Sistemas de Comunicaciones

Master en Ingeniería de Telecomunicación

Unit 2. Modulation and AWGN channel

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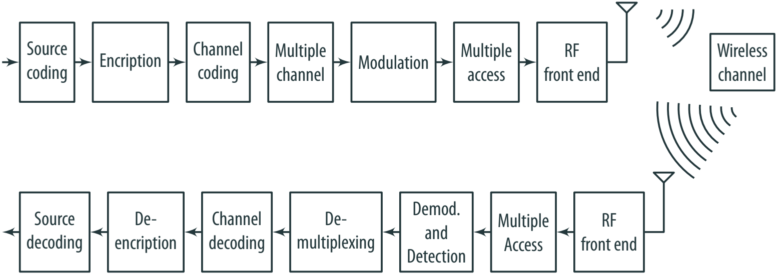
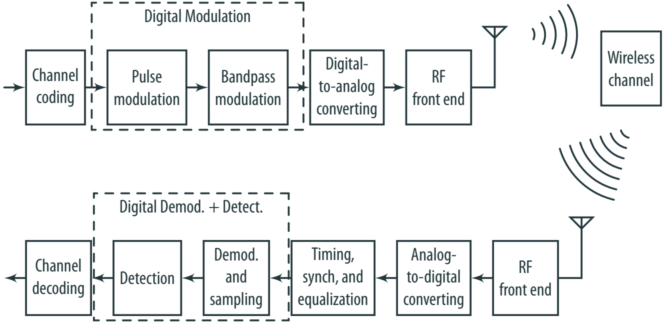
Modulation in Digital Communications Systems

Digital Communications Systems

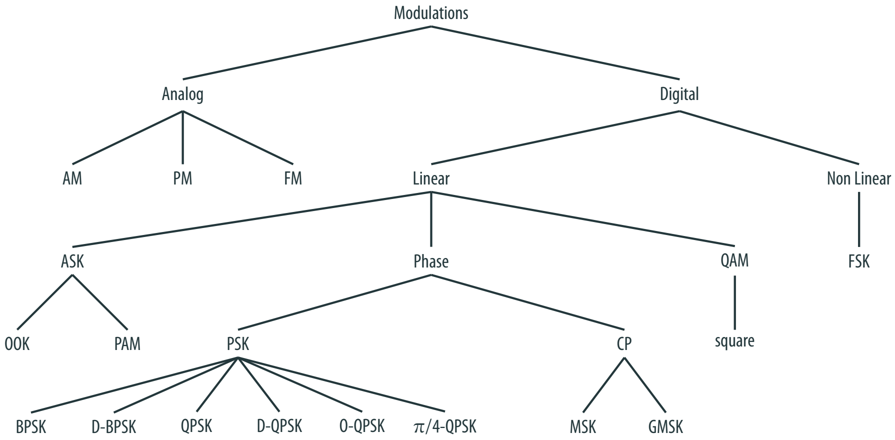


Modulation and Demodulation



- Demodulation can be *coherent or non-coherent*, if the carrier is recovered and used or not, respectively.

Classification



Modulations: Notes

- Within the analogs, the FM remains in radio broadcast and some simple private mobile radio (walkie-talkie)
- *Linear modulations* can be expressed as a sequence of symbols, from a given constellation, multiplied by a pulse.
  - In wireless communication *root raised cosine* pulses are quite extended: bandwith is $B = (1 + \alpha)/T$, with T symbol time and α the roll-off factor
- *OFDM* (orthogonal frequency division modulation): downlink of WiMAX, LTE, 5G, IEEE 802.11,...,DSL
  - Used to avoid ISI
  - A set of subcarriers (tipycally around 7.5-15 kHz) are modulated
  - Usually M-QAM is used in every subcarrier
- *FH* (frequency hopping): 2G-2.5G with GMSK and 8-PSK modulations
  - changes the carrier to avoid problems in a given channel
  - it is within the spread spectrum (SS) family
- *DS-SS* (direct sequence spread spectrum): IEEE 802.11b, GPS
  - a linear modulation with very much bandwith than needed
  - robust to ISI with a rake receiver
- *UWB* (ultra wide band): wireless sensors
  - A pulse of huge bandwith (GHz) and no carrier

Different Modulations

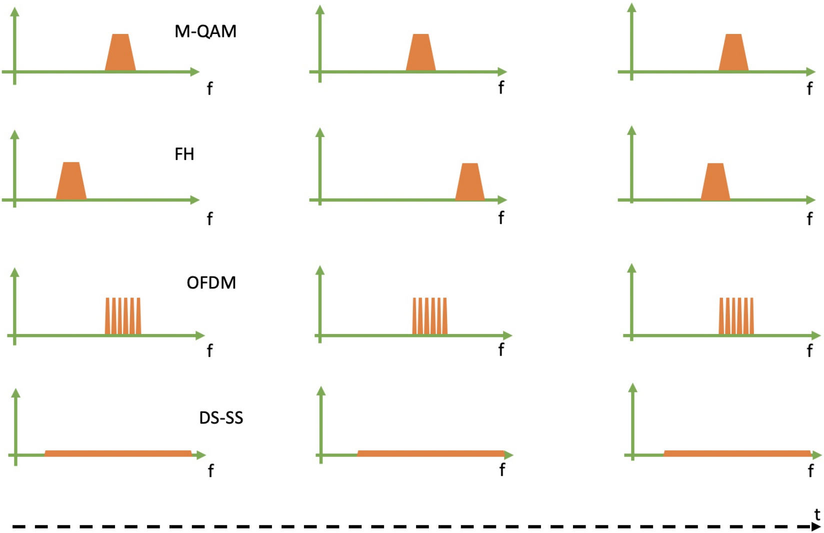


Figure: Example of modulations and their spectra

We have an incoming sequence of bits, at rate $R_b = 1/T_b$, and gather them in groups of $\log_2 M$ to output a symbol m_i, then we modulate it to generate a time signal, $s_i(t)$, that it is added to the signals already generated.

$$R_b = \frac{1}{T_b} = \frac{\log_2 M}{T} = R_s \log_2 M. \tag{1}$$

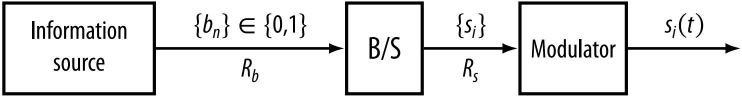


Figure: Modulator

Two main types of transmission attending to the carrier:

- We have low-pass or band-pass
- Low-pass can be converted to band-pass by multiplying by a sinusoid, i.e., a carrier of frequency f_c.

In band pass, how $s_i(t)$ is generated leads to different modulation schemes:

- If we modify the phase of the carrier: PSK
- If we modify the frequency of the carrier: FSK
- If we modify the amplitude of the carrier: ASK, including OOK
- If we modify the amplitude and phase: M-QAM

PAM-ASK modulation

- He have the following signals in digital amplitude modulation

$$s_i(t) = s_i\phi(t) = \begin{cases} s_i g(t), & i = 1, \dots, M \quad \text{PAM low-pass,} \\ s_i\sqrt{2}g(t)\cos(\omega_c t), & i = 1, \dots, M \quad \text{ASK band-pass.} \end{cases} \tag{2}$$

  where $g(t)$ is a unit energy (low-pass) conformation pulse and s_i is one out of M posibilities.
  - The OOK (on-off keying) is just an on-off transmission of a signal.
- The transmitted signal yields

$$x(t) = \sum_{n=-\infty}^{\infty} s[n]\phi(t - nT) \tag{3}$$

  where $s[n]$ is the discrete sequence of transmitted symbols s_i.
- In radio communications the $g(t)$ is usually a root-raised cosine filter.

