Coherent detection

- General Model: a flat Rayleigh fading channel:

$$y[m] = h[m]x[m] + w[m] \tag{1}$$

where $w[m] \sim \mathcal{CN}(0, n_0)$.

Definition: SNR, received signal-to-noise ratio per (complex) symbol time

$$snr \stackrel{\text{def}}{=} \frac{\text{average Rx signal energy per (complex) symbol time}}{\text{noise energy per (complex) symbol time}} \tag{2}$$

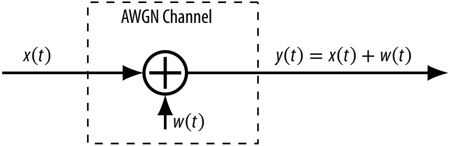
- We compute the average energy as average power by symbol time
  - We work on the equivalent low pass model
  - For noise, the psd is n_0, power is $n_0 W$ and energy $n_0 W \cdot T$
    - since $W = 1/T$, we have energy n_0.

Important: Note that, in general, the *snr* is not the e_b/n_0

AWGN and coherent detection

1. Case $h[m] = 1, x[m] = \pm a$, we have the AWGN channel, the detector yields
$$y = x + w \Rightarrow r = \mathrm{Re}[y] = x + z \tag{3}$$
that is a sufficient statistic, where $z \sim \mathcal{N}(0, n_0/2)$.
2. We have
   - Average Rx Signal Energy: a^2
   - Average Rx Noise Enery: n_0
   - The *snr* yields: a^2/n_0
   - Average Rx Noise in the detector (for BER computation): $n_0/2$
3. The BER
$$p_e = Q\left(\sqrt{\frac{a^2}{n_0/2}}\right) = \left\{snr = \frac{a^2}{n_0}\right\} \Rightarrow p_e = Q\left(\sqrt{2snr}\right) \tag{4}$$

Note: here the *snr* has the same value as the E_b/N_0

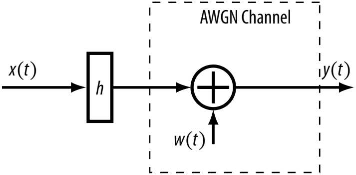


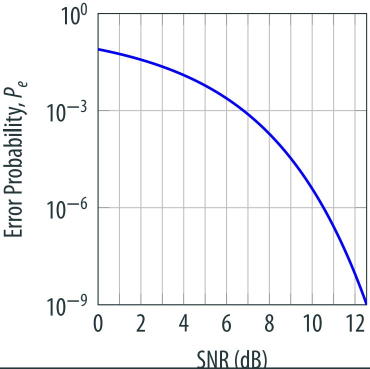
Flat LTI known channel

Case $h[m]=h$, *constant and known*

1. Case $h[m]=h$, *constant and known*, we have a similar detection as in the AWGN channel, for one symbol, $y=hx+w \Rightarrow r=\mathrm{Re}[(h/|h|)^*y]=|h|x+z$ is a sufficient statistic, where $z\sim\mathcal{N}(0,n_0/2)$, the BER,
2. The $snr=(|h|a)^2$
3. The BER

$$p_e=Q\left(\sqrt{\frac{(|h|a)^2}{n_0/2}}\right)=\left\{snr=\frac{(|h|a)^2}{n_0}\right\}\Rightarrow p_e=Q\left(\sqrt{2snr}\right) \tag{5}$$





Non-Coherent Detection in Flat Fading

- Model: a flat Rayleigh fading channel:

$$y[m] = h[m]x[m] + w[m] \tag{6}$$

  where $w[m] \sim \mathcal{CN}(0, n_0)$, $h[m] \sim \mathcal{CN}(0, 1)$, and
  1. we do not specify the relation between $h[m]$ for different ms
  2. we do not have any a priori information on $h[m]$: *Non-coherent* communication

  Note that the channel is *flat*.
- First consider a BPSK with amplitude $x[m] = \pm a$, independent along time
  - this scheme in this Rayleigh channel always fails
  - even if $w[m] = 0$, the phase of $y[m]$ is uniform distributed in 0,2π regardless of $x[m]$.
- *Orthogonal signaling* (a special coding):
  - we use two samples (*channel uses*) to transmit

$$\mathbf{x}_A \stackrel{\text{def}}{=} \begin{pmatrix} x[0] \\ x[1] \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \text{ or } \mathbf{x}_B \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ a \end{pmatrix}; \text{ and detect with } \begin{pmatrix} y[0] \\ y[1] \end{pmatrix} \tag{7}$$

Detection in orthogonal signaling

1. Detection

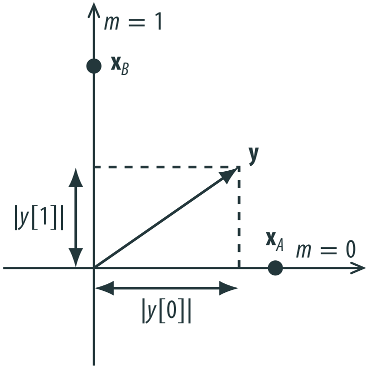
- The detection yields:

$$\ln\left\{\frac{f(\mathbf{y}|\mathbf{x}_A)}{f(\mathbf{y}|\mathbf{x}_B)}\right\} = \frac{\left(|y[0]|^2 - |y[1]|^2\right)a^2}{(a^2+n_0)n_0} \underset{\mathbf{x}_B}{\overset{\mathbf{x}_A}{\gtrless}} 0 \quad (8)$$

- it follows

$$|y[0]|^2 \underset{\mathbf{x}_B}{\overset{\mathbf{x}_A}{\gtrless}} |y[1]|^2 \quad (9)$$

- To compute the error rate (ER) we have in $|y[0]|^2$ and $|y[1]|^2$ signal plus noise to the square, i.e. we compare the energies: *energy* or *square-law* detector.



