Project 1: Image Processing & Edge Detection

Report

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Part 1: Discussion

1). Convolution Theorem:

Consider a linear time invariant system with input $I(x)$ and impulse response $h(x)$. Recall that the impulse response is defined as the output of the system when the input is an impulse $\delta(x)$. We may sketch the proof for the formula given by $h(x) * I(x) = \Sigma h(x - u)I(u)$ as below:

$$I(x) = \sum_{u=-\infty}^{\infty} h(x-u)I(u) \ \ (*)$$

Now, with $\delta(x)$ as an input, we have h(x) as the output. We can draw that $\delta(x) \to$ LTI System $\to h(x)$.
Since the system is time invariant, with $\delta(x\text{-}u)$, the output is $h(x\text{-}u)$.
Then by $(*)$ & Linearity, with input $I(x)$, the output is

$$\sum_{u=-\infty}^{\infty} I(u)\, h(x-u) = h(x) * I(x)$$

2). Laplacian Operator

Consider 2D image $I(x, y)$ and its Laplacian given by $\Delta \mathrm{I} = \frac{\partial^2 I}{\partial \mathrm{x}^2} + \frac{\partial^2 I}{\partial \mathrm{y}^2}$, we will show that the Laplacian is rotationally invariant.

Consider an arbitrary direction r which results in an angle θ with the x-axis. It follows that $(\mathrm{x}, \mathrm{y}) = (r\cos\theta,\ r\sin\theta)$. Then,

$$\frac{\partial I}{\partial r} = \frac{\partial I}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial r}$$

By Chain Rule, we have

$$\frac{\partial^2 I}{\partial \mathrm{r}^2} = (\frac{\partial^2 I}{\partial \mathrm{x}^2}\cos\theta + 0)\cos\theta + (\frac{\partial^2 I}{\partial \mathrm{x}\,\partial \mathrm{y}}\sin\theta + 0)\cos\theta + (\frac{\partial^2 I}{\partial \mathrm{x}\,\partial \mathrm{y}}\cos\theta + 0)\sin\theta + (\frac{\partial^2 I}{\partial \mathrm{y}^2}\sin\theta + 0)\sin\theta$$

i.e.

$$\frac{\partial^2 I}{\partial \mathrm{r}^2} = \frac{\partial^2 I}{\partial \mathrm{x}^2}\cos^2\theta + \frac{2\,\partial^2 I}{\partial \mathrm{x}\,\partial \mathrm{y}}\cos\theta\sin\theta + \frac{\partial^2 I}{\partial \mathrm{y}^2}\sin^2\theta$$

Now, for r' which is perpendicular to r, we have similarly from Chain Rule that

$$\frac{\partial^2 I}{\partial \mathrm{r}'^2} = \frac{\partial^2 I}{\partial \mathrm{x}^2}\sin^2\theta - \frac{2\,\partial^2 I}{\partial \mathrm{x}\,\partial \mathrm{y}}\cos\theta\sin\theta + \frac{\partial^2 I}{\partial \mathrm{y}^2}\mathrm{con}^2\,\theta$$

Adding the above two equations, we obtain that

$$\frac{\partial^2 I}{\partial \mathrm{r}^2} + \frac{\partial^2 I}{\partial \mathrm{r}'^2} = \frac{\partial^2 I}{\partial \mathrm{x}^2}(\cos^2\theta + \sin^2\theta) + \frac{\partial^2 I}{\partial \mathrm{y}^2}(\sin^2\theta + \cos^2\theta) = \frac{\partial^2 I}{\partial \mathrm{x}^2} + \frac{\partial^2 I}{\partial \mathrm{y}^2} = \Delta \mathrm{I}$$

Since r is selected arbitrarily, we conclude that Laplacian is invariant to rotations as the sum of any two orthogonal directions.

3) Condition of Linear Variation

By Marr and Hildreth, given a twice differentiable function f(x, y), for which (i.) $f_{xx} = 0$ along y direction & (ii.) f varies linearly along y direction, i.e. f_y has constant slope in y direction (the condition of linear variation). Then we have the following theorems:

(1.) The slope of $\frac{\partial^2 f}{\partial x^2}$ is greater than the slope of $\frac{\partial^2 f}{\partial r^2}$ for r any other direction.

(2.) $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ on ℓ is equivalent to $\frac{\partial^2 f}{\partial x^2} = 0$ if and only if $f(0, y)$ is constant or linear on ℓ, for ℓ an open line segment.

These theorems imply that the zero-crossings of Laplacians are equivalent to those of $\frac{\partial^2 f}{\partial x^2}$, which is also the direction of maximal slope of the second directional derivatives.