QAM (and PSK)

- If, in band-pass, we transmit another sequence in quadrature we have a M-QAM:

$$\begin{aligned} s_i(t) &= R_i g(t)\sqrt{2}\cos(\omega_c t + \theta_i) & (4)\\ &= R_i\sqrt{2}g(t)\cos(\theta_i)\cos(\omega_c t) - R_i\sqrt{2}g(t)sin(\theta_i)sin(\omega_c t) \\ &= s_i^I\sqrt{2}g(t)\cos(\omega_c t) - s_i^Q\sqrt{2}g(t)sin(\omega_c t) & (5)\end{aligned}$$

  where note that the $k = \log_2(M)$ bits are translated into one pair (R_i, θ_i) or, equivalently, (s_i^I, s_i^Q), denoted as in-phase and quadrature transmissions.
- We review here the square QAM where M is of the form $M = 2^k$ with k even.
- In the QAM, R_i and θ_i are such that s_i^I and s_i^Q are values of a M'-PAM with $M' = 2^{k/2}$.
- Other values for R_i and θ_i are possible, in particular, if R_i is constant, we have a *PSK modulation*, where usually the θ_i are equidistant.
  - A QPSK has its first value at $\theta_i = 0$, hence it can be seen, in the IQ plane, as a 4-QAM rotated $\pi/4$.
- Usually this is represented in the low-pass (complex valued) equivalent form...

Examples of QAM and PSK

- 16-QAM

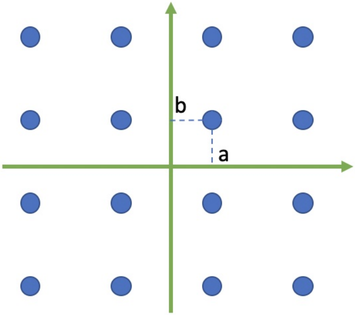


Figure: 16-QAM constellation

- 16-PSK

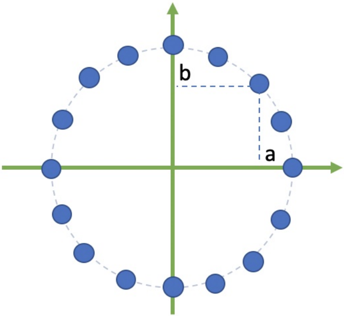


Figure: 16-PSK

- where $a = R_i \cos(\theta_i)$ and $b = R_i \sin(\theta_i)$

Demodulation

- The scheme of the receiver is of the form

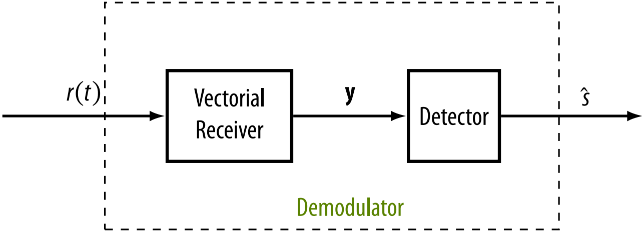


Figure: Demodulator

- Where we clearly observe two parts
  - The *vectorial receiver* provides $\mathbf{x}$: it should be a vector value from which easily extract the transmitted symbols, s_i, but it should not loss any valuable information on the symbol, i.e., it must be a *sufficient statistic*.
  - The *detector*: that decides on the transmitted symbol, s_i, upon the incoming $\mathbf{y}$ value.

Vectorial Receiver

- If we assume the channel to be just an attenuation plus noise, it can be proved that the vectorial receiver maximizing the signal-to-noise ratio (snr) is a couple of *matched filters* to the in-phase and quadrature pulses:
  - $\phi_I(t) = \sqrt{2}g(T-t)\cos(\omega_c(t-T))$
  - $\phi_Q(t) = -\sqrt{2}g(T-t)\sin(\omega_c(t-T))$
- The demodulator, of an IQ modulation can be described as

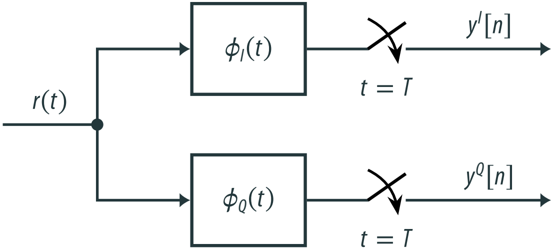


Figure: Vectorial receiver

where after the filters we sample at rate $T = 1/R_s$.

Vectorial Receiver for Linear I-Q Modulations

- If we have at the input $a \cdot x(t) + n(t)$ for some attenuation a and some noise $n(t)$, and given the couple of matched filters to the in-phase and quadrature pulses:
  - $\phi_I(t) = \sqrt{2}g(T-t)\cos(\omega_c(t-T))$
  - $\phi_Q(t) = -\sqrt{2}g(T-t)\sin(\omega_c(t-T))$
- The output of the vectorial receiver yields

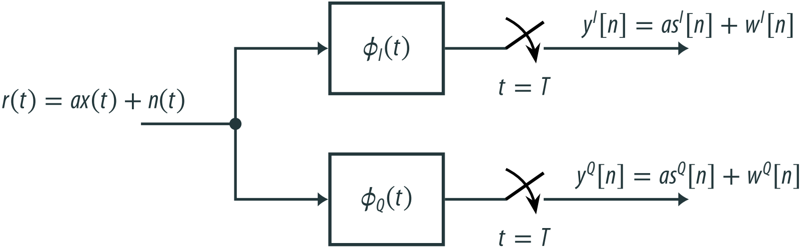


Figure: Vectorial receiver

Task 2.1 (Vectorial Receiver for QAM)

Prove that the output of the vectorial receiver follows the expressions in the figure and provide the values of the noise $w^I[n], w^Q[n]$ as a function of $n(t)$.

Equivalent discrete-time lowpass channel model

Motivation

- The mathematical development with the in phase and quadrature parts is involved
- Even when programming the transmitted and received in a digital receiver it is handy to have both parts in a compact form
- The complex valued representation, in digital communications, is quite useful to simplify the developments
- The low-pass equivalent can be defined as

$$s(t) = \sqrt{2}\,\mathrm{Re}\left[s_b(t)\mathrm{e}^{\mathrm{j}2\pi f_c t}\right] \tag{6}$$

  where $s_b(t)$ is the low-pass equivalent of $s(t)$.
- Let us briefly develop these ideas

Equivalent low-pass model

Task 2.2 (Equivalent low-pass model)

1. Determine the equivalent low-pass model of
$$x(t) = a(t)\cos(2\pi f_c t) \tag{7}$$
where $a(t)$ is a real low-pass signal.
2. Determine the equivalent low-pass model of
$$x(t) = b(t)\sin(2\pi f_c t) \tag{8}$$
where $b(t)$ is a real low-pass signal. Hint: $\cos(\theta + \pi/2) = -\sin(\theta)$ and $\mathrm{e}^{\mathrm{j}\pi/2} = \mathrm{j}$.
3. Determine the equivalent low-pass model of
$$x(t) = a(t)\cos(2\pi f_c t) + b(t)\sin(2\pi f_c t) \tag{9}$$

Low-pass M-QAM equivalent

The equation

$$s_i(t) = s_i^I\sqrt{2}g(t)\cos(\omega_c t) - s_i^Q\sqrt{2}g(t)sin(\omega_c t) \tag{10}$$

in low-pass equivalent yields

$$s_i(t) = \sqrt{2}\,\mathrm{Re}\left[s_{i,b}g(t)\mathrm{e}^{\mathrm{j}2\pi f_c t}\right] \tag{11}$$

where

$$s_{i,b}(t) = s_{i,b}g(t) = (s_i^I + \mathrm{j}s_i^Q)g(t) \tag{12}$$

Low-pass equivalent noise would fit $n(t) = \sqrt{2}\,\mathrm{Re}\left[n_b(t)\mathrm{e}^{\mathrm{j}2\pi f_c t}\right]$, complex-valued.

