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Task 1:

From the solutions we know that,

$$CLRB = \sigma^2 * \frac{1}{\sum_{n=0}^{N-1} r^{2n}}, \ \widehat{A} = \frac{\sum_{n=0}^{N-1} x[n] * r^n}{\sum_{n=0}^{N-1} r^{2n}}$$

In the matlab code we compute R matrix where it's the row matrix of r^n because we will use it multiple times. Then, we calculate the $CRLB$ according to formula above with the given values. Initialize the estimations matrix and start the Monte-Carlo simulation.

In each loop we simulate X with the given model $x[n] = A * r^n + w[n]$. Then feed these X matrix to our estimator $\widehat{A}$ and store them in a row matrix. Then, we calculate the variance of our estimated values, and compare it with $CRLB$.

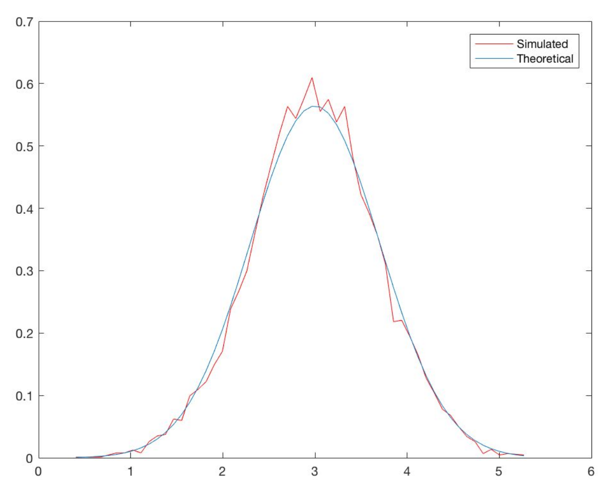
In our tests, we saw that the variance of the estimator is very close to $CRLB$. Also, to check if our estimator is unbiased:

$$E(\widehat{A}) = E\left(\frac{\sum_{n=0}^{N-1} x[n] * r^n}{\sum_{n=0}^{N-1} r^{2n}}\right) = \frac{\sum_{n=0}^{N-1} E(x[n]) * r^n}{\sum_{n=0}^{N-1} r^{2n}} = \frac{\sum_{n=0}^{N-1} A * r^n * r^n}{\sum_{n=0}^{N-1} r^{2n}} = A * \frac{\sum_{n=0}^{N-1} r^{2n}}{\sum_{n=0}^{N-1} r^{2n}} = A$$

$$bias = \frac{1}{M} * \sum_{i=1}^{M} (\widehat{A} - A_{true})$$

We also tested the bias of the estimator in our matlab code using bias equation and it comes out as close to 0 and increasing Monte-Carlo loop count yields in much more smaller values.

In conclusion, we see that our estimator $\widehat{A}$ reaches $CRLB$ in the Monte-Carlo simulation. For our estimator to be unbiased $E(\widehat{A}) = A$ condition satisfies.

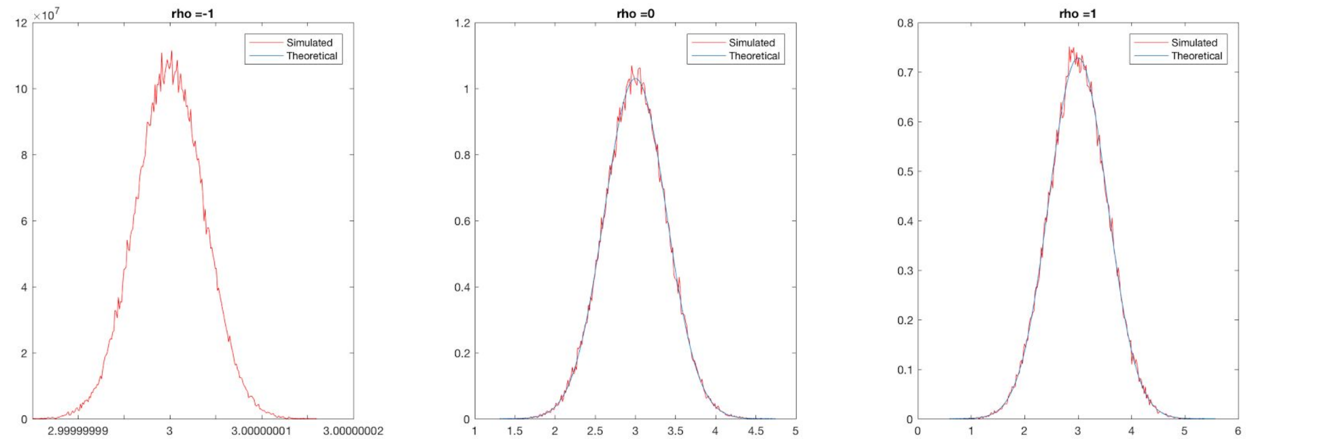


Task 2:

In the code we first find the variance and the estimator using the symbolic toolbox and we get the simplified version of these equations as follows:

$$var(\widehat{A}) = \frac{\sigma^2 * (\rho + 1)}{2}, \ \widehat{A} = \frac{x[0] + x[1]}{2}$$

Then we perform Monte-Carlo simulation where we compute correlated gaussian noise using Cholesky Decomposition[1] for different values of $\rho = [-1, \ 0, \ 1]$ and calculate the estimated value of A and plot 3 theoretical and simulated pdf for these three values of ρ.