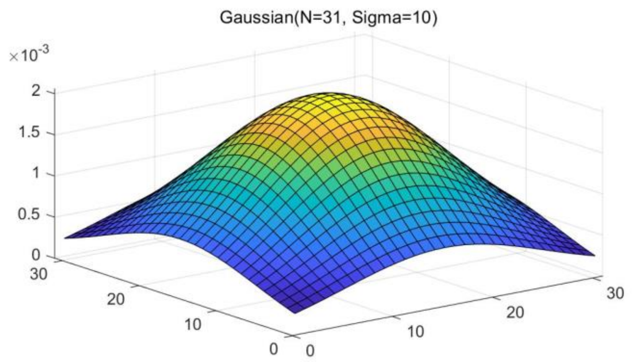
Part 2: Filters

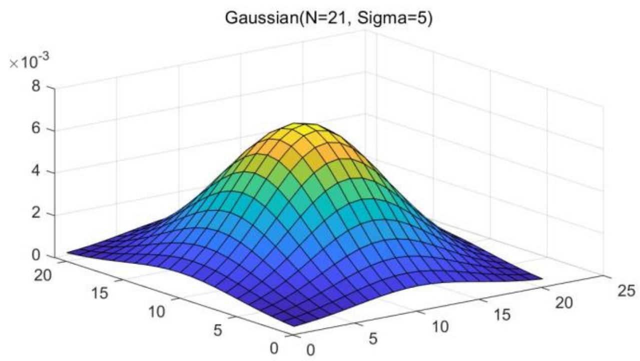
1). 2D-Gaussian:

The function *make2DGaussian(N, σ)* takes 2 parameters: *N* the size & σ the standard deviation of Gaussian Distribution. In 2D, we have

$$G(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

For practical use, we need to normalize the filter to 1



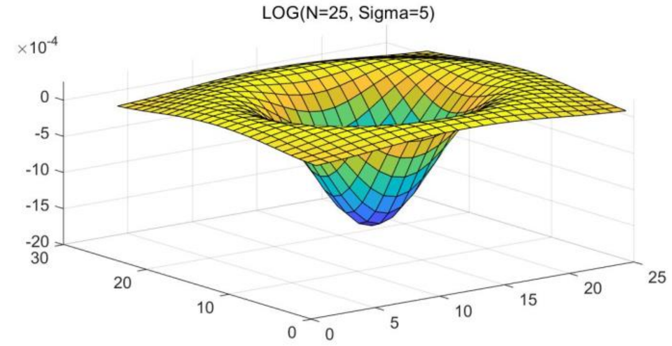


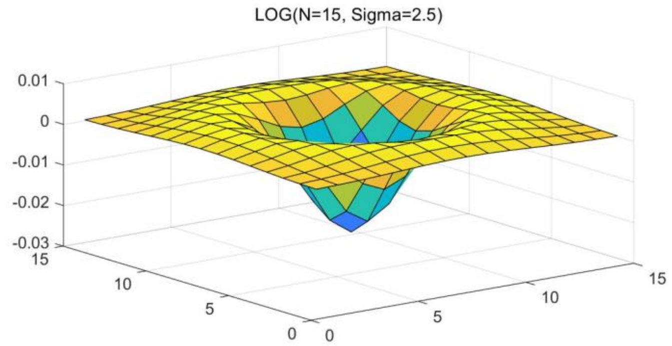
2). 2D Laplacian of Gaussian

Function *make2DLOG(N, σ)* takes 2 parameters as well, *N* the size & σ the standard deviation of Gaussian Distribution (of Gaussian Part). In 2D, we have

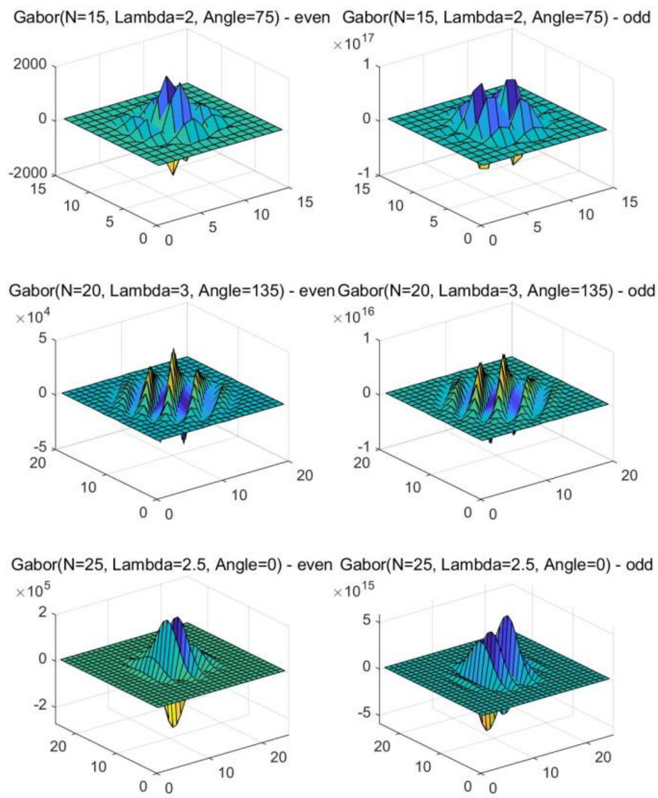
$$\nabla^2 G(x, y) = -\frac{1}{\pi\sigma^4}\left(1 - \frac{x^2 + y^2}{2\sigma^2}\right)e^{-\frac{x^2+y^2}{2\sigma^2}}$$