LP-equivalent Channel in the Frequency Domain

- We know that at the receiver we are filtering the signal with a bandwidth of the order of W
- Hence we are just interested on the channel frequency response from $f_c - \frac{W}{2}$ to $f_c + \frac{W}{2}$
- Note that the response in this bandwidth is not symmetric, in general, with respect to f_c
- Also, it is tedious to work with the $\cos(2\pi f_c)$ and $\sin(2\pi f_c)$ in the maths
- We could think of working with the right part moved to the 0 frequency: this is the lowpass equivalent model

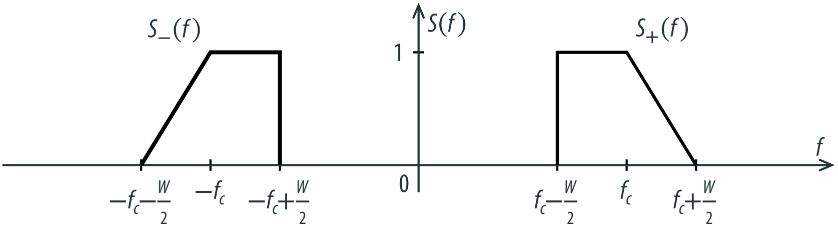


Figure: Example of power spectral density (PSD) of a modulation

Equivalent lowpass signal model

- We define

$$S_b(f) \stackrel{\text{def}}{=} \sqrt{2} S_+(f + f_c)$$

- We can reconstruct $S(f)$ and $s(t)$ from $S_b(f)$

$$S(f) = \frac{1}{\sqrt{2}} \left(S_b(f - f_c) + S_b^*(-f - f_c) \right)$$

$$s(t) = \frac{1}{\sqrt{2}} \left(s_b(t) \mathrm{e}^{\mathrm{j}2\pi f_c t} + s_b^*(t) \mathrm{e}^{-\mathrm{j}2\pi f_c t} \right) = \sqrt{2}\, \mathrm{Re}\left[s_b(t) \mathrm{e}^{\mathrm{j}2\pi f_c t} \right]$$

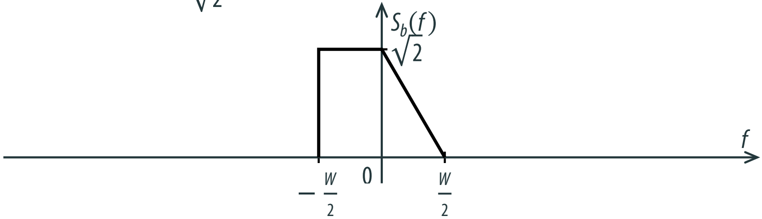


Figure: Positive part of PDS shifted to the left and scaled

Note 1: the following transforms are used: $s^*(t) \longleftrightarrow S^*(-f)$ and $s(t)\mathrm{e}^{\mathrm{j}2\pi f_c t} \longleftrightarrow S(f - f_c)$.
Note 2: *asymmetric spectrum involves complex-valued signals in time*: $s_b(t)$ will be, in general, complex-valued.

AWGN Channel

Let us first focus on a real-valued AWGN channel modelling, e.g., a PAM transmission system:

- x can be one out of the possible s_i values
- w is the noise $n(t)$ at the output of the matched filter at the vector receiver
- We transmit a symbol, X that takes finite values, i.e. is a discrete r.v., and receive it as Y with Gaussian noise, $W \sim \mathcal{N}(0, N_0/2)$, added: $Y = X + W$. A sequence of samples of the r.v., are given by

$$y[n] = x[n] + w[n]$$

- The samples of W are iid, i.e. its white $\Rightarrow$ additive white Gaussian noise (AWGN)

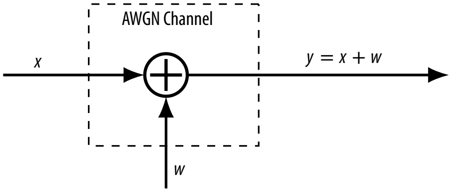


Figure: Model of an AWGN channel

LP-equivalent AWGN channel

Low-pass AWGN channel

In a AWGN channel, we can model the system by

- transmitting the low-pass equivalent
- adding channel low-pass equivalent noise
- use a low-pass equivalent vectorial receiver $\phi_b(t) = g(t - T)^*$

At the output we have $s_{i,b} + w_b$, where $W_b \sim \mathcal{N}(0, N_0/2) + \mathrm{j}\mathcal{N}(0, N_0/2)$ is the noise, w_b, at the output of the *vectorial receiver*.

Task 2.3 (Equivalent Low-Pass Vectorial Receiver)

Write the vectorial receiver in Figure 7 in equivalent low-pass form indicating the input, the output and value(s) of the corresponding matched filter(s), all of them in equivalent low-pass form.

Detection in AWGN

Detection in AWGN

- We have the maximum a posteriori criterium (MAP), that equals the maximum likelihood (ML) if the symbols are equally likely.
- Assume $x = \{x_A, x_B\}$, and $\Pr(X = x_A) = \Pr(X = x_B) = 0.5$:

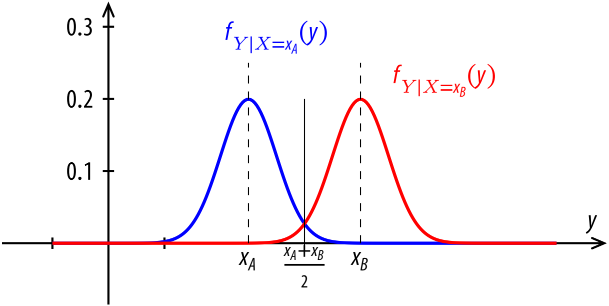


Figure: Probability density function at reception of the two possible symbols

- ML rule: choose x_A if $\|y - x_A\| < \|y - x_B\| \Rightarrow p_e = Q\left(\sqrt{\frac{(\|x_A - x_B\|/2)^2}{N_0/2}}\right)$

Q function

Function Q: integral from x to infinity of an standard Gaussian (*standard norma distribution*: zero mean, unit variance)

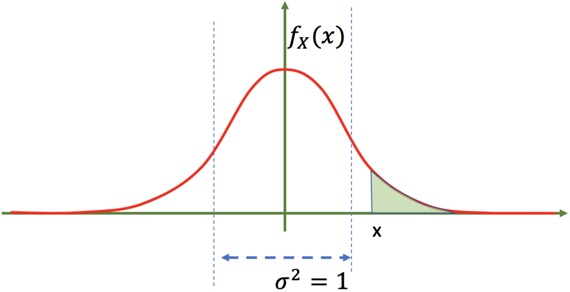


Figure: Integral of a standard normal pdf from x to ∞

$Q(x)$ function

- Main role in the bit error rate (BER) computation in AWGN channels.

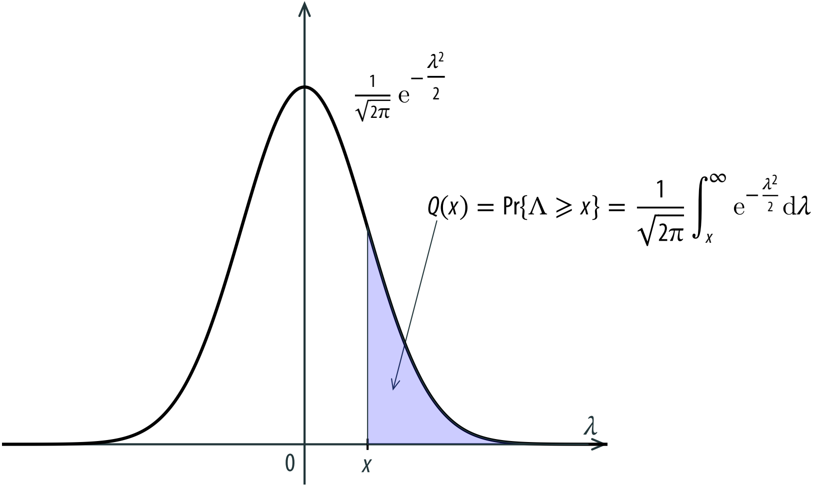


Figure: Function $Q(\cdot)$ as the integral of a Gaussian from x to ∞

Bounds for $Q(x)$

Function $Q(x)$ is hard to deal with, some bounds are usually used

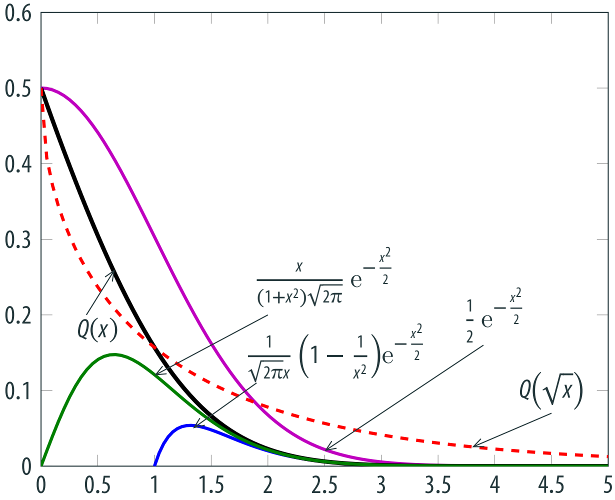


Figure: Bounds of function $Q(\cdot)$, used in theoretical developments to bound BER

Detecction

- Low distance between X_A and X_B but low noise variance

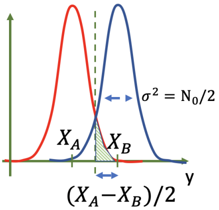


Figure: Shorter distance between points with lower variance

- High distance between X_A and X_B but high noise variance

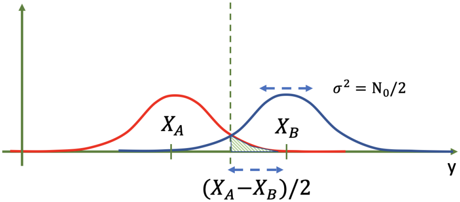


Figure: Larger distance between points but larger variance

- We have the same probability of error!!
- BER does depend on the ratio between energy of the signal and the variance of the noise.

