

广义有限差分法求解 Kirchhoff 和 Winkler 薄板弯曲问题^{*}

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摘要 论文将广义有限差分法用于数值计算 Kirchhoff 板和 Winkler 板的弯曲问题。广义有限差分法是基于最小二乘原理的一种区域型无网格方法。相比于传统的网格类数值解法, 广义有限差分法无需网格生成且无需数值积分。通过数值实验结果表明, 广义有限差分法可以有效地求解两类薄板在不同横向荷载作用下的弯曲问题。

关键词 广义有限差分法, Kirchhoff 板, Winkler 板, 薄板弯曲

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0 引言

板是实际工程中很重要的一类基本结构, 它被广泛应用于桥梁、航空、造船、冶金、车辆及零件结构等工程领域。由于板的厚度通常远远小于其长和宽的尺寸, 因此在计算板的弯曲问题时经常引入一些假设将原来的三维弹性力学问题简化为二维问题求解。基于 Kirchhoff 基本假定, 即忽略剪切变形的影响, 简化得到所谓的 Kirchhoff 薄板弯曲问题; 而基于 Winkler 基本假定, 即假设地基表面所受压力与表面的沉陷成正比, 简化得到 Winkler 薄板弯曲问题。这两类薄板弯曲问题是板结构中最常见且基础的问题。它们的数学模型都为四阶偏微分方程的边值问题。为了求解这一类高阶偏微分方程的边值问题, 往往采用有限差分法^[1]、有限单元法^[2]、边界元法^[3]等网格类数值方法进行求解。

此外, 由于无需网格划分的特点, 众多无网格方法^[4-7]也被用于薄板弯曲分析。1993 年, Hein 采用弥散单元法(Diffuse Element Method)对 Kirchhoff 板进行分析^[8]; 1996 年, Belytschko 和 Krysl 采用无网格伽辽金法(Element-Free Galerkin Method)对弹性薄板和薄壳的静定问题进行分析^[9]; 随后, Liu 等进一步将无网格伽辽金法应用于层合板的计算^[10]; 此外, Fu 等采用边界粒子法(Boundary Particle Method, BPM)对 Kirchhoff 板^[11] 和 Winkler

板^[12]弯曲问题进行求解。

本文将采用一种新型的无网格法——广义有限差分法^[13] (Generalized Finite-Difference Method, GFDM)。其基本思想早在 1975 年就被学者 Perrone 提出^[14], 但直到 2001 年, Benito 等对其改进后才被学者们重视。广义有限差分法是一种基于泰勒展开式的区域型无网格法, 打破了传统有限差分法需要正交网格的限制, 保留了数值算法中局部化的优势, 在保证计算精度的同时也提高了计算效率。自 Benito 等推导得到了改进后广义有限差分法的显式表达式^[15], 并研究了影响该方法的几项因素^[16] 后, 陆续也有许多学者利用广义有限差分法求解其他问题。在 2012 年, Gavete 等^[17]首次将广义有限差分法应用于求解四阶偏微分方程; 在 2015 年, Fan 等^[18-21]将广义有限差分法用于稳定求解二维柯西反算问题以及双调和方程反算问题等。基于这些研究, 本文将采用广义有限差分法求解 Kirchhoff 和 Winkler 薄板弯曲问题。

第二节通过数值算例验证广义有限差分法计算 Kirchhoff 和 Winkler 薄板弯曲问题的有效性, 第三节给出了结论。

1 广义有限差分法求解 Kirchhoff 板和 Winkler 板弯曲问题

本节主要介绍两类板在横向荷载下的弯曲问题

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以及推导得到的广义有限差分法的显式差分表达式.

1.1 板弯曲问题

Kirchhoff 板在横向荷载 $q(x, y)$ 作用下的挠度控制方程^[22]:

$$\nabla^2 W(x, y) = \frac{q(x, y)}{D} \quad ((x, y) \in \Omega) \quad (1)$$

Winkler 板在横向荷载 $q(x, y)$ 作用下的挠度控制方程^[22]:

$$(\nabla^2 + \frac{k}{D})W(x, y) = \frac{q(x, y)}{D} \quad ((x, y) \in \Omega) \quad (2)$$

其中, $W(x, y)$ 为板的挠度, k 是 Winkler 地基系数, $q(x, y)$ 为施加的横向荷载, Ω 为计算区域. $D = Eh^3 / [12(1-\nu^2)]$ 为板的抗弯刚度, h 为板的厚度. E 为板的弹性模量, ν 为泊松比.

为了得到唯一的挠度解, 需要在薄板的边界施加上一定的约束, 以下列出三种常见的薄板约束方式以及其所对应的边界条件:

(1) 固支边: $W=0, \theta_n=0$;

(2) 简支边: $W=0, M_n=0$;

(3) 自由边: $M_n=0, V_n=0$;

上述的边界条件可以用挠度表示如下.

