Hull & White (1990)

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1 Introduction

The Hull-White (1990) one-factor model is a no-arbitrage short-rate model that fits exactly today's term structure by inserting a time-dependent drift into an OU short rate:

$$dr_t = (\theta_t - ar_t)\,dt + \sigma\,dW_t^{\mathbb{Q}}, \qquad a > 0,\, \sigma > 0. \tag{1}$$

Unlike equilibrium models such as Vasicek, the initial curve is an input. The function θ_t is chosen so that model bond prices match $B_0(t)$ exactly. Forward-rate volatility declines with the forward's time to maturity, a distinctive feature relative to Ho-Lee.

2 Distribution and Intuition

Solving (1) gives

$$r_t = r_s e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta_u\,du + \sigma \int_s^t e^{-a(t-u)} dW_u,$$

so that, conditional on r_s, r_t is Gaussian with

$$\mathbb{E}_s[r_t] = r_s e^{-a(t-s)} + \gamma_t - \gamma_s e^{-a(t-s)}, \qquad \mathrm{Var}_s(r_t) = \frac{\sigma^2}{2a}\big(1 - e^{-2a(t-s)}\big),$$

where $\gamma_t = f_0(t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$ and $f_0(t)$ is the instantaneous forward curve implied by $B_0(t)$. The mean reverts at speed a while variance saturates at $\sigma^2/(2a)$ as in Vasicek. Intuitively, the time-dependent shift θ_t is

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exactly what forces the model to pass through today's curve, while (a, σ) govern dynamics around it.

3 No-arbitrage drift and pricing PDE

Matching $B_0(\cdot)$ implies the drift

$$\theta_t = \frac{\partial f_0(t)}{\partial t} + a f_0(t) + \frac{\sigma^2}{2a}\big(1 - e^{-2at}\big). \tag{2}$$

This makes the model arbitrage-free and curve-consistent. For any claim $V(r, t)$ the pricing PDE under $\mathbb{Q}$ is

$$\frac{\partial V}{\partial t} + (\theta_t - ar)\frac{\partial V}{\partial r} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} - rV = 0.$$

4 Bond pricing

Zero-coupon prices are exponential-affine in r_t:

$$B(t, T) = \alpha(t, T) \exp\{-\beta(t, T) r_t\}, \qquad \beta(t, T) = \frac{1 - e^{-a(T-t)}}{a}. \tag{3}$$

The pre-factor is

$$\alpha(t, T) = \frac{B_0(T)}{B_0(t)} \exp\Big\{\beta(t, T) f_0(t) - \frac{\sigma^2}{4a^3}\big(e^{-aT} - e^{-at}\big)^2\big(e^{2at} - 1\big)\Big\} = \frac{B_0(T)}{B_0(t)} \exp\Big\{\beta(t, T) f_0(t) - \frac{\sigma^2}{4a}\big(1 - e^{-2at}\big)\beta(t, T)^2\Big\}. \tag{4}$$

These formulas make it straightforward to boot-strap $B_0(\cdot)$ and then evaluate $B(t, T)$ at any future time given r_t.

5 Bond options

Under the T-forward measure the forward bond $B(T, S)$ is lognormal with the same variance structure as in Vasicek. As a result, the Black-style closed form for a call on a S-maturity ZC bond, expiring at $T < S$, is identical in form to Vasicek:

$$c_t^B = A\, B(t, T)\, \Phi(h) - X\, B(t, T)\, \Phi(h - \sigma_P), \qquad p_t^B = X\, B(t, T)\, \Phi(-h + \sigma_P) - A\, B(t, T)\, \Phi(-h),$$

with $A = \frac{B(t,S)}{B(t,T)}$, X the strike in forward-bond units, $h = \frac{1}{\sigma_P} \ln \frac{A}{X} + \frac{1}{2}\sigma_P$, and

$$\sigma_P^2 = \frac{\sigma^2}{a^2}\left(e^{-a\delta} - 1\right)^2 \frac{1 - e^{-2a(T-t)}}{2a}, \qquad \delta = S - T. \tag{5}$$

Drift differences between Vasicek and Hull-White do not affect these option formulas because the forward measure removes drift.

6 Caps, floors, caplets, floorlets

A caplet on $[T, T+\delta]$ with strike K is a put on $B(T, T+\delta)$, hence

$$\text{Caplet}(t) = (1 + K\delta)\, p_t^B,$$
$$\text{Floorlet}(t) = (1 + K\delta)\, c_t^B,$$

with the same σ_P as in (5). Equivalently, one can write the Black-like form

$$\text{Caplet}(t) = (1 + K\delta) B_0(T) \big(K\Phi(-d_2) - B_0(T, T+\delta)\Phi(-d_1)\big),$$

with

$$d_1 = \frac{\ln \frac{B_0(T,T+\delta)}{K} + \frac{1}{2}\nu^2}{\nu}, \qquad d_2 = d_1 - \nu, \qquad \nu^2 = \frac{\sigma^2}{a^2}\left(e^{-a\delta} - 1\right)^2 \frac{1 - e^{-2aT}}{2a}.$$

This is the standard calibration formula relating caplet Black volatility to (a, σ).

7 Swaptions

There is no closed form for swaptions in one-factor Hull-White. Two tractable routes are common.

- Jamshidian's trick: decompose the coupon-bond option into a sum of ZC options using the unique r^* that makes the fixed leg equal to par. This gives an exact price but requires solving for r^* and summing ZC options.

- Couderc's Black-vol approximation: for ATM swaptions,

$$\left(\sigma^{\mathrm{HW}}_{\mathrm{swaption}}(T,n)\right)^2 \approx \frac{\sigma^2}{a^2}\left[\sum_{i=1}^{n}\omega_i(T)\left(1-e^{-ai\delta}\right)+\omega'_n(T)\left(1-e^{-an\delta}\right)\right]^2\frac{1-e^{-2aT}}{2aT},$$

  with weights $\omega_i(T)=\frac{B_0(T+i\delta)}{\sum_{j=1}^{n}B_0(T+j\delta)}$ and $\omega'_n(T)=\frac{B_0(T+n\delta)}{B_0(T)-B_0(T+n\delta)}$. This yields a fast least-squares fit of (a,σ) to ATM swaption volatilities.

Closed-form approximations to payer and receiver swaption prices based on forward-measure Gaussians are also available in the Couderc note (including the ATM condition defining the d adjustment).

8 Calibration

Two standard calibration passes are used.

1. Fit the curve exactly via (2) and (3) using $B_0(\cdot)$ from market instruments; extract $f_0(t)=-\partial_t \ln B_0(t)$.
2. Fit (a,σ) to option data. For caplets one can match Black caplet volatilities using

$$\left(\sigma^{\mathrm{HW}}_{\mathrm{caplet}}(T,\delta)\right)^2=\frac{\sigma^2}{a^2}\left(e^{-a\delta}-1\right)^2\frac{1-e^{-2aT}}{2aT}.$$

   For ATM swaptions use Couderc's relation above. Calibrating to prices rather than implied volatilities is often more stable.

9 Exposure of an IRS

Given calibrated (a,σ) and θ_t fixed by the curve, simulate r_t on a grid using the exact normal transition

$$r_{t+\Delta}=\mathbb{E}[r_{t+\Delta}\mid r_t]+\sqrt{\mathrm{Var}[r_{t+\Delta}\mid r_t]}\,Z,$$

with the mean and variance above, then revalue the IRS at each time using closed-form bond prices (3)-(4). From the panel of simulated values compute PFE, EE and EPE across time; this is the procedure in the exercise brief.