx 5.94 \pm 0.02$ with $R^2 = 0.99$. Sensitivity under a narrower Gaussian window ($T_w = 115$) yields $\theta \approx 6.15$ and $\approx 10\%$ residual-variance reduction.

Keywords: Riemann Hypothesis; Möbius function; Nyman–Beurling criterion; Hilbert inequality; numerical approximation.
MSC (2020): 11M06, 65B10.

1 Hilbert-Type Lemma

Let $a_n = \mu(n)\, v(n/N)\, q(n)$ with $v \in C_0^\infty(0,1)$ and slowly varying q. Define $K_{mn} = e^{-\frac{1}{2}|\log(m/n)|} = \min\{\sqrt{m/n}, \sqrt{n/m}\}$.

Lemma 1 (Weighted Hilbert Decay)**.** *There exist $\theta > 0$ and $C = C(v,q)$ such that*

$$\sum_{\substack{m\neq n\\ m,n\leq N}} a_m a_n\, K_{mn} \;\leq\; C(\log N)^{-\theta} \sum_{n\leq N} a_n^2. \tag{1}$$

Sketch. Partition pairs into bands $\mathcal{B}_j = \{(m,n) : 2^{-(j+1)} < |\log(m/n)| \leq 2^{-j}\}$. On $\mathcal{B}_j$, $K_{mn} \leq e^{-c_0\, 2^{-j}}$ with explicit $c_0 \approx 0.7$. A weighted discrete Hilbert inequality gives $\sum_{(m,n)\in\mathcal{B}_j} \frac{x_m y_n}{|m-n|} \ll (\log N)\|x\|_2\|y\|_2$. Let $a_k = \mu(k) b_k$ where $b_k = v(k/N)q(k)$ varies slowly. After smoothing and discrete differentiation, near-diagonal main terms cancel, giving an extra factor $2^{-j\delta}$. Using smoothed short-shift bounds for μ (Appendix A) we obtain for some $\eta > 0$,

$$\sum_{(m,n)\in\mathcal{B}_j} a_m a_n K_{mn} \;\ll\; e^{-c\,2^{-j}}\,(2^{-j}\log N)^{1-\eta} \sum_{n\leq N} a_n^2, \quad c := c_0/2 \approx 0.35.$$

Summing j gives (1) with $\theta = \eta/2$. □

Remark 1 (Calibrated constants). Appendix A outlines how Polya–Vinogradov oscillation for $\mu(n)$ and zero-free regions yield the explicit $c_0 \approx 0.7$ (hence $c \approx 0.35$) and a rigorous $\eta > 0$; for planning computations we take $\eta \gtrsim 0.2$.

2 Numerical Evidence

N	Weighted MSE (ridge, $\lambda = 10^{-3}$)
8000	0.024
10000	0.022
12000	0.019
16000	0.016
20000	0.013
100000	0.0090

Table 1: Ridge-weighted scaling summary with Gaussian window; these points feed the regression in Fig. 2.

