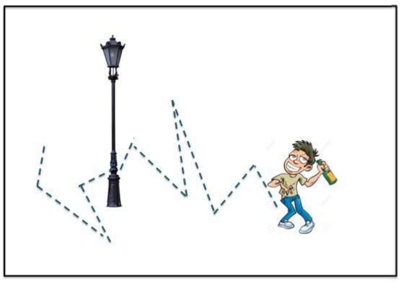
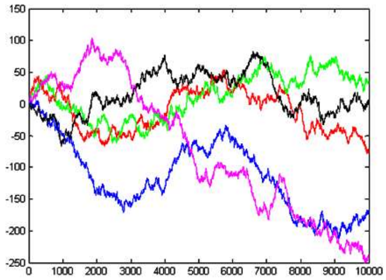
Chapter 4.
Examples of Markov Chains





Example 1. Random walk. Let $\xi_1, \xi_2, \ldots$ be independent integer-valued random variables having common density f. Let X_0 be an integer-valued random variable that is independent of the ξ_i's and set $X_n = X_0 + \xi_1 + \cdots + \xi_n$. The sequence X_n, $n \geq 0$, is called a *random walk*. It is a Markov chain whose state space is the integers and whose transition function is given by

To verify this, let π_0 denote the distribution of X_0. Then

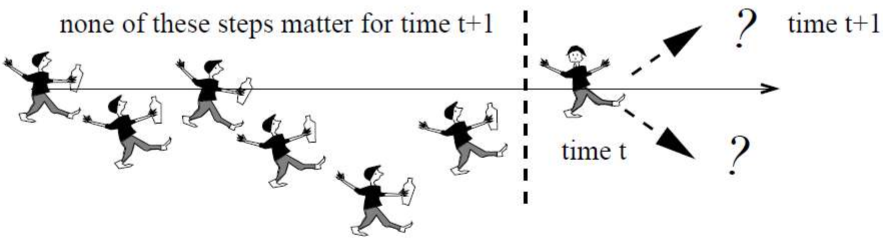
$$
\begin{aligned}
P(X_0 = x_0, \ldots, X_n = x_n) \\
&= P(X_0 = x_0, \xi_1 = x_1 - x_0, \ldots, \xi_n = x_n - x_{n-1}) \\
&= P(X_0 = x_0)P(\xi_1 = x_1 - x_0) \cdots P(\xi_n = x_n - x_{n-1}) \\
&= \pi_0(x_0) f(x_1 - x_0) \cdots f(x_n - x_{n-1}) \\
&= \pi_0(x_0) P(x_0, x_1) \cdots P(x_{n-1}, x_n),
\end{aligned}
$$

and thus (14) holds.

Suppose a "particle" moves along the integers according to this Markov chain. Whenever the particle is in x, regardless of how it got there, it jumps to state y with probability $f(y-x)$.

As a special case, consider a *simple random walk* in which $f(1)=p$, $f(-1)=q$, and $f(0)=r$, where p, q, and r are nonnegative and sum to one. The transition function is given by

Let a particle undergo such a random walk. If the particle is in state x at a given observation, then by the next observation it will have jumped to state $x+1$ with probability p and to state $x-1$ with probability q; with probability r it will still be in state x.

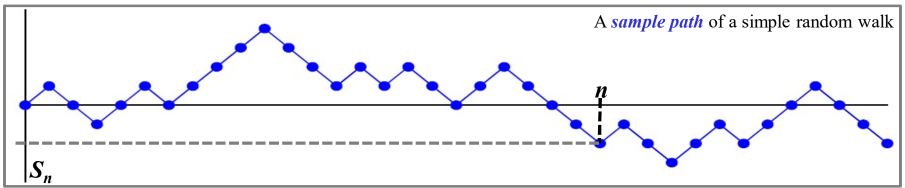


In a Markov chain, the future depends only upon the present: NOT upon the past.

