markowitz

July 5, 2025

[144]:
```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
from scipy.optimize import minimize # for optimization
from scipy.cluster.hierarchy import linkage, dendrogram, fcluster # for␣
↪hierarchical clustering
from scipy.spatial.distance import squareform # to convert distance matrix to␣
↪condensed form
from sklearn.metrics import silhouette_score # for calculating silhouette scores
from kneed import KneeLocator # for finding the knee point in the silhouette␣
↪score curve
from itertools import combinations # for generating combinations of stocks
import edhec_risk_kit as erk # for financial risk metrics
```

[145]:
```
import yfinance as yf # for fetching stock data
```

[146]:
```
symbols = [
    "AKBNK.IS", "ARCLK.IS", "ASELS.IS", "BIMAS.IS", "EKGYO.IS",
    "EUPWR.IS", "FROTO.IS", "GARAN.IS", "GUBRF.IS",
    "ISCTR.IS", "KCHOL.IS", "KOZAA.IS", "KOZAL.IS", "PGSUS.IS",
    "SAHOL.IS", "SASA.IS", "SISE.IS", "TCELL.IS", "THYAO.IS",
    "TKFEN.IS", "TOASO.IS", "TUPRS.IS", "VAKBN.IS",
    "YKBNK.IS", "ALARK.IS", "SOKM.IS", "ODAS.IS"
]
# Creating a dataframe named symbols, with Yahoo Finance tickers for BIST␣
↪equities
```

[147]:
```
data = yf.download(symbols, start="2010-01-01", end="2025-07-02",␣
↪auto_adjust=False)['Adj Close'] # Downloading adjusted close prices for the␣
↪specified symbols from Yahoo Finance
```

```
[*********************100%***********************]  27 of 27 completed
```

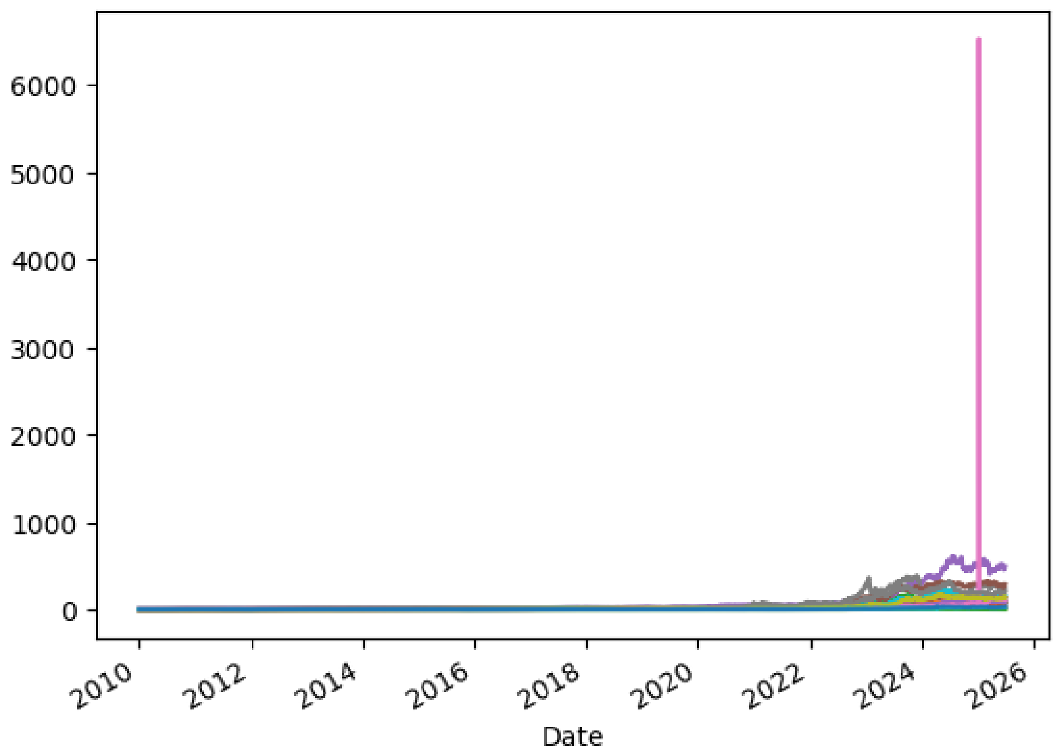
[148]:
```
min_valid_ratio = 0.999 # Minimum valid ratio for data cleaning, to keep␣
↪columns with at least 99.9% valid data
# This means we will drop columns with more than 0.1% missing values
data_clean = data.loc[:, data.isnull().mean() < (1 - min_valid_ratio)] # Drop␣
↪columns with more than 0.1% missing values
```

```
data_clean.head()
data_clean.columns
```

[148]: Index(['AKBNK.IS', 'ALARK.IS', 'ARCLK.IS', 'ASELS.IS', 'BIMAS.IS', 'FROTO.IS',
       'GARAN.IS', 'GUBRF.IS', 'ISCTR.IS', 'KCHOL.IS', 'KOZAA.IS', 'SAHOL.IS',
       'SASA.IS', 'SISE.IS', 'TCELL.IS', 'THYAO.IS', 'TKFEN.IS', 'TOASO.IS',
       'TUPRS.IS', 'VAKBN.IS', 'YKBNK.IS'],
      dtype='object', name='Ticker')

