Free space EM

- Under *free space* (f.s.) propagation and in far field ($r >> \lambda$), the electric field in the location of the received antenna behaves as a plain wave

$$e(f,t,\mathbf{r}) = \frac{\alpha(\varphi,\theta,f)\cos(2\pi f(t - r/c))}{r} \tag{1}$$

where $\mathbf{r}$ is a vector given by spheric coordinates $\mathbf{r} = [r,\varphi,\theta]^\top$, pointing to the receiving antenna, and $\alpha(\varphi,\theta,f)$ depends on the transmitting antenna used and the power delivered by it.

- The average power in the same point is given by the average Poynting's vector

$$\langle \mathcal{S} \rangle = \frac{e^2}{\eta} = \frac{eirp}{4\pi r^2} \tag{2}$$

where η is the medium characteristic impedance (120π in free space) and we made use of the definition of $eirp = p_t{}' g_t$, where $p_t{}'$ is the delivered power to the antenna and g_t its isotropic gain. And here $e^2 = e^2_{rms} = e^2_{max}/2$.

Through the course, when needed, we will used lower case letters to denote magnitudes in natural units and capital letters to denote magnitudes in decibels).

eirp: equivalent isotropic radiated power (eirp), *pire* in Spanish

Friis

- The received power, the power delivered by the receiving antenna, is the product of $\langle\mathcal{S}\rangle$ by its effective area

$$p_r{}' = \langle\mathcal{S}\rangle S_{eq} = \langle\mathcal{S}\rangle \frac{\lambda^2}{4\pi} g_r = \frac{eirp(\varphi_t,\theta_t)}{4\pi r^2} \frac{\lambda^2}{4\pi} g_r(\varphi_r,\theta_r) = eirp(\varphi_t,\theta_t) \frac{1}{l_{bf}} g_r(\varphi_r,\theta_r) \quad (3)$$

- This expression, in dB, leads to the well-known *Friis* formula

$$P_r'(\text{dBm}) = EIRP(\text{dBm}) - L_{bf}(\text{dB}) + G_r(\text{dB}). \quad (4)$$

where L_{bf} are the *basic free space (bf) losses*, and G_t,G_R are the gains of antennas (dB).

- The e_b/n_0 received is a function of the power of the envelop, p_r, and the bit time, T_b

$$e_b/n_0 = \frac{e_b}{n_0} = \frac{p_r T_b}{n_0} = \frac{p_r}{n_0 R_b} = \frac{p_r}{KT_0 f_s R_b} \quad (5)$$

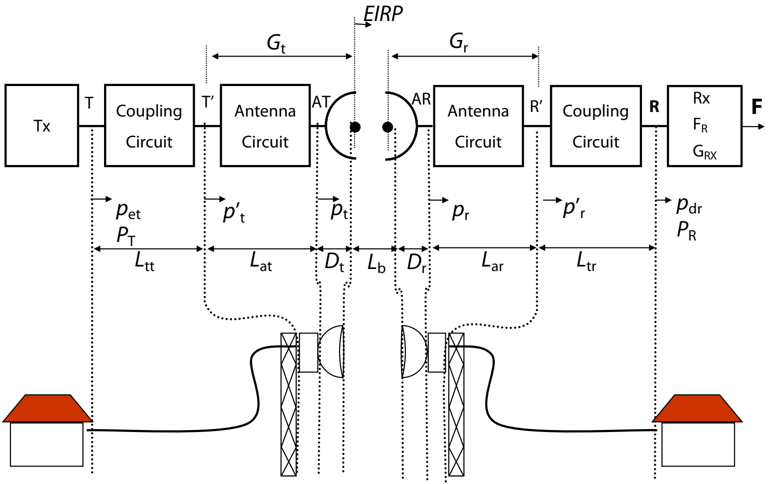
- In dB, we get the sensitivity

$$P_r(\text{dBm}) \geq E_b/N_0(\text{dBm}) + F_s(\text{dB}) + 10\log(R_b) - 174(\text{dBm/Hz}) \quad (6)$$

where K is the Boltzmann constant, $T_0 = 290$ K is the *room* temperature, F_s is the noise figure of the system and E_b/N_0 is given by the desired BER and the modulation used.

Notation

As in Hernando Rábanos 2006,



Simple flat LTI system

Most important to note that:

1. $e(f,t,\mathbf{r})$ is the value at a given frequency, if we *define*,

$$H(f) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \frac{\alpha(\varphi,\theta,f)\mathrm{e}^{-\mathrm{j}2\pi fr/c}}{r} \tag{7}$$

   and $e(f,t,\mathbf{r}) = \sqrt{2}\,\mathrm{Re}[H(f)e^{\mathrm{j}2\pi ft}]$
2. the propagation in f.s. (free space) just introduces a loss of $1/r$ to the field and $1/r^2$ to the power
3. the antenna frequency response is almost flat in its frequency bandwidth
4. there is a remarkable dependency on the antennas radiation patterns
5. the antennas and propagation are considered to be lineal, so we can integrate over the frequency responses to get the overall powers, it is easy given the previous assumptions
6. when computing $\langle\mathcal{S}\rangle$ we integrate over all transmitted frequencies, using $e(f,t,\mathbf{r})$

LTI

The systems behaves as *linear and time invariant, LTI*: the received signal is built as a linear combination of delayed versions of the transmitted one and the weights of the linear combinations and the delays are constant through time.

Additional losses

In propagation (f>100MHz) and *fixed service* we have additional *constant in time* losses to L_{bf} as follows

1. $+L_a$ if the transmitting or/and receiving directions are not the maximum gain ones
2. $+L_g$ attenuation due to gasses, quite strong above 20 GHz
3. $+L_d$ diffraction losses (obstacles), above 20 GHz, *Line Of Sight* (LOS) is a necessary
4. $+L_v$ due to vegetation, similar to gases (as diffraction for large frequencies)
5. $+L_r$ due to *permanent* reflections, see *example of Flat Earth*. (Reflections may also provoke selective fading)

and also *slow variant* ones, so the system can be considered LTI

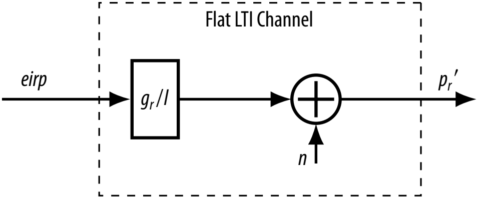
6. $+L_l$ due to rain, strong above 10 GHz
7. those due to additional diffraction losses caused by variations on the refraction phenomena (grazing incidence).

Flat channel

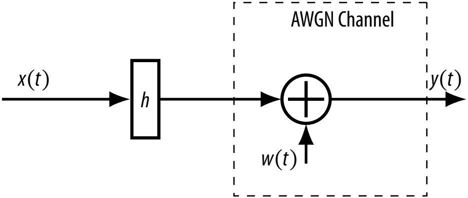
The received power can be written as $p_r{}' = eirp \cdot g_r/l$, where l depends on f_c, the carrier frequency, the distant r,... *but It does not change for frequencies in the Tx bandwith*.

