Introduction to Math 120

- Course Setup and Expectations
- Grades
- Monty Hall Problem

Welcome to Math 120, Introduction to College Mathematics.

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This class is designed to be a practical math course where we learn how to get better at life.

We will answer questions such as:

- How does money get used in daily life, and how do we use money wisely?
- What strategies are optimal when gambling at a casino?
- How can we use data to make predictions about outcomes that we can't test on a small scale?

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The first thing you will need to know is how to access the course on Canvas, our course management system. To get to Canvas go to `www.nsc.instructure.com` and sign in using your student id and password.

For each day of class there are two assignments to be completed.

- Prior to coming to class each student must complete a pre-class assignment that will prepare you for the group activities you will work on during class. The pre-class assignment is designed to baby-step you through some of the key ideas that will be covered during class.
- After class you will complete a homework assignment that is to be handed in at the beginning of the next class period.

Class time will be devoted to group work and discussion with a small bit of lecture between problems. It is essential that students come prepared to work hard during class. There is no course textbook, so the only way to learn the material is to participate in class and to complete the pre-class assignments.

The course is broken up into 3 topics:

- Consumer Mathematics
- Probability and Statistics
- Algebra and Modeling

Grades will be weighted as follows:

10% Daily Quizzes

15% Pre-Class Work

10% In-Class Group Work

10% Homework

15% Probability Exam

15% Modeling Exam

10% Consumer Math Portfolio

15% Final Project

Grades will be weighted as follows:

- 10% Daily Quizzes (5-min at the start of class)
- 15% Pre-Class Work
- 10% In-Class Group Work
- 10% Homework
- 15% Probability Exam
- 15% Modeling Exam
- 10% Consumer Math Portfolio
- 15% Final Project

Grades will be weighted as follows:

10% Daily Quizzes (5-min at the start of class)

15% Pre-Class Work (Completed on Ximera in Canvas)

10% In-Class Group Work

10% Homework

15% Probability Exam

15% Modeling Exam

10% Consumer Math Portfolio

15% Final Project

Grades will be weighted as follows:

10% Daily Quizzes (5-min at the start of class)

15% Pre-Class Work (Completed on Ximera in Canvas)

10% In-Class Group Work (a daily write-up, handed in at the end of class)

10% Homework

15% Probability Exam

15% Modeling Exam

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- 15% Pre-Class Work (Completed on Ximera in Canvas)
- 10% In-Class Group Work (a daily write-up, handed in at the end of class)
- 10% Homework (Due at the beginning of class, must be neat)
- 15% Probability Exam
- 15% Modeling Exam
- 10% Consumer Math Portfolio
- 15% Final Project

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15% Modeling Exam

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10% Homework (Due at the beginning of class, must be neat)

15% Probability Exam (In-class exam)

15% Modeling Exam (In-class exam)

10% Consumer Math Portfolio (Completed as small investigations across the semester)

15% Final Project (a group activity completed in the last few weeks of the course, culminating in a portfolio and video presentation)

Consumer Mathematics

Probability and Statistics

Consumer Mathematics

Algebra and Modeling

Final Project

A typical class period will be structured as we have displayed to the right. The quiz will be at the start of class, so you can't afford to be late. Homework from the previous class period will be handed in along with the quiz.

Most of the time will be devoted to exploring problems in small groups. Some days and topics will require more instruction than others, and some activities will be shorter than others.

At the end of the class period each group will hand in a write-up of their activities for the day. We will show an example of an activity write-up later in this class period.

We will be using calculators and computers in this class. Please bring your laptops and tablets to class each day.

$$\text{laptop} > \text{tablet} > \text{smartphone} > \text{calculator}$$

You are encouraged to install the WolframAlpha app on your device. Make sure that your device is connected to the NSC wifi.

The homework will also rely heavily on web resources.

The rest of today's class will be devoted to clarifying expectations for the homework and group activities.

Homework must be completed on time. It will be due at the beginning of class. No late work will be accepted (without prior permission).

- Write in full sentences—even when dealing with mathematical expressions.
- If you have scratch work, then box it off to the side so that it doesn't mix with the main work and conclusions.
- Staple all pages together.

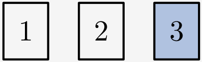
Each day in class students will break into groups of 2 or 3 to work on problems as a group for 10–15 minutes at a time. Each group will complete a write-up of the group activities for the day that includes the names of each member of the group.

These write-ups will be graded very generously, but it is important to do neat work because the group activities are meant to prepare us for homework. If we don't practice doing neat work, then we will definitely fail to do neat work when exam time comes.

We will use the rest of class time to complete a group activity.

Monty Hall Problem

On a game show, the host shows the contestant three doors. Behind one of the doors is a prize, while nothing is behind the other two doors. The contestant chooses door 3.



Instead of opening up door 3, the host opens door 1 and shows the contestant that nothing is behind it.

He then gives the contestant the chance to switch his choice to door 2. Should the contestant switch?

Monty Hall Problem Simulation

Get in groups of 2. Write both of your names on a new sheet of paper, and follow the instructions below.

1. Simulate the game by playing 15 rounds. One player will be the host and will secretly pick a door. The other player will be the contestant, who will make a guess. The host will then cross off a door that doesn't have the prize. The player should stick with the original guess. Record the number of times the player won the prize.
2. Simulate the game for 15 more rounds where the player switches their guess each time. Record the number of times the player won the prize.
3. Decide whether the player should switch or not (or if it matters at all). Write a brief explanation of your conclusion.

Here is a sample solution.

Bill S. Preston Esquire, Ted Theodore Logan
Monty Hall Problem Simulation

1. Outcomes for original guess: `WLLLWLLLLWWLLLW`
   The contestant won 5 times when sticking with the initial guess.
2. Outcomes for switching: `LWWWWWLWLLWLWLW`
   The contestant won 9 times when switching from the initial guess.
3. It seems from the simulation that switching is the better choice. However, this seems unlikely, so we probably need to do more simulations. The difference is probably due to luck.

Observe how the sample solution had the name of the students, and the name of the group activity. The work was written in complete sentences wherever possible.

The answer to part 3 was only semi-correct, but that doesn't matter at all. The answer did support the results from parts 1 and 2, and it showed that the students were thinking critically about the problem.

Class time is limited, so group activity time is time to get down to business with no delay. Most class periods will involve multiple group activities as well as some discussion and new content, so it is important that we use our time well.

You can use the same page for all of the group activities in one class period a given day. We encourage you to switch groups often to get to know your classmates and learn from them.

Homework I

There is no homework to hand in at the beginning of class next time. However, you must complete each item below before the start of the next class.

1. Go to the class discussion board on Piazza (in Canvas) and introduce yourself on the thread titled "Introductions".
2. Bookmark the website http://desmos.com and play around with the calculator a little bit to get a feel for it.
3. Be sure to complete the pre-class activities before the beginning of the next class period.

Percentages

- Consumer Math Portfolio Assignments
- Reference Amounts
- Consecutive Percentages
- Increase and Decrease

As part of the unit on Consumer Mathematics you will create a portfolio of written assignments by investigating various financial scenarios. The homework assignment at the end of each class period in the Consumer Math portion of the course is designated a Consumer Math Portfolio Assignment.

Unlike the other homework assignments, a Consumer Math Portfolio Assignment must be nicely typed, and it must be submitted to Canvas before the start of the next class period. The Consumer Math Portfolio will actually consist of 5 smaller assignments that will be graded separately, but you are encouraged to use the same theme and layout for each of the Consumer Math Portfolio Assignments.

In the preview work, we reviewed the concept of percentages. The key formula for calculating with percentages is

$$\text{Percentage} = \frac{\text{Part}}{\text{Whole}} \times 100$$

This equation will continue to play an important role in our future calculations with percentages.

In 2012, a rain gauge in Boulder City recorded a total annual rainfall of 3.17 inches. In 2013, the same rain gauge recorded a total annual rainfall of 2.45 inches.

Channel 4 news reports that rainfall in 2013 was 23% lower than in 2012.

Channel 6 news reports that rainfall in 2012 was 29% higher than in 2013.

In 2012, a rain gauge in Boulder City recorded a total annual rainfall of 3.17 inches. In 2013, the same rain gauge recorded a total annual rainfall of 2.45 inches.

Channel 4 news reports that rainfall in 2013 was 23% lower than in 2012.

Channel 6 news reports that rainfall in 2012 was 29% higher than in 2013.

Can both reports be accurate? Discuss!

When calculating a percentage, it is important to understand which amount is being referenced because this amount will be the whole used in calculating the percentage.

The actual difference in rainfall between the two years was 0.72 inches. If the rainfall from 2012 (3.17 inches) is the amount being referred to for comparison, then

$$\frac{0.72}{3.17} \times 100 \approx 23\%$$

and so 2013 did have 23% less rainfall *than* 2012.

But if the rainfall from 2013 (2.45 inches) is the amount being referred to for comparison, then

$$\frac{0.72}{2.45} \times 100 \approx 29\%$$

and so 2012 did have 29% more rainfall *than* 2013.

Down and then Up

Suppose your employer tells you that, due to a downturn in sales, your pay will be cut by 10% this month. If sales pick up next month, however, you will then receive a 10% raise. Discuss whether this scenario would result in your eventual salary being higher, lower, or the same as your original salary.

Up and then Down

Suppose your employer tells you that you've earned a 10% raise this month due to strong sales. If sales drop next month, however, your pay will then be cut by 10%. Discuss whether this scenario would result in your eventual salary being higher, lower, or the same as your original salary.

In everyday conversation, you might refer to a pay increase or a pay reduction as simply a change in pay. The word change in common use does not indicate a direction for the change.

Mathematical language has a strong emphasis on precision. The ambiguity of a "directionless" change is not precise. For this reason, the words increase and decrease are often used instead to indicate which type of change took place. No one will confuse a 10% increase in pay with a 10% decrease in pay. If the word change is used, the direction of the change should be indicated by using positive and negative numbers to indicate the direction of the change.

For example, if Maury made $\$100$ last week and $\$90$ this week, we can say that his weekly earnings decreased by 10% or we can say that his weekly earnings changed by -10%.

Changes in Pricing

The cost of an unlocked iPhone 4s at the end of 2011 was $\$649$. At the end of 2012, the same iPhone 4s could be bought for $\$549$. Find the percent change in the cost of the iPhone 4s over that time period.

Shifting References

The average American spent $\$8,508$ on health care in 2011. The cost of health care is projected to rise about 5.8% per year for the years 2012–2022. Calculate the projected amount the average American will spend on health care in 2014.
Hint: Note that the cost of health care in 2011, 2012, *and* 2013 *all serve as reference amounts in this problem.*

Consumer Math Portfolio Assignment 1 I

1. Write up a 1-page personal bio-sketch that explains who you are and what your strengths are. Think of your audience as the class members, but be sure to write it in a tone that would impress a prospective employer. It might help to think about how someone might introduce you if your were the keynote speaker at an event. (If that doesn't help, think about what your friends might say at your funeral.) Be sure to include at least some details on your personal mathematics history.

   Submit this assignment in Canvas. Remember to follow the guidelines on screen 1. (No hollow platitudes. By the end of the bio-sketch I should have a good feel for who you are.)

Interest

- Simple Interest
- Compound Interest
- APY

Simple Interest

In the preview work, we introduced the idea of simple interest. This is interest that is calculated as a straightforward percentage of the principal annually. We can use the idea of simple interest to get a feel for how good a deal actually is.

