Algebraic and Topological Methods in Promise Constraint Satisfaction Problems

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Submitted as the Part B Extended Essay as part of my
Master of Mathematics and Computer Science degree

Trinity Term
23rd of April 2023

Pentru Lulu, Didi, Monica și Carmen,
femeile care m-au crescut cu toată dragostea din lume.

Contents

Conventions and Notation followed throughout this paper

- All graphs are finite and undirected unless otherwise specified.
- Given a graph G, we denote its vertex set by $V(G)$ and edge set by $E(G)$.
- All functions between topological spaces are continuous.
- **Top** denotes the category whose objects are topological spaces with continuous maps as morphisms between them, which are composed as usual functions.
- **Grp** denotes the category whose objects are with group homomorphisms as morphisms between them, which are composed as usual functions.
- **Ab** denotes the category whose objects are Abelian groups with group homomorphisms as morphisms between them, which are composed as usual functions.
- Given a natural number $n \in \mathbb{N}_{\geqslant 1}$, $[n]$ denotes the set $\{1, \ldots, n\}$.
- Given two integers $i, j \in \mathbb{Z}$, $[i \ldots j]$ denotes the set $\{i, i+1, \ldots, j-1, j\}$.
- C_n denotes the undirected graph consisting of a single cycle with n vertices: $V(C_n) = [n]$ and $E(C_n) = \{(i, i+1), (i+1, i) : i \in [n-1]\} \cup \{(1, n), (n, 1)\}$.
- K_n denotes the undirected complete graph with n vertices: $V(K_n) = [n]$, $E(K_n) = \{(i, j) : i, j \in [n]\,;\ i \neq j\}$.
- L_n denotes the undirected chain with n vertices: $V(L_n) = [n]$, $E(L_n) = \{(i, i+1), (i+1, i) : i \in [n-1]\}$.

1 Promise Constraint Satisfaction Problems

One of the earliest fundamental goals set for computer science is finding out what are the inherent mathematical properties of a computational problem that make it tractable (polynomially solvable) [15]. This question has attracted interest to the study of constraint satisfaction problems (CSPs) for two reasons: the first one being the generality of CSPs, given the sheer number of problems that can be expressed as such and the second being the general richness of the mathematical structure and the plethora of tools that we have to disect this structure [9]. In this section, we formally define CSPs and their generalization: promise constraint satisfaction problems (PCSPs).

1.1 Relational Structures

A relational structure is just a tuple consisting of a finite set and some relations over that set. Formally, that is:

Definition 1.1. We call any finite tuple of positive integers $\sigma = (\sigma_1, \ldots, \sigma_n)$ a **signature** and define a **relational structure** $\boldsymbol{A}$ of signature σ over a set A to be a tuple $(A, R_1, \ldots, R_n)$, where $R_i \subseteq A^{\sigma_i}$ for all i (i.e. R_i is a σ_i-ary relation over A).

We will follow the established convention and denote a relational structure and its underlying set by the same letter, with the sole distinction of using boldface for the former; with the exception of graphs, where the standard notation is used.

For example, a structure $\mathbf{A} = (A; R_1, R_2, R_3)$, where R_1 is a unary relation over A, R_2 is a ternary relation over A and R_3 is a binary relation over A would have signature $(1, 3, 2)$. We'll use the notation ar for the arity of a relation and σ for the signature of a relational structure so that in the above example $\sigma(\mathbf{A}) = (\mathrm{ar}(R_1), \mathrm{ar}(R_2), \mathrm{ar}(R_3)) = (1, 3, 2)$.

The fundamental recurring question in the algebraic theory of constraint satisfaction is whether given two relational structures $\mathbf{A}, \mathbf{B}$ there exists a relational structure homomorphism from $\mathbf{A}$ to $\mathbf{B}$. We define this formally:

Definition 1.2. Given two relational structures $\mathbf{S_1} = (S_1; Q_1, \ldots, Q_n)$ and $\mathbf{S_2} = (S_2; R_1, \ldots, R_n)$ of the same signature, we define a homomorphism from $\mathbf{S_1}$ to $\mathbf{S_2}$ as a function $f : S_1 \to S_2$ which "preserves relations". Formally, f need obey that for any $i \in [1 \ldots n]$ and tuple $(x_1, \ldots, x_{\mathrm{ar}(Q_i)}) \in Q_i$, it holds that $(f(x_1), \ldots, f(x_{\mathrm{ar}(Q_i)})) \in R_i$.

A family of relational structures of great importance is that of directed graphs, as a graph (V, E) is fully described by its underlying set of vertices V and a single binary relation over that set ($E \subseteq V \times V$) determining its edges.

1.2 Constraint Satisfaction Problems

We provide a definition of constraint satisfaction problems in terms of homomorphisms.

Definition 1.3. Given a relational structure $\mathbf{A}$, CSP($\mathbf{A}$) is the decision problem:

- Input: a relational structure $\mathbf{X}$ of the same signature as $\mathbf{A}$
- Output: `yes` if there exists a homomorphism from $\mathbf{X}$ to $\mathbf{A}$, `no` if there is none

In fact, CSPs come in two variants: the decision variant (exactly as above) and the search variant, where one has to find a homomorphism from $\mathbf{X}$ to $\mathbf{A}$. In the case of CSPs, the complexity of the decision and search variants of CSP(A) for some relational structure $\mathbf{A}$ are equivalent [3]. We may thus restrict our focus to decision CSPs.

Notation 1.1. Throughout this paper we will use the notation $A \to B$ to denote the statement *there exists a homomorphism* $f : A \to B$ and $A \nrightarrow B$ to denote the statement *there exists no homomorphism* $f : A \to B$

To better motivate the above definitions, consider the following example:

Example 1.1. Given a complete graph K_n, CSP(K_n) is the same as the problem of n-colourability. To see this, note that for any graph G and homomorphism $f : V(G) \to K_n$, we may interpret f as an n-colouring which sends each vertex to its colour (think what would happen if f would send two adjacent vertices to the same colour, given that K_n has no loops)

1.3 The Complexity of CSPs

The ubiquity of constraint satisfaction problems motivated the development of a large body of literature on the subject ([3, 4, 7, 14, 17]), covering striking results such as:

Theorem 1.1 (Bulatov-Zhuk - [4, 17])**.** For any relational structure $\mathbf{A}$, CSP($\mathbf{A}$) is either NP-hard or polynomially solvable (in particular, CSP($\mathbf{A}$) cannot be NP-intermediate).

In the case of graphs, the hardness criterion turns out to be very simple:

Theorem 1.2 (Hell-Nešetřil - [7])**.** For any graph G, CSP(G) is polynomially solvable if and only if G is 2-colourable or contains a loop, otherwise it is NP-hard.

As will be elaborated below, a direction of interest in the study of constraint satisfaction is the generalization of results such as the ones above to PCSPs.

1.4 Promise Constraint Satisfaction Problems

Definition 1.4. A PCSP template is a pair of relational structures of the same signature $(\mathbf{A}, \mathbf{B})$ such that $\mathbf{A} \to \mathbf{B}$.

Definition 1.5. Given a PCSP template $(\mathbf{A}, \mathbf{B})$, the decision problem $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ is:

- Input: A structure $\mathbf{X}$ of the same signature as $\mathbf{A}$ and $\mathbf{B}$ such that either $\mathbf{X} \to \mathbf{A}$ or $\mathbf{X} \nrightarrow \mathbf{B}$. Equivalently, a structure X of the same signature as $\mathbf{A}$ and $\mathbf{B}$ such that it is false that $\mathbf{X} \to \mathbf{B}$ and $\mathbf{X} \nrightarrow \mathbf{A}$ (i.e. the promise part of the acronym PCSP is that this condition will hold for all inputs).
- Output: `yes` if $\mathbf{X} \to \mathbf{A}$, `no` if $\mathbf{X} \nrightarrow \mathbf{B}$

For example, the problem $\mathrm{PCSP}(K_3, K_5)$ can be described as follows: *I give you a graph G, output **yes** if it is 3-colourable, output **no** if it is not even 5-colourable.*

As with CSPs, PCSPs have a decision (as above) and a search variant. In the search variant, one is given a PCSP template $(\mathbf{A}, \mathbf{B})$ and for an input $\mathbf{I}$ such that $\mathbf{I} \to \mathbf{A}$ is asked to find a homomorphism $I \to B$. However, unlike for CSPs, it is unknown whether or not the two problems are equivalent modulo polynomial time reduction. Throughout this paper, we shall solely discuss the decision variant of PCSPs.

Lastly, we note that for any relational structure $\mathbf{A}$, as $\mathbf{A} \to \mathbf{A}$ through the identity homomorphism, we may define the problem $\mathrm{PCSP}(\mathbf{A}, \mathbf{A})$. That PCSPs are a generalization of CSPs can be interpreted as the fact that $\mathrm{CSP}(\mathbf{A})$ is precisely the same problem as $\mathrm{PCSP}(\mathbf{A}, \mathbf{A})$.

1.5 The Complexity of PCSPs

Most of the results on the complexity of CSPs stem from application of the heavy mathematical machinery of universal algebra. However, this machinery breaks down in the case of PCSPs [4, 17]. An exponent of this is that while we know that $\mathrm{PCSP}(K_3, K_5)$ is NP-hard, we do not know anything about the complexity of the problem $\mathrm{PCSP}(K_3, K_6)$.

Consider for example the following conjecture, which serves as a generalization of the Hell-Nešetřil theorem:

Conjecture 1.1 (Barkensiek-Guruswami)**.** Let G and H be any non-bipartite loopless graphs such that $H \to G$. Then $\mathrm{PCSP}(H, G)$ is NP-hard.

A well-known equivalent statement of this conjecture is the following: for any odd i and $j \geqslant 3$, $\mathrm{PCSP}(C_i, K_j)$ is NP-hard. Significant progress has been made in [10] towards the proof of this theorem, by showing that for any odd $i \geqslant 3$, $\mathrm{PCSP}(C_i, K_3)$ is NP-hard using topology in conjunction with the more general theory of polymorphisms of relational structures, restated in the language of category theory (the topic of the next section).

2 The Algebraic Theory of PCSPs

The study of the complexity of promise constraint satisfaction problems is essentially based on the study of poylmorphism minions arising from templates. We first give the standard definitions of these structures:

Definition 2.1. Given two functions $f : A^n \to B$ and $g : A^m \to B$, we call f a **minor** of g and write $f = g^\mu$ if there exists a function $\mu : [m] \to [n]$ such that $f(x_1, \ldots, x_n) = g(x_{\mu(1)}, \ldots, x_{\mu(m)})$.

One can think that f is a minor of g if it can be obtained by permuting, identifying and introducing dummy variables.

Definition 2.2. Let $\mathcal{O}(A, B) = \{f : A^n \to B : n \geqslant 1\}$. A **concrete minion** $\mathcal{M}$ on a pair of sets (A, B) is a subset of $\mathcal{O}(A, B)$ such that for any $f : A^n \to B$ in $\mathcal{M}$ and $\pi : [n] \to [m]$, we have that f^π is also in $\mathcal{M}$ (i.e. $\mathcal{M}$ is closed under taking minors). We further denote $\mathcal{M}^{(n)} := \{f : A^n \to B : f \in \mathcal{M}\}$.

We note that what we call a concrete minion is known simply as a *minion* in [2]. The reason why we add the qualifying word *concrete* is that we will draw a distinction between concrete minions as described above and their categorical generalization (abstract minions) in the next subsection.

Definition 2.3. Let $\mathcal{M}$ and $\mathcal{N}$ be concrete minions. We call a function $\xi : \mathcal{M} \to \mathcal{N}$ a homomorphism if:

- ξ preserves arities - i.e. for any $n \geqslant 1$ ξ sends n-ary functions to n-ary functions
- ξ commutes with minor taking - i.e. for any $n, m \geqslant 1$, $\mu : [n] \to [m]$ and n-ary function $f \in \mathcal{M}$, we have that $\xi(f)^\mu = \xi(f^\mu)$

Definition 2.4. We denote by **Min** the category of concrete minions, i.e.:

- The objects of **Min** are minions

- For two minions $\mathcal{M}, \mathcal{N} \in \mathrm{ob}(\mathbf{Min})$, $\mathrm{Hom}_{\mathbf{Min}}(\mathcal{M}, \mathcal{N})$ is given by the set of minion homomorphisms from $\mathcal{M}$ to $\mathcal{N}$.
- Homomorphism composition in **Min** is given by regular function composition.

Given a PCSP template $(\mathbf{A}, \mathbf{B})$, one can obtain a minion called its **polymorphism minion** which, as shown in [2], contains enough information to determine the complexity of $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$. Intuitively, a polymorphism over two mathematical objects $\mathbf{A}$ and $\mathbf{B}$ is a minion over the underlying sets of $\mathbf{A}$ and $\mathbf{B}$ restricted to structure-preserving functions. For examples, for two groups G and H, we have:

$$\mathrm{Pol}(G, H) := \{f : G^n \to H : n \geqslant 1, \text{f is a group homomorphism}\}$$

The idea of *structure preserving functions* and the study of such functions is formalized by the language of category theory. This section restates the definition of polymorphisms in the language of category theory.

2.1 The Category of Relational Structures

A natural first step in translating the algebraic theory of PCSPs into categorical language is defining the category in which relational structures live as objects.

Definition 2.5. Given a signature σ, we define Rel_σ as the category whose objects are relational structures of signature σ and $\mathrm{Hom}(\mathbf{A}, \mathbf{B}) := \{f : \mathbf{A} \to \mathbf{B} : f \text{ is a relational homomorphism}\}$ for any two relational structures $\mathbf{A}$, $\mathbf{B}$ with composition and identities being the natural ones.

