INVERSE

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Singular (not invertible)
* If $Ax = 0$ exists with $x \neq 0$
* If $\det(A) = 0$

Invertability of $n \times n$
* A square matrix is invertible if full rank / all columns are indie.
* The rref of A is I
* $Ax = 0$ has one solution.
* $Ax = b$ has one solutions.

Gauss-Jordan
$(A \mid I) \to (I \mid A^{-1})$
This is like solving $Ax = \begin{pmatrix}1\\0\end{pmatrix}$ and $Ax = \begin{pmatrix}0\\1\end{pmatrix}$ at the same time.

ELIMINATION

E matrix
Subtract 3 row 1 from row 2

$$E_{21} = \begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

NB: Matmul is associative.
$E_{32}(E_{21}(A)) = (E_{32}E_{21})A$

Inverse
$E^{-1}E = I$

VECTOR SPACE
* $v + w$ and cv are in the space
* all combinations of $cv + dw$ are in the space
* must go through origin

SOLVE $Ax = b$

* $x_{\text{particular}}$: set all free variables to 0, solve $Ax = b$ for pivots.
* $x = x_p + x_n$ because $Ax_p = b$ and $Ax_n = 0$ means $A(x_p + x_n) = b$

RANK

* Number of linearly independent rows or columns.
* Number of pivots in $R = \text{rref}(A)$

If a matrix in $\mathbb{R}^n$ has full rank, it spans all of $\mathbb{R}^n$.

Full column rank
* $r = n$ (no free variables)
* $N(A)$ is the zero vector.
* Solution $Ax = b$ is $x = x_p$
* 0 or 1 solutions.

Full row rank
* $r = m$
* $Ax = b$ has a solution for every b.
* $n - r = n - m$ free variables.

Full rank square matrix
* $r = m = n$
* invertible
* $N(A) = \{0\}$
* 1 solution

FOUR FUNDAMENTAL SUBSPACES

A is $m \times n$

$C(A^T) \perp N(A)$
$C(A) \perp N(A^T)$

Column space $C(A)$
* All linear combinations of $C(A)$
* In $\mathbb{R}^m$
* $Ax = b$ has a solution if b is in $C(A)$
* $\dim C(A) = r$

Nullspace N(A^T)
* $\dim N(A^T) = m - r$

Row space $C(A^T)$
* All combinations of rows of A is same as combinations of cols of A^T
* $\dim C(A^T) = r$
* $C(A^T) = C(A^T_{rref})$ because row operations don't change span of row space

Nullspace $N(A)$
* All solutions x to $Ax = 0$
* $N(A)$ always contains 0
* $\dim N(A) = \#\text{ free vars} = n - r$

DETERMINANTS

Properties
* $\det I = 1$
* exchange rows: reverse sign of det.
* $\begin{bmatrix}ta & tb\\ c & d\end{bmatrix} = t\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ ⚠ first row only
* $\begin{bmatrix}a + a\prime & b + b\prime\\ c & d\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix} + \begin{bmatrix}a\prime & b\prime\\ c & d\end{bmatrix}$
* 2 equal rows: $\det = 0$
* subtract $l \times$ row 1 from row k: det doesn't change.
* row of zeros: $\det = 0$
* $U = \begin{bmatrix}d_1 & . & .\\ 0 & d_2 & .\\ 0 & 0 & d_n\end{bmatrix}$: $\det U = d_1 d_2 \ldots d_n$
* $\det A = 0$: singular
* $\det AB = (\det A)(\det B)$
* $\det A^T = \det A$

Cofactors
$\det A = a_{11}C_{11} - a_{12}C_{12} + \ldots$
$+$ if $(i + j)$ is even, else $-$

Inverse
$$A^{-1} = \frac{1}{\det A}C^T$$

Cramer's rule
$Ax = b$
$B_j = (\text{A with col j replaced by b})$
$$x_j = \frac{\det B_j}{\det A}$$

Volume
$v = \text{abs}(\det \text{A})$

MATMUL

Not **cummutative**
In general $AB \neq BA$

Matmul
$AB + AC = A(B + C)$

$AB = C$
* Cols of C are combinations of cols of A
* Rows of C are combinations of rows of B
* AB is sum of (cols A)(cols B)

Block matmul
$$\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}\begin{bmatrix}B_1 & B_2\\ B_3 & B_4\end{bmatrix} = \begin{bmatrix}A_1B_1 + A_2B_3 & .\\ . & .\end{bmatrix}$$

PERMUTATION

Exchange rows 1 and 2

$$PA = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}A$$

Multiplying on the left does row operations, on the right does column operations.

SYMMETRIC

$A^T = A$

A^TA is always symmetric, regardless of A

ORTHOGONALITY

Vectors
$x^Ty = x \cdot y = 0$

Subspaces
Every vector in S is orthogonal to every vector in T.