A local rent-to-own store offers a 6-piece bedroom set for $\$1,773.91$. If a person is not able or does not wish to pay the full price upfront, the store offers a financing plan where you instead pay $\$29.99$ per week for 91 weeks. What is the annual simple interest rate this store is charging?

Simple Interest

$$\$29.99 \times 91 = \$2,729.09$$

This is the total cost of the set using the installment plan.

$$\$2,729.09 - \$1,773.91 = \$955.18$$

This is the amount of interest being charged.

$$\$955.18 = \$1,773.91 \times r \times \frac{91}{52}$$

This is the formula for simple interest, using 52 weeks in a year.

$$r \approx 0.308 = 30.8\%$$

The store is charging almost 31% annual interest!

More commonly used than simple interest is compound interest. When interest is *compounded*, it is added to the principal. Then, the next time interest is calculated the principal is larger and so even more interest is charged. Compound interest means you earn (or are charged) interest on interest!

Example: Mary loans Phillip $\$100$, with the stipulation that he will owe 10% interest for each week he waits to pay it back. This is an example of simple interest. If he paid the loan back after 1 week, he'd owe $\$100 + 10\% \times \$100 = \$100 + \$10 = \$110$. If he paid the loan back after 2 weeks he'd owe,
$\$100 + 10\% \times \$100 + 10\% \times \$100 = \$100 + \$10 + \$10 = \$120$.

If the interest were compounded, Phillip would still owe $\$100 + 10\% \times \$100 = \$100 + \$10 = \$110$ after one week, but then the $\$10$ is added to the principal for a new principal of $\$110$. Therefore, after two weeks, Phillip would owe $\$110 + 10\% \times \$110 = \$110 + \$11 = \$121$. This $\$121$ would then be the principal for the 3^{rd} week.

Computing Compound Interest

Find how much Phillip would have to pay back if he didn't pay until after the following number of weeks using each type of interest.

	Compound Interest (10%)	Simple Interest (10%)
After 1 week	$\$110$	$\$110$
After 2 weeks	$\$121$	$\$120$
After 3 weeks		
After 4 weeks		
After 5 weeks		

Computing Compound Interest

Find how much Phillip would have to pay back if he didn't pay until after the following number of weeks using each type of interest.

	Compound Interest (10%)	Simple Interest (10%)
After 1 week	$110	$110
After 2 weeks	$121	$120
After 3 weeks	$133.10	$130
After 4 weeks	$146.41	$140
After 5 weeks	$161.05	$150

There *is* a specific formula for computing compound interest all at once rather than in multiple steps. Because calculations involving compound interest are typically done with years, rather than weeks or months, as the basic unit of time, the explanation for the equation is in terms of years.

Number of compoundings per year × number of years

Total **A**mount

$$A = P \times \left(1 + \frac{r}{n}\right)^{nt}$$

Principal

Original amount plus bits of interest from each compounding

The Power of Compound Interest

When Benjamin Franklin died in 1790, his will left $\pounds 1000$ (British currency known as pounds) each to the cities of Boston and Philadelphia. Each city was to establish an account that would generate interest for 200 years, at which time the cities could withdraw the money to use for public works projects. If the Boston account compounded monthly at 4% and the Philadelphia account compounded monthly at 3.5%, how much more (in pounds) would Boston have at the end of the 200 years than Philadelphia? Write down a guess before calculating an exact result.

$$A = P \times \left(1 + \frac{r}{n}\right)^{nt}$$

One type of simple interest that is used widely is something called the annual percentage yield (APY).

An annual percentage yield is a hypothetical interest rate that *would have earned* the same amount as a lesser compound interest rate.

APY

Computing APY

A bank offers a CD (a type of investment account) that pays 2.27% interest compounded daily. The bank also lists this account has having an annual percentage yield of 2.30%.

Compute the compound interest earned on a one-year $\$100$ investment with the rate 2.27%, compounded daily, using the formula $A = P \times \left(1 + \frac{r}{n}\right)^{nt}$.

Compute the simple interest earned on a one-year $\$100$ investment with the annual rate 2.30% using the formula $I = P \times r \times t$.

Hint: Remember that A and I represent different things.

Consumer Math Portfolio Assignment 2 I

1. Visit the site `http://www.census.gov/govs/local/historical_data.html` and select a year of historical finance data. Within the data file, choose a government (federal or a specific state) and locate the amount of money spent by that government on debt interest for that year.
    1. Use `bankrate.com` to locate a long-term investment account (such as a CD, savings account, money market account, or bonds) paying a high rate of interest and calculate how much money the state or federal government would have earned to date if the money paid for debt interest was instead invested to earn interest in that account.
    2. Write a report of your work and your findings as if you were going to present the results to government officials. Include a paragraph on something you learned or thought about because of this exercise.

Consumer Math Portfolio Assignment 2 II

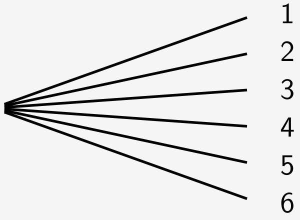
2. Planning for retirement can be daunting. In 2012, Fidelity Investments suggested[1] a person will need to save $8\times$ their final salary before retirement to live comfortably post-retirement!
   1. Use the retirement calculator at http://www.bankrate.com/calculators/retirement/roi-calculator.aspx to explore several investment scenarios. Find a table of actual historical return rates on various types of investment, such as http://pages.stern.nyu.edu/~adamodar/New_Home_Page/datafile/histretSP.html and compute three scenarios for investing that seem reasonable with your own current and projected future finances.
   2. Keep a log of your calculations and write a summary of the results as well as a paragraph about what you learned or thought about as a result of this exercise.

[1] http://usatoday30.usatoday.com/money/business/story/2012/09/12/fidelity-issues-new-retirement-savings-guidelines/57756922/1

Introduction to Probability and Statistics

- Probability Trees
- The Product Rule and Addition Rule
- Complementary Events

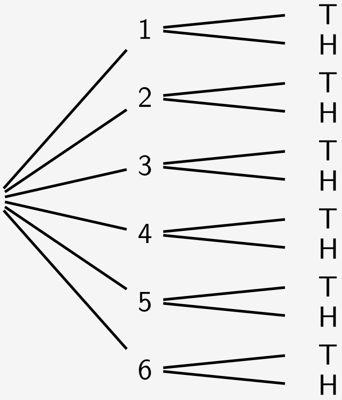
A *probability tree* is a visual representation of the events of an experiment. The following picture is a probability tree for rolling a standard 6-sided die:



When used with a single event, probability trees seem rather useless. We might as well just think of the sample space (sample space $=$ the set of possible outcomes) as a list.

However, they can help to organize more complex information such as sequential events.

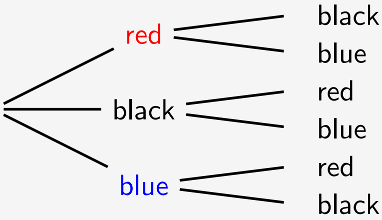
The following probability tree shows the outcomes when rolling a die followed by flipping a coin.



Notice that the tree gives an intuitive flow from left to right for sequential events and that the tree can help us list the entire sample space by listing the branch/sub-branch combinations.

The power of the probability tree is that not all events at deeper levels need to be identical.

Consider an urn containing a red ball, a black ball, and a blue ball. The following probability tree describes the possibilities when drawing two balls without replacing the first one.



Probability trees can handle even more complex situations.

Probability Trees

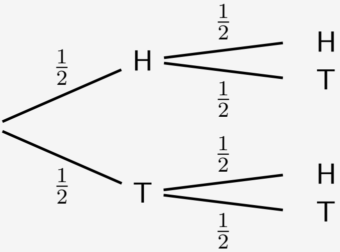
Draw a probability tree that fits the following situation:

- First, flip a coin.
- If the coin comes up heads, draw a ball from an urn containing a white ball and a black ball.
- If the coin comes up tails, roll a standard 6-sided die.

You might be wondering, “Where is the probability?”

The probability is hidden in the lines connecting the various events. What we normally do is put the probability of the event above or below the line.

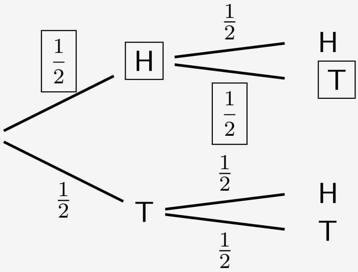
For example, the following is the probability tree for back-to-back coin flips.



Notice the probabilities at each level of branches adds to 1. This must always happen.

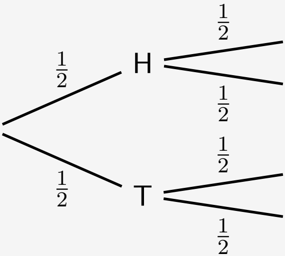
What's the probability of flipping a head followed by a tail?

To determine the probability of a specific set of sequential events, you multiply the probabilities listed on the path taken in the probability tree. This is a special case of a more general rule known as the *product rule* for probabilities.



$$P(\mathsf{H},\ \mathsf{T}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

We can do this for each possibility and list the probabilities at the end of each path.



$$\text{H} \to P(\text{H, H}) = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}$$

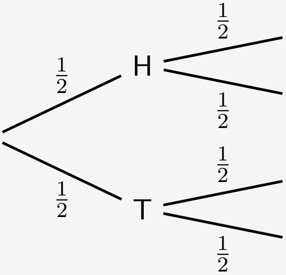
$$\text{T} \to P(\text{H, T}) = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}$$

$$\text{H} \to P(\text{T, H}) = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}$$

$$\text{T} \to P(\text{T, T}) = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}$$

What is the probability of flipping one head and one tail?

Notice that this question is different from the previous one. There is only one path that is a head followed by a tail, but there are two paths that result in one head and one tail.



H $\rightarrow P(\text{H, H}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\boxed{\text{T} \rightarrow P(\text{H, T}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}}$

$\boxed{\text{H} \rightarrow P(\text{T, H}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}}$

T $\rightarrow P(\text{T, T}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

To combine the probability of different paths, we add the probabilities together. This is a special case of the *addition rule* for probabilities.

$$P(\text{one head and one tail}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

The Product Rule and Addition Rule

Coins A and B are biased in the following manner:

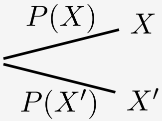
- Coin A: $P(\mathsf{H}) = \frac{2}{3}$, $P(\mathsf{T}) = \frac{1}{3}$
- Coin B: $P(\mathsf{H}) = \frac{1}{4}$, $P(\mathsf{T}) = \frac{3}{4}$

Do the following:

- Draw the probability tree for flipping coin A followed by coin B, including the probabilities at each step.
- Determine the probability of each branch.
- Determine the probability of flipping one head and one tail.
- Determine the probability of flipping the same side twice.

An event X can either **happen** or it can **not happen**. The event that is X **not happening** is called the *complement* of X and we write it as X' (read as "X prime").

We can draw a probability tree for this.



Since we know something happens 100% of the time, we have the following relationship:

$$P(X) + P(X') = 1.$$

This can be rewritten as

$$P(X') = 1 - P(X).$$

The complement rule only applies to either-or situations. The following are some examples of this:

- If a team won $\frac{3}{4}$ of their games and there were no ties, then they lost $\frac{1}{4}$ of their games.
- If a student got 90% of the multiple choice problems on a test correct, then the student got 10% of the problems wrong.

Complementary Events

Write down 6 sets of complementary events and make up possible probabilities for the event and the complement. Use fractions for 2 of them, decimals for 2 of them, and percents for 2 of them.