Flat LTI model and AWGN channel

- From the power point of view, the model reduces to



- From the signal point of view,

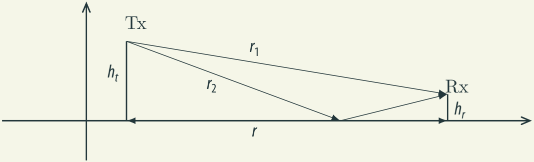


that it is equivalent to an AWGN channel: $y(t)' = y(t)/h = x(t) + w'(t)$ with $w'(t) = w(t)/h$, and ${\sigma'}_w^2 = \sigma_w^2/h^2$, the variance of the noise.

Flat Earth

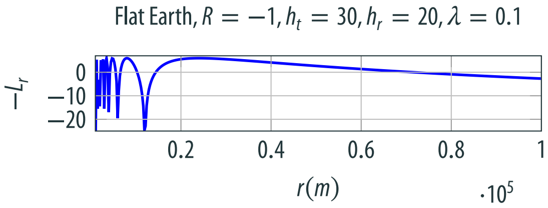
Example: Flat Earth

- Also, we may have strong fixed reflected path



- The effect is a loss, $+L_r$, for $r \geq 12 h_t h_r / \lambda$, *proportional to r^4*

$$L_{bf} + L_r = 40 \log(r\,(\text{km})) - 20 \log(h_t h_r) + 120 \tag{8}$$



A generalized model: Loss exponent

- An extended expression for the loss is as follows,

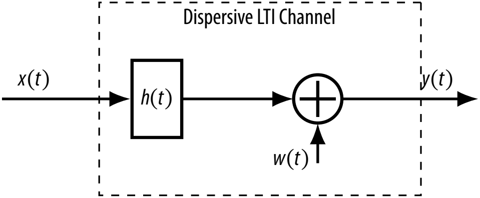
$$L = 10\log(kd^n) \tag{9}$$

  where n is the so called, loss exponent, and k models all constant losses
  - Propagation models for flat losses can be rewritten using this simple formula: $n = 2$ in f.s., $n = 4$ in flat Earth, $3 < n < 5$ in urban areas
  - Empirical methods such as the Okumura-Hata can be rewritten in this form
- The models: free-space plus additional losses, loss exponent model, empirical approaches such as
  1. Okumura-Hata
  2. Cost 231
  3. ITU-R P.1546
  4. Longley-Rice

  or the point to point software analysis *provide a mean value for the losses*, do not explain
  - variations along distance (not due to r)
  - variations along frequency
  - variations along time (moving or not)
- a *log-normal* model can be assumed to explain variations around the mean value

Dispersive/Selective channel

- Whenever multi-path is present, we may have a channel frequency response changing with frequency
- If the modulation bandwidth, W, is large enough, one frequency components experience a different attenuation than another: *selective* channel
- In time, we have a sum of delayed copies of the signal, causing ISI: *dispersive* channel



$$y(t) = x(t) * h(t) + w(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau + w(t)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega + w(t)$$

Static two ray model: frequency response

Two ray model

The channel impulse response yields

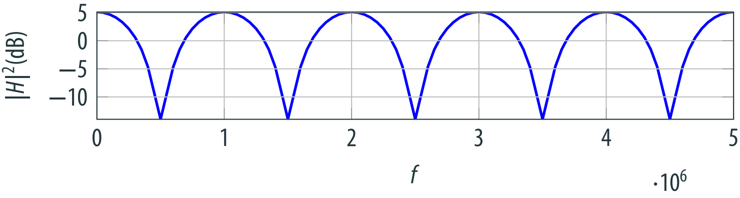
$$h(t) = a(\delta(t) + b\delta(t - \tau_1)),\ -1 < a,b < 1 \tag{10}$$

with frequency response $H(\omega) = a\left[1 + b\mathrm{e}^{-\mathrm{j}\omega\tau_1}\right]$ with module

$$10 \log |H(\omega)|^2 = 10 \log(a^2[1 + b^2 - 2|b| \cos(\omega\tau_1 - \omega_0\tau_1)]),\ \ |\omega_0\tau_1| = \begin{cases} 0, & b < 0 \\ \pi, & b > 0 \end{cases}$$

Nulls at *notches* with period $1/\tau_1$, with maximum null for $b = 1$

Two ray model, $a = 1, b = 0.8, \tau_1 = 10^{-6}$



Channel Frequency Response

Channel Frequency Response

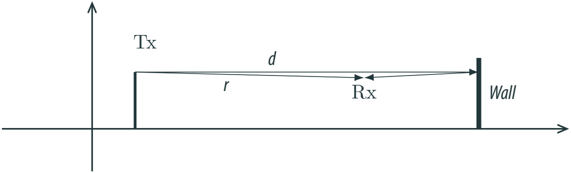
- Observe selectiveness and dispersion. Try changing *b*, its module, sign and delay. What is the effect of changing *a*?

```
close all; clear all; rng(30)
Nfft=2048;U=1/sqrt(Nfft)*exp(-1j*2*pi*([0:Nfft-1]')*([0:Nfft-1])/Nfft);%DFT matrix
%% Channel
a=1;b=+0.8;
h=a*[1,0,0,0,0,b,0,0,0,0]; L=10; %L:channel length;
%% Filtering, using DFT
ht=U*[h'; zeros(Nfft-L,1)]; %DFT of channel

%% Representation
Tm = 1e-6/5; %tau=Tm*5=1e-6
T = 1e-3;
Ht = 20*log10(abs(ht));
figure(1),plot([1:Nfft]/Nfft*1/Tm,20*log10(abs(ht))), grid on, hold on,
plot(ones(1,2)*(1/(2*T)),[min(Ht),max(Ht)],'--k')
title('Channel Frequency Response'), xlabel('Hz'), legend('a[\delta(t)+b\delta(t-\
     tau)]','h''W')
```

Reflection on a wall: response along distance

- In time, the two ray model can be analyzed in the following reflection scenario,



- if we assume the receive antenna to be omnidirectional in the horizontal plane, and the reflection coefficient to be $R = -1$, the received electric field yields

$$e(f,t,r) = \frac{\alpha(f)\cos(2\pi f(t - r/c))}{r} - \frac{\alpha(f)\cos(2\pi f(t - (2d - r)/c))}{2d - r} \quad (11)$$

- the phase difference is as follows, *with peaks for $e(f,t,r)$ at multiples of 2π and valleys (nulls) for odd multiples of π*

$$\Delta\theta = \left(\frac{2\pi f(2d - r)}{c} + \pi\right) - \left(\frac{2\pi fr}{c}\right) = \frac{4\pi f}{c}(d - r) + \pi = \frac{4\pi}{\lambda}(d - r) + \pi \quad (12)$$

- The Δr that makes $\Delta\theta$ vary π (peak to a valley) is the *coherence distance*,

