The Legendre Pseudorandom Function as a Multivariate Quadratic Cryptosystem

Security and Applications

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   - An OPPRF from the Legendre PRF
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- Their cryptanalysis is still an active and ongoing research! For instance, see [ACG+19, LP19].

The Legendre Symbol and PRF

Definition (Legendre Symbol)

Let p be an odd prime. The Legendre Symbol of a and p is

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & a \ has \ modular \ square - roots \mod p \\ -1, & a \ has \ no \ modular \ square - roots \mod p \\ 0, & if \ a \equiv 0 \mod p \end{cases}$$

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Definition (Sequential Legendre PRF)

Damgård proposed using the sequence of consecutive Legendre symbols with respect to a large prime p as a pseudorandom generator [Dam88]. Let $\{a\}_K$ denote the following sequence.

$$\{a\}_K := \left(\frac{K}{p}\right), \left(\frac{K+1}{p}\right), \ldots, \left(\frac{K+a-1}{p}\right)$$

Pseudo-randomness of the Legendre PRF

There is a vast literature on asserting the high level of pseudo-randomness of quadratic (and higher order) characters.

- Pólya-Vinogradov inequality for character sums. Consecutive values of $\left(\frac{a}{p}\right)$ mimic a random variable i.e.

$$\sum_{a=M+1}^{M+N} \left(\frac{a}{p}\right) \leq \sqrt{p} \log p$$

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- Gyarmati&Mauduit&Sárközy: good cross correlation of the Legendre symbol sequences [GMS14]

Hard problems and cryptographic assumptions

Definition (Shifted Legendre Symbol Problem)

Let K be uniformly sampled from $\mathbb{F}_p$, and define $\mathcal{O}_{Leg}$ to be an oracle that takes $x \in \mathbb{F}_p$ and outputs $\left(\frac{K+x}{p}\right)$. Then the Shifted Legendre Symbol (SLS) problem is to find K given oracle access to $\mathcal{O}_{Leg}$ with non-negligible probability.

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Definition (Multivariate Quadratic (MQ) problem)

Given a random system of quadratic polynomials $\mathbf{f} = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) \in \mathbb{F}[x_1, \ldots, x_n]^m$, find a common zero $\mathbf{x_0} \in \mathbb{F}^n$ of the polynomials $f_1, \ldots, f_m$.

State of the art Legendre PRF key-recovery attacks

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Relevant research directions:

- Could we have sub-exponential key-recovery attacks?

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- Could we have sub-exponential key-recovery attacks?
- What about provable security of the Legendre PRF?
- Would it be possible to connect it to other cryptographic assumptions?

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- Examples

The MQ instance in the undetermined case

- Let's fix an aribtrary $r \in \mathbb{F}^*$ quadratic non-residue. Let $b_i := \left(\frac{K+i}{p}\right)$ and x_i be the corresponding unknown.

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Therefore we have the following four cases.If $b_i = b_{i+1} = 1$, then we know that $x_{i+1}^2 = K + i + 1$ and $x_i^2 = K + i$, hence

$$x_{i+1}^2 - x_i^2 = 1.$$

The MQ instance in the undetermined case (contd.)

If $b_i = b_{i+1} = -1$, then we have that $x_{i+1}^2 = r(K+i+1)$ and $x_i^2 = r(K+i)$, hence

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Finally if $b_i = 1 = -b_{i+1}$ or $b_i = -1 = -b_{i+1}$ then we obtain the following two quadratic equations:

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Remarks:

- Sparse MQ instance without linear terms
- Very peculiar polynomial structure unlike regular MQ instances
- "Minimality" of the sparseness

Example

Let $p = \texttt{0xffffffffffffffffffffdd}$ and $K = \texttt{0x27aaa97c746c22e12d0f}$.

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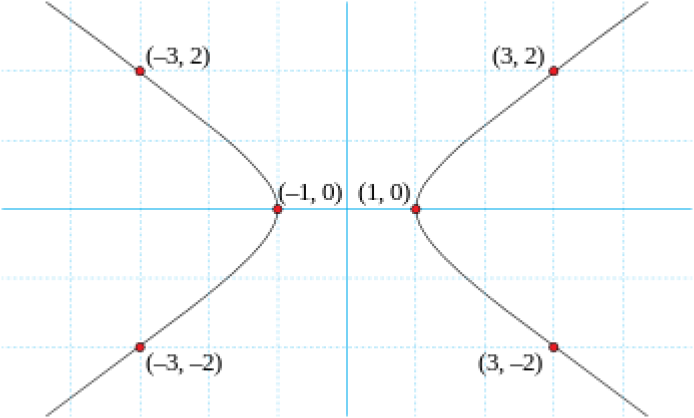
Example