Homework 1 I

1. For a certain biased coin, the probability of flipping a head is $\frac{2}{7}$. What is the probability of flipping a tail?

2. A ball is drawn from an urn containing 3 white balls, 6 yellow balls, and 1 blue ball. What is the probability of drawing each color of ball?

3. The following experiment is performed: First, a standard 6-sided die is rolled. After that, a ball is drawn from an urn containing 2 red balls and 3 green balls.

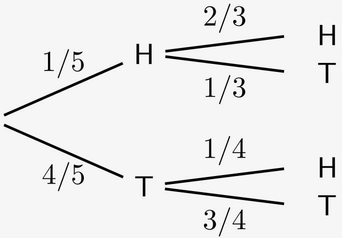
   - Draw a probability tree for this experiment (including the probabilities).
   - Determine the sample space for this experiment.
   - Determine the probability of each possible outcome.
   - Determine the probability that an even number will be rolled, followed by a red ball being drawn.

Homework 1 II

4. Urn 1 contains 4 red balls and 2 green balls. Urn 2 contains 5 red balls and 3 green balls. An experiment is performed in which a person draws a ball from urn 1 followed by a ball from urn 2.
   - Draw a probability tree for the experiment.
   - Determine the probability of each outcome.
   - Determine the probability of drawing two balls of the same color.
   - Determine the probability of drawing two balls of different colors. Use two different methods to get this value.

Homework 1 III

5. Consider the probability tree below for flipping some biased coins:



- Determine the probability of each outcome.
- Determine the probability of getting one head and one tail.
- Determine the probability of getting *at least* one tail.

Independent Events and the Multiplication Rule

- With or Without Replacement
- The Birthday Paradox
- Coincidences

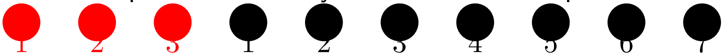
A common framework for thinking about independent and dependent events is to draw balls from an urn with or without replacement. Putting the ball back into the urn means that every new draw will have the same probabilities as the original one. But if you do not replace the ball, the probabilities change.

For example, consider an urn with one red ball and one black ball. If you draw two balls without replacement, you will always have one red ball and one black ball. But if you draw with replacement, it's possible to draw the red ball both times.

Balls and Urns - Part 1

Suppose you have an urn that contains 3 red balls and 7 black balls. Using your intuition, record your answers to the following questions.

- Are you more likely to draw two red balls if you draw with replacement or if you draw without replacement?
- Are you more likely to draw two black balls if you draw with replacement or if you draw without replacement?
- Are you more likely to draw one red ball and one black ball if you draw with replacement or if you draw without replacement?



Balls and Urns - Part 2

Suppose you have an urn that contains 3 red balls and 7 black balls.

1. Create a probability tree for drawing two balls with replacement and a tree for drawing two balls without replacement.
2. Determine the probability of drawing two red balls in each situation. Was your guess correct?
3. Determine the probability of drawing two black balls in each situation. Was your guess correct?
4. Determine the probability of drawing one red ball and one black ball in each situation. Was your guess correct?

If you pick a random person somewhere on the planet, how likely is it that this person will have the same birthday (month and day) as you do?

In order to answer this question, we will frame the question using probabilistic language. We can think about this as an experiment in which we're drawing balls from an urn, and the balls are labeled with all possible birthdays. Drawing a ball at random is similar to the randomness of the person's birthday.

For now, we will ignore February 29 as a possibility, but we'll come back to it later.

Since there are 365 balls in the urn and only one of the balls has your birthday on it, we have that

$$P(\text{drawing your birthday}) = \frac{1}{365} \approx 0.27\%$$

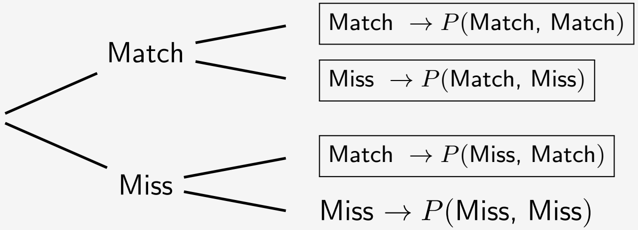
This is a very unlikely!

If you picked two people, how likely is it that at least one of them will have the same birthday as you?

- Frame this question in terms of balls and urns. What would you do if you actually wanted to conduct an exmperiment to answer this question?

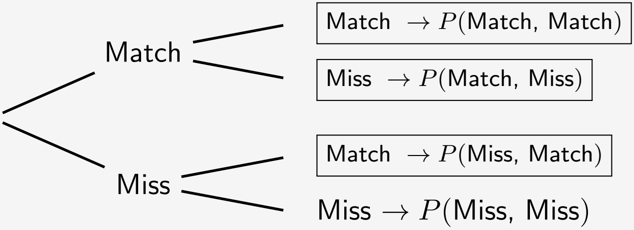
The experiment is that you would be drawing two balls with replacement, and checking whether either ball has your birthday on it.

We can put this into a probability tree and look to see which branches result in a match:



If we compute the probability directly, we would need to add up all three outcomes that have at least one match.

However, notice that there is only one path that doesn't have a match. We can use complements to make the calculation easier.



$$P(\text{At least one match}) = 1 - P(\text{Miss, Miss})$$

Since we are doing this experiment with replacement, the probability of missing doesn't change from one draw to the next. This means that we can use the multiplication rule to get

$$\begin{aligned} P(\text{Miss, Miss}) &= P(\text{Miss}) \cdot P(\text{Miss}) \\ &= P(\text{Miss})^2 \end{aligned}$$

Since we know that $P(\text{Miss}) = \frac{364}{365}$, we can very easily do this calculation:

$$\begin{aligned} P(\text{At least one match}) &= 1 - P(\text{Miss, Miss}) \\ &= 1 - P(\text{Miss})^2 \\ &= 1 - \left(\frac{364}{365}\right)^2 \\ &= 0.55\% \end{aligned}$$

This is still very unlikely!

What happens if we pick three people?

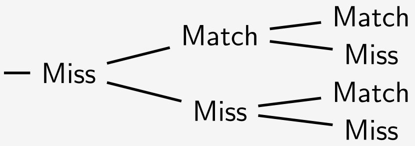
Sharing Your Birthday

1. Create a tree that shows all of the possibilities for picking three random people and determine which branches lead to at least one person sharing your birthday.
2. Use the property of complements and the multiplication rule to compute the probability that at least one person out of three will share your birthday.
3. Look for a pattern that you can use to compute the probability that at least one preson out of four will share your birthday.
4. How many people would you need to pick to have a 50% chance of at least one of them sharing your birthday?

We will now take the last question and see what happens if we change the language slightly: How many people would you need to pick to have a 50% chance that someone will have a shared birthday?

As before, we will frame this problem in terms of balls and urns. But this time, "miss" and "match" will mean missing or matching *any* of the previous draws, not some specific draw.

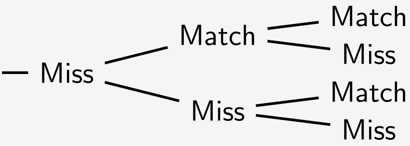
The start of the probability tree is different this time, because after one draw there's only one outcome because there's nothing to match. The following is the probability tree for three draws:



The Birthday Paradox

1. Fill in the probabilities for the probability tree below. Note that "Miss" and "Match" mean that it misses *all* previously drawn birthdays or matches *some* previously drawn birthday.
2. Determine the probability of at least two people sharing a birthday out of three people.
3. Based on the change from the previous example, take a guess as to the number of people it would take to have a 50% chance that two people would share a birthday.

- The answer to $\#3$ is surprisingly small, which is why this problem is known as the birthday paradox. You will do this calculation for homework. The key branch is the Miss-Miss-Miss branch!



A *coincidence* is when two or more unusual events happen at the same time when there's no connection between them. Because our minds tend to draw relevance between events when this happens, we sometimes read more facts into coincidences than are really there.

For example, a man might meet a woman who happens to have the same birthday. Starting from that commonality, they build a relationship and eventually marry. Some people might say that "fate" brought them together.

But even though the chances of two people sharing the same birthday is about 0.27%, the fact that there are billions of married people in the world means the chances that some couple would have the same birthday is very high.

The difficulty in interpreting events like this is that this calculation does not show that "fate" did not have a role bring this *specific* couple together. All it says is that if everything happened randomly, we would expect *some* couples to share a birthday.

The distinction between a *specific* couple and *some* couple is similar to the distinction in the birthday problems. In one case, we have a *specific* person whose birthday we're trying to match. In the other, we simply said that *some* person's birthday matches another's in the collection of people.

Failing to specify certain details in advance, like *some* instead of *specific*, can dramatically change the probability calculations and can lead people to believe certain events are less likely to occur than their intuition tells them, and because of that they might incorrectly attribute an underlying reason to the relationship between the two events rather than a chance relationship between them.

Coincidences

1. Determine the probability of flipping heads 10 times in a row in 10 flips.
2. Suppose every NSC student flipped a coin 10 times in a row. How many of them would you expect to get all heads? (There are approximately 3300 NSC students.)

Homework 2 I

1. In class, you estimated the number of people it would take to have a 50% chance of someone having a shared birthday. This was called the birthday paradox problem because the number of people is surprisingly small. We will do this calculation explicitly by taking advantage of complements. The complement of "somebody shares a birthday" is "nobody shares a birthday." This is useful for simplifying calculations.

    1. If the probability that "nobody shares a birthday" is 20%, what is the probability that "somebody shares a birthday"? If the probability that "nobody shares a birthday" is x, what is the probability that "somebody shares a birthday"?

    For the next set of problems, ignore February 29.

    1. Given two people, determine the probability that nobody shares a birthday. (Hint: The first person can have any birthday, but the second person can only have one of 364 birthdays. Why?)

Homework 2 II

2. Given three people, determine the probability that nobody shares a birthday.
3. Given four people, determine the probability that nobody shares a birthday.
4. Extend the pattern and determine the number of people there must be in order for there to be less than a 50% chance that nobody shares a birthday.
5. Explain why this corresponds to there being a greater than a 50% chance that somebody shares a birthday.

Homework 2 III

2. (Optional) For the birthday calculations we did in class, we ignored February 29.
    1. At first, it may seem that to include February 29, we would simply increase the number of days to 366. This turns out to be incorrect. Why? (Hint: How often does February 29 occur?)
    2. Instead of working in one-year cycles, leap years compel us to work in 4-year cycles. In a 4-year cycle, how many balls would be labeled "January 1"? How many would be labeled "February 29"? How many balls would we need in the urn?
    3. Determine the number of draws it would take to have a 50% chance of drawing February 29.
    4. This calculation still isn't quite right because every 100 years we skip a leap year. Would including this fact increase or decrease the number of draws it would take to have a 50% chance of drawing February 29? (Do not calculate this. Just think!)

Homework 2 IV

- Note: This still isn't completely accurate because every 400 years, we don't skip the 100 year leap year!

Homework 2 V

3. (Optional) Watch the video at `http://youtu.be/jY6F49oRiJM`. For the following problems, assume you have a fair coin.
   1. What is the probability of flipping 5 tails in a row in 5 flips?
   2. If two people flip a coin 5 times in a row, what is the probability that at least one of them will get 5 tails in a row?
   3. How many people would it take to have a 75% chance that someone would flip 5 tails in a row?
   4. How many people would it take to have a 75% chance that two people would have the same sequence (same results in same order) of 5 coin flips?

