Vasicek (1977)

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1 Introduction

The Vasicek (1977) model is the prototypical *short-rate equilibrium model*. It specifies the dynamics of the instantaneous short rate r_t as a mean-reverting Ornstein–Uhlenbeck process:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^{\mathbb{Q}}, \tag{1}$$

where $\kappa > 0$ is the speed of mean reversion, θ is the long-run mean level under the risk-neutral measure, and $\sigma > 0$ is the volatility. This specification ensures that the short rate is Gaussian at all times. Unlike geometric Brownian motion, it admits negative interest rates — a limitation in practice, but analytically convenient.

The model is an *equilibrium model*: parameters (κ, θ, σ) are postulated from economic reasoning rather than fitted to match exactly the initial yield curve (as in Heath–Jarrow–Morton or affine no-arbitrage models). Nevertheless, Vasicek is historically important as the first tractable affine term structure model.

2 Distribution and Intuition

Equation (1) is a linear SDE with Gaussian solution. Solving explicitly yields

$$r_T \mid r_t \sim \mathcal{N}\big(m(t,T), v(t,T)\big),$$

$$m(t,T) = \theta + (r_t - \theta)e^{-\kappa(T-t)},$$

$$v(t,T) = \tfrac{\sigma^2}{2\kappa}\big(1 - e^{-2\kappa(T-t)}\big).$$

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Thus the conditional mean reverts exponentially towards θ at rate κ, while the variance converges to its stationary value $\sigma^2/(2\kappa)$. Intuitively, in the short run rates are anchored by the current r_t, but in the long run they stabilize around θ with Gaussian uncertainty.

3 Pricing PDE

Let $V(r,t)$ denote the value of a derivative contingent on the short rate r_t. Applying Itô's lemma and constructing a locally riskless portfolio with a zero-coupon bond as numéraire yields the Vasicek PDE:

$$\frac{\partial V}{\partial t} + \kappa(\theta - r)\frac{\partial V}{\partial r} + \tfrac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} - rV = 0. \tag{2}$$

This is analogous to the Black–Scholes PDE, but with the short rate as underlying. The Feynman–Kac theorem then gives the risk-neutral valuation:

$$V(r,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\ell(r_T) \;\middle|\; r_t = r\right],$$

where $\ell(r_T)$ is the payoff at maturity T.

4 Bond Pricing

4.1 Zero-Coupon Bond

The affine structure of (2) implies that the time-t price of a zero-coupon bond maturing at T has the exponential-affine form:

$$P(t,T) = A(t,T)\exp\big(-B(t,T)r_t\big), \tag{3}$$

with

$$\begin{aligned} B(t,T) &= \frac{1-e^{-\kappa(T-t)}}{\kappa}, \\ A(t,T) &= \exp\left((\theta - \tfrac{\sigma^2}{2\kappa^2})(B(t,T)-(T-t)) - \tfrac{\sigma^2}{4\kappa}B(t,T)^2\right). \end{aligned}$$

Intuitively, $B(t,T)$ represents the effective duration under mean reversion, while $A(t,T)$ ensures no-arbitrage.

4.2 Coupon Bonds

A coupon-bearing bond with coupons c_i at dates T_i and notional N is priced as a linear combination of zeros:

$$P^{\text{coupon}}(t) = \sum_i c_i P(t, T_i) + N P(t, T_n).$$

Thus the analytical tractability of zero bonds extends to all fixed-income instruments with deterministic cash flows.

5 Options Pricing

5.1 Bond Options

A European call option on a zero-coupon bond with strike K and maturity T written on a bond maturing at $S > T$ has payoff $(P(T, S) - K)^+$. The underlying bond price $P(T, S)$ is lognormal under the T-forward measure associated with $P(t, T)$, because discounting by $P(t, T)$ removes drift. The variance of $\ln P(T, S)$ under this measure is

$$\sigma_P^2 = \frac{\sigma^2}{2\kappa^3}(1 - e^{-\kappa(T-t)})^2(1 - e^{-2\kappa(S-T)}),$$

obtained from integrating the variance of the short rate process over $[t, T]$ and projecting onto the affine $B(T, S)$ term. The closed-form price is then Black-like:

$$C(t) = P(t, S)\Phi(d_1) - KP(t, T)\Phi(d_2), \tag{4}$$

where Φ is the standard normal cdf and

$$d_{1,2} = \frac{\ln \frac{P(t,S)}{KP(t,T)} \pm \frac{1}{2}\sigma_P^2}{\sigma_P}.$$