Coherence distance, $\Delta r \;:\; \Delta\theta' = \Delta\theta + \pi$

$$\Delta r \stackrel{\text{def}}{=} \frac{\lambda}{4} \tag{13}$$

- As a function of f the distance between a peak to a valley is given by a change in f, make $f' = f + \Delta f$, and estimate Δf such that $\Delta\theta' = \Delta\theta + \pi$,

$$\Delta f = \frac{1}{2}\left(\frac{2d - r}{c} - \frac{r}{c}\right)^{-1} = \frac{c}{4(d-r)} \tag{14}$$

being the *delay spread*

Delay Spread: T_d

$$T_d \stackrel{\text{def}}{=} \frac{2d - r}{c} - \frac{r}{c} = 2\,\frac{d - r}{c} \tag{15}$$

the difference between the propagation delay along the two paths and

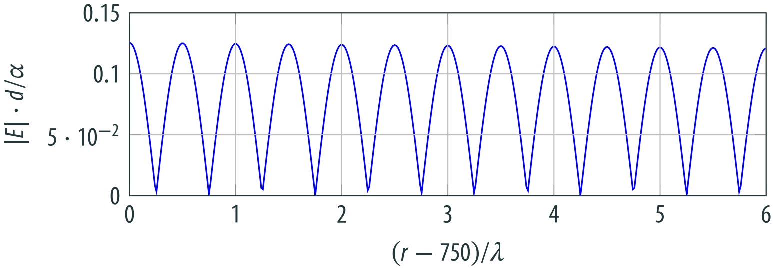
Coherence bandwidth: W_c

$$W_c \stackrel{\text{def}}{=} 1/(2T_d) = \Delta f \tag{16}$$

- This model, for a given distance r has equal frequency response as the static two ray model.
  - The coherence bandwidth is the inverse of the delay spread: we have nulls in frequency every $1/T_d = 1/\tau$
- The interesting point now is to analyze the variation with the position r

How would the field change if I were at some other points?

f=900 MHz, t=0, λ=0.3 m, d=800 m



Approximation

- We can approximate the summation of the two sinusoids (Tse pp.17):

$$e(f,t,r) = \frac{\alpha(f)\cos(2\pi f(t - r/c))}{r} - \frac{\alpha(f)\cos(2\pi f(t - (2d - r)/c))}{2d - r} \tag{17}$$

- We first recall that

$$\cos(u) - \cos(v) = -2\sin\left(\frac{u-v}{2}\right)\sin\left(\frac{u+v}{2}\right) \tag{18}$$

- Then in the denominator we assume $r \approx 2d - r$, i.e. we are near the wall, and for simplicity denote $\alpha(f) = \alpha$
- And approximate

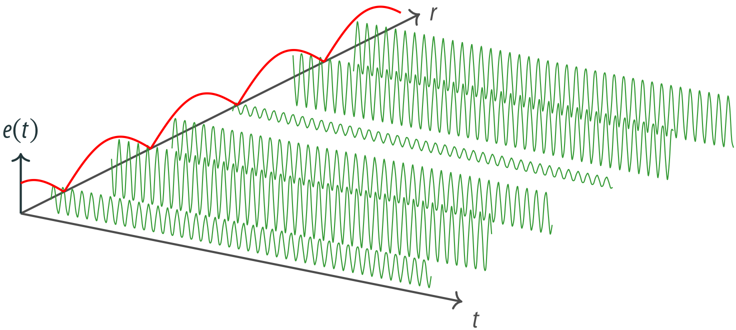
$$e(f,t,r) = -\frac{2\alpha\sin(2\pi f(d - r)/c)\sin(2\pi f(t - d/c))}{r} \tag{19}$$

where we have the carrier multiplied by amplitude $2\alpha\sin(2\pi(d - r)/\lambda)/r$

LTI Channel in a wall reflection: representation

At every distance we receive a sinusoidal form whose amplitude depends on the distance: the distance between a maximum and minimum amplitud value is $\lambda/4$

- red: $2\alpha \sin(2\pi(d-r)/\lambda)/r$
  - the module is depicted
- green: red$\cdot \sin(2\pi f(t-d/c))$



Example: Fixed Service

- In fixed point to point we have atmospheric multi path, called *scattering*, due to reflections on atmospheric layers
- This effect grows with frequency and approximately the cube of the distance, r^3
- The channel is modeled as a two components:

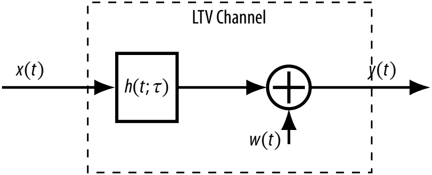
1. *flat fading*, can be seen as a AWGN channel where the E_b/N_0 varies slowly with time
2. *selective/fading*, that vanishes as the bandwidth decreases; it must be modeled as a LTI system with ISI that varies slowly with time

Exercise: delay spread

Consider a channel where the largest path has no more than double the length of the shortest one, compute the delay spread, and the coherence bandwidth, if

1. the shortest path has 30 km
2. the shortest path has 3 km
3. the shortest path has 3 m

Linear time variant channel (LTV)



The response depends on time t. At any $t = t_0$, we have a different channel impulse $h(t; t_0)$

$$y(t) = x(t) * h(t; t_0) + w(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau; t_0)\mathrm{d}\tau + w(t) \tag{20}$$

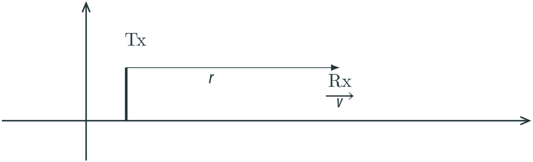
$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega; t_0)X(\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega + w(t) \tag{21}$$

We assume that the channel is under spread

under spread

An LTV channel is *under spread* if the channel can be considered LTI during a time larger than the channel response in τ, i.e. its delay spread.

Doppler: free space, moving antenna



- If the receiver antenna moves away the transmitter with velocity v and $r(t) = r_0 + vt$, the delay changes from r/c to $(r_0 + vt)/c$, and (1) yields,

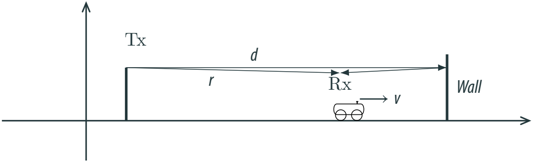
$$e(f,t,r_0 + vt) = \frac{\alpha(f)\cos(2\pi f(t - r_0/c - vt/c))}{r_0 + vt} \tag{22}$$

  - Note that as the antenna moves, the variations on the phase in the numerator are quite fast compared to the effect in the denominator
  - By rewriting $f(t - r_0/c - vt/c)) = f[(1 - v/c)t - r_0/c]$
- It follows

$$e(f,t,r_0 + vt) = \frac{\alpha(f)\cos(2\pi f[(1 - v/c)t - r_0/c])}{r_0 + vt} \tag{23}$$

  where we observe frequency deviation, a *Doppler shift*, of $-fv/c = -v/\lambda$
- It cannot be modeled as *linear time invariant*.