Conditional Probability

- Conditional Probability
- The False Positive Paradox

In the pre-class work, we used conditional probability as a probability calculation in which there is a restriction on the sample space. This is the formula.

$$P(X|Y) = \frac{\#\text{ of successes for } X \text{ where } Y \text{ also happens}}{\#\text{ of possibilities where } Y \text{ happens}}$$

In the video, we saw that if we know a man has two children and at least one of them is a boy, then the probability the other one is a boy is $\frac{1}{3}$ instead of $\frac{1}{2}$.

This was computed by looking at the sample space for two children and removing the combination that had no boys.

Boy-Boy	Boy-Girl
Girl-Boy	Girl-Girl

Knowing that there is at least one boy leads to two combinations where the other sibling is a girl and only one where the other sibling is a boy.

We will practice some calculations involving conditional probabilities using sample spaces.

Conditional Probabilities 1

For each of the situations below, write out the entire sample space, remove portions based on the condition, then compute the probability.

1. When rolling two 6-sided dice, determine the probability of rolling a 5 if the total is 9.
2. When rolling two 6-sided dice, determine the probability of rolling a 1 or a 6 if the total is at least 7.
3. When rolling a 4-sided die and a 6-sided die together, determine the probability of rolling a 5 given that one of the dice is a 3.

The method of counting the elements in a sample space works when the events are equiprobable. But when this is not the case, we need to be a little more sophisticated in our approach.

For these situations, we use a more general formula for computing conditional probabilities:

$$P(X|Y) = \frac{P(X \text{ when } Y \text{ also happens})}{P(Y)}$$

We will be using probability trees to organize our information. Just as before, we may need to add up multiple paths to get the full probability.

Conditional Probabilities 2

For the following problem, assume that you have a biased coin and $P(\mathsf{H}) = \frac{1}{3}$.

1. Draw the probability tree for two coin tosses and compute the probability of each outcome.
2. Determine the probability of flipping at least one head.
3. Determine the probability of flipping one head and one tail.
4. Determine the probability of flipping one tail given that at least one head was flipped. In other words, determine the probability of having $\{H, T\}$ if you know you have $\{H, ?\}$.

- Note: This is $P(X|Y)$ where $X =$ "Flip one tail$''$ and $Y =$ "Flip at least one head."

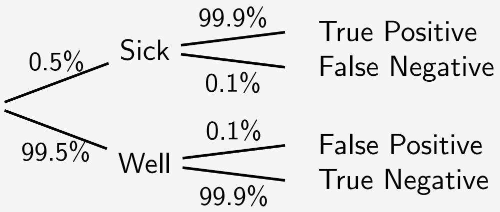
Suppose you show up to a walk-in clinic to be tested for a very rare disease (only 0.5% of the population has it) and the test is 99.9% accurate. This means that if you have the disease the test will correctly diagnose you 99.9% of the time, and if you don't have the disease the test will correctly diagnose you 99.9% of the time.

The test comes back positive. What is the probability that you have the disease?

- 99%
- 99.9%
- 99.99%
- Better than 99.99%
- Worse than 99%

This problem can be answered using conditional probability.

We begin with a probability tree. The first branch will correspond to whether the person has the illness (sick or well) and the second branch will correspond to the test result (positive or negative).



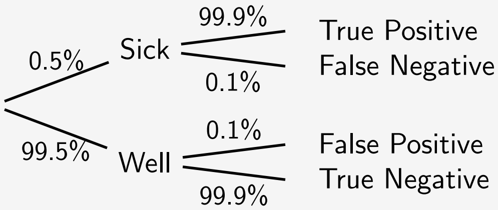
Notice that some people who are sick will come back with a negative test. This is known as a false negative. There are also some people who are well whose test will come back positive. This is called a false positive.

False Positive Paradox

Using the probability tree below, what is the probability that you have the disease if you test positive?

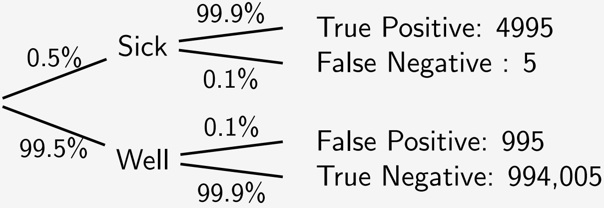
- Determine the probability of receiving a positive diagnosis.
- Determine the probability that you are sick and that you get a positive diagnosis.
- Determine the probability of being sick given that you received a positive diagnosis.

$$P(X|Y) = \frac{P(X \text{ when } Y \text{ also happens})}{P(Y)}$$



The underlying intuition of the false positive paradox is that if the disease is very rare, then the test will result in a large number of false positives. Instead of using percents, we will look at concrete numbers.

Suppose there are $1,000,000$ people in the population and all get tested. The probability tree below shows the number of people in each category.



As you can see, nearly $1,000$ people will receive false positives. These false positives make it so that a random person testing positive is less likely to have the disease than the accuracy rate of the test would lead us to believe.

This calculation is part of the reason why doctors don't regularly test everyone for rare diseases all the time. If we blindly test people, we create large numbers of false positives that can lower the overall effectiveness of the test and create unnecessary stress for patients and their families.

Instead of testing blindly, doctors look for other risk factors before administering a test for a rare disease. In a sense, they run multiple tests and only conclude that a person has the disease if all or most of the tests come back positive.

Homework 3 I

1. Suppose a family has three children and at least one of them is a boy. Hint: Three children give rise to eight possibilities, $\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$.
   - What is the probability that at least one of the other children is a girl?
   - What is the probability that at least one of the other children is a boy?
   - What is the probability that both other children are girls?
   - What is the probability that both other children are boys?
2. Suppose there are two urns. One urn contains two black balls and the other contains one white ball and one black ball, but you don't know which is which. You walk over to an urn and pull out one ball. It's black. What is the probability that the other urn contains two black balls?

Homework 3 II

3. To mitigate the false positive paradox for a rare disease, a doctor implements a cheap pre-screening test that is 90% accurate (for both positive and negative results). The doctor only runs the second test if the first test comes back positive. The second test is 99.9% accurate (for both positive and negative results). Only 0.5% of the population has the disease.

   - What is the probability that someone who tests positive for the disease with the pre-screening test actually has the disease?
   - What is the probability that someone has the disease if the pre-screening comes back positive but the follow-up test is negative?
   - What is the probability that someone has the disease if both tests come back positive?

Hint: Construct a probability tree with three levels: one level for sick/well, the next level for pre-screening test positive/negative, and the final level for second test positive/negative.

Expected Value

- Roulette
- Multi-Step Games
- House Edge

In the pre-class work, we used the expected value formula to determine how much we would expect to win or lose on average when playing a probabilistic game:

$$E[X] = \sum P(X_i) \cdot V(X_i)$$

where

- X_i is a notation that represents each possible outcome of the game X.
- $P(X_i)$ is the probability of the event X_i.
- The $V(X_i)$ is the *value* of the event X_i.

We used this formula with probability trees to compute the expected value of some simple games.

In casino games, the player needs to pay money up front in order to play. This can sometimes cause people to misinterpret events.

For example, in the game of roulette the players have an opportunity to bet on red or black. There are 18 red numbers, 18 black numbers, and (usually) 2 green numbers. If you bet on red or black, you put a chip in the appropriate area. If you win, you get two chips pushed back to you. However, you have only won one chip because you paid one chip up front and it's just getting pushed back to you.

When doing expected value calculations, you need to pay attention to this because you care about your *net* winnings (winnings after accounting for anything you paid up front to play). The money that is pushed towards you is called your *gross* winnings. Both concepts are useful, so you will need to pay attention to the distinctions.

The words net and gross are also used in finance with the same meaning as usual. Your net pay is the actual amount of money you have after taxes, deductions, and other expenses. Your gross pay is the total amount of money you earned.

Payout odds in a casino are usually stated in the form $A{:}B$ (read "A to B"), which means that you win A dollars net when you wager B dollars.

In roulette, the payout odds for a bet on red or black is $1{:}1$. This means that if you wager $\$1$ on red and a red number is spun, then you win $\$1$ (net). These values scale, so that if you wagered $\$50$ on red and red is spun, then you win $\$50$ (net). If you wager on a specific number, the payout is $35{:}1$, which means that if you wager $\$1$, you would win $\$35$ (net).

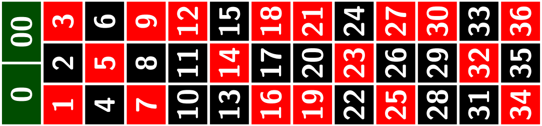
The payouts are not always relative to a $\$1$ bet. In blackjack, casinos may pay 3:2 or 6:5, depending on the specific game you're playing.

- Which is a better payout, 3:2 or 6:5?

Roulette Calculations

A roulette wheel consists of 18 red numbers, 18 black numbers, and 2 green numbers. Each number is equiprobable.

1. Betting on red pays 1:1. Determine the expected value of betting \$1 on red.
2. Betting on green pays 17:1. Determine the expected value of betting \$5 on green.
3. Betting on any specific number pays 35:1. Determine the expected value of betting \$25 on a specific number.



Some games involve multiple rounds of events. For example, a modern slot machine will sometimes have a regular payout, but it will sometimes also go into a bonus game in which you have an additional round of payouts.

We can keep track of these multiple rounds by using a probability tree. The probabilities are computed in exactly the same manner as before, and we simply use the payouts at the end of each branch to determine the value of each outcome.

Multi-Step Games

A gambling game is constructed in the following manner. One ball is chosen at random from an urn that contains one red ball, one white ball, and one blue ball. If it's the red ball, the player loses \$20. If it's the white ball, the player flips a coin; heads wins \$10 and tails loses \$10. If it's the blue ball, the player rolls a 4-sided die and wins \$5 times the number of the die (so rolling a 2 wins \$10).

1. Draw the probability tree for this game and compute the probabilities for each outcome.
2. Determine the value of each outcome.
3. Determine the expected value of this game.

Casino games virtually always have a negative expected value. An important quantity to know is the *house edge* for a game. The house edge is stated as a percent and represents the percent of the wager that the house wins on average.

Because the house edge is relative to the initial wager, it is useful to think of the game in multiple steps, where the first step is the wager and each step afterwards is treated as a gross value instead of a net value. For example, when betting on red, it's best to think of the first step as losing \$1, and then winning \$2 if it comes up red and winning \$0 otherwise. The probability tree below displays this.

-\$1	Wager on Red	+\$2	Red: Net \$1
		+\$0	Black/Green: Net -\$1

This setup is also useful for games that are more complex because it allows you to break it down into smaller steps.

We computed earlier that the expected value of a \$1 bet on red is about -\$0.053. Since the house wins all the money you lose, the house wins \$0.053. To get the percent, we divide the house winnings by the original wager.

$$\text{House Edge} \approx \frac{0.053}{1} = 0.053 = 5.3\%$$

We also computed that the expected value of a \$5 bet on green is about -\$0.263. This would give

$$\text{House Edge} \approx \frac{0.263}{5} \approx 0.053 = 5.3\%$$

It turns out that betting on red, black, or green all have the exact same house edge. If you bet on red, you win more often, but you win less when you win. If you bet on green, you don't win very often, but when you win you win a lot more. But in the end, it averages out to be the same.

Not all games are like this. For example, the house edge on craps varies dramatically depending on the type of bet you place.

The size of the house edge is used to determine how good or bad a game is for the player. Players prefer small house edges, but casinos prefer large ones.