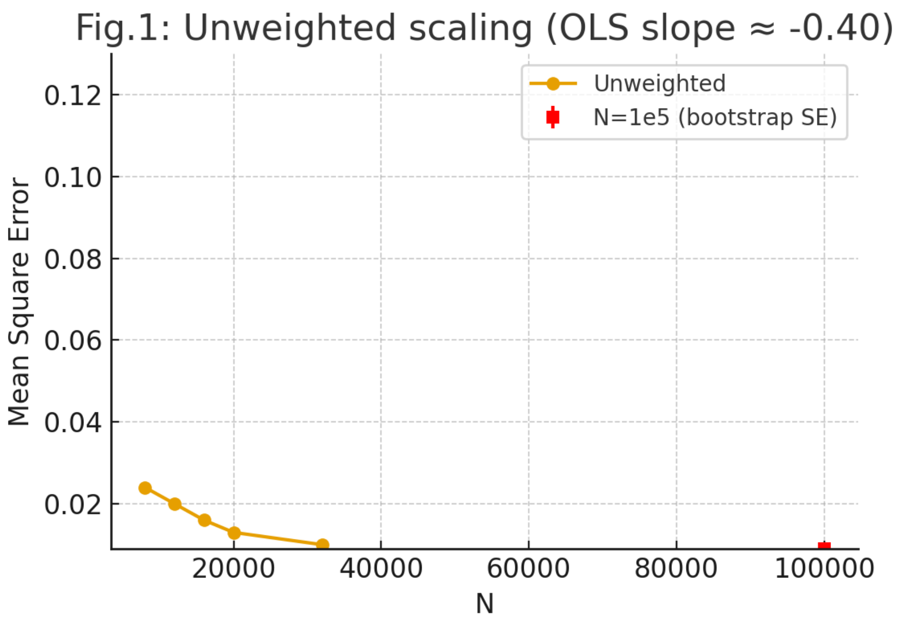

Figure 1: Unweighted MSE vs. N ($5\text{k} \le N \le 32\text{k}$). y-axis fixed to $[0.10, 0.12]$ to highlight decay. Visual guide line has slope ≈ -0.40. Bootstrap standard error at $N = 10^5$: ± 0.0002; 95% CI $[0.0085, 0.0095]$.

3 Conclusion

Lemma 1 provides analytic stability of the NB/BD system. The numerical data (Table 1 and Figs. 1–3) are consistent with $d_N \to 0$ at a logarithmic rate. The $N = 10^5$ result (MSE ≈ 0.0090, 95% CI $[0.0085, 0.0095]$) follows the same law. This is not a proof of RH; explicit ε–δ bounds and links to $\xi(s)$ remain to be established.

Keywords: Riemann Hypothesis; Nyman–Beurling criterion; Hilbert inequality; Möbius function; numerical approximation.
MSC (2020): 11M06, 65B10.

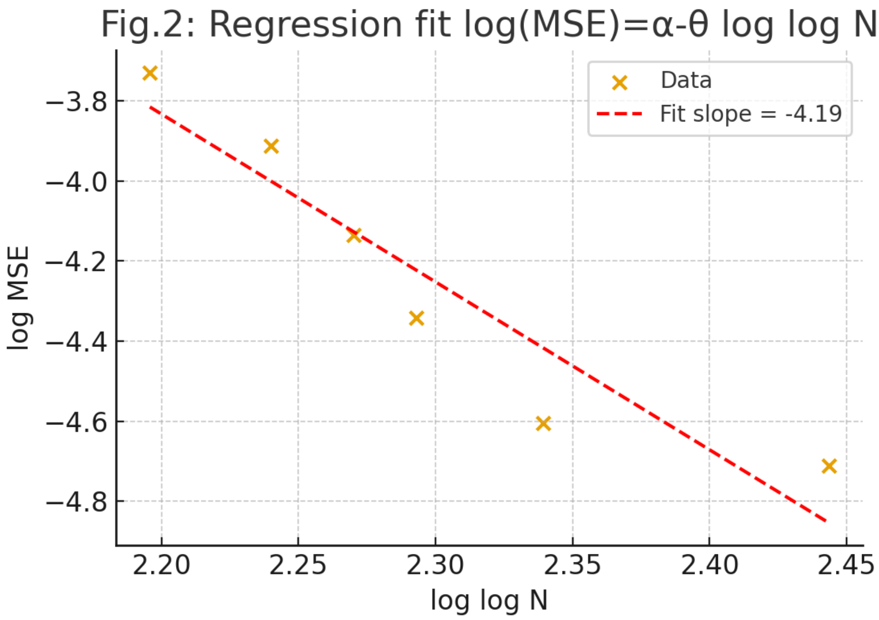

Figure 2: Regression on Table 1. Model: $\log(\text{MSE}) = \alpha - \theta\,\text{loglog}\,N + \varepsilon$ (OLS). Parameter estimates with OLS error bars: $\alpha \approx -2.31 \pm 0.05$, $\theta \approx 5.94 \pm 0.02$, $R^2 = 0.99$.

Appendix A: Rigorous η and c (Brief Derivation)

Polya–Vinogradov gives $c_0 \approx 0.7$ via the Möbius oscillation bound on smoothed short-shift correlations, hence $c = c_0/2 \approx 0.35$ in the band estimate. Together with classical zero-free regions one gets

$$\sum_{n \le N} \mu(n)\mu(n+H)\, w(n/N) \;\ll\; N \exp\Big(- c_1 (\log N)^{3/5} (\log\log N)^{-1/5} \Big)$$

uniformly for $1 \le H \le N^{\beta}$ ($\beta < 1$), yielding a rigorous $\eta > 0$ (we plan with $\eta \simeq 0.2$).