X_0 , X_1 , X_2 , … are ***i.i.d.*** random variables with $P(X_i = +1) = P(X_i = -1) = 1/2$.

$$\boldsymbol{S_n = X_0 + X_1 + \cdots + X_n}$$

$\{S_n\ , n \geq 1\} = \{S_1, S_2, \ldots\}$: simple random walk (Markov chain)

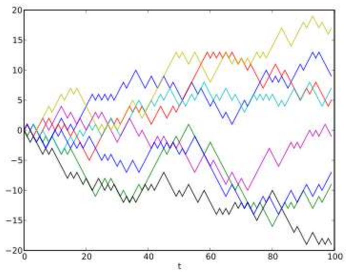


A ***sample path*** of a simple random walk

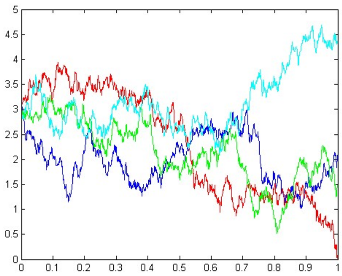
Definition. The discrete random variables $X_1, X_2, \ldots$ on $\mathbb{Z}^d$ are called steps of the random walk and have the following probability distribution: $\forall i \in \mathbb{N} : P(X_i = e) = \frac{1}{2d}$ if $e \in \mathbb{Z}^d$ and $\|e\| = 1$, and $P(X_i = e) = 0$ otherwise.

Definition. $S_0 = 0 \in \mathbb{Z}^d$ and $S_n = X_1 + \ldots + X_n$ for $n \in \mathbb{N}$ is called the position of the random walk at time n.

Polýa's Theorem. Simple random walks of dimension $d = 1, 2$ are recurrent, and of $d \geq 3$ are transient.

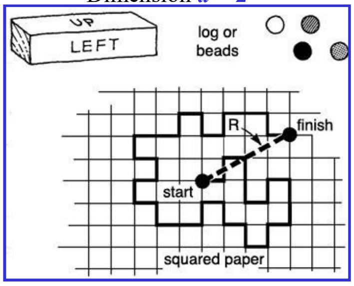


Dimension
***d* = 1**

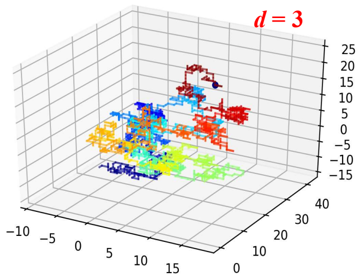


Recurrence vs ***Transience***

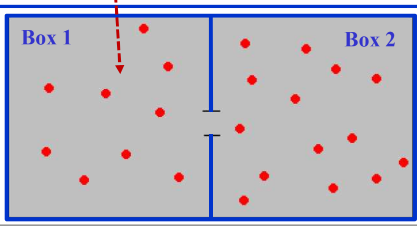
Dimension ***d* = 2**



Dimension
***d* = 3**



Example 2. Ehrenfest chain. The following is a simple model of the exchange of heat or of gas molecules between two isolated bodies. Suppose we have two boxes, labeled 1 and 2, and d balls labeled $1, 2, \ldots, d$. Initially some of these balls are in box 1 and the remainder are in box 2. An integer is selected at random from $1, 2, \ldots, d$, and the ball labeled by that integer is removed from its box and placed in the opposite box. This procedure is repeated indefinitely with the selections being independent from trial to trial. Let X_n denote the number of balls in box 1 after the nth trial. Then X_n, $n \geq 0$, is a Markov chain on $\mathscr{S} =$



The transition function of this Markov chain is easily computed. Suppose that there are x balls in box 1 at time n. Then with probability x/d the ball drawn on the $(n+1)$th trial will be from box 1 and will be transferred to box 2. In this case there will be $x - 1$ balls in box 1 at time $n + 1$. Similarly, with probability $(d - x)/d$ the ball drawn on the $(n+1)$th trial will be from box 2 and will be transferred to box 1, resulting in $x + 1$ balls in box 1 at time $n + 1$. Thus the transition function of this Markov chain is given by

Note that the Ehrenfest chain can in one transition only go from state x to $x - 1$ or $x + 1$ with positive probability.

A state a of a Markov chain is called an *absorbing state* if $P(a, a) = 1$ or, equivalently, if $P(a, y) = 0$ for $y \neq a$. The next example uses this definition.

Example 3. Gambler's ruin chain. Suppose a gambler starts out with a certain initial capital in dollars and makes a series of one dollar bets against the house. Assume that he has respective probabilities p and $q = 1 - p$ of winning and losing each bet, and that if his capital ever reaches zero, he is ruined and his capital remains zero thereafter. Let X_n, $n \geq 0$, denote the gambler's capital at time n. This is a Markov chain in which 0 is an absorbing state, and for $x \geq 1$

(15)

Such a chain is called a *gambler's ruin chain* on $\mathscr{S} = \{0, 1, 2, \ldots\}$. We can modify this model by supposing that if the capital of the gambler increases to d dollars he quits playing. In this case 0 and d are both absorbing states, and (15) holds for $x = 1, \ldots, d - 1$.