Noise

- The noise energy at the output of the matched filter in the vectorial receiver is computed as its power multiplied by the symbol time, T.
- Since the power is the integral of its power spectral density within the bandwidth, $W = 1/T$:

$$P_n = 2 * W * n_0/2 \tag{13}$$

- Its integral along time, that could be "generously" interpreted as energy, is a r.v. with a given variance (or power): $$P_n \cdot T = P_n/W = n_0 \tag{14}$$

  This is the power of the noise in both the in-phase and the quadrature, in any of them we have half this power, i.e., $n_0/2$

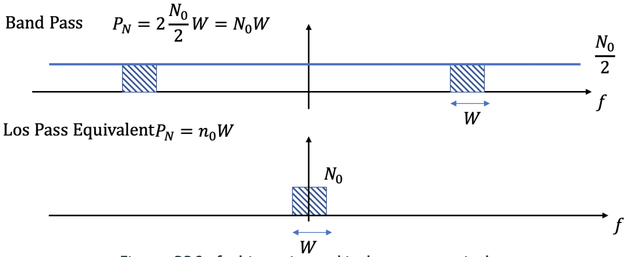


Figure: PDS of white noise and its low-pass equivalent

Note: we assume unit energy filters in the vectorial receiver.

Monte Carlo Simulation of AWGN (Matlab/Octave)

Exercise 2.1 (BPSK Monte Carlo Simulation (Matlab/Octave))

Run the following code. What is the needed E_b/N_0 for BER$= 10^{-3}$? What is the effect of changing N_p? Do you get points for every E_b/N_0 simulated? Why? What are the units and scales of the axis?

```
clear all, close all
Np=1e5; %Number of bits (BPSK) Tx
eb=1; a=sqrt(eb); Es=eb; %XA=-a, XB=a
SNRdB=0:1:15; %Range of Eb/NO to simulate
for k1=1:length(SNRdB)
    Bn=randi([0 1],1,Np); %Generar bits
    xn=Bn*2*a-a; %BPSK % Symbol (-a,a)
    ebno=10.^(SNRdB(k1)/10); %ebno, unid. nat.
    no=eb/ebno;
    sigma=sqrt(no/2); %sdt del ruido
    wn=sigma*randn(1,Np);
    yn=xn+wn; %Channel AWGN
    bn=sign(yn)/2+0.5; %Detection
    BER(k1)=sum(bn~=Bn)/Np; %MC error analysis
end
semilogy(SNRdB,BER), title('BER BPSK')
xlabel('E_b/N_o (dB)'), ylabel('BER')
```

Bidimensional in AWGN

- If we do not use the low-pass equivalent for a M-QAM or M-PSK we have a two dimensional detection problem.
- Suppose we want to detect between two possible points of a constellation: to compute the BER.
- The first thing to do is to model the multidimensional Gaussian noise.

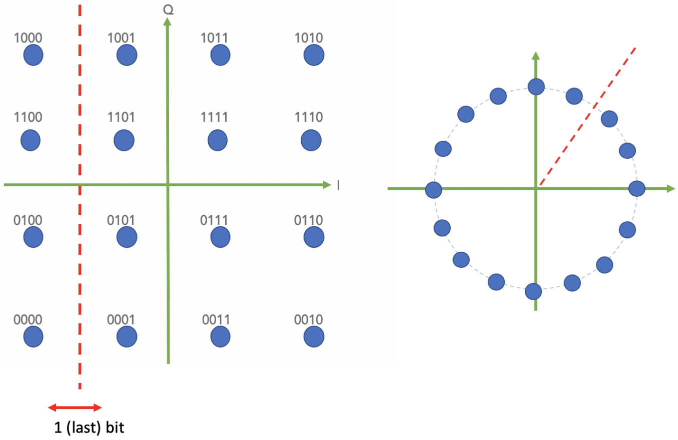


Figure: One bit decision in 16-QAM and 16-PSK

Jointly Gaussian random variables (2D)

- We face a model $\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{W}$ where $\boldsymbol{W}$ is complex valued.
- Given $X \sim \mathcal{N}(m_X, X)$ and $Y \sim \mathcal{N}(m_Y, Y)$ and correlation coefficient $\rho \stackrel{\text{def}}{=} \rho_{XY} = \frac{c_{XY}}{\sigma_X \sigma_Y}$, X and Y are *jointly normal* or *jointly Gaussian* if (and only if, iif) its joint pdf is given by:

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\, \mathrm{e}^{\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-m_X)^2}{X} - 2\rho\frac{(x-m_X)(y-m_Y)}{\sigma_X\sigma_Y} + \frac{(y-m_Y)^2}{Y}\right]\right\}}$$

- By defining the *random vector* $\boldsymbol{Z}$, the *mean vector* $\mathbf{m}_Z$ and the *covariance matrix* $\mathbf{C}_Z$:

$$\boldsymbol{Z} = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \mathbf{m}_Z = \mathbb{E}[\boldsymbol{Z}] = \begin{bmatrix} \mathbb{E}[Z_1] \\ \mathbb{E}[Z_2] \end{bmatrix}, \quad \mathbf{C}_Z = \mathbb{E}\left[(\boldsymbol{Z} - \mathbf{m}_Z)(\boldsymbol{Z} - \mathbf{m}_Z)^{\mathsf{T}}\right]$$

  where $\mathbf{C}_Z(i,j) = \mathbb{E}\left[(Z_i - m_{Z_i})(Z_j - m_{Z_j})\right]$
- $Z_1 = X$ and $Z_2 = Y$ are *jointly Gaussian* iff:

$$f_Z(\mathbf{z}) = f_Z(x,y) = \frac{1}{2\pi|\mathbf{C}_Z|^{1/2}}\, \mathrm{e}^{-\frac{1}{2}(\mathbf{z}-\mathbf{m}_Z)^{\mathsf{T}}\mathbf{C}_Z^{-1}(\mathbf{z}-\mathbf{m}_Z)}$$

Jointly Gaussian random variables

- In the general case, for n r.v.,

$$f_X(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\mathbf{C}_X|^{1/2}}\, \mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\mathbf{m}_X)^\top \mathbf{C}_X^{-1}(\mathbf{x}-\mathbf{m}_X)}$$

Definition

A *white Gaussian random vector*, **W**, is a collection of independent and identically distributed (iid) Gaussian r.v.
A *standard Gaussian random vector*, **W**, is a collection of zero-mean, unit-variance, iid Gaussian r.v.

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{(2\pi)^{n/2}}\, \mathrm{e}^{-\frac{1}{2}\|\mathbf{w}\|^2}$$

- The white condition implies that the covariance matrix is $\sigma^2\mathbf{I}$.
- The standard condition implies that the mean is a zero vector and the diagonal is the identity matrix, $\mathbf{C}_W = \mathbf{I}$

Multidimensional AWGN

- In communications, many system models involve zero mean and a covariance matrix of the form $\mathbf{C}_W = \sigma^2 \mathbf{I}$

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \, \mathrm{e}^{-\frac{\|\mathbf{w}\|^2}{2\sigma^2}}$$

  where
  - every entry of the vector has zero mean and variance $\sigma^2 = N_0/2$
  - and is independent of the other entries
- It follows

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{(2\pi \cdot N_0/2)^{n/2}} \, \mathrm{e}^{-\frac{\|\mathbf{w}\|^2}{2(N_0/2)}}$$

- From now on, we will focus on this case and denote this noise by *aditive white Gaussian noise, (AWGN)*.
  - When the channel is just an addition of AWGN, the channel is referred to as *AWGN channel*.