转角:

$$\theta_n = \frac{\partial W}{\partial x} \frac{dx}{dn} + \frac{\partial W}{\partial y} \frac{dy}{dn} = \frac{\partial W}{\partial x} \cos\alpha + \frac{\partial W}{\partial y} \sin\alpha \quad (3)$$

弯矩:

$$M_n = -D \left\{ \nu \nabla^2 W + (1-\nu) \left[\cos^2 \alpha \frac{\partial^2 W}{\partial x^2} + \sin^2 \alpha \frac{\partial^2 W}{\partial y^2} + \sin(2\alpha) \frac{\partial^2 W}{\partial x \partial y} \right] \right\} \quad (4)$$

剪力:

$$V_n = -D \left\{ \left(\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y} \right) \nabla^2 W + (1-\nu) \left(-\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y} \right) \times \left[\frac{1}{2} \sin(2\alpha) \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right) + \cos(2\alpha) \frac{\partial^2 W}{\partial x \partial y} \right] \right\} \quad (5)$$

其中, α 为外法线与 x 轴的夹角, $n = [\cos\alpha, \sin\alpha]$ 为单位外法线向量.

1.2 广义有限差分法

类似于其他无网格法, 广义有限差分法计算板的弯曲问题时首先需要对求解区域布置节点进行离散, 并筛选每个节点最近的 ns 个节点用于表示该节点的差分表达式. 如图 1 中找到离 i 点最近的 ns 个

点用于构造 i 点处的差分表达式. 此外, 由于板边界必须同时满足两种边界条件^[22,23], 因此形成的系数矩阵是一个细长型矩阵, 为了保证系数矩阵的稳定性, 可以在边界外布置一层虚假点, 使得构建的系数矩阵为一个稀疏方阵. 在计算四阶偏微分方程时, 一般选取的 $ns \geq 30$, 且物理边界与虚假边界的间距一般取为: $d = \min(d_{ij})/2$. 其中, d_{ij} 为离 i 点最近的点与 i 点的欧氏距离.

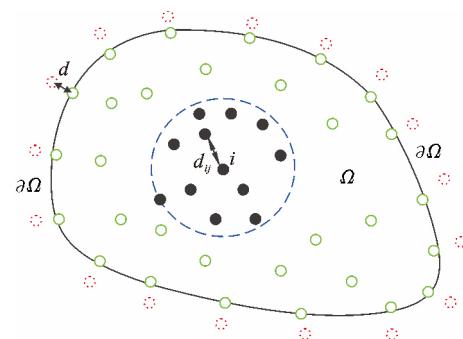


图 1 广义有限差分法配点布置示意图

Fig. 1 Schematic diagram of collocation nodes in the GFDM

由于是四阶偏微分方程计算, 一般来说需要将泰勒展开式展开至四阶偏导项:

$$\begin{aligned} W_i^j = & W_i + h_{ij} \frac{\partial W}{\partial x} \Big|_i + k_{ij} \frac{\partial W}{\partial y} \Big|_i + \frac{1}{2} h_{ij}^2 \frac{\partial^2 W}{\partial x^2} \Big|_i + \\ & h_{ij} k_{ij} \frac{\partial^2 W}{\partial x \partial y} \Big|_i + \frac{1}{2} k_{ij}^2 \frac{\partial^2 W}{\partial y^2} \Big|_i + \frac{1}{6} h_{ij}^3 \frac{\partial^3 W}{\partial x^3} \Big|_i + \\ & \frac{1}{2} h_{ij}^2 k_{ij} \frac{\partial^3 W}{\partial x^2 \partial y} \Big|_i + \frac{1}{2} h_{ij} k_{ij}^2 \frac{\partial^3 W}{\partial x \partial y^2} \Big|_i + \frac{1}{6} k_{ij}^3 \frac{\partial^3 W}{\partial y^3} \Big|_i + \\ & \frac{1}{24} h_{ij}^4 \frac{\partial^4 W}{\partial x^4} \Big|_i + \frac{1}{6} h_{ij}^3 k_{ij} \frac{\partial^4 W}{\partial x^3 \partial y} \Big|_i + \frac{1}{4} h_{ij}^2 k_{ij}^2 \frac{\partial^4 W}{\partial x^2 \partial y^2} \Big|_i + \\ & \frac{1}{6} h_{ij} k_{ij}^3 \frac{\partial^4 W}{\partial x \partial y^3} \Big|_i + \frac{1}{24} k_{ij}^4 \frac{\partial^4 W}{\partial y^4} \Big|_i + \dots \end{aligned} \quad (6)$$

随后, 根据最小二乘原理, 用上述 ns 个泰勒展开式构造得到如下的泛函 $B(W)$:

$$\begin{aligned} B(W) = & \sum_{j=1}^{ns} \left[(W_j - W_i^j + h_{ij} \frac{\partial W}{\partial x} \Big|_i + k_{ij} \frac{\partial W}{\partial y} \Big|_i + \right. \\ & \left. \frac{1}{2} h_{ij}^2 \frac{\partial^2 W}{\partial x^2} \Big|_i + h_{ij} k_{ij} \frac{\partial^2 W}{\partial x \partial y} \Big|_i + \frac{1}{2} k_{ij}^2 \frac{\partial^2 W}{\partial y^2} \Big|_i + \right. \\ & \left. \frac{1}{6} h_{ij}^3 \frac{\partial^3 W}{\partial x^3} \Big|_i + \frac{1}{2} h_{ij}^2 k_{ij} \frac{\partial^3 W}{\partial x^2 \partial y} \Big|_i + \frac{1}{2} h_{ij} k_{ij}^2 \frac{\partial^3 W}{\partial x \partial y^2} \Big|_i + \right. \\ & \left. \frac{1}{6} k_{ij}^3 \frac{\partial^3 W}{\partial y^3} \Big|_i + \frac{1}{24} h_{ij}^4 \frac{\partial^4 W}{\partial x^4} \Big|_i + \frac{1}{6} h_{ij}^3 k_{ij} \frac{\partial^4 W}{\partial x^3 \partial y} \Big|_i + \right. \\ & \left. \frac{1}{4} h_{ij}^2 k_{ij}^2 \frac{\partial^4 W}{\partial x^2 \partial y^2} \Big|_i + \frac{1}{6} h_{ij} k_{ij}^3 \frac{\partial^4 W}{\partial x \partial y^3} \Big|_i + \frac{1}{24} k_{ij}^4 \frac{\partial^4 W}{\partial y^4} \Big|_i \right) \end{aligned}$$

$$w(h_{ij}, k_{ij}) \quad (7)$$

其中, W_i 代表坐标为 (x_i, y_i) 处的解, W_j^i 代表坐标为 (x_j^i, y_j^i) 处的解, $h_{ij} = x_i - x_j^i, k_{ij} = y_i - y_j^i$, 表示第 i 个点和第 j 个点之间 x 方向和 y 方向上的距离, 权函数的定义为:

$$w(d_{ij}) = \begin{cases} 1 - 6 \left(\frac{d_{ij}}{d_{mi}} \right)^2 + 8 \left(\frac{d_{ij}}{d_{mi}} \right)^3 - 3 \left(\frac{d_{ij}}{d_{mi}} \right)^4 & (d_{ij} \leq d_{mi}) \\ 0 & (d_{ij} > d_{mi}) \end{cases} \quad (8)$$

随后分别对 $B(\omega)$ 求关于 $\partial W_i / \partial x, \partial W_i / \partial y, \partial^2 W_i / \partial x^2, \partial^2 W_i / \partial y^2, \partial^2 W_i / \partial x \partial y \dots \partial^4 W_i / \partial y^4$ 的变分, 得到如下线性方程组:

$$AD_w = b \quad (9)$$

其: 中

$$D_w = \begin{bmatrix} \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial^2 W}{\partial x^2} & \frac{\partial^2 W}{\partial x \partial y} & \frac{\partial^2 W}{\partial y^2} \\ \cdots & \frac{\partial^4 W}{\partial x^2 \partial y^2} & \frac{\partial^4 W}{\partial x \partial y^3} & \frac{\partial^4 W}{\partial y^4} \end{bmatrix}^T \quad (10)$$

$$A = \begin{bmatrix} \sum_{j=1}^{ns} h_{ij}^2 w^2 & \sum_{j=1}^{ns} h_{ij} k_{ij} w^2 & \cdots & \sum_{j=1}^{ns} \frac{1}{24} h_{ij} k_{ij}^4 w^2 \\ \sum_{j=1}^{ns} k_{ij}^2 w^2 & \cdots & \sum_{j=1}^{ns} \frac{1}{24} k_{ij}^5 w^2 \\ \text{SYM} & \ddots & \vdots \\ \cdots & \sum_{j=1}^{ns} \frac{1}{576} k_{ij}^8 w^2 \end{bmatrix} \quad (11)$$

$$b = \begin{bmatrix} \sum_{j=1}^{ns} (-W_i + W_j^i) h_{ij} w^2 \\ \sum_{j=1}^{ns} (-W_i + W_j^i) k_{ij} w^2 \\ \sum_{j=1}^{ns} (-W_i + W_j^i) \frac{h_{ij}^2}{2} w^2 \\ \vdots \\ \sum_{j=1}^{ns} (-W_i + W_j^i) \frac{k_{ij}^4}{24} w^2 \end{bmatrix} \quad (12)$$

通过对以上公式的推导求解, 可以得到各阶偏导项的差分表达式:

$$\begin{cases} \frac{\partial W}{\partial x} \Big|_i = w_0^{x,i} W_i + \sum_{j=1}^{ns} w_j^{x,i} W_j^i \\ \frac{\partial W}{\partial y} \Big|_i = w_0^{y,i} W_i + \sum_{j=1}^{ns} w_j^{y,i} W_j^i \\ \frac{\partial^2 W}{\partial x^2} \Big|_i = w_0^{xx,i} W_i + \sum_{j=1}^{ns} w_j^{xx,i} W_j^i \\ \frac{\partial^2 W}{\partial x \partial y} \Big|_i = w_0^{xy,i} W_i + \sum_{j=1}^{ns} w_j^{xy,i} W_j^i \\ \frac{\partial^2 W}{\partial y^2} \Big|_i = w_0^{yy,i} W_i + \sum_{j=1}^{ns} w_j^{yy,i} W_j^i \\ \vdots \\ \frac{\partial^4 W}{\partial y^4} \Big|_i = w_0^{yyyy,i} W_i + \sum_{j=1}^{ns} w_j^{yyyy,i} W_j^i \end{cases} \quad (13)$$

通过以上计算, 可以将离散后每一个点的偏导值用一个差分表达式替代, 然后代入控制方程和边界条件, 得到一个线性代数方程组, 从而解得数值解.