Orthonormal
Orthogonal length 1.
* $Q^TQ = I$ always
* $QQ^T = I$ if Q is square
* $Q^{-1} = Q^T$

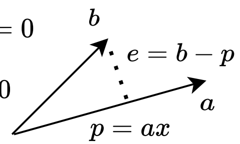
Gram-Schmidt
given indie a, b, c we want orthogonal A, B, C and then orthonormal $q_1 = \dfrac{A}{\parallel A \parallel}$...

$$A = a \qquad B = b - \frac{A^Tb}{A^TA}A$$
$$C = c - -\frac{A^Tb}{A^TA}A - \frac{B^Tc}{B^TB}B$$

PROJECTIONS

$A^TA\widehat{x} = A^Tb$
$\widehat{x} = (A^TA)^{-1}A^Tb$
$P = A(A^TA)^{-1}A^T$
$p = A\widehat{x} = Pb$

Derivation
$a \perp e \to a^Te = 0$
$a^T(b - p) = 0$
$a^T(b - xa) = 0$
$a^Tb = xa^Ta$
$$x = \frac{a^Tb}{a^Ta}$$



Properties
$P^T = P$; $P^2 = P$; $\det = -1 \text{or} 1$

Least squares
Solving $Ax = b$ when no solution: project b onto $C(A)$. Then there is a solution. This is "least" because orthogonality.

TRACE
Trace is the sum of diagonals. This is also the sum of eigenvalues.

DIAGONALIZING

$A = X\Lambda X^{-1}$

X: Eigenvector matrix
Λ: diagonal eigenvalue matrix.

Powers
$A^k = X\Lambda^kX^{-1}$

Diagonizability
Any matrix with no repeated λ is diagonizable.

Symmetric
Diagonalizable if symmetric.

TRANPOSE

$(A^T)^T = A$
$(A + B)^T = A^T + B^T$
$(kA)^T = kA^T$
$(AB)^T = B^TA^T$
$(A^{-1})^T = (A^T)^{-1}$

TRIANGULAR

Invertible
If and only if no element on the diagonal is 0.

A=LU

Without row exchanges
E is elimination, U is upper triangular, L is inverse of E.

$EA = U$
$A = LU$

With row exchanges
$PA = LU$

BASIS

sequence of indie vectors that span a space

EIGENVALUES & EIGENVECTORS

$Ax = \lambda x$, $n \times n$ matrix has n eigenvalues.

Singular
$\lambda = 0$ is an eigenvalue (nullspace)

Projection
* Any x in the plain $Px = x$ is eigenvector with $\lambda = 1$
* Any $x \perp$ to plain is $Px = 0$ and $\lambda = 0$

Trace
Sum of eigenvalues.

Determinant
Product of eigenvalues.

Solve $Ax = \lambda x$
* Solve $\det(A - \lambda I) = 0$ to find λ
* Plug λ and solve $(A - \lambda I)x = 0$ to find eigenvectors x

$B + kI$
* Eigenvectors stay the same, eigenvalues are $+ = k$

B^2
* Same eigenvectors, square the eigenvalues

B^{-1}
* Invert eigenvalues: $\dfrac{1}{\lambda}$

$\lambda = 0$
* If $\lambda = 0$ is an eigenvalue, B is singular.

MARKOV MATRIX

Rows sum to 1 and all entries ≥ 0

$\lambda_1 = 1$ always. $|\lambda_i| < 1$ always.

The trace/determinant is a factor of eigenvalues, so for a $2x2$ we can work out other eigenvalue easily.

Steady state
Vector $x_1 = \vec{1}$ connected to λ_1.
$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k \_ 2 \ldots$
$= c_1 x_1$

A^T same eigenvalues as A because $\det(A - \lambda I) = \det(A^T - \lambda I) = 0$

DIFFERENTIAL EQUATIONS

$(e^{\lambda t})\prime = \lambda e^{\lambda t}$
$\dfrac{du}{dt} = Au$

A is constant $\rightarrow \dfrac{du}{dt} = Au$ is linear with constant coefficients.

Solution
Choose $u = e^{\lambda t}x$, when $Ax = \lambda x$
$\Rightarrow \dfrac{du}{dt} = \lambda e^{\lambda t}x = Ae^{\lambda t}x = Au.$

We want to find $u(t)$ s.t. $\dfrac{du}{dt} = Au$, if we can do that we can easily find $u(t)$ for any t.

1) Find the eigenvalues
2) Find the eigenvectors
3) Write $u(0)$ as a combination of $c_1x_1 + c_2x_2 \ldots$
4) $u(t) = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2 \ldots$

Second order equations
$my\prime\prime + by\prime + ky = 0$
$\dfrac{dy}{dt} = y\prime$
$\dfrac{dy\prime}{dt} = -ky - by\prime$
$\Rightarrow \dfrac{d}{dt} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} y \\ y\prime \end{bmatrix} = Au$

Stability
$u(t) \rightarrow 0$ when $Re(\lambda) < 0$
A is "stable" and $u(t) \rightarrow 0$ when all λ have negative real parts.

Exponential of a matrix
$e^{At} = I + At + \dfrac{1}{2}(At)^2 \ldots = Ae^{At}$
$e^{At} = Xe^{\Lambda t}X^{-1}$
e^{At} has the same eigenvectors as A.

