Homotopy Analysis Method (HAM) in Neuroscience

serena di santo

(Dated: March 2021)

CONTENTS

I. INTRODUCTION TO HAM

We are interested in finding an analytic (approximate) solution for a nonlinear (differential or algebraic) problem. HAM was proposed by Shijun Liao in 1992. One good reference to learn more about it is Shijun Liao's book Homotopy Analysis Method in Nonlinear Differential Equations, Springer 2011.

HAM is based on the concept of **homotopy** in topology. Two continuous functions are homotopic if one can be **continuously deformed** into the other. The classical example is that a mug and a doughnut are the same thing in a topology because they can be deformed into one another. In the same way a sphere, a cube and a pyramid are the same thing. However a doughnut cannot be continuously deformed into a sphere.

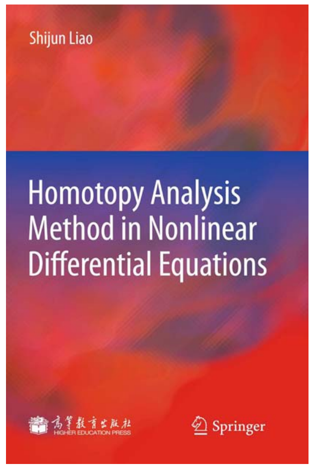

FIG. 1. Book cover

A. The pipeline

The idea is the following:

- choose a linear problem to start with, solve it.
- define a homotopy that transforms the nonlinear problem into the linear one as a parameter q (homotopy parameter) varies continuously from 0 to 1.
- the solution of the homotopy can be written as a series of the parameter q. The m-th coefficient of the series can be found by solving a linear problem and gives the m-th order approximation of the solution of the nonlinear problem.
- ensure that the series converges when $q = 1$.

B. The advantages

The advantages of HAM are:

- it is independent of any small/large physical parameter (unlike other perturbative methods).
- it provides a convenient way to guarantee the convergence of the solution series even with very strong nonlinearities
- provides a great freedom in the choice of the initial problem. For a smarter the choice, the expression of the m-th order solution will be simpler and the series will converge faster.

II. FIRST STEPS WITH HAM

Given a continuous (nonlinear) equation

$$\mathcal{N} = 0 \tag{1}$$

(let's assume that it has at least one solution). Let $\mathcal{L}(x)$ denote an auxiliary (linear) equation with the property

$$\mathcal{L}(x_0) = 0 \tag{2}$$

(meaning that its solution is x_0). If $\mathcal{N}$ can be deformed continuously into a $\mathcal{L}$, one can construct a homotopy:

$$\mathcal{H}(x, q) = (1-q)\mathcal{L}(x) + q\mathcal{N}(x) \tag{3}$$

$q \in [0, 1]$ is called the embedding parameter or the homotopy parameter.

We will choose $\mathcal{L}(x)$ later, for now we just note that we have a great freedom to choose it (so that there exist an homotopy between $\mathcal{N}$ and $\mathcal{L}$). We can notice that when we construct the homotopy as in Eq.3, we have

$$\begin{aligned} \mathcal{H}(x, 0) &= \mathcal{L}(x) \\ \mathcal{H}(x, 1) &= \mathcal{N}(x) \end{aligned} \tag{4}$$

Thus, as q increases from 0 to 1, the homotopy $\mathcal{H}(x, q)$ continuously changes (or deforms) from the nonlinear function $\mathcal{N}(x)$ to the linear function $\mathcal{L}(x)$.

Then enforcing

$$\mathcal{H}(x, q) = 0 \tag{5}$$

we have a family of equations

$$(1-q)\mathcal{L}(\tilde{x}(q)) + q\mathcal{N}(\tilde{x}(q)) = 0 \tag{6}$$

where we have made explicit that we have a family of solutions that vary continuously from the solution of the linear problem to the solution of the nonlinear problem as q varies continuously from 0 to 1. Indeed if we set $q = 0$ we have $\mathcal{L}(\tilde{x}(0)) = 0$, and comparing with Eq.2 we find

$$\tilde{x}(0) = x_0 \tag{7}$$

and when we set $q = 1$ we have $\mathcal{N}(\tilde{x}(1)) = 0$

$$\tilde{x}(1) = x \tag{8}$$

that is the solution of our initial nonlinear problem. Now let us expand $\tilde{x}(q)$ in a Taylor series around $q = 0$:

$$\begin{aligned} \tilde{x}(q) &= x_0 + \sum_{n=1}^{\infty} x_n q^n \\ x_n &= \frac{1}{n!} \frac{\partial^n \tilde{x}(q)}{\partial q^n}\bigg|_{q=0} \end{aligned} \tag{9}$$