Projection on one dimension

- The projection of a multidimensional noise into one direction, or normal vector, is most important
  - it permits removing noise that it is not in the dimension where the information is transmitted, improving the SNR
  - it allows to reduce the problem to the one-dimensional AWGN channel
- If the systems allows us to project in one dimension, not loosing any useful information, we will do it.
- How does the projected noise yield?

Normalized Combination of Independent Gaussians

- Given a AWGN random vector, **w**, multiplied by a normal vector **c** (unit module), we get a univariate zero-mean Gaussian r.v. (AWGN) of variance $\sigma^2 = N_0/2$,

$$\tilde{w} = \mathbf{c}^{\mathsf{T}}\mathbf{w} \sim \mathcal{N}(0, N_0/2)$$

$\Rightarrow$Prove it. Hint: use the result on sum of gaussians where, since **c** is unit module, $\sum_{i=1}^{n} c_i^2 = 1$.

Sum of Gaussians: Given two Gaussian r.v., $\boldsymbol{X} \sim \mathcal{N}\left(m_X, \sigma_X^2\right)$, $\boldsymbol{Y} \sim \mathcal{N}\left(m_Y, \sigma_Y^2\right)$

- If they are *independent* the sum $aX + bY$ is distributed as

$$\mathcal{N}\left(am_X + bm_Y, a^2\sigma_X^2 + b^2\sigma_Y^2\right)$$

Outline of the detection in multimensional AWGN

- We transmit an n-dimensional symbol, $\boldsymbol{X}$
- We transmit a binary signal B, $b = \{\pm 1\}$ with equal probability: $\boldsymbol{X}$ has 2 points in the n-dimensional space: $\mathbf{x}_A$ and $\mathbf{x}_B$
  - $\mathbf{x} = \{\mathbf{x}_A, \mathbf{x}_B\}$, and $\Pr(\boldsymbol{X} = \mathbf{x}_A) = \Pr(\boldsymbol{X} = \mathbf{x}_B) = 0.5$
- A random additive white Gaussian noise (AWGN), $\boldsymbol{W}$, is added
  - $\boldsymbol{W}$ is a vector of n iid zero mean Gaussian noises of variance $N_0/2$, $\boldsymbol{W} \sim \mathcal{N}(\mathbf{0}, N_0/2 \cdot \mathbf{I})$
- We receive the random vector $\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{W}$
- How we do demodulate and detect B?
  - Solution: we project the n dimensional received vector $\mathbf{y}$ onto the line between $\mathbf{x}_A$ and $\mathbf{x}_B$ (*signal subspace*)
- What the BER is?
  - by *projecting*, the problem is equivalent to the one-dimensional scenario:
    - square of the half of the distance between points gives the average power
    - noise power is that of the n dimensional noise projected onto 1 dimension: $N_0/2$
  - by applying the ML detector

$$p_e = Q\left(\sqrt{\frac{(\|\mathbf{x}_A - \mathbf{x}_B\|/2)^2}{N_0/2}}\right) \tag{15}$$

Vector space

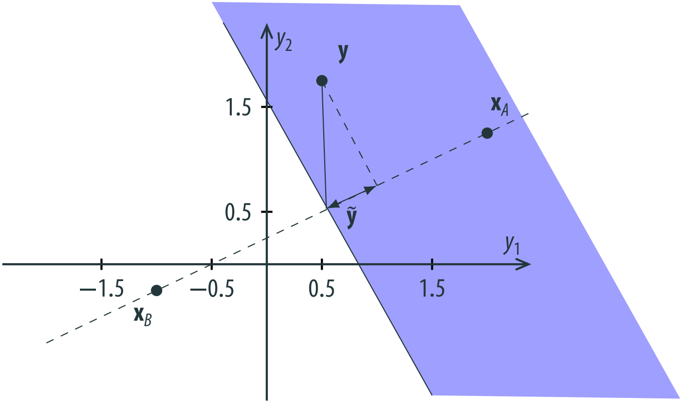


Figure: Two bidimensional transmitted symbols, $\mathbf{x}_A$ and $\mathbf{x}_B$. Noise of the channel is added to the transmitted symbol, this is the received vector $\mathbf{y}$. The shaded area is the one where received points are detected as A.

Detection in N-D AWGN

- Suppose we transmitted $\mathbf{x}_A$, and due to the added AWGN we have $\mathbf{y}$.

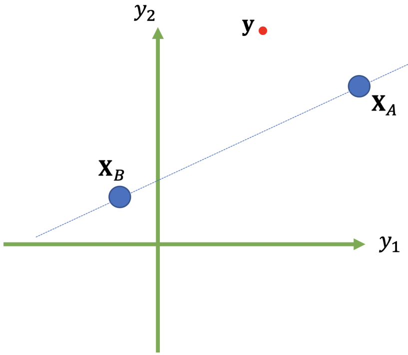


Figure: Two bidimensional transmitted symbols, $\mathbf{x}_A$ and $\mathbf{x}_B$, joined by a line, and all possible received points.

Detection in N-D AWGN

- We could design a detector using the threshold in the figure

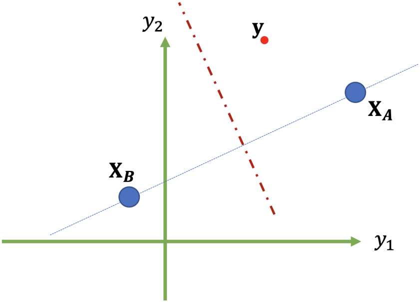


Figure: Received point after AGWN channel, y, and decision boundary (dash-dotted)

Let us write the maths...

Algebraic problem

- The whole problem reduces to an algebraic issue:

$$\mathbf{y} = (\mathbf{X}_A + \mathbf{X}_B)/2 + b(\mathbf{X}_A - \mathbf{X}_B)/2 + \mathbf{w} \quad (16)$$

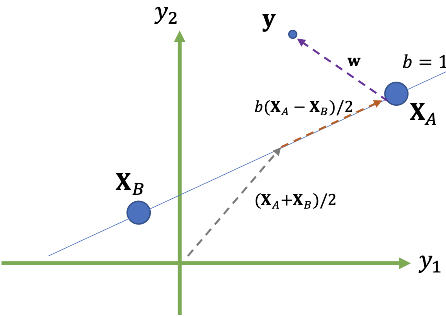


Figure: Received vector as a the sum of vectors

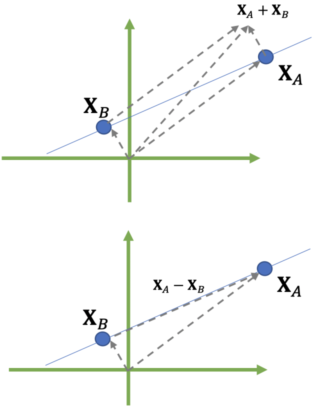


Figure: Sum and subtraction of vectors

Detection in 2-D AWGN

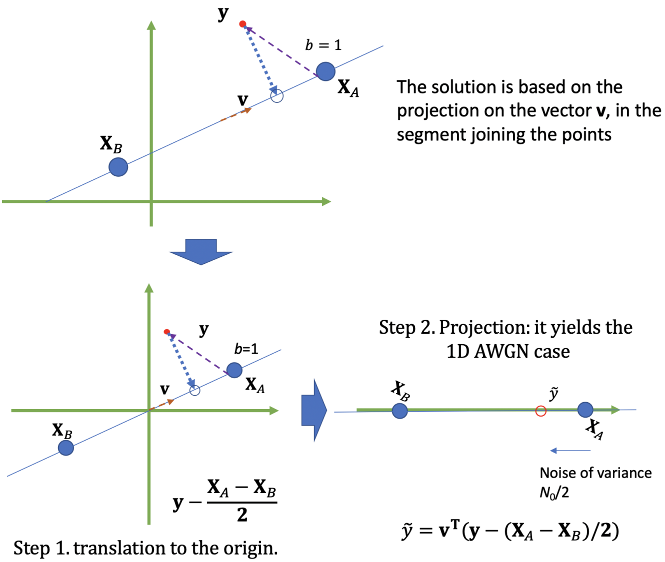


Figure: From multidimensional to one-dimension detection

Detection for AWGN in vector space

- The decision boundary is now a line in a plane (instead a point in a line), either $\mathbf{x}_A$ or $\mathbf{x}_B$ can be written as
$$\mathbf{x} = \frac{1}{2}(\mathbf{x}_A + \mathbf{x}_B) + \frac{b}{2}(\mathbf{x}_A - \mathbf{x}_B) \Rightarrow \mathbf{y} = \mathbf{x} + \mathbf{w} \tag{17}$$
where b is an scalar which is equally likely to be ± 1, encoding the information.