Options on coupon bonds can be priced using *Jamshidian's trick*. Suppose we have a coupon bond paying coupons c_i at times T_i $(i = 1, \ldots, m)$ and face value F at T_m. The payoff of a European call option with maturity $T < T_1$ and strike K is

$$\left(\sum_{i=1}^{m} c_i P(T, T_i) + F P(T, T_m) - K \right)^+ .$$

Jamshidian (1989) shows that in one-factor models (like Vasicek) there exists a unique short rate r^* such that

$$\sum_{i=1}^{m} c_i P(t, T_i; r^*) + F P(t, T_m; r^*) = K.$$

Then, the option on the coupon bond can be decomposed into a portfolio of zero-coupon bond options:

$$\text{Option}(t) = \sum_{i=1}^{m} c_i \, C_i(t) + F \, C_m(t),$$

where $C_i(t)$ is the price at time t of a call option on a zero-coupon bond maturing at T_i with strike $P(t, T_i; r^*)$.

Intuitively, the trick works because in one-factor models the bond prices are monotonic functions of the short rate. Thus the strike condition admits a unique r^*, which allows decomposing the nonlinear coupon-bond payoff into a sum of linear zero-bond option payoffs.

5.2 Futures Options

An option on an interest rate futures contract (such as Eurodollar or SOFR futures) differs from an option on a forward rate because futures are settled daily. This daily mark-to-market feature changes the measure under which the futures rate is a martingale, and in Gaussian short-rate models such as Vasicek it leads to a convexity adjustment: the expected futures rate exceeds the corresponding forward rate when volatility is nonzero.

Formally, the futures rate $F(t, T)$ delivering at T is defined as the risk-neutral expectation of the short rate at T:

$$F(t, T) = \mathbb{E}_t^{\mathbb{Q}}[r_T].$$

From the explicit solution of the Vasicek process,

$$r_T = \theta + (r_t - \theta)e^{-\kappa(T-t)} + \int_t^T \sigma e^{-\kappa(T-s)} dW_s,$$

the conditional expectation is

$$\mathbb{E}_t^{\mathbb{Q}}[r_T] = \theta + (r_t - \theta)e^{-\kappa(T-t)}.$$

On the other hand, the forward rate $f(t,T)$ is defined from zero-coupon bonds by

$$f(t,T) = -\frac{\partial}{\partial T}\ln P(t,T),$$

which, using the exponential-affine bond price representation (3), yields

$$f(t,T) = e^{-\kappa(T-t)}r_t + \theta(1 - e^{-\kappa(T-t)}) - \tfrac{\sigma^2}{2\kappa^2}(1 - e^{-2\kappa(T-t)}).$$

Subtracting these two expressions shows that the futures rate lies above the forward rate by the *convexity adjustment*:

$$F(t,T) - f(t,T) = \frac{\sigma^2}{2\kappa^2}\big(1 - e^{-2\kappa(T-t)}\big). \tag{5}$$

Intuitively, the adjustment arises because discounting and the short rate are correlated; under the money-market numéraire the futures rate reflects not just the mean of the forward rate but also its covariance with discounting. In Gaussian models this extra term is explicit.

Since $F(t,T)$ is an affine function of the Gaussian short rate, it is normally distributed. Therefore, under Vasicek, options on futures rates are priced in the *Bachelier* framework. A European call option at time t with expiry $T_o < T$ and strike K has price

$$C^{\text{fut}}(t) = P(t,T_o)\Big((F(t,T) - K)\,\Phi(d) + \sigma_F\sqrt{T_o - t}\,\phi(d)\Big), \tag{6}$$

where Φ and ϕ denote the standard normal cdf and pdf, and

$$d = \frac{F(t,T) - K}{\sigma_F\sqrt{T_o - t}}, \qquad \sigma_F^2 = e^{-2\kappa(T-T_o)}\frac{\sigma^2}{2\kappa}\Big(1 - e^{-2\kappa(T_o-t)}\Big).$$

Thus, the Gaussian nature of the Vasicek short rate leads naturally to normal (Bachelier) option prices, in contrast to the lognormal Black formula often used in practice. If one insists on a lognormal specification, one can instead apply the Black formula with adjusted forward $F(t,T)$ and an implied lognormal volatility calibrated to the Vasicek variance, but (6) is the model-consistent representation.