Remark 2.1. A special instance of such a category is the category Gr of directed graphs.

To be able to work with polymorphisms of relational structures, we first need to know what the product in that category is:

Lemma 2.1. Given a tuple of integers $\sigma = (\sigma_1, \ldots, \sigma_n)$ and two relational structures $\mathbf{A} = \{A, S_1, \ldots, S_n\}$ and $\mathbf{B} = \{B, T_1, \ldots, T_n\}$ of signature σ, the categorical product of $\mathbf{A} \times \mathbf{B}$ is given by a structure $\mathbf{A} \times \mathbf{B} = (A \times B, R_1, \ldots, R_n)$ where for each $i \in [n]$, we have that $((a_1, b_1), \ldots, (a_{\sigma_i}, b_{\sigma_i})) \in R_i$ iff $(a_1, \ldots, a_{\sigma_i}) \in S_{\sigma_i}$ and $(b_1, \ldots, b_{\sigma_i}) \in T_{\sigma_i}$

2.2 Abstract Minions

Before stating the categorical formulation of polymorphism minions, we first provide a few technical definitions from the language of category theory:

Definition 2.6. Given a category $\mathcal{C}$, and a family of objects $\{A_i : i \in [n], A_i \in \text{ob}(\mathcal{C})\}$, we say that $\prod_{i\in[n]} A_i$ is a **product** of the family if there exists a family of **projection mappings**
$\{\pi_k : \prod_{i\in[n]} A_i \to A_k \ : \ k \in [n]\}$ such that for any object $X \in \text{ob}(C)$ and set of maps
$\{f_k : X \to A_k \ : \ k \in [n]\}$ there exists a *unique* mapping $(f_1 \times \cdots \times f_n) : X \to \prod_{i\in[n]} A_i$ such that for any $k \in [k]$, we have that $f_k = \pi_k \circ (f_1 \times \cdots \times f_n)$.

In this context, we say that the family of mappings $\{\pi_k : \prod_{i\in[n]} A_i \to A_k \ : \ k \in [n]\}$ **exhibits $\prod_{i\in[n]} A_i$ as a product in $\mathcal{C}$.**

If we consider a finite family of objects $\{A_i : i \in [n]\}$, we denote its corresponding product by $A_1 \times \cdots \times A_n$.

We note that while the categorical product is generally not unique, it is unique up to isomorphism, formally, that is:

Lemma 2.2 ([12])**.** Given a category $\mathcal{C}$ and two objects $A, B \in \text{ob}(\mathcal{C})$, for any two products P, Q of A and B with corresponding projection mappings $\pi_A^P : P \to A, \pi_B^P : P \to B$ and $\pi_A^Q : Q \to A, \pi_B^Q : Q \to B$, there exists an isomorphism $\varphi_{PQ} : P \to Q$.

Definition 2.7. A category $\mathcal{C}$ is said to **have all finite products**, if for all objects $A, B \in \text{ob}(\mathcal{C})$, we have that the product $A \times B$ exists.

Remark 2.2. Grp, **Ab**, **Top**, and $\mathbf{Rel}_\sigma$ (for all signatures σ) are categories with all finite products.

Definition 2.8. For any category $\mathcal{C}$ with all finite products and $n \in \mathbb{N}_{\geqslant 1}$, we write A^n for the product $\underbrace{A \times \cdots \times A}_{n \text{ times}}$.

Lastly, we generalize the notion of taking minors: Let A be an object in a category with all powers $\mathcal{M}$. We denote the projection mappings of A^n for $n \geqslant 1$ onto its components by $\pi_1^n, \ldots, \pi_n^n$ so that for example; for $(1, 2, 3) \in \mathbb{N}^3$, we'd have $(\pi_2^3 \times \pi_1^3 \times \pi_3^3 \times \pi_2^3)(1, 2, 3) = (2, 1, 3, 2)$. A more general form of this example is the definition below:

Definition 2.9. Let $\mathcal{C}$ be a category with all powers, A, B be two arbitrary objects in $\mathcal{C}$ and consider some arbitrary $f : A^n \to B$, $g : A^m \to B$ for some $n, m \geqslant 1$. We say that f is a minor of g and write $f = g^\mu$ if there exists a function $\mu : [m] \to [n]$ such that $f = g \circ (\pi^n_{\mu(1)} \times \cdots \times \pi^n_{\mu(m)})$

Definition 2.10. We denote by $\omega_{\geqslant 1}$ the skeletal category of non-empty finite sets. That is, the objects of $\omega_{\geqslant 1}$ are the sets [1], [2], [3], [4], ... and the homomorphisms between them are functions, which are composed as usual.

Definition 2.11. An **abstract minion** $\mathcal{M}$ is a functor from $\omega_{\geqslant 1}$ to **Set**. Given two minions $\mathcal{M}$ and $\mathcal{N}$, a minion homomorphism from $\mathcal{M}$ to $\mathcal{N}$ is defined as a natural transformation from $\mathcal{M}$ to $\mathcal{N}$.

Now we may finally define minion polymorphisms categorically:

Definition 2.12. Given a category with all products $\mathcal{C}$, we define $\mathrm{Pol}_{\mathcal{C}}$ as a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to [\omega_{\geqslant 1}, \mathbf{Set}]$, such that for any two $A, B \in \mathcal{C}$:

$$\mathrm{Pol}_{\mathcal{C}}(A, B) : [n] \mapsto \mathrm{Hom}(A^n, B)$$

$$\mathrm{Pol}_{\mathcal{C}}(A, B)(\mu) : f \mapsto f^{\mu}$$

for every $n, m \geqslant 1$, $\mu : [n] \to [m]$ and $f \in \mathrm{Hom}(A^n, B)$. Throughout this text, we will write $\mathrm{Pol}_{\mathcal{C}}(A, B)^{(n)}$ for $\mathrm{Pol}_{\mathcal{C}}(A, B)([n])$. We note that A^n may denote any object satisfying the condition in Definition 2.5 and may be chosen arbitrarily.

Moreover, any two maps $f : X \to A$, $g : B \to Y$, correspond to a mapping sending

$$\mathrm{Pol}_{\mathcal{C}}(f, g) : \mathrm{Pol}_{\mathcal{C}}(A, B) \to \mathrm{Pol}_{\mathcal{C}}(X, Y)$$

such that

$$\mathrm{Pol}_{\mathcal{C}}(f, g) : h \mapsto g \circ h \circ ((f \circ \pi_1^n) \times \cdots \times (f \circ \pi_n^n))$$

for every $h \in \mathrm{Pol}_{\mathcal{C}}(A, B)^{(n)}$.

Very often in practice we drop the category indexing when it is clear from context, writing just $\mathrm{Pol}(A, B)$ rather than $\mathrm{Pol}_{\mathcal{C}}(A, B)$.

Whether or not the notions of concrete and abstract minions are equivalent (formally, that is whether or not the categories **Min** or $[\omega_{\geqslant 1}, \mathbf{Set}]$ are equivalent) is an open question. However, there exists a faithful functor from **Min** to $[\omega_{\geqslant 1}, \mathbf{Set}]$ as in Construction 2.1. Put into words, that is that every concrete minion may be expressed as an abstract one, making abstract minions a generalization of concrete minions that conveys all of their algebraic properties.

Construction 2.1. Given a minion $\mathcal{M} \subseteq \mathcal{O}(A, B)$ for some sets A and B, we consider the functor F sending $[n]$ to $\mathcal{M}^{(n)}$ and $\mu : [n] \to [m]$ to the function $(-)^{\mu} : \mathcal{M}^{(m)} \to \mathcal{M}^{(n)}$ given by minor-taking. It is easy to see that this mapping is functorial.

For the remainder of this paper we will refer to abstract minions simply by minions.

2.3 Hardness from Polymorphisms

All of the concrete hardness results in this paper are derived from the Theorem 2.2 and the reductions presented in this subsection.

Theorem 2.1 ([2] - Theorem 3.1)**.** Given two PCSP templates $(\mathbf{A_1}, \mathbf{B_1})$, $(\mathbf{A_2}, \mathbf{B_2})$, if $\mathrm{Pol}(\mathbf{A_1}, \mathbf{B_1}) \to \mathrm{Pol}(\mathbf{A_2}, \mathbf{B_2})$, then $\mathrm{PCSP}(\mathbf{A_2}, \mathbf{B_2})$ is log-space reducible to $\mathrm{PCSP}(\mathbf{A_1}, \mathbf{B_1})$.

Definition 2.13. A coordinate i of a function $f : A^n \to B$ is called essential if there exist $a_1, \ldots, a_n \in A$ and $a_i' \in A$ such that

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1} \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_n)$$

we say that f has essential arity m if it has m essential coordinates.

Definition 2.14. A minion $\mathcal{M}$ is said to have bounded essential arity if there exists a constant $k \geqslant 0$ such that for all $f \in \mathcal{M}$, f has essential arity $\leqslant k$.

Theorem 2.2 ([2] - Theorem 5.15)**.** Given a PCSP template $(\mathbf{A}, \mathbf{B})$, and a minion $\mathcal{M}$ such that $\mathcal{M}$ has bounded essential arity and no constant functions (i.e. functions of essential arity zero); if $\mathrm{Pol}(\mathbf{A}, \mathbf{B}) \to \mathcal{M}$, then $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.

Definition 2.15. The minion $\mathcal{Z}_{\leqslant N}$ is a sub-minion of $\mathrm{Pol}(\mathbb{Z}, \mathbb{Z})$ (i.e. for every $k \geqslant 1$, $\mathcal{Z}_{\leqslant N}^{(k)} \subseteq \mathrm{Pol}(\mathbb{Z}, \mathbb{Z})^{(k)}$) consisting of all maps f of the form

$$f(x_1, \ldots x_n) = \sum_{k=1}^{n} c_k x_k$$

such that $\sum_{k=1}^{n} c_k$ is odd and $\sum_{k=1}^{n} |c_k| \leqslant N$.

Lemma 2.3 ([10])**.** For any $N \in \mathbb{N}_{\geqslant 1}$, $\mathcal{Z}_{\leqslant N}$ has bounded essential arity and no constant functions.

In Section 4, for various PCSP templates $(\mathbf{A}, \mathbf{B})$, we will make repeated use the following result to derive a sequence of minion homomorphisms the composition of which being a homomorphism $\mathrm{Pol}(\mathbf{A}, \mathbf{B}) \to \mathcal{Z}_{\leqslant \mathbf{N}}$ to deduce the complexity of the problem $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$:

Definition 2.16. Let $\mathcal{C}$ and $\mathcal{D}$ be categories with all finite products and $F : \mathcal{C} \to \mathcal{D}$ a functor. We say that F **preserves finite products** if for any product $\prod_{i \in I} A_i$ of a finite family of objects $\{A_i \in \mathrm{ob}(\mathcal{C}) : i \in I\}$ with projections $\pi_k : \prod_{i \in I} A_i \to A_k$, we have that the family of maps $F(\pi_k) : F(\prod_{i \in I} A_i) \to FA_k$ exhibits $F(\prod_{i \in I} A_i)$ as a product in $\mathcal{D}$.

Theorem 2.3. Let $\mathcal{C}, \mathcal{D}$ be two categories with all products, $A, B \in \mathrm{ob}(C)$ and $F : \mathcal{C} \to \mathcal{D}$ be a product-preserving functor. Then there exists a minion homomorphism $\mathrm{Pol}(A, B) \to \mathrm{Pol}(FA, FB)$.

Proof. We define $\mu : \mathrm{Pol}(A,B) \to \mathrm{Pol}(FA,FB)$ as the natural transformation such that $\mu_{[n]}$ sends f to Ff for any $f \in \mathrm{Pol}(A,B)^{(n)}$. To show that this transformation is indeed natural, we must show that for any $\eta : [n] \to [m]$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Pol}(A,B)^{(n)} \arrow[r, "(-)^{\eta}"] \arrow[d, "\mu_{[n]}"'] & \mathrm{Pol}(A,B)^{(m)} \arrow[d, "\mu_{[m]}"] \\
\mathrm{Pol}(FA,FB)^{(n)} \arrow[r, "(-)^{\eta}"] & \mathrm{Pol}(FA,FB)^{(m)}
\end{tikzcd}$$

where $(-)^{\pi}$ denotes the image of π through the minion $\mathrm{Pol}(A,B)$ (i.e. the operation of taking minors).

Note that in the definition of polymorphism minions we have chosen the product object arbitrarily. In this case we may consider $(FA)^n = F(A^n)$ with projection mappings (denoted as in Definition 2.9) corresponding to the functorial images of those of A^n, i.e. $\{F(\pi_i^n) : F(A^n) \to FB\}$, thus making the naturality condition above a matter of definition unwrapping, i.e. showing that for any $f \in \mathrm{Pol}(A,B)^{(n)}$, we have that:

$$\mu_{[m]}(f^{\eta}) = F(f^{\eta}) = F(f \circ (\pi^m_{\eta(1)} \times \cdots \times \pi^m_{\eta(n)})) = F(f) \circ F(\pi^m_{\eta(1)} \times \cdots \times \pi^m_{\eta(n)}) =$$

$$F(f) \circ (F(\pi^m_{\eta}(1) \times \cdots \times \pi^m_{\eta}(n)) = F(f)^{\eta} = (\mu_{[n]}f)^{\eta}$$

$\square$

Remark 2.3. Note that in fact, the condition that F is a product-preserving functor is overly strong for determining a reduction as in Theorem 2.3, in fact, the condition that F preserves powers of A (i.e. $F(A^n) \simeq F(A)^n$) being sufficient, the proof being analogous.