Energy detector

Step 2. The *snr*

- When $\mathbf{x}_A$ is transmitted, we receive $ha + w$ in one axis and w in the other one.
- These are independent and circular Gaussian, with variance $a^2 + n_0$ and n_0.
- The square values, $|y[0]|^2$ and $|y[1]|^2$, are exponentially distributed with mean $a^2 + n_0$ and n_0
- We have
  1. Average Rx Signal Energy: $a^2/2 \Leftarrow$ we average $|h|^2 a^2$ over the possible values of $|h|^2$, for 1/2 of the symbol time.

$$\text{av. Rx sig. energy per sym. time} = \int |h|^2 a^2/2 f_{|H|^2}(|h|^2) d|h|^2 = a^2/2 \tag{10}$$

     since $|H|^2 \sim \mathrm{Exp}(1) = \exp(-|h|^2)$ and its mean is 1
  2. Average Rx Noise Enery: n_0
  3. The *snr* yields: $a^2/(2n_0)$

Step 3. P_e: the pb of error can be computed by direct integration:

$$p_e = \Pr\{|y[1]|^2 > |y[0]|^2 | \mathbf{x}_A\} = \left[2 + \frac{a^2}{n_0}\right]^{-1} \tag{11}$$

- For the energy detector: $snr = \frac{a^2/2}{n_0} \Rightarrow p_e = \frac{1}{2(1+snr)} \overset{snr \gg 0}{\approx} \frac{1}{2snr}$
  - If $p_e \leq 10^{-3}$ we need $snr \approx 500 \rightarrow SNR \approx 27$ dB !!!

Notes on the error

- If $\mathbf{x}_A$ is transmitted we receive two values $y[0] = ah[0] + w[0]$ and $y[1] = w[1]$ and measure their energies:

$$\begin{array}{lll} y[0] \sim \mathcal{CN}\left(0, a^2 + n_0\right) & |y[0]| \sim \mathrm{Rayleigh}(a^2 + n_0) & |y[0]|^2 \sim \mathrm{Exp}(a^2 + n_0) \\ y[1] \sim \mathcal{CN}(0, n_0) & |y[1]| \sim \mathrm{Rayleigh}(n_0) & |y[1]|^2 \sim \mathrm{Exp}(n_0) \end{array}$$

  where $h \sim \mathcal{CN}(0, 1)$, the exponential pdf $\mathrm{Exp}(a) = 1/a \cdot \exp\left(-x/a\right)$ for a r.v. X and the Rayleigh pdf $\mathrm{Rayleigh}(a) = 2x/a \cdot \exp\left(-x^2/a\right)$
- The probability of error[1] is the probability that $z_1 = |y[1]|^2$ be larger than $z_0 = |y[0]|^2$,

$$\begin{aligned} p_e = \Pr(Z_1 > Z_0|\mathbf{X}_A) &= \int_0^\infty \int_{z_0}^\infty f_{Z_1Z_0|\mathbf{X}_A}(z_1, z_0|\mathbf{x}_A) dz_1 dz_0 \\ &= \{Z_1 \perp\!\!\!\perp Z_0 | \mathbf{X}_A\} = \int_0^\infty \left[\int_{z_0}^\infty f_{Z_1}(z_1) dz_1\right] f_{Z_0}(z_0) dz_0 \\ &= \int_0^\infty \left[\int_{z_0}^\infty \frac{1}{n_0}\, \mathrm{e}^{-\frac{z_1}{n_0}}\, \mathrm{d}z_1\right] \frac{1}{n_0 + a^2}\, \mathrm{e}^{-\frac{z_0}{n_0 + a^2}} dz_0 = \frac{n_0}{2n_0 + a^2} \end{aligned}$$

[1] See Pb 2.15 in Problemas de CD by Payán

Non-coherent vs. AWGN channel

- It is interesting to compare the non-coherent case with the AWGN channel (no fading) already seen.
- If $x = \pm a$, $snr = a^2/n_0$, and

$$p_e = Q\left(\sqrt{\frac{a^2}{n_0/2}}\right) = Q\left(\sqrt{2snr}\right) \tag{12}$$

  - That decays exponentially

$$Q(x) < \frac{1}{2}\,\mathrm{e}^{-x^2/2}, \qquad x > 0 \tag{13}$$

- Hence it decays exponentially with *snr* in the AWGN channel while just inversely with *snr* in the fading channel.
  - For $p_e = 10^{-3}$ we need $SNR \approx 7$ dB

Coherent detection

- We could think of estimating the channel, $h[m]$, by using some pilot or training sequence.
  - The accuracy of the estimation depends on the coherence time of the channel:
    - for North-American Mobile system IS-136, 30 kHz channel, and $D_s = 100$ Hz, $T_c \approx 80$ symbols ($T = 1/30$ ms, $T_c = 1/(4D_s) = 2.5$ ms)
- *Assume perfect channel estimation*