Example 4. Birth and death chain. Consider a Markov chain either on $\mathscr{S} = \{0, 1, 2, \ldots\}$ or on $\mathscr{S} = \{0, 1, \ldots, d\}$ such that starting from x the chain will be at $x - 1$, x, or $x + 1$ after one step. The transition function of such a chain is given by

where p_x, q_x, and r_x are nonnegative numbers such that $p_x + q_x + r_x = 1$. The Ehrenfest chain and the two versions of the gambler's ruin chain are examples of *birth and death chains*. The phrase "birth and death" stems from applications in which the state of the chain is the population of some living system. In these applications a transition from state x to state $x + 1$ corresponds to a "birth," while a transition from state x to state $x - 1$ corresponds to a "death."

Example 5. Queuing chain. Consider a service facility such as a checkout counter at a supermarket. People arrive at the facility at various times and are eventually served. Those customers that have arrived at the facility but have not yet been served form a waiting line or queue. There are a variety of models to describe such systems. We will consider here only one very simple and somewhat artificial model.

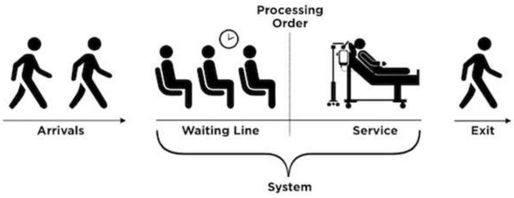
Let time be measured in convenient periods, say in minutes. Suppose that if there are any customers waiting for service at the beginning of any given period, exactly one customer will be served during that period, and that if there are no customers waiting for service at the beginning of a period, none will be served during that period. Let ξ_n denote the number of new customers arriving during the nth period. We assume that $\xi_1, \xi_2, \ldots$ are independent nonnegative integer-valued random variables having common density f.

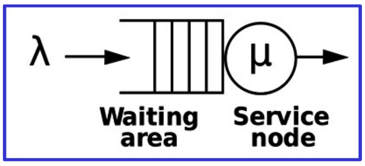
Let X_0 denote the number of customers present initially, and for $n \geq 1$, let X_n denote the number of customers present at the end of the nth period. If $X_n = 0$, then $X_{n+1} = \xi_{n+1}$; and if $X_n \geq 1$, then $X_{n+1} = X_n + \xi_{n+1} - 1$. It follows without difficulty from the assumptions on ξ_n, $n \geq 1$, that X_n, $n \geq 0$, is a Markov chain whose state space is the nonnegative integers and whose transition function P is given by

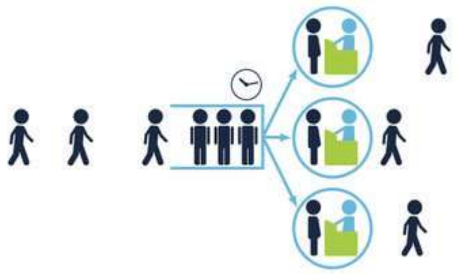
$$P(0, y) = f(y)$$

and

$$P(x, y) = f(y - x + 1), \qquad x \geq 1.$$



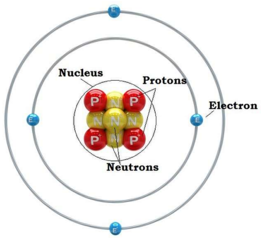


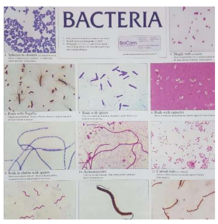


Multi-Server System

Example 6. Branching chain. Consider particles such as neutrons or bacteria that can generate new particles of the same type. The initial set of objects is referred to as belonging to the 0th generation. Particles generated from the nth generation are said to belong to the $(n + 1)$th generation. Let X_n, $n \geq 0$, denote the number of particles in the nth generation.

Nothing in this description requires that the various particles in a generation give rise to new particles simultaneously. Indeed at a given time, particles from several generations may coexist.