3 Homomorphism Complexes

3.1 Categorical Formulation of Equivariance

Definition 3.1. Given an object X in a category $\mathcal{C}$, the **automorphism group** of X is denoted $\mathrm{Aut}(X)$ and consists of the set $\{f \in \mathrm{Hom}_{\mathcal{C}}(X,X) : f \text{ is an isomorphism}\}$ with group operation given by morphism composition.

The automorphism group enables us to generalize the notion of a group action on a mathematical object that is compatible with the object's structure. We formalize this as follows:

Definition 3.2. Given a group G and an object X in some category $\mathcal{C}$, we call a **group action** of G on X any group homomorphism $\rho : G \to \mathrm{Aut}(X)$.

Example 3.1. An action of a group G on a set S is essentially a homomorphism from G to $\mathrm{Sym}(S)$ $(= \mathrm{Aut}(S))$.

Example 3.2. An action of a group G on a vector space V is also known as a linear group representation and is a homomorphism from G to $\mathrm{GL}(V)$ $(= \mathrm{Aut}(V))$.

A central notion in that of an equivariant function between two objects acted upon by the same group. Formally, that is:

Definition 3.3. Given a group G and two objects X, Y in some category $\mathcal{C}$, both acted on by G by $\rho_X : G \to \mathrm{Aut}(X)$, $\rho_Y : G \to \mathrm{Aut}(Y)$, we call a map $f : X \to Y$ **equivariant** if it commutes with the action of the group, that is, for every $g \in G$, we have that $f \circ \rho_X(g) = \rho_Y(g) \circ f$.

It turns out that these two concepts of group action and equivariance have a very nice formalization in terms of category theory, which starts with the following observation:

Remark 3.1. A group is a category with a single object $*$ such that all elements of $\mathrm{Hom}(*, *)$ are isomorphisms.

In light of this new perspective on groups, we may rephrase the definition of a group action as follows:

Definition 3.4. Given a group G and an object X in some category $\mathcal{C}$, we call a **group action** a functor F from G (seen as a category with single object $*$) to $\mathcal{C}$ such that $F(*) = X$.

The reason why this coincides with Definition 3.2 stems from the following: Having fixed which object F maps $*$ to, F needs to map $\mathrm{Aut}_G(*)$ $(= \mathrm{Hom}_G(*, *))$ to $\mathrm{Hom}_{\mathcal{C}}(F(*), F(*))$ functorially (i.e. F is essentially just a monoid homomorphism); F need map isomorphisms is G to isomorphisms in $\mathcal{C}$ thus the image of $\mathrm{Aut}_G(*, *)$ through F is contained in $\mathrm{Aut}_{\mathcal{C}}(X)$.

The beauty of this formalism is how easily related concepts can be rephrased in terms of it. For example, we may redefine equivariant maps as follows:

Definition 3.5. Given a group G, two objects X, Y in some category $\mathcal{C}$, both acted on by some group G by $A : G \to \mathcal{C}$ (where $A(*) = X$) and $B : G \to \mathcal{C}$ (where $B(*) = Y$); we call a natural transformation from A to B an **equivariant map**.

The reason why this coincides with Definition 3.3 stems from the following: the naturality condition of a natural transformation $\eta : A \to B$ as in the definition above that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "A(g)"] \arrow[d, mapsto, "\eta_*"] & X \arrow[d, mapsto, "\eta_*"] \\ Y \arrow[r, "B(g)"] & Y \end{tikzcd}$$

Equivalently, $\eta_* \circ A(g) = B(g) \circ \eta_*$ - i.e. η_*, a map from X to Y needs to commute with the group action. Conversely it is easy to see that a group action as in Definition 3.1 determines a natural transformation between two functors as in Definition 3.2.

Remark 3.2. In light of Definitions 3.2 and 3.3, we remark that given a group G and a category $\mathcal{C}$, the functor category $[G, \mathcal{C}]$ is essentially the category of G-actions on objects in $\mathcal{C}$, that is:

- Every object in $[G, \mathcal{C}]$ corresponds to an action of G on an object in $\mathcal{C}$ (and vice-versa)
- Given two objects A and B in $[G, \mathcal{C}]$, $\mathrm{Hom}(A, B)$ consists of the set of equivariant maps from A to B seen as group-actions.

Objects acted upon by a group G, are usually call them G-objects. For example, objects of $[G, \mathbf{Top}]$ are called **G-spaces** and objects of $[G, \mathbf{Spx}]$ are called **G-complexes**.

3.2 Prerequisite Notions of Topology

Definition 3.6. An (abstract) **simplicial complex** K consists of a set of **vertices** $V(K)$ and a set of **faces** consisting of a family of *finite* subsets of $V(K)$ denoted $\Sigma(K) \subseteq \mathcal{P}(V(K))$ such that for any face $\sigma \in \Sigma(K)$ and nonempty subset τ $(\subseteq \sigma)$, we have that $\tau \in \Sigma(K)$.

Definition 3.7. A **simplicial map** $f : K_1 \to K_2$ where K_1 and K_2 are simplicial complexes is a map $f : V(K_1) \to V(K_2)$ such that for any face $\sigma = \{\sigma_1, \ldots, \sigma_n\} \in \Sigma(K_1)$, we have that $f(\sigma) = \{f(\sigma_1), \ldots, f(\sigma_n)\} \in \Sigma(K_2)$. Put into words, that is: f is a face-preserving function between the sets of vertices of two simplicial complexes.

Definition 3.8. Let K be a simplicial complex with $V(K) = \{v_1, \ldots, v_n\}$. We define its **geometric realisation** $|K|$ as the following subspace of $\mathbb{R}^N$ (identifying $v_1 = (1, 0, \ldots, 0, 0)$,

...and $v_n = (0, 0, \dots, 0, 1)$):

$$|K| := \bigcup_{\sigma \in \Sigma(K)} \left\{ \sum_{v_i \in \sigma} \lambda_i v_i : \lambda_i \geqslant 0, \sum_{v_i \in \sigma} \lambda_i = 1 \right\}$$

Definition 3.9. Let K_1, K_2 be simplicial complexes and $f : K_1 \to K_2$ be a simplicial map. Then we may define the **affine extension** (or also the **geometric realisation**) of f as the function $|f| : |K_1| \to |K_2|$ given by:

$$|f| : \sum_i \lambda_i v_i \mapsto \sum_i \lambda_i f(v_i)$$

Definition 3.10. We define **Spx** as the category whose objects are simplicial complexes and for $A, B \in \mathrm{ob}(\mathbf{Spx})$, $\mathrm{Hom}(A, B)$ is the set of simplicial maps from A to B, with composition of morphisms being just function composition.

Remark 3.3. We may consider the geometric realisation as a functor $| - | : \mathbf{Spx} \to \mathbf{Top}$.

Definition 3.11. Given two topological spaces X and Y and functions $f, g : X \to Y$, we say that f and g are **homotopic** and write $f \simeq g$ if there exists a function $H : [0, 1] \times X \to Y$ such that for all $x \in X$, $H(0, x) = f(x)$ and $H(1, x) = g(x)$. We call the map H a **homotopy**.

Lemma 3.1. Homotopy is an equivalence relation. That is, given any two topological spaces X and Y, for any three maps $f, g, h, : X \to Y$, we have:

- (reflexivity): $f \simeq f$.
- (symmetry): $f \simeq g$ implies that $g \simeq f$.
- (transitivity): $f \simeq g$ and $g \simeq h$ imply that $f \simeq h$.

Lemma 3.2. Homotopy is an equivalence compatible with function composition. Formally, for any three spaces X, Y and Z, maps $f, f' : X \to Y$ and $g, g' : Y \to Z$, we have that if $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$.

Definition 3.12. We say that two spaces X and Y are homotopy equivalent if there exist maps $f : X \to Y$, $g : Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$.

Definition 3.13. We define **hTop** as the following category:

- The objects of **hTop** are topological spaces.
- For any two spaces X, Y, $\mathrm{Hom}_{\mathbf{hTop}}(X, Y)$ is $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ modulo homotopy, that is $\mathrm{Hom}_{\mathbf{hTop}}(X, Y) := \{[f] : f \in \mathrm{Hom}_{\mathbf{Top}}(X, Y)\}$ where $[f]$ represents the equivalence class of f relative to homotopy.

- For any $f : X \to Y$, $g : Y \to Z$, we have that $[g] \circ [f] := [g \circ f]$, which is well-defined as per Lemma 3.2.

We now turn our attention to topological spaces endowed with the action of a group G compatible with their topology as in Subsection 3.1. For the rest of this subsection, G will denote an arbitrary finite group.

Definition 3.14. Given G-spaces $X, Y \in \mathrm{ob}([G, \mathbf{Top}])$ and two maps $f, g : X \to Y$, we say that f and g are G-homotopic and write $f \simeq_G g$ if there exists a homotopy $H : [0,1] \times X \to Y$ such that for every $t \in [0,1]$, we have that the function $f_t : X \to Y$ given by $f_t : x \mapsto H(t, x)$ is G**-equivariant**.

We have that Lemmas 2.1 and 2.2 have equivalent formulations in the theory of equivariant homotopy.

Lemma 3.3. G-Homotopy is an equivalence relation. That is, given any two topological spaces X and Y, for any three maps $f, g, h, : X \to Y$, we have:

- (reflexivity): $f \simeq f$.
- (symmetry): $f \simeq g$ implies that $g \simeq f$.
- (transitivity): $f \simeq g$ and $g \simeq h$ imply that $f \simeq h$.

Lemma 3.4. G-Homotopy is an equivalence compatible with function composition. Formally, that is: for any G-spaces X, Y and Z, maps $f, f' : X \to Y$ and $g, g' : Y \to Z$, we have that if $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$.

Definition 3.15. We define $\mathbf{hTop}_G$ as the following category:

- The objects of $\mathbf{hTop}_G$ are G-topological spaces, i.e. $\mathrm{ob}(\mathbf{hTop}_G) = \mathrm{ob}([G, \mathbf{Top}])$.
- For any two spaces X, Y, $\mathrm{Hom}_{\mathbf{hTop}_G}(X, Y)$ is $\mathrm{Hom}_{[G,\mathbf{Top}]}(X, Y)$ modulo homotopy, that is $\mathrm{Hom}_{\mathbf{hTop}}(X, Y) := \{[f] : f \in \mathrm{Hom}_{[G,\mathbf{Top}]}(X, Y)\}$ where $[f]$ represents the equivalence class of f relative to G-homotopy.
- For any two $\mathbf{hTop}_G$ morphisms $f : X \to Y$, $g : Y \to Z$, we have that $[g] \circ [f] := [g \circ f]$, which is well-defined as per Theorem 3.4.

Definition 3.16. We define Q_G as the functor from $[G, \mathbf{Top}]$ to $\mathbf{hTop}_G$ such that:

- For any G-space X, Q_G maps X to itself.
- For G-equivariant map $f : X \to Y$, Q_G maps f to its G-homotopy class.

3.3 Homomorphism Complexes

The central topological notion in the theory of promise constraint satisfaction problems is that of a homomomorphism complex. The theory of homomorphism complexes originates from Lovasz's celebrated proof of the Knesser conjecture [11], thus giving rise to the field of topological combinatorics.

Definition 3.17. Given two graphs G, H, we may associate the ordered pair (G, H) a simplicial complex called the **homomomorphism complex** of (G, H) whose vertices are the homomorphisms from G and H and a set of homomorphisms $\sigma = \{f_1, \dots, f_n\}$ forms a face if for any edge (u, v) in G, we have that:

$$\{f_1(u), \dots, f_n(u)\} \times \{f_1(v), \dots, f_n(v)\} \subseteq E(H)$$

We denote the simplicial complex simply by $\mathrm{Hom}(G, H)$ and its geometric realization by $|\mathrm{Hom}|(G, H)$.

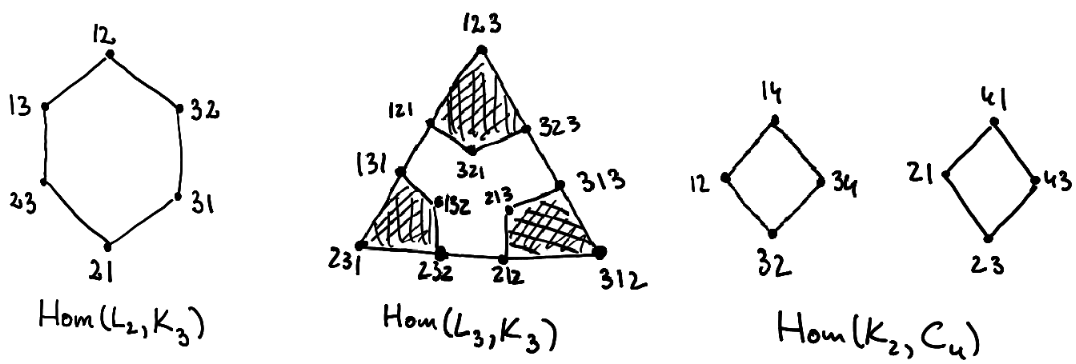

Figure 3.1

Example 3.3. In the above figure, specifically in the depiction of $\mathrm{Hom}(L_3, K_3)$, each shaded quadrilateral represents a 3-simplex.