Let $p = \texttt{0xffffffffffffffffffdd}$ and $K = \texttt{0x27aaa97c746c22e12d0f}$.The smallest quadratic non-residue mod p is 2.We display the MQ instance induced by the evaluation of the linear Legendre PRF, $\{6\}_K = (1, 1, 1, -1, -1, 1)$.The complete MQ instance corresponding to $\{6\}_K$ has the following form:

$$\begin{aligned} x_1^2 - x_0^2 &= 1 \\ x_2^2 - x_1^2 &= 1 \\ x_3^2 - 2x_2^2 &= 2 \\ x_4^2 - x_3^2 &= 2 \\ 2x_5^2 - x_4^2 &= 2 \end{aligned}$$

A somewhat cripple analogy



$$x^2 - 2y^2 = 1$$

Gröbner-basis of the ideal

Theorem

Given $\{n\}_K = (b_0, \ldots, b_{n-1})$ and its corresponding ideal $I = \langle f_1, f_2, \ldots, f_m \rangle$, where $m = n - 1$ as defined by the equations from the previous slides. Its Gröbner basis consists of the polynomials g_i, for $i \in [0, n-2]$ such that,

$$g_i = \begin{cases} x_i^2 - x_{n-1}^2 + (n-i), \textit{if} \quad b_{n-1} = 1 \wedge b_i = 1 \\ x_i^2 - r x_{n-1}^2 + r(n-i), \textit{if} \quad b_{n-1} = 1 \wedge b_i = -1 \\ x_i^2 - r^{-1} x_{n-1}^2 + (n-i), \textit{if} \quad b_{n-1} = -1 \wedge b_i = 1 \\ x_i^2 - x_{n-1}^2 + r(n-i), \textit{if} \quad b_{n-1} = -1 \wedge b_i = -1 \end{cases} \tag{1}$$

Specifically, $I = \langle g_0, \ldots, g_{n-2} \rangle$ and $G := (g_i)_{i=0}^{n-2}$ is a reduced Gröbner-basis.

Example (contd.)

The Gröbner-basis of the previous example of $\{6\}_K$ consists of the following quadratic bi-variate polynomials:

$$x_0^2 - x_5^2 + 5$$
$$x_1^2 - x_5^2 + 4$$
$$x_2^2 - x_5^2 + 3$$
$$x_3^2 - 2x_5^2 + 4$$
$$x_4^2 - 2x_5^2 + 2$$

Adding new, independent polynomials to the MQ problem

Observe that in these cases, we can express the modular square root function $\mathrm{sqrt}_p : \mathbb{F}_p^* \to \mathbb{F}_p^*$ as a polynomial function as follows:

$$\mathrm{sqrt}_p(x) = \begin{cases} \pm x^{\frac{p+1}{4}} \mod p, & \text{if} \quad p \equiv 3 \mod 4 \\ \pm x(2x)^{\frac{p-5}{8}}(4x^{\frac{p-1}{4}} - 1) \mod p, & \text{if} \quad p \equiv 5 \mod 8 \end{cases}$$

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By this observation, we can obtain $\mathcal{O}(\log^2 p)$ new polynomials, one for each quadratic term $x_i x_j$:

$$x_i x_j = \mathrm{sqrt}_p(r^{L_0(x_i)+L_0(x_j)}(K+i)(K+j)).$$

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In a similar fashion, we can add new polynomials involving the linear terms of the unknowns for every $i \neq j$:

$$x_i = \mathrm{sqrt}_p(r^{L_0(x_i)-L_0(x_j)}(x_j^2 - r^{L_0(x_j)}(j-i)))$$

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Solving directly the MQ problem

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- In the undetermined case, the Gröbner-basis seemingly does not help at all! It seems that we can't do better than just brute-force.
- New polynomials in the overdetermined case doesn't help. The degree of regularity behaves just like in random MQ systems;

m	n	d_{reg} Random MQ	d_{reg} Legendre MQ
7	7	3	3
8	8	4	4
9	9	4	4
10	10	5	5
11	11	5	5

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$$F_K(a) = \sum_{i=0}^{a-1} 2^{a-1-i}(K+i)^{\frac{p-1}{2}} \mod p$$

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$$F_K(a) = \sum_{i=0}^{a-1} 2^{a-1-i}(K+i)^{\frac{p-1}{2}} \mod p$$

Note that $deg(F_K(a)) = \frac{p-1}{2}$, i.e. the degree of the polynomial representing the Legendre PRF has almost full degree over $\mathbb{F}_p$, that is exponential in the security parameter.