[149]:
```
data_clean.plot(legend=False)
```

[149]: <Axes: xlabel='Date'>



[150]:
```
data_clean["TKFEN.IS"]["2025-01-01":"2025-01-31"]
```

[150]: Date
2025-01-02   64.849998
2025-01-03   64.500000
2025-01-06   65.300003
2025-01-07   67.550003
2025-01-08   65.400002
2025-01-09   65.199997

```
2025-01-10    6520.000000
2025-01-13      62.150002
2025-01-14      64.599998
2025-01-15      66.099998
2025-01-16      66.250000
2025-01-17      65.050003
2025-01-20      64.199997
2025-01-21      63.549999
2025-01-22      63.849998
2025-01-23      64.449997
2025-01-24      63.750000
2025-01-27      63.549999
2025-01-28      64.000000
2025-01-29      63.000000
2025-01-30      64.199997
2025-01-31      63.750000
Name: TKFEN.IS, dtype: float64
```

[151]:

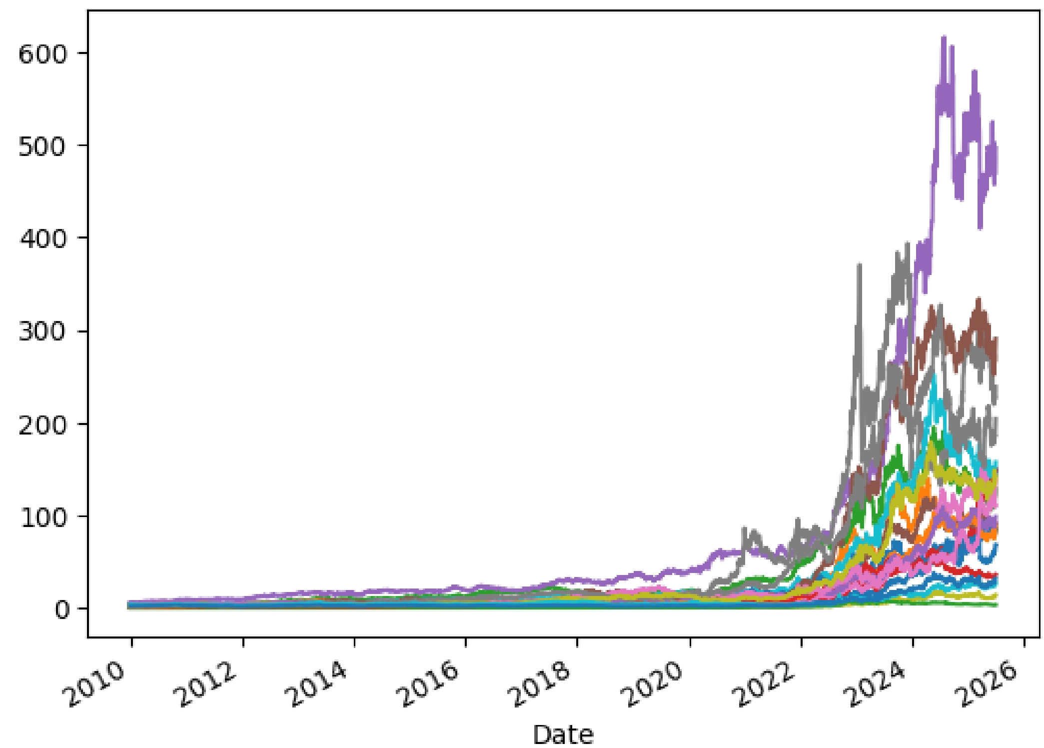
```python
data_clean.loc["2025-01-10", "TKFEN.IS"] = data_clean.loc["2025-01-10", "TKFEN.IS"] / 100
#Due to an error in the incoming data, the price was being calculated as 6520. We corrected this by dividing the price by 100. We can also use the mean or median of the previous and next days to correct this error.
```

[152]:

```python
data_clean.plot(legend=False)
```

[152]:

```
<Axes: xlabel='Date'>
```



Global Return Calculation

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

[153]:
```
returns = data_clean.pct_change(fill_method=None).dropna() # Calculating daily␣
↪returns
mean_returns = returns.mean() # Calculating mean returns
cov_matrix = returns.cov() # Calculating covariance matrix of returns
corr_matrix = returns.corr() # Calculating the correlation matrix of daily␣
↪returns
distance_matrix = np.sqrt(2 * (1 - corr_matrix)) # Calculating the distance␣
↪matrix using the correlation matrix
distance_vector = squareform(distance_matrix.values, checks=False) # Converting␣
↪the distance matrix to a condensed distance vector
periods_per_year = 252
annual_rf = 0.4707 # Annual risk-free rate for Turkey, as of 2025-07-02
```