Therefore, if Eq.9 converges in $q = 1$, then a full solution for our initial nonlinear problem, called the homotopy series solution will be given by:

$$x = x_0 + \sum_{n=1}^{\infty} x_n \tag{10}$$

and an m-th order approximation to the solution is given by:

$$x \simeq x_0 + \sum_{n=1}^{m} x_n. \tag{11}$$

We can calculate the coefficients x_n either differentiating Eq.6 n times with respect to q and then setting $q = 0$, or alternatively (i find this method more intuitive), by substituting the Taylor series solution back into Eq.6 and then equating the coefficients of the like powers of q. It is easy to see and important to note that the resulting equations are linear. Therefore, as anticipated, we have transformed the solution of a nonlinear problem to the solution of an infinite number of linear problems. But how do we make sure that the series converges in $q = 1$? It can be shown that this convergence can be ensured by introducing a convergence parameter c_0 and with an appropriate choice of a convergence parameter c_0 and of the linear problem $\mathcal{L}(x)$. For a thorough demonstration please see the referenced book, e.g. Chapter2, while here we will offer empirical proof of convergence for the examples considered. Moreover the great freedom that we have in the choice of the homotopy, allows to optimize the choice of c_0 and $\mathcal{L}(x)$ so that the solution converges in as little as the first HAM order, i.e. $x \simeq x_0 + x_1$ to a good approximation.

the author of the book shared a mathematica code to solve nnlin problems through HAM.

III. FIRST HAM PROBLEM IN NEUROSCIENCE: EI RATE SYSTEM WITH QUADRATIC TRANSFER FUNCTION

We set up a supralinear mean field system of E and I populations with:

$$\dot{r} = -r + \phi(Wr + h) \tag{12}$$

where, $r = (r_E, r_I)$ are the firing rates of the 2 populations, W is the connectivity matrix, h is the external current and ϕ is a nonlinear function, which in this section we fix to $\phi(x) = x^2$ The steady state solution reads:

$$r = \phi(Wr + h) \tag{13}$$

Equivalently we can write a self-consistency equation for the currents u, with $r = \phi(u)$:

$$u = W\phi(u) + h \tag{14}$$

. This is our nonlinear problem $\mathcal{N}$:

$$\mathcal{N}(u) = -u + Wu^2 + h = 0. \tag{15}$$

One obvious choice for the auxiliary linear problem is:

$$\mathcal{L}(u_0) = -u_0 + Wu_0 + h = 0. \tag{16}$$

Thus we can define the homotopy:

$$\begin{aligned} \mathcal{H}(\tilde{u}(q)) &= (1-q)\mathcal{L}(\tilde{x}(q)) + q\mathcal{N}(\tilde{x}(q)) = 0 \\ &= (1-q)(-\tilde{u}(q) + W\tilde{u}(q) + h) + q(-\tilde{u}(q) + W\tilde{u}(q)^2 + h) = 0 \end{aligned} \tag{17}$$

Now, substituting the series $\tilde{u}(q) = \sum_{m=0}^{\infty} u_k q^k = u_0 + u_1 q + u_2 q^2 + ...$ we get:

$$\begin{aligned} &(1-q)[-(u_0 + u_1 q + u_2 q^2 + ...) + W(u_0 + u_1 q + u_2 q^2 + ...) + h] + \\ &+q[-(u_0 + u_1 q + u_2 q^2 + ...) + W(u_0 + u_1 q + u_2 q^2 + ...)^2 + h] = 0 \end{aligned} \tag{18}$$

and equating the terms with the same power of q and setting $q = 0$ we get:

$$\begin{aligned} -u_0 + Wu_0 + h + 0 &= 0 \\ &\Rightarrow u_0 = (1-W)^{-1}h \\ -u1 + Wu_1 + u_0 - Wu_0 - h - u_0 + Wu_0^2 + h &= 0 \\ &\Rightarrow u_1 = (1-W)^{-1}(-u_0 + Wu_0^2 + h) \\ -u_2 + W_u 2 + u_1 + Wu_1 - u_1 + 2Wu_0u_1 &= 0 \\ &\Rightarrow u_2 = (1-W)^{-1}(Wu_1 + 2Wu_0u_1) \\ -u_3 + Wu_3 + u_2 - Wu_2 - u_2 + Wu_1^2 + 2Wu_0u_2 &= 0 \\ &\Rightarrow u_3 = (1-W)^{-1}(-Wu_2 + Wu_1^2 + 2Wu_0u_2) \end{aligned} \tag{19}$$

Therefore we have an approximate solution to the nonlinear problem (up to third order):

$$u \simeq u_0 + u_1 + u_2 + u_3. \tag{20}$$

The file HAM-handson.nb shows that HAM1 can be a good solution already for system with $\alpha \sim 1$ where α is controlling the relative weight of recurrent versus feedforward contributions. Low order solutions can be very bad for small α (i.e. feedforward dominated regime). Conversely, if we start from the purely feedforward linear problem $\mathcal{L} = -u_0 + h = 0$ we observe a quick convergence for small α. What is guaranteed is that the full infinite series HAM∞ is the solution. If we find a good linear problem to start with (and eventually a good convergence parameter c_0), we can optimize the chances that low order HAM solution is a good solution, as in this case.

