Bonus on Models

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This Version: May 25, 2022

1 Introduction

This note compares the derivations of the Bachelier and Black models for caplet pricing, and the Vasicek model for the zero-coupon bond option. Bachelier and Black are not internally consistent models for the full term structure of interest rates: they consider one forward rate, one accrual period δ, and one maturity at a time, without linking options on the same rate across maturities. They are best viewed as option price converters, translating an observed option price into an implied volatility. In this context, σ_N denotes the normal (Bachelier) volatility, while σ_V denotes the Black lognormal volatility. Later, we denote by σ the volatility implied by the Vasicek model.

The fundamental difference between Bachelier (or Black) and Vasicek lies in the scope of their parameters. In Vasicek, a single volatility σ and mean-reversion speed κ jointly determine the prices of all options across maturities, ensuring consistency with the term structure. In contrast, in the Bachelier or Black frameworks, each option requires its own fitted volatility parameter σ_N or σ_B, which cannot be used to price another option. Hence, Bachelier and Black provide pricing consistency within a given option but not across the curve, whereas Vasicek offers a structural link between maturities through its single-factor dynamics.

2 Bachelier Model (Normal)

In the Bachelier model, the forward or simple rate evolves as an arithmetic Brownian motion under the risk-neutral measure:

$$dF_t = \sigma_N \, dW_t^{\mathbb{Q}}, \qquad F_0 = f_0. \tag{1}$$

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The rate distribution at maturity T is normal:

$$F_T \sim \mathcal{N}\big(f_0, \sigma_N^2 T\big). \tag{2}$$

A European call option with strike K and maturity T has discounted expected payoff

$$C_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[(F_T - K)^+] = e^{-rT}\big((f_0 - K)\Phi(d) + \sigma_N\sqrt{T}\,\phi(d)\big), \tag{3}$$

where

$$d = \frac{f_0 - K}{\sigma_N\sqrt{T}}. \tag{4}$$

This is the *normal model* (or *Bachelier formula*). It allows for negative rates and treats volatility as an absolute change per unit time, appropriate when rates fluctuate symmetrically around a mean level.

3 Black Model (Lognormal)

Black's framework assumes a geometric Brownian motion for the forward rate:

$$dF_t = \sigma_B F_t\, dW_t^{\mathbb{Q}}, \qquad F_0 = f_0 > 0. \tag{5}$$

Integrating gives

$$F_T = f_0 \exp\left(-\tfrac{1}{2}\sigma_B^2 T + \sigma_B W_T\right), \tag{6}$$

so F_T is lognormally distributed with strictly positive support. The call price is

$$C_0 = e^{-rT}\big(f_0\Phi(d_1) - K\Phi(d_2)\big), \qquad d_{1,2} = \frac{\ln(f_0/K) \pm \frac{1}{2}\sigma_B^2 T}{\sigma_B\sqrt{T}}. \tag{7}$$

This *Black formula* remains the industry standard for caplets, floors, and swaptions. It assumes that volatility scales with the rate level, thereby precluding negative rates but providing a more realistic description when rates are far from zero.

4 Vasicek Model (Gaussian Short Rate)

Vasicek (1977) introduced a short-rate process with mean reversion:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^{\mathbb{Q}}, \tag{8}$$

where $\kappa > 0$ measures reversion speed, θ is the long-run mean, and σ is volatility. The solution is Gaussian:

$$\begin{aligned}
r_T | r_t &\sim \mathcal{N}(m(t,T), v(t,T)), \\
m(t,T) &= \theta + (r_t - \theta)e^{-\kappa(T-t)}, \\
v(t,T) &= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(T-t)}).
\end{aligned}$$

The zero-coupon bond price retains the exponential-affine form:

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}, \tag{9}$$

with

$$\begin{aligned}
B(t,T) &= \frac{1 - e^{-\kappa(T-t)}}{\kappa}, \\
A(t,T) &= \exp\left((\theta - \tfrac{\sigma^2}{2\kappa^2})(B(t,T) - (T-t)) - \tfrac{\sigma^2}{4\kappa}B(t,T)^2\right).
\end{aligned}$$

Under the forward measure associated with $P(t,T)$, the bond price ratio $P(T,S)/P(T,T)$ is lognormal, leading to a Black-style option price on bonds or caplets.

5 Caplets and Floorlets

A caplet with payoff on $[T,T+\delta]$ is given by

$$\Pi_T^{caplet} = \delta(L(T,T+\delta) - K)^+ = \frac{1}{\delta}\Big(\frac{P(T,T)}{P(T,T+\delta)} - 1 - K\delta\Big)^+. \tag{10}$$

Because $P(T,T+\delta)$ follows a lognormal-like process in Vasicek, the caplet price can be expressed as a bond option:

$$C(t) = \delta\big(P(t,T)\Phi(d_1) - KP(t,T+\delta)\Phi(d_2)\big), \tag{11}$$

with variance term derived from the short-rate dynamics.

6 Implied Volatility as a Price Measure

Although the Bachelier and Black models yield different implied volatilities for the same market price, the *law of one price* ensures equivalence once the volatility is interpreted as a derived quantity rather than a primitive parameter. Implied volatility acts as a convenient *unit of comparison*: it converts the derivative's price into a standardized measure of moneyness and risk.

In the Bachelier framework, implied volatility reflects the absolute uncertainty of the rate, while in the Black framework it scales with the rate level. Each model uses volatility as its own "price converter." If both models are internally consistent (no arbitrage), then although their volatilities differ numerically, they encode the same information: how expensive the derivative is relative to its forward.

In practice, volatility provides a natural metric for comparing derivatives across strikes and maturities. Quoting volatility instead of raw price allows traders to interpret option richness or cheapness across different instruments and models.

7 Remarks

The three models: Bachelier, Black, and Vasicek, – represent stages in the evolution of interest rate option theory. Bachelier offers a linear world suitable near zero rates; Black ensures positivity through lognormality; Vasicek introduces mean reversion and a term-structure foundation. Implied volatility, regardless of model, remains the universal price language linking them.