This time, to normalize, we want the sum of values in the filter to be zero.


LOG(N=25, Sigma=5)
×10^-4
0
-5
-10
-15
-20
30
20
10
0
0
5
10
15
20
25


LOG(N=15, Sigma=2.5)
0.01
0
-0.01
-0.02
-0.03
15
10
5
0
0
5
10
15

3). 2D Gabor


Gabor(N=15, Lambda=2, Angle=75) - even
Gabor(N=15, Lambda=2, Angle=75) - odd
×10^17
Gabor(N=20, Lambda=3, Angle=135) - even
Gabor(N=20, Lambda=3, Angle=135) - odd
×10^4
×10^16
Gabor(N=25, Lambda=2.5, Angle=0) - even
Gabor(N=25, Lambda=2.5, Angle=0) - odd
×10^5
×10^15

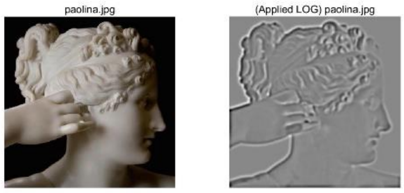
Even & Odd Gabor Functions take size N, wavelength λ and Angle θ as parameters, as we set by default phase offset to be zero, standard deviation of Gaussian part to be equal to λ and ellipticity to be 1.
Then in 2D, we have

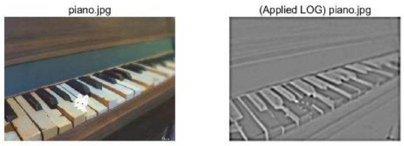
$$G_{even} = e^{-\frac{x^2+y^2}{2\sigma^2}} \cdot \cos\frac{2\pi(x\cos\theta + y\sin\theta)}{\lambda}$$
$$G_{odd} = e^{-\frac{x^2+y^2}{2\sigma^2}} \cdot \sin\frac{2\pi(x\cos\theta + y\sin\theta)}{\lambda}$$

To normalize, we set them to 1.

Part 2: Filters

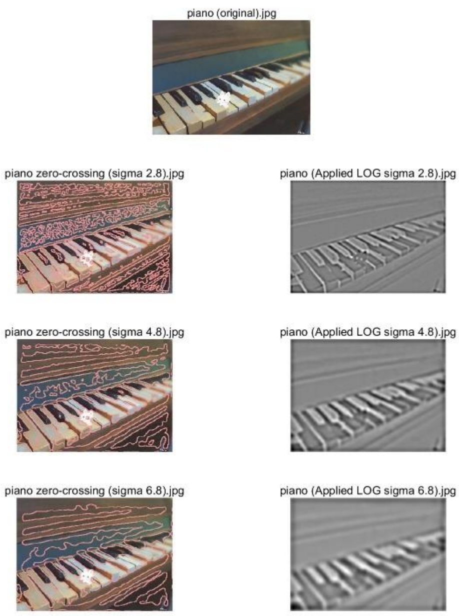
1). Convolution of LOG

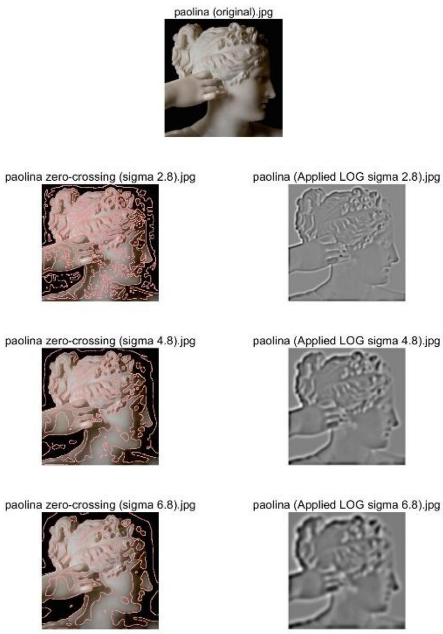




Note that the image size (in pixel) is approximately 300-400. We take N=500 as size of filter.
For controlling most of pixels, the parameter σ is set by 2.8