The MinRank attack

We rewrite each generator polynomial f_i in the ideal $I = \langle f_1, \ldots, f_m \rangle$ induced by the Legendre PRF, as folllows:

$$f_i(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + c = \mathbf{x}^T A_i \mathbf{x} + B\mathbf{x} + c,$$

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Each polynomial f_i can be represented in the extension field, in the following form:

$$\mathcal{F}_i(X) = \sum_{i,j=1}^{n} \alpha_{ij} X^{q^{i-1}+q^{j-1}} + \sum_{i=1}^{n} \beta_i X^{q^{i-1}} + \gamma = \mathbf{X}^T M_i \mathbf{X} + N_i \mathbf{X} + \gamma, \qquad (2)$$

Group structure of the solutions of a Legendre key-recovery attack

- We saw that every K such that the symbol of K and $K+1$ is a certain Legendre symbol sequence (e.g., $(1,1)$) lies on a Pell-conic.

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Group structure of the solutions of a Legendre key-recovery attack

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- The solutions of a Legendre key-recovery attack ***lack a group structure***!!!

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Efficient VRF from the Legendre PRF

The Legendre PRF evaluator wants to prove that the following binary relation $\mathcal{R} : \{0,1\}^* \times \{0,1\}^*$ holds:

$$\mathcal{R}_{PRF} = \left\{ \left(\{n\}_K, K \right) : \{n\}_K = \left(\left(\frac{K}{p} \right), \left(\frac{K+1}{p} \right), \ldots, \left(\frac{K+n-1}{p} \right) \right) \right\},$$

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which is equivalent to the relation:

$$\mathcal{R}^*_{PRF} = \left\{ \left(\{n\}_K, \mathbf{x} \right) : (f_1(\mathbf{x}) = 0, f_2(\mathbf{x}) = 0, \ldots, f_m(\mathbf{x}) = 0) \right\},$$

where the multivariate quadratic polynomials $(f_i)_{i=1}^m$ are defined on the previous slides.

The arithmetic circuit representation of the Legendre VRF statement

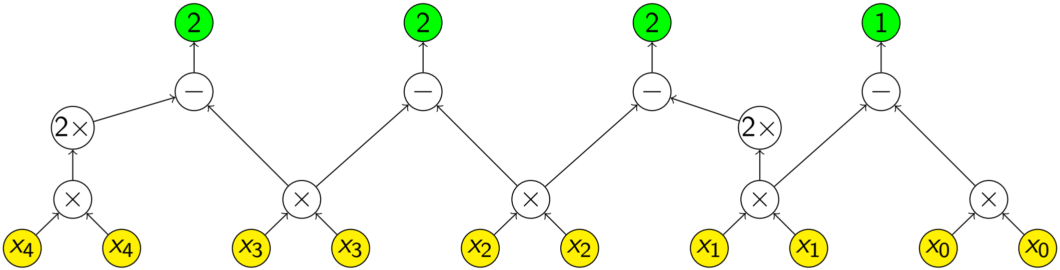

Figure: Arithmetic circuit representation of the ZKP statement that proves the relation $\mathcal{R}_{PRF} = \{\{5\}_K = (1, 1, -1, -1, 1), K\}$, where 2 is the least quadratic non-residue. Applying our arithmetization the PRF evaluator proves that it knows the zeros of the following polynomials $(2x_4^2 - x_3^2 = 2, x_3^2 - x_2^2 = 2, x_2^2 - x_1^2 = 2, x_1^2 - x_0^2 = 1)$. Nodes with $2x$ denote a multiplication gate, where one of the inputs is the constant quadratic non-residue 2. Note, that the arithmetic circuit has a constant multiplicative depth of two.

Overview of the most important VRF constructions

		Time complexity		
	$\lvert\pi\rvert$	Prove	Verify	Assumption
[GNP+15]	$1\mathbb{G}$	$1\mathrm{H} + 1\mathbb{G}$	$1\mathrm{H} + 1\mathbb{G}$	Factoring
[PWH+17]	$1\mathbb{G} + 2\mathbb{F}_p$	$3\mathrm{H} + 2\mathbb{G}$	$3\mathrm{H} + 4\mathbb{G}$	EC-DDH
[BGLS03]	$1\mathbb{G}$	$2\mathrm{H} + 1\mathbb{G}$	$1P$	co-DH
[DY05]	$1\mathbb{G}$	$1\mathbb{G} + 1\mathbb{F}_p$	$2\mathbb{G} + 2P$	q-DBDHI
[LBM20]	$1\mathbb{G}$	$1\mathbb{G}$	$1P$	q-DDHE
[EKS+20]	$\mathcal{O}(k + l)$	$\mathcal{O}(kl)$	$\mathcal{O}(kl)$	Module-SIS
Legendre†	$3\mathbb{G}$	$9n\mathbb{G}$	$n\mathbb{G} + 3P$	SLS, KEA
Legendre*	$\mathcal{O}(\log(n))\mathbb{G}$	$\mathcal{O}(n\log(n))\mathbb{G}$	$\mathcal{O}(\log(n))\mathbb{G}$	SLS

Table: Overview of various VRF constructions. Hashing, group operations, exponentiation and pairings are denoted as $\mathrm{H}, \mathbb{G}, \mathbb{F}_p, P$ respectively. n is the length of the Legendre sequence.