IV. SECOND HAM PROBLEM IN NEUROSCIENCE: EI RATE SYSTEM WITH GENERIC TRANSFER FUNCTION

Here we illustrate a variation of the HAM method, by applying it to solve an EI system with generic transfer function. This will give the reader an idea of the flexibility of the method. Given the nonlinear problem

$$\mathcal{N}(u) = -u + W\phi[u] + h = 0, \tag{21}$$

with solution u, and given an initial guess for our problem:

$$u_0 = h(1-W)^{-1} \tag{22}$$

we can define an homotopy $\mathcal{H}(u,q)$ and set it to 0:

$$\mathcal{H}(\tilde{u},q) = (1-q)[\phi(\tilde{u}(q)) - \phi(u_0)] + qc_0\phi(\tilde{u}(q)) = 0 \tag{23}$$

Where we have added the constant c_0 which is called the convergence parameter. Notice that again:

$$\begin{aligned} q=0: &\quad \phi(\tilde{u}(0)) = \phi(u_0) \Rightarrow \tilde{u}(0) = u_0; \\ q=1: &\quad \phi(\tilde{u}(1)) = 0 \Rightarrow \tilde{u}(1) = u. \end{aligned} \tag{24}$$

Again, let us take the Taylor series of the solution of the homotopy equation: $\tilde{u}(q) = u_0 + \sum_{q=1}^{\infty} u_k q^k$, and let us take the first derivative with respect to q of Eq.23:

$$\begin{aligned} \frac{d}{dq}\mathcal{H}(\tilde{u},q) &= \frac{d}{dq}\left[(1-q+c_0q)\phi(\tilde{u}(q)) - \phi(u_0)\right] = 0 \\ &= (c_0-1)\phi(\tilde{u}) + \phi(u_0) + (1-q+c_0q)\phi'(\tilde{u})\frac{d\tilde{u}}{dq} = 0 \end{aligned} \tag{25}$$

where $\phi' = d\phi(\tilde{u})/d\tilde{u}$. If we now evaluate Eq25 in $q=0$ we obtain:

$$(c_0-1)\phi(u_0)+\phi(u_0) + \phi'(u_0)u_1 = 0 \Rightarrow u_1 = -\frac{c_0\phi(u_0)}{\phi'(u_0)} \tag{26}$$

where we used the definition $u_1 = d\tilde{u}/dq$ (see Eq.9). This is also called the Newtonian Iteration Formula. Thus we have the first order approximation of the solution $u \simeq u^{(1)} = u_0 + u_1$. Note that we just had to solve one linear problem. We can keep going and find the next order by taking the second derivative of Eq.23:

$$\begin{aligned} \frac{d^2}{dq^2}\mathcal{H}(\tilde{u},q) &= \\ = (c_0-1)\phi'(\tilde{u})\frac{d\tilde{u}}{dq} + (c_0-1)\phi'(\tilde{u})\frac{d\tilde{u}}{dq} &+ \\ +(1-q+c_0q)\phi''(\tilde{u})\left(\frac{d\tilde{u}}{dq}\right)^2 + (1-q+c_0q)\phi'(\tilde{u})\frac{d^2\tilde{u}}{dq^2} &= 0. \end{aligned} \tag{27}$$

Evaluating it in $q=0$ we have:

$$\begin{aligned} 2(c_0-1)\phi'(u_0)u_1 + \phi''(u_0)x_1^2 + 2\phi'(u_0)u_2 = 0 \\ \Rightarrow u_2 = -\frac{2(c_0-1)\phi'(u_0)u_1 + \phi''(u_0)u_1^2}{2\phi'(u_0)}. \end{aligned} \tag{28}$$