When $\rho = 1$, the noise is positively correlated and the variance is σ^2.

When $\rho = 0$, correlation matrix is identity matrix, thus the noise will be uncorrelated. So our variance becomes $\frac{\sigma^2}{N} = \frac{\sigma^2}{2}$ which we can recall from the earlier examples where we had same model but when we were dealing with uncorrelated WGN.

When $\rho = -1$, the noise becomes negatively correlated meaning that it will cancel on the summation $x[0] + x[1]$ so the average of this summation gives us the exact estimate for A.

$$\widehat{A} = \frac{s[0]+s[1]+w[0]+w[1]}{2}$$

$$when: \ w[0] = -w[1], \ \widehat{A} = \frac{s[0]+s[1]}{2} = mean(S)$$

Task 3:

$$f(x;\theta) = \frac{1}{\theta},\ x \in [0,\ \theta]$$
$$f(x;\theta) = 0,\ x \notin [0,\ \theta]$$

Let x belong to order statistics where $x_{(0)} \leq x_{(1)} \leq ... \leq x_{(n)}$, then the likelihood function is:

$$L(\theta;x) = \prod_{n=1}^{n} \frac{1}{\theta} = \theta^{-n}$$

In our matlab code we have calculated MLE estimator for the likelihood of IID $U[0,\ \theta]$. Let us make a small recap here:

$$ln(L(\theta;x)) =- nln(\theta),\ \frac{\delta ln(L(\theta;x))}{\delta\theta} =- \frac{n}{\theta}$$
$$-\frac{n}{\theta} = 0$$

Therefore; we conclude that to this equation will approach to 0 when θ is as big as possible. Since $\theta \geq x_{(n)}$ is known and $L(\theta;x)$ maximized at $\theta = x_{(n)}$, MLE is:

$$\widehat{\theta} = x_{(n)}$$

For MLE estimator, in $U(0,\ 1)$, $E(x_{(k)}) = \frac{k}{n+1}$, but since we are working on $U(0,\theta)$ expectation becomes $E(x_{(k)}) = \frac{k}{n+1} * \theta$:

$$E(\widehat{\theta}) = E(x_{(n)}) = \frac{n}{n+1} * \theta \Rightarrow \lim_{n \to +\infty} \frac{n}{n+1} * \theta = \theta$$

If the n is large enough the bias goes to 0 so MLE estimator is <u>asymptotically unbiased</u>.

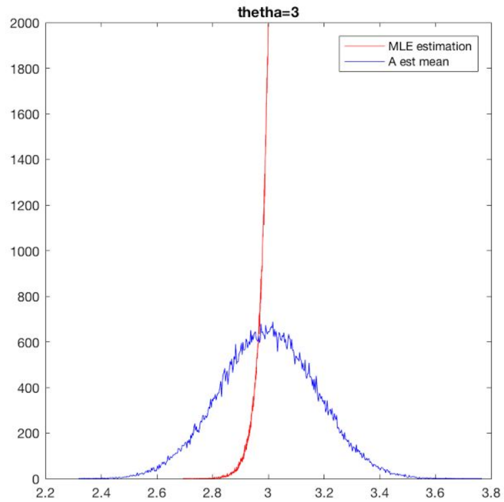
For the second estimator:

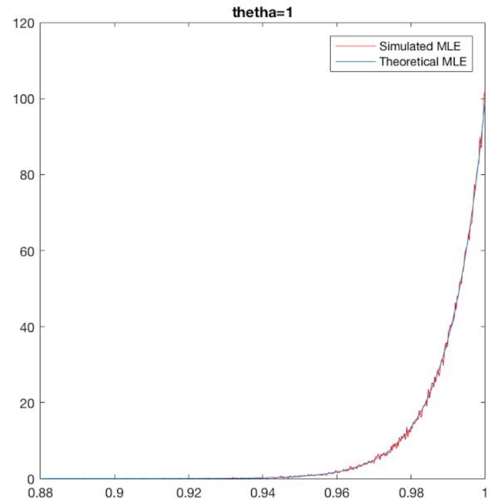
$$A\_est\_mean = \bar{x} * 2 \Rightarrow E(A\_est\_mean) = E(\bar{x}) * 2$$
$$E(A\_est\_mean) = \frac{\theta}{2} * 2 = \theta$$

So we see that the second estimator is unbiased as well.

In our matlab code we have compared the bias values and variances of the both estimators, while the bias was higher for MLE estimator, the variance was much lower. So we conclude the MLE estimator is better than A_est_mean estimator.

When it is given that $U(0,1)$ and $x_{(k)}$ is the kth orders statistic from the sample, then probability density distribution of $x_{(k)}$ is a beta distribution[2] with parameters k and $n-k+1$. In our problem $k = n$ so we obtain $Beta(n,\ 1)$ as a theoretical pdf. But we were unable to generalize this for $U(0,\ \theta)$. So in our matlab code we plotted the case where $\theta = 1$ as a separate plot. Unfortunately we were not able to come up with a pdf of the second estimator.





References:

[1] Matlab randn documentation, mathworks.com, retrieved on 02.10.2017.
[2] Gentle, James E. (2009), Computational Statistics, Springer, p. 63, ISBN 9780387981444.