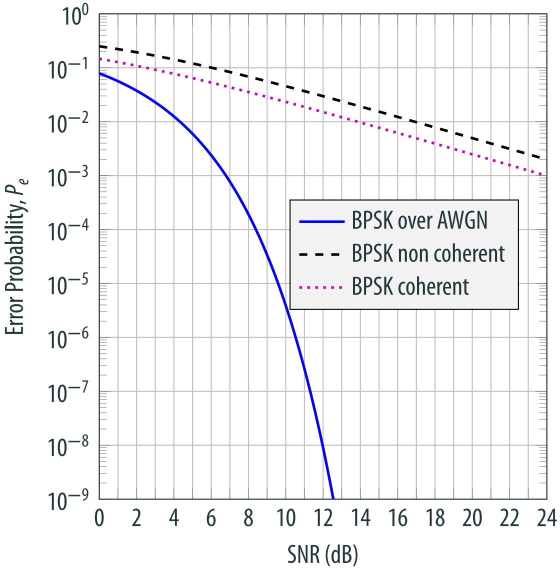
1. For one symbol, $y = hx + w \Rightarrow r = \text{Re}[(h/|h|)^* y] = |h|x + z$ is a sufficient statistic, where $z \sim \mathcal{N}(0, n_0/2)$
2. We have
   - **2.1.** Average Rx Signal Energy: $a^2 \Leftarrow$ we average $|h|^2 a^2$ over the possible values of $|h|^2$.
     - Similar case than before, but we always transmit a signal $\pm a$ hence $1/2$ vanishes
   - **2.2.** Average Rx Noise Energy: n_0
   - **2.3.** The *snr* yields: a^2/n_0
3. The BER depends on the value of h, given (*conditioned to*) a value of h we have

$$p_e(h) = Q\left(\sqrt{\frac{(a|h|)^2}{n_0/2}}\right) = Q\left(\sqrt{2|h|^2 snr}\right) \tag{14}$$

Averaging over the values of h. Hint: integration by parts, Pb 5.3 in Problemas de CD by Payán.

$$p_e = \mathbb{E}_H\left[Q\left(\sqrt{2|h|^2 snr}\right)\right] = \frac{1}{2}\left(1 - \sqrt{\frac{snr}{1+snr}}\right) \overset{snr \gg 0}{\approx} \frac{1}{4snr} \tag{15}$$

Comparison



- The coherent and non-coherent are 3 dB one from each other
- The coherent is 17 dB from the AWGN at 10^{-3}
- Note: the non-coherent uses orthogonal modulation, with two channel uses

Monte Carlo Simulation of Complex AWGN (Matlab/Octave)

Exercise 6.1 (Complex BPSK Monte Carlo Simulation (Matlab/Octave))

Complete an run the following code

```
clear all, close all
Np=1e5; %Number of bits (BPSK) Tx
eb=1; a=sqrt(eb); Es=eb; %XA=-a, XB=a
SNRdB=0:1:15; %Range of Eb/NO to simulate
NdB=length(SNRdB); sqNdB=ceil(sqrt(NdB));
for k1=1:NdB
    Bn=randi([0 1],1,Np); %Generate bits
    xn=(Bn*2*a-a); %BPSK % Symbols (-a,a)
    ebno=10.^(SNRdB(k1)/10); %ebno, nat. unit.
    no=eb/ebno; sigma=sqrt(no/2); %sdt del ruido
    hn=FILL...; %Flat Rayleigh Channel
    wn = sigma*(randn(1,Np))+1j*sigma*(randn(1,Np));%Re & Im are
         independent
    yn = hn.*xn+wn; %Channel AWGN
    bn = (sign(real(FILL...*yn))+1)/2;%Detection
    bnAWGN = (sign(real(xn+wn))+1)/2;%Detection
    BER(k1)=sum(bn~=Bn)/Np; %MC error analysis
    BERAWGN(k1)=sum(bnAWGN~=Bn)/Np; %MC error analysis
end
figure(1), semilogy(SNRdB,BERAWGN), title('BER BPSK')
xlabel('E_b/N_o (dB)'), ylabel('BER'), grid on, hold on
semilogy(SNRdB,BER), legend('AWGN','Rayleigh')
```

Fading

- The problem is not the knowledge of the channel
- But the high probability of having a deep fade
- By inspection of (14) we conclude that p_e is high when $|h|^2 snr$ is near to one or below
- We compute the probability of having a *deep fade* as

Deep Fade

- Deep fade event: $|h|^2 < \frac{1}{snr}$
- $\Pr(\text{deep fade})$:

$$\Pr(|h|^2 snr < 1) = \Pr\left(|h|^2 < \frac{1}{snr}\right) = \int_0^{1/snr} e^{-x} dx \approx \frac{1}{snr}$$

Hint: the aproximation is for high snr, where for $x \gg 0$, then $1 - \exp(-1/x) \approx 1/x$

- *Even if the snr is 10^2, the probability of having a deep fading is about 10^{-2}*
- *It has the same order of magnitude as the error probability itself*
  - *Indicating that the p_e is driven by the pb of having a fading*

BPSK and QPSK

Exploit the degrees of freedom

The main objective in the system design is to exploit all the available degrees of freedom

- In the coherent BPSK we could sent the same throughput modulating the quadrature
  - Since both transmissions are orthogonal we have the same Pb of error for a bit, p_e,
  - Transmitting double energy per symbol (same per bit)
- This is the QPSK, with symbols $\pm a \pm ja$
  - its $snr = 2a^2/n_0$, hence, *in AWGN*, $p_e = Q\left(\sqrt{2a^2/n_0}\right) = Q\left(\sqrt{snr}\right)$
  - while for the BPSK, *in AWGN*, $snr = a^2/n_0$ and $p_e = Q\left(\sqrt{a^2/n_0}\right) = Q\left(\sqrt{2snr}\right)$
- *In Rayleigh fading* channel

$$p_e = \mathbb{E}_H\left[Q\left(\sqrt{|h|^2 snr}\right)\right] = \frac{1}{2}\left(1 - \sqrt{\frac{snr}{2+snr}}\right) \overset{snr \gg 0}{\approx} \frac{1}{2snr} \tag{16}$$

  where the approximation is for high snr.
- Hint: same as in the BPSK replacing $2snr$ by snr