Doppler and multipath: moving antenna and wall reflection



- We have the same case that in (11) but with a frequency variation due to the doppler effect, described in (22): combining both equations:

$$e(f,t,r_0+vt) = \frac{\alpha(f)\cos(2\pi f[(1-v/c)t - r_0/c])}{r_0+vt} - \frac{\alpha(f)\cos(2\pi f[(1+v/c)t-(2d-r_0)/c])}{2d-r_0-vt}$$

  - assume we are near the wall, so $r_0+vt \approx 2d-r_0-vt$ in the denominator and we can sum the numerators:

- It follows

$$e(f,t,r_0+vt) \approx -\frac{2\alpha(f)\sin(2\pi f[-vt/c+(d-r_0)/c])\sin(2\pi f[t-d/c])}{r_0+vt} \qquad (24)$$

Doppler spread in the wall scenario

- We add two sinusoids
  - The first one has as Doppler shift of $D_1 = -fv/c$,
  - The second one has a Doppler shift of $D_2 = fv/c$
  - The overall Doppler is $D_s = D_2 - D_1 = 2fv/c$
  - For f=900 MHz and v=60 km/h we have D_s=100 Hz

Doppler Spread: D_s

$$D_s \overset{\text{def}}{=} D_2 - D_1 \tag{25}$$

- that can be rewritten as the product of two sinusoids, one of frequency $D_s/2$ and the other with the original frequency f
- the effect can be seen as a modulation of the carrier by a sinusoid of Doppler frequency: *we have a fading every $1/D_s$ seconds !!.* From (24),

$$2\pi f v \Delta t/c = \pi \Rightarrow \Delta t = \frac{c}{2fv} = \frac{1}{D_s}$$

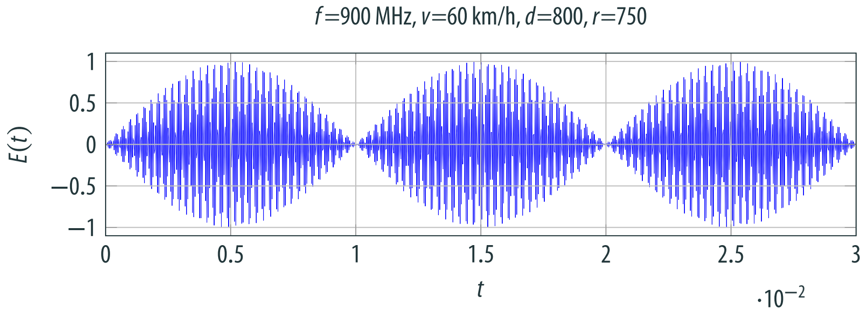
Note that in space, $\Delta r = v\Delta t = \lambda/2$.

Fading due to Doppler in the wall scenario

Coherence time: T_c

$$T_c \stackrel{\text{def}}{=} 1/(4D_s) \tag{26}$$

- While moving we may have fading due to
  1. Multi path+Doppler effect, with significant variations of the order of ms (T_c), or $\lambda/4$ m (Δr).
     - Same coherence distance than in the LTI case!!
  2. *Shadowing* effect, the obstacles in the environment change causing fading, of the order of s or m: *large scale*.



Two ray impulse response of the moving Rx and wall

- The scenario can be modeled with an LTV system given in (20) with response $h(t;t_0)$,
  - but where the $h(t;t_0)$ is given by deltas delayed $\tau_i(t_0)$ and values $a_i(t_0)$ that vary with time
  - and the integral yields a sum

$$y(t) = \sum_i a_i(t_0)x(t-\tau_i(t_0)) \tag{27}$$

- In the wall case we have two deltas with values

$a_1(t) = \frac{\lvert\alpha(f)\rvert}{r_0+vt}$	$a_2(t) = \frac{\lvert\alpha(f)\rvert}{2d-r_0-vt}$
$\tau_1(t) = \frac{r_0+vt}{c}$	$\tau_2(t) = \frac{2d-r_0-vt}{c} - \frac{\pi}{2\pi f}$

- The doppler effect is not readily seen in this representation
  - In the table above, the delay for a path is changing with

$$\frac{d\tau_i(t)}{dt} = \tau_i'(t) \Rightarrow \tau_1'(t) = v/c \Rightarrow \Delta\tau_1(t) = \tau_1'(t)t \tag{28}$$

  - the Doppler shift can be rewritten as

$$-f\tau_i'(t) \tag{29}$$

Impulse response of N rays in a moving channel

- If we model the multi path channel by several rays, from $i = 1$ to $i = N$, and consider a moving Rx, we have similar definitions, for a carrier frequency f_c,

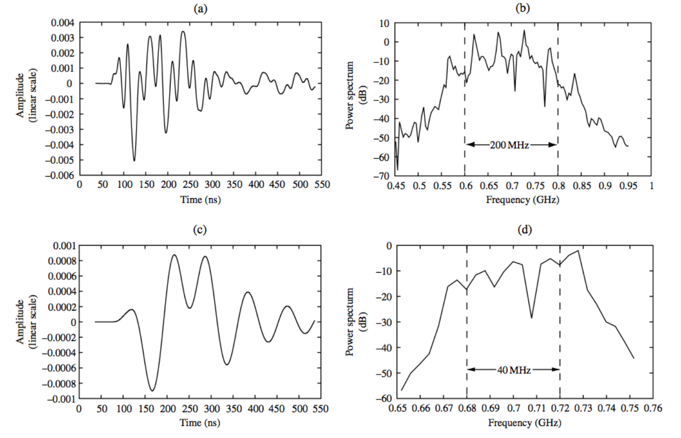
Moving LTV

- Doppler Spread: $D_s \stackrel{\text{def}}{=} \max_{i,j} f_c|\tau_i'(t) - \tau_j'(t)|$
- Coherence time: $T_c = \frac{1}{4D_s}$
- Delay Spread: $T_d \stackrel{\text{def}}{=} \max_{i,j} |\tau_i(t) - \tau_j(t)|$
- Coherence Bandwidth: $W_c = \frac{1}{2T_d}$
- Underspread Channel : if $T_c >> T_d$

Example: a moving multi path channel

Frequency selective

We show a bandpass channel for $f_c = 0.7$ GHz for 200 MHz and 40 MHz bandwidth



Example: a moving multi path channel

Example of values

Key channel parameters and time-scales	Symbol	Representative values
Carrier frequency	f_c	1 GHz
Communication bandwidth	W	1 MHz
Distance between transmitter and receiver	d	1 km
Velocity of mobile	v	64 km/h
Doppler shift for a path	$D = f_c v/c$	50 Hz
Doppler spread of paths corresponding to a tap	D_s	100 Hz
Time-scale for change of path amplitude	d/v	1 minute
Time-scale for change of path phase	$1/(4D)$	5 ms
Time-scale for a path to move over a tap	$c/(vW)$	20 s
Coherence time	$T_c = 1/(4D_s)$	2.5 ms
Delay spread	T_d	1 µs
Coherence bandwidth	$W_c = 1/(2T_d)$	500 kHz

Types of wireless channels

Types of wireless channels

Types of channel	Defining characteristic
Fast fading	T_c <<delay requirement
Slow fading	T_c >> delay requirement
Flat fading	$W << W_c$
Frequency-selective fading	$W >> W_c$
Underspread	$T_d << T_c$

The delay requirement depends on the transmission, it may be

- the frame length
- the burst length
- ...

Important: wall example

In the wall examples above we did transmit a tone of amplitude $\alpha(f)$. If a signal is transmitted instead, we must include the dependence with time in the $\alpha(f,t)$ term.