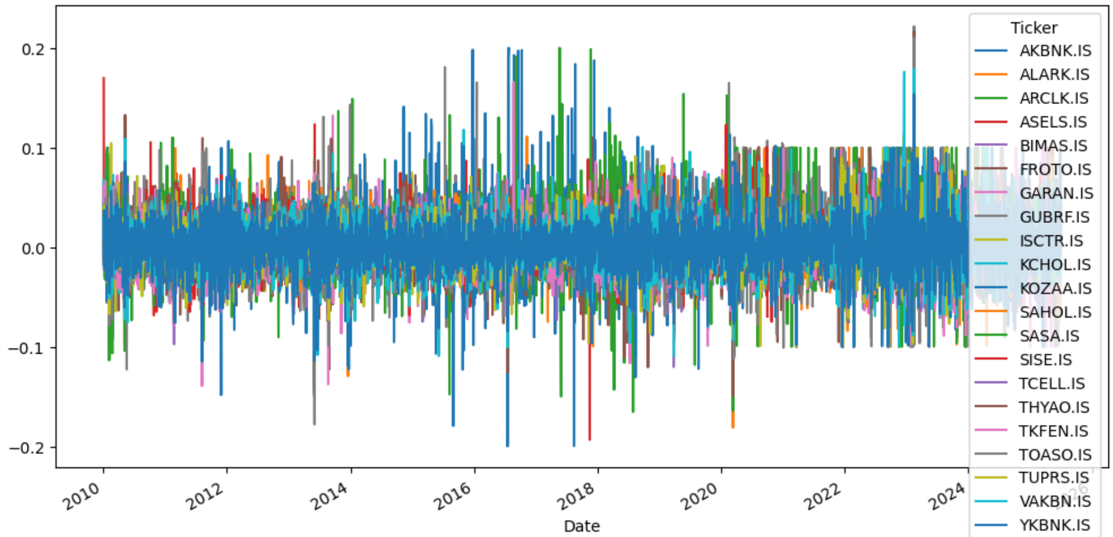
[154]:
```
returns.head()
```

[154]:
```
Ticker      AKBNK.IS  ALARK.IS  ARCLK.IS  ASELS.IS  BIMAS.IS  FROTO.IS  \
Date
```

```
2010-01-04 -0.010417  0.005025  0.000000  0.169645  0.000000  0.022222
2010-01-05  0.010526  0.025000  0.025424  0.007633  0.007299  0.021739
2010-01-06 -0.005208  0.058536  0.033058 -0.022727  0.000000  0.005319
2010-01-07  0.000000  0.009217 -0.008000  0.062016  0.000000  0.010582
2010-01-08  0.000000  0.000000 -0.016129  0.000000  0.000000  0.026178

Ticker      GARAN.IS  GUBRF.IS  ISCTR.IS  KCHOL.IS  …  SAHOL.IS   SASA.IS  \
Date                                                …
2010-01-04 -0.007874  0.006250       0.0  0.004525  …  0.008772  0.034020
2010-01-05  0.023810  0.049689       0.0  0.067568  …  0.008696  0.000000
2010-01-06  0.000000  0.029586       0.0  0.004219  … -0.017241 -0.016450
2010-01-07  0.015504  0.017242       0.0 -0.004202  …  0.000000  0.033130
2010-01-08  0.007633 -0.022599       0.0  0.000000  …  0.026316  0.048568

Ticker       SISE.IS  TCELL.IS  THYAO.IS  TKFEN.IS  TOASO.IS  TUPRS.IS  \
Date
2010-01-04  0.037635  0.009434  0.017699 -0.008097 -0.004237  0.016806
2010-01-05  0.000000  0.037383  0.000000  0.061224  0.042553  0.041322
2010-01-06  0.005182  0.027027 -0.008696  0.028846 -0.004082  0.000000
2010-01-07  0.000000  0.026316 -0.017544  0.000000  0.036885  0.000000
2010-01-08  0.005154 -0.017094 -0.017857  0.018692  0.017787 -0.007937

Ticker      VAKBN.IS  YKBNK.IS
Date
2010-01-04  0.009434 -0.018182
2010-01-05  0.032710  0.037037
2010-01-06 -0.009050  0.017857
2010-01-07 -0.004566  0.011696
2010-01-08  0.004587  0.005780

[5 rows x 21 columns]
```

[155]:
```
returns.plot(figsize=(12,6))
```

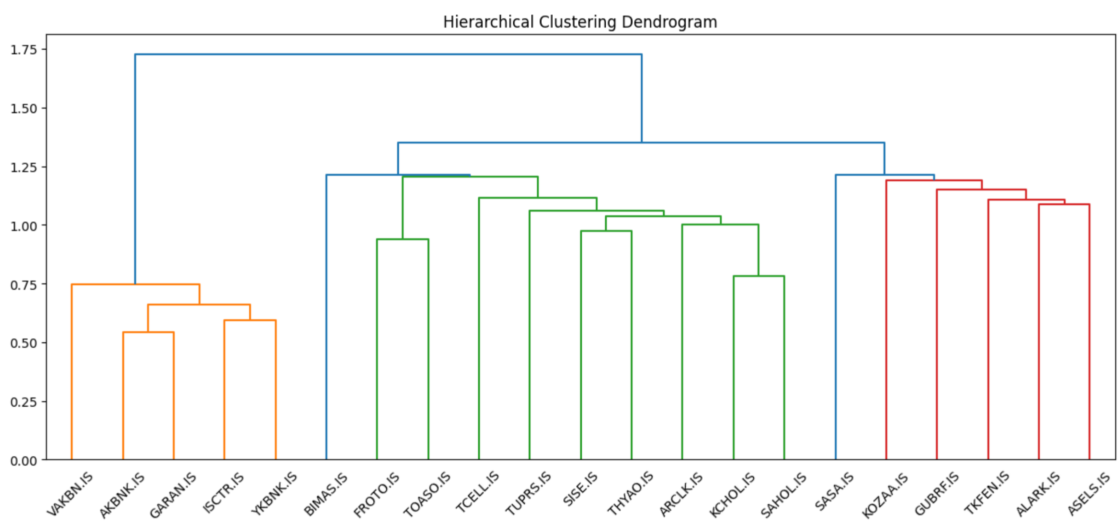
[155]:
```
<Axes: xlabel='Date'>
```



1 Clustering

[156]:
```
dend = linkage(distance_vector, method='ward') # Performing hierarchical␣
↪clustering using Ward's method
# Ward's method minimizes the variance of clusters being merged.
```