4-PAM versus QAM

- A 4-PAM could be used instead QAM, with alphabet $\{-3b, -1b, 1b, 3b\}$ for some b ($E_{av} = 5b^2$).
  - The p_e in AWGN channel yields

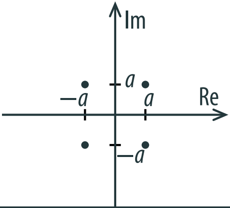
$$p_e = \frac{3}{2} Q\left(\sqrt{\frac{2b^2}{n_0}}\right) = \frac{3}{2} Q\left(\sqrt{\frac{2 \cdot 5b^2}{5n_0}}\right) = \frac{3}{2} Q\left(\sqrt{\frac{2snr}{5}}\right) \tag{17}$$

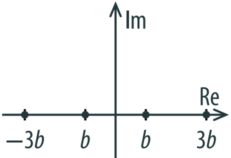
  - The p_e in Rayleigh fading yields

$$p_e \overset{snr \gg 0}{\approx} \frac{5}{4snr} \tag{18}$$

  - To achieve approximately the same p_e than the QAM we need $b = a$
  - By comparing (16) and (18) for the same snr we have a factor of 2.5 $\rightarrow$ 4 dB loss.
  - This loss grows for larger constellations

It is more efficient to pack, for desired minimum distance, a given number of constellation points in a higher dimension than in a lower-dimensional space: *exploit the degrees of freedom*





DBPSK

- The *differential* BPSK (DBPSK), $u[m] = \pm 1$,

$$x[m] = u[m]x[m-1] \tag{19}$$

- where we assume the channel does not changes very much from one symbol to another
- have a loss of 3 dB compared with coherent BPSK,
  - The detector is similar to the non-coherent detector previously analyzed, where we check for the reception in two consecutive values. Hence the BER is the same one.
- hence same performance than non-coherent orthogonal one $\Rightarrow$ but it uses *just one real dimension per single symbol time*

Results on constellations

Table 3.1 Performance of coherent and non-coherent schemes under Rayleigh fading. The data rates are in bits/s/Hz, which is the same as bits per complex symbol time. The performance of differential QPSK is derived in Exercise 3.5. It is also 3-dB worse than coherent QPSK.

Scheme	Bit error prob. (High SNR)	Data rate (bits/s/Hz)
Coherent BPSK	1/(4SNR)	1
Coherent QPSK	1/(2SNR)	2
Coherent 4-PAM	5/(4SNR)	2
Coherent 16-QAM	5/(2SNR)	4
Non-coherent orth. mod.	1/(2SNR)	1/2
Differential BPSK	1/(2SNR)	1
Differential QPSK	1/SNR	2

Source: Tse and Viswanath

Some are more spectrally efficient than other

BUT

all have a p_e that decays very slowly: like $1/snr$

Diversity

- To avoid deep fades we may send the same information along severals signal paths
  - every path fading independently
  - a reliable communication is achieved if at least there is one strong path
- There are several types of diversity. The most basic ones being:
  - *time diversity* can be achieved through *coding* and *interleaving*
  - *frequency diversity* is deployed transmitting a signal through different frequencies
  - *space diversity* is based on transmitting using arrays of antennas, i.e. multiple-input multiple-output (MIMO) antennas
- Other diversity involve
  - macrodiversity: if a terminal is connected to two or more central nodes, e.g. base estations in PLMN
  - path: if the signal is transmitted by two different infrastructure, different Tx and/or Rx locations
- Other solutions such as *frequency hopping, FH,* use different paths along time, but not two of them at the same time, hence is not diversity in strict sense.

Time Diversity

- In its simplest form the idea is to repeat the transmission
  - Time diversity is achieved by averaging the fading over time
  - Typically the coherence time, T_c, is of the order of tens to hundreds symbols
  - Hence *we should wait a time longer than T_c to repeat the transmission*
- We can also use channel coding and interleaving
  - the output of a channel coder is a mixture of the k input bits to a higher dimension n
  - the $n - k$ extra dimension provide robustness to errors in the n bits
  - if the n bits are transmitted consecutively and $nT_b \approx T_c$, no protection is achieved
  - solution: after coding a *interleaving* is performed: bits of one or several frames of length n are scrambled

Time diversity model

- In the following we assume coherence reception and *flat fading*
- We transmit a codeword where the coded bits are mapped into L symbols $\mathbf{x} = [x_1, \ldots, x_L]^T$
  - Where we omit time index m to keep the notation uncluttered
- The received signal is given by

$$y_l = h_l x_l + w_l, \quad l = 1, \ldots, L \tag{20}$$

- *Assuming ideal interleaving*, two consecutive symbols experience independent fading $\Rightarrow$ h_l are i.i.d. and distributed as $\mathcal{CN}(0, 1)$
- Noise w_l is i.i.d. $\mathcal{CN}(0, n_0)$

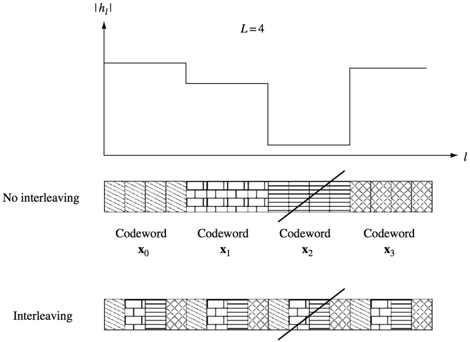
Repetition Coding

- The simplest channel coding is the repetition, where $x_l = x, l = 1, \ldots L$ for some L
- Parameter L can be denoted as *diversity branches*
- Several words are then interleaved
- In vector form

$$\mathbf{y} = \mathbf{h}x + \mathbf{w} \tag{21}$$

where $\mathbf{h} = [h_1, h_2, \ldots, h_L]^T$

Figure 3.5 The codewords are transmitted over consecutive symbols (top) and interleaved (bottom). A deep fade will wipe out the entire codeword in the former case but only one coded symbol from each codeword in the latter. In the latter case, each codeword can still be recovered from the other three unfaded symbols.