Task 2.4 (Vector Space)

If $\mathbf{u}_1 = (\mathbf{x}_A + \mathbf{x}_B)/2$, $\mathbf{u}_2 = (\mathbf{x}_A - \mathbf{x}_B)/2$ and $\mathbf{u}_3 = \mathbf{w}$, represent $\mathbf{u}_1$, $\mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$.

- We project the received vector onto the line joining $\mathbf{x}_A$ and $\mathbf{x}_B$, the *signal direction* given by the **normalized** *signal vector*, $\mathbf{v} = (\mathbf{x}_A - \mathbf{x}_B)/\|(\mathbf{x}_A - \mathbf{x}_B)\|$:
  1. the noise, being isotropic, can be viewed as two independent components, in the signal direction (or subspace) and in the orthogonal one (the problem is rotationally invariant)
  2. by projection we suppress the noise orthogonal to the signal direction
  3. the resulting value, $\tilde{y}$, is a so-called *sufficient statistic*
  4. the projection on the signal vector, $\mathbf{v}$, is the so-called *matched filter (MF)*
  5. we arrive at the equivalent problem $\hat{b} = \mathrm{sign}(\tilde{y})$ where
$$\tilde{y} \stackrel{\text{def}}{=} \mathbf{v}^{\top}(\mathbf{y} - (\mathbf{x}_A + \mathbf{x}_B)/2) = b\|(\mathbf{x}_A - \mathbf{x}_B)\|/2 + \tilde{w} \tag{18}$$
where $\tilde{w}$ is distributed as $\mathcal{N}(0, N_0/2)$, since the projection is to a normalized vector.

Detection in complex-valued AWGN

Detection in circular complex-valued AWGN: the easy way

- For the case of circular complex-valued AWGN the case $\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{W}$ can be rewritten as

$$\begin{bmatrix} \boldsymbol{Y}_\mathrm{r} \\ \boldsymbol{Y}_\mathrm{j} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_\mathrm{r} \\ \boldsymbol{X}_\mathrm{j} \end{bmatrix} + \begin{bmatrix} \boldsymbol{W}_\mathrm{r} \\ \boldsymbol{W}_\mathrm{j} \end{bmatrix} \quad (19)$$

- where

$$\begin{bmatrix} \boldsymbol{W}_\mathrm{r} \\ \boldsymbol{W}_\mathrm{j} \end{bmatrix} \sim \mathcal{N}\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}\right) \quad (20)$$

- Hence we can apply the theory for a *multimensional real-valued AWGN channel*.
  - where we have twice the dimensions of the complex-valued problem
  - and we can transmit one stream of bits using $\boldsymbol{X}_\mathrm{r}$ and another with $\boldsymbol{X}_\mathrm{j}$
  - i.e. *we double the data rate at the same BER*.

Application to low-pass equivalent AWGN channel

- Imagine we have an AWGN channel and we transmit a BPSK, PAM, QAM, PSK,...
- At reception we go through the vectorial receiver, that provides the output for both matched filters, to the in-phase and quadrature parts.
- Then we can perform detection

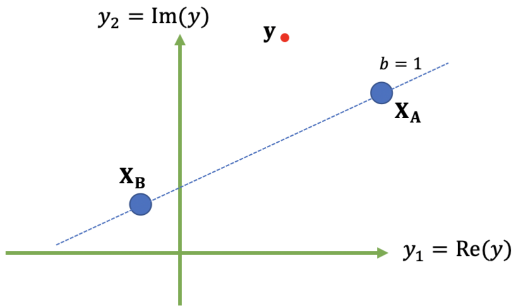


Figure: One-dimensional complex-valued reception

Complex valued random variables

- A complex random variable $W = W_{\mathrm{r}} + \mathrm{j}\, W_{\mathrm{j}}$, can be seen as a vector of two variables $W_{\mathbb{R}} = [\, W_{\mathrm{r}},\, W_{\mathrm{j}}]^{\top}$ described statistically by its joint pdf.
- In complex-valued AWGN channels the real and imaginary parts are zero-mean and independent of variance $N_0/2$
  - In this case the Gaussian rv is said to be *circular*
- A complex-valued AWGN is usually denoted as

Gaussian complex random variable

$W = W_{\mathrm{r}} + \mathrm{j}\, W_{\mathrm{j}} \sim \mathcal{CN}(0,N_0,0)$ where

$$\begin{aligned} W_{\mathrm{r}} &\sim \mathcal{N}(0, N_0/2), \\ W_{\mathrm{j}} &\sim \mathcal{N}(0, N_0/2), \end{aligned} \tag{21}$$

and $\mathcal{CN}\left(m,\sigma^2,\tilde{\sigma}^2\right)$ is a complex valued Gaussian r.v. of mean m, variance σ^2 and pseudo-variance $\tilde{\sigma}^2$, if real and imaginary parts are independent $\tilde{\sigma}^2 = 0$.

- Notice that the variance is double of the one of the real, or imaginary, part
- The extension to multidimensional complex-valued AWGN is straightforward $\boldsymbol{W} = \boldsymbol{W}_{\mathrm{r}} + \mathrm{j}\, \boldsymbol{W}_{\mathrm{j}} \sim \mathcal{CN}(\mathbf{0},N_0\mathbf{I},\mathbf{0})$

Detection in circular complex AWGN: complex-valued form

- There are *3 changes with respect to the real case*
  1. Noise is now $W \sim \mathcal{CN}(\mathbf{0}, N_0 \cdot \mathbf{I})$
     - note that at each dimension we have now a complex valued noise of variance N_0
  2. Projection is similar but with Hermitian, $(\cdot)^{\mathrm{H}}$ denotes transpose conjugate:

$$\tilde{y} \stackrel{\text{def}}{=} b\|(\mathbf{x}_A - \mathbf{x}_B)\|/2 + w = \mathbf{v}^{\mathrm{H}}(\mathbf{y} - (\mathbf{x}_A + \mathbf{x}_B)/2) \tag{22}$$

  3. If b is binary and it is modulated into a real vector

$$\begin{aligned}\hat{b} &= \mathrm{sign}(\mathrm{Re}[\tilde{y}]) = \mathrm{sign}(\mathrm{Re}[b\|(\mathbf{x}_A - \mathbf{x}_B)\|/2 + \tilde{w}_\mathrm{r} + \mathrm{j}\tilde{w}_\mathrm{j}]) \\ &= \mathrm{sign}(b\|(\mathbf{x}_A - \mathbf{x}_B)\|/2 + \tilde{w}_\mathrm{r})\end{aligned}$$

  where the variance of the noise is $N_0/2$.

- As a result, we have the same BER, see (15)
- *However*, note that we could transmit another binary signal $b^* = \pm\mathrm{j}$ as two imaginary points $\mathbf{x}_C$ and $\mathbf{x}_D$ as an orthogonal transmission to the real one: *we have double rate and same BER*,
  - we would detect it as $\hat{b}^* = \mathrm{sign}(\mathrm{Im}[\tilde{y}]) = \mathrm{sign}(b\|(\mathbf{x}_C - \mathbf{x}_D)\|/2 + \tilde{w}_\mathrm{j})$

Particularization to Complex-valued 1D

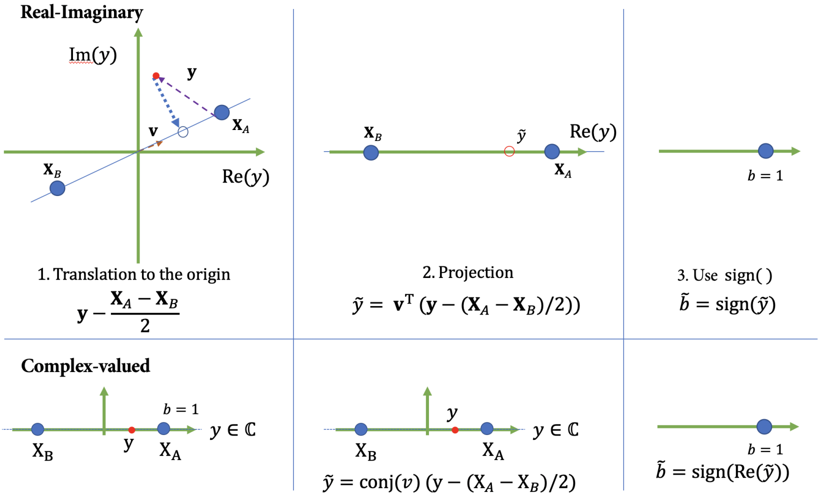


Figure: Reception as two-dimensional real or one-dimensional complex valued

Important: Note that by multiplying by $\mathrm{conj}(v)$, being $|v| = 1$, we just *undo the rotation* ...