Excersises

Exercise 5.1 (Wall reflection LTI)

Given a radio transmission of a sinusoid at frequency $f = 900$ MHz, this signal gets the receiver at a distance 750 m. The transmitter power is 1 W, the transmitter (Tx) is omnidirecctional with gain 10 dBi and it is assumed it transmits with this gain into the direction of the receiver (Rx). The receiver antennas is isotropic

a) Compute the received field in the surroundings of the Rx, and the value of α in (1)

b) If a wall, at 800 m from the Tx in the direction Tx-Rx reflects the transmitted signal with coefficient -1, compute the amplitude of the field from 750 to $750 + 6\lambda$ meters. Hint: use Equation (2.13) in Tse and Vis. with $v = 0, t = 0$ and changing distance, r.

c) In the previous scenario compute the signal along t at 750, at $750 + \lambda/8$ and $750 + \lambda/4$, and its Fourier transform. For the Fourier transform you may read the documentation and use fftshift. For example

```
f=1e9; Tm=1/f/20; fm=1/Tm; %,carrier, sampling time and frequency
t=[0:Tm:200/f]; %observed time period
y= 2*cos(2*pi*f*t);
L=length(t); Nfft = 4*1024;
Y = fftshift(fft(y,Nfft)/L);
fq = fm/2*linspace(-Nfft/2,Nfft/2-1,Nfft)/(Nfft/2);
figure, plot(fq,abs(Y),'r')
```

Exercises

Exercise 5.2 (Wall reflection LTV)

In the same scenario than in the previous excercise the Rx moves towards the wall at a constant speed $v = 50$ km/h.

a) Compute the averaged received field in the surrounds of the Rx

b) Compute the Fourier transform

Equivalent lowpass signal model (Review)

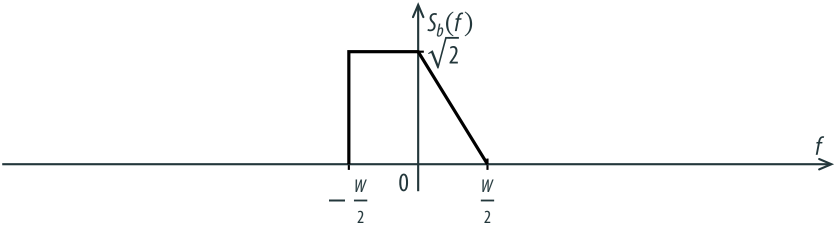
- We define

$$S_b(f) \stackrel{\text{def}}{=} \sqrt{2} S_+(f + f_c)$$

- We can reconstruct $S(f)$ and $s(t)$ from $S_b(f)$

$$S(f) = \frac{1}{\sqrt{2}}\left(S_b(f - f_c) + S_b^*(-f - f_c)\right)$$

$$s(t) = \frac{1}{\sqrt{2}}\left(s_b(t)\mathrm{e}^{\mathrm{j}2\pi f_c t} + s_b^*(t)\mathrm{e}^{-\mathrm{j}2\pi f_c t}\right) = \sqrt{2}\,\mathrm{Re}\left[s_b(t)\mathrm{e}^{\mathrm{j}2\pi f_c t}\right]$$



Note: the following transforms are used: $s^*(t) \longleftrightarrow S^*(-f)$ and $s(t)\mathrm{e}^{\mathrm{j}2\pi f_c t} \longleftrightarrow S(f - f_c)$.

Step 1. Channel lowpass equivalent model

- We can rewrite the channel model to lowpass equivalent, given N multi paths and the bandpass signal

$$
\begin{aligned}
y(t) &= \sum_{i=1}^{N} a_i(t)x(t-\tau_i(t)) \\
&= \sum_{i=1}^{N} a_i(t)\sqrt{2}\,\mathrm{Re}\left[x_b(t-\tau_i(t))\mathrm{e}^{\mathrm{j}2\pi f_c(t-\tau_i(t))}\right] \\
&= \sqrt{2}\,\mathrm{Re}\left[\left\{\sum_{i=1}^{N} a_i(t)\mathrm{e}^{-\mathrm{j}2\pi f_c\tau_i(t)}x_b(t-\tau_i(t))\right\}\mathrm{e}^{\mathrm{j}2\pi f_c t}\right]
\end{aligned}
$$

- by comparison with $y(t) = \sqrt{2}\,\mathrm{Re}\left[y_b(t)\mathrm{e}^{\mathrm{j}2\pi f_c t}\right]$

$$
y_b(t) = \sum_{i=1}^{N} a_i^b(t)x_b(t-\tau_i(t)); \text{ where: } a_i^b(t) \stackrel{\text{def}}{=} a_i(t)\mathrm{e}^{-\mathrm{j}2\pi f_c\tau_i(t)}, \tag{30}
$$

$$
\text{and we may define: } h_b(\tau;t) = \sum_{i=1}^{N} a_i^b(t)\delta(\tau-\tau_i(t)) \tag{31}
$$

Step 2. Discrete-time equivalent lowpass channel model

Discrete-time lowpass equivalent

To get a sampled version, $x_b[m]$, of the lowpass equivalent signal we

1. filter with lowpass filter of width $W/2$: to get $x_b(t)$
2. sample at rate $T_m = T = 1/W$: $x_b[m] = x_b(m/W)$

- $x_b(t)$ yields, by the Nyquist–Shannon sampling theorem, with $\mathrm{Sinc}(t) = \sin(\pi t)/\pi t$,

$$x_b(t) = \sum_m x_b[m]\mathrm{Sinc}(t/T_m - m) \tag{32}$$

- After the channel, using the discrete-time lowpass channel model, we have

$$y_b[m] = \sum_{l=1}^{L} h_b[l;m]x_b[m-l] \tag{33}$$

for the sake of simplicity we will denote $h_l[m] = h_b[l;m]$ and $y[m] = y_b[m]$, where, as explained in the following slide,

$$h_b[l;m] \stackrel{\text{def}}{=} h_l[m] = \sum_{i=1}^{N} a_i^b[m]\mathrm{Sinc}[l - \tau_i[m]/T_m] \tag{34}$$

Notes on discrete samples of the channel

- From (30), (31) and the sampling theorem

$$y_b(t) = \sum_{i=1}^{N} x_b(t) a_i^b(t) \delta(t - \tau_i(t)) = \sum_{i=1}^{N} \left(\sum_n x[n] \mathrm{Sinc}(Wt - n) \right) a_i^b(t) \delta(t - \tau_i(t))$$

$$y_b(t) = \sum_n x[n] \sum_{i=1}^{N} a_i^b(t) \mathrm{Sinc}(Wt - \tau_i(t)W - n)$$

- The samples at multiples of $T_m = 1/W, y[m] \stackrel{\text{def}}{=} y_b(m/W)$

$$y[m] = \sum_n x[n] \sum_{i=1}^{N} a_i^b(m/W) \mathrm{Sinc}[m - n - \tau_i(m/W)W] \tag{35}$$

- Let $l \stackrel{\text{def}}{=} m - n$, then

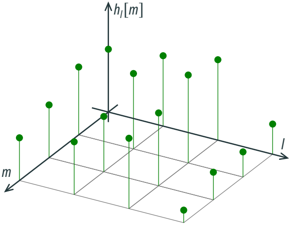
$$y[m] = \sum_l x[m - l] \sum_{i=1}^{N} a_i^b(m/W) \mathrm{Sinc}[l - \tau_i(m/W)W] \tag{36}$$

and we define

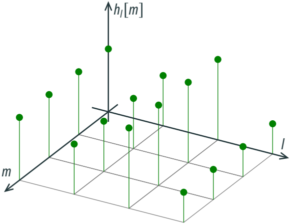
$$h_l[m] \stackrel{\text{def}}{=} \sum_{i=1}^{N} a_i^b(m/W) \mathrm{Sinc}[l - \tau_i(m/W)W] \tag{37}$$

