Linear Interest Rates Instruments

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This Version: April 28, 2023

Introduction

Interest rate options are options written on *linear* IR instruments: futures, FRAs, swaps, and bonds. The guiding recipe is: identify the right underlying (forward rate, futures price, swap rate, or forward bond price), pick the natural numeraire (discount factor or swap annuity), and value the option with a *Black-type* (lognormal) or *Bachelier* (normal) formula under the corresponding forward measure. Discounting (for collateralised deals) is on the OIS curve; forwards are taken from the appropriate forwarding curve (dual-curve setup).

1 Futures Options (on STIR futures)

A call/put on a futures contract with expiry T and futures price H_t has maturity payoff $(H_T - K)^+$ / $(K - H_T)^+$. Under the Black-76 framework for futures, the option value at t is

$$C_t^{\text{Fut}} = B_{t,T}\big(H_t N(d_1) - K N(d_2)\big), \qquad P_t^{\text{Fut}} = B_{t,T}\big(K N(-d_2) - H_t N(-d_1)\big),$$

with $d_{1,2} = \dfrac{\ln\big(\frac{H_t}{K}\big) \pm \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{T-t}}$, Black vol σ, and $B_{t,T}$ the (possibly, OIS) discount factor. For STIR futures quoted as *price* $H = 100 - h$ (with rate h), a call on price is a put on the rate and vice versa. *Intuition:* futures are martingales under their measure, so valuation reduces to a standard Black price with discounting to t.

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Caplets and Floorlets

A caplet is an option on the simple-compounded forward rate for $[T, T+\delta]$, with payoff at $T+\delta$ given by $\delta A\,(L_T - K)^+$ where L_T is the fixing at T. Under the $T{+}\delta$-forward measure (numeraire $B_{t,T+\delta}$), the forward $F_t \equiv f_{t;T,T+\delta}$ is lognormal in the Black setup, and the Black caplet price is

$$\text{Caplet}_t = A\,\delta\,B_{t,T+\delta}\Big(F_t N(d_1) - K N(d_2)\Big),$$

with $d_{1,2} = \dfrac{\ln\!\big(\frac{F_t}{K}\big) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$. The floorlet is obtained by put-call symmetry:

$$\text{Floorlet}_t = A\,\delta\,B_{t,T+\delta}\Big(K N(-d_2) - F_t N(-d_1)\Big).$$

Intuition: discount back from payment $T+\delta$, then apply a call/put on the *forward* rate F_t because that rate is fixed at T but known in distribution at t.

2 Caps and Floors

A cap (floor) is a strip of caplets (floorlets) over dates $T_1, \ldots, T_N$ with common strike K and accrual δ_i. Therefore

$$\text{Cap}_t(K) = \sum_{i=1}^{N} A\,\delta_i\,B_{t,T_i+\delta_i}\Big(F_{t,i}N(d_{1,i}) - K N(d_{2,i})\Big),$$

with the obvious notation; floors sum the floorlets. *Intuition:* each period is an independent “roof tile...” protection stacks additively.

3 Swaptions

A payer (receiver) swaption gives the right at expiry T to enter a swap paying (receiving) fixed K on dates $t_1, \ldots, t_N$. Let the *swap annuity* (PVBP) be $A_{\text{swap}}(t) = \sum_{j=1}^{N} \delta_j B_{t,t_j}$ and the *forward swap rate* be $S_t = \dfrac{B_{t,t_0} - B_{t,t_N}}{A_{\text{swap}}(t)}$ (for $t_0 = T$ at option expiry). Under the annuity measure, S_t is lognormal; the Black swaption prices are

$$\text{Swpt}_t^{payer} = A_{\text{swap}}(t)\Big(S_t N(d_1) - K N(d_2)\Big), \qquad \text{Swpt}_t^{receiver} = A_{\text{swap}}(t)\Big(K N(-d_2) - S_t N(-d_1)\Big),$$

with $d_{1,2} = \dfrac{\ln(\frac{S_t}{K}) \pm \frac{1}{2}\sigma_S^2(T-t)}{\sigma_S\sqrt{T-t}}$. *Intuition:* the annuity plays the role of the discounting numeraire, turning the swap rate into the priced "underlying" of a standard Black call/put.