This is a generalization of the Newtonian Iteration Formula. But we can keep going, the third order gives:

$$\begin{aligned} 2(c_0-1)\phi''(\tilde{u})\left(\frac{d\tilde{u}}{dq}\right)^2 + 2(c_0-1)\phi(\tilde{u})\frac{d^2\tilde{u}}{dq^2}+ \\ +(c_0-1)\phi''(\tilde{u})\left(\frac{d\tilde{u}}{dq}\right)^2 + (1-q+c_0q)\phi'''(\tilde{u})\left(\frac{d\tilde{u}}{dq}\right)^3 + 2(1-q+c_0q)\phi''(\tilde{u})\frac{d^2\tilde{u}}{dq^2}+ \\ +(c_0-1)\phi'(\tilde{u})\frac{d^2\tilde{u}}{dq^2} + (1-q+c_0q)\phi''(\tilde{u})\frac{d^2\tilde{u}}{dq^2} + (1-q+c_0q)\phi'(\tilde{u})\frac{d^3\tilde{u}}{dq^3} \end{aligned} \tag{29}$$

and setting $q=0$ we get:

$$\begin{aligned} 3(c_0-1)\phi''(u_0)u_1^2 + 6(c_0-1)\phi'(u_0)u_2 + \phi'''(u_0)u_1^3 + 6\phi''(u_0)u_1u_2 + 6\phi'(u_0)u_3 = 0 \\ \Rightarrow u_3 = -\frac{3(c_0-1)\phi''(u_0)u_1^2 + 6(c_0-1)\phi'(u_0)u_2 + \phi'''(u_0)u_1^3 + 6\phi''(u_0)u_1u_2}{6\phi'(u_0)} \end{aligned} \tag{30}$$

A third order approximation of the solution of $\mathcal{N}(u)$ is $u \simeq u^{(3)} = u_0 + u_1 + u_3$. Note that at every perturbative order m the convergence constant c_0 can be chosen so that $\mathcal{N}u^{(m)}$ takes its minimum value. It can be shown that this guarantees the convergence of the series. Note that it is the great freedom in the choice of the homotopy that allows to introduce the convergence parameter.

V. THIRD HAM PROBLEM IN NEUROSCIENCE: 1 POPULATION RATE MODEL WITH A FEATURE SPACE

We consider a population of neurons $u(x)$ with a feature selectivity (e.g. respond to a stimulus in a certain region of the visual space or with a certain orientation preference). For simplicity we consider a quadratic transfer function. The equation for the dynamics reads:

$$\dot{u}(x) = -u(x) + \int_{-\infty}^{\infty} W(x-y)u^2(y) + h(x) \tag{31}$$

and the steady state equation is our nonlinear problem:

$$\mathcal{N} = -u(x) + \int_{-\infty}^{\infty} W(x-y)u^2(y) + h(x) = 0 \tag{32}$$

Let us assume that the connectivity decays with distance in the feature space as a Gaussian function and that the input is also a Gaussian function:

$$\begin{aligned} W(x-y) = W_0\mathcal{G}(x-y, \sigma_w) \\ h(x) = h_0\mathcal{G}(x, \sigma) \end{aligned} \tag{33}$$

Let us take $\mathcal{L}[u_0(x)] = -u_0(x) + \alpha\mathcal{G}(\sigma_u) = 0$. We keep the freedom to set α and σ_u later on. Analogously to Section III we define the homotopy:

$$\mathcal{H} = (1-q)(-\tilde{u}(x) + \mathcal{G}(\sigma)) + q(-\tilde{u}(x) + W_0\int_{-\infty}^{\infty}\mathcal{G}(\sigma_w)\tilde{u}^2(y) + h(x)) = 0 \tag{34}$$

where for brevity we omit to write the feature dependence on $\mathcal{G}$. We equate the terms with q^1 and set $q=0$:

$$\begin{aligned} u_0 - u_1 - \alpha\mathcal{G}(\sigma_u) - u_0(x) + W_0\alpha^2\int_{-\infty}^{\infty}\mathcal{G}(\sigma_w)\mathcal{G}^2(\sigma_u) + h_0\mathcal{G}(\sigma) = 0 \\ u_1(x) = -\alpha\mathcal{G}(\sigma_u) + \frac{\alpha^2 W_0}{2\sqrt{\pi\sigma_u}}\mathcal{G}(\sigma_w + \frac{\sigma_u}{2}) + h_0\mathcal{G}(\sigma) \end{aligned} \tag{35}$$

The first order solution is:

$$u^{(1)} = u_0 + u_1 = \frac{\alpha^2 W_0}{2\sqrt{\pi\sigma_u}}\mathcal{G}(\sigma_{wu}) + h_0\mathcal{G}(\sigma) \tag{36}$$

where we defined $\sigma_{wu}^{-1} = \sigma_w^{-1} + (\sigma_u/2)^{-1}$. Now, we decide that the initial problem had σ_u such that $\sigma_w^{-1} + (\sigma_u/2)^{-1} = \sigma^{-1}$ and (in absence of better indications) $\alpha = 1$:

$$u^{(1)} = \left(\frac{W_0}{2\sqrt{\pi \sigma_{wu}}} + h_0 \right) \mathcal{G}(\sigma) \tag{37}$$