Channel

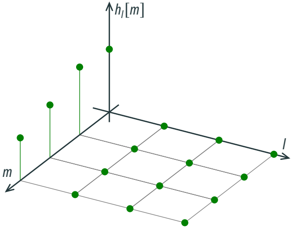
- LTV channel



- LTI channel



- Case $1/W >> \max \tau_i(t)$



Discrete low pass equivalent model

- As the Rx moves the delay spread changes
  - when the Rx moves a distance $\Delta r = c \cdot T_m = c/W$ the path moves from one tap to the next one (equivalently we have $\Delta\tau = \Delta r/c$ equal to the sampling time, $1/W$).
  - since $\Delta r = v\Delta t \Rightarrow \Delta t = c/(vW)$
  - this is the *time-scale for a path to change over a tap*
- As the Rx moves the path phase changes
  - when the distance shift, Δr, is $\lambda/4$ the phase difference changes π
  - since $\Delta r = v\Delta t$, we conclude $\Delta r = \lambda/4 = v\Delta t \Rightarrow \Delta t = \lambda/(4v) = 1/(4D)$
  - this is the *time-scale for change of path phase*

Task 2.2: discrete equivalent low-pass model

Rewrite the equations for the discrete equivalent low-pass model for the LTI scenario.

Task 2.3: discrete equivalent low-pass model

Determine and represent the discrete equivalent low-pass model at time $t = 4$ s of the channel given by

$$h(t;\tau) = a_1(t)\delta(t-\tau_1(t)) + a_2(t)\delta(t-\tau_2(t)) \tag{38}$$

if $a_1(4) = 1, a_2(4) = -0.5, \tau_1(4) = 0\,\mu s, \tau_2(4) = 2\,\mu s, f_c = 900$ MHz and $W = 1$ MHz.

AWGN; Degree of Freedom

- When AWGN noise is present we may easily model it by either
  1. adding a noise of power N_0, $\mathcal{CN}(0, N_0)$, (*complex-valued Gaussian*) to the equivalent lowpass signal model,
  2. adding two iid noises of power $N_0/2$, $\mathcal{N}(0, N_0/2)$ to the real (in phase) part of the equivalent lowpass signal model and the imaginary part (in quadrature)
- Degrees of Freedom

Degree of freedom

Every $1/W$ we transmit a (complex) symbol: we say that it represents *one (complex) dimension or degree of freedom*

- A continuous-time band-limited signal can be represented by W complex *dimensions* per second
- The received signal $y(t)$ is also, due to the channel, band-limited to approximately W and has W *dimensions* per second
  - We have also Doppler effect
- We say that the (received) signal space has dimension the *dimension or degrees of freedom of the channel*

Motivation and Introduction

- Motivation
  - Real channels are not deterministic: they do not have a given known channel response
  - Even point-to-point radio links change greatly with time
  - We need to provide an statistical characterization
  - This allows to design, simulate, implement and test
- Introduction
  - If the channel is given by just one time-invariant tap (i.e. is flat LTI) we may characterize it by a random variable (r.v.)
  - However, if the tap changes with time or we have several taps, *we need random processes (r.p.)* to characterize the channel.
- One tap is computed as the contributions of every multipath to that sample time: *Gaussian approximations are good for one tap...*

Review: Complex valued random variables

- A complex random variable $W = W_{\mathrm{r}} + \mathrm{j}\, W_{\mathrm{j}}$, can be seen as a vector of two variables $W_{\mathbb{R}} = [\, W_{\mathrm{r}}, \, W_{\mathrm{j}}]^{\top}$ described statistically by its joint pdf.
- In complex-valued AWGN channels the real and imaginary parts are zero-mean and independent of variance $N_0/2$
  - In this case the Gaussian rv it is said to be *circular*
- A complex-valued AWGN is usually denoted as

Gaussian complex random variable

$W = W_{\mathrm{r}} + \mathrm{j}\, W_{\mathrm{j}} \sim \mathcal{CN}(0, N_0, 0)$ where

$$\begin{aligned} W_{\mathrm{r}} &\sim \mathcal{N}(0, N_0/2), \\ W_{\mathrm{j}} &\sim \mathcal{N}(0, N_0/2), \end{aligned} \tag{39}$$

- Note that the variance is double of the one of the real, or imaginary, part
- The extension to multidimensional complex-valued AWGN is straight-forward $\boldsymbol{W} = \boldsymbol{W}_{\mathrm{r}} + \mathrm{j}\, \boldsymbol{W}_{\mathrm{j}} \sim \mathcal{CN}(0, N_0 \mathbf{I}_{\boldsymbol{W}}, 0)$

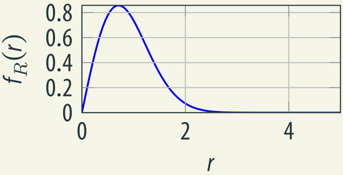
Complex valued random variable to model one tap

Gaussian complex circular random variable

$$X_\mathrm{r} \sim \mathcal{N}(0, \sigma_X^2/2), \quad X_\mathrm{j} \sim \mathcal{N}(0, \sigma_X^2/2)$$
$$X = X_\mathrm{r} + \mathrm{j}X_\mathrm{j} \sim \mathcal{CN}(0, \sigma_X^2)$$

- If we transform $[X_\mathrm{r}, X_\mathrm{j}]^\top$ into polar coordinates $[|X|, \angle X]^\top$
  - $\angle X$ is a uniform v.a. in $[0, 2\pi]$
  - $R = |X|$ is a Rayleigh r.v. with pdf $f_R(r) = \frac{2r}{\sigma^2} \mathrm{e}^{\frac{-r^2}{\sigma^2}} \sim \mathit{Rayleigh}(\sigma/\sqrt{2})$.

Rayleigh pdf, $\sigma = 1$



Mode	σ
Median	$\sigma\sqrt{2\ln 2} = 1.18\sigma$
Mean	$\sigma\sqrt{\pi/2} = 1.25\sigma$
RMS value	$\sigma\sqrt{2} = 1.41\sigma$
Standard dev.	$\sigma\sqrt{2-\pi/2} = 0.655\sigma$

- The power, $P = |X|^2$, is an exponential r.v. $P \sim \mathit{Exp}(1/\sigma^2)$.
- The sum of the power of *standard* Gaussian r.v. is chi-squared distributed.
- The sum of the power of zero mean Gaussian r.v. is Gamma distributed.

