Linear Interest Rates Instruments

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1 Introduction

It does first makes sense to introduce a reminder on spot/forward rates:

- *Spot rate* $y_{0,T}$ ("zero rate"): annualised yield from today to T, implied by a zero-coupon bond price $B_{0,T}$.
- *Forward rate* $f_{0;T,T+\delta}$: rate for borrowing over $[T, T+\delta]$ agreed today, so that to think *roll a short loan at* T vs *lock it now.*
- Instantaneous view: $B_{0,T} = e^{-\int_0^T f_0(u)du}$, so the spot is an *average* of instantaneous forwards: $r(T) = \frac{1}{T}\int_0^T f_0(u)du$.
- *Shape intuition*: if the curve slopes up, forwards lie *above* spots; if down, forwards lie *below* spots.

Identities to keep in mind:

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\begin{aligned}
B_{0,T} &= e^{-y_{0,T}T}, \qquad y_{0,T} = -\frac{1}{T}\ln B_{0,T},\\
B_{0,T} &= B_{0,T+\delta}e^{\delta f_{0;T,T+\delta}}, \qquad (1+y_{0,2})^2 = (1+f_{0,0})(1+f_{0,1}).
\end{aligned}
$$

These relations simply say that the discount factor $B_{0,T}$ is the price today of one unit at T, which can be viewed as compounding at a single flat continuously compounded rate $y_{0,T}$. The link $B_{0,T} = B_{0,T+\delta}e^{\delta f_{0;T,T+\delta}}$ shows that discounting from $T+\delta$ back to T must occur at the forward rate agreed today. Finally, $(1+y_{0,2})^2 = (1+f_{0,0})(1+f_{0,1})$ expresses that a two-year spot investment must deliver the same return as rolling one-year

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investments where the second year's rate is locked in today; hence multi-period spots are geometric averages of period forwards, and the curve's slope tells you whether forwards lie above or below the corresponding spots.

The most fundamental building block is the zero (or spot) rate $y_{0,T}$, which is the single constant rate that, if compounded continuously over $[0, T]$, reproduces the observed discount factor. Formally, the price of a zero-coupon bond maturing at T is $B_{0,T} = e^{-y_{0,T}T}$. From coupon-bearing bonds we instead speak of a yield-to-maturity, denoted y, which is the single internal rate of return that equates the bond price P to the present value of all coupons and redemption:

$$P = \sum_i \frac{c_i}{(1 + y/m)^{n_i}} + \frac{A}{(1 + y)^T}.$$

where m denotes the number of coupon payments per year (e.g. $m = 2$ for semi-annual coupons), and n_i is the number of coupon periods between today and the i-th coupon date. A closely related notion is the par yield, namely the coupon rate y^* that sets a newly issued bond exactly at par. Algebraically, par pricing means $100 = 100\, y^* \sum_{s=1}^{N} B_{0,s\delta} + 100\, B_{0,N\delta}$, so that

$$y^* = \frac{1 - B_{0,N\delta}}{\sum_{s=1}^{N} B_{0,s\delta}}.$$

Finally, the plain-vanilla swap rate is nothing more than the par yield expressed in swap form: the fixed rate S that makes the present value of fixed payments equal to that of the floating leg. It has the closed form

$$S = \frac{1 - B_{0,N\delta}}{\sum_{s=1}^{N} \delta\, B_{0,s\delta}},$$

which shows that swaps are priced off the same discount factors as bonds, and that the fixed swap rate is simply a weighted average of the underlying forward rates. Conceptually, zeros are the primitives, bond yields are averages over them, and swap rates are par yields expressed in the language of fixed-for-floating exchanges.

I finish the introduction with the word of the term structure. When practitioners speak of the "yield curve" they may actually mean several related objects. One can plot zero (spot) rates $y_{0,T}$ against maturity, or alternatively forward rates $f_{0;T,T+\delta}$, or else par yields and swap rates. In practice the zero curve is the fundamental one, and traders typically bootstrap it from a mixture of cash deposits and short-term futures for the front end, and from interest rate swaps for the medium and long end.