[157]:
```
plt.figure(figsize=(15, 6))
dendrogram(dend, labels=distance_matrix.index)
plt.title("Hierarchical Clustering Dendrogram")
plt.show()
```



[158]:

```
scores = []
for n_clusters in range(2, 11):  # min 2, max 10 cluster dener
    labels = fcluster(dend, n_clusters, criterion='maxclust') # Assigning␣
 ↪cluster labels based on the hierarchical clustering
    # fcluster creates flat clusters from the hierarchical clustering defined␣
 ↪by the linkage matrix.
    score = silhouette_score(distance_matrix, labels, metric='precomputed') #␣
 ↪Calculating the silhouette score for the clustering
    # The silhouette score measures how similar an object is to its own cluster␣
 ↪compared to other clusters
    scores.append(score)
    print(f"Cluster: {n_clusters}, Silhouette Score: {score:.4f}")
```

```
Cluster: 2, Silhouette Score: 0.0629
Cluster: 3, Silhouette Score: 0.0701
Cluster: 4, Silhouette Score: 0.0731
Cluster: 5, Silhouette Score: 0.0756
Cluster: 6, Silhouette Score: 0.0805
Cluster: 7, Silhouette Score: 0.0825
Cluster: 8, Silhouette Score: 0.0844
Cluster: 9, Silhouette Score: 0.0848
Cluster: 10, Silhouette Score: 0.0866
```

[159]:

```
cluster_range = list(range(2, len(scores) + 2)) # Creating a range for the␣
 ↪number of clusters
# This will be used for plotting the silhouette scores against the number of␣
 ↪clusters
plt.figure(figsize=(8,5))
plt.plot(cluster_range, scores, marker='o')
plt.xlabel('Number of Clusters')
plt.ylabel('Silhouette Score')
plt.title('Number of Clusters vs Silhouette Score')
plt.grid(True)
plt.show()
```

[160]:
```
kneedle = KneeLocator(cluster_range, scores, S=1.0, curve="concave",␣
 ↪direction="increasing")
optimal_clusters = int(kneedle.elbow)
print("Knee/elbow detected at:", optimal_clusters)

# KneeLocator automatically detects the "elbow point" in the silhouette score␣
 ↪curve,
# which corresponds to the most meaningful and interpretable number of clusters␣
 ↪in the data,
# as recommended by Hastie, Tibshirani, and Friedman (2009) and Tsay (2010).
# The elbow method ensures that we avoid excessive fragmentation and select a␣
 ↪parsimonious clustering solution.
```

```
Knee/elbow detected at: 6
```

[161]:
```
labels = fcluster(dend, optimal_clusters, criterion='maxclust')
clusters_df = pd.DataFrame({
    'Stock': distance_matrix.index,
    'Cluster': labels
})
print(clusters_df.sort_values('Cluster'))
# The clusters_df DataFrame shows the stocks and their assigned cluster labels.
```

```
       Stock  Cluster
```

```
0    AKBNK.IS        1
6    GARAN.IS        1
8    ISCTR.IS        1
19   VAKBN.IS        1
20   YKBNK.IS        1
17   TOASO.IS        2
5    FROTO.IS        2
13    SISE.IS        3
15   THYAO.IS        3
14   TCELL.IS        3
9    KCHOL.IS        3
2    ARCLK.IS        3
18   TUPRS.IS        3
11   SAHOL.IS        3
4    BIMAS.IS        4
1    ALARK.IS        5
16   TKFEN.IS        5
10   KOZAA.IS        5
3    ASELS.IS        5
7    GUBRF.IS        5
12    SASA.IS        6
```

2 Markowitz

[162]:
```
def annualize_rets(r, periods_per_year):
    """
    Annualizes the returns of a Series or DataFrame
    """
    compounded_growth = (1 + r).prod()
    n_periods = r.shape[0]
    return compounded_growth**(periods_per_year/n_periods) - 1
# This function calculates the annualized return based on the compounded growth␣
↪of returns.
```

[163]:
```
def annualize_vol(r, periods_per_year):
    """
    Annualizes the volatility of a Series or DataFrame
    """
    return r.std()* (periods_per_year**0.5)
# This function calculates the annualized volatility based on the standard␣
↪deviation of returns.
```

[164]:
```
def sharpe_ratio(r, riskfree_rate, periods_per_year):
    """
    Computes the Sharpe Ratio of a Series or DataFrame
    """
```

```
    rf_per_period = (1 + riskfree_rate)**(1/periods_per_year) - 1
    excess_ret = r - rf_per_period
    ann_ex_ret = annualize_rets(excess_ret, periods_per_year)
    ann_vol = annualize_vol(r, periods_per_year)
    return ann_ex_ret / ann_vol
# This function calculates the Sharpe Ratio, which is a measure of␣
↪risk-adjusted return.
```