Rayleigh (NLOS) and Rician (LOS)

- The channel in the discrete-time lowpass model yields a set of L taps

$$h_l[m] = \sum a_i(m/W)\mathrm{e}^{-\mathrm{j}2\pi f_c \tau_i(m/W)}\mathrm{Sinc}[l - \tau_i(m/W)W] \tag{40}$$

1. Rayleigh channel (NLOS)
   - if no path is clearly stronger than the others: we have *non line-of-sight (NLOS)*
   - each tap $h_l[m]$ can be modeled as $\mathcal{CN}\left(0, \sigma_l^2\right)$, here the module is Rayleigh distributed and its phase is uniform in $[0,2\pi]$
   - this way we model several random paths contributing to one tap
2. Rician channel (LOS)
   - If one path is stronger than the other, the so-called *specular* path, we are modeling a *LOS* scenario, at least one of the taps is as follows,

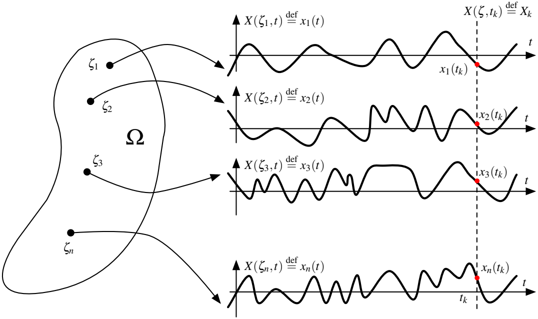
$$h_l[m] = \sqrt{\frac{\kappa}{\kappa+1}}\sigma_l \mathrm{e}^{\mathrm{j}\theta} + \sqrt{\frac{1}{\kappa+1}}s \tag{41}$$

     where Θ is a r.v. distributed uniformly between 0 and 2π and $S \sim \mathcal{CN}\left(0, \sigma_l^2\right)$, and κ, the K factor, adjusts the overall power, σ_l^2, distribution.
   - the Rician pdf can be written using Bessels functions.

Review: Random Process

Process

1. A *random process* (r.p.) $X(t)$ can be seen a sequence of random variables where each outcome is a *sample function*.
   - We use $x[k]$ for a sample function in the discrete case.
2. In a random process $X(a)$, $a \in \mathcal{T}$, is a family of indexed r.v. $\mathcal{T}$ represents the set of indexes and $X(a) \equiv X_a$ the associated r.v.



Review: Random process

- We have to specify
  1. the statistical description of the r.p. at any given time, t: the pdf/pmf of a $X(t)$
  2. the statistical description of the relation between a r.v. at a given time, $X(t)$, and any other r.v. at other time, $X(t+\Delta t), \forall \Delta t$.

In the continuous r.v. discrete time case, we have to specify the joint probability density function $f_{X_1 X_2 \cdots X_n}(x_1 x_2 \cdots x_n) \stackrel{\text{def}}{=} f_{\mathbf{X}}(\mathbf{x})$ for all indexes in $\mathcal{T}$, $n = |\mathcal{T}|$.

- This is a formidable task in the general case: we may limit the random process by imposing some constraints, in the hope they model the real systems:
  1. *Stationary*: if we assume first that
     - the statistical description of all r.v., $x[k]$ in the discrete case, at any time and $\forall k$, is the same one. We have $f_{X(k)}(\mathbf{x}(k)) = f_{X(k+\tau)}(\mathbf{x}(k+\tau)) \qquad \forall \tau, k, n$
     - accordingly the relation between two samples of the sample function, $x[k]$ and $x[l]$, only depends on the distance between samples $d = k - l$.
  2. *Stationary in wide sense (WSS)*: in practice we only focus on the stationary of the mean and covariance

Review: Random process

Definition

A random process $X[k]$ is *wide sense stationary* if and only if (iff):

1. $\mathbb{E}[X[k]] = m_X[k] = m_X$
2. $\mathbb{E}[X[k+\tau]X[k]] = R_X[k+\tau, k] = R_X[\tau]$

Given a stationary r.p. with $X[k] \sim \mathcal{N}(m, \sigma^2)$ and $R_X[\tau]$ we have a *Gaussian random process*.

- $R_X[0] = \sigma^2$

Given a stationary r.p. with $X[k] \sim \mathcal{N}(m, \sigma^2)$ and $R_X[\tau \neq 0] = 0$ we have a *white Gaussian* random process. Used in many cases, e.g., to model noise.

- Note that if we have a finite set of indices, we have a vector of independent Gaussian r.v., it is said that the vector is a *white Gaussian random vector*.
- Note also that in this case, if $m = 0$ and $\sigma^2 = 1$, n samples of a sample function, is distributed as a standard Gaussian.

Doubled time and frequency: auto-correlations

- In our models we have two times axis
  1. in continuous-time we have t,τ
  2. in discrete-time we have m,l (Let us focus on the discrete-time case)
- In m we may define the *tap gain auto-correlation function*

$$R_l[\Delta m] \stackrel{\text{def}}{=} \mathbb{E}[h_l^*[m]h_l[m+\Delta m]] = \mathbb{E}[h_l^*[0]h_l[\Delta m]] \tag{42}$$

where we assume we have a *wide sense stationary (WSS)* process (in t or m)

- as WSS, the response does not depends on time m, but on distance between times,Δm.

- In l we might define the *power-delay profile (PDP)*

$$R_m[l,l+\Delta l] \stackrel{\text{def}}{=} \mathbb{E}[h_l^*[m]h_{l+\Delta l}[m]] = \begin{cases} R_l[0] & \text{if } \Delta l = 0 \\ 0 & \text{if } \Delta l \neq 0 \end{cases}$$

- we assume that $h_l[m]$ and $h_q[m']$ are independent r.v. for any $l \neq q$ and m,m'.
- this assumption is denoted as *uncorrelated scattering (US)*

The overall resulting model is denoted as *WSSUS*

Further properties

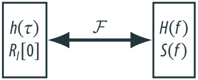
- Tap-gain autocorrelation function
  - This is the evolution of a tap along time. We expect that for times much shorter than the *coherence time*, T_c, the autocorrelation remains approximately constant
  - the *coherence time* can be also defined as the smallest value of $\Delta m/W$ such that $R_l[\Delta m]$ is statistically *different* from $R_l[0]$
  - the energy received in the l-th path is proportional to $R_l[0]$
  - Its Fourier transform, $S_l(\nu)$, provides a notion of the variation along frequency ν: *in the moving channel is due to Doppler* and measured through the *Doppler spread*, D_s
- Power-delay profile (PDP)
  - the *multi path spread* can be defined as the smallest value L/W, such that $\sum_{l=1}^{L} R_l[0]$ contains *most* of the total energy $\sum_{l=1}^{\infty} R_l[0]$: L is related to T_d
  - it is not stationary:
    - it cancels for l/W larger than the delay spread
    - it may have different values for every l up to the delay spread

    and we can not compute the Fourier transform of the autocorrelation
  - we compute the Fourier transform, $S_m(f)$, of the sequence $R_l[0]$: $R_1[0], R_2[0], R_3[0], \ldots$
  - these values are used to compute W_c, that it is related to T_d

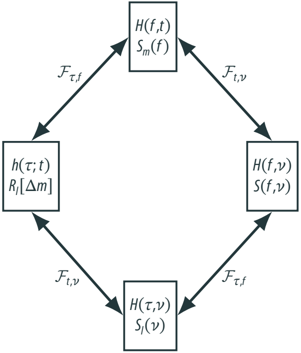
Note: theory states that we can compute the Fourier transform of a stationary process

Doubled time and frequency: deterministic and probabilistic

- In LTI systems we have time τ or taps l, and its frequency response in f
  - No dependence with m



- In LTV systems we have time τ or taps l, and its frequency response in f; but also the variation along time t or m, and the corresponding frequency response in ν



- Note that we
  - use h and H for deterministic models
  - use R and S for stochastics ones

Example: Clark's Model

Clarke's Model (Description)