We note that any homomorphism complex $\mathrm{Hom}(G, H)$ comes endowed with a natural symmetry. This stems from the fact that given any automorphism $\alpha : G \to G$ of G, we may consider the following simplicial map given by pre-composition, which we'll denote by $\rho(\alpha)$:

$$\rho(\alpha) : \mathrm{Hom}(G, H) \to \mathrm{Hom}(G, H)$$
$$\rho(\alpha) : f \longmapsto f \circ \alpha$$

It is easy to see that $\rho(\alpha)$ is a simplicial map for any $\alpha \in \mathrm{Aut}(G)$. Moreover, for any $\beta \in \mathrm{Aut}(G)$, $\rho(\alpha) \circ \rho(\beta) = \rho(\alpha \circ \beta)$, so we may take ρ to be a group action $\rho : \mathrm{Aut}(G) \to \mathrm{Aut}_{\mathbf{Spx}}(\mathrm{Hom}(G, H))$.

Everything in this subsection up until here enables us to define the following functor:

Definition 3.18. Given a graph G, we may consider a functor from the category of graphs to the category of $\mathrm{Aut}(G)$-simplicial complexes $\Delta(G, -) : \mathbf{Gr} \to [\mathrm{Aut}(G), \mathbf{Spx}]$:

- For any graph H, $\Delta(G, H) := \mathrm{Hom}(G, H)$, where $\mathrm{Hom}(G, H)$ is endowed with the $\mathrm{Aut}(G)$-group action as described above.
- For any graph homomorphism $f : H_1 \to H_2$, $\Delta(G, f)$ is the simplicial map given by pre-composition with f:

$$\begin{aligned} \Delta(G, f) &: \mathrm{Hom}(G, H_1) \to \mathrm{Hom}(G, H_2), \\ \Delta(G, f) &: v \mapsto f \circ v. \end{aligned}$$

It is easy to show that $\Delta(G, -)$ above is indeed functorial and that $\Delta(G, f)$ is an equivariant map for any graph homomorphism f.

To enter the realm of topology, we need to transform these simplicial complexes into topological spaces by considering their geometric realizations. We thus define the following two functors:

Definition 3.19. Given a graph G, we may consider a functor from the category of graphs to the category of $\mathrm{Aut}(G)$-topological spaces $|\Delta|(G, -) : \mathbf{Gr} \to [\mathrm{Aut}(G), \mathbf{Top}]$:

- For any graph H, $|\Delta|(G, H) := |\mathrm{Hom}(G, H)|$, where $|\mathrm{Hom}(G, H)|$ endowed with the following group action: Let ρ denote the group action of $\mathrm{Aut}(G)$ on $\mathrm{Hom}(G, H)$. The the action $|\rho| : \mathrm{Aut}(G) \to |\mathrm{Hom}(G, H)|$ of $\mathrm{Aut}(G)$ on $|\mathrm{Hom}(G, H)|$ defined such that $|\rho|(g)$ is simply the affine extension of $\rho(g)$ for all $g \in \mathrm{Aut}(G)$.
- For any graph homomorphism $f : H_1 \to H_2$, $|\Delta|(G, f)$ is defined to be the affine extension of $\Delta(G, f)$.

Notation 3.1. For any graph G, we denote the composition of functors $Q_G \circ |\Delta|(G, -)$ by $\Delta^h(G, -)$.

3.4 K_2-complexes and their Properties

So far, we've considered homomorphism complexes in full generality, however, all known complexity theoretical results stem from the study of homomorphism complexes arising from fixing the domain to K_2, thus, we will restrict our interest to the functors $\Delta(K_2, -)$, $|\Delta|(K_2, -)$ and $\Delta^h(K_2, -)$ which we will shorten to $\Delta(-)$, $|\Delta|(-)$ and $\Delta^h(-)$ respectively (we will however redefine $\Delta(-)$ and $\Delta^h(-)$ to simplify the proofs of the results in Section 4). We call any simplicial complex K endowed with a $\mathbb{Z}_2$-action such that $K = \Delta(G)$ for some graph G a $\mathbf{K_2}$**-complex** and its geometric realization a $\mathbf{K_2}$**-space**. Note however that we will be refering both to K_2-spaces and complexes (as defined above) and to $\mathbb{Z}_2$-spaces and complexes, that is topological spaces and simplicial complexes endowed with an action of $\mathbb{Z}_2$ which is compatible to their structure.

Notation 3.2. Given any $\mathbb{Z}_2$-space or complex, we denote the action of $\mathbb{Z}_2$ by $(-)$. For example, given two $\mathbb{Z}_2$-spaces X and Y and a map $f : X \to Y$, the equivariance condition on f would be that for any point x in X, we have that $f(-x) = -f(x)$.

Notation 3.3. We often denote the vertices of a K_2-complex by a pair — i.e. for a vertex $f : K_2 \to G$ of $\Delta(G)$, we write $(f(1), f(2))$ rather than f.

Before moving on, we will alter the definition of $|\Delta|(-)$ in a manner that will greatly ease the reasoning of the majority of the proofs in section 4.

Definition 3.20. Given a group G, a set S and an action of G on S - $\rho : G \to \mathrm{Aut}(S)$, we say that ρ is a **free action** if $\rho(g)(x) = x$ implies that $g = e_G$. The condition is identical when G is acting on a topological space (i.e. only the identity may fix any points) or when it is acting on a simplicial complex (i.e. only the identity may fix any vertices).

Theorem 3.1. Given a loopless graph G, $\Delta(G)$ is a free $\mathbb{Z}_2$-complex.

Definition 3.21. Given a finite and free $\mathbb{Z}_2$-complex K, with $V(K) = \{v_1, \dots, v_n, -v_1, \dots, -v_n\}$, we define its geometric realization $|K|$ of K as the following subset of $\mathbb{R}^n$: After identifying $v_1 = (1, \dots, v_n)$ ($\in \mathbb{R}^n$), $-v_1 = (-1, \dots, 0)$, ..., $v_n = (0, \dots, 1), -v_n = (0, \dots, -1)$, we set

$$|K| := \bigcup_{\sigma \in \Sigma(K)} \{\sum_{v_i \in \sigma} \lambda_i v_i : \lambda_i \geqslant 0, \sum_{v_i \in \sigma} \lambda_i = 1\}$$

and define the $\mathbb{Z}_2$-action as: for any $v \in |K|$ $(-)_{\mathbb{Z}_2}(v) := -v$.

Remark 3.4. We note for any graph G, the redefinition does not change the $\mathbb{Z}_2$-homotopy type of $|\Delta|(G)$.

K_2-spaces have been the subject of thorough study which led to a large body of literature on the subject [1, 5, 13, 16, 18]. These specific homomorphism complexes are also known by the name of **box complexes**, usually given with a slightly different definition. However, for the scope of this paper, it is only relevant to know that the geometric realisation of the box complex of a graph G is $\mathbb{Z}_2$-homotopy equivalent to $|\Delta|(G)$ ([10] - Appendix A). For the remainder of this section we will concern ourselves with unveiling the structure of K_2-spaces arising from this construction and the interplay between the topological and combinatorial notions involved. We do so by first exhibiting a notion of universality of K_2-complexes amongst general $\mathbb{Z}_2$-complexes, that is:

Remark 3.5. We note that if a $\mathbb{Z}_2$-complex is free, so is its geometric realization.

Theorem 3.2. (Universality - [5, 18]) For any free $\mathbb{Z}_2$-complex K, there exists a graph G such that $|\Delta|(G)$ is $\mathbb{Z}_2$-homotopy equivalent to $|K|$.

Another surprising and powerful result, rediscovered by the author, which lies at the centre of our proof that (non-approximate) graph-colouring is NP-complete is the folowing:

Theorem 3.3 (Kozlov)**.** For any $n \geqslant 2$ $|\Delta|(K_n) \simeq_{\mathbb{Z}_2} \mathcal{S}^{n-2}$

We provide an original proof of the theorem above in Appendix A.

To define polymorphisms of simplicial complexes, in order to use them in proving hardness, one needs to consider products of simplicial complexes.

Theorem 3.4 ([12])**.** Given two simplicial complexes K_1 and K_2, their categorical product $K_1 \times K_2$ is the following simplicial complex: The vertex set of $K_1 \times K_2$ is $V(K_1) \times V(K_2)$ and a subset $\sigma = \{(x_1, y_1), \ldots, (x_n, y_n)\} \subseteq V(K_1) \times V(K_2)$ forms a face if and only if $\sigma_x \in \Sigma(K_1)$ and $\sigma_y \in \Sigma(K_2)$.

3.5 Barycentric Subdivision for Graphs

A truly remarkable property of K_2-complexes is the fact that there exists a construction Ω_k sending graphs to graphs which behaves in a manner very similar to barycentric subdivision, in some sense resulting in a combinatorial counterpart to the simplicial approximation theorem. The discovery of Ω_k and the results around it are owed to Wrochna [16].

Theorem 3.5. (Approximation - [16]) For any graphs G, H there exists a $\mathbb{Z}_2$-map from $|\Delta|(G)$ to $|\Delta|(H)$ if and only if $\Omega_k G \to H$ for some odd integer k.

Moreover, similarly to barycentric subdivision, we also have the following:

Theorem 3.6. (Equivalence - [16]) For any graph G, $|\Delta|(G)$ and $|\Delta|(\Omega_k G)$ are $\mathbb{Z}_2$-homotopy equivalent for all odd integers k.

The definition of Ω_k is however very involved, and the number of vertices of $\Omega_k G$ for some graph G is exponential in the number of vertices of G, making it hard to visualise in practice.

Definition 3.22. Given two categories $\mathcal{C}$, $\mathcal{D}$ we a **pre-functor** a mapping from the objects of $\mathcal{C}$ to the objects of $\mathcal{D}$.

Definition 3.23. For any odd integer $k \geqslant 1$, Ω_k is a pre-functor from **Gr** to itself, sending graphs G such that: writing $k = 2l + 1$, for any graph G, $\Omega_k G$ is defined as follows: the vertex set of $\Omega_k G$ is the set of tuples $(A_0, \ldots, A_l)$ of vertex subsets $A_i \subseteq V(G)$ such that A_0 consists of a single vertex. Two tuples $A = (A_0, \ldots, A_l)$ and $B = (B_0, \ldots, B_l)$ form an edge (A, B) in $\Omega_k G$ if $A_l \times B_l \subseteq E(G)$ and for every $i \in [0 \ldots l-1]$, $A_i \subseteq B_{i+1}$ and $B_i \subseteq A_{i+1}$.

3.6 Adjoint Functors in Posetal Categories

Definition 3.24. We call a category $\mathcal{C}$ **thin** if for any two objects $A, B \in \mathrm{ob}(\mathcal{C})$, $\mathrm{Hom}(A, B)$ has cardinality at most one. Such categories are also called **posetal**, which stems from the fact that the notion of a thin category is equivalent to that of a preodered (set-theoretical) class. Given a category $\mathcal{C}$, we define its **thinning** as the category $\mathcal{C}^t$, whose objects coincide with those of $\mathcal{C}$ and the hom-set between any two objects $A, B \in \mathrm{ob}(\mathcal{C}^t)$ is given by:

$$\mathrm{Hom}_{\mathcal{C}^t}(A, B) := \begin{cases} *_{AB} & \text{if } \mathrm{Hom}_{\mathcal{C}}(A, B) \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases} .$$

Composition of morphisms in $\mathcal{C}^t$ is defined in the only possible way.

While in **Top**, whether or not $X \to Y$ for some spaces $X, Y \in \mathrm{ob}(\mathbf{Top})$ is obvious (one can always take a constant map taking all points in X to a specific point in Y), the situation changes dramatically in the case of $[G, \mathbf{Top}]$. Whether or not $X \to Y$ for some G-spaces $X, Y \in \mathrm{ob}([G, \mathbf{Top}])$ is a highly intricate problem. For example, a consequenced of the Borsuk-Ulam Theorem can be stated as: considering two spheres $\mathcal{S}^n$ and $\mathcal{S}^m$ as objects in $[G, \mathbf{Top}]$ endowed with with the natural $\mathbb{Z}_2$-action given by reflection, we have that $\mathcal{S}^n \to \mathcal{S}^m$ if and only if $n \leqslant m$ [16].

We note that a functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ induces a functor from $\mathcal{C}^t$ to $\mathcal{D}^t$, sending any object $X \in \mathrm{ob}(\mathcal{C})$ to FX and morphism $f : X \to Y$ to $*_{(FX)(FY)}$. We denote this thinning of F by F^t or by F as well, the distinction being made through context. Moreover, if $F^t \dashv G^t$, we say that F and G are **thin-adjoint**.

The order-theorical counterpant of thin-adjoint functors is known as a **Galois connection**. That is, given two posets $(A, \leqslant)$ and $(B, \leqslant)$, a pair of functions $f : A \to B$ and $g : B \to A$ such that for any two $x \in A$, $y \in B$, we have that $x \to g(y)$ if and only if $f(x) \to y$ is called a Galois connection between A and B.

In the theory of K_2-complexes we often do not have the luxury of being offered functors to work with, however, pre-functors such as Λ_k and Γ_k as defined below and Ω_k are abundant in the theory. It turns out that is often the case that these pre-functors correspond to functors in the thinning of their associated categories. Formally, if F is a monotone pre-functor as defined below, it is clear that F^t may be considered as a functor.

Definition 3.25. Given a pre-functor $F : \mathcal{C} \to \mathcal{D}$, if for any two objects $X, Y \in \mathcal{C}$ we have that if $X \to Y$ then $FX \to FY$ we say that F is **monotone**.

Definition 3.26. Given two pre-functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ are **adjoint** and write $F \dashv G$ if for any $x \in \mathrm{ob}(\mathcal{C})$ and $y \in \mathrm{ob}(\mathcal{D})$ we have that $FX \to Y$ iff $X \to GY$.

