Hull-White (1990) One-Fractor Model Calibration with Swaption Volatilities

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1 Introduction

Hull-White (1990) one-factor model is a model describing stochastic evolution of interest rates. The important point is that the model is arbitrage free ie it can fit today's term structure of interest rates. In other words, term structure is an input to the model (in contrast to Vasicek (1977) model and other equilibrium models). The interest rate evolves as follows:

$$dr_t = (\theta_t - \alpha r_t)dt + \sigma_r dW_t \tag{1}$$

where α and σ_r are constant mean reversion rate and volatility of an interest rate, and θ_t is a time dependent drift. Interestingly, in order to calibrate the model we first need to analytically derive the value of θ_t – it is indeed the one of two key points here.

2 θ_t

In a one-factor term structure, a zero-coupon bond price in the traditional risk-neutral world must have a return equal to the short rate r. Then

$$dB(t,T) = r_t B(t,T)dt + \sigma_B(t,T)B(t,T)dW_t \tag{2}$$

is the zero-coupon bond price evolution with $\sigma_B(t,T)$ being its volatility. Itô's Lemma shows that

$$d\ln B(t,T) = \left(r_t - \frac{\sigma_B^2(t,T)}{2}\right)dt + \sigma_B(t,T)dW_t \tag{3}$$

so the time t forward rate for the period between T_1 and T_2 ($T_2 > T_1$) is

$$F(t,T_1,T_2) = -\frac{\ln B(t,T_2) - \ln B(t,T_1)}{T_2 - T_1}$$

and from (3)

$$dF(t,T_1,T_2) = \frac{\sigma_B^2(t,T_2) - \sigma_B^2(t,T_1)}{2(T_2 - T_1)}dt - \frac{\sigma_B(t,T_2) - \sigma_B(t,T_1)}{T_2 - T_1}dW_t \tag{4}$$

We then define the risk-free rate over the period $R(t,T)$ from t to T as

$$R(t,T) = F(0,t,T) + \int_0^t dF(0,t,T)$$

so substituting (4) we get

$$R(t,T) = F(0,t,T) + \int_0^t \frac{\sigma_B^2(\tau,T) - \sigma_B^2(\tau,t)}{2(T-t)} d\tau - \int_0^t \frac{\sigma_B(\tau,T) - \sigma_B(\tau,t)}{T-t} dW_\tau$$

When t approaches T, $R(t,T)$ becomes an instantaneous risk-free rate, r_t, and $f(0,t,T)$ becomes instantaneous forward rate, f_t. Consequently, we obtain the following:

$$r_t = f_t + \int_0^t \frac{\partial \sigma_B^2(\tau,t)}{2\partial t} d\tau - \int_0^t \frac{\partial \sigma_B(\tau,t)}{\partial t} dW_t \tag{5}$$

and differentiating with respect to t we find that

$$dr_t = \left[\frac{\partial f_t}{\partial t} + \int_0^t \left(\sigma_B(\tau,t) \frac{\partial^2 \sigma_B(\tau,t)}{\partial t^2} + \left(\frac{\partial \sigma_B(\tau,t)}{\partial t} \right)^2 \right) d\tau - \int_0^t \frac{\partial^2 \sigma_B(\tau,t)}{\partial t^2} dW_\tau \right] dt$$
$$- \left. \frac{\partial \sigma_B(\tau,t)}{\partial t} \right|_{\tau=t} dW_t$$

where one should recall that bond price volatility converges to zero as time gets closer to maturity.

We then check the original paper of Hull-White (1990) to see that the authors use Black-Scholes (1973) option pricing framework to show[1] that

$$\sigma_B = \frac{\sigma_r(1 - e^{-\alpha(T-t)})}{\alpha} \tag{6}$$

allowing us to restate instantaneous risk-free rate from (5) as

$$r_t = f_t + \frac{\sigma_r^2}{\alpha^2}(1 - e^{-\alpha t}) - \frac{\sigma_r^2}{2\alpha^2}(1 - e^{-2\alpha t}) - \int_0^t \sigma_r e^{-\alpha(t-\tau)} dW_\tau \tag{7}$$

and

$$dr_t = \left[\frac{\partial f_t}{\partial t} + \frac{\sigma_r^2}{\alpha}(e^{-\alpha t} - e^{-2\alpha t}) + \int_0^t \alpha \sigma_r e^{-\alpha(t-\tau)} dW_\tau \right] dt - \sigma_r dW_t$$

Then taking expression for $\int_0^t \sigma_r e^{-\alpha(t-\tau)} dW_\tau$ from (7) we have

$$dr_t = \left[\frac{\partial f_t}{\partial t} + \frac{\sigma_r^2}{\alpha}(e^{-\alpha t} - e^{-2\alpha t}) - \alpha(r_t - f_t) + \frac{\sigma_r^2}{\alpha}(1 - e^{-\alpha t}) - \frac{\sigma_r^2}{2\alpha}(1 - e^{2\alpha t}) \right] dt - \sigma_r dW_t$$
$$= \left[\frac{\partial f_t}{\partial t} + \alpha(f_t - r_t) + \frac{\sigma_r^2}{2\alpha}(1 - e^{-2\alpha t}) \right] dt - \sigma_r dW_t \tag{8}$$

One may have already noted that (8) is same as (8) with

$$\theta_t = \frac{\partial f_t}{\partial t} + \alpha f_t + \frac{\sigma_r^2}{2\alpha}(1 - e^{-2\alpha t})$$

where we recall Wiener process is symmetric around zero, so we change the sign freely.