More formally, the term structure is the entire collection $\{B_{0,T}\}$ or equivalently $\{y_{0,T}\}$ for all maturities

T. Economists have long debated why this structure takes on the shapes it does. The *expectations hypothesis* interprets the forward rate as the market's expected future short rate; under this view an upward-sloping curve reflects an expectation of rising rates. The *liquidity preference theory*, often associated with Keynes, argues instead that investors demand a premium for holding longer maturities, since they are less liquid and more exposed to risk; forward rates therefore exceed expected future spot rates by a term premium. A third perspective is the *market segmentation* or *preferred habitat* theory, which stresses that different investor clienteles operate at different horizons, so supply and demand imbalances in each "segment" of the curve influence yields independently of pure expectations.

In modern markets, curve construction also reflects conventions of collateralisation. Discounting of collateralised cash flows is performed on the overnight indexed swap (OIS) curve, while separate forwarding curves are built for projecting future floating rates. This has led to the common "dual-curve" setup, where discounting and forwarding are disentangled. I discuss the dual curve in the context of swaps.

2 Single-period contracts

A forward rate agreement (FRA) is the simplest interest rate derivative. It allows two parties to lock today the interest rate that will apply over a future accrual period $[T, T+\delta]$ on a notional amount A. In practice, one party pays a fixed contractual rate k while receiving the floating rate i_T that will prevail at the start of the period. Settlement is typically in arrears, meaning the payoff is exchanged at T rather than at $T+\delta$, and discounted accordingly. Under simple compounding, the payoff at time T is

$$\text{FRA payoff} = \frac{A(i_T - k)\delta}{1 + i_T\delta}.$$

The denominator reflects the fact that the interest difference, which truly accrues over $[T, T+\delta]$, is settled upfront at T rather than in arrears at $T+\delta$.

Valuation is the heart of FRA analysis. At time $t < T$, the present value of the FRA struck at k is[1]

$$\mathrm{FRA}_t(k) = A\big(B_{t,T} - (1+k\delta)B_{t,T+\delta}\big) = A\, B_{t,T+\delta}\left(\frac{B_{t,T}}{B_{t,T+\delta}} - (1+k\delta)\right).$$

Here $B_{t,T}$ is the discount factor from t to T. The ratio $\frac{B_{t,T}}{B_{t,T+\delta}} = f_{t;T,T+\delta}$ is precisely the no-arbitrage forward factor over $[T, T+\delta]$, so setting $k = f_{t;T,T+\delta}$ ensures that the FRA has zero value at inception. More generally, away from par, the value can be interpreted as the notional multiplied by the difference between the market forward rate and the contractual strike, weighted by the appropriate discount factor. One can also encounter *arrears FRAs*, where the floating rate is observed and paid at $T + \delta$ rather than T, shifting the timing of cash flows and requiring an additional convexity correction in valuation.

A close cousin is the short-term interest rate (STIR) futures contract, such as Eurodollar or SOFR futures. A futures contract can be thought of as a standardised, exchange-traded analogue of the FRA, but there are two crucial differences. First, the settlement convention: FRAs settle at the beginning of the accrual period (T), whereas futures are marked to market daily throughout their life. Second, the price quotation: if $h_{t,T}$ denotes the implied futures rate for period $[T, T+\delta]$, the quoted futures price is $H_{t,T} = 100 - h_{t,T}$ per \$100 of notional. Because of daily marking to market, a one basis point move in the futures rate corresponds to a fixed cash change (e.g. \$25 for a standard \$1 million, three-month Eurodollar contract).

These structural differences imply that futures rates and forward rates are not identical. Since futures settle daily, the correlation between interest rates and discounting introduces a convexity bias: in general, the quoted futures rate lies above the corresponding forward rate when rates are volatile or upward-trending. A common adjustment formula in continuous compounding is

$$f_{t;T,T+\delta} \approx h_{t;T,T+\delta} - \tfrac{1}{2}\sigma^2 T(T+\delta),$$

where σ is the volatility of the short rate. The correction is small for short maturities or low volatility, but non-negligible further out. From a valuation perspective, therefore, one must be careful: forwards are priced by discount factors, futures by daily resettlement, and the two diverge systematically.