[165]:
```
def portfolio_return(weights, returns):
    """
    Compute the return of a portfolio given weights and returns.
    """
    weights = np.array(weights)  # Ensure weights is a numpy array
    return weights.T @ returns
# This function calculates the portfolio return based on the weights and␣
↪returns of the assets.
```

[166]:
```
def portfolio_vol(weights, covmat):
    """
    Compute the variance of a portfolio given weights and the covariance matrix.
    """
    weights = np.array(weights)  # Ensure weights is a numpy array
    return (weights.T @ covmat @ weights)**0.5
# This function calculates the portfolio volatility based on the weights and␣
↪covariance matrix of returns.
```

[167]:
```
def minimize_vol(target_return, er, cov):
    """
    Minimize the portfolio volatility for a given target return.
    """
    n = er.shape[0]
    init_guess = np.repeat(1/n, n)
    bounds = ((0.0, 1.0), )*n
    return_is_target = {
        'type': 'eq',
        'args':(er, ),
        'fun': lambda weights, er: target_return - portfolio_return(weights, er)
    }
    weights_sum_to_1 = {
        'type': 'eq',
        'fun': lambda weights: np.sum(weights) - 1
    }
    results = minimize(portfolio_vol,init_guess,
                       args=(cov, ), method='SLSQP',
                       options={'disp': False},
                       constraints=(return_is_target, weights_sum_to_1),
                       bounds=bounds)
```

```
    return results.x
# This function minimizes the portfolio volatility for a given target return␣
 ↪using the Sequential Least Squares Programming (SLSQP) method.
```

[168]:
```
def optimal_weights(n_points, er, cov):
    """
    Generate optimal portfolio weights for a given number of points.
    """
    target_rs = np.linspace(er.min(), er.max(), n_points)
    weights = [minimize_vol(target_return, er, cov) for target_return in␣
 ↪target_rs]
    return weights
# This function generates optimal portfolio weights for a range of target␣
 ↪returns, which will be used to plot the efficient frontier.
```

[169]:
```
def msr(riskfree_rate, er, cov):
    """
    Minimize the portfolio volatility for a given riskfree_rate.
    """
    n = er.shape[0]
    init_guess = np.repeat(1/n, n)
    bounds = ((0.0, 1.0), )*n
    weights_sum_to_1 = {
        'type': 'eq',
        'fun': lambda weights: np.sum(weights) - 1
    }
    def neg_sharpe_ratio(weights, riskfree_rate, er, cov):
        """
        Compute the negative Sharpe ratio for a given set of weights.
        """
        r = portfolio_return(weights, er)
        vol = portfolio_vol(weights, cov)
        return -(r - riskfree_rate) / vol

    results = minimize(neg_sharpe_ratio, init_guess,
                       args=(riskfree_rate, er, cov), method='SLSQP',
                       options={'disp': False},
                       constraints=(weights_sum_to_1),
                       bounds=bounds)
    return results.x
# This function calculates the weights of the portfolio that maximizes the␣
 ↪Sharpe Ratio for a given risk-free rate, expected returns, and covariance␣
 ↪matrix.
```

[170]:
```
def gmv(cov):
    """
    Compute the Global Minimum Variance portfolio weights.
```

```
    """
    n = cov.shape[0]
    return msr(0, np.repeat(1/n, n), cov)
# This function calculates the weights of the Global Minimum Variance␣
↪portfolio, which minimizes the portfolio variance for a given covariance␣
↪matrix.
```

[171]:
```
def plot_ef(n_points, er, cov, style = ".-", show_cml=False, riskfree_rate=0.0,␣
↪show_ew = False, show_gmv = False):
    """
    Plots the efficient frontier for a given number of points.
    """
    # Generate random portfolio weights
    weights = optimal_weights(n_points, er, cov)
    rets = [portfolio_return(w, er) for w in weights]
    vols = [portfolio_vol(w, cov) for w in weights]
    ef = pd.DataFrame({'Returns': rets, 'Volatility': vols})
    ax = ef.plot.line(x='Volatility', y='Returns', style=style)
    if show_ew: # Plotting the equal-weighted portfolio
        n = er.shape[0]
        w_ew = np.repeat(1/n, n)
        r_ew = portfolio_return(w_ew, er)
        vol_ew = portfolio_vol(w_ew, cov)
        ax.plot(vol_ew, r_ew, color='goldenrod', marker = "o", markersize = 12)
    if show_gmv: # Plotting the Global Minimum Variance portfolio
        w_gmv = gmv(cov)
        r_gmv = portfolio_return(w_gmv, er)
        vol_gmv = portfolio_vol(w_gmv, cov)
        ax.plot(vol_gmv, r_gmv, color='midnightblue', marker = "o", markersize␣
↪= 10)

    if show_cml: # Plotting the Capital Market Line (CML)
        ax.set_xlim(left=0)
        w_msr = msr(riskfree_rate, er ,cov)
        r_msr = portfolio_return(w_msr, er)
        vol_msr = portfolio_vol(w_msr, cov)
        cml_x =[0, vol_msr]
        cml_y = [riskfree_rate, r_msr]
        ax.plot(cml_x, cml_y, color='green', marker = "o", linestyle =␣
↪"dashed", markersize=12, linewidth=2, label='CML')
    return ax
```

