#2.06 Let $*$ be defined on $\mathbb{C}$ by letting $a * b = |ab|$
Ans: This is not a group because Axiom 2 fails:
There is no identity element. $a*e = |ae|$ the magnitude returns $\mathbb{R}$ so $a*e \neq a$.

#2.18 G $= \{e, a, b\}$ where

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad b = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

Show whether $(G, \times)$ where $\times$ is standard matrix multiplication is a group.

Ans:

a. It is associative (proved in math18).

b. There is an identitiy element: e.

c. There is an inverse: $a * b = b * a = e$.

#2.28 An element $a \neq e$ in a group is said to have order 2 if $a * a = e$. Prove that if G is a group and $a \in G$ has order 2, then for any $b \in G$, $b^{-1} * a * b$ also has order 2.

Ans: We try to prove $b^{-1} * a * b = e$ is order two, so let's try computing $(b^{-1} * a * b) * (b^{-1} * a * b)$. By associativity:

$$(b^{-1} * a * b) * (b^{-1} * a * b)$$
$$= b^{-1} * a * (b * b^{-1}) * a * b$$
$$= b^{-1} * a * e * a * b$$
$$= b^{-1} * a * a * b$$
$$= b^{-1} * e * b \quad (a * a = e)$$
$$= e$$

Hence $b^{-1} * a * b$ is order 2.

#2.31 If $*$ is a binary operation on a set S, an element x of S is an *idempotent for* $*$ if $x * x = x$. Prove that a group has exactly one idempotent element.
Ans: By Theorem 2.16, we can use left cancellation to prove this statement.
Let e be the identity element of the group.
$e * e = e$ so we know that e is idempotent. By proof of contradiction, suppose there is another idempotent element $x \neq e$.
Then $x * x = x$ and $x * e = x$, so $x * x = x * e$
By left cancellation, $x = e$ which contradicts our assumption.
Therefore, a group has exactly one idempotent element.

#2.38 Let G be a group and let $a, b \in G$. Show that $(a * b)' = a' * b'$ if and only if $a * b = b * a$.
Corollary 2.19 states that $(a * b)' = b' * a'$
Ans: We first prove forward direction: $(a * b)' = a' * b' \Rightarrow a * b = b * a$.
Since $(a * b)' = b' * a'$:

$$\begin{aligned} b' * a' &= a' * b' \\ (b' * a')' &= (a' * b')' \\ (a')' * (b')' &= (b')' * (a')' \\ a * b &= b * a \end{aligned}$$

Now we prove the reverse direction: $a * b = b * a \Rightarrow (a * b)' = a' * b'$.
Since $a * b = b * a$:

$$\begin{aligned} (b * a)' &= a' * b' \\ a' * b' &= a' * b' \quad \text{(By Corollary 2.19)} \end{aligned}$$

So we proved both sides of the statement is true.

#3.26 Compute the given expression using the indicated modular addition.

$$\frac{3\pi}{4} +_{2\pi} \frac{3\pi}{2}$$

Ans:

$$\frac{3\pi}{4} + \frac{6\pi}{4} \equiv \frac{1\pi}{4} \pmod{2\pi}$$

#3.27 Compute the given expression using the indicated modular addition.

$$2\sqrt{2} +_{\sqrt{32}} 3\sqrt{2}$$

Ans:

$$2\sqrt{2} + 3\sqrt{2} \equiv \sqrt{2} \pmod{4\sqrt{2}}$$

#4.11 Convert the following permutations in S_8 from disjoint cycle notation to two-row notation.

(a) (1, 4, 5)(2, 3)
(b) (1, 8, 5)(2, 6, 7, 3, 4)
(c) (1, 2, 3)(4, 5)(6, 7, 8)

Ans:

$$\text{(a)} \quad (1,4,5)(2,3) \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 5 & 1 & 6 & 7 & 8 \end{pmatrix}$$

$$\text{(b)} \quad (1,8,5)(2,6,7,3,4) \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 6 & 4 & 2 & 1 & 7 & 3 & 5 \end{pmatrix}$$

$$\text{(c)} \quad (1,2,3)(4,5)(6,7,8) \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 5 & 4 & 7 & 8 & 6 \end{pmatrix}$$