2 数值算例

本节采用广义有限差分法求解了 Winkler 板以及 Kirchhoff 薄板弯曲问题, 并且将结果与其他方法求得的结果作了对比.

为了便于计算结果比较, 本节选取薄板的参数为: $E = 2.1 \times 10^{11}, h = 0.01, \nu = 0.3, q_0 = 10^6, k = \pi^4 D$. Rerr 表示平均相对误差, Merr 表示最大绝对误差, Aerr 表示平均绝对误差, 定义如下:

$$\text{Rerr}(W) = \sqrt{\frac{1}{N} \sum_{i=1}^N \left(\frac{W(i) - \widetilde{W}(i)}{W(i)} \right)^2} \quad (14)$$

$$\text{Merr}(W) = \max_{1 \leq i \leq N} |W(i) - \widetilde{W}(i)| \quad (15)$$

$$\text{Aerr}(W) = \sqrt{\frac{1}{N} \sum_{i=1}^N (W(i) - \widetilde{W}(i))^2} \quad (16)$$

其中, $W(i)$ 和 $\widetilde{W}(i)$ 分别表示 $x_i = (x_i, y_i)$ 处的解析解和数值解, N 表示区域内部的计算点个数.

2.1 均布荷载作用下板弯曲问题的收敛性分析

算例 1 受均布荷载的方形四边简支 Kirchhoff 薄板^[22] (SSSS):

$$\begin{cases} \nabla^2 W = \frac{q_0}{D} & ((x, y) \in \Omega) \\ W = 0 & ((x, y) \in \partial\Omega) \\ M_n = 0 & ((x, y) \in \partial\Omega) \end{cases} \quad (17)$$

其中, $\Omega = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$;

解析解表达式为:

$$W = \frac{16q_0}{\pi^6 D} \sum_m^{\infty} \sum_n^{\infty} \frac{\sin(m\pi x) \sin(n\pi y)}{mn (m^2 + n^2)^2} \quad (m=1, 3, 5, \dots, n=1, 3, 5, \dots) \quad (18)$$

本节选取前 25 项作为精确解来计算误差. 该问题的计算域示意图见图 2.

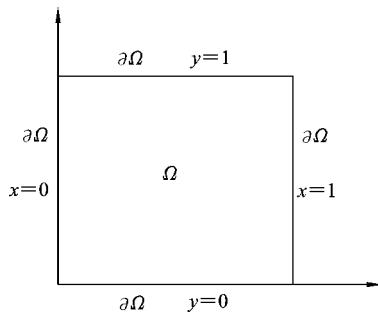


图 2 方形薄板计算域示意图

Fig. 2 Schematic diagram of computational domain for square plate

从图 3 中可以看出, 随着计算点数的增加, 广义有限差分法的最大绝对误差、相对误差和平均绝对误差都随之降低.

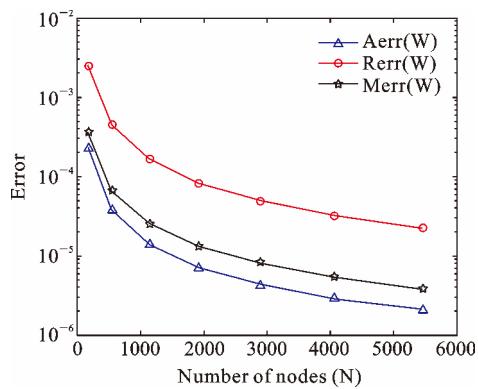


图 3 广义有限差分法计算算例 1 的误差收敛图

Fig. 3 Convergence rate of the GFDM in Case 1

算例 2 受均布荷载的方形四边简支 Winkler 薄板^[12] (SSSS):

$$\begin{cases} \left(\nabla^2 + \frac{k}{D}\right)W = \frac{q_0}{D} & ((x, y) \in \Omega) \\ W = 0 & ((x, y) \in \partial\Omega) \\ M_n = 0 & ((x, y) \in \partial\Omega) \end{cases} \quad (19)$$

其中, $\Omega = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

解析解表达式为:

$$W = \frac{16q_0}{\pi^6 D} \sum_m^{\infty} \sum_n^{\infty} \frac{\sin(m\pi x)\sin(n\pi y)}{mn \left[(m^2 + n^2)^2 + \frac{k}{\pi^4 D} \right]} \quad (20)$$

$m = 1, 3, 5, \dots, n = 1, 3, 5, \dots$

仍然选取前 25 项作为精确解来计算误差. 该问

题的计算域和布点个数与算例 1 一致. 从图 4 中可以看出, 随着计算点数的增加, 广义有限差分法的最大绝对误差、相对误差和平均绝对误差都随之降低.

