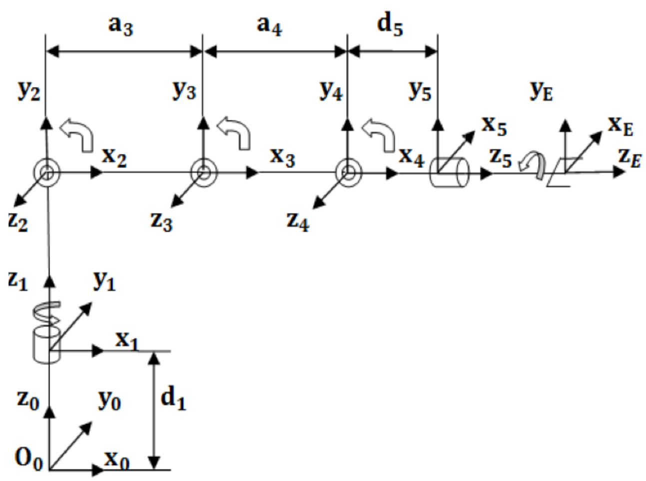
6 Degree of Freedom Robotic Arm

FORWARD KINEMATICS

Table 1: The DH parameters of the DFROBOT

Joint	Link	a_{i-1} min	α_{i-1} degree	d_i mm	θi degree
0-1	1	0	0	45	θ_1
1-2	2	0	90	0	θ_2
2-3	3	90	0	0	θ_3
3-4	4	90	0-90	0	θ_4
4-5	5	0	-90	30	θ_5
5-6	6	0	0	0	gripper



The robot coordinate frame

$$Ai = \begin{bmatrix} C\theta i & -S\theta iC\alpha i & S\theta iS\alpha i & aiC\theta i \\ S\theta i & C\theta iC\alpha i & -C\theta iS\alpha i & aiS\theta i \\ 0 & S\alpha i & C\alpha i & di \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where,

a_i : The length distance from z_i to z_{i+1} measured along z_i

α_i: The twist angle between z_i and z_{i+1} measured about x_i

d_i: The offset distance from x_i to x_{i+1} measured along z_i

θ_i: The angle between x_i and x_{i+1}measured about z_i

The transformation matrices for A1 -> A6:

$$A_1^0 = A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2^1 = A_2 = \begin{bmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3^2 = A_3 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4^3 = A_4 = \begin{bmatrix} C_4 & -S_4 & 0 & a_4C_4 \\ S_4 & C_4 & 0 & a_4S_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5^4 = A_5 = \begin{bmatrix} C_5 & 0 & -S_5 & 0 \\ S_5 & 0 & C_5 & 0 \\ 0 & -1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6^5 = A_6 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$S_i = \sin\theta_i$ $\quad C_i = \cos\theta_i$

Multiplying the six transformation matrices to find the nth frame of the base:

$$A_n^0 = A_1^0 \, . \, A_2^1 \, \ldots\ldots \, A_n^{n-1} = \prod_{i=1}^{n} \, A_i^{i-1} = \begin{bmatrix} R_n^0 & P_n^0 \\ 0 & 1 \end{bmatrix}$$

Rn: matrix is a 3x3 matrix represent the rotation.
Pn: is a 3x1matrix represent the position in the three axis x, y, z.

$$A_6^0 = A_1^0 * A_2^1 * A_3^2 * A_4^3 * A_5^4 * A_6^5$$
$$= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$nx = \cos(\theta_6)\cos(\theta_1+\theta_2) - \sin(\theta_6)\sin(\theta_1+\theta_2)$$

$$ny = \cos(\theta_6)\sin(\theta_1+\theta_2)\cos(\theta_3+\theta_4+\theta_5) + \cos(\theta_1+\theta_2) \cdot \sin(\theta_6)$$

$$nz = \cos(\theta_6)\sin(\theta_3+\theta_4+\theta_5)$$

$$Ox = -\cos(\theta_1)\sin(\theta_6)\cos(\theta_3+\theta_4+\theta_5) - \sin(\theta_1+\theta_2)\cos(\theta_6)$$

$$Oy = -\sin(\theta_6)\sin(\theta_1+\theta_2)\cos(\theta_3+\theta_4+\theta_5) + \cos(\theta_1+\theta_2)\cos(\theta_6)$$

$$Oz = -\sin(\theta_6) \cdot \sin(\theta_3+\theta_4+\theta_5)$$

$$ax = -\cos(\theta_1+\theta_2) \cdot \sin(\theta_3+\theta_4+\theta_5)$$

$$ay = -\sin(\theta_1+\theta_2)\sin(\theta_3+\theta_4+\theta_5)$$

$$az = \cos(\theta_3 + \theta_4 + \theta_5)$$

Position matrix:

$$
\begin{aligned}
Px = {} & a_4 \cos(\theta_1+\theta_2)\cos(\theta_3)\cos(\theta_4) \\
& - a_4 \cos(\theta_1+\theta_2)\sin(\theta_3)\sin(\theta_4) \\
& + \sin(\theta_1+\theta_2)\, d_5 \\
& + a_3 \cos(\theta_1+\theta_2)\cos(\theta_3
\end{aligned}
$$

$$
\begin{aligned}
Py = {} & a_4 \sin(\theta_1+\theta_2)\cos(\theta_3)\cos(\theta_4) \\
& - a_4 \sin(\theta_1+\theta_2)\sin(\theta_3)\sin\theta_4 \\
& - \cos(\theta_1+\theta_2)\, d_5 \\
& + a_3 \sin(\theta_1+\theta_2)\cos(\theta_3)
\end{aligned}
$$

$$
\begin{aligned}
Pz = {} & a_4 \sin(\theta_3)\cos(\theta_4) + a_4 \cos(\theta_3)\sin(\theta_4) \\
& + a_3 \sin(\theta_3) + d_1
\end{aligned}
$$