A typical situation is illustrated in Figure 1: one initial particle gives rise to two particles. Thus $X_0 = 1$ and $X_1 = 2$. One of the particles in the first generation gives rise to three particles and the other gives rise to one particle, so that $X_2 = 4$. We see from Figure 1 that $X_3 = 2$. Since neither of the particles in the third generation gives rise to new particles, we conclude that $X_4 = 0$ and consequently that $X_n = 0$ for all $n \geq 4$. In other words, the progeny of the initial particle in the zeroth generation become extinct after three generations.

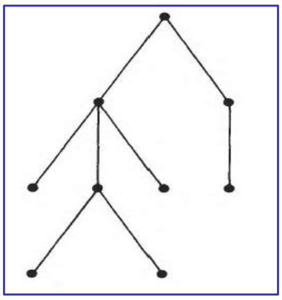


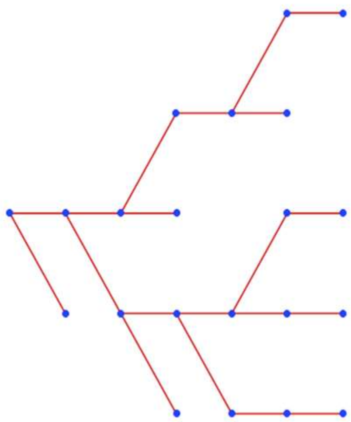
Figure 1

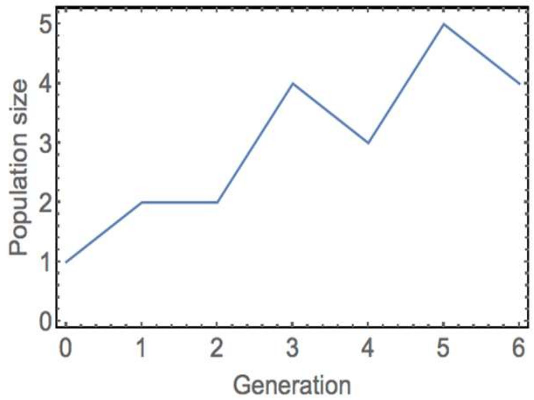
In order to model this system as a Markov chain, we suppose that each particle gives rise to ξ particles in the next generation, where ξ is a non-negative integer-valued random variable having density $\boxed{f}$. We suppose that the number of offspring of the various particles in the various generations are chosen independently according to the density f.

Under these assumptions X_n, $n \geq 0$, forms a Markov chain whose state space is the nonnegative integers. State 0 is an absorbing state. For if there are no particles in a given generation, there will not be any particles in the next generation either. For $x \geq 1$

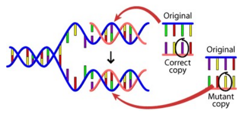
where $\xi_1, \ldots, \xi_x$ are independent random variables having common density $\boxed{f}$. In particular, $P(1, y) = f(y)$, $y \geq 0$.

If a particle gives rise to $\xi = 0$ particles, the interpretation is that the particle dies or disappears. Suppose a particle gives rise to ξ particles, which in turn give rise to other particles; but after some number of generations, all descendants of the initial particle have died or disappeared (see Figure 1). We describe such an event by saying that the descendants of the original particle eventually become *extinct*. An interesting problem involving branching chains is to compute the probability $\boxed{\rho}$ of eventual extinction for a branching chain starting with a single particle or, equivalently, the probability that a branching chain starting at state 1 will eventually be absorbed at state 0. Once we determine ρ, we can easily find the probability that in a branching chain starting with x particles the descendants of each of the original particles eventually become extinct. Indeed, since the particles are assumed to act independently in giving rise to new particles, the desired probability is just $\boxed{\rho^x.}$





Example 7. Consider a gene composed of d subunits, where d is some positive integer and each subunit is either normal or mutant in form. Consider a cell with a gene composed of m mutant subunits and $d - m$ normal subunits. Before the cell divides into two daughter cells, the gene duplicates. The corresponding gene of one of the daughter cells is composed of d units chosen at random from the $2m$ mutant subunits and the $2(d - m)$ normal subunits. Suppose we follow a fixed line of descent from a given gene. Let X_0 be the number of mutant subunits initially present, and let X_n, $n \geq 1$, be the number present in the nth descendant gene. Then X_n, $n \geq 0$, is a Markov chain on $\mathscr{S} = \{0, 1, 2, \ldots, d\}$ and



States 0 and d are absorbing states for this chain.