SABR Model (2002)

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1 The Model

Hagan et al. (2002) so-called *stochastic alpha beta rho* model is a parsimonious stochastic-volatility model that produces realistic smiles/skews for options on forwards, with fast asymptotic implied-vol formula for calibration and pricing. It is mainly used for FX and rates.

I work under the risk-neutral probability measure, i.e. T-forward measure where forward F_t is a martingale. The model dynamics are

$$dF_t = \alpha_t F_t^{\beta} dW_t^F \tag{1}$$

$$d\alpha_t = \nu \alpha_t dW_t^{\alpha} \tag{2}$$

where α is the initial vol level which scales the whole smile[1]. Important β parameter is the elasticity of the forward: controls backbone/structural skew[2]. Also we commonly assume that there is a non-zero correlation between forward and volatility: $d\langle W^F, W^{\alpha}\rangle_t = \rho dt$ – so that to have implied vol slope. Lastly, ν is the vol-of-vol parameter (kurtosis of the return distribution or curvature of the volatility smile parameter). Intuitively, if $\beta = 0$, then it would correspond somewhat to a Bachelier (1900) model (*normal*, i.e moves are in absolute terms), while $\beta = 1$ would result is somewhat of a Black (1976) model (*log-normal*, i.e. returns are percentage-like). The most common case of $0 < \beta < 1$ would roughly refer to CEV model[3]. More specifically, it allows the

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[1]Unlike Heston, Bates, Hull and White, etc. here we give the SDE for volatility and not the variance.

[2]Recall the Constant Elasticity of Variance model (Cox and Ross, 1976).

[3]The implication would be that with $\beta = 1$ changing the F_0 doesn't mechanically change Black vol at fixed moneyness, any skew/smile you see comes from ρ (slope, skewness) and ν (curvature, kurtosis), not from level effects. Conversely, with $\beta = 0$, if, say, F_0 halves, the same absolute vol looks twice as large in Black-vol terms (while normal vols stay stable, Black vols scale around $1/F_0$) – generating a strong structural left-skew in Black vols (common in low-rate regimes). In the intermediate case, at

implied-volatility smile to vary in level, slope, and curvature via a CEV backbone (β) and stochastic volatility (α, ρ, ν) – with α_0 setting the overall level, ρ the (ATM) slope, ν the curvature, β level-elasticity (at fixed moneyness, changing the forward level F_0 by factor c, rescales Black vols by $c^{\beta-1}$). The beauty is that that model matches the volatility smile very well with just 3 parameters[4]

2 Partial Differential Equation

Let $U(F, \alpha, t)$ be the undiscounted option value under the T-forward measure (so F_t has zero drift). The infinitesimal generator (e.g. by Feynman–Kac theorem) yields the parabolic PDE

$$\frac{\partial U}{\partial t} + \frac{1}{2}\alpha^2 F^{2\beta}\frac{\partial^2 U}{\partial F^2} + \rho\nu\alpha F^{\beta}\frac{\partial^2 U}{\partial F \partial \alpha} + \frac{1}{2}\nu^2\alpha^2\frac{\partial^2 U}{\partial \alpha^2} = 0$$

with terminal condition $U(F, \alpha, T) = \text{payoff}(F)$. If I instead price under the money-market numeraire with constant r, I would use $V = e^{-r(T-t)}U$ or equivalently add a rV term on the right-hand side.

For $\beta < 1$, $F = 0$ is attainable; standard practice is absorbing boundary for calls ($U \to 0$ as $F \to 0$). For rate products with potential negatives, use $\beta \approx 0$ or a shifted SABR $\tilde{F} = F + s$ with $\beta \in$ (0,1].

3 Option Prices

For plain-vanilla calls/puts we do not solve the SABR PDE; instead we (i) get an *implied volatility* from SABR's asymptotics and (ii) plug it into a standard pricing formula.

- Choose the backbone: pick β (and a shift s if rates may be near/through zero, using $\tilde{F}_0 = F_0 + s$, $\tilde{K} = K + s$).
- Compute an implied vol from SABR (see the next section for that one) at each strike K and expiry T.
- Price with Black/Bachelier: use Black if you work with lognormal vols, or Bachelier if you work with normal vols (or shifted-$\tilde{F}$).

fixed moneyness the entire Black smile rescales approximately as $F_0^{\beta-1}$, so that there is an interaction between level and slope effects. The case of $\beta > 1$ is not considered as it tends to generate unrealistic level dependence

[4]Practitioners tend to pick β for a product/asset first (e.g. conventions are β close to 1 for FX, β close to zero for rates, β somewhat around 0.5 or little more for commodities etc.) by calibrating α_0, ρ, ν per maturity for a set of βs minimises smile error and yields smooth, moderate ρ and ν across maturities. Each maturity yields a smile fit. All smiles fit result in surface. Lovely!

Compactly:

$$\text{Black price:}\quad C = P(0,T)\big(F_0 N(d_1) - K N(d_2)\big),\; d_{1,2} = \frac{\ln(F_0/K) \pm \frac{1}{2}\sigma_{\mathrm{BS}}^2 T}{\sigma_{\mathrm{BS}}\sqrt{T}},$$

$$\text{Bachelier price:}\quad C = P(0,T)\big((F_0 - K)N(d) + \sigma_{\mathrm{N}}\sqrt{T}\, n(d)\big),\; d = \frac{F_0 - K}{\sigma_{\mathrm{N}}\sqrt{T}}.$$

(Use put–call parity for puts.) In practice the SABR step gives you σ_{BS} (or σ_{N} if $\beta = 0$ / shifted-setup), and the pricing step is then trivial.[5]

4 Implied Volatility

This is the workhorse: a fast, accurate short-time expansion that maps the SABR parameters $(\alpha_0, \beta, \rho, \nu)$ to a Black (or normal) implied volatility. Intuition: α_0 sets level, ρ sets ATM slope (skew direction), ν sets curvature and how the smile grows with T, while β controls the backbone (how the smile rescales with the level).