Detection in Repetition Code

- The coherent detection is the one of "vector Gaussian detection problem" in Unit 2:
  - the *matched filter* is used
  - the scalar

$$r = \frac{\mathbf{h}^*}{\|\mathbf{h}\|}\mathbf{y} = \|\mathbf{h}\|x + \frac{\mathbf{h}^*}{\|\mathbf{h}\|}\mathbf{w} \tag{22}$$

    is a sufficient statistic
- and we have a scalar detection problem with noise $v = (\mathbf{h}^*/\|\mathbf{h}\|)\mathbf{w} \sim \mathcal{CN}(0, n_0)$
- the matched filter is also called a *maximal ratio combiner* or *coherent combining*:
  - it aligns the phases in each branch l
  - it weights each branch according to its strength (quality)
  - maximizing the *SNR*
- For BPSK, $x = \pm a$ and $snr = a^2/n_0$ ($\mathbb{E}[|h_i|^2] = L$, *per symbol time* $= 1$), given $\mathbf{h}$,

$$p_e(\mathbf{h}) = Q\left(\sqrt{2\|\mathbf{h}\|^2 snr}\right) \tag{23}$$

where the received snr is $\|\mathbf{h}\|^2 snr$, i.e., we average over $\|\mathbf{h}\|^2$ to estimate p_e,

$$z = \|\mathbf{h}\|^2 = \sum_1^L |h_l|^2 \tag{24}$$

Detection in Rayleigh fading with Repetition Code

- In Rayleigh fading $h_l \sim \mathcal{CN}(0,1)$, each term $|h_l|^2$ being the sum of squares of the *real and imaginary parts* ($2L$ terms)
- And $\|\mathbf{h}\|^2$ is the sum of the squares of $2L$ i.i.d. real Gaussian r.v.
- It can be shown that $\|\mathbf{h}\|^2$ is distributed as a chi-square with $2L$ degrees of freedom with pdf

$$f_Z(z) = \frac{1}{(L-1)!} z^{L-1} e^{-z}, \quad z \geq 0 \tag{25}$$

- The average p_e over $v = \|\mathbf{h}\|^2$,

$$p_e = \int_0^\infty Q\left(\sqrt{2z \cdot snr}\right) f_Z(z) dz = \left(\frac{1-\mu}{2}\right)^L \sum_{l=0}^{L-1} \binom{L-1+l}{l} \left(\frac{1+\mu}{2}\right)^l \tag{26}$$

where

$$\mu \stackrel{\text{def}}{=} \sqrt{snr/(1+snr)} \tag{27}$$

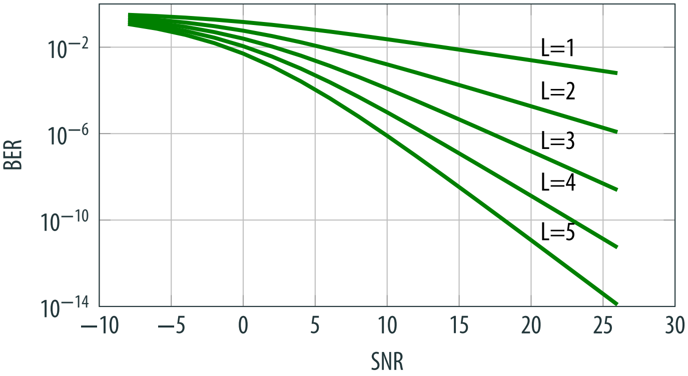
- *At high* snr, using Taylor series

$$\frac{1-\mu}{2} \approx \frac{1}{4snr}, \quad \frac{1+\mu}{2} \approx 1, \quad \sum_{l=0}^{L-1} \binom{L-1+l}{l} = \binom{2L-1}{L}$$

Probability of error in Rayleigh flat channel and repetition code

- hence, for high snr,

$$p_e \approx \binom{2L-1}{L} \frac{1}{(4snr)^L} \Rightarrow \text{slope of } -L \text{ in the curve} \tag{28}$$



Analyzing fading: Notes on chi-square

- For $L = 1$ we know we have an exponential
- For larger L's the distributions tends to be Gaussian, as many independent r.v. are added
- The probability of deep fade, if $z = \|\mathbf{h}\|^2$,

$$\Pr\left(\|\mathbf{h}\|^2 < 1/snr\right) = \int_0^{1/snr} f_Z(z)dz \approx \frac{1}{L!}\frac{1}{snr^L} \tag{29}$$

where exponent L explains that for $\sum_{l=1}^{L} |h_l|^2$ to be low $\Rightarrow$ every value $|h_l|^2$ must be low

L is called the *diversity gain*

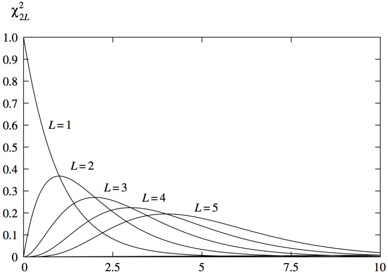


Figure 3.7 The probability density function of $\|\mathbf{h}\|^2$ for different values of L. The larger the L, the faster the probability density function drops off around 0.