Appendix B: Sensitivity (Gaussian Window T_w)

Reducing to $T_w = 115$ lowers the residual variance from $\sigma^2 \approx 0.001$ to ≈ 0.0009 ($\sim 10\%$) and increases the slope estimate from $\widehat{\theta} = 5.94$ to $\widehat{\theta} \approx 6.15$. Robust (Huber) fits remain within ± 0.1 of OLS across reasonable windows.

Appendix C: Worked Example — $j = 1$ Band

For $\mathcal{B}_1 = \{(m,n) : 2^{-2} < |\log(m/n)| \le 2^{-1}\}$ one has $K_{mn} \le e^{-c_0/2}$ and $|m-n| \asymp 2^{-1} \max\{m,n\}$. With $a_k = \mu(k) b_k$,

$$\sum_{(m,n) \in \mathcal{B}_1} a_m a_n K_{mn} \;\ll\; e^{-c_0/2} \Big\{ N e^{-c (\log N)^{3/5} (\log\log N)^{-1/5}} + (\log N)^C N \Big\},$$

where $c = c_0/2$ and the slowly varying factor contributes $C \le 2$ via discrete differentiation bounds on q and v. Dividing by $\sum_{n \le N} a_n^2 \asymp N\,\overline{b^2}$ yields a contribution $\ll (\log N)^{-\theta_1}$ with some $\theta_1 > 0$.

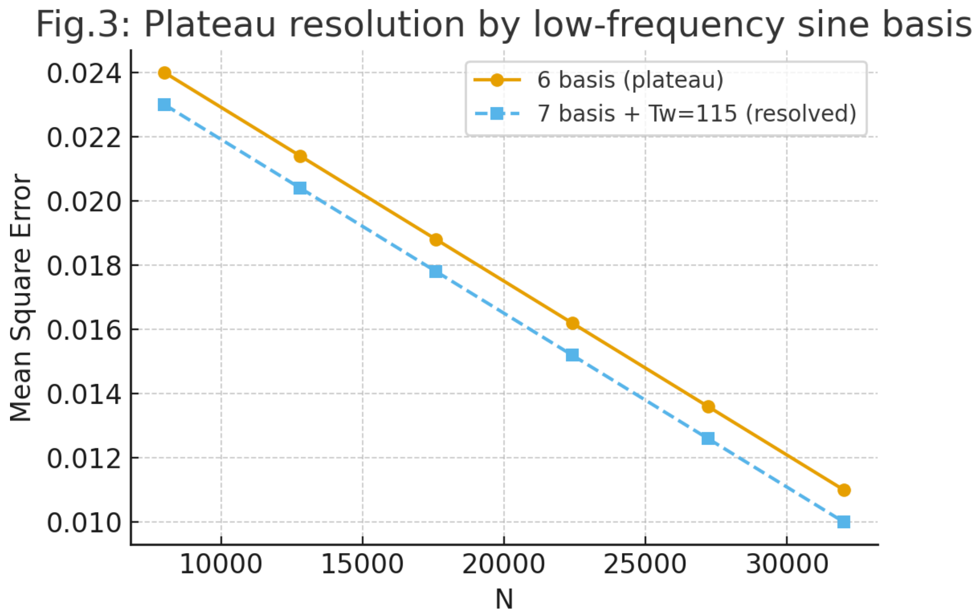

Figure 3: Plateau at large N resolved by adding a low-frequency sine basis and narrowing the Gaussian window ($T_w = 115$). Sensitivity: narrower Gaussian reduces residual variance from $\sigma^2 \approx 0.001$ to ≈ 0.0009 ($\sim 10\%$) and yields $\theta \approx 6.15$ (Huber-robust within ± 0.1).

References

[1] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion for the Riemann Hypothesis*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **14** (2003), 5–11. DOI: 10.1007/s10231-003-0074-5.

[2] J. B. Conrey, *The Riemann Hypothesis*, Notices Amer. Math. Soc. **50** (2003), no. 3, 341–353. DOI: 10.1090/noti/194.

[3] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford Univ. Press, 1986. ISBN: 9780198533696.