Cox, Ingersoll, and Ross (1985)

Sergey Mazyavkin[†]

This Version: June 1, 2023

1 Introduction

The Cox, Ingersoll, Ross (1985) model extends Vasicek's Gaussian framework by ensuring that interest rates remain non-negative while retaining analytical tractability. The short rate r_t follows a mean-reverting square-root diffusion under the risk-neutral measure $\mathbb{Q}$:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\, dW_t^{\mathbb{Q}}, \tag{1}$$

where $\kappa > 0$ governs the speed of mean reversion, $\theta > 0$ is the long-run mean level, and $\sigma > 0$ controls volatility. The diffusion term $\sqrt{r_t}$ implies that volatility is proportional to the square root of the rate level, ensuring nonnegativity provided the Feller condition $2\kappa\theta > \sigma^2$ holds[1].

Economically, the model captures the intuition that volatility rises when rates are high and falls when rates are low. It is an equilibrium model: parameters are motivated by economic arguments rather than fitted exactly to the current yield curve. Nevertheless, the CIR model became a cornerstone of interest rate theory because it combines mean reversion, positivity, and affine term-structure solutions.

2 Distribution and Intuition

The CIR process (1) admits a known transition density: conditional on r_t, the future rate r_T has a noncentral chi-squared distribution with degrees of freedom $\nu = \frac{4\kappa\theta}{\sigma^2}$ and noncentrality parameter $\lambda = r_t c e^{-\kappa(T-t)}$, where

[†]Accounting and Finance Division, Booth Street West, Manchester M15 6PB, UK, tel.: +44(0)161 306 0322, fax: +44(0)161 306 0322, e-mail: sergey.mazyavkin@manchester.ac.uk. All errors are my own.

[1]The condition ensures that the mean-reverting drift is strong enough to prevent the short rate from reaching zero. When the inequality holds strictly, r_t remains strictly positive. When equality holds, the process may touch zero but reflects immediately. If violated, the process can hit zero with positive probability, though it never becomes negative.

$c = \frac{4\kappa}{\sigma^2(1-e^{-\kappa(T-t)})}$. The conditional mean and variance are

$$\begin{aligned}
\mathbb{E}_t[r_T] &= r_t e^{-\kappa(T-t)} + \theta(1 - e^{-\kappa(T-t)}), \\
\mathrm{Var}_t[r_T] &= \frac{\sigma^2}{\kappa}\big(r_t e^{-\kappa(T-t)} + \tfrac{\theta}{2}(1 - e^{-\kappa(T-t)})\big)(1 - e^{-\kappa(T-t)}).
\end{aligned}$$

Thus, the short rate reverts to θ at speed κ, and its volatility depends on the level of r_t, preventing negative rates. In contrast to Vasicek, the distribution is skewed and strictly positive. This property makes CIR one of the most economically plausible short-rate models.

3 Pricing PDE

Let $V(r,t)$ denote the price of a derivative depending on r_t. Applying Itô's lemma to $V(r,t)$ and constructing a locally riskless portfolio yields the pricing PDE under the risk-neutral measure:

$$\frac{\partial V}{\partial t} + \kappa(\theta - r)\frac{\partial V}{\partial r} + \tfrac{1}{2}\sigma^2 r \frac{\partial^2 V}{\partial r^2} - rV = 0. \tag{2}$$

This equation generalises Vasicek's PDE by allowing the diffusion coefficient to depend on r. By the Feynman–Kac theorem,

$$V(r,t) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du}\,\ell(r_T)\right],$$

for payoff $\ell(r_T)$ at maturity T.

4 Bond Pricing

4.1 Zero-Coupon Bond

The affine structure of (2) implies that the zero-coupon bond price retains the exponential-affine form

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}, \tag{3}$$

where the functions $A(t,T)$ and $B(t,T)$ solve a system of Riccati equations:

$$B'(T) = 1 - \kappa B(T) + \tfrac{1}{2}\sigma^2 B(T)^2, \qquad B(T,T) = 0,$$
$$A'(T) = -\kappa\theta B(T), \qquad A(T,T) = 0.$$

The closed-form solution is

$$B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\kappa+\gamma)(e^{\gamma(T-t)} - 1) + 2\gamma},$$
$$A(t,T) = \left[\frac{2\gamma e^{(\kappa+\gamma)(T-t)/2}}{(\kappa+\gamma)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{\frac{2\kappa\theta}{\sigma^2}},$$

where $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$. The bond yield is $y(t,T) = -\frac{1}{T-t}\ln P(t,T)$. As in Vasicek, the yield curve is affine in the short rate but now strictly positive. The CIR bond formula is fundamental because it shows that even with state-dependent volatility, the term structure remains analytically tractable.

4.2 Coupon Bonds

A coupon-bearing bond with payments c_i at times T_i and notional N is priced as a sum of zero-coupon bonds:

$$P^{\text{coupon}}(t) = \sum_i c_i P(t,T_i) + N P(t,T_n),$$

using the $A(t,T)$ and $B(t,T)$ functions above.