PROJECTIONS ORTHONORMAL BASIS

"expanding the vector in the basis"

because orthonormal:
$q_1^T v = x_1 q_1^T q_1 + x_2 q_1^T q_2 \ldots = x_1$
$Qx = v \Rightarrow x = Q^T v$

FOURIER SERIES

$f(x) = a_0 \cdot 1 + a_1 \cos x + b_1 \sin x$
$+ a_2 \cos 2x + b_2 \sin 2x \ldots$
infinitely many $f(x)$ because it's a function $\Rightarrow$ infinite basis $(1, \cos x, \sin x \ldots)$

the infinitely many basis are orthogonal

Dot/inner product of functions
$v^T w = v_1 w_1 \ldots + v_n w_n$
$f^T g = \displaystyle\int_0^{2\pi} f(x)g(x)dx = 0$
if orthonormal

Find a_1
Inner product of $\cos x$ with $f(x)$:
$\displaystyle\int_0^{2\pi} f(x)\cos x dx = a_1 +$ (a lot of zeros)

FFT

Multiplying $n \times n$ is n^2; FFT $\rightarrow n \log_n$

Complex numbers
length given by $\overline{z}^T z = z^H z$
symmetry: $A^H = A$
perpendicular: $\overline{Q}^T Q = I = Q^H Q$

Fourier matrix
$F_n =$
$$\begin{bmatrix} 1 & 1 & 1 & . & 1 \\ 1 & w & w^2 & . & w^{n-1} \\ . & . & . & . & . \\ 1 & w^{n-1} & w^2(n-1) & . & w^{n-1} \,\hat{}\, 2 \end{bmatrix}$$

$(F_n)_{ij} = w^{ij}$

$w^n = 1$; $w = \exp\left(i\dfrac{2\pi}{n}\right)$

$F_{64} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} P$
P: separates between even and odd components, interlaces.
D: $\begin{bmatrix} 1 & 0 & 0 & \ldots \\ 0 & w & 0 & \ldots \\ 0 & 0 & w^2 & \ldots \end{bmatrix}$

SYMMETRIC

$A = A^T$
eigenvalues of real symmetric matrix are real; eigenvectors of a symmetric matrix are orthogonal.

Spectral theorem
$A = Q\Lambda Q^{-1} = Q\Lambda Q^T = \lambda_1 q_1 q_1^T + \ldots$

Signs
$\#\text{positive pivots} = \#\text{positive}\lambda\prime s$

POSITIVE DEFINITE MATRICES

⚠symmetric

Positive definite if (any sufficient):
* all $\lambda_i > 0$
* $a > 0$; $ac - b^2 > 0$
* pivots $a > 0$; $(ac - b^2)a > 0$
* $x^T Ax > 0$

Semi-definite allows $\lambda_i = 0$

Second derivative matrix
* 1st derivative = 0: extrema
* 2nd derivative > 0: goes up so minimum
If the matrix of 2nd derivatives is pos def then the function has a global minimum.

$A^T A$ is positive definite if cols are indie (else semi-definite)

SIMILAR MATRICES

A, B similar if for some matrix M:
$B = M^{-1}AM$

They have same eigenvalues (not same eigenvectors, but same number of eigenvectors)

e.g. $S^{-1}AS = \Lambda$ - A and Λ are similar.

JORDAN BLOCK

$J_i = \begin{bmatrix} \lambda_i & 1 & . & . \\ . & \lambda_i & 1 & . \\ . & . & . & \lambda_i \end{bmatrix}$

good case: J is Λ

Jordan theorem
square A is similar to a Jordan matrix
$J = \begin{bmatrix} J_1 & . \\ . & J_2 \end{bmatrix}$ with #blocks=#eigenvecs

SINGULAR VALUE DECOMPOSITION

$A = U\Sigma V^T$
for any matrix!

$A^T A = V\Sigma^T U^T U \Sigma V^T = V\Sigma^2 V^T$
V: eigenvectors, Σ^2: eigenvalues (positive)

Then use $AA^T = U\Sigma^2 U^T$

* $v_1 \ldots v_r$ orthonormal basis for rowspace
* $u_1 \ldots u_r$ " " for column space
* $v_{r+1} \ldots v_n$ " " for nullspace
* $u_{r+1} \ldots u_m$ " " $N(A^T)$

TRANSFORMATIONS

$T(v + w) = T(v) + T(w)$
$T(cv) = cT(v)$

Basis
if we know $T(v_i)$ for all vecs in basis, we know everything about the transformation
$T(v) = c_1 T(v_1) + \ldots$

Matrix A
want a matrix A that tells us $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$
* choose input basis $\mathbb{R}^n$ and output basis $\mathbb{R}^m$
* i^{th} col of A:
$T(v_i) = a_{1i}w_1 + \ldots + a_{mi}w_m$

LEFT AND RIGHT INVERSES

Left inverse
Full column rank $r = n$ and $N(A) = \{0\}$
0 or 1 solutions to $Ax = b$
$A^T A$ is $n \times n$ and full rank, invertible, so $(A^T A)^{-1} A^T A = BA = I$ and B is left inverse