- The house edge for slot machines usually run between 2% and 15%, depending on the game.
- The house edge for Keno is between 25% and 29%, depending on the wager.
- The house edge for blackjack is about 0.5%, as long as you make good decisions.

Computing the house edge for blackjack is extremely complicated because it depends on the decisions that the player makes, the type of blackjack payout (3:2 or 6:5), the number of decks, and several other factors. For very complex games like this, computers are used to simulate player decisions and estimate the house edge.

The House Edge

Craps is played with two six-sided dice. It is possible to make wagers on the outcome of a single roll (called proposition bets).

- Betting on any 7 pays 4:1. Determine the house edge on this wager.
- Betting on craps (any 2, 3, or 12) pays 7:1. Determine the house edge on this wager.
- What would the house edge be if the payout odds for craps were 7.5:1 instead of 7:1? (As they are in some places.)

Homework 4 I

1. Determine the amount of money won given the initial wager and the payout odds.
    1. $\$5$ wager paying out $2:1$
    2. $\$10$ wager paying out $4:1$
    3. $\$20$ wager paying out $5:2$
    4. $\$50$ wager paying out $6:5$
    5. $\$90$ wager paying out $8:3$

Homework 4 II

2. A game is played with a 6-sided die. If the numbers 1 or 2 are rolled, the player loses $\$10$. If the numbers 3 or 4 are rolled, the player wins $\$5$. If the numbers 5 or 6 are rolled, the player wins $\$7$. What is the expected value of this game?

3. A coin is flipped three times. If the player gets at least two heads in a row, he wins $\$25$. If the player gets three tails in a row, he wins $\$50$. Otherwise, he loses $\$20$. Determine the expected value of this game.

Homework 4 III

4. A player pays $\$5$ to play a game. In this game, he has a 50% chance of losing all of his money, a 20% chance of a *gross* winning of $\$1$, a 20% chance of a *gross* winning of $\$5$, and a 10% chance of a *gross* winning of $\$25$. Determine the expected value of this game.

Homework 4 IV

5. A roulette wheel has the numbers 1–36, plus a 0 and a 00 for a total of 38 numbers. There are many bets a player may make.
   1. Betting on 1–12 wins whenever the number is 1–12. This bet pays $2:1$.
      1. What is the expected value of a $50 wager?
      2. What is the house edge?
   2. A line bet means betting that any one of three specific numbers will come up. This pays $11:1$.
      1. What is the expected value of a $\$75$ wager?
      2. What is the house edge?

Homework 4 V

6. A simplified slot machine is set with the following rules:
   - 90% of the time, the player loses
   - 9% of the time, the machine pays $1 : 1$
   - 1% of the time, the player enters a bonus round. In the bonus round, 1% of the time it pays $1000 : 1$, and the rest of the time it pays $5 : 1$.

   Determine the house edge.

Introduction to Statistics

- Samples
- Coin Flip Frequencies
- Distribution Shapes

In the pre-class materials, we talked about three special values, known as measures of central tendency.

- Mean: Add the values and divide by the number of data points.
- Median: This is the middle value when the numbers are arranged in order
- Mode: This is the most common value.

Each of these values describes some feature about the distribution of the data.

We used a model of drawing numbered balls from an urn without knowing how many balls are in the urn and what numbers are on the balls as a way to introduce the idea of statistical sampling.

You probably wondered why we couldn't just dump out all of the balls from the urn and count them up to see what was inside. The reason is that in applications of statistics, it's rarely possible to get complete information on the population. For example, to find the average height of a person, we would need to measure the height of every person on the planet, and then take the average. This is a logistical impossibility.

(The urn would represent the world, the balls would represent people, and the numbers on the balls would represent their heights.)

Instead, we need to take a *sample* of the population and use the information we gathered there to try to draw conclusions about the larger population from that. We always need to be careful when trying to *infer* information about a population from a sample because there are many types of errors that can be made.

This is the chart that was used in the pre-class materials:

Number	Frequency
7	2
8	5
9	2
10	1

We clearly see that 8 is the most common number in the sample. But does this mean that 8 is the most common number in the urn? Not necessarily.

- Name some reasons why 8 might not be the most common number in the urn.
- If you were to take a new sample from the urn, what actions could you take to account for the issues just discussed?

In more advanced courses in statistics, you can learn about different techniques that are used to reduce the effects of these types of biases. The goal is to have a *representative* sample, meaning that the distribution in the sample reflects the distribution of the larger population.

Statistical Sampling 1

1. Gather data on the age of all the people in the classroom.
2. Determine the mean, median, and mode of the data.
3. Explain why this is probably not a representative sample of the ages of people living in the state of Nevada.

Instead of making lists, we can also use a frequency table to keep track of the data. This is especially useful when the same numbers appear repeatedly. This tends to be much more compact and easier to work with. The following is the frequency table from the pre-class materials.

Number	Frequency
7	2
8	5
9	2
10	1

This table says that the number 7 was drawn 2 times, the number 8 was drawn 5 times, and so on.

Flipping a coin is another example of taking a statistical sample. The entire population would be the collection of all results from all coin flips, perhaps even all future coin flips.

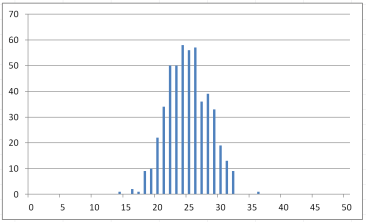
Statistical Sampling 2

Everyone will contribute to the data by flipping a coin 10 times. (Catch the coin in the air!) You will report the number of heads that are flipped.

1. Create a frequency table for the coin-flipping data collected by the class.
2. Determine the mean, median, and mode of the data.

As the amount of data gets larger, frequency tables can become hard to read. For such data sets we often use a graphical representation to help us understand the data.

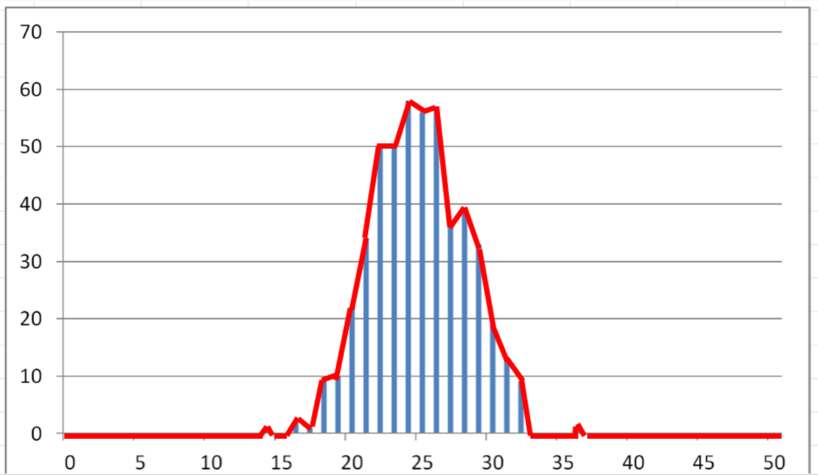
The following chart was constructed using 500 sets of 50 simulated coin tosses. The horizontal axis shows the number of heads flipped, out of 50, and the height of each bar is the frequency.



We can clearly see that the data peaks in the middle and tapers off on both sides. Shapes like this are referred to as normal distributions.

Distributions

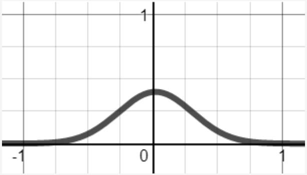
Often, we represent distributions by drawing just the outline of the bars, rather than the bars themselves. Thus, our simulation of 500 sets of 50 coin flips could be represented like this:

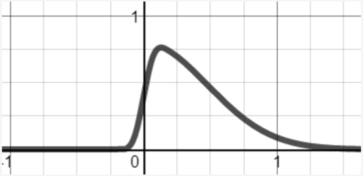


The normal distribution (also called the bell-shaped curve) is one of the most important statistical distributions because it describes so many natural phenomena.

The normal curve is characterized by having a peak in the middle with the data tapering off symmetrically on both sides.

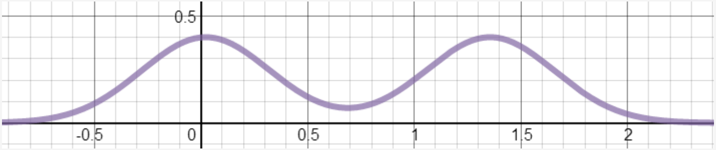
If the tapering is not symmetric, then we have a skewed distribution.

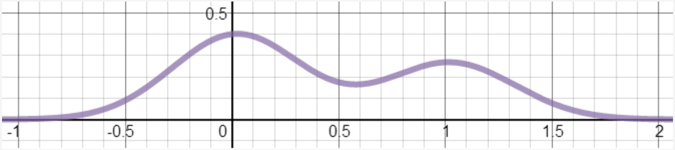




The number of children in families would form a skewed distribution because many families have small numbers of children while a few families have large numbers of children.

So far, the distributions have been centered around a single value. But sometimes the data is clustered around multiple values. Both of the following images show bimodal distributions (two peaks). An example of a bimodal data set would be the heights of people in an elementary school building. The adults and children would be clustered separately.





Sometimes the peaks are completely separate, sometimes they blend into each other, and sometimes they're not quite the same height. There are many shapes that distributions can take.

The graphs were created using an online graphing program called Desmos. It's free software at `desmos.com` that has a number of useful features that we will be using a few times throughout the course.

Homework 5 I

1. For the following data set

$$\{3, 4, 4, 5, 5, 5, 5, 6, 6, 6, 7, 8, 8, 9, 10\}$$

- Create a frequency table for the data.
- Create a graph of the frequency of the data.
- Is the data normally distributed? Skewed? Bimodal?
- Determine the mean, median, and mode of the data. (Round to the nearest 0.1.)

Homework 5 II

2. Create a set of data with the following conditions. Use between 5 and 10 data points.
    - The mean is 5 and the median is 7.
    - The mode is 9 and the mean is 8.
    - The median is 4 and the mode is 7.
    - A bimodal distribution where the mean and median are both 5.

Homework 5 III

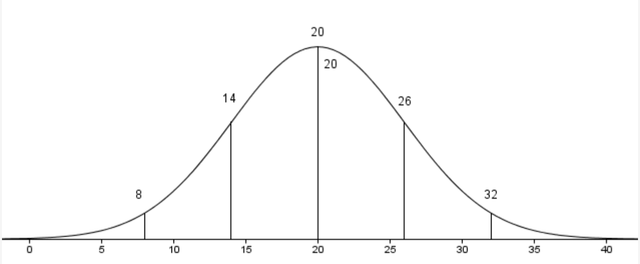
3. Using a web browser, go to `https://www.desmos.com/calculator/6g19grieuu`. You will see a graph with five sliders. Describe how the following sliders change the shape of the curve.

   - The a slider
   - The b slider
   - The c slider
   - The d slider
   - The l slider

Normal Curves

- Computing Standard Deviations
- Percentiles

In the pre-class materials, we looked at normal curves obtained by picking μ and σ to be different values. The picture below is an example of a normal curve created by the GeoGebra program.



- Determine the values of μ and σ from the picture.

We used WolframAlpha to compute the standard deviation of small data sets, but didn't talk about the actual calculation. The calculation is not difficult, but it can get long if there are lots of data points.

