#4.12 Compute the permutation products.

(a) $(1, 5, 2, 4)(1, 5, 2, 3)$

(b) $(1, 5, 3)(1, 2, 3, 4, 5, 6)(1, 5, 3)^{-1}$

(c) $[(1, 6, 7, 2)^2(4, 5, 2, 6)^{-1}(1, 7, 3)]^{-1}$

(d) $(1, 6)(1, 5)(1, 4)(1, 3)(1, 2)$

Ans:

(a) We start with 3:

$$\sigma(3) = 5, \sigma(5) = 4, \sigma(4) = 1, \sigma(1) = 2, \sigma(2) = 3$$

Thus, $(1, 5, 2, 4)(1, 5, 2, 3) = (3, 5, 4, 1, 2) = (1, 2, 3, 5, 4)$

(b) First calculate $(1, 5, 3)^{-1} = (1, 3, 5)$. Then we can calculate the product of $(1, 5, 3)(1, 2, 3, 4, 5, 6)(1, 3, 5)$:

$$\sigma(1) = 4, \sigma(4) = 3, \sigma(3) = 6, \sigma(6) = 5, \sigma(5) = 2, \sigma(2) = 1$$

Thus $(1, 5, 3)(1, 2, 3, 4, 5, 6)(1, 3, 5) = (1, 4, 3, 6, 5, 2)$

(c) First calculate $[(1, 6, 7, 2)^2(4, 5, 2, 6)^{-1}(1, 7, 3)]^{-1}$
$= (2, 7, 6, 1)(2, 7, 6, 1)(4, 5, 2, 6)(3, 7, 1)$ Then:

$$\sigma(1) = 3, \sigma(3) = 1, \sigma(2) = 2, \sigma(4) = 5, \sigma(5) = 6, \sigma(6) = 4, \sigma(7) = 7,$$

Thus $[(1, 6, 7, 2)^2(4, 5, 2, 6)^{-1}(1, 7, 3)]^{-1} = (1, 3)(2)(4, 5, 6)(7)$

(d) $(1, 6)(1, 5)(1, 4)(1, 3)(1, 2)$

$$\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 4, \sigma(4) = 5, \sigma(5) = 6, \sigma(6) = 1$$

Thus $(1, 6)(1, 5)(1, 4)(1, 3)(1, 2) = (1, 2, 3, 4, 5, 6)$

#4.26 $f_3 : \mathbb{R} \to \mathbb{R}$ defined by $f_3(x) = -x^3$. Is f_3 a permutation?

Ans:

Recall that a permutation is one-to-one and onto function. Thus, we check these two properties.

(a) f_3 is one-to-one:
Assume $f_3(x) = f_3(y)$, then $-x^3 = -y^3$ implies $x = y$. Thus, f_3 is one-to-one.

(b) f_3 is onto:
For any $y \in \mathbb{R}$, we can find $x = -\sqrt[3]{y} \in \mathbb{R}$ such that
$f_3(x) = -x^3 = -(-\sqrt[3]{y})^3 = y$. Thus, f_3 is onto.

So since f_3 is one-to-one and onto, it is a permutation.

#4.35 Give a careful proof using the definition of isomorphism that if G and G' are both groups with G abelian and G' not abelian, then G and G' are not isomorphic.

Ans:

Recall that a group isomorphism needs to satisfy the following:

(a) ϕ is a homomorphism, i.e., $\forall a, b \in G, \phi(a * b) = \phi(a) * \phi(b)$.

(b) ϕ is bijective, i.e., ϕ is both injective (one-to-one) and surjective (onto).

Since G is abelian, we can apply the homomorphism ϕ, we know

$$\phi(a * b) = \phi(b * a) = \phi(a) * \phi(b) = \phi(b) * \phi(a)$$

Since ϕ is an isomorphism and $\phi(a) * \phi(b) = \phi(b) * \phi(a)$, we know G' is abelian. However we already know G' is not abelian so we have a contradiction.
So, G and G' are not isomorphic.

#5.08 Is the n $\times$ n matrices with determinant greater than or equal than 1 a subgroup of $GL(n, \mathbb{R})$?

Ans:

We check the properties of a subgroup:

(a) Closure:
Let A, B be two $n \times n$ matrices with det ≥ 1. Then $\det(AB) = \det(A)\det(B) \geq 1$. So it is closed under $GL(n, \mathbb{R})$.

(b) Identity:
The identity matrix has determinent 1, which is in $GL(n, \mathbb{R})$.

(c) Inverse:
Let A be an $n \times n$ matrix with det ≥ 1. Then $\det(A^{-1}) = \frac{1}{\det(A)} \leq 1$. So it fails the inverse property so it is NOT a subgroup.

#5.22 Describe all the elements in the cyclic subgroup of $GL(2, \mathbb{R})$ generated by the given 2×2 matrix.

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Ans:

We try multiply the matrix by itself:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Which is the identity matrix, so the cyclic subgroup generated is:

$$\left\{ \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

#5.30 Find the order of the cyclic subgroup of $\mathbb{Z}_{10}$ generated by 8

Ans:

We can calculate the order of the cyclic subgroup generated by 8:

n	$8^n \mod 10$
1	8
2	4
3	2
4	6
5	8

We repeated at n=5 so the order of by 8 is 4.

#5.56 Show that if $a \in G$, where G is a finite group with identity e, then there exists $n \in \mathbb{Z}^+$ such that $a^n = e$.

Ans:

We can use the idea of the pigeonhole principle. Since G is finite, there is only a finite number of elements in G. So there must be two powers of a that are equal, for exmaple, $a^i = a^j$ for some $i > j$. Then we can multiply both sides by a^{-i} to get $a^{j-i} = e$. So there exists $n = j - i$ such that $a^n = e$.

#5.61 For sets H and K, we define the *intersection* $H \cap K$ by

$$H \cap K = \{x \mid x \in H \text{ and } x \in K\}.$$

Show that if $H \leq G$ and $K \leq G$, then $H \cap K \leq G$. (Remember: $\leq$ denotes "is a subgroup of," not "is a subset of.")

Ans:

We need to show that $H \cap K$ is a subgroup of G. We can check the properties of a subgroup:

(a) Closure:
Let $a, b \in H \cap K$, then by definition of intersection $a, b \in H$ and $a, b \in K$.
Since H and K are subgroups of G, $ab \in H$ and $ab \in K$.
Thus, $ab \in H \cap K$.

(b) Identity:
Since H and K are subgroups of G, they both have e in them.
Thus, $e \in H \cap K$.

(c) Inverse:
Let $a \in H \cap K$, then $a \in H$ and $a \in K$. Since H and K are subgroups of G, $a^{-1} \in H$ and $a^{-1} \in K$.
Thus, $a^{-1} \in H \cap K$.
So all three properties are satisfied, and $H \cap K$ is a subgroup of G.