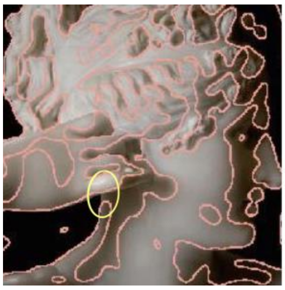
2). Zero-Crossings with LOG

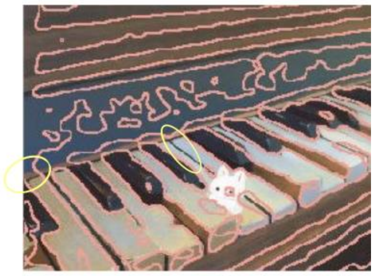




Take size N=500, choose sigma = 2.8, 4.8, 6.8, apply LOG filter respectively, the results are shown below.
For both pictures, we used the same parameters here.
Recall that the strategy to find zero-crossings is based on checking the sign on neighborhood of each point.

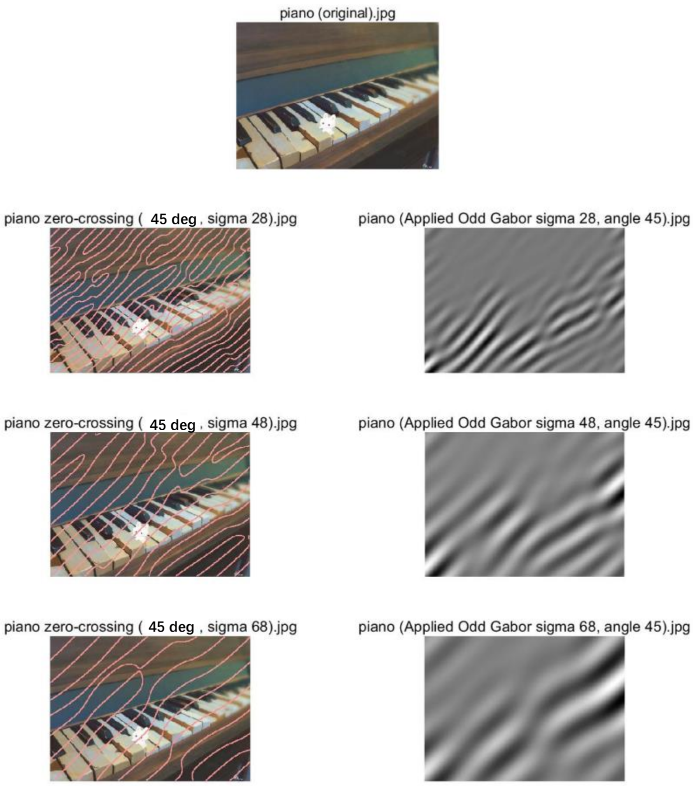
3). Some Discussions





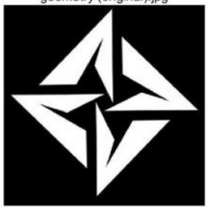
Note that the algorithm generally works well with both images. But when there are intersections of 2 orthogonal lines, some errors may occur.
Moreover, a smaller σ results in a clearer image with more obvious edges and yields more strict distinction between edges. But on the other hand, it's more possible to contain some erroneous edges.

4) Gabor Filters

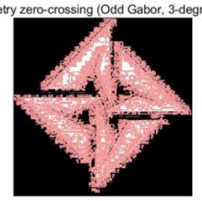


Here we provide results focusing on just 45 degrees. It works well in general wrt. Convoluted graphs.
However, if we tried in 3 orientations with Gabor, it becomes a bit of messy, for both images, especially when choosing relatively small σ.
Moreover, we notice that a small wavelength can result in more effective edge detection.
Here we also show provide another example with some simpler figures to validate the accuracy of our Gbaor filter. The results are shown as below.

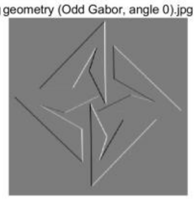
geometry (original).jpg



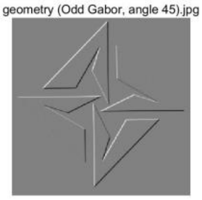
geometry zero-crossing (Odd Gabor, 3-degree).jpg



geometry (Odd Gabor, angle 0).jpg



geometry (Odd Gabor, angle 45).jpg



geometry (Odd Gabor, angle 90).jpg