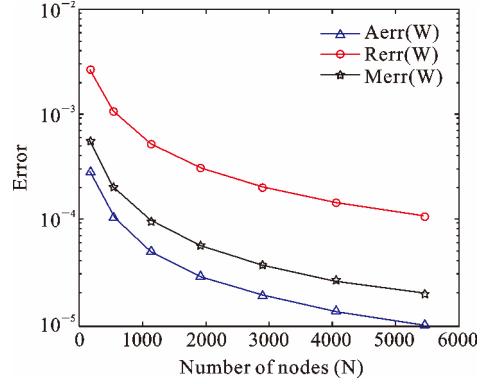


图 4 广义有限差分法计算算例 2 的误差收敛图

Fig. 4 Convergence rate of the GFDM in Case 2

从以上两个算例可以看出, 广义有限差分法在求解薄板弯曲问题中具有较好的计算收敛性.

2.2 复杂荷载作用下板的弯曲问题

算例 3 受指数荷载的方形四边固支 Kirchhoff 薄板^[11] (CCCC):

$$\begin{cases} \nabla^2 W = \frac{q(x, y)}{D} & ((x, y) \in \Omega) \\ W = 0 & ((x, y) \in \partial\Omega) \\ \theta_n = 0 & ((x, y) \in \partial\Omega) \end{cases} \quad (21)$$

其中, $\Omega = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$; 布点示意图见图 5, 边界点个数为 $29 \times 4 = 116$ 个, 内部点个数为 $29 \times 29 = 841$ 个. 且横向荷载为:

$$q(x, y) = 2q_0 \exp[-5(x - y)] \quad (22)$$

由图 6 可知, 广义有限差分法在计算含指数函

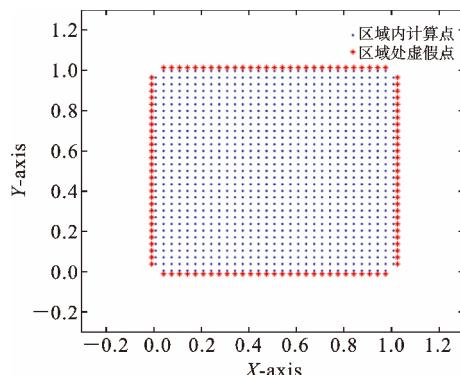


图 5 配点布置示意图

Fig. 5 Schematic diagram of collocation nodes

型横向荷载作用的方形四边固支 Kirchhoff 薄板 (CCCC) 时计算结果与边界粒子法的计算结果吻合很好。

在指数荷载作用下, 基于广义有限差分法、边界粒子法求得的数值解如图 6。

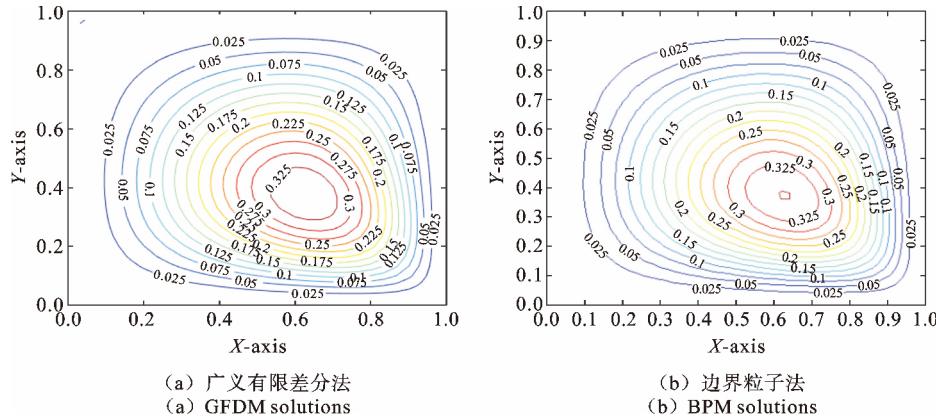


图 6 算例 3 的挠度数值解

Fig. 6 Numerical deflections in Case 3

算例 4 受复杂荷载的方形四边固支 Winkler 薄板^[12] (CCCC) :

$$\begin{cases} \left(\nabla^2 + \frac{k}{D} \right) W = F(x, y) & ((x, y) \in \Omega) \\ W = 0 & ((x, y) \in \partial\Omega) \\ \theta_n = 0 & ((x, y) \in \partial\Omega) \end{cases} \quad (23)$$

其中, $\Omega = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$, 布点形式与算例 3 一致, 且:

$$F(x, y) = \frac{q_0}{100D} \left(\left(64\pi^4 + \frac{k}{D} \right) [\cos(2\pi x) - 1] \right.$$

$$\times [\cos(2\pi y) - 1] + 48\pi^4 [\cos(2\pi x) + \cos(2\pi y)] - 64\pi^4 \}$$
(24)

解析解表达式为:

$$W = \frac{q_0}{100D} \frac{[\cos(2\pi x) - 1][\cos(2\pi y) - 1]}{100D} \quad (25)$$