When computing the standard deviation of a data set derived from taking a sample of a population, we use the following formula:

$$\sigma = \sqrt{\frac{1}{N-1}\sum(x_i - \mu)^2}$$

This formula looks more complicated than it really is, but it will require a bit of explanation.

$$\sigma = \sqrt{\frac{1}{N-1}\sum(x_i - \mu)^2}$$

- N represents the total number of data points in the data set.
- x_i represent the actual data points.
- $\sum$ means that you're going to add together several terms. In this case, you will add together the $(x_i - \mu)^2$ terms.

Mathematical formulas are very often written like this, and it takes time to learn to read them just like learning to read any other language.

To begin to understand how to use this formula, we will work through an example.

Example: Compute the standard deviation of the data set $\{3, 6, 10, 8, 13\}$.

We can see right away that $N = 5$ for this data set because there are 5 data points.

Next, we compute μ:

$$\mu = \frac{3 + 6 + 10 + 8 + 13}{5} = 8$$

We need to compute the differences $x_i - \mu$ and square the results. In order to facilitate the organization of this, we will put the information into a simple table.

i	x_i	μ	$x_i - \mu$	$(x_i - \mu)^2$
1	3	8	-5	25
2	6	8	-2	4
3	10	8	2	4
4	8	8	0	0
5	13	8	5	25

To get $\sum(x_i - \mu)^2$, we simply add the values in the last column to get 58. From here, we can simply plug it into the formula:

$$\begin{aligned}
\sigma &= \sqrt{\frac{1}{N-1}\sum(x_i - \mu)^2} \\
&= \sqrt{\frac{1}{4} \cdot 58} \\
&= \sqrt{14.5} \\
&\approx 3.81
\end{aligned}$$

Computing Standard Deviations

Consider the following sample data set:

$$\{22, 18, 19, 23, 23, 15\}$$

1. Determine N and μ for this data set.
2. Create a chart with the columns i, x_i, μ, $x_i - \mu$, and $(x_i - \mu)^2$.
3. Compute σ.
4. Construct a frequency table for the data.
5. Plot the distribution. Is the distribution normal or skewed?

We often use the standard deviation when measuring data even if the distribution is slightly skewed. As the skewing becomes more and more prevalent, we need to choose more sophisticated tools to describe the shape of the curve.

At some point, you may see a formula for standard deviation that uses N instead of $N - 1$ in the denominator. The reasons are technical, but if you are finding the standard deviation of a sample (as we are) you use $N - 1$. If you are finding the standard deviation of the entire population, then you use N.

Percentiles are a way of describing the location of a data point in a distribution. You often hear this term used when talking about data points on the higher end of the distribution.

To say that a data point is "in the 90^{th} percentile" means that data point *is larger than* 90% of the data. In general, for a data point to be "in the n^{th} percentile" means that the data point is larger than $n\%$ of the data.

We will find percentiles of some data points in a small list of data in order to build some intuition.

Percentile Calculations

Example: Consider the following list of data, listed from smallest to largest for convenience:

$$\{2, 6, 8, 10, 14, 16, 19, 19, 20, 100\}$$

1. How many data points is the number 10 greater than?
2. 10 is in what percentile?
3. What percentile is 19 in? (Hint: 19 is not greater than 19.)
4. What numbers are in the 100$^{\text{th}}$ precentile?

For the normal curve, there are tables or formulas that can be used to work out the percentiles. The tables are known as z-tables.

The reason they're called z-tables relates back to the idea of a z-score, which is a number that describes the location of a data point using the mean and the standard deviation. Each data point has a z-score associated with it and that z-score is the number of standard deviations from the mean the data point is located.

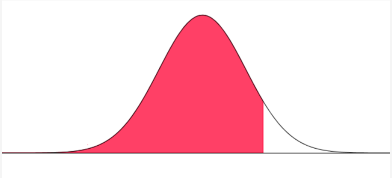
To find the z-score of the data value x, you would use the formula:

$$z = \frac{x - \mu}{\sigma}$$

The formula for locating a data point, x, with a particular z-score is:

$$x = \mu + \sigma \cdot z$$

Since percentiles represent the percent of data that is below a certain value, the following picture is commonly associated with a z-table. A z-table is a chart of values that give percentiles for normally distributed data based on their z-scores.



The shaded portion represents the data that is below a particular value (the value right at the boundary of the red and white regions), so the numbers obtained from the z-table represent the percent of data to the left of that data point.

The table below is a z-table. The percents are listed as decimals, so that $.500$ is the 50$^{\text{th}}$ percentile.

z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
-2	.023	.018	.014	.012	.008	.006	.005	.003	.003	.002
-1	.159	.136	.115	.097	.081	.067	.055	.045	.036	.029
-0	.500	.460	.420	.372	.344	.308	.274	.242	.212	.184
0	.500	.540	.580	.618	.656	.692	.726	.758	.788	.816
1	.841	.864	.885	.903	.919	.933	.945	.955	.964	.971
2	.977	.982	.986	.988	.992	.994	.995	.997	.997	.998

For example, a data point with a z-score of 1.2 is in the 88.5$^{\text{th}}$ percentile, and if a data point has a z-score of -0.8 it is in the 21.2$^{\text{nd}}$ percentile.

- Determine the percentile of data points with z-scores of -1.5, 1.9, and -0.3.

z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
-2	.023	.018	.014	.012	.008	.006	.005	.003	.003	.002
-1	.159	.136	.115	.097	.081	.067	.055	.045	.036	.029
-0	.500	.460	.420	.372	.344	.308	.274	.242	.212	.184
0	.500	.540	.580	.618	.656	.692	.726	.758	.788	.816
1	.841	.864	.885	.903	.919	.933	.945	.955	.964	.971
2	.977	.982	.986	.988	.992	.994	.995	.997	.997	.998

We can also use the table to work backwards from a percentile to the approximate z-score.

To find the z-score of the 70th percentile, we find the value closest to 0.700 in the chart and pick the corresponding z-score. The closest value is $.692$, which corresponds to the z-score 0.5, so we know that the 70$^{\text{th}}$ percentile has a z-score of about 0.5 (slightly above that).

If the value is exactly in the middle, take the average of the two z-scores.

- Determine the z-score of the 65th percentile, the 10th percentile, and the 99th percentile.

Another use of the z-table is to approximate the percent of data that falls between two points on the normal curve. This process is best explained by a picture.

In other words, to determine the percent of data between two points, we take the percent of data to the left of the larger value and subtract off the percent of data to the left of the lower value.

z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
-2	.023	.018	.014	.012	.008	.006	.005	.003	.003	.002
-1	.159	.136	.115	.097	.081	.067	.055	.045	.036	.029
-0	.500	.460	.420	.372	.344	.308	.274	.242	.212	.184
0	.500	.540	.580	.618	.656	.692	.726	.758	.788	.816
1	.841	.864	.885	.903	.919	.933	.945	.955	.964	.971
2	.977	.982	.986	.988	.992	.994	.995	.997	.997	.998

Percent of Data Between Two Values

1. Determine the percent of data that falls between $z = -1$ and $z = 1$.
2. Determine the percent of data that falls between $z = -2$ and $z = 2$.
3. Determine the percent of data that falls between $z = 0$ and $z = 1$.

The first two values obtained in the previous exercise reflect what is known as the 68-95-99.7 rule. For normally distributed data:

- 68% of the data falls within 1 standard deviation of the mean.
- 95% of the data falls within 2 standard deviations of the mean.
- 99.7% of the data falls within 3 standard deviations of the mean.

Practically speaking, this means that in most circumstances, finding a data point outside of 3 standard deviations of the mean warrants further investigation. It could be just chance, but it could also be a sign of something more sinister like fraud. We will look into this possibility in the next section.

Homework 6 I

1. Consider the following set of data:

$$\{21, 29, 33, 25, 28\}$$

   1. Determine the mean, median, and mode of the data set.
   2. Determine the standard deviation of the data set by hand.
   3. Use `WolframAlpha.com` to determine the standard deviation of the data set. (Your answers should match up.)
   4. The data point 29 is in which percentile?
   5. Approximately what is the z-score for the data point 29?
   6. According to the z-table, approximately what is the percentile of the data point 29?

Homework 6 II

- Note: These two values should be somewhat close. The gap between the values is related to the fact that we have a small data set.

Homework 6 III

2. The results of IQ tests are normally distributed with $\mu = 100$ and $\sigma = 15$.
    1. Determine the IQs corresponding to the following z-scores: $z = -2$, $z = -0.8$, $z = 0.3$, and $z = 1.7$.
    2. Determine the z-scores for the following IQs: 90, 105, 135, 72.
    3. Determine the the percentiles corresponding to the following IQs: 94, 85, 100, 142.
    4. Determine the percent of the population that has an IQ in the following ranges: between 70 and 100, between 94 and 115, below 88, above 130.

Homework 6 IV

3. In the next class, we are going to talk about detecting fraud. Your assignment is to make up two lists of 75 random coin tosses. The first list will be made up off the top of your head. Try to make it as random as possible. After making your first list obtain a second list by flipping a coin 75 times in a row and recording the results. Flip the coin higher than your head and catch it in the air.

Detecting Frauds

- 75 Coin Flips
- Swap Analysis
- Runs Analysis

For the homework, you were asked to create two lists of 75 coin flips, one fake and one real. We are going to run a series of analyses in order to try to find different ways of picking out the fake.

This is part math and part psychology. Remember that while we can draw some mathematical conclusions about the data, the math does not give us a definitive answer. Rare events happen when there are lots of samples drawn, so we need to be careful to not overstate our conclusions.

When trying to fake data, most people try to give things at least a superficial appearance of being legitimate, so it doesn't make sense for us to do a straight heads/tails count and look for one to be out of the ordinary. However, people tend not to analyze their lists for more complex distributions.

- Consider the following sets of three-coin sequences. In a set of three flips, which should be the most common?

HHH	HHT	HTH	THH
TTT	TTH	THT	HTT

In fact, these are all equally likely to occur in any particular set of three flips. If we draw the probability tree, we will see that each three-coin outcome has a $\frac{1}{8}$ chance of appearing. If we take the 75 coin flips and break them into 25 three-coin sequences, we expect to see each one appearing about 3 times.

Three-Coin Sequences

1. Analyze both lists of 75 tosses by breaking them each into 25 three-coin sequences and creating a frequency table for each possibility. Keep the two lists separate.
2. If every outcome is equally likely, what is the probability that a random list of 25 three-coin sequences will not have any HHH or TTT combinations? (Hint: Think about a hit-miss probability tree.)
3. Does one of the two distributions appear to be more even than the other? If so, which one looks more even?
4. Compare your results with others. Is there a general pattern that is emerging?

Another way to measure sequences is to count the number of swaps in the sequence. A swap in a sequence is when a streak changes from head to tail or tail to head. For example, the sequence HTTTTHHHTT contains three swaps.

Initially, it may not seem obvious how to estimate swaps. But it turns out that it's very similar to just flipping coins.

- Suppose the previous coin flip was a head. What must the next coin flip be in order to get a swap? What is the probability of this happening?
- Suppose the previous coin flip was a tail. What must the next coin flip be in order to get a swap? What is the probability of this happening?

Since we have 75 coin tosses and the first flip can't count as a swap (why not?), there are only 74 chances to get a swap. Since each one has a 50% chance of giving a swap, we expect to have 37 swaps on average.