5.3 Caps, Floors, Caplets, Floorlets

A caplet on the accrual period $[T, T+\delta]$ with strike K and notional N pays at time $T+\delta$ the amount

$$N\delta\,(L(T,T+\delta)-K)^+,$$

where $L(T,T+\delta)$ is the forward rate fixed at T for borrowing over $[T,T+\delta]$. By no-arbitrage this forward rate is linked to bond prices through

$$1+\delta L(T,T+\delta)=\frac{1}{P(T,T+\delta)} \qquad \Rightarrow \qquad L(T,T+\delta)=\frac{1}{\delta}\Big(\frac{1}{P(T,T+\delta)}-1\Big).$$

The caplet payoff can therefore be rewritten as

$$N\delta(L(T,T+\delta)-K)^+=N\Big(1-(1+K\delta)P(T,T+\delta)\Big)^+,$$

which is exactly the payoff of a European put option on the zero-coupon bond maturing at $T+\delta$ with strike $(1+K\delta)^{-1}$. In other words, a caplet is equivalent to a bond option. By the same reasoning a floorlet payoff,

$$N\delta(K-L(T,T+\delta))^+=N\big((1+K\delta)P(T,T+\delta)-1\big)^+,$$

is equivalent to a call on the same bond. This duality immediately implies the put–call parity relationship between caplets and floorlets.

Since bond options have closed-form solutions under Vasicek, the same is true for caplets and floorlets. The price at time $t<T$ of a caplet on $[T,T+\delta]$ is

$$\text{Caplet}(t)=N\delta\Big(P(t,T)\Phi(d_1)-(1+K\delta)P(t,T+\delta)\Phi(d_2)\Big), \tag{7}$$

with

$$d_{1,2}=\frac{\ln\frac{P(t,T)}{(1+K\delta)P(t,T+\delta)}\pm\frac{1}{2}\sigma_{cap}^2}{\sigma_{cap}},$$

and variance term

$$\sigma_{cap}^2=\frac{\sigma^2}{2\kappa^3}\,(1-e^{-\kappa(T-t)})^2\,(1-e^{-2\kappa\delta}).$$

A floorlet has the analogous formula, either derived directly or obtained from put–call parity.

A cap is constructed as a portfolio of caplets spanning a sequence of accrual periods, and a floor as a portfolio of floorlets. Thus the valuation of these standard market instruments reduces to the repeated application of the caplet formula (7). Intuitively, the key step is the reduction of a nonlinear rate payoff to an option on a zero-coupon bond; in the Vasicek model the Gaussian nature of the short rate then delivers analytic tractability.

5.4 Swaptions

A payer swaption is the right, at expiry T, to enter into a fixed–for–floating interest rate swap with payment dates $T_1, \dots, T_N$, accrual factors δ_j and notional N. The value at T of paying fixed rate K and receiving floating is

$$N\Big(1 - P(T, T_N) - K\sum_{j=1}^{N} \delta_j P(T, T_j)\Big),$$

since the floating leg collapses to par. The payoff of the payer swaption is therefore

$$\Pi_T^{\text{payer}} = N\big(1 - P(T, T_N) - KA(T)\big)^+, \qquad A(T) = \sum_{j=1}^{N} \delta_j P(T, T_j).$$

Equivalently, writing $S(T) = \frac{1-P(T,T_N)}{A(T)}$ as the forward swap rate, the payoff can be expressed as

$$\Pi_T^{\text{payer}} = NA(T)\,(S(T) - K)^+.$$

Hence a swaption is an option on the forward swap rate with annuity $A(T)$ as scaling factor.

To value the swaption one can regard the payoff as an option on a coupon bond. Define

$$B_{\text{fix}}(T; K) = P(T, T_N) + K\sum_{j=1}^{N} \delta_j P(T, T_j),$$

so that $\Pi_T^{\text{payer}} = (1 - B_{\text{fix}}(T; K))^+$. This is a put on a coupon bond. In a one–factor affine model such as Vasicek, there exists a unique r^* such that $B_{\text{fix}}(T; K; r^*) = 1$, and Jamshidian's trick applies: the option on the coupon bond can be decomposed into a sum of zero–coupon options. Writing the coupon payments explicitly, the payer swaption price at time $t < T$ is

$$\text{Swpt}^{\text{payer}}(t) = N\left(\sum_{j=1}^{N-1} K\delta_j\, P^{B,\text{put}}(t; T, T_j, K_j) + (1 + K\delta_N)\, P^{B,\text{put}}(t; T, T_N, K_N)\right),$$

where $K_j = P(T,T_j;r^*)$ and $P^{B,\mathrm{put}}$ denotes the zero–bond put price

$$P^{B,\mathrm{put}}(t;T,U,K) = KP(t,T)\Phi(-d_2) - P(t,U)\Phi(-d_1),$$

with

$$d_{1,2} = \frac{\ln \frac{P(t,U)}{KP(t,T)} \pm \frac{1}{2}\sigma_B^2}{\sigma_B}, \qquad \sigma_B = B(T,U)\,\sigma\sqrt{\tfrac{1-e^{-2\kappa(T-t)}}{2\kappa}}.$$