2.1 Selected Max Return Portfolio

[172]:
```
selected_stocks = []
for cl in clusters_df['Cluster'].unique():
    stocks_in_cluster = clusters_df[clusters_df['Cluster'] == cl]['Stock']
    ann_returns = returns[stocks_in_cluster].apply(lambda r: annualize_rets(r,␣
 ↪periods_per_year))
    best_stock = ann_returns.idxmax() # Maximizing the annualized return to␣
 ↪select the representative stock for each cluster
    selected_stocks.append(best_stock)
print(selected_stocks)
# The selected_stocks list contains the representative stocks for each cluster␣
 ↪based on the highest annualized return.
# These stocks can be used for further analysis or portfolio construction.
```

```
['GARAN.IS', 'ASELS.IS', 'TUPRS.IS', 'BIMAS.IS', 'FROTO.IS', 'SASA.IS']
```

[173]:
```
er = annualize_rets(returns[selected_stocks],␣
 ↪periods_per_year=periods_per_year) # Annualized returns for the selected␣
 ↪stocks
cov = returns[selected_stocks].cov() * periods_per_year # Covariance matrix of␣
 ↪the selected stocks' returns, annualized
er
```

[173]:
```
Ticker
GARAN.IS    0.243984
ASELS.IS    0.487845
TUPRS.IS    0.334462
BIMAS.IS    0.328331
FROTO.IS    0.371083
SASA.IS     0.559699
dtype: float64
```

[174]:
```
annualize_vol(returns[selected_stocks], periods_per_year=periods_per_year) #␣
 ↪Annualized volatility of the selected stocks
```

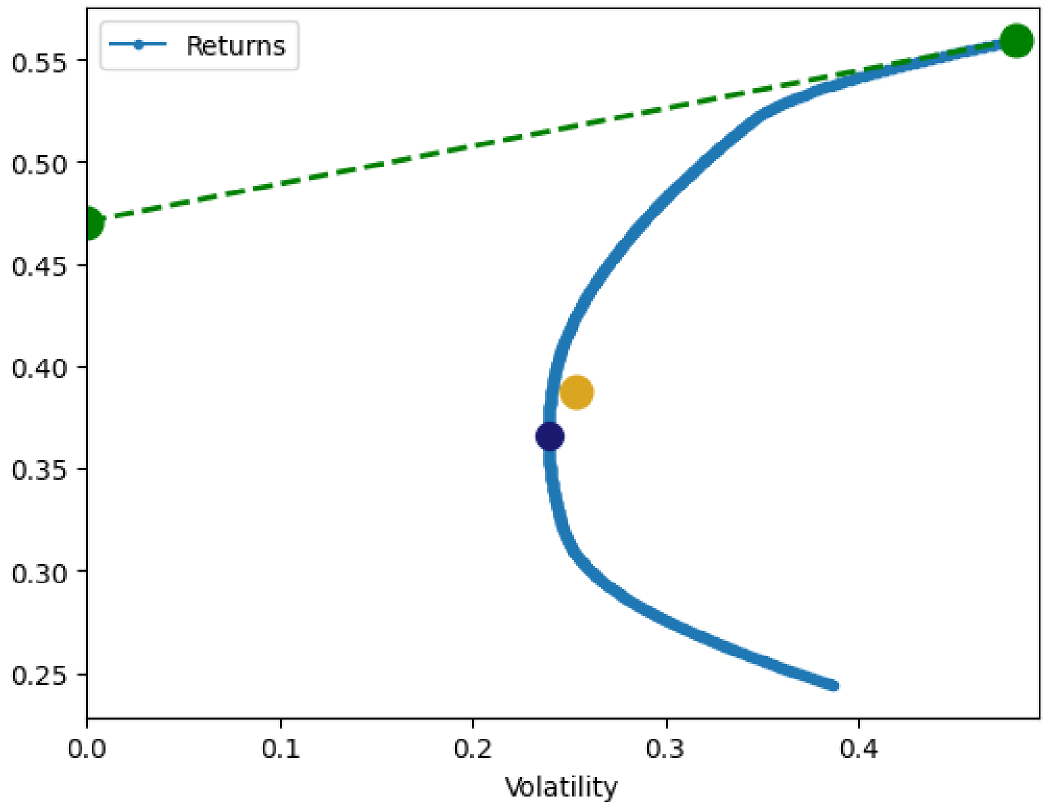
[174]:
```
Ticker
GARAN.IS    0.387340
ASELS.IS    0.377558
TUPRS.IS    0.340309
BIMAS.IS    0.302693
FROTO.IS    0.372812
SASA.IS     0.481874
dtype: float64
```

[175]:
```
sharpe_ratio(returns[selected_stocks], riskfree_rate=annual_rf,␣
 ↪periods_per_year=periods_per_year) # Sharpe Ratio of the selected stocks,␣
 ↪which measures the risk-adjusted return of the portfolio
```

[175]: Ticker
```
GARAN.IS   -0.398795
ASELS.IS    0.030633
TUPRS.IS   -0.272840
BIMAS.IS   -0.320483
FROTO.IS   -0.182219
SASA.IS     0.125396
dtype: float64
```

[176]:
```
plot_ef(n_points=1000, er=er, cov=cov,
           riskfree_rate=annual_rf, show_ew=True, show_gmv=True, show_cml=True)␣
 ↪# Plotting the Efficient Frontier (EF) with the selected stocks
# The plot shows the efficient frontier, the equal-weighted portfolio, the␣
 ↪global minimum variance portfolio
```