由图 7 可知, 广义有限差分法在计算含周期函数型横向荷载作用的方形四边固支 Winkler 薄板 (CCCC) 时计算结果与解析解吻合很好。

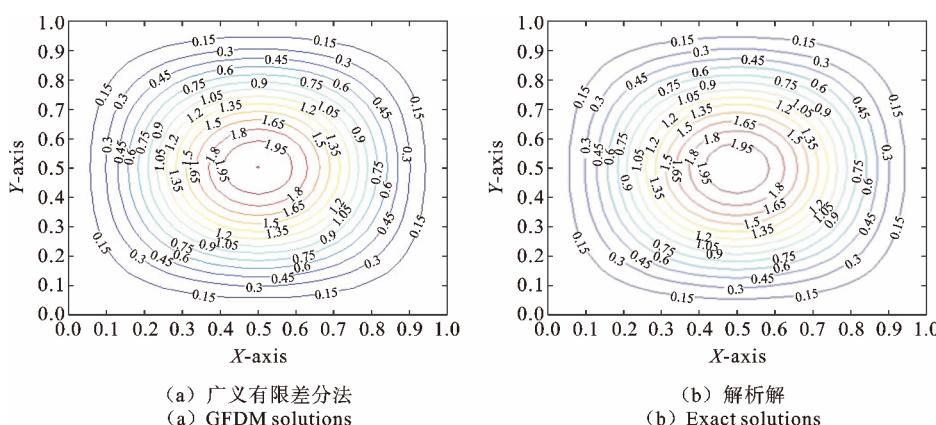


图 7 算例 4 的挠度解

Fig. 7 Deflections in Case 4

算例 5 受均布荷载的直角扇形简支 Kirchhoff 薄板 (SSS)

其中, 广义有限差分法计算节点数为 247; COMSOL 软件中单元数为 494。

$$\begin{cases} \nabla^2 W = \frac{q_0}{D} & ((x, y) \in \Omega) \\ W = 0 & ((x, y) \in \partial\Omega) \\ M_n = 0 & ((x, y) \in \partial\Omega) \end{cases} \quad (26)$$

从图 8 和表 1 可以看出, 相比于有限元法, 广义有限差分法通过更少的节点就能算出满足工程精度的结果。 (1.0×10^{-2})

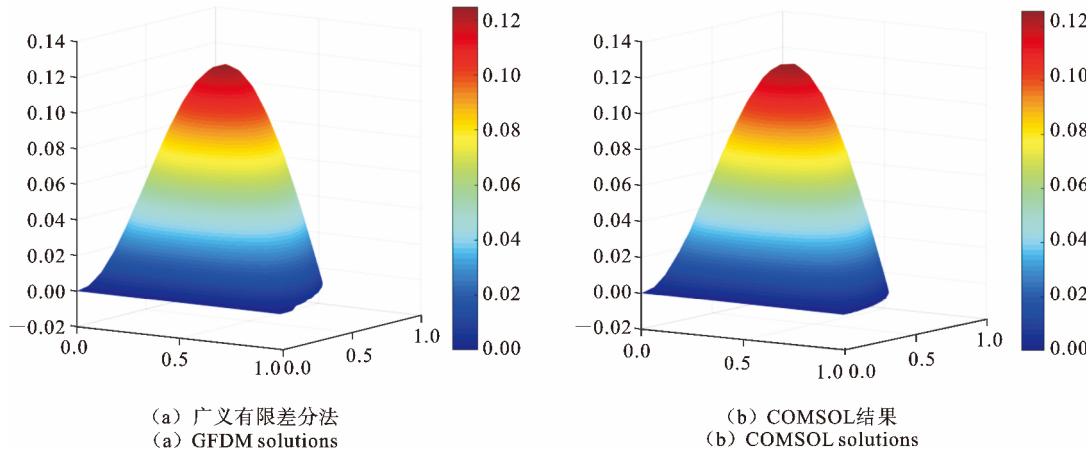


图 8 算例 5 的挠度解

Fig. 8 Deflections in Case 5

表 1 均布荷载下的直角扇形简支板计算结果

Table 1 Numerical results of simply supported sector plates under uniformly distributed load

坐标	挠度值 \tilde{W} (GFDM)	挠度值 W (文献[22])	挠度相对误差	挠度值(COMSOL)	挠度相对误差
0.3536, 0.3536	0.1176	0.1170	5.1×10^{-3}	0.1179	2.6×10^{-3}

算例 6 受静水压力荷载的方形四边简支 Kirchhoff 薄板(SSSS)以及方形四边固支 Kirchhoff 薄板^[22](CCCC)：

$$\begin{cases} \nabla^2 W = \frac{q(x)}{D} & ((x, y) \in \Omega) \\ W = 0 & ((x, y) \in \partial\Omega) \\ M_n = 0 & ((x, y) \in \partial\Omega) \end{cases} \quad (27)$$

$$\begin{cases} \nabla^2 W = \frac{q(x)}{D} & ((x, y) \in \Omega) \\ W = 0 & ((x, y) \in \partial\Omega) \\ \theta_n = 0 & ((x, y) \in \partial\Omega) \end{cases} \quad (28)$$

其中, 计算区域和布点形式与算例 3 相同, 且:

$$q(x) = q_0 x \quad (29)$$

表 2 和 3 分别给出了广义有限差分法计算静水压力荷载作用下的方形四边简支和固支 Kirchhoff 薄板弯曲问题的结果。

由表 2 和 3 可以看出, 广义有限差分法在计算算例 6 时仅需 900 多个计算节点即可得到满足工程精度的计算结果。

表 2 静水压力荷载下的方形简支板计算结果

Table 2 Numerical results of simply supported square plates under hydrostatic pressure loading