Swap Analysis

1. Analyze both lists of 75 tosses by counting the number of swaps.
2. For this number of coin tosses, the number of swaps is normally distributed with $\mu = 37$ and $\sigma \approx 4.3$. Determine the z-score for the number of swaps in both lists.
3. Does either list fall outside of 2 standard deviations from the mean on this measure?
4. Only 5% of the data should fall outside of 2 standard deviations from the mean. Determine the percent of data in the class that falls outside of 2 standard deviations from the mean. If this is significantly above 5%, then this is an indication that fraud exists in the data.

- Note: Being a suspect is not the same as being guilty. In a classroom of 20 students, we expect to have some student's legitimate data fall into the list of suspects. This is just another warning to not draw conclusions too quickly.

Our last analysis will be an analysis of runs. For any given list of 75 flips, we can count up the total number of runs of each length. For example, HTTTHHHTT has 1 run of 1, 1 run of 2, and 2 runs of 3.

This gives us two pieces of data on the lists:

- A distribution of runs
- The size of the longest run

We will now analyze the lists based on these measures.

Runs Analysis

1. Analyze both lists of 75 tosses by counting the number of runs of each length and determining the longest run.
2. Below are the runs results of $10,000$ simulated lists of 75 coin tosses each with the average number of runs of a given length falling into each category. How closely does your distribution match the simulation?

Length	1	2	3	4	5	6	7	8	9
Runs	18.7	9.4	4.8	2.3	1.1	0.6	0.3	0.1	0.1

3. Below is the percent of lists that had different length longest runs. Determine the percentile for the length of the longest run for both of your lists.

Length	≤ 3	4	5	6	7	8
Percent	0.2	6.8	22.6	26.8	19.6	11.6
Length	9	10	11	12	≥ 13	
Percent	6.4	3.2	1.3	0.8	0.7	

Homework 7 I

1. Write a short (one paragraph) summary of the analysis of the two lists you generated. Be sure to answer the following questions:
   - Which was determined to be the fraudulent list?
   - What types of analysis did you use?
   - Did the analysis lead to the correct conclusion?
2. Write a paragraph describing how you might test whether a 6-sided die is a fair die.
3. A 6-sided die is rolled 20 consecutive times without the number 1 showing up. What is the probability of this happening? (Remember that if the events are independent, then you can use the product property.)

Credit

- Installment Credit
- Credit Cards

As we saw in the preview work, consumer credit plays a large role in the modern economy. Those who use credit wisely and prove themselves creditworthy can save thousands of dollars on future purchases that must be financed.

Consumer credit generally comes in two types:

Installment credit provides for one-time purchases and usually includes fixed repayment terms.

Revolving credit allows for the on-going issuance of (usually smaller) amounts of credit that are repaid over time according to the amount borrowed and the specific terms of the credit.

Monthly Car Payments

A car has a price of $\$12,000$. It may be purchased on installment plans of several lengths. The longer the plan, the more total interest the borrower pays. For each of the loan terms shown below, calculate the simple interest rate the borrower will be charged.

Auto Loan

Loan term	**Total Interest**	**Interest Rate**
48 *months*	$\$3,500$	
60 *months*	$\$5,000$	
72 *months*	$\$6,500$	

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Auto Loan

Loan term	**Total Interest**	**Interest Rate**
48 *months*	$\$3,500$	7.3%
60 *months*	$\$5,000$	8.3%
72 *months*	$\$6,500$	9.0%

Usually, a car loan comes with a set compound interest rate instead of a preset amount of interest to be charged. This can make computing the monthly payment a little more involved. This is because the interest is compounded on the remaining balance each month. Thus, you are charged more in interest on the first month of your loan than you are on the last month of your loan.

This also means that at the beginning of the loan, more of your payment goes toward interest and less towards the principal of the loan. An amortization table is a table showing the details of each payment over the life of the loan and can help illustrate this concept.

A small amortization table

$1,000 loan at 4% over 12 months

Month	Payment	Principal paid	Interest paid	Balance
Jan.	$85.15	$81.82	$3.33	$918.18
Feb.	$85.15	$82.09	$3.06	$836.09
Mar.	$85.15	$82.36	$2.79	$753.73
Apr.	$85.15	$82.64	$2.51	$671.09
May	$85.15	$82.91	$2.24	$588.18
June	$85.15	$83.19	$1.96	$504.99
July	$85.15	$83.47	$1.68	$421.52
Aug.	$85.15	$83.74	$1.41	$337.78
Sep.	$85.15	$84.02	$1.13	$253.76
Oct.	$85.15	$84.30	$0.85	$169.45
Nov.	$85.15	$84.59	$0.56	$84.87
Dec.	$85.15	$84.87	$0.28	$0.00
—	$1,021.80	$1,000.00	$21.80	—

In practice, the amount of principal paid with early payments can be dwarfed by the amount of interest paid. This means it is advantageous to make additional principal payments (if possible) to reduce the total amount of interest you will be charged over the life of the loan.

Amoritization Tables

Fill out the remainder of the first 5 entries in this amortization table showing a home loan of $\$165,000$ over 360 months (30 years) at 4.5%.

Month	Payment	Interest paid	Principal paid	Balance
Month 1	$\$836.03$	$\$618.75$	$\$217.28$	$\$164,782.72$
Month 2	$\$836.03$			
Month 3	$\$836.03$			
Month 4	$\$836.03$			
Month 5	$\$836.03$			

Hint: The interest paid in month 2 is the amount found when the annual rate of 4.5% is applied to $\$164,782.72$ for one month.

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Month 1	$\$836.03$	$\$618.75$	$\$217.28$	$\$164,782.72$
Month 2	$\$836.03$	$\$617.94$	$\$218.09$	$\$164,564.63$
Month 3	$\$836.03$	$\$617.12$	$\$218.91$	$\$164,345.71$
Month 4	$\$836.03$	$\$616.30$	$\$219.73$	$\$164,125.98$
Month 5	$\$836.03$	$\$615.47$	$\$220.56$	$\$163,905.42$

Hint: The interest paid in month 2 is the amount found when the annual rate of 4.5% is applied to $\$164,782.72$ for one month.

Using simple interest

The second kind of credit is revolving credit, which means the term is open-ended and additional amounts may be borrowed or repaid according to specific terms over time. The most common form of consumer revolving credit is that provided by a credit card.

Credit cards often charge an *annual* percentage rate (APR) on debt accumulated on the card, but assess those charges at the end of each *monthly* billing cycle. To do this accurately, the APR is divided by 365 to get a daily interest rate, and this daily rate is multiplied by the number of days in the billing cycle to compute the interest rate.

In fact, the formula for computing the amount of interest charged by a credit card company bears a striking resemblance to that used for calculating simple interest.

Credit card calculations

$$I = P \times r \times t$$

$$\text{FC} = \text{ADB} \times \frac{\text{APR}}{365} \times \text{BC}$$

- **F**inance **C**harge — The amount of interest charged by the credit card company.
- **A**verage **D**aily **B**alance — The average of how much debt was carried on the credit card each day.
- Daily interest rate — The APR divided by the number of days in a year.
- **B**illing **C**ycle — The number of days in the billing cycle.

A credit card example

Suppose a card's billing cycle began Aug. 8 and carried a balance of $\$57.54$. The card user purchases $\$63.44$ of gasoline on Aug. 12, $\$32.30$ for entertainment on Aug. 19, and $\$66.00$ for jeans on Sep. 1. The card user also makes a $\$100$ partial payment to the card company on Aug. 25.
Find the card's balance for each day of the BC. Then find the ADB.

Aug. 8 \$57.54	Aug. 9	Aug. 10	Aug. 11	Aug. 12	Aug. 13	Aug. 14
Aug. 15	Aug. 16	Aug. 17	Aug. 18	Aug. 19	Aug. 20	Aug. 21
Aug. 22	Aug. 23	Aug. 24	Aug. 25	Aug. 26	Aug. 27	Aug. 28
Aug. 29	Aug. 30	Aug. 31	Sep. 1	Sep. 2	Sep. 3	Sep. 4
Sep. 5	Sep. 6	Sep. 7				

A credit card example

Suppose a card's billing cycle began Aug. 8 and carried a balance of $57.54. The card user purchases $63.44 of gasoline on Aug. 12, $32.30 for entertainment on Aug. 19, and $66.00 for jeans on Sep. 1. The card user also makes a $100 partial payment to the card company on Aug. 25.

Find the card's balance for each day of the BC. Then find the ADB.

Aug. 8 $57.54	Aug. 9 $57.54	Aug. 10 $57.54	Aug. 11 $57.54	Aug. 12 $120.98	Aug. 13 $120.98	Aug. 14 $120.98
Aug. 15 $120.98	Aug. 16 $120.98	Aug. 17 $120.98	Aug. 18 $120.98	Aug. 19 $153.28	Aug. 20 $153.28	Aug. 21 $153.28
Aug. 22 $153.28	Aug. 23 $153.28	Aug. 24 $153.28	Aug. 25 $53.28	Aug. 26 $53.28	Aug. 27 $53.28	Aug. 28 $53.28
Aug. 29 $53.28	Aug. 30 $53.28	Aug. 31 $53.28	Sep. 1 $119.28	Sep. 2 $119.28	Sep. 3 $119.28	Sep. 4 $119.28
Sep. 5 $119.28	Sep. 6 $119.28	Sep. 7 $119.28				

Computing a Finance Charge

Using the formula, compute the finance charge for the ADB and BC you just found with an APR of 27.99%.

$$\text{FC} = \text{ADB} \times \frac{\text{APR}}{365} \times \text{BC}$$

What will the balance be on the credit card on Sep. 8?

Computing a Finance Charge

Using the formula, compute the finance charge for the ADB and BC you just found with an APR of 27.99%. ADB $= \$103.37$, FC $= \$2.46$

$$\text{FC} = \text{ADB} \times \frac{\text{APR}}{365} \times \text{BC}$$

What will the balance be on the credit card on Sep. 8? $\$121.74$

Amortization tables can help you know how long it will take to pay off a debt and how much interest you will pay in doing so. Paying a larger amount each month can drastically reduce the amount of interest over the term of the loan.

For example, a $\$3,000$ credit card balance with a 17.0% APR will take 40 months to pay off and cost $\$934$ in interest with a $\$100$ monthly payment. However it will take only 24 months to pay off and cost $\$553$ in interest with a $\$150$ monthly payment.

Consumer Math Portfolio Assignment 3 I

1. Explore the credit card calculators at `http://www.creditcards.com/calculators/`. For a specific loan amount and interest rate of your choosing, use the calculators to find out how much interest you could save by paying more than the minimum payment. (You will probably need to use more than one of the calculators for this problem.) Write a report of your findings.
2. Find three cars for sale in the surrounding area (you should search dealership ads, used car websites, etc.) that you would like to own. For each of these, use the car loan calculator at `http://www.edmunds.com/calculators/car-loan.html` to compute financing. Write an in-depth report listing the features of each vehicle and financing details as if you were presenting options for purchase to your significant other. Make a recommendation based on your findings and provide logical reasoning to support your conclusion.

Insurance

- Randomness
- Normal Curve to the Rescue

As we saw in the preview work, calculating insurance premiums involves assessments of risk and cost. These assessments are based on ideas of probability and so have some inherent randomness that cannot be completely eliminated.

In the coming activities we will see some of the issues facing those who make calculations involving randomness. We will draw on our knowledge of probability and the normal distribution to answer simplified questions about assessing risk, but the focus will be on understanding the what rather than the how.

Recall the problem from the preview work involving bicycle thefts. The insurer, Phoebe, estimated the proportion of bicycles that would be stolen in her neighborhood and set her premium accordingly. Presumably, she used past data about local bicycle thefts in making her estimates. However, it is possible that Phoebe's estimate will not match what actually occurs.

