Ho & Lee (1986)

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1 Introduction

The Ho-Lee (1986) model is the simplest no-arbitrage short-rate model that fits exactly today's term structure. It assumes a linearly drifting Gaussian short rate under the risk-neutral measure:

$$dr_t = \theta_t \, dt + \sigma \, dW_t^{\mathbb{Q}}, \qquad \sigma > 0. \tag{1}$$

The time-dependent drift θ_t is chosen so that model bond prices match the initial discount curve $B_0(T)$ exactly. Relative to Hull-White, Ho-Lee has no mean reversion; forward-rate volatility grows linearly with the time to maturity. This makes the model analytically very transparent but can produce wide tails for long horizons.

2 Distribution and Intuition

Integrating (1) from 0 to t gives

$$r_t = r_0 + \int_0^t \theta_u \, du + \sigma W_t, \qquad r_t \sim \mathcal{N}\big(m(t), \sigma^2 t\big).$$

Curve consistency determines θ_t from the initial instantaneous forward curve $F(0,t)$:

$$\theta_t = \frac{\partial F(0,t)}{\partial t} + \sigma^2 t. \tag{2}$$

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Intuitively, the first term forces the model through today's curve; the $\sigma^2 t$ term offsets the convexity created by the growing Gaussian uncertainty.

3 Pricing PDE

For any claim $V(r,t)$ the usual no-arbitrage argument yields

$$\frac{\partial V}{\partial t} + \theta_t \frac{\partial V}{\partial r} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} - rV = 0,$$

with terminal condition given by the payoff at maturity. The Feynman-Kac representation is unchanged:

$$V(t) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\,\ell(r_T)\right].$$

4 Bond pricing

Zero-coupon bonds are exponential-affine in r_t with a *linear* loading on the short rate:

$$P(t,T) = A(t,T)\,\exp\big(-(T-t)r_t\big) \tag{3}$$

The pre-factor matches the initial curve and compensates for diffusion:

$$\ln A(t,T) = \ln\frac{P(0,T)}{P(0,t)} + F(0,t)\,(T-t) - \frac{1}{2}\sigma^2\,t\,(T-t)^2. \tag{4}$$

These expressions follow from the bond-volatility specification $v(t,T) = \sigma\,(T-t)$ and the identities in the Hull's note.

5 Bond options

Consider a European option with expiry T written on a zero-coupon bond maturing at $S > T$. Under the T-forward measure the forward bond $P(T,S)$ is lognormal with variance

$$\sigma_P^2 = \int_t^T [v(u,S) - v(u,T)]^2 du = \sigma^2\,(S-T)^2\,(T-t). \tag{5}$$

Hence Black's formula applies to $P(T, S)$ in the usual way:

$$C(t) = P(t,T)\left(\frac{P(t,S)}{P(t,T)}\Phi(d_1) - K\Phi(d_2)\right), \qquad d_{1,2} = \frac{\ln\frac{P(t,S)}{KP(t,T)} \pm \frac{1}{2}\sigma_P^2}{\sigma_P}.$$

Put prices follow by parity.

6 Caps, floors, caplets, floorlets

A caplet on $[T, T+\delta]$ with strike K is a put on $P(T, T+\delta)$, so the caplet price at $t < T$ is

$$\text{Caplet}(t) = N\delta\left(P(t,T)\Phi(d_1) - (1+K\delta)P(t,T+\delta)\Phi(d_2)\right),$$

with $d_{1,2}$ as above and variance

$$\sigma_{\text{cap}}^2 = \sigma^2\delta^2(T-t).$$

A floorlet is the corresponding call on $P(T, T+\delta)$; caps and floors are portfolios of caplets and floorlets. The linear growth $\propto (T-t)$ is the characteristic Ho-Lee term-structure of caplet variance.

7 Swaptions

A payer swaption with expiry T and fixed dates $T_1, \ldots, T_N$ has payoff

$$\Pi_T^{\text{payer}} = N\Big(1 - P(T,T_N) - K\sum_{j=1}^{N}\delta_j P(T,T_j)\Big)^+.$$

Since Ho-Lee is one-factor and $P(T, \cdot)$ is monotone in r_T, Jamshidian's trick applies: solve for the unique r^* such that

$$P(T,T_N;r^*) + K\sum_{j=1}^{N}\delta_j P(T,T_j;r^*) = 1,$$

then decompose the payoff into a sum of zero-bond options with strikes $K_j = P(T, T_j; r^*)$ and use the bond-option formula above. Alternatively, under the annuity measure the forward swap rate is Gaussian in Ho-Lee, so a Bachelier representation with analytically computed variance provides the same price. The two descriptions are equivalent because the model is one-factor and affine in r.

8 Remarks

Ho-Lee shares with Hull-White the ability to fit today's curve exactly, but it lacks mean reversion. As a result, forward-rate volatility grows linearly with horizon and long-dated option volatilities scale like $\sqrt{T-t}$ times the accrual length, which can be too strong in practice. On the positive side, the linear structure makes lattice construction and calibration extremely straightforward; Ho-Lee often serves as a baseline for building trees or for stress testing Gaussian dynamics.