静水压力荷载	坐标	挠度值 \tilde{W} (GFDM)	挠度值 W (文献[22])	挠度相对误差	弯矩值 M (GFDM)	弯矩值 M (文献[22])	弯矩相对误差
$q(x) = q_0 x$	0.25, 0.5	0.06810	0.06812	2.9×10^{-4}	13094	13200	8.0×10^{-3}
$q(x) = q_0 x$	0.5, 0.5	0.10563	0.10560	2.8×10^{-4}	23946	23900	1.9×10^{-3}
$q(x) = q_0 x$	0.75, 0.5	0.08448	0.08423	3.0×10^{-3}	25754	25900	5.6×10^{-3}

表 3 静水压力荷载下的方形固支板计算结果

Table 3 Numerical results of clamped square plates under hydrostatic pressure loading

静水压力荷载	坐标	挠度值 \tilde{W} (GFDM)	挠度值 W (文献[22])	挠度相对误差	弯矩值 M (GFDM)	弯矩值 M (文献[22])	弯矩相对误差
$q(x) = q_0 x$	0.5, 0.5	0.0329	0.0328	3.2×10^{-3}	11454	11500	4.0×10^{-3}

2.3 不同约束情况下薄板的弯曲问题

算例 7 不同约束情况下的方形 Kirchhoff 薄板

如前文中描述,不同的约束方式对应着不同的边界条件,此算例计算了一些其他边界条件情况下

的板弯曲问题.为了简化讨论,方形板所受荷载为均布荷载.且布点形式和算例 3 一致.表中 C 表示固支约束,S 表示简支约束,F 表示自由约束.

具体结果如下表 4.

表 4 不同边界情况下的薄板弯曲问题计算结果

Table 4 Numerical results of plate bending problems under different boundary conditions

边界条件	坐标	挠度值 \tilde{W} (GFDM)	挠度值 W (文献[22,24])	挠度相对误差	弯矩值 M (GFDM)	弯矩值 M (文献[22,24])	弯矩相对误差
CCCC	0.5, 0.5	0.06581	0.06552	4.4×10^{-3}	22910	23100	8.2×10^{-3}
SSSS	0.5, 0.5	0.21127	0.21112	7.1×10^{-4}	47892	47900	1.7×10^{-4}
CSCS	0.5, 0.5	0.09971	0.09984	1.3×10^{-3}	24391	24400	3.7×10^{-4}
SSSC	0.5, 0.5	0.14487	0.14560	5.0×10^{-3}	39185	39000	4.7×10^{-3}
SFSF	0.5, 0.5	0.68023	0.68067	6.5×10^{-4}	122444	122500	4.8×10^{-4}
SCSF	0, 0.5	0.58316	0.58759	7.5×10^{-3}	97021	97200	1.8×10^{-3}
SSSF	0, 0.5	0.66716	0.66871	2.3×10^{-3}	111526	112000	4.2×10^{-3}
CFFF	0.5, 1	6.57337	6.63512	9.3×10^{-3}	/	/	/
CFFF	0.5, 0	/	/	/	446425	445500	2.1×10^{-3}

注:“/”表示在该点处的值是 0.

2.4 不同边界形状板的弯曲问题

算例 8 四边固支且边界形状为平行四边形的 Kirchhoff 薄板(CCCC)

图 9 为四边固支的平行四边形板示意图, θ 为平行四边形边与 y 轴的夹角, 平行四边形底和高均为 1, 即面积不变.

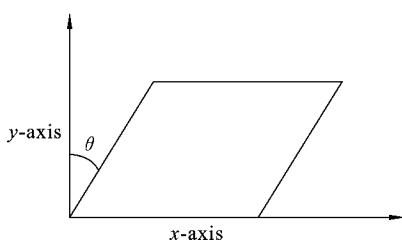


图 9 平行四边形板示意图

Fig. 9 Schematic diagram of parallelogram plate

在均布荷载 q_0 作用下,不同夹角对应的平行四边形板的弯曲情况也不一样,布点情况如下图 10.

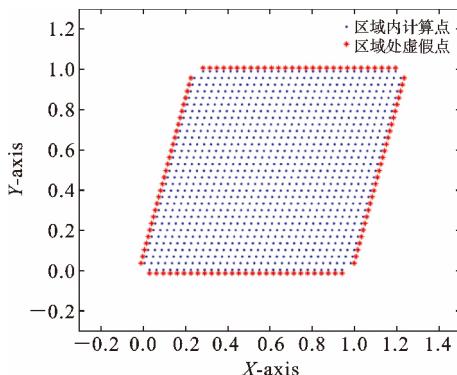


图 10 平行四边形板布点示意图

Fig. 10 Schematic diagram of nodes of parallelogram plate

不同夹角平行四边形板在固支情况下对应的弯曲情况如下图 11. 不同夹角平行四边形板的最大挠度值如下表 5.