Time diversity in GSM

- In GSM we have convolutional codes and interleaving
- The channels are 200 kHz in a FDD band of 25 MHz, with TDMA frames of 8 slots of 577 μs each $\times 8 = 4.615$ ms
- The voice over 20 ms is encoded with a convolutional code to give 456 bits
- The 456 bits are interleaved into 8 frames: the delay introduced by interleaving is roughly 40 ms
- The maximum possible time diversity gain is 8
  - For $f_c = 900$ MHz, the coherence time is, since $D_s = 2f_c v/c$,

$$T_c \stackrel{\text{def}}{=} 1/(4D_s) = c/(8f_c v) \tag{30}$$

  - For $T_c \leq 5$ ms hence we have independent fading $\Rightarrow v \geq 30$ km/h
  - For speeds below 30 km/h the fading is correlated: GSM uses *frequency hopping (FH)*
    - with *frequency hopping* every frame we change the channel within the bandwith available.
    - The typical delay spread is $T_d = 1\ \mu$s, and the coherence bandwidth $W_c \stackrel{\text{def}}{=} 1/(2T_d) = 500$ kHz, of the order of the 200 kHz of the channels.
    - We have the same effect than in time diversity

Summary Probability of Error

- We can summarize the main results on the probabilities computations, as they will be reused further in this Unit
- For a Rayleigh channel and the time diversity schemes,

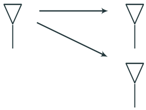
	$p_{e\|\mathbf{h}}$	p_e, high snr	$L=2$
Repetition type	$Q\left(\sqrt{2snr\sum_l^L \|h_l\|^2}\right)$	$\approx \binom{2L-1}{L}\frac{1}{(4snr)^L}$	$\frac{3/16}{snr^2}$
Rotation type	$Q\left(\sqrt{\frac{snr}{2}\sum_l^L \|h_l\|^2\|d_l\|^2}\right)$	$\leq \frac{4^L}{snr^L\prod_l^L \|d_l\|^2}$	$\frac{15}{snr^2}$

- In the case of the rotation code we just computed the error between two symbols, as a term of the union bound
- It is assumed that h_i and h_j are independent $i \neq j$, e.g. by using interleaving

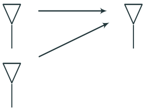
If we write the p_e as $p_e \leq \frac{\alpha}{snr^L}$, we say that
L is the *diversity gain* while α provides the *coding gain*.

Introduction

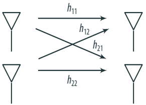
- When we have a strict delay constraint, limited bandwidth and a large coherence time, and FH is not possible,
- *Spatial diversity* can be used: we place multiple antennas at the transmitter (Tx) and/or the receiver (Rx)
- Antennas must be placed far apart to ensure that the paths between Tx and Rx antenna/s fade independently
- The needed separation depends on
  - *scattering environment*
  - *wavelength*
  - Near the ground with many scatterers around: we have separations of $0.5 - 1$ times λ
  - In a base station (BS) several meters high: several to tens λ
- We will look at *receive diversity* and *transmit diversity*: SIMO, MISO (including *space-time codes*) and MIMO
- MIMO channels also provide additional degrees of freedom



SIMO



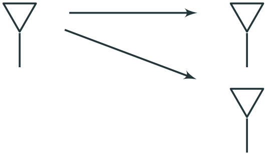
MISO



MIMO

Receive diversity

- We transmit one signal, and get L receptions, in $\mathbf{y}$, from L receiving antennas $\mathbf{y} = \mathbf{h}x_1 + \mathbf{w}$



- The p_e equals that of the repetition code and

$$p_{e|\mathbf{h}} = Q\left(\sqrt{2\|\mathbf{h}\|^2 snr}\right) \tag{31}$$

with $\|\mathbf{h}\|^2 = \sum_{l=1}^{L} |h_l|^2$ and at high snr

$$p_e \approx \binom{2L-1}{L} \frac{1}{(4snr)^L} \Rightarrow \text{slope of } -L \text{ in the curve} \tag{32}$$

Receive Diversity: Power and Diversity Gain

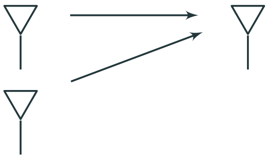
- We can rewrite the argument as

$$\|\mathbf{h}\|^2 snr = \underbrace{L snr}_{\text{power gain or array gain}} \cdot \underbrace{\frac{1}{L}\|\mathbf{h}\|^2}_{\text{reflects diversity gain}} \tag{33}$$

- By coherent combining L paths we get an increase in power
  - In the repetition code this term is at the expense of repetitions: not a power gain
- By averaging over the paths, $\sum_{l=1}^{L} |h_l|^2/L$, if the channels gains h_l are
  - *independent*: then as $L \to \infty$, the normalized diversity gain $\to 1$
  - *correlated*: then we have no *diversity gain*
    - We always get a power gain.
- Note that we get the same result than in the repetition code but with just one channel use.