4 Bond Options

For a zero-coupon bond option expiring at T on bond maturing $U > T$, the forward bond price is $F_t^B = \dfrac{B_{t,U}}{B_{t,T}}$. Under the T-forward measure the Black bond option is

$$C_t^B = B_{t,T}\Big(F_t^B N(d_1) - KN(d_2)\Big), \qquad P_t^B = B_{t,T}\Big(KN(-d_2) - F_t^B N(-d_1)\Big),$$

with $d_{1,2} = \dfrac{\ln(\frac{F_t^B}{K}) \pm \frac{1}{2}\sigma_B^2(T-t)}{\sigma_B\sqrt{T-t}}$. Options on coupon bonds are valued as portfolios of zero-bond options on each cash flow (Jamshidian decomposition when applicable).

5 Parity Relations

Put-call parity under the relevant forward measure yields a family of "cap–floor–swap" identities:

Caplet-Floorlet parity	$\text{Caplet}_t - \text{Floorlet}_t = A\,\delta\,B_{t,T+\delta}\,(F_t - K)$ (equals an FRA at strike K)
Cap Floor parity (strip)	$\text{Cap}_t(K) - \text{Floor}_t(K) = A_{\text{swap}}(t)\,\big(S_t - K\big)$ (equals the PV of a payer swap at K)
Swaption parity	$\text{Swpt}_t^{payer} - \text{Swpt}_t^{receiver} = A_{\text{swap}}(t)\,\big(S_t - K\big)$ (call–put parity on the swap rate)
Bond option parity	$C_t^B - P_t^B = B_{t,U} - K\,B_{t,T} = B_{t,T}\big(F_t^B - K\big)$.
Futures option parity	$C_t^{\text{Fut}} - P_t^{\text{Fut}} = B_{t,T}\,\big(H_t - K\big)$.

Intuition: a call minus a put at the same strike reproduces the forward payoff; for caps/floors this aggregates period-by-period to a swap, for swaptions it reproduces the forward-starting swap PV at strike K.

6 Bachelier (Normal) Model

When forwards or swap rates can be near/less than zero, the Bachelier (normal) model is often preferred. If the forward X_t (rate, swap rate, or forward bond price) is *normal* with volatility σ_N under its forward measure, the call/put prices read

$$C_t^{\text{Bach}} = N(t)\Big((X_t - K)\,N(d) + \sigma_N\sqrt{\tau}\,\phi(d)\Big), \quad P_t^{\text{Bach}} = N(t)\Big((K - X_t)\,N(-d) + \sigma_N\sqrt{\tau}\,\phi(d)\Big),$$

where $\tau = T - t$, $d = \dfrac{X_t - K}{\sigma_N\sqrt{\tau}}$, $N(\cdot)$ and $\phi(\cdot)$ are the standard normal CDF and PDF, and $N(t)$ is the appropriate numeraire: $B_{t,T+\delta}$ for caplets, the swap annuity $A_{\text{swap}}(t)$ for swaptions, and $B_{t,T}$ for bond and futures options. *Rule of thumb:* Bachelier prices behave linearly around the strike and remain well-defined for negative rates; Black prices scale with the level of the underlying.

7 Hedging

The natural hedges mirror the underlyings: caplets/floorlets hedge with FRAs or STIR futures on the same period; caps/floors hedge as the sum of their period hedges; swaptions hedge with swaps matched to the option's annuity (PVBP); bond options hedge with the corresponding bonds or strips. Conceptually, the aim is to neutralize exposure to the driving forward or swap rate under the option's measure while recognizing that vega/vol-of-vol risks remain (ignored here by design). Parity relations provide static hedges and consistency checks: e.g., a cap minus a floor at strike K should match a payer swap at K valued on the same curves.