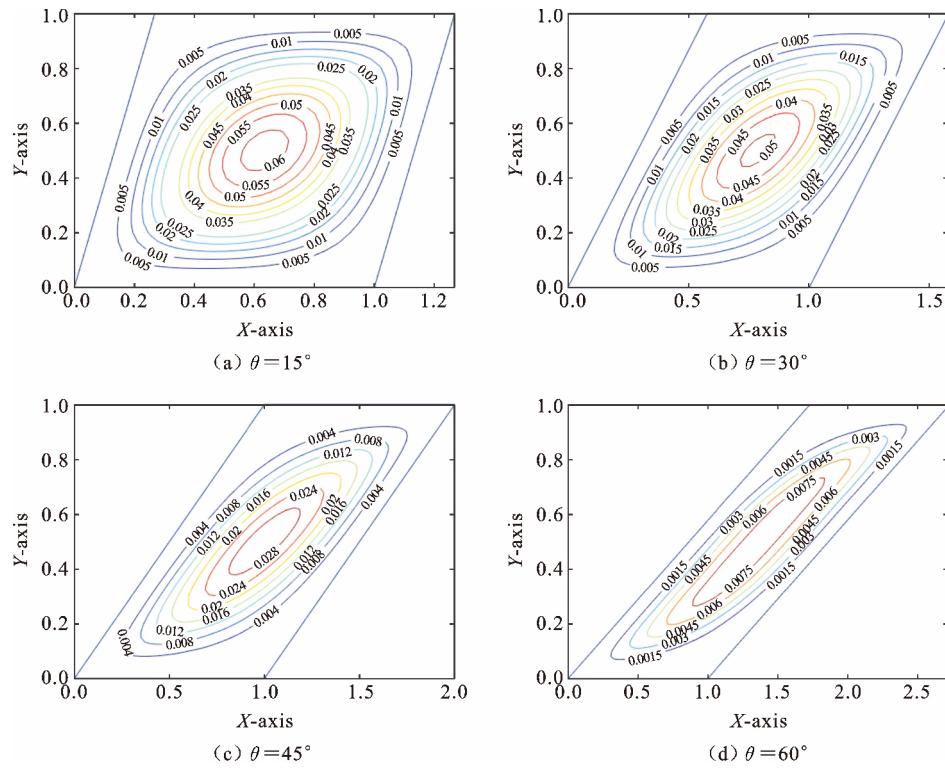


图 11 不同夹角平行四边形固支板的挠度数值解

Fig. 11 Numerical deflections of parallelogram clamped plates

表 5 不同夹角平行四边形板的最大挠度值

Table 5 Maximum deflection of a parallelogram plate with different angles

夹角 θ	15°	30°	45°	60°
广义有限差分法 最大挠度值(m)	0.0625	0.0513	0.0305	0.0085
COMSOL 最大挠度值(m)	0.0629	0.0515	0.0306	0.0085

从上面结果可以明显看出,同等面积的四边固支平行四边形板,在受均布荷载作用下的最大挠度随着 θ 角的增大而减小.

3 结论

本文采用广义有限差分法求解 Kirchhoff 型和 Winkler 型薄板的弯曲问题. 广义有限差分法是一种新型的区域型无网格法,其算法简单,无需网格划分以及数值积分. 由本文的数值算例可知,广义有限差分法可以有效求解薄板弯曲问题,并且具有较高的计算精度和良好的收敛性. 在计算含指数函数型

和周期函数型横向荷载的薄板弯曲问题时,广义有限差分法仅需少量计算节点即可得到与边界粒子法计算结果一致的挠度解. 在计算直角扇形简支薄板弯曲问题时,同样精度条件下,广义有限差分法所需的计算节点数比 COMSOL 软件中的单元数要少.

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Generalized Finite Difference Method for Solving Kirchhoff Plate and Winkler Plate Bending Problems

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Abstract A newly-developed domain-type meshless collocation method, the generalized finite difference method (GFDM), is applied to solve the Kirchhoff and Winkler plate bending problems under different types of loading and varied boundary conditions. In comparison with the traditional finite difference method, the GFDM is free from mesh generation and numerical integration, which can be more flexible when the shape of the computational domain is irregular. Besides, the GFDM keeps the merits of simplicity and wide applicability in the classical finite difference method. In this study, the Kirchhoff and Winkler plate bending problems are governed by the fourth-order partial differential equations under Kirchhoff and Winkler assumptions, respectively. To solve these thin-plate bending problems, the moving least squares theory and fourth-order Taylor series expansion are used to construct the GFDM approximation formulation. Then the derivatives at each node in the computational domain are expressed by the linear combinations of nearby function values and weighting coefficients. The quartic spline is adopted as the weighting coefficient function in the GFDM. Several numerical examples have been tested to verify the accuracy and efficiency of the proposed GFDM. It is found that the generalized finite difference method with fewer discretization nodes provides accurate numerical results for solving Kirchhoff and Winkler plate bending problems under different types of transverse loading (e.g., uniformly distributed loading, hydrostatic pressure loading, exponential-function type loading and periodic-function type loading) and varied boundary conditions (e.g., simply-supported BC, clamped BC and free BC) in various computational domains (e.g., square domain, sectoral domain and parallelogram plate). The convergence analysis shows that the GFDM has a rapid convergence rate of solving thin-plate bending problems. In comparison with the boundary particle method (BPM) and the COMSOL software (the finite element method), the results of the GFDM are in good agreement with those of the BPM, and the GFDM may require fewer nodes to obtain acceptable deflections compared with the COMSOL software.

Key words generalized finite difference method, Kirchhoff plate, Winkler plate, plate bending