Lemma 3.5 ([10]). Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two adjoint pre-functors. Then for all graphs X, Y, we have that:

1. $X \to GFX$
2. $FGX \to X$
3. F and G are monotone

Proof. We have that $FX \to FX$. Because $F \dashv G$, it follows that $X \to GFX$, analogously, to prove (2), note that as $GX \to GX$ it that $FGX \to X$ from the adjunction. To prove monotony, consider two arbitrary graphs X and Y such that $X \to Y$. Then, as $Y \to GFY$, we have that $FX \to FY$ and thus by adjunction we have that $Y \to GY$. The proof for the monotony of G is analogous. □

Remark 3.6. It follows from the monotony in the lemma that adjoint pre-functors correspond to adjoint functors in the corresponding thin categories.

We will exhibit an adjunction of pre-functors of great importance in the theory of topological combinatorics, which underlies the key result in Subsection 4.3.

Definition 3.27. Given an integer $k \geqslant 1$, Λ_k is a pre-functor mapping a graph G to a graph $\Lambda_k G$ which is obtained by replacing each edge in G to a path of length k.

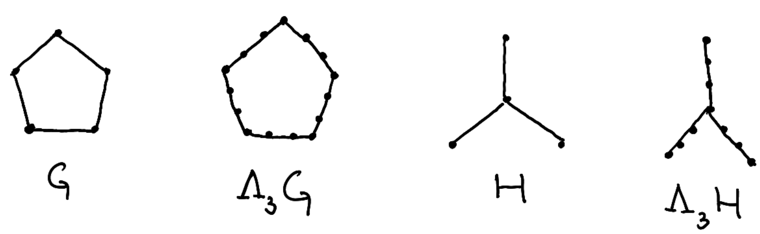

Figure 3.1

Definition 3.28. Given an integer $k \geqslant 1$, Γ_k is a pre-functor mapping a graph G to a graph $\Gamma_k G$ such that $\Gamma_k G$ has the same vertex set as G and two vertices are adjacent in Γ_k iff there exists a path of length k between them in G.

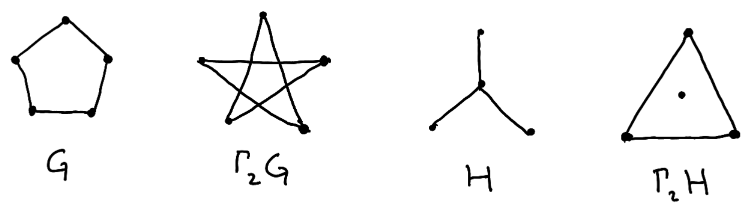

Figure 3.2

Theorem 3.7 (Wrochna - [16])**.** For any odd $k \geqslant 1$, we have that $\Lambda_k \dashv \Gamma_k \dashv \Omega_k$.

3.7 Hopelessness is Topological

Definition 3.29. Given a graph G, we say that G is **hopeless** if for any non-bipartite H such that $H \to G$, we have that $\mathrm{PCSP}(H, G)$ is NP-hard.

By the Hell-Nešetřil theorem, we know that in the context above, $\mathrm{CSP}(G)$ is NP-hard. The notion of hopelessness as defined above underlies the Barkensiek-Guruswami conjecture, which equivalently states that all loopless non-bipartite graphs are hopeless (i.e. that $\mathrm{CSP}(G)$ doesn't get any easier by restricting its inputs to the domain of a "non-trivial" PCSP template (G, H)). In this section we describe one the key results in [10].

Lemma 3.6 ([10])**.** Let $F, G : \mathbf{Gr} \to \mathbf{Gr}$ be pre-functors such that $F \dashv G$, where F is log-space computable. Then for any (graph) PCSP template (A, B) and (X, Y), F determines a log-space reduction from $\mathrm{PCSP}(A, B)$ to $\mathrm{PCSP}(X, Y)$ whenever $A \to GX$ and $FY \to B$.

Proof. Let I denote an arbitrary graph. We have that if $I \to A$, then also $I \to GX$, considering the assumed morphism $A \to GX$. From the adjunction $F \dashv G$, we get that $FI \to Y$. Conversely, suppose that $FI \to Y$. By the adjunction, we have that $I \to GY$. Composing this with the assumed morphism $GY \to B$, we get that $I \to B$. Thus:

- If $I \to A$, then $FI \to X$,
- If $I \nrightarrow B$, then $FI \nrightarrow Y$. □

Lemma 3.7 ([10])**.** For any odd k and graph G, $\Gamma_k \Lambda_k G \to G$

Proof. Note that $V(G) \subseteq V(\Lambda_k G) = V(\Gamma_k \Lambda_k G)$. We define $h : \Gamma_k \Lambda_k G \to G$ as a map sending each $v \in V(G)$ to v and each path in $\Gamma_k \Lambda_k G$ corresponding to an edge $(u, v) \in E(G)$ to a path of length k alternating between u and v. $\square$

Lemma 3.8 ([10])**.** Let k be odd and G be a graph. If $\Omega_k G$ is hopeless, then so is G.

Proof. Let H be an arbitrary non-bipartite graph. By Lemma 3.6, as Γ_k is log-space computable and as by Lemma 3.5, $H \to \Omega_k \Gamma_k H$ and $\Gamma_k \Omega_k G \to G$, we have that $\mathrm{PCSP}(H, G)$ is log-space reducible to $\mathrm{PCSP}(\Omega_k H, \Omega_k G)$. By the hopelesness of $\Omega_k G$, if we show that Ω_k is non-bipartite, it would follow that $\mathrm{PCSP}(\Omega_k H, \Omega_k G)$ is NP-hard, and so, considering the reduction, it would follow that $\mathrm{PCSP}(H, G)$ is NP-hard as well.

To see that $\Omega_k H$ is non-bipartite, note that as k is odd and H is non-bipartite, then $\Lambda_k H$ is non-bipartite as well. Noting that $\Gamma_k \Lambda_k H \to H$ by Lemma 3.7, if follows from adjunction we have that $\Lambda_k H \to \Omega_k H$, which by the non-bipartiteness of $\Lambda_k H$, implies that Ω_k is also non-bipartite. $\square$

Theorem 3.8. For any loopless non-bipartite graphs G, H such that $|\Delta|(G) \to |\Delta|(H)$, we have that if H is hopeless, then G is hopeless as well.

Proof. By Theorem 3.5, we have that $\Omega_k(G) \to H$ for some odd $k \geqslant 1$. Since H is hopeless, it follows that $\Omega_k(G)$ need be as well (formally, by the functoriality of Pol and Theorem 2.3.2). The claim follows from the previous lemma. $\square$

The moral of Theorem 3.8 is that the Brakensiek-Guruswami conjecture is an intrinsically-topological one. It is the goal of the next section to unveil how the solution of the conjecture might lie hidden in the structure of $\mathbb{Z}_2$-homotopy classes of $\mathbb{Z}_2$-maps between hyperspheres.

4 Hardness from Topology

In this section, we discuss one of the main results in [10], namely Theorem 4.4 ([10] - Theorem 1.4) and and we provide a novel topological proof of the known fact that (non-approximate) graph colouring is NP-complete.

We note that all of the results in this Subsections 4.1 and 4.2 and the next may be found in Krokhin et al. ([10]).

4.1 Topological Prerequisites

Theorem 4.1 ([6] - Proposition 2B.6)**.** The degree of any $\mathbb{Z}_2$-map (i.e. any odd map) of spheres $f : \mathcal{S}^n \to \mathcal{S}^n$ is odd.

Theorem 4.2 (Künneth's Formula)**.** Let X and Y be two topological spaces. Then for any $k \in \mathbb{N}_{\geqslant 1}$, we have that there exists a short exact sequence:

$$0 \longrightarrow \bigoplus_{i+j=k} H_i(X) \otimes H_j(Y) \longrightarrow H_k(X \times Y) \longrightarrow \bigoplus_{i+j=k-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_i(X), H_j(Y)) \longrightarrow 0$$

Lemma 4.1 ([6])**.** Given any integer $n \geqslant 1$, the (singular) homology groups of the sphere $\mathcal{S}_n$ are given by:

$$H_k(\mathcal{S}^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = n \\ 0 & \text{otherwise} \end{cases}$$

4.2 Half of the Brakensiek-Guruswami conjecture

The entirety of this subsection is dedicated to proving the following result:

Theorem 4.3 ([10], Theorem 1.4)**.** Let G and H be non-bipartite loopless graphs such that $G \to H$ and $|\Delta|(H) \to \mathcal{S}^1$. Then $\mathrm{PCSP}(G, H)$ is NP-hard.

All of the results in this subsection are owed to [10]

The idea behind the proof is that we will exhibit a chain of minion morphisms ending in $\mathcal{Z}_{\leqslant n}$ for some n:

$$\mathrm{Pol}(G, H) \xrightarrow{\mu_1} \mathrm{Pol}(\Delta(G), \Delta(H)) \xrightarrow{\mu_2} \mathrm{Pol}(\Delta^h(G), \Delta^h(H)) \xrightarrow{\mu_3} \mathrm{Pol}(\mathcal{S}^1, \mathcal{S}^1) \xrightarrow{\mu_4} \mathcal{Z}_{\leqslant N}$$

for some $N \in \mathbb{N}_{\geqslant 1}$, which by Theorem 2.2 implies that $\mathrm{PCSP}(G, H)$ is NP-hard.

Lemma 4.2. For any odd $k \geqslant 3$, we have that $|\Delta|(C_k) \simeq_{\mathbb{Z}_2} \mathcal{S}^1$.

Lemma 4.3. For any non-bipartite graph G, we have that $\mathcal{S}^1 \to |\Delta|(G)$.

Proof. As G is non-bipartite, it must contain a cycle C_k for some odd K, thus $C_k \to G$. As $|\Delta|(C_k) = \mathcal{S}^1$ by the previous lemma, we have by the functoriality of $|\Delta(-)|$ that $\mathcal{S}^n \to |\Delta|(G)$. $\square$

Lemma 4.4. $\Delta(-)$ is product preserving and thus, for any two graphs G and H, there exists a minion homomorphism $\mu_1 : \mathrm{Pol}(G, H) \to \mathrm{Pol}(\Delta(G), \Delta(H))$ by Theorem 2.3

Proof. We note that $\Delta(G) \times \Delta(H)$ and $\Delta(G \times H)$ have the same vertex set and that the identity vertex mapping $(v_1, v_2) \mapsto (v_1, v_2)$ yields an isomorphism between the two. The the graph product projection mappings $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ are also mapped to simplicial maps $\Delta(\pi_G)$, $\Delta(\pi_H)$, which correspond to precisely the same vertex mappings as the projections from the product $\Delta(G) \times \Delta(H)$. $\square$

Lemma 4.5. For all graphs G, H, there exists a minion homomorphism $\mu_2 : \mathrm{Pol}(\Delta(G), \Delta(H)) \to \mathrm{Pol}(\Delta^h(G), \Delta^h(H))$.

Proof. We define μ_2 as follows: for any $n \in \mathbb{N}_{\geqslant 1}$, and $f : \Delta(G)^n \to \Delta(H)$, we set

$$\mu_2(f) \coloneqq \left[(\sum_{v \in \sigma_1} \lambda_{1,v} v, \ldots, \sum_{v \in \sigma_n} \lambda_{n,v} v) \mapsto \sum_{v_1 \in \sigma_1, \ldots, v_n \in \sigma_n} \lambda_{1,v_1} \ldots \lambda_{n,v_n} (v_1, \ldots, v_n) \right]$$

for any point in $\Delta^{\sigma_1} \times \cdots \times \Delta^{\sigma_n}$ (where Δ^σ denotes the convex hull of the vertices of σ in the geometric realization).

To conclude the proof, we need to check that μ_2 is indeed a minion polymorphism (i.e. that the naturality condition is satisified). Let $n, m \in \mathbb{N}_{\geqslant 1}$, $\eta : [n] \to [m]$ and $f : \Delta(G)^n \to \Delta(H)$ be chosen arbitrarily. For an arbitrary point $x \in |\Delta(G)|^m$, lying in $\Delta^{\sigma_1} \times \cdots \times \Delta^{\sigma_m}$ for some $\sigma_1, \ldots, \sigma_m \in \Sigma(\Delta(G))$, such that

$$x = (\sum_{v \in \sigma_1} \lambda_{1,v} v, \ldots, \sum_{v \in \sigma_m} \lambda_{m,v} v),$$

denoting $[g] = \mu_2(f^\eta)$ and $[h] = \mu_2(f)^\eta$ (where $g, h : |\Delta|(G)^m \to |\Delta|(H)$), we have that:

$$g : x \longmapsto \sum_{v_1 \in \sigma_1, \ldots, v_m \in \sigma_m} \lambda_{1,v_1} \ldots \lambda_{n,v_m} f(v_{\eta(1)}, \ldots, v_{\eta(n)}),$$

$$h : x \longmapsto \sum_{v_1 \in \sigma_{\eta(1)}, \ldots, v_m \in \sigma_{\eta(m)}} \lambda_{\eta(1),v_1} \ldots \lambda_{\eta(n),v_m} f(v_1, \ldots, v_n).$$

Writing $\sigma \coloneqq \{f(v_1, \ldots, v_n) : v_i \in \sigma_{\eta(i)}, i \in [n]\}$, we note that both $h(x)$ and $g(x)$ lie in Δ^σ, thus we may consider the linear homotopy between h and g, which one may easily check that is a $\mathbb{Z}_2$-homotopy. Thus $g \simeq_{\mathbb{Z}_2} h$ and so $\mu_2(f^\pi) = \mu_2(f)^\pi$, thus the naturality condition of μ_2 is satisfied. $\square$

Lemma 4.6. Let G and H be non-bipartite loopless graphs such that $G \to H$ and $|\Delta|(H) \to \mathcal{S}^1$; then there exists a minion homomorphism $\mu_3 : \mathrm{Pol}(\Delta^h(G), \Delta^h(H)) \to \mathrm{Pol}_{\mathbf{hTop}_{\mathbb{Z}_2}}(\mathcal{S}^1, \mathcal{S}^1)$ for some integer n.