Oblivious Pseudorandom Functions (OPRFs)

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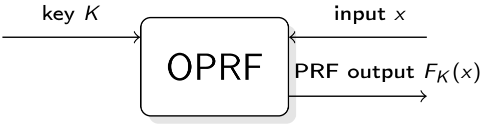

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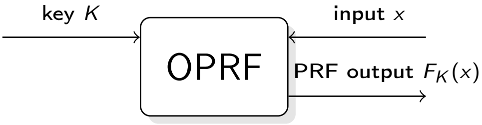

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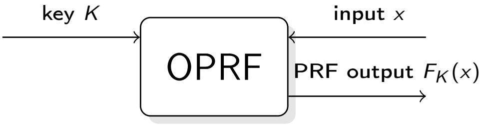

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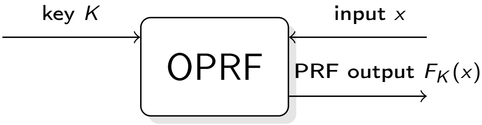

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- Immediate (inefficient) idea: take `_insert your favourite PRF here_` and evalute it generically in a two-party setting. This works for any PRF, like AES, SHA-3 etc.
- The MPC realisation of the Legendre PFF by [GRR+16], implies an OPRF protocol.

The Legendre OPRF

Participants: sender $\mathcal{S}$, receiver $\mathcal{R}$.

Preprocessing:

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Oblivious **Programmable** PRFs (O**P**PRFs)

- In addition to being an OPRF, sender can also program the output of the OPRF at certain points. Specifically, sender can choose $F_K(\cdot)$ such that it holds for prescribed x_i and y_i, that $y_i = F_K(x_i)$, for some $i \in [0, n]$, where n is the number of programmed points.

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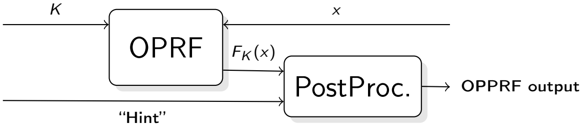
- Kolesnikov et al [KMP+17] introduces three *generic* transformations to transform any OPRF generically to being an O**P**PRF.



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- Luckily, in a PSI application, only a handful programmed points are needed...

Programming the Legendre PRF

- The programming constraints can be expressed as follows: find a p prime, s.t. it holds for all $i \in [0, n)$: $y_i = \left(\frac{x_i}{p}\right) = \left(\frac{p}{x_i}\right)(-1)^{\frac{(p-1)(x_i-1)}{4}}$.

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- Note, p is a solution of a simultaneous congruence system: $p \equiv m_i \mod x_i$, for all $i \in [0, n)$, where $m_i \in M_i$. Solve this by the Chinese-Remainder Theorem.

Overview of the state-of-the-art O**P**PRFs

OPPRF	Programming complexity	Hint size	Online communication complexity	Constraint on no. of programmed points	No. of evaluations
Lagrange	$O(n^2)$	$O(n)$	$(n+kn)\ \mathbb{G}$	space-efficiency	any
Garbled BF.	$O(n\lambda_{\mathrm{BF}})$	$n\lambda_{\mathrm{BF}}$	$(60n+kn)\ \mathbb{G}$	space-efficiency	any
Table-based	$O(n)$	$O(n)$	$(n+kn)\ \mathbb{G}$	space-efficiency	1
Legendre CRT	$O(n \log n)$	1	$\mathcal{O}(n)\ \mathbb{G}$	depends on λ	any
Legendre bruteforce	$O(2^n)$	1	$1\ \mathbb{G}$	time-efficiency	any

Table: Comparison of our results with the generic O**P**PRF constructions of [KMP+17] (relying on the OPRF of [KKRT16]). The number of programmed input positions is denoted as n, λ_{BF} is the soundness parameter of the Bloom filter, while k denotes the number of base-OTs, typically $k \approx 4\lambda$.

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- More cryptographic assumptions?
- Real-world deployments of the Legendre PRF? Let us know!

Acknowledgements and Q&A

We thank **Gergő Zábrádi** for insightful discussions.

Thanks for the attention!
Questions?

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