4.1 Lognormal (general $0 < \beta \leq 1$): Black implied vol

Define

$$z = \frac{\nu}{\alpha_0}\,(F_0K)^{\frac{1-\beta}{2}} \ln\frac{F_0}{K}, \qquad x(z) = \ln\left(\frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1-\rho}\right),$$

$$\mathcal{D} = (F_0K)^{\frac{1-\beta}{2}}\left[1 + \frac{(1-\beta)^2}{24}\left(\ln\frac{F_0}{K}\right)^2 + \frac{(1-\beta)^4}{1920}\left(\ln\frac{F_0}{K}\right)^4\right].$$

Then the (first–order in T) Black implied vol is

$$\sigma_{\mathrm{BS}}(F_0,K;T) \;\approx\; \frac{\alpha_0}{\mathcal{D}}\,\frac{z}{x(z)}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha_0^2}{(F_0K)^{1-\beta}} + \frac{\rho\,\beta\,\nu\,\alpha_0}{4\,(F_0K)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24}\,\nu^2\right]T\right\}.$$

Sanity check (ATM): at $K = F_0$,

$$\sigma_{\mathrm{BS}}^{\mathrm{ATM}} \approx \alpha_0 F_0^{\beta-1}\left[1 + \left(\tfrac{(1-\beta)^2}{24}\alpha_0^2 F_0^{2\beta-2} + \tfrac{\rho\,\beta\,\nu\,\alpha_0}{4} F_0^{\beta-1} + \tfrac{2-3\rho^2}{24}\nu^2\right)T\right],$$

which makes the roles of the parameters transparent (level/slope/curvature and the β-backbone scaling).

[5] Greeks come for free by differentiating Black/Bachelier with respect to F_0, K, and the implied vol, applying the chain rule back to SABR parameters.

4.2 Normal backbone ($\beta = 0$) or shifted setup: normal implied vol

If you prefer normal vols (Bachelier)—either because $\beta = 0$ or because you are using a shift s for rates—use the analogous expansion

$$z_N = \frac{\nu}{\alpha_0}\,(\tilde{F}_0 - \tilde{K}), \qquad x(z_N) = \ln\left(\frac{\sqrt{1 - 2\rho z_N + z_N^2} + z_N - \rho}{1-\rho}\right),$$

$$\sigma_{\mathrm{N}}(\tilde{F}_0,\tilde{K};T) \;\approx\; \alpha_0\,\frac{z_N}{x(z_N)}\left\{1 + \frac{2-3\rho^2}{24}\,\nu^2 T\right\},$$

where $\tilde{F}_0 = F_0 + s$, $\tilde{K} = K + s$. Intuition is the same: ρ tilts, ν fattens, β (or the shift choice) determines the backbone you are reporting the smile in.

For each (K, T): compute σ_{BS} (or σ_{N}), plug into Black/Bachelier, price, done. The formula is extremely fast and robust for short–to–medium maturities; for far wings/long T use arbitrage-controlled tweaks or a numerical PDE/MC if you need high accuracy.[6]

5 Few Notes

SABR vs. other models:

- vs Black–Scholes (constant vol): SABR creates skew/smile via (β, ρ, ν) while preserving analytic-style speed through an implied-vol formula.
- vs Local Vol (Dupire, 1994): local vol can exactly fit today's surface but often gives poor forward-smile dynamics. SABR captures more realistic dynamics (stochastic vol) with far fewer degrees of freedom; not an exact fitter but robust and fast.
- vs Heston: Heston has mean-reverting variance and (semi-)analytic prices via Fourier methods. SABR is typically faster to calibrate to rates/FX smiles and simpler to implement; Heston gives a fuller dynamic (term structure in variance) and richer joint dynamics for equities.
- vs Bates (Heston+jumps): jumps help short-maturity matching. If the issue is primarily skew shape and term-structure rather than jump risk, SABR is usually sufficient; use Bates when you need jump-driven tails or gap risk.

[6] Tiny consistency note with the PDE: the cross term in the generator is $a_{F\alpha} = \rho\nu\alpha^2 F^\beta$, hence the $\rho\beta\nu$ and $F_0^{\beta-1}$ factors that appear in the ATM expansion.

Also important to keep in mind:

- The basic asymptotic vol can produce static arbitrage in far wings/long maturities; use arbitrage-free refinements or wing extrapolations in production[7]

- For exotics or high accuracy, solve the 2D PDE (ADI schemes) or simulate (MC with variance-reduction). The asymptotic vol is best for vanillas[8].

- For negative rates, replace F with $F + s$ to support negative rates with $\beta > 0$

In general, I would use SABR when I need a minimal-parameter, fast, stable smile engine for caps/floors and swaptions, FX vanillas, or equity options where a CEV backbone is natural and negative rates are handled via $\beta \approx 0$ or a shift.

[7] e.g., enforce monotone/convex total variance, clamp wing slopes to Lee bounds, use an "arbitrage-free SABR" variant, or extrapolate the wings with a simple, convex, bound-respecting tail (often linear in log-strike for total variance) that matches the fitted smile at the last quoted strikes.

[8] For path-dependent and/or American options, vanilla expansion may be not accurate enough, go for numerical methods instead.