- The Clarke's model assumes that
  1. the transmitter is fixed and the receiver is *moving at speed v*
  2. the objects causing scattering are statics
  3. there are K paths, indexed with $k = 1,\ldots,K$
  4. in the simplest Clarke's model, each one coming from direction $\theta = 2\pi/k$, with delay τ_θ, and equal amplitude $a_\theta = a/\sqrt{K}$
  5. the total received power from all paths is a^2
  6. we also assume a *flat channel*, $T_d < T = 1/W$
- The model yields

$$y(t) = \sum_{i=1}^{K} a_{\theta_i} x(t - \tau_{\theta_i}(t)) + w(t)$$

$$y[m] = h_0[m]x[m] + w[m]$$

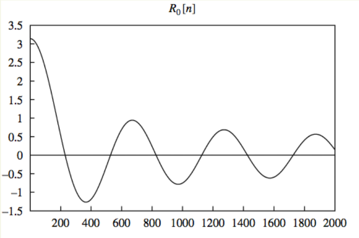
Clarke's Model (Solution)

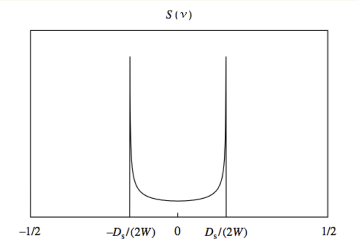
- if the phase of each path is iid and uniform in $[0, 2\pi]$ the tap $h_0[m]$ is the sum of many small independent components
- we may model $h_0[m]$ as a Gaussian *stationary* process with auto-correlation

$$R_0[n] = 2a^2\pi J_0(n\pi D_s/W) \tag{43}$$

where

1. $J_0(x) = 1/\pi \int_0^{2\pi} \mathrm{e}^{jx\theta} d\theta$ is the zeroth-order Bessel function of the first kind
2. its Fourier transform, the power spectral density, is given by ... see figure
3. The time n/W at which $R_0[n] = 0.05R_0[0]$ is the *coherence time* $T_c = \frac{J_0^{-1}(0.05)}{\pi D_s}$

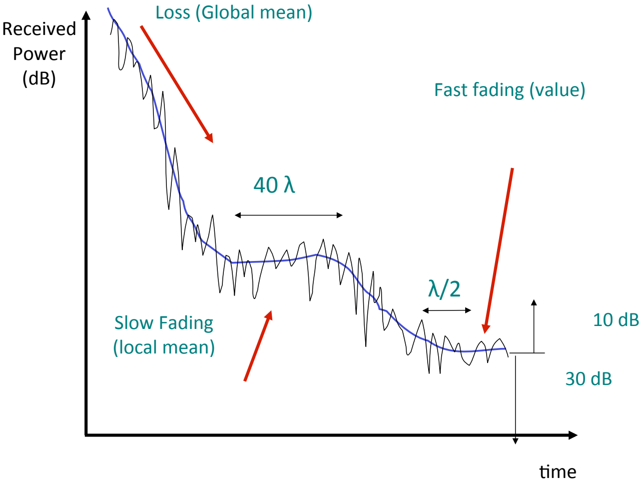




Fast and slow variations

Given a traveling speed

- while the effect of diffraction (*shadowing*) evolves slowly with time
- the changes due to destructive and constructive sums of multiple path changes fast



Non AWGN

- The noise added to the channel models above is additive white Gaussian noise.
- In some channels the noise may be non Gaussian and non white.
- Non Gaussian noise:
  - Impulse noise: we add to AWGN a noise, $u[m]$, the product of a Bernouilli r.v., $b[m]$, and a constant or another r.v. such as a Gaussian, $v[m]$

    $$w[m] = u[m] + b[m]v[m] \tag{44}$$

    hence $w[n]$ is AWGN if $b[m] = 0$ and the sum of an AWGN and a given value or r.v. if $b[m] = 1$.
- Non white noise: the noise is non white stationary random process, that can be obtained by filtering AWGN

  $$w[m] = u[m] * \psi[m] \tag{45}$$

  where the filter response, $\Psi(f)$, the so called *noise shaping filter*, defines the frequency response of the noise.

DSL

In DSL the cable can be modeled as a LTI system with non white (*coloured*) noise.

Non linear channels

- In the path from bits generation in the Tx to bits estimation in the Rx the system design aims at offering a *linear* channel
  - There are some frequency up and down converters, that are considered linear as long as they perfectly translate the central frequency
- However, amplifiers and converters exhibit high non linearity
  - Other elements such as connectors or antennas may exhibit some non-linearity
- The effect of non linearity in the low pass equivalent model is a deformation of the constellation: with different rotations and attenuations for every point
  - In general, constant envelop signals are more robust to non-linearity
    - That is why 8-PSK is used in EDGE
- We may use a high back-off (*BO*) to ensure we work in the linear range, or some techniques as pre-distorters to *linearize* it by inverting the non linear response.

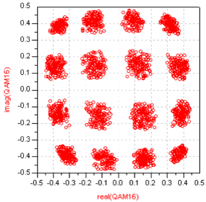
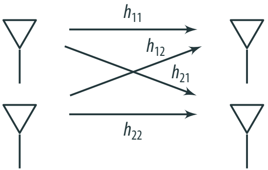


Figure by Agilent

MIMO: Introduction

- Needs motivation? 3.5G, 4G, 5G mobile and WiFi already include them in the standards...
- We focus on a 2×2



- We have 4 channels where the output of each transmitted pair is added at each receiver antenna.
- We can apply every concept above to model this channel.
- For example, if flat, linear and invariant, we have a system model

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{46}$$

- Or equivalently, if h_1 is the i-th column,

$$\mathrm{y} = \mathrm{h}_1 x_1 + \mathrm{h}_2 x_2 + \mathrm{w} = \mathbf{Hx} + \mathbf{w} \tag{47}$$

that it is widely used in MIMO systems.

Flat linear invariant MIMO

- Given N_t and N_r transmit and receive antennas, respectively, and a flat linear invariant MIMO channel
  - what it is the impact of the MIMO system?
- *Multiplexing:* if $N_t = N_r = N$ and matrix $\mathbf{H}$ is full rank (i.e. invertible) then we can sent N bits streams in parallel, in *SU-MIMO (single user MIMO)*
  - this is the *multiplexing gain*
  - the BER will depend on the E_b/N_0, that depends on the noise an the entries, $h_{i,j}$, of $\mathbf{H}$.
  - for $\mathbf{H}$ to be invertible $\Rightarrow$ different entries $\Rightarrow$ *antennas must be far one from each other (in terms of λ)*
  - the detection problem is not straightforward, but one simple approach is multiply the observation $\mathbf{y}$ by $\mathbf{H}^{-1}$: *inverse, canceller or zero-forcing detector*
  - if $\mathbf{H}$ is known to the transmitter, we can easy the detection...*could you figure it out how?*
  - if $N_t \neq N_r$, it can be concluded that at most $\min(N_t, N_r)$ bit streams can be multiplexed.
- *Diversity:* we could decide not to multiplex to just improve the E_b/N_0 at reception by combining the observations, in SU-MIMO.
- *Beamforming:* we could use the N_t transmitting antennas to point the *beam* to given directions, where the users are, in *MU-MIMO (multiuser-user MIMO)*

MIMO can be used *either* to multiplex several throughputs, improve signal to noise ratio (diversity) *or* beamforming. But not all of them at the same time!