Transmit Diversity

- Main problem is that if all Tx antenna transmit at the same time, how do I separate the transmissions?
  - If $h_i \sim \mathcal{CN}(0,1)$ and we Tx the same symbol x at the same time we get $y = (\sum_{i=1}^{L} h_i)x + w$ and, as L grows, $y \to w$.
- We could fully emulate the repetition code by repeating the transmission over the L antennas during L symbol times
  - This is the repetition code for MISO
- However we have a waste of degree of freedoms
- We should try to transmit using the L antennas at a symbol time: *space-time coding*
- We analyze the Alamouti space-time coding for two transmit antennas



Alamouti scheme

- We assume flat fading, and that the channel *does not change* along two symbol times
- The receive channel model:

$$y[m] = h_1[m]x_1[m] + h_2[m]x_2[m] + w[m] \tag{34}$$

where the h_i is the channel form the i-th antenna

- Two consecutive symbols are transmitted, $u_i = \pm a$,

$$\begin{aligned} x_1[1] &= u_1, \qquad & x_1[2] &= -u_2^* \\ x_2[1] &= u_2 & x_2[2] &= u_1^* \end{aligned} \tag{35}$$

- The receive signal at a given m

$$\begin{bmatrix} y[1] & y[2] \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix} + \begin{bmatrix} w[1] & w[2] \end{bmatrix} \tag{36}$$

- that *Alamouti's scheme allows to rewrite it as a mixture of the inputs*

$$\begin{bmatrix} y[1] \\ y[2]^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w[1] \\ w[2]^* \end{bmatrix} \tag{37}$$

where the *columns are orthogonal*, we project **y** onto these columns

Detection in Alamouti

- Projecting onto the columns we get $r_1 = [h_1^*, h_2][y_1, y_2^*]^\top/\|\mathbf{h}\|$ and $r_2 = [h_2^*, -h_1][y_1, y_2^*]^\top/\|\mathbf{h}\|$

$$r_i = \|\mathbf{h}\| u_i + v_i \quad i = 1{,}2 \tag{38}$$

  where $\mathbf{h} = [h_1, h_2]^\top$ and $v_i \sim \mathcal{CN}(0, n_0)$
- Note that the system is equivalent to an orthogonal modulation with pulses the columns of the resulting matrix.
- *The P_e yields as in the repetition code*, with diversity gain 2, *but with double Tx power* needed (2 antennas)
- Comparing the repetition code to using one antenna at a time, with two antennas,
  - same scenario as in time diversity but with transmitting from one antenna at a time
  - we need a 4-PAM $\{\pm b, \pm 3b\}$ if not to loose espectral efficiency
  - in a 4-PAM, to get same distance, we need 5 times the energy of a BPSK
  - we do not transmit the two times from the two antennas, save 1/2 of the power
  - we have a loss of 5/2 (4 dB)
- Again repetition code is inefficient, not exploiting the degrees of freedom
  - The Alamouti scheme spreads the information onto two orthogonal directions

Space-time codes, definition

- A general scheme for a *space-time code* is of the form X_i where for each combination of Tx symbols, u_i we get a codeword of dimensions $L \times N$
  - L is the number of antennas
  - N is the *block length* of the code
  - In the Alamouti scheme the codewords are 2×2

$$\begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix} \tag{39}$$

  - In the repetition code the codewords are 2×2

$$\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \tag{40}$$

  - Codes coming from time diversity with codes $\{x_i\}$ can be used as space-time codes as

$$X_i = \mathrm{diag}(x_{i1},\ldots x_{iL}) \tag{41}$$

Conditioned Probability of error of Space-Time Codes

- To approximate p_e we first *normalize the codewords* to unit average energy per symbol time: $snr = 1/n_0$
- Then model the system in matrix form

$$\mathbf{y}^{\mathsf{T}} = \mathbf{h}^{\mathsf{H}}\mathbf{X} + \mathbf{w}^{\mathsf{T}} \tag{42}$$

where

$$\mathbf{y} \overset{\text{def}}{=} \begin{bmatrix} y[1] \\ \vdots \\ y[N] \end{bmatrix}, \qquad \mathbf{h} \overset{\text{def}}{=} \begin{bmatrix} h_1^* \\ \vdots \\ h_L^* \end{bmatrix}, \qquad \mathbf{w} \overset{\text{def}}{=} \begin{bmatrix} w[1] \\ \vdots \\ w[N] \end{bmatrix} \tag{43}$$

- At reception, when $\mathbf{X}_A$ is Tx, we receive $\mathbf{h}^{\mathsf{H}}\mathbf{X}_A + \mathbf{w}$ and need to compare to the $\mathbf{h}^{\mathsf{H}}\mathbf{X}_i$
- This is the problem of detecting a vector in circular Gaussian noise, with sufficient statistics (BPSK) $\mathrm{Re}[\mathbf{v}^{\mathsf{H}}\mathbf{y}]$ where $\mathbf{v} \overset{\text{def}}{=} \left(\mathbf{h}^{\mathsf{H}}\mathbf{X}_A - \mathbf{h}^{\mathsf{H}}\mathbf{X}_B\right)/\left\|(\mathbf{h}^{\mathsf{H}}\mathbf{x}_A - \mathbf{h}^{\mathsf{H}}\mathbf{x}_B)\right\|$,(Unit 2)
- We use the union bound to compute the P_e conditioned to the transmission of symbol A and a given channel

$$\Pr(\mathbf{X}_A \to \mathbf{X}_B|\mathbf{h}) = Q\left(\sqrt{\frac{\left(\left\|\mathbf{h}^{\mathsf{H}}(\mathbf{X}_A - \mathbf{X}_B)\right\|/2\right)^2}{n_0/2}}\right) \tag{44}$$

Probability of Error of Space-Time codes for high SNR

- *After some maths*, for high *snr*, we have

$$\Pr(\mathbf{X}_A \to \mathbf{X}_B) \leq \frac{4^K}{snr^K \prod_{l=1}^{K} \lambda_l^2} \quad (45)$$

where λ_l are the non null eigenvalues of $(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^{\mathrm{H}}$

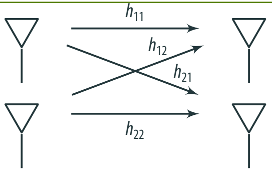
- if $\lambda_l^2 > 0 \; \forall l$ we have *maximal diversity* gain $K = L$, and $K < L$ otherwise.