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[1]The authors discuss bond option valuation

3 Calibration with Black Vol

Given the form of the model in (8), we can move to model calibration. Since we are doing it with swaption, we enjoy more information (relatively to caplets) on the maturity dimension. Nonetheless, the process is not so simple as the Black volatility of forward swap rate is a stochastic function – this goes for all Black volatilities derived from any short-rate model, so it would require Monte Carlo simulation.

Let us then define the annuity starting at some time in the future and providing n[2] 1\$ payments with δ frequency as $A_{n,\delta}(t,T) = \sum_{i=1}^{n} B(t, T + i\delta)$. Applying Itô's Lemma we can see that annuity's volatility is

$$\sigma_A = -\frac{\sigma_r}{\alpha} \sum_{i=1}^{n} \frac{B(t, T + i\delta)(1 - e^{-\alpha(T + i\delta - t)})}{A_{n,\delta}(t,T)} \tag{9}$$

Then the volatility of the "bond-to-annuity ratio" $\left(\frac{(1 - e^{-\alpha(T-t)}/\alpha)}{A_{n,\delta}(t,T)} \right)$ is

$$\sigma_{\text{ratio}} = \frac{\sigma_r}{\alpha} \sum_{i=1}^{n} \frac{B(t, T + i\delta)(e^{-\alpha(T-t)} - e^{-\alpha(T + i\delta - t)})}{A_{n,\delta}(t,T)}$$

because $\sum_{i=1}^{n} \frac{B(t, T+i\delta)}{A_{n,\delta}(t,T)} = 1$. Then recall that time t forward swap rate for a contract starting at T and maturing at $T + n\delta$ is conventionally defined as

$$s(t, T, T + n\delta) = \frac{B(t,T) - B(t, T + n\delta)}{\sum_{i=1}^{n} \delta B(t, T + i\delta)} \tag{10}$$

Next, we find its associated volatility to be

$$\sigma_s = \frac{\sigma_r}{\alpha} \left(\sum_{i=1}^{n} \frac{B(t, T + i\delta)(e^{-\alpha(T-t)} - e^{-\alpha(T + i\delta - t)})}{A_{n,\delta}(t,T)} + \frac{B(t, T + n\delta)(e^{-\alpha(T-t)} - e^{-\alpha(T + n\delta - t)})}{B(t,T) - B(t, T + n\delta)} \right)$$

and its Black equivalent (in variance terms) by integrating over t to take the form of

$$\sigma_{s,\text{Black}}^2(T, T + n\delta) = \frac{1}{T} \int_0^T \sigma_s^2 dt \tag{11}$$

that is to be done via simulation as bond prices are stochastic. Couderc (2006) suggests using approximation

$$(11) \approx \frac{\sigma_r^2}{\alpha^2} \left(\sum_{i=1}^{n} \frac{B(0, T + i\delta)(1 - e^{-\alpha i\delta})}{A_{n,\delta}(0,T)} + \frac{B(0, T + n\delta)(1 - e^{-\alpha n\delta})}{B(0,T) - B(0, T + n\delta)} \right)^2 \frac{1 - e^{-2\alpha T}}{2\alpha T} \tag{12}$$

where $t = 0$ and T is the time-to-maturity. With the help of this approximation we can numerically find α and σ_r to be consistent with today's term structure of interest rates[3].

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[2]Where n is the number of payments rather than years, ie $n = \text{years}/\delta$

[3]We traditionally take representative volatility as being an input to today's ATM swaptions' values

4 Calibration with Normal Vol

Aa new normal market conditions like ultra-low or negative interest rates is approaching, swaption volatilities with some range of maturities are not quoted. It may be that its calculation is impossible or excess volatility is produced. It is not surprising, therefore, that market practitioners barely use this problematic maturities' swaption volatility for pricing or hedging. Against this backdrop, Barchelier's normal model is used as an alternative model.

Nonetheless, it seems unreasonable to destroy Hull and White (1900) model assumptions in attempt of closed-form derivation. Instead, we are going to calibrate the model with swaption by first building volatility converter, ie to move from Bachelier's to Black's volatility and then apply the methodology described in the previous section.