Proof. By Lemma 4.3 we have that there exists a $\mathbb{Z}_2$-map $\mathcal{S}^1 \to |\Delta|(G)$ and by assumption we have that $|\Delta|(H) \to \mathcal{S}^1$. The claim follows from the functoriality of $\mathrm{Pol}_{\mathbf{hTop}_{\mathbb{Z}_2}}$. □

We now move towards finally describing a homomorphism $\mathrm{Pol}(G, H) \to \mathcal{Z}_{\leqslant N}$ for graphs G and H.

Definition 4.1. We say that a minion $\mathcal{M}$ is **small** if for any $n \in \mathbb{N}_{\geqslant 1}$, we have that $\mathcal{M}^{(n)}$ is a finite set.

Remark 4.1. Note that for any two graphs G and H such that $G \to H$, $\mathrm{Pol}(G, H)$ is a small minion. In general, polymorphisms of finite relational structures are small minions.

Lemma 4.7 ([10] - Lemma 3.27)**.** Let $\mathcal{M}$ be a small minion and $\xi : \mathcal{M} \to \mathrm{Pol}(\mathbb{Z}, \mathbb{Z})$ a homomorphism. Then there exists N such that for any $f \in \mathcal{M}^{(n)}$, writing $\xi(f)(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i$, we have that $\sum_{i=1}^{n} |c_i| \leqslant N$.

Proof of Theorem 4.3. Composing the minion homomorphisms from Lemmas 4.4, 4.5 and 4.6, we get a minion homomorphism $\mu_3 \circ \mu_2 \circ \mu_1 : \mathrm{Pol}(G, H) \to \mathrm{Pol}_{\mathbf{hTop}_{\mathbb{Z}_2}}(\mathcal{S}_1, \mathcal{S}_1)$. Noting that the fundamental group functor π_1 (technically, π_1 is a functor from **Top** to **Grp**, however, as π_1 is homotopy-invariant and **hTop** may be regarded as a quotient category of Top by homotopy, we may correctly use it regardlessly) is product preserving and maps $\mathcal{S}^1$ to $\mathbb{Z}$, we obtain a homomorphism $\mathrm{Pol}(G, H) \to \mathrm{Pol}(\mathbb{Z}, \mathbb{Z})$. By Lemmas 4.7 (the arity of functions is bounded) and Theorem 4.1 (there are no constant functions), we have that the image of this homomorphism lies in $\mathcal{Z}_{\leqslant N}$ for some $N \geqslant 1$, and so $\mathrm{Pol}(G, H) \to \mathcal{Z}_{\leqslant N}$, thus implying by Theorem 2.2 and Lemma 2.3 that $\mathrm{PCSP}(G, H)$ is NP-hard. □

The relevance of Theorem 4.3. lies in the title of this subsection, bearing in mind the equivalent statement of the Brakensiek-Guruswami conjecture given in Subsection 1.4. Noting that for any odd $k \geqslant 1$, $\Delta(C^k) \simeq_{\mathbb{Z}_2} \mathcal{S}^1$ and that $K_3 = C_3$, it immediately follows from the functoriality of Pol that:

Theorem 4.4. For any odd $k \geqslant 1$, $\mathrm{PCSP}(C_k, K_3)$ is NP-hard.

4.3 Hardness of Graph Colouring

In this subsection, we present a novel topological proof of the hardness of graph colouring using topological methods.

Lemma 4.8. There exists a minion homomorphism $\mu_1 : \mathrm{Pol}(K_n, K_n) \to \mathrm{Pol}(\mathcal{S}^{n-2}, \mathcal{S}^{n-2})$

Proof. By Lemmas 4.5 and 4.6 we have that there are minion homomorphism $\mathrm{Pol}(K_n, K_n) \to \mathrm{Pol}(\Delta(K_n), \Delta(K_n))$ and $\mathrm{Pol}(\Delta(K_n), \Delta(K_n)) \to \mathrm{Pol}_{\mathbf{hTop}_{\mathbb{Z}_2}}(\Delta^h(K_n), \Delta^h(K_n))$. By Theorem 3.3, we have that $|\Delta|(K_n) \simeq_{\mathbb{Z}_2} \mathcal{S}^{n-2}$. Considering the $\mathbb{Z}_2$-homotopy equivalence maps between the two spaces, we gain a homomorphism $\mathrm{Pol}(\Delta^h(K_n), \Delta^h(K_n)) \to \mathrm{Pol}_{\mathbf{hTop}_{\mathbb{Z}_2}}(\mathcal{S}^{n-2}, \mathcal{S}^{n-2})$. The claim follows from considering the composition of these three minion homomorphisms. $\square$

Lemma 4.9. For any $n \in \mathbb{N}_{\geqslant 1}$, the n^{th} (singular) homology functor $H_n(-)$ preserves powers of $\mathcal{S}^n$.

Proof. Just for the span of this proof, we say that a space X is n-homologous if X is path-connected and $H_1(X) = H_2(X) = \cdots = H_{k-1}(X) = 0$. By Lemma 4.1, we have that $\mathcal{S}^n$ is n-homologous.

As for any Abelian group G, $\mathrm{Tor}_1^{\mathbb{Z}}(G, 0) = \mathrm{Tor}_1^{\mathbb{Z}}(0, G) = 0$, it follows that in the Künneth formula, the first nonzero map need be an isomorphism for all $k \leqslant n$. Thus for any two n-homologous spaces X, Y, we have that their product is also n-homologous and that $H_n(X \times Y) \simeq (H_n(X) \otimes H_0(Y)) \oplus (H_n(Y) \otimes H_0(X)) \simeq (H_n(X) \otimes \mathbb{Z}) \oplus (H_n(Y) \otimes \mathbb{Z}) \simeq H_n(X) \oplus H_n(Y)$. $\square$

Lemma 4.10. For every $n \geqslant 1$, there exists a minion homomorphism $\mathrm{Pol}(\mathcal{S}^n, \mathcal{S}^n) \to \mathscr{Z}_{\leqslant N}$ for some $N \in \mathbb{N}_{\geqslant 1}$.

Proof. We note that by Lemmas 4.9, Theorem 4.10 and Remark 2,3, $H_{n-2}(-)$ detemines a minion homomorphism $\mathrm{Pol}(\mathcal{S}^n, \mathcal{S}^n) \to \mathrm{Pol}(\mathbb{Z}, \mathbb{Z})$. Analogous to the proof of Theorem 4.3, we have that the image of this homomorphism lies in $\mathscr{Z}_{\leqslant N}$ for some $N \in \mathbb{N}_{\geqslant 1}$. $\square$

Theorem 4.5. For any $n \geqslant 3$, $\mathrm{CSP}(K_n)$ is NP-hard.

Proof. Composing the homomorphisms from Lemma 4.8 and Lemma 4.10, we get a homomorphism $\mathrm{Pol}(K_n, K_n) \to \mathscr{Z}_{\leqslant N}$ for some $N \in \mathbb{N}_{\geqslant 1}$ and so $\mathrm{CSP}(K_n) = \mathrm{PCSP}(K_n, K_n)$ is NP-hard by Theorem 2.2. $\square$

A Box complexes of complete graphs

This section is dedicated to proving that for every $n \geqslant 2$, $|\Delta|(K_n)$ is $\mathbb{Z}_2$-homotopy equivalent to $\mathcal{S}^{n-2}$. We will do this in the following steps:

1. We show that for all $n \in \mathbb{N}_{\geqslant 2}$ and $k \in \mathbb{N}_{\geqslant 1}$, $|\Delta|(K^n)$ is simply connected and the k-th (simplicial) homology groups of $|\Delta(K_n)|$ and $\mathcal{S}^{n-2}$ coincide.
2. We construct a simplicial map η_n from $\Delta(K_n)$ to a certain triangulation of $\mathcal{S}^{n-2}$ ($\partial\Delta^{n-1}$). This turns out to help us explicitly define a cycle Γ_n in $\Delta(K_n)$ which corresponds to a generator of its $n-2^{\text{nd}}$ homology group, which enables us to show that η_n induces an isomorphism in all homology groups and is thus a homotopy equivalence by Whitehead's Theorem.
3. We construct a $\mathbb{Z}_2$-simplicial map ε_n from $\Delta(K^n)$ to a certain triangulation of $\mathcal{S}^{n-2}$ ($S^{n-2}B$) which induces an isomorphism on all homology groups and is thus a $\mathbb{Z}_2$-homotopy equivalence by a similar argument.

Without any further ado, we proceed with the first step:

Definition A.1. We define $C_0(K_n)$ as the maximal subsimplex with vertex set $[n] \times [n+1]$ in $\Delta(K_{n+1})$ and $C_1(K_n)$ as the maximal subsimplex with vertex set $[n+1] \times [n]$ in $\Delta(K_{n+1})$.

We can think of $C_0(K_n)$ and $C_1(K_n)$ as the upper and lower hemisphere of $\Delta(K_n)$, as in Figure A.1.

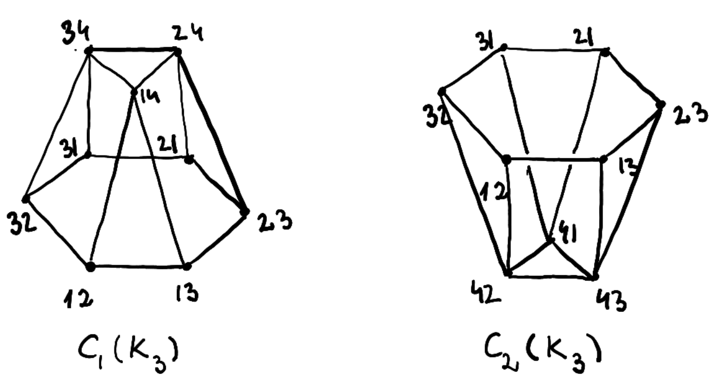

Figure A.1

We note that in all depictions of $\Delta(K_4)$ and its subsimplices, triangles such as $(12, 13, 14)$ above represent 2-faces and quadrilaterals such as $(12, 14, 34, 32)$ represent 3-faces.

Lemma A.1. Given a simplicial complex K and simplicial map f from it to itself such that for every simplex $\sigma \in \Sigma(K)$, we have that $f(\sigma) \cup \sigma \in \Sigma(K)$, then f is (linearly) homotopic to the identity.

Lemma A.2. For any $n \geqslant 2$, $C_0(K_n)$ and $C_1(K_n)$ are contractible.

Proof. We prove the lemma for $C_0(K_n)$, the proof for $C_1(K_n)$ being analogous. We do so by considering the following simplicial map — $R_n : C_0(K_n) \to C_0(K_n)$ which sends each vertex $(a, b) \in [n+1] \times [n]$ to $(n+1, b)$. One can see that R_n is a retraction of $C_0(K_n)$ onto the simplex $\{(n+1, 1) \ldots, (n+1, n)\}$, which is in turn a deformation retraction as by Lemma A.1 it is linearly homotopic to the identity on $C_0(K_n)$. As the simplex $\{(n+1, 1) \ldots, (n+1, n)\}$ is clearly contractible, so is $C_0(K_n)$. □

Lemma A.3. The homology groups of $|\Delta|(K^n)$ and $\mathcal{S}^{n-2}$ coincide for all $n \geqslant 3$.

Proof. We will show this by directly computing the simplicial homology groups of $\Delta(K^{n+1})$ by applying the Mayer-Vietoris Theorem. As $C_0(K_n)$ and $C_1(K_n)$ form a cover of $\Delta(K^{n+1})$, we get the long exact sequence:

$$\begin{aligned} \ldots \to H_{k+1}(\Delta(K_{n+1})) \longrightarrow H_k(C_0(K_n) \cap C_1(K_n)) \longrightarrow H_k(C_0(K_n)) \oplus H_k(C_1(K_n)) \\ \longrightarrow H_k(\Delta(K_{n+1})) \longrightarrow H_{k-1}(C_0(K_n) \cap C_1(K_n)) \to H_{k-1}(C_0(K_n)) \oplus H_{k-1}(C_1(K_n)) \to \ldots \end{aligned}$$

As $C_0(K_n)$ and $C_1(K_n)$ are contractible, we have that $H_k(C_0(K_n)) \oplus H_k(C_1(K_n))$ is zero for all k, which coupled with the fact that $C_0(K_n) \cap C_1(K_n)$ is homotopy equivalent to $\Delta(K_n)$ yields the family of exact sequences:

$$0 \longrightarrow H_k(\Delta(K_{n+1})) \longrightarrow H_{k-1}(\Delta(K_n)) \longrightarrow 0.$$

Due to exactness, the middle map need be an isomorphism, thus we have that for all k, $H_{k+1}(\Delta(K_{n+1})) \simeq H_k(\Delta(K_n))$. As $|\Delta|(K_n) \simeq \mathcal{S}^1$, the claim follows by induction. □

Lemma A.4. $|\Delta|(K^n)$ is simply-connected for $n \geqslant 4$.

Proof. This follows directly from van Kampen's Theorem on the cover $C_0(K_n)$, $C_1(K_n)$, noting that for $n \geqslant 4$, $C_0(K_n) \cap C_1(K_n) = \Delta(K_{n-1})$ is path-connected and that $C_0(K_n)$, $C_1(K_n)$ are contractible. □

We now move towards completing the second step of our proof.