[176]: <Axes: xlabel='Volatility'>



[177]:
```
weights_gmv = gmv(cov) # Calculating the Global Minimum Variance (GMV)␣
 ↪portfolio weights
weights_msr = msr(riskfree_rate=annual_rf, er=er, cov=cov) # Calculating the␣
 ↪Maximum Sharpe Ratio (MSR) portfolio weights
```

```
print("\nGMV Weights", weights_gmv, "\n\nMSR Weights", weights_msr)
```

```
GMV Weights [0.07249995 0.13720329 0.20286335 0.39415326 0.12624865 0.0670315 ]

MSR Weights [4.51028104e-17 0.00000000e+00 2.42861287e-17 0.00000000e+00
 3.46944695e-16 1.00000000e+00]
```

[178]:

```
gmv_return = portfolio_return(weights_gmv, er)
gmv_vol = portfolio_vol(weights_gmv, cov)
msr_return = portfolio_return(weights_msr, er)
msr_vol = portfolio_vol(weights_msr, cov)

print("GMV Portfolio Return:", gmv_return)
print("GMV Portfolio Volatility:", gmv_vol)
print("MSR Portfolio Return:", msr_return)
print("MSR Portfolio Volatility:", msr_vol)
```

```
GMV Portfolio Return: 0.3662516773480772
GMV Portfolio Volatility: 0.23936359259400983
MSR Portfolio Return: 0.5596992448262768
MSR Portfolio Volatility: 0.4818736585393083
```

2.2 Min Risk Portfolio

[179]:

```
selected_stocks_min_risk = []
for cl in clusters_df['Cluster'].unique():
    stocks_in_cluster = clusters_df[clusters_df['Cluster'] == cl]['Stock']
    volatilities = annualize_vol(returns[stocks_in_cluster], periods_per_year)
    min_risk_stock = volatilities.idxmin()
    selected_stocks_min_risk.append(min_risk_stock)
print( selected_stocks_min_risk)
#   The selected_stocks_min_risk list contains the representative stocks for␣
 ↪each cluster based on the lowest annualized volatility.
```

```
['ISCTR.IS', 'ALARK.IS', 'TCELL.IS', 'BIMAS.IS', 'FROTO.IS', 'SASA.IS']
```

[180]:

```
er_min_risk = annualize_rets(returns[selected_stocks_min_risk],␣
 ↪periods_per_year)
cov_min_risk = returns[selected_stocks_min_risk].cov() * periods_per_year
print(er_min_risk)
```

```
Ticker
ISCTR.IS    0.240287
ALARK.IS    0.297335
TCELL.IS    0.199129
BIMAS.IS    0.328331
FROTO.IS    0.371083
SASA.IS     0.559699
```

```
dtype: float64
```

[181]:
```
annalize_vol(returns[selected_stocks_min_risk],␣
    ↪periods_per_year=periods_per_year)
```

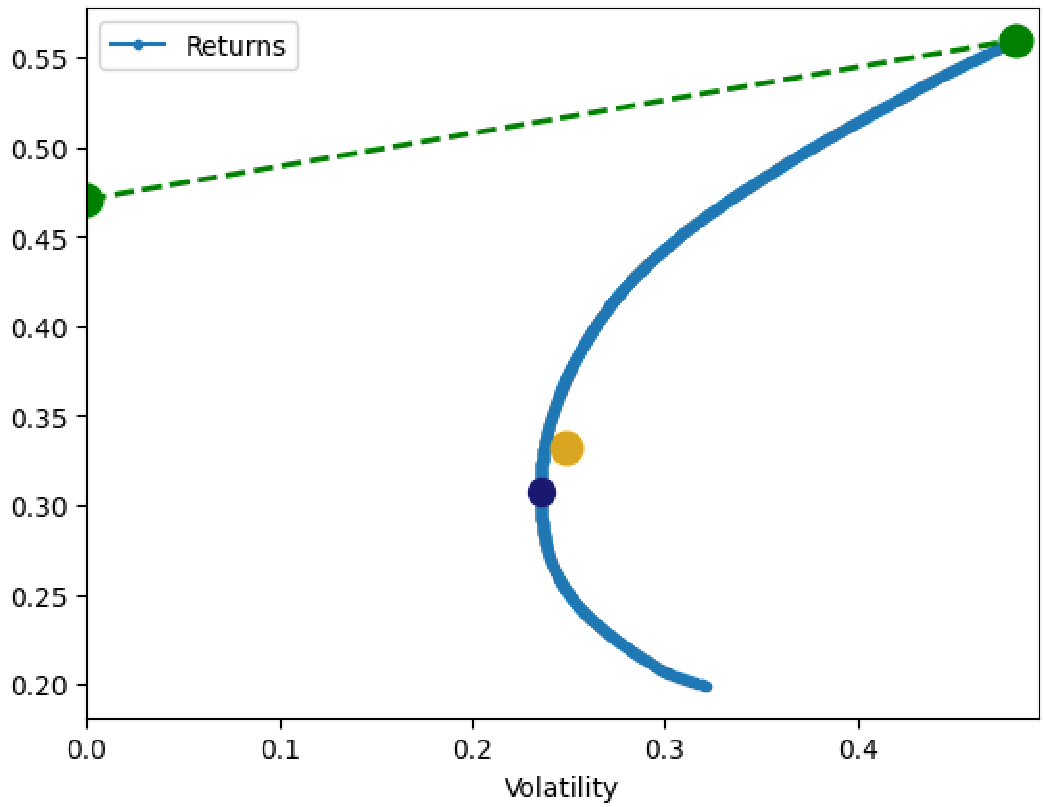
[181]:
```
Ticker
ISCTR.IS    0.375020
ALARK.IS    0.367545
TCELL.IS    0.321979
BIMAS.IS    0.302693
FROTO.IS    0.372812
SASA.IS     0.481874
dtype: float64
```