First, recall that in Bachelier's setup the value of a payer swaption (with the notional 1\$ for simplicity) is defined as

$$\mathcal{S}(t, T, T+n\delta) = A_{n,\delta}(t, T)[(s(t, T, T+n\delta) - k)\Phi(d) + \sigma_{s,\text{Normal}}\sqrt{T-t}\phi(d)] \qquad (13)$$
$$d = \frac{s(t, T, T+n\delta) - k}{\sigma_{s,\text{Normal}}\sqrt{T-t}}$$

with k being a strike rate, Φ and ϕ being a standard normal cdf and pdf respectively. Similarly, Black's setting gives us the price of the same payer swaption as

$$\mathcal{S}(t, T, T+n\delta) = A_{n,\delta}(t, T)[s(t, T, T+n\delta)\Phi(d_1) - k\Phi(d_2)] \qquad (14)$$
$$d_1 = \frac{\ln\frac{s(t,T,T+n\delta)}{k} + \frac{1}{2}\sigma_{s,\text{Black}}^2(T-t)}{\sigma_{s,\text{Black}}\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma_{s,\text{Black}}\sqrt{T-t}$$

Then the procedure is very simple. We equate (13) and (14) and, given $\sigma_{s,\text{Normal}}$ we numerically solve for $\sigma_{s,\text{Black}}$. Since the prices are indeed same regardless the model, we should be able to get correct implied volatilities. ATM nature implies that $s(t, T, T+n\delta) = k$ and $s(t, T, T+n\delta)$ is observed on the market that is enough for implementing current procedure.

Finally, after obtaining $\sigma_{s,\text{Black}}$ we numerically solve (12) as described in the previous section.

Another approach to go with ATM swaptions is to realise that $s(t, T, T+n\delta) - k = 0$, thus
$$(13) = A_{n,\delta}(t, T) \times \sigma_{s,\text{Normal}}\phi(0)\sqrt{T-t} = (14)$$
with $\phi(0) = 0.3989$. Consequently we have

$$\sigma_{s,\text{Normal}} \approx \frac{s(t, T, T+n\delta)\Phi(d_1) - k\Phi(d_2)}{\phi(0)\sqrt{T-t}} \qquad (15)$$

where we substitute (12) in the expressions for d_1 and d_2. We then can solve this numer-

ically to obtain α and σ_r. The overall formula takes a bulky form of

$$\sigma_{s,\text{Normal}} \approx \frac{s(0,T,T+n\delta)}{\phi(0)\sqrt{T}}$$

$$\times \Phi \left(\frac{\sigma_r^2}{2\alpha^2} \left(\sum_{i=1}^{n} \frac{B(0,T+i\delta)(1-e^{-\alpha i\delta})}{A_{n,\delta}(0,T)} + \frac{B(0,T+n\delta)(1-e^{-\alpha n\delta})}{B(0,T)-B(0,T+n\delta)} \right)^2 \frac{1-e^{-2\alpha T}}{2\alpha\sqrt{T}} \right)$$

$$- \frac{k}{\phi(0)\sqrt{T}}$$

$$\times \Phi \left(\frac{\sigma_r^2}{2\alpha^2} \left(\sum_{i=1}^{n} \frac{B(0,T+i\delta)(1-e^{-\alpha i\delta})}{A_{n,\delta}(0,T)} + \frac{B(0,T+n\delta)(1-e^{-\alpha n\delta})}{B(0,T)-B(0,T+n\delta)} \right)^2 \frac{e^{-2\alpha T}-1}{2\alpha\sqrt{T}} \right)$$

with $t = 0$ and T being a time to maturity. A more compact form of $\sigma_{s,\text{Normal}}$ is

$$\left[2\Phi \left(\frac{\sigma_r^2}{2\alpha^2} \left(\sum_{i=1}^{n} \frac{B(0,T+i\delta)(1-e^{-\alpha i\delta})}{A_{n,\delta}(0,T)} + \frac{B(0,T+n\delta)(1-e^{-\alpha n\delta})}{B(0,T)-B(0,T+n\delta)} \right)^2 \frac{1-e^{-2\alpha T}}{2\alpha\sqrt{T}} \right) - 1 \right]$$

$$\times \frac{k}{\phi(0)\sqrt{T}} = \sigma_{s,\text{Normal}}$$

where we recall that we work with an ATM swaption and $\Phi(-x) = 1 - \Phi(x)$. Probably, an important point to note one more time is that *the Law of One Price* implies that $(13) = (14)$. By taking into account ATM nature one can find that

$$\sigma_{s,\text{Black}} = \frac{2}{\sqrt{T-t}} \Phi^{-1} \left(\frac{1}{2k} \phi(0) \sigma_{s,\text{Normal}} \sqrt{T-t} + \frac{1}{2} \right)$$

so, technically, anything inside $\Phi^{-1}(\cdot)$ should be within $(0,1)$ interval. Thus,

$$s(t,T,T+n\delta) > \phi(0)\sigma_{s,\text{Normal}}(T,T+n\delta)\sqrt{T-t}$$

From (10) it is clearly seen that one should be very careful calibrating the model with negative interest rates as equation constraint is unlike to be satisfied in such a case. The problem is that, despite it is the Normal model, the procedure is basically a conversion from Black model – when interest rates are negative, (14) and, consequently, (12) do not exist. Therefore, (15) does not exist in the negative interest rates environment as well.

A famous approach in pricing interest rate derivatives in such situations is to simply shift the lognormal distribution assumed by Black model, however, it is not helpful for calibration as Black volatilities are not quoted – another approach in needed.