Monte Carlo Simulation of Complex AWGN (Matlab/Octave)

Exercise 2.2 (Complex BPSK Monte Carlo Simulation (Matlab/Octave))

Run the following code. What is the needed E_b/N_0 for BER$= 10^{-3}$? Does it change? Why?

```
clear all, close all
Np=1e5; %Number of bits (BPSK) Tx
eb=1; a=sqrt(eb); Es=eb; %XA=-a, XB=a
SNRdB=0:1:15; %Range of Eb/NO to simulate
h=1*exp(3*pi/4*1j); %unit norm
NdB=length(SNRdB); sqNdB=ceil(sqrt(NdB));
for k1=1:NdB
    Bn=randi([0 1],1,Np); %Generate bits
    xn=h*(Bn*2*a-a); %BPSK % Symbols (-a,a)
    ebno=10.^(SNRdB(k1)/10); %ebno, nat. unit.
    no=eb/ebno; sigma=sqrt(no/2); %sdt del ruido
    wn=sigma*(randn(1,Np))+1j*sigma*(randn(1,Np));%Re & Im are independent
    yn=xn+wn; %Channel AWGN
    bn=WRITE here your answer %Detection
    BER(k1)=sum(bn~=Bn)/Np; %MC error analysis
    figure(1),subplot(sqNdB,sqNdB,k1),
    plot(real(yn(1:200)),imag(yn(1:200)),'ro'),hold on
    plot(real(xn(1:20)),imag(xn(1:20)),'bs','MarkerSize',10)
end
figure(2), semilogy(SNRdB,BER), title('BER BPSK')
xlabel('E_b/N_o (dB)'), ylabel('BER')
```

Task: BER of a QAM

Task 2.5 (BER of a QAM)

Use the previous concepts to compute the BER of a QAM where we transmit $s = s_\mathrm{r} + \mathrm{j}s_\mathrm{j}$ where $s_\mathrm{r} = \pm 1$ and $s_\mathrm{j} = \pm 1$, and hence we have four points $\mathbf{x}_A$, $\mathbf{x}_B$, $\mathbf{x}_C$ and $\mathbf{x}_D$ given by $\mathbf{x} = r \cdot s$ where r is some amplitude constant. Solve it

1. Assuming you have a detection in real vector space where real and imaginary part are two dimensions
2. Assuming you have the detection in the complex vector space with one dimension

Task 2.6 (MC for the BER of a QAM)

Modify the Matlab/Octave code for the complex case and BPSK to the 4-QAM.

Modulation Comparison

SER of M-QAM vs. M-PSK

We include some results on SER of M-QAM vs. M-PSK in coherent detection, AWGN channel.

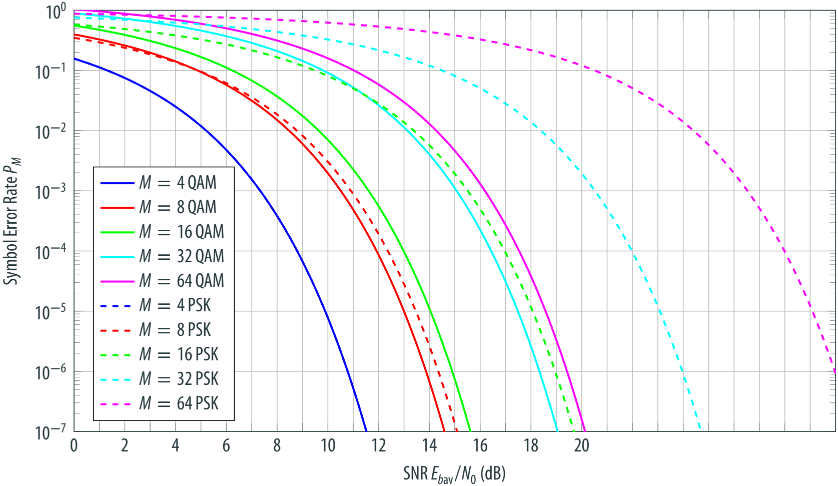


Figure: Symbol error rate for some digital linear constellation

BER for M-QAM

Here we include the results for the BER and M-QAM, again in AWGN channel.

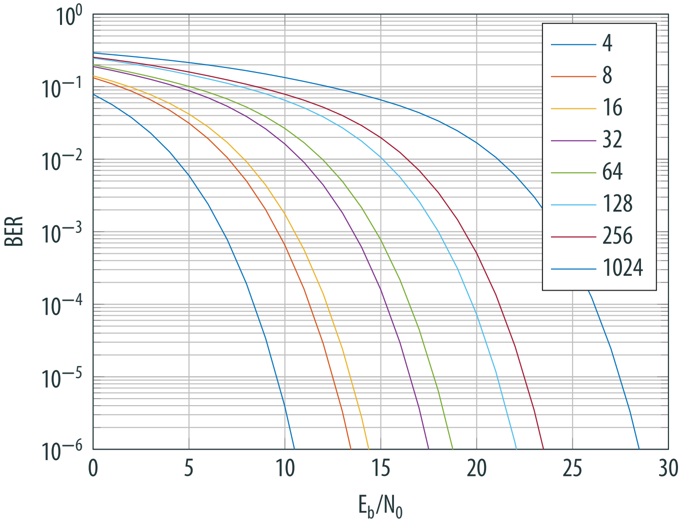


Figure: Bit error rate for M-QAM

Comparison between modulations

- We focus on three parameters: bandwith, spectral efficiency and BER (or SER).
- Assuming coherent MFSK modulation we have the following table, with $\gamma_b = E_{bav}/N_0$:

	B (Hz)	η((bit/s)/Hz)	SER
M-PSK	$\frac{1}{T}$	$\log_2 M$	$P_M \approx 2Q\left[\sqrt{2\gamma_b \log_2 M sin\left(\frac{\pi}{M}\right)}\right]$
M-QAM	$\frac{1}{T}$	$\log_2 M$	$P_M \approx 4Q\left(\sqrt{\frac{3\log_2 M}{(M-1)}\gamma_b}\right)$
M-FSK	$\frac{M}{2T}$	$\frac{2\log_2 M}{M}$	$P_M \approx (M-1)Q\left(\sqrt{\gamma_b \log_2 M}\right)$

Exercise 2.3

Throughput and bandwith For a given bandwith, e.g. 1 MHz, provide the gain of throughput for a BPSK, 4-QPSK, 16-QAM, 64-QAM and 256-QAM, with respect to a BPSK and provide the needed increase in energy to keep the BER.