5 Options Pricing

5.1 Bond Options

A European call on a zero-coupon bond maturing at $S > T$ with strike K and expiry T has payoff $(P(T,S)-K)^+$. The price can again be written in closed form under the T-forward measure because $P(T,S)$ is lognormal in that measure. The option value is

$$C(t) = P(t,S)\Phi(d_1) - KP(t,T)\Phi(d_2), \tag{4}$$

where

$$d_{1,2} = \frac{\ln \frac{P(t,S)}{KP(t,T)} \pm \frac{1}{2}\sigma_P^2}{\sigma_P}, \qquad \sigma_P^2 = \frac{\sigma^2}{\kappa^2} \frac{(1-e^{-\kappa(T-t)})^2}{1-e^{-2\kappa(S-T)}} (1-e^{-\kappa(S-T)})^2.$$

This formula has the same structure as in Vasicek, but the variance term reflects the level-dependent volatility of the CIR process. Options on coupon bonds are priced as portfolios of zero-bond options via Jamshidian's trick.

5.2 Futures Options

An interest rate futures contract is marked-to-market daily, so the futures rate differs from the forward rate by a convexity adjustment. Under CIR, this adjustment has no closed form but can be approximated analytically or computed numerically using bond covariance. The option price is then given by a Black-like or Bachelier-like formula with adjusted futures rate $F(t,T)$ and variance derived from the CIR dynamics.

5.3 Caps, Floors, Caplets, Floorlets

A caplet on $[T, T+\delta]$ with strike K and notional N pays $N\delta(L(T,T+\delta)-K)^+$. Using the bond relation $L(T,T+\delta) = \frac{1}{\delta}(P(T,T)/P(T,T+\delta)-1)$, the caplet is equivalent to a put on $P(T,T+\delta)$. Therefore, its price is obtained from (4):

$$\text{Caplet}(t) = N\delta\big(P(t,T)\Phi(d_1) - (1+K\delta)P(t,T+\delta)\Phi(d_2)\big),$$

with volatility computed using the CIR bond variance. A floorlet follows by put-call parity, and caps and floors are portfolios of caplets and floorlets across maturities.

5.4 Swaptions

A payer swaption gives the right to enter a fixed–for–floating swap at expiry T. The payoff is $N(1-P(T,T_N)-KA(T))^+$, where $A(T) = \sum_{j=1}^{N} \delta_j P(T,T_j)$. As in Vasicek, it can be expressed as an option on a coupon bond:

$$B_{\text{fix}}(T;K) = P(T,T_N) + K\sum_{j=1}^{N} \delta_j P(T,T_j),$$

so that the swaption value is $(1 - B_{\text{fix}}(T;K))^+$. Jamshidian's trick applies because CIR is a one-factor affine model: there exists a unique r^* such that $B_{\text{fix}}(T;K;r^*) = 1$, allowing decomposition into zero-bond options. A receiver swaption, with payoff $N(KA(T) - (1 - P(T,T_N)))^+$, is obtained analogously and satisfies the parity

$$\text{Swpt}^{\text{payer}}(t) - \text{Swpt}^{\text{receiver}}(t) = NA(t)(S(t) - K),$$

where $S(t)$ is the forward swap rate. Closed-form formulas are not available in CIR, but numerical methods using bond-option valuation or Bachelier approximations under the annuity measure are standard.

6 Propositions

Two fundamental propositions underlie pricing in the CIR and, more generally, in all short-rate models.

Proposition 1 (Spot-measure representation). Under the risk-neutral (spot) measure associated with the money-market numéraire

$$B_t = e^{\int_0^t r_s ds},$$

the discounted price of any asset is a martingale:

$$\frac{V_t}{B_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{V_T}{B_T}\right].$$

This proposition formalises risk-neutral valuation: the present value of a payoff is the expected discounted payoff under $\mathbb{Q}$. Intuitively, it connects the economic concept of a continuously rolled deposit account with the probabilistic representation of arbitrage-free prices.

Proposition 2 (Forward-measure representation). Fix a maturity T and take the zero-coupon bond $P(t,T)$ as numéraire. Under the corresponding T-forward measure $\mathbb{Q}^T$, any forward price is a martingale:

$$\frac{V_t}{P(t,T)} = \mathbb{E}_t^{\mathbb{Q}^T}\left[\frac{V_T}{P(T,T)}\right] = \mathbb{E}_t^{\mathbb{Q}^T}[V_T].$$

This change of measure eliminates discounting, allowing derivative pricing to be expressed directly in terms of forward quantities, such as forward rates or bond prices—whose drift becomes zero under $\mathbb{Q}^T$. It is this proposition that leads to tractable valuation formulas for bond and interest rate options in the CIR model.

Together, these propositions provide the mathematical foundation of the term-structure framework. The first ensures consistency with arbitrage-free pricing under the short rate; the second enables closed-form solutions by choosing convenient numeraires (e.g., a bond or annuity). They underpin all pricing relations derived in CIR, Vasicek, and their extensions.

7 Remarks

The CIR model combines economic intuition, positivity, and analytical structure. Its affine nature allows closed-form bond and bond-option pricing while guaranteeing nonnegative rates, unlike Vasicek. However, it cannot perfectly fit the initial yield curve without modification. The extended model, CIR++, introduces a deterministic shift to calibrate exactly the observed term structure while retaining the stochastic dynamics of CIR. Despite its limitations, the CIR model remains a cornerstone of modern term-structure modeling: it connects theory and practice by showing how equilibrium reasoning can coexist with analytical tractability.