Determinant criterion

The *coding gain* is given by the minimum of the determinant over all codewords pairs: *determinant criterion*

MIMO: Introducction

- We focus on a 2×2



- We have a system model

$$\begin{bmatrix} y[1] \\ y[2] \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w[1] \\ w[2] \end{bmatrix} \tag{46}$$

- Or equivalently

$$\mathbf{y} = \mathbf{h}_1 x_1 + \mathbf{h}_2 x_2 + \mathbf{w} \tag{47}$$

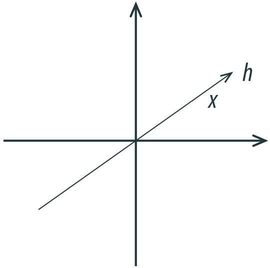
- With repetition from two Tx antennas we may exploit spatial diversity
  - If we Tx from 1 antena a symbol we have 2 copies, by repeating the Tx from the other antenna, we have 2 more copies, all of them with independent fading $\Rightarrow$ we have diversity 4
- The Alamouti scheme could be used,
  - In each received antenna we have the transmitted symbols with diversity 2. Since we have 2 receive antennas, we get diversity 4, and more efficient than the repetition code

Exercise 6.2 (Alamouti in MIMO)

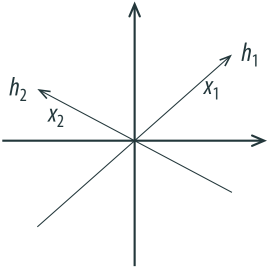
Analyze the Alamouti scheme used in MIMO, computing its p_e.

MIMO: spacial multiplexing

- In the SIMO we project the transmission over just 1 channel vector $\mathbf{h}$
- In the MIMO we project the transmission over 2 channel vectors $\mathbf{h}_1$ and $\mathbf{h}_2$
- We wonder what the diversity of the scheme is if we transmit just one time two symbols
  - We can transmit 2 different symbols, $x_i = u_i$, each one from a different antenna, and recover them
  - Since each symbol reaches each Rx antenna, *we have diversity 2 and 2 degrees of freedom*
- This scheme is also called *V-BLAST*
- In the general $n_t \leq n_r$ case each antenna transmit a different symbol that is received by n_r antennas, *hence the diversity gain is n_r*, provided that $h_{1,i}, h_{2,i}, \ldots h_{n_r,i}$ are statistically independent.



1×2 Case



2×2 Case

Detectors for V-BLAST MIMO

- The straight-forward detector is inverting the system. This is the so-called *zero-forcing, decorrelator or interference nuller*
- By inverting we have a perfect copy of the transmitted codeword $\mathbf{x}_i$

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w} \quad \Rightarrow \quad \tilde{x} = \mathbf{H}^{-1}\mathbf{y} = \mathbf{x} + \mathbf{H}^{-1}\mathbf{w} = \mathbf{x} + \tilde{\mathbf{w}} \tag{48}$$

- *Corrupted by noise* with variance, e.g. for output 1,

$$\sigma^2_{\tilde{w}_1} = \frac{|h_{22}|^2 + |h_{21}|^2}{|h_{11}h_{22} - h_{21}h_{12}|^2} n_0 \tag{49}$$

- We have diversity due to $|\mathbf{h}_1|$ having two elements: if one of them cancels, the denominator does not
  - In this sense we have diversity gain 2
- However, the denominator cancels if $h_{11}h_{22} = h_{21}h_{12}$, reducing the diversity gain to 1
- This can also be explained by vectors $|\mathbf{h}_1|$ and $|\mathbf{h}_2|$ having similar directions
  - In this case a linear (multiplying by a matrix) detector fails
  - Non linear detectors, and in particular the optimum *maximum likelihood, (ML)* detector, retain the 2 diversity gain

2x2 MIMO solutions

- For a 2×2 system we have

	Diversity Gain	Degrees of Freedom used per symbol time
Repetition	4	1/2
Alamouti	4	1
V-BLAST (ML)	2	2
V-BLAST (nulling)	1	2
Channel itself	4	2

Note: channel itself is the maximum achievable value *BUT they cannot be achieved at the same time*

- Question: what are these values for a 2×4 MIMO? and for a 4×2?

Summary: Benefits of MIMO

- *Array gain*: increase in the SNR that results from the coherent combining effect of the wireless signals at the receiver
- *Spatial diversity gain*: As already defined, the ability to transmit a signal when fading is present. It is given by the number of independent copies one can get trough.
  - In Rayleigh fading and for high SNR we showed that this number is the exponent of the *snr* in the denominator of the bound for the P_e.
  - If n_t and n_r Tx and Rx antennas are available, the maximum available diversity gain is $n_t n_r$.
- *Spatial multiplexing gain*: MIMO offers a linear increase in the data rate, transmitting multiple, independent, data streams in the same bandwidth.
  - In general, the number of data streams that can be reliably supported is $\min(n_t, n_r)$
  - Hence, MIMO increases capacity
- *Interference reduction and avoidance*: first, improvement in signal quality makes the system more robust to interference as it does for the noise. Second, we can design the MIMO system to cancel interference.

We cannot get all advantages at the same time, we must prioritize according with our needs.