INVERSE KINEMATICS

In the inverse kinematic the final position is known but we need to calculate the joint angles.

$$A_6^0 = A_1^0 * A_2^1 * A_3^2 * A_4^3 * A_5^4 * A_6^5$$

1- Find the inverse matrix of all six transformation matrices.

$$A_1^{-1} = \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ -S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2^{-1} = \begin{bmatrix} C_1 & S_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S_2 & -C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3^{-1} = \begin{bmatrix} C_3 & S_3 & 0 & -a_3 \\ -S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4^{-1} = \begin{bmatrix} C_4 & S_4 & 0 & -a_4 \\ -S_4 & C_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5^{-1} = \begin{bmatrix} C_5 & S_5 & 0 & 0 \\ 0 & 0 & -1 & d_5 \\ -S_5 & C_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6^{-1} = \begin{bmatrix} C_6 & S_6 & 0 & 0 \\ -S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2- Find the joint angles $\theta_1, \theta_2, \theta_6$:

$$A_2^{-1} \cdot A_1^{-1} \cdot A_6^{0} = A_3 \cdot A_4 \cdot A_5 \cdot A_6$$

$$\cos\theta_2 \, P_x - \cos\theta_2 \, P_y - d_1 = d_5$$

$$P_x \sin\theta_2 - P_y \cos(\theta_2) = d_5 + d_1$$

Using Pythagoras and trigonometric relationship:

$$\sin(\theta + A) = \frac{(d_5 + d_1)\sqrt{Px^2 + Py^2}}{Px^2 + Py^2}$$

$$A = \tan^{-1}\left(\frac{-Py}{Px}\right)$$

$$\therefore \theta_2 = \sin^{-1}\left(\frac{(d_5+d_1)\sqrt{Px^2+Py^2}}{Px^2+Py^2}\right) - \tan^{-1}\left(-Py/Px\right)$$

Finding θ_1:

$$ax = -\cos(\theta_1 + \theta_2)\sin(\theta_3 + \theta_4 + \theta_5)$$
$$ay = -\sin(\theta_1 + \theta_2)\sin(\theta_3 + \theta_4 + \theta_5)$$

$$\theta_{12} = \tan^{-1}\left(\frac{ay}{ax}\right) \qquad \therefore \theta_1 = \theta_{12} - \theta_2$$

Finding θ_6:

$$-\sin\theta_6 = \sin\theta_2\, nx - \cos\theta_2\, ny$$
$$-\cos\theta_6 = \sin\theta_2\, ox - \cos\theta_2\, oy$$

$$\therefore \theta_6 = \tan^{-1}\left(\frac{-\cos\theta_2\, ny + \sin\theta_2\, nx}{-\cos\theta_2\, oy + \sin\theta_2\, ox}\right)$$

Finding θ_3

$$A_3^{-1} \cdot A_1^{-1} \cdot A_2^{-1} \cdot A_6^{0} = A_4 \cdot A_5 \cdot A_6$$

$$\sin(\theta_2 + \theta_3)\, px - \cos(\theta_2 + \theta_3) - a_3 \sin\theta_2 - d_1 = d_5$$

$$\sin(\theta_2+\theta_3)p_x - \cos(\theta_2+\theta_3)p_y = d_5 + a_3\sin\theta_2 + d_1$$

$$\theta_{23} = \tan^{-1}\left(-\frac{p_y}{p_x}\right) \mp \tan^{-1}\left(\frac{\sqrt{p_x^2+p_y^2-(a_3S_2+d_1+d_5)^2}}{(a_3S_2+d_1+d_5)}\right)$$

$$\therefore \theta_3 = \theta_{23} - \theta_2$$

Finding θ_4: $A_1^{-1} \cdot A_6^0 = A_1 \cdot A_2 A_3 \cdot A_4 A_5 A_6 A_1^{-1}$

$$p_z - d_1 = a_4\sin(\theta_3+\theta_4) + a_3\sin\theta_3$$

$$\theta_{34} = \sin^{-1}\left(\frac{-a_3\sin(\theta_3) + p_z - d_1}{a_4}\right)$$

$$\theta_4 = \theta_{34} - \theta_3$$

Finally θ_5: $a_z = \cos(\theta_3+\theta_4+\theta_5)$

$$\theta_{345} = \cos^{-1}(a_z)$$

$$\therefore \theta_5 = \theta_{345} - \theta_3 - \theta_4$$