Definition A.2. Given a $\mathbb{Z}_2$-simplicial complex K, we define its **suspension** SK as:

$$V(SK) \coloneqq V \sqcup \{\uparrow_K, \downarrow_K\}$$

$$\Sigma(SK) \coloneqq \Sigma(K) \cup \{(\sigma \cup \{\uparrow\}), (\sigma \cup \{\downarrow_K\}) : \sigma \in \Sigma(K)\},$$

with the $\mathbb{Z}_2$-action given by $(-)_{SK}(x) \coloneqq (-)_K(x)$ (for all $x \in V$) and $(-)_{SK}(\uparrow_K) \coloneqq \downarrow_K$, $(-)_{SK}(\uparrow_K) \coloneqq \downarrow_K$.

Lemma A.5. For any $\mathbb{Z}_2$-simplicial complex K, we have that $S|K| \simeq_{\mathbb{Z}_2} |SK|$.

Lemma A.6. Let $B \coloneqq (V, \Sigma)$ be the simplicial complex with $V \coloneqq \{1, 2\}$ and $\Sigma \coloneqq \{\{1\}, \{2\}\}$. We have that for any $k \in \mathbb{N}_{\geqslant 0}$ $|S^k B| \simeq \mathcal{S}^k$ (i.e. applying suspension k times to B results in a space $\mathbb{Z}_2$-homotopy equivalent to the k-sphere).

For ease of notation, we will write $(k, \uparrow)$ for $\uparrow_{S^{(k-1)}B}$ and $(k, \downarrow)$ for $\downarrow_{S^{(k-1)}B}$ when refering to any such vertices of a complex $S^n B$ ($n \in \mathbb{N}_{\geqslant 1}$, $k \leqslant n$).

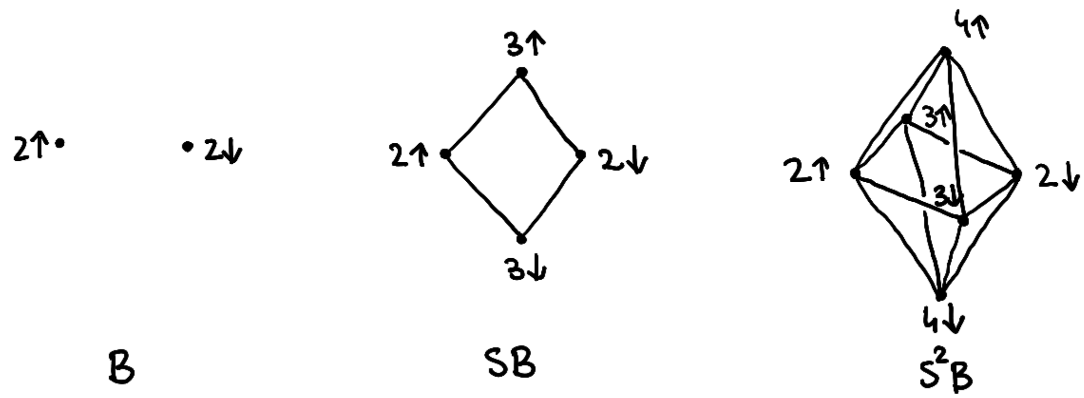

Figure A.2

Lemma A.7 ([6])**.** Given a simplicial complex K, we note that $|K| \times I$ is triangulated by the following simplicial complex, which we denote by $K \times I$. The vertices of $K \times I$ are given by $V(K) \times \{0, 1\}$ and $\Sigma(K \times I)$ consists of the following union:

$$\{(\sigma_0, 0), \ldots, (\sigma_i, 0), (\sigma_i, 1), \ldots, (\sigma_n, 1) : (\sigma_0, \ldots, \sigma_n) \in \Sigma(K), \sigma_0 < \cdots < \sigma_n\}$$
$$\cup \{\sigma \times \{0, 1\} : \sigma \in \Sigma(K)\}.$$

Notation A.1. We denote by $\iota_K^0, \iota_K^1 : K \to K \times I$ the mappings:

$$\iota_K^0 : x \mapsto (x, 0), \qquad \iota_K^1 : x \mapsto (x, 1)$$

Definition A.3. Given a simplicial complex K, we denote by $P_K : C_\bullet(K) \to C_{\bullet+1}(K \times I)$ the degree one chain map called the **prism operator** given by the linear extension of the following mapping:

$$P_K : (\sigma_0, \dots, \sigma_n) \mapsto \sum_{i=0}^{n} (-1)^i ((\sigma_0, 0) \dots, (\sigma_i, 0), (\sigma_i, 1), \dots (\sigma_n, 1))$$

In practice we will drop the indexing, infering the type of K from context.

Lemma A.8 ([6])**.** For any simplicial complex K, P corresponds to a chain homotopy between ι_K^0 and ι_K^1, that is: $\partial P + P\partial = \iota_K^1 - \iota_K^0$.

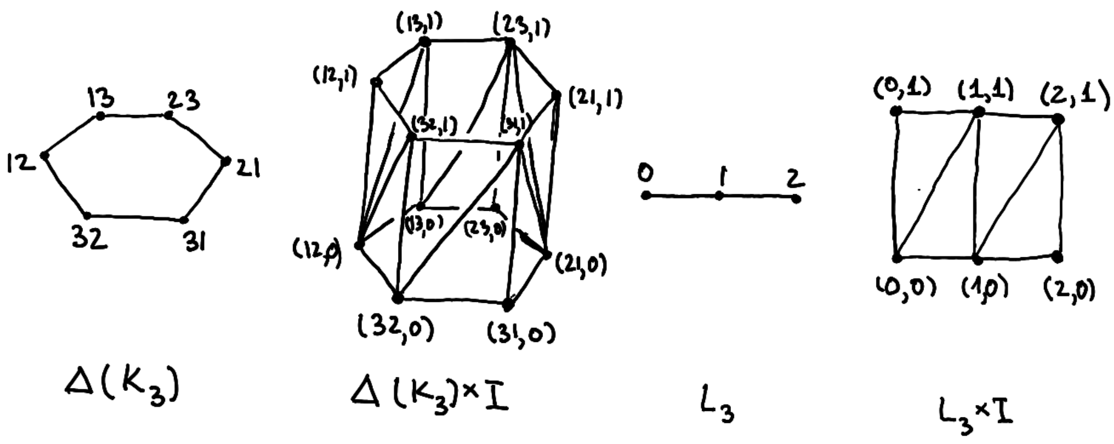

Figure A.3

We may now finally give a description of Γ_n and show that $H_k(\varepsilon_n)$ is an isomorphism for all $k \geqslant 0$ and $n \geqslant 3$.

Construction A.1. We define $\Gamma_2 \coloneqq (1,2) - (2,1)$ and define $\Gamma_{n>2}$ inductively. Similarly to R_n in the proof of Lemma A.2, we define the mappings $q_n^0, q_n^1 : \Delta(K_{n-1}) \times I \to \Delta(K_n)$ as following the simplicial maps:

$$\begin{aligned} q_n^0 &: ((x,y),0) \mapsto (x,y), & q_n^0 &: ((x,y),1) \mapsto (x,n), \\ q_n^1 &: ((x,y),0) \mapsto (x,y), & q_n^1 &: ((x,y),1) \mapsto (n,y). \end{aligned}$$

We define Γ_n as $\Gamma_n^0 - \Gamma_n^1$, where

$$\Gamma_n^0 \coloneqq (q_n^0 \circ P)(\Gamma_{n-1}) + t_n((1,n), \dots, (n-1,n)),$$

$$\Gamma_n^1 \coloneqq (q_n^1 \circ P)(\Gamma_{n-1}) + t_n((n,1),\ldots,(n,n-1)),$$
where $t_2 = -1$ and for $n > 2$ $t_n = (-1)^n t_{n-1}$

The intuition behind this construction is that in an analogy based off the fact that $|\Delta(K_n)| \simeq_{\mathbb{Z}_2} \mathcal{S}^n$, we may think of Γ_n as a chain representing the whole sphere and of Γ_n^0 and Γ_n^1 as its upper and lower hemispheres, meeting in the equator $\Gamma_{n-1} = -\partial\Gamma_n^0 = -\partial\Gamma_n^1$, as we will see later.

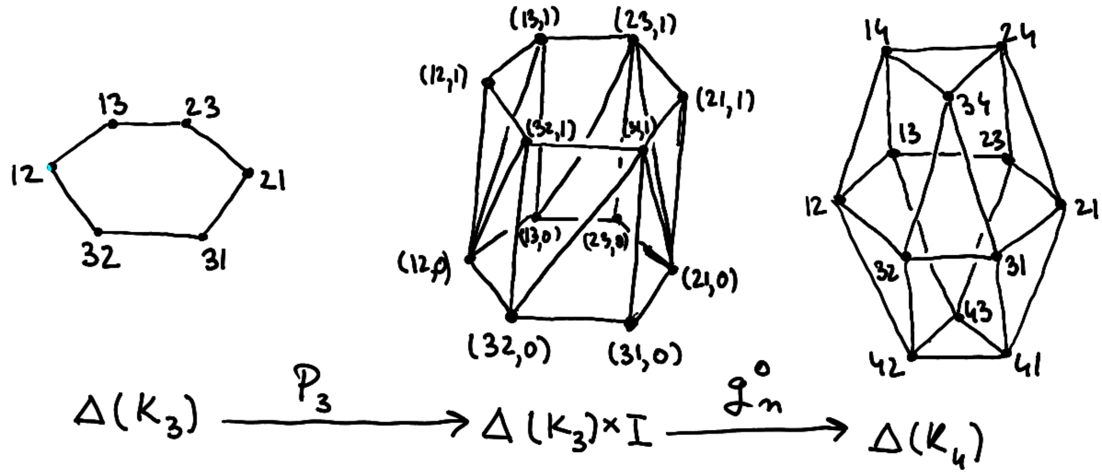

Figure A.4

Definition A.4. We denote by $\partial\Delta^{n-1}$ the usual triangulation of $\mathcal{S}^{n-2}$, where $V(\partial\Delta^{n-1}) = [n]$ and $\Sigma(\partial\Delta^{n-1})$ consists of all non-empty subsets of $V(\partial\Delta^{n-1})$ except $V(\partial\Delta^{n-1})$ itself. We further define Λ_n as:

$$\Lambda_n = t_n \sum_{k=1}^{n} (-1)^k (1,\ldots,k-1,k+1,\ldots,n)$$

where t_n is as in Construction A.2 and note that $[\Lambda_n]$ visibly generates $H_{n-2}(\partial\Delta^{n-1})$.

Construction A.2. Let $\eta_n : \Delta(K_n) \to \partial\Delta^{n-1}$ be the simplicial map given by:

$$\eta_n : (x,y) \mapsto x.$$

Notation A.2. Given a simplicial complex K, we write $\Sigma^{(n)}(K)$ to denote the set of n-simplices in K, i.e. $\Sigma^{(n)}(K) \coloneqq \{\sigma \in \Sigma(K) : |\sigma| = n\}$.

Notation A.3. We denote the terms in the chain Γ_n as follows:

$$\Gamma_n = \sum_{\sigma \in \Sigma^{(n-2)}(\Delta(K_n))} \gamma_\sigma \sigma.$$

Theorem A.1. [6, Theorem 4.33] Given simply connected CW-complexes X, Y and a map $f : X \to Y$ such that $H_n(f)$ is an isomorphism for all $n \geqslant 2$, then f need be a homotopy equivalence.

Lemma A.9. For all $n \geqslant 3$ and $k \geqslant 0$, we have that:

1. Γ_n and $\eta_n(\Gamma_n)$ are cycles,
2. $\eta_n(\Gamma_n) = \Lambda_n$,
3. $H_k(\eta_n)$ is an isomorphism,
4. $|\eta_n|$ is a homotopy equivalence.

Proof. We prove all of the above by induction. For the base case $n = 3$, the claims are clear from figure A.5. For the inductive case, we prove the results in order:

1. It suffices to show that Γ_n is a cycle as simplicial maps send cycles to cycles. For brevity, we write T_n for $((1, n), \dots, (n-1, n))$. We have that

$$\begin{aligned}
\partial\Gamma_n^0 &= \partial(q_0 \circ P)(\Gamma_{n-1}) + t_{n-1}\partial T_n \\
&= (q_n^0 \circ \iota^1_{\Delta(K_{n-1})})(\Gamma_{n-1}) - (q_n^0 \circ \iota^0_{\Delta(K_{n-1})})(\Gamma_{n-1}) + (P \circ \partial)(\Gamma_{n-1}) + t_{n-1}\partial T_n \\
&= -t_{n-1}\partial T_n - \Gamma_{n-1} + t_{n-1}\partial T_n \\
&= -\Gamma_{n-1}.
\end{aligned}$$

   In the second to last equality we have assumed the following:

   - $(P \circ \partial)(\Gamma_{n-1}) = 0$: this holds as by the induction hypothesis, we have that Γ_{n-1} is a cycle.
   - $(q_n^0 \circ \iota^1_{\Delta(K_{n-1})})(\Gamma_{n-1}) = t_{n-1}\partial((1, n), \dots, (n-1, n))$: this stems from the following: identifying $(1, n)$ with 1, ... and $(n-1, n)$ with $n-1$ and restricting the domain of $(q_n^0 \circ \iota^1)$ to $\Delta(K_n) \to \partial\Delta^{n-1}$, we have that $(q_n^0 \circ \iota^1)$ coincides with η_{n-1}, thus, by the induction hypothesis, Γ_{n-1} is mapped to the cycle corresponding to Λ_{n-1} under the identification, that is:

$$\begin{aligned}
&t_{n-1}\sum_{i=1}^{n-1}(-1)^i((1, n), \dots, (i-1, n), (i+1, n) \dots, (n-1, n)) \\
&\qquad\qquad\qquad\qquad = -t_{n-1}\partial((1, n), \dots, (n-1, n))
\end{aligned}$$

   By an analogous computation it follows that also $\partial\Gamma_n^1 = -\Gamma_{n-1}$ and so

$$\partial\Gamma_n = \partial\Gamma_n^0 - \partial\Gamma_n^1 = -\Gamma_{n-1} + \Gamma_{n-1} = 0$$

2. Writing σ_i for $((1,i),\ldots,(i-1,i),(i+1,i),\ldots,(n,i))$ ($\in C_{n-2}(\Delta(K_n))$) and σ_i' for $(1,\ldots,i-1,i,\ldots,n)$ ($\in C_{n-2}(\partial\Delta^{n-1})$), we note that for any $i \in [n]$, the only $(n-2)$-*face* σ in $\Delta(K_n)$ such that $\eta_n(\sigma_i) = \sigma_i'$ is σ. In particular, this means that

$$\eta_n(\Gamma_n) = \eta_n\left(\sum_{\sigma\in\Sigma^{(n-2)}(\Delta(K_n))} \gamma_\sigma \sigma\right) = \sum_{i=1}^{n} \gamma_{\sigma_i}\sigma_i'$$

as all other faces in a chain can only be mapped to smaller faces, which are zero in a simplicial chain.