In summary, FRAs illustrate the central principle of valuation: take the present value of the difference

[1] There are two standard ways to view the valuation of an FRA. The so-called *principal (or direct) method* prices the contract as the present value of two opposite loans: lending the notional until T and borrowing it until $T + \delta$ at the contractual rate. The difference in present values of these two positions gives the FRA's value. The *reverse (or indirect) method* instead asks: what forward rate is implied by today's discount factors, and how does it compare with the contractual strike? The FRA is then valued as the discounted notional times this rate difference over the accrual period. Both perspectives rely on the same no-arbitrage logic and, though they emphasise different intuition—cash-flow replication versus forward-rate comparison—they lead to the identical numerical result.

between contractual and market-implied cash flows, discounted at the appropriate curve. Futures implement the same economic idea but with altered cash-flow timing due to daily settlement. Both instruments therefore provide windows onto forward rates, but the precise mapping depends on whether valuation is under the forward measure (for FRAs) or under futures' daily-marked cash flows (requiring convexity adjustment).

3 Multi-period: Swaps

An interest rate swap is essentially a strip of forward rate agreements packaged together. In the standard "plain vanilla" form one party pays a fixed rate k on a notional A while receiving floating interest indexed to a benchmark (e.g. LIBOR, SOFR), reset in arrears, on payment dates $t_1, \dots, t_N$. The *par swap rate* is defined as the fixed rate S that makes the present value of the fixed leg equal to that of the floating leg at inception. Since the floating leg is always worth par at initiation (because it is equivalent to a floating-rate bond priced at par), the swap rate emerges as

$$S = \frac{1 - B_{0,t_N}}{\sum_{i=1}^{N} \delta\, B_{0,t_i}},$$

which is exactly the par yield on a coupon bond with the same payment dates. Intuitively, the fixed rate is a weighted average of forward rates, where the weights are the discount factors that determine how much each period's payment matters in present-value terms.

Once the swap is running, its mark-to-market value can be viewed from two equivalent angles. The *principal method* treats the floating leg as a bond whose value at any time t is the notional plus accrued floating interest, while the fixed leg is just a schedule of discounted fixed payments; the difference is the swap's value. The *reverse method* decomposes the swap into a portfolio of FRAs, each representing the exchange of a fixed rate k against the prevailing forward rate on a single accrual period. Adding these FRA values yields the same overall mark-to-market, since the two approaches are simply different replications of the same cash flows. As with FRAs, the principal method stresses cash-flow replication, while the reverse method highlights the forward-rate content of the swap; both agree numerically. At time t (between two payment dates), the value of a payer swap on notional A is the present value of the floating leg minus the fixed leg. In the principal method this reads

$$V_t^{\text{payer}} = A\Big(B_{t,t_j}\big(1 + \delta\, i_{t_{j-1}}\big) - k \sum_{n=j}^{N} \delta B_{t,t_n} - B_{t,t_N}\Big),$$

where t_{j-1} is the last reset date, $i_{t_{j-1}}$ the floating rate set then, and $B_{t,u}$ the discount factor to time u.

Equivalently, in the reverse (FRA-strip) method we write

$$V_t^{\text{payer}} = \sum_{i=j}^{N} \mathrm{FRA}_t(k; t_{i-1}, t_i),$$

that is, the sum of the values of all remaining forward rate agreements struck at the fixed swap rate k. The receiver swap has the opposite sign. Both methods yield the same result because they are just two ways of expressing the same replication: either as bonds (principal method) or as a collection of single-period forwards (reverse method).

Directionality is straightforward: a *payer swap* (pay fixed, receive floating) benefits when floating rates rise above the fixed strike, whereas a *receiver swap* profits when rates fall. Beyond the vanilla form, there are non-standard swaps that change timing or structure: *in-arrears swaps*, where floating payments are set and paid at the same time rather than set in advance, require convexity adjustments; others include amortising swaps, accreting swaps, and basis swaps (fixed vs floating, or floating vs floating on two different indices).

Modern practice also distinguishes between the curve used for discounting and the curve used for projecting future floating rates. Since the crisis, collateralised swaps are discounted using the overnight indexed swap (OIS) curve, reflecting the funding rate of collateral. However, the floating leg may still reference a term rate such as LIBOR or SOFR, whose forwards are derived from a separate "forecasting curve." This *dual-curve* framework means that valuation involves discounting expected floating payments (computed on the forwarding curve) at OIS-based discount factors. The swap rate itself is still the fixed rate that equalises fixed and floating legs, but it now depends explicitly on which curve is used for discounting, reinforcing the importance of collateral and funding conventions in modern derivative pricing.