An alternative representation treats the swap rate $S(T)$ directly as underlying. Under the annuity measure with numéraire $A(T)$ the swap rate is a martingale, and in Vasicek it is Gaussian. Its variance is

$$\sigma_S^2 = \frac{\sigma^2}{2\kappa}\int_t^T \left(\frac{\sum_{j=1}^N \delta_j B(s,T_j)P(s,T_j)}{A(s)}\right)^2 e^{-2\kappa(s-t)}\,ds.$$

Hence the Bachelier formula applies:

$$\mathrm{Swpt}^{\mathrm{payer}}(t) = NA(t)\Big((S(t)-K)\Phi(d) + \sigma_S\sqrt{T-t}\,\phi(d)\Big), \qquad d = \frac{S(t)-K}{\sigma_S\sqrt{T-t}}.$$

Thus, swaptions in the Vasicek model admit two equivalent closed-form descriptions: either as a sum of zero–bond options via Jamshidian's trick, or directly as a Bachelier option on the forward swap rate.

A receiver swaption, by contrast, gives the right to receive fixed and pay floating. Its payoff at expiry is

$$\Pi_T^{\mathrm{receiver}} = N\big(KA(T) - (1-P(T,T_N))\big)^+ = NA(T)\,(K-S(T))^+,$$

so it is a put on the forward swap rate with annuity $A(T)$ as scaling factor. Its price can be expressed in two equivalent ways. By Jamshidian's trick, it is a portfolio of zero–coupon bond calls, obtained by decomposing the coupon–bond payoff using the unique short rate r^* solving $B_{\mathrm{fix}}(T;K;r^*) = 1$. Alternatively, under the annuity measure the swap rate is Gaussian, and the receiver swaption is a Bachelier put with variance σ_S^2 as above:

$$\mathrm{Swpt}^{\mathrm{receiver}}(t) = NA(t)\Big((K-S(t))\Phi(-d) + \sigma_S\sqrt{T-t}\,\phi(d)\Big), \qquad d = \frac{S(t)-K}{\sigma_S\sqrt{T-t}}.$$

Importantly, both representations — the coupon bond option decomposition via Jamshidian's trick and the direct Bachelier formulation on the swap rate — yield the same arbitrage–free swaption price in the Vasicek model. The equivalence follows because the model is one–factor affine: bond prices and swap rates are monotone in the short rate, ensuring that the decomposition point r^* is unique, and that both approaches are consistent.

Since the payer swaption is a call on $S(T)$ and the receiver swaption is the put, they are linked by the parity relation

$$\mathrm{Swpt}^{\mathrm{payer}}(t) - \mathrm{Swpt}^{\mathrm{receiver}}(t) = NA(t)(S(t) - K),$$

completely analogous to cap–floor parity. In practice, one, therefore, speaks of payer and receiver swaptions rather than calls and puts on swaps, though mathematically the equivalence is exact.

6 Remarks

The Vasicek model is the canonical Gaussian short-rate model: simple, tractable, and analytically elegant. Its key features are mean reversion and closed-form solutions for bonds and standard interest rate derivatives. However, it allows negative rates and cannot perfectly fit the initial yield curve without extension. For this reason, practitioners often prefer the Hull–White (extended Vasicek) or CIR models. Still, Vasicek remains the foundation of modern interest rate theory.

Vasicek model illustrates the distinction between *equilibrium* and *no-arbitrage* approaches: its parameters are postulated on economic grounds rather than calibrated to fit exactly today's term structure. This makes it elegant for theory and pedagogy, but less accurate for practice. Nevertheless, the model's analytical tractability underpins techniques such as Jamshidian's decomposition, convexity adjustments, and normal (Bachelier) option pricing, which extend far beyond the Gaussian setting. In this sense Vasicek is both limited and indispensable: too simple to be final, but essential as a starting point.