As $\eta_n(\Gamma_n)$ is a cycle and Λ_n generates $H_{n-2}(\partial\Delta^{n-1})$, we have that $\eta_n(\Gamma_n) = k\Lambda_n + \beta$ where k is some integer and β is either zero or the boundary of some $(n-1)$-chain. As $\Sigma^{(n-1)}(\partial\Delta^{n-1}) = \varnothing$, we have that $\beta = 0$ and so $\eta_n(\Gamma_n) = k\Lambda_n$. Thus, by the observation in the previous paragraph, we have that:

$$\eta_n(\Gamma_n) = \sum_{i=1}^{n} \gamma_{\sigma_i}\sigma_i' = k\Lambda_n = \sum_{i=1}^{n}(-1)^i k t_n \sigma_i'.$$

It is however clear from Construction A.2 that γ_{σ_n} is t_{n-1}, thus equating the corresponding terms in the equation above, it follows that $t_{n-1} = (-1)^n k t_{n-1}$, which by the recurrent definition of t_n, immediately implies that $k = 1$ and so $\eta_n(\Gamma_n) = \Lambda_n$ as desired.

3. As η_n maps a cycle in $\Delta(K^n)$ to Λ_n, which corresponds to a generator of $H_{n-2}(\partial\Delta^{n-1})$, it follows that $H_n(\eta_n) : H_{n-2}(\Delta(K_n)) \to H_{n-2}(\partial\Delta^{n-1})$ need be surjective. Moreover, as $H_{n-2}(\Delta(K_n)) \simeq H_{n-2}(\partial\Delta^{n-1}) \simeq \mathbb{Z}$ and as a surjective group homomorphism $\mathbb{Z} \to \mathbb{Z}$ need be an isomorphism, it follows that $H_{n-2}(\eta_n)$ is also an isomorphism. Finally, we have that for $k \in \mathbb{N}\setminus\{0, n-2\}$, $H_k(\partial\Delta^{n-2}) = H_k(\Delta(K_n)) = 0$, thus $H_k(\eta_n)$ is trivially an isomorphism.

4. This is an immediate corollary of Theorem A.1, as by Lemma A.4 $\Delta(K_n)$ corresponds to a simply connected space and $|\partial\Delta^{n-1}| \simeq \mathcal{S}^{n-2}$ is also simply connected. □

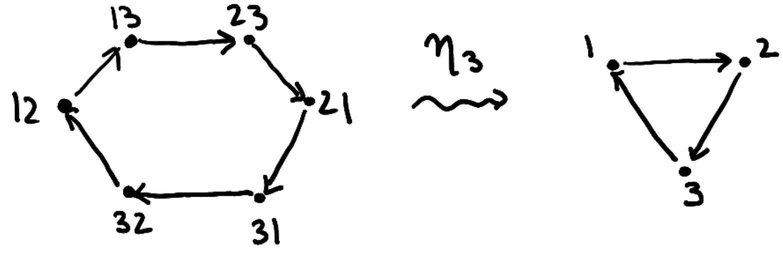

Figure A.5

We may now proceed with the final step of our proof:

Construction A.3. Let $\varepsilon_n : \Delta(K_n) \to S^{n-2}B$ be given by:

$$\varepsilon_n : (x, y) \longmapsto \begin{cases} (\max(x, y), \uparrow) & \text{if } x < y \\ (\max(x, y), \downarrow) & \text{if } x > y \end{cases}$$

Definition A.5. For some $n \geqslant 0$, and simplicial complex K, we say that a chain

$$\Gamma = \sum_{\sigma \in \Sigma^{(n)}(K)} \lambda_\sigma \sigma \in C_n(K)$$

is **primal** if $\lambda_\sigma \in \{-1, 0, 1\}$ for all $\sigma \in \Sigma^{(n)}(K)$ and $\Gamma \neq 0$.

Lemma A.10. For all $n \geqslant 3$, Γ_n and $\varepsilon_n(\Gamma_n)$ are primal.

Proof. Given a chain $\Gamma \in C_{n-2}(\Delta(K_n))$, writing

$$\Gamma = \sum_{\sigma \in \Sigma^{(n-2)}(\Delta(K_n))} \lambda_\sigma \sigma,$$

we define the support of Γ (written as $\mathrm{supp}(\Gamma)$) as the set of faces that occur as nonzero terms in the sum above and are mapped by ε_n to $(n-2)$-faces in $S^{n-2}B$ - i.e. $\mathrm{supp}(\Gamma) = \{\sigma \in \Sigma^{(n-2)}(\Delta(K_n)) : \lambda_\sigma \neq 0, \varepsilon_n(\sigma) \in \Sigma^{(n-2)}(S^{n-2}B)\}$. We note that by definition, we have that:

$$\varepsilon_n(\Gamma) = \sum_{\sigma \in \Sigma^{(n-2)}(\Delta(K_n))} \lambda_\sigma \varepsilon_n(\sigma) = \sum_{\sigma \in \mathrm{supp}(\Gamma)} \lambda_\sigma \varepsilon_n(\sigma).$$

We will prove the claim by induction. We will keep two induction invariants:

1. All of the faces in the support of Γ_n are mapped by ε_n to distinct faces in S^{n-2B} — i.e. for any two distinct $\sigma_1, \sigma_2 \in \mathrm{supp}(\Gamma_n)$, we have that $\varepsilon_n(\sigma_1) \neq \varepsilon_n(\sigma_2)$,
2. $\varepsilon_n(\Gamma_n)$ is primal.

Resolving the base case ($n = 3$) is a matter of straightfoward computation, however the result is visible from Figure A.6.

For the general case, it follows from the induction hypotheses that $\mathrm{supp}(\Gamma_{n-1})$ is a primal cycle. Furthermore, writing $\sigma = (\sigma_0, \ldots, \sigma_m)$ for any m-face σ, we have that:

$$\varepsilon_n(\Gamma_n^0) = \varepsilon_n(q_n^0 \circ P)(\Gamma_{n-1}) + t_n T_n) \tag{1}$$

$$
\begin{aligned}
&= (\varepsilon_n \circ q_n^0 \circ P)(\Gamma_{n-1}) && (2)\\
&= \sum_{\sigma \in \Sigma^{(n-3)}(\Delta(K_{n-1}))} \lambda_\sigma (\varepsilon_n \circ q_n^0 \circ P)(\sigma) && (3)\\
&= \sum_{\sigma \in \Sigma^{(n-3)}(\Delta(K_{n-1}))} \lambda_\sigma \sum_{i=0}^{n-3} (-1)^i (\varepsilon_n \circ q_n^0)((\sigma_0, 0), \ldots, (\sigma_i, 0), (\sigma_i, 1), \ldots, (\sigma_{n-3}, 1)) && (4)\\
&= \sum_{\sigma \in \Sigma^{(n-3)}(\Delta(K_{n-1}))} \lambda_\sigma \sum_{i=0}^{n-3} (-1)^i (\varepsilon_n(\sigma_0), \ldots, \varepsilon_n(\sigma_i), (n, \uparrow), \ldots, (n, \uparrow)) && (5)\\
&= \sum_{\sigma \in \Sigma^{(n-3)}(\Delta(K_{n-1}))} \lambda_\sigma (-1)^{n-3} (\varepsilon_n(\sigma_0), \ldots, \varepsilon_n(\sigma_{n-3}), (n, \uparrow)) && (6)\\
&= \sum_{\sigma \in \Sigma^{(n-3)}(\Delta(K_{n-1}))} \lambda_\sigma (-1)^{n+1} (\varepsilon_n(\sigma) \mathbin{+\!\!+} (n, \uparrow)) && (7)\\
&= \sum_{\sigma \in \operatorname{supp}(\Gamma_{n-1})} \lambda_\sigma (-1)^{n+1} (\varepsilon_{n-1}(\sigma) \mathbin{+\!\!+} (n, \uparrow)) && (8)
\end{aligned}
$$

(where $+\!\!+$ denotes simply adding a vertex at the end of a face tuple - e.g. $(1, 2) \mathbin{+\!\!+} 3 = (1, 2, 3)$).

By the induction hypothesis, all of the terms from the sum in (8) above are distinct. Moreover, by an analogous proof, the same holds for Γ_n^1, thus, noting that the images through ε_n of the supports of Γ_n^0 and Γ_n^1 are disjoint, it follows that the first inductive invariant holds for n. Finally, we note that as $\varepsilon_n(\Gamma_n^0)$ is visibly primal as in (8) above, and that $\varepsilon_n(\Gamma_n^1)$ is also primal by an analogous argument, it follows that $\varepsilon_n(\Gamma_n) = \varepsilon_n(\Gamma_n^0) + \varepsilon_n(\Gamma_n^1)$ is primal. $\square$

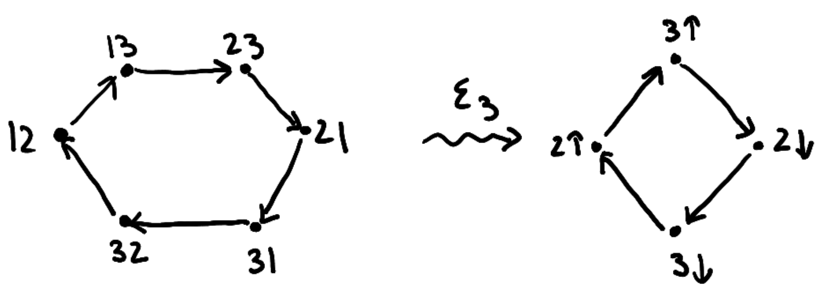

Figure A.6

Lemma A.11. $H_k(\varepsilon_n)$ is an isomorphism for all $n \geqslant 3$ and $k \geqslant 0$.

Proof. As $\varepsilon_n(\Gamma_n)$ is a cycle, writing Θ_n for an $(n-2)$-cycle corresponding to the generator of $H_{n-2}(S^{n-2}B) \simeq \mathbb{Z}$, we have that $\varepsilon(\Gamma_n) = k\Theta_n + \beta$ for some $k \in \mathbb{Z}$ and β is either zero or the boundary of some $(n-1)$-chain. As $\Sigma(S^{n-2}B)^{n-1} = \varnothing$, we have that $\beta = 0$ and so $\varepsilon(\Gamma_n) = k\Theta_n$. Finally, as $\varepsilon(\Gamma_n)$ is a primal cycle, by the Lemma A.10 we have that $k \in \{-1, 1\}$. Thus, as ε_n maps a cycle corresponding to a generator of $H_{n-2}(\Delta(K_n)) \simeq \mathbb{Z}$ to one corresponding to a generator of $H_{n-2}(S^{n-2}B) \simeq \mathbb{Z}$, it follows that $H_{n-2}(\varepsilon_n)$ need be an isomorphism. As in Lemma A.9 (3), $H_k(\varepsilon_n)$ is trivially an isomorphism for $k \in [0 \dots n-3]$. □

We now arm ourselves with the following result from algebraic topology, which coupled with our results so far turns the desired $\mathbb{Z}_2$-homotopy between $|\Delta|(K_n)$ and $\mathcal{S}^{n-2}$ into a mere corollary.

Theorem A.2 ([8])**.** If G is a discrete group and $f : X \to Y$ is a G-map between G-complexes which induces homotopy equivalences $X^H \to Y^H$ between the H-fixed subspaces for all subgroups $H \leqslant G$, then f is a G-homotopy equivalence.

Corollary A.2.1. ε_n is a $\mathbb{Z}_2$-homotopy equivalence

Proof. Note that ε_n is visibly a $\mathbb{Z}_2$-equivariant map and apply Theorem A.2. □

Corollary A.2.2. $\Delta(K_n)$ is $\mathbb{Z}_2$-homotopy equivalent to $\mathcal{S}^{n-2}$

Proof. Note that $\mathbb{Z}_2$ acts freely on both $\Delta(K_n)$ and $S^{n-2}B$, thus their fixed spaces are empty. By Theorem A.2, it follows that ε_n is a $\mathbb{Z}_2$-homotopy equivalence. As $S^{n-2}B$ is $\mathbb{Z}_2$-homotopy equivalent to $\mathcal{S}^{n-2}$, the result is proved. □

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