A figure to compare them all

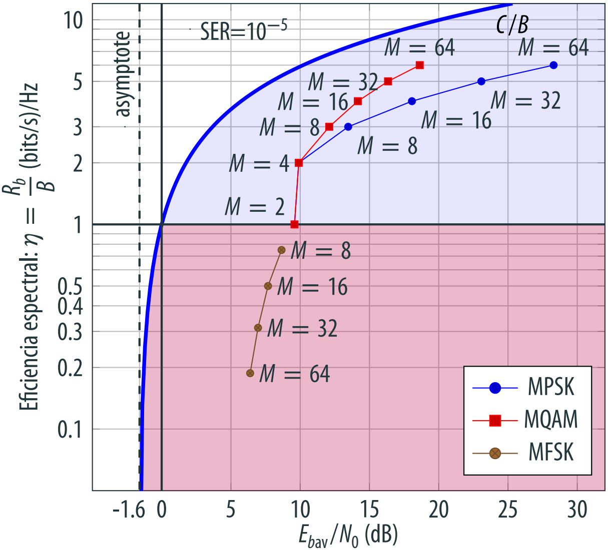


Figure: Bit error rate for M-QAM

Appendices

Appendix I. Notes on equivalent lowpass model

Since the signal $s(t)$ is real,

$$S(f) = S^*(-f) \tag{23}$$

We have

$$S_b(f) = \begin{cases} S_+(f+f_c) = S_+^*(-f+f_c) & f+f_c > 0 \\ 0 & f+f_c \le 0 \end{cases} \tag{24}$$

by manipulating this expression, first conjugating,

$$S_b^*(f) = \begin{cases} S_+^*(f+f_c) = S_+(-f+f_c) & f+f_c > 0 \\ 0 & f+f_c \le 0 \end{cases} \tag{25}$$

then computing the symmetric,

$$S_b^*(-f) = \begin{cases} S_+^*(-f+f_c) = S_+(f+f_c) & -f+f_c > 0 \\ 0 & -f+f_c \le 0 \end{cases} \tag{26}$$

and shifting by $-f_c$, we have

$$S_b^*(-f-f_c) = \begin{cases} S_+(-f) & f < 0 \\ 0 & f \ge 0 \end{cases} \tag{27}$$

that shows that the left part of the spectrum, $S_+(-f)$, can be replaced by $S_b^*(-f-f_c)$.

Appendix II. Circular complex-valued random variables

Definition

Circularity: The r.v. X is *circular symmetric* if $e^{j\theta} X$ has the same distribution than X, $\forall\theta$. It implies $m_X = 0$.

- For a complex (column) random vector we have $\boldsymbol{X} = \boldsymbol{X}_r + j\boldsymbol{X}_j$, with $\boldsymbol{X}_{\mathbb{R}} = [\boldsymbol{X}_r, \boldsymbol{X}_j]^{\mathsf{T}}$ described statistically by their joint pdf.
- The definition of *circular* holds for vectors. For a Gaussian distributed $\boldsymbol{X}$ to be circular:
  - $\mathbf{m}_{\boldsymbol{X}} = 0$
  - $\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^{\mathsf{H}}] = 0$. Proof: $\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^{\mathsf{H}}] = \mathbb{E}[e^{j\theta}\boldsymbol{X}(e^{j\theta}\boldsymbol{X})^{\mathsf{T}}] = e^{j2\theta}\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^{\mathsf{T}}] \Rightarrow \ldots$
  - But needn't to be independent: $\mathbf{C}_{\boldsymbol{X}} = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^{\mathsf{H}}] \neq \mathbf{I}$
- A random vector being circular Gaussian distributed is denoted as $\boldsymbol{X} \sim \mathcal{CN}(0, \mathbf{C}_{\boldsymbol{X}})$.

Exercise 2.4 (Circularity)

Discuss which of the following are circular: a 4-QAM, a 16-QAM, a 16-PSK, a complex r.v. with constant modulus and uniformly distributed phase, a complex r.v. with iid real and imaginary parts that are zero mean Gaussian distributed.

Appendix III: Complex valued random variables

- A complex random variable $X = X_\mathrm{r} + \mathrm{j}X_\mathrm{j}$, can be seen as a vector of two variables $X_\mathbb{R} = [X_\mathrm{r}, X_\mathrm{j}]^\mathsf{T}$ described statistically by its joint pdf.
- Which second central moment defines the r.v.? $\mathbb{E}\left[(X - \mathbf{m}_X)(X - \mathbf{m}_X)^\mathsf{T}\right]$? $\mathbb{E}\left[(X - \mathbf{m}_X)(X - \mathbf{m}_X)^\mathsf{H}\right]$?
  - Answer is BOTH, we define the covariance, $\sigma_X = \mathbb{E}\left[(X - \mathbf{m}_X)(X - \mathbf{m}_X)^\mathsf{H}\right]$, and the pseudocovariance, $\tilde{\sigma}_X = \mathbb{E}\left[(X - \mathbf{m}_X)(X - \mathbf{m}_X)^\mathsf{T}\right]$

Gaussian complex random variable

$$X = X_\mathrm{r} + \mathrm{j}X_\mathrm{j} \sim \mathcal{CN}\left(m_X, \sigma^2_X, \tilde{\sigma}^2_X\right)$$

Gaussian complex random variable with zero mean iid real and imaginary parts

$$\begin{aligned} X_\mathrm{r} &\sim \mathcal{N}(0, N_0/2), \\ X_\mathrm{j} &\sim \mathcal{N}(0, N_0/2), \\ X = X_\mathrm{r} + \mathrm{j}X_\mathrm{j} &\sim \mathcal{CN}(0, N_0, 0) \end{aligned} \Rightarrow \begin{aligned} \tilde{\sigma}_X &= \mathbb{E}[X_\mathrm{r}^2] - \mathbb{E}[X_\mathrm{j}^2] + 2\mathrm{j}\mathbb{E}[X_\mathrm{r}X_\mathrm{j}] = 0 \\ \sigma_X &= \mathbb{E}[X_\mathrm{r}^2] + \mathbb{E}[X_\mathrm{j}^2] = N_0 \end{aligned}$$

- A complex r.v. is *proper* iff $\tilde{\sigma}_X = 0$ the
  - if X_r and X_j are iid the random variable is proper

Appendix IV: Complex-valued random variables

- For a complex (column) random vector we have $\boldsymbol{X} = \boldsymbol{X}_\mathrm{r} + \mathrm{j}\boldsymbol{X}_\mathrm{j}$, with $\boldsymbol{X}_\mathbb{R} = [\boldsymbol{X}_\mathrm{r}, \boldsymbol{X}_\mathrm{j}]^\top$ described statistically by their joint pdf.
- It is *proper* iff its pseudo-covariance $\tilde{\mathbf{C}}_{\boldsymbol{X}} = \mathbb{E}\left[(\boldsymbol{X} - \mathbf{m}_{\boldsymbol{X}})(\boldsymbol{X} - \mathbf{m}_{\boldsymbol{X}})^\top\right]$ cancels.
  - If $\boldsymbol{X}_\mathrm{r}$ and $\boldsymbol{X}_\mathrm{j}$ are iid then $\boldsymbol{X}$ is proper.
- The definition of *circular* holds for vectors. For a Gaussian distributed $\boldsymbol{X}$ to be circular:
  - $\mathbf{m}_{\boldsymbol{X}} = 0$
  - $\tilde{\mathbf{C}}_{\boldsymbol{X}} = 0$ Proof: $\tilde{\mathbf{C}}_{\boldsymbol{X}} = \mathbb{E}\left[\mathrm{e}^{\mathrm{j}\theta}\boldsymbol{X}(\mathrm{e}^{\mathrm{j}\theta}\boldsymbol{X})^\top\right] = \mathrm{e}^{\mathrm{j}2\theta}\mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^\top\right] \Rightarrow \tilde{\mathbf{C}}_{\boldsymbol{X}} = 0$
  - But needn't to be independent: $\mathbf{C}_{\boldsymbol{X}} = \mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^\mathsf{H}\right] \neq \mathbf{I}$
- A random vector being circular Gaussian distributed is denoted as $\boldsymbol{X} \sim \mathcal{CN}(0, \mathbf{C}_{\boldsymbol{X}})$.

Appendix V: Sum of Gaussians

$$
\begin{aligned}
f_Z(z) = f_X(x) * f_Y(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-(z-x)^2/(2)}\mathrm{e}^{-x^2/(2)}\mathrm{d}x = \\
= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\left(z^2+2x^2-2zx\right)^2/(2)}\mathrm{d}x \\
= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-z^2/(2\cdot 2)}\mathrm{e}^{-\left(\frac{z^2}{2}+2x^2-2zx\right)^2/(2)}\mathrm{d}x = \{u = \sqrt{2}x\} = \\
= \frac{1}{2\pi}\mathrm{e}^{-z^2/(2\cdot 2)}\int_{-\infty}^{\infty} \mathrm{e}^{-\left(u-\frac{z}{\sqrt{2}}\right)^2/(2)}\mathrm{d}u/\sqrt{2} \\
= \frac{1}{2\pi}\mathrm{e}^{-z^2/(2\cdot 2)}\frac{\sqrt{2\pi}}{\sqrt{2}} \\
= \frac{1}{\sqrt{2\pi}\sqrt{2}}\mathrm{e}^{-\frac{z^2}{2\cdot 2}}
\end{aligned}
$$