[182]:
```
sharpe_ratio(returns[selected_stocks_min_risk], riskfree_rate=annual_rf,␣
    ↪periods_per_year=periods_per_year)
```

[182]:
```
Ticker
ISCTR.IS   -0.418590
ALARK.IS   -0.321431
TCELL.IS   -0.574491
BIMAS.IS   -0.320483
FROTO.IS   -0.182219
SASA.IS     0.125396
dtype: float64
```

[183]:
```
plot_ef(n_points=1000, er=er_min_risk, cov=cov_min_risk,
           riskfree_rate=annual_rf, show_ew=True, show_gmv=True, show_cml=True)
```

[183]:
```
<Axes: xlabel='Volatility'>
```



[184]:
```
weights_gmv_min_risk = gmv(cov_min_risk)
weights_msr_min_risk = msr(riskfree_rate=annual_rf, er=er_min_risk,␣
 ↪cov=cov_min_risk)
print("GMV Weights Min Risk:\n", weights_gmv_min_risk, "\n\nMSR Weights Min␣
 ↪Risk:\n", weights_msr_min_risk)
```

```
GMV Weights Min Risk:
 [0.05783126 0.14180785 0.25190803 0.35509755 0.12391332 0.069442  ]

MSR Weights Min Risk:
 [1.21430643e-17 0.00000000e+00 2.25514052e-17 8.50014503e-17
 0.00000000e+00 1.00000000e+00]
```

[185]:
```
gmv_return_min_risk  = portfolio_return(weights_gmv_min_risk, er_min_risk) #␣
 ↪Expected return of the GMV portfolio
gmv_vol_min_risk = portfolio_vol(weights_gmv_min_risk, cov_min_risk) # Expected␣
 ↪volatility of the GMV portfolio
msr_return_min_risk = portfolio_return(weights_msr_min_risk, er_min_risk) #␣
 ↪Expected return of the MSR portfolio
msr_vol_min_risk = portfolio_vol(weights_msr_min_risk, cov_min_risk) # Expected␣
 ↪volatility of the MSR portfolio
```

```
print("GMV Portfolio Return:", gmv_return_min_risk)
print("GMV Portfolio Volatility:", gmv_vol_min_risk)
print("MSR Portfolio Return:", msr_return_min_risk)
print("MSR Portfolio Volatility:", msr_vol_min_risk)
```

```
GMV Portfolio Return: 0.30766094582648307
GMV Portfolio Volatility: 0.23534676596851387
MSR Portfolio Return: 0.5596992448262768
MSR Portfolio Volatility: 0.48187365853930836
```

[186]:
```
portfolio_metrics = {
    "Max Ret. GMV Port.": {
        "Return": gmv_return,
        "Risk": gmv_vol
    },
    "Max Ret. MSR Port.": {
        "Return": msr_return,
        "Risk": msr_vol
    },
    "Min Vol. GMV Port.": {
        "Return": gmv_return_min_risk,
        "Risk": gmv_vol_min_risk
    },
    "Min Vol. MSR Port.": {
        "Return": msr_return_min_risk,
        "Risk": msr_vol_min_risk
    }
}
portfolio_df = pd.DataFrame(portfolio_metrics).T
portfolio_df.columns = ["Return", "Risk"]
print(portfolio_df)
# The portfolio_df DataFrame contains the expected returns and risks of the␣
↪portfolios
```

```
                      Return      Risk
Max Ret. GMV Port.  0.366252  0.239364
Max Ret. MSR Port.  0.559699  0.481874
Min Vol. GMV Port.  0.307661  0.235347
Min Vol. MSR Port.  0.559699  0.481874
```

[187]:
```
plt.figure(figsize=(12, 7))
colors = ['blue', 'green', 'orange', 'red']
plt.scatter(portfolio_df["Risk"], portfolio_df["Return"], s=200, c=colors,␣
↪edgecolors='k')
offsets = {
    "Max Ret. GMV Port.": (0, 10),
    "Max Ret. MSR Port.": (-70, -20),
    "Min Vol. GMV Port.": (0, 10),
```

```
    "Min Vol. MSR Port.": (70, -20)
}
for idx in portfolio_df.index:
    plt.annotate(idx,
                 (portfolio_df.loc[idx, "Risk"], portfolio_df.loc[idx,␣
 ↪"Return"]),
                 textcoords="offset points",
                 xytext=offsets[idx],
                 ha='center',
                 fontsize=11,
                 fontweight='bold',
                 arrowprops=dict(arrowstyle="->", color='gray', lw=0.5))
plt.title('Portfolios: Return vs Risk', fontsize=16)
plt.xlabel('Annualized Volatility (Risk)', fontsize=14)
plt.ylabel('Annualized Return', fontsize=14)
plt.xlim(left=0)
plt.ylim(bottom=0)
plt.grid(True, linestyle='--', alpha=0.7)
plt.show()

# The final plot shows the portfolios with their respective returns and risks,␣
 ↪annotated with their names.
# Each point represents a portfolio, and the colors differentiate them.
```