There are several possible reasons the proportion Phoebe calculated could be an inaccurate predictor of future bicycle thefts.

1. Perhaps Phoebe made a mistake in her calculations.
2. Perhaps the proportion of bicycles stolen in the future is different from the proportion of bicycles stolen in the past. This could happen for many reasons, such as:
    - Increased local police patrols driving thefts down or a new gang of bicycle thieves stealing more bicycles.
    - A temporary random fluctuation in the proportion of bicycles stolen. Perhaps unusually hot weather in the coming summer will discourage potential thieves, or a faulty batch of bike locks sold locally will result in several additional thefts.
    - A stabilization of the proportion of bicycles stolen at a value not suggested by prior data. Perhaps limited historical data on bicycle thefts was available to analyze or a series of random events dramatically skewed recent thefts from the underlying "stable" values.

If Phoebe regularly overestimates the the proportion of bicycles that will be stolen, her premium will be too high (relative to the risk) and she could lose business to a competitor or people simply dropping their coverage. On the other hand, if she consistently underestimates the proportion, her premium will be too low (relative to the risk) and she could be forced to pay out more in claims than she collects in premiums.

It seems that Phoebe not only needs an estimate of the proportion of bicycles that will be stolen, but also a sense of how likely her estimate is of being substantially incorrect.

To understand how Phoebe might get a sense of how accurate or inaccurate her estimates are, let's turn to a testable example of an event with built-in randomness.

Two friends have a bet involving coin flipping. Null believes that if you flip a coin successively and record the results, on average, it will take about the same number of flips for the sequences HTT and HTH to occur. Alt believes one of them will fairly consistently take more flips to produce.

Null vs. Alt

Discuss whether you agree with Null or with Alt. Provide justification for your answer.

To settle the bet, the two friends decide to flip a coin until each sequence appears 100 times and record how many flips it took each time. They get the following:

Number of coin flips until HTT appears:

7	10	5	9	20	4	5	10	6	4	5	5	15	9	10	3	12	17	5	7
4	13	4	6	6	9	19	7	4	11	15	3	4	6	8	6	4	7	6	7
18	6	6	11	5	6	8	4	6	8	5	6	10	12	8	14	8	9	5	8
4	3	7	5	3	7	11	3	3	5	6	12	4	11	4	9	6	14	6	13
9	19	9	14	13	13	6	6	4	8	4	5	19	6	9	3	10	5	7	3

Number of coin flips until HTH appears:

10	21	8	16	4	5	7	20	3	8	3	3	4	25	7	4	8	5	21	8
10	11	4	10	9	8	6	12	7	6	3	10	5	38	9	3	19	3	3	7
3	4	6	11	4	10	8	6	7	14	19	9	6	7	7	8	8	3	4	32
3	11	7	8	9	12	14	4	3	16	24	10	6	14	5	5	7	5	5	15
12	16	8	19	4	3	6	22	4	42	9	13	6	12	12	10	8	14	7	19

It's difficult to see just what these numbers might mean, but it can be made clearer by finding the average of each set. Null and Alt calculate the average of each list and find that for HTT to appear it took an average of 7.88 flips but for HTH it took an average of 9.78 flips.

HTT vs. HTH

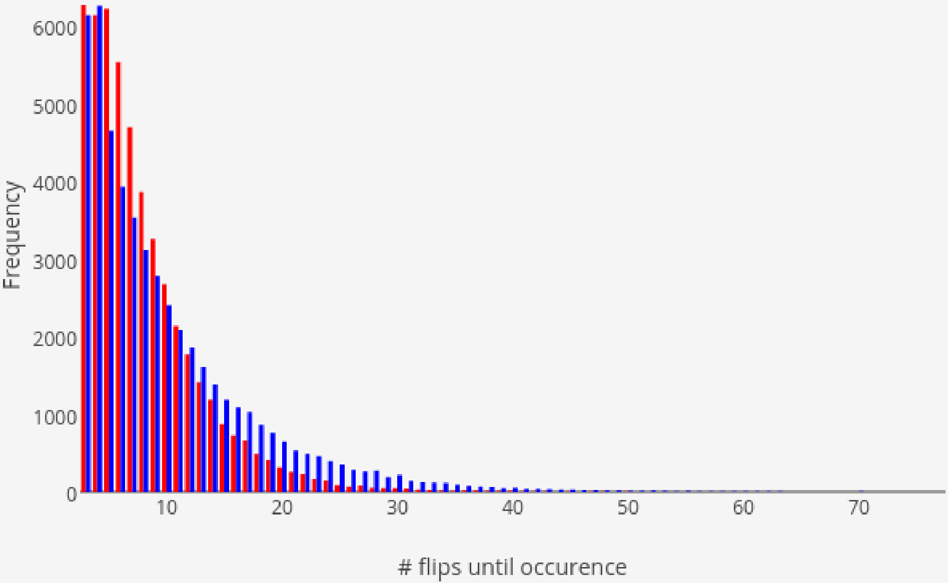
In pairs, gather some data similar to Null and Alt's. Flip a coin until HTT appears and record the number of flips. Repeat 5 times and then do the same for HTH.

Discuss the compiled class data. Does this influence your answer about whether Null or Alt is correct?

So far, it looks like it might take fewer flips, on average for the sequence HTT to appear than for the sequence HTH to appear. After 100 trials for each, Null and Alt found that it took an average of 7.88 flips for HTT and 9.78 for HTH and clearly $7.88 < 9.78$.

But wait! Just because Null and Alt's data had a lower number of flips needed for HTT than HTH doesn't mean that would always happen, does it?

To test their idea further, Null and Art find a computer program that simulates the flipping of coins. The program counts the number of flips it takes for a specific pattern to appear and records the answer.Using the program, they simulate $50,000$ sets of flips for both patterns. A graph of their results is shown on the next slide.


6000
5000
4000
3000
2000
1000
0
Frequency
10
20
30
40
50
60
70
flips until occurence

HTT vs. HTH

Discuss the results of the simulation. Do these results influence your answer about whether Null or Alt is correct?

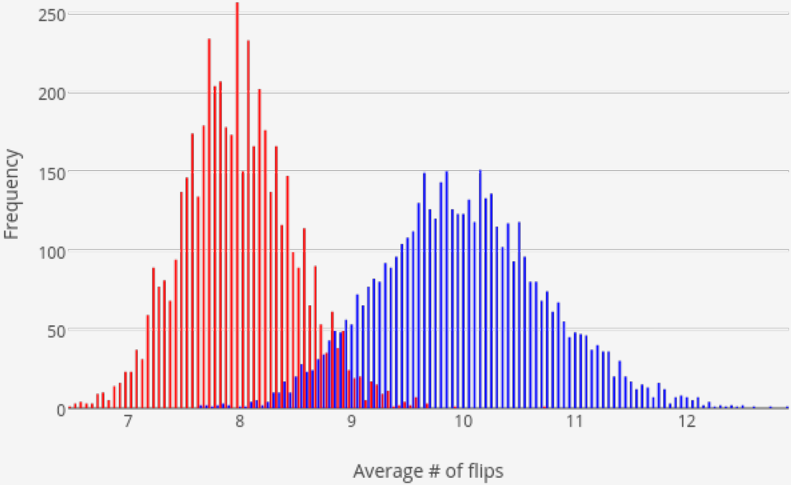
	Average
HTT	8.02356
HTH	10.04084

Null believes that, on average, both patterns should take about the same number of flips to occur. Alt disagrees.

So far, the average number of flips for HTT occur has definitely been lower than the average number of flips for HTH to occur. But could this be due to chance? What's the actual (as opposed to our measured) average?

This is where things get very interesting. Null and Alt decide to change their program a little bit and simulate coin flips in order to find a large set of averages for how long it takes each pattern to occur. They collect 5000 runs of 100 trials each for both patterns and examine how the averages of HTT and HTH stack up when they are simulated and computed so many times.

What they see really surprises them.



The averages for each pattern are grouped fairly nicely into two "bell-shaped" groupings, with HTT pretty clearly grouped around 8 and HTH nicely balanced around 10.

After thinking about things for a while, Alt has an idea. If you were to watch the coin flips as they happened, waiting for HTH to appear, you might get pretty excited if HT happened. You'd just need an H to finish the sequence! But, if you got a T instead, you have to start all the way over.

HTH	Success!
HTT	Drat. Must start over and get an H, then a T, then an H.

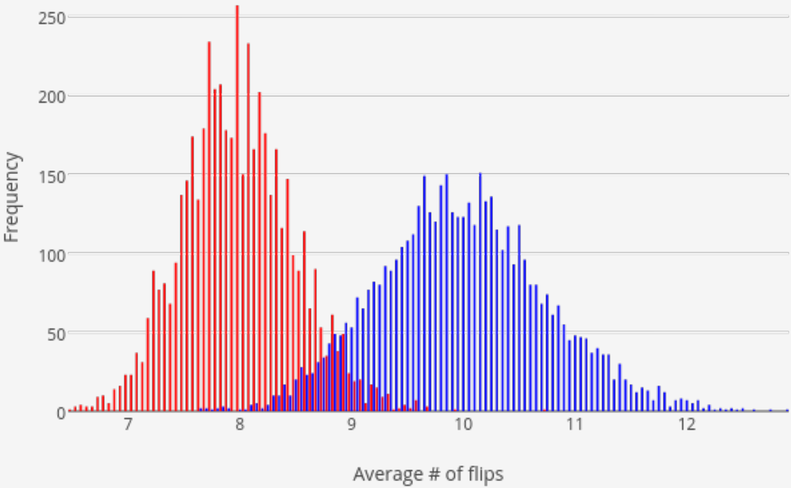
However, if you were waiting for HTT to appear and you saw HT, even if you don't get a T the next time and instead get an H, you don't have to start over because you've already got an H and that's a third of the way there.

HTT	Success!
HTH	Hey, already got an H. Just need a T and another T.

It seems there is a reason HTT might show up sooner than HTH. Even if HTT fails to show up after an initial start of HT, the failure is part way to a future success. With HTH, a failure after HT means you have to start completely over. So Alt is right, but it took running a lot of trials to make a convincing case.

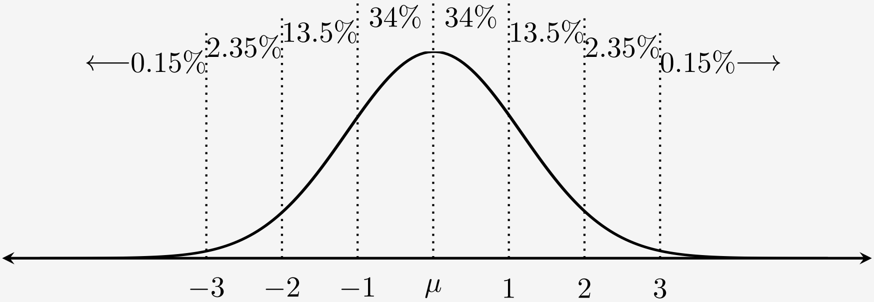
Furthermore, the graph of averages showed interesting groupings that hint at something deeper going on.

So what does this all have to do with Phoebe's estimates of the proportion of bicycles that will be stolen? The "bell-shaped" nature of the averages data allows us to predict just how likely it is that, say, a set of flips to 100 occurrences will have a lower average for HTH than for HTT. Look again at the plot and notice that this only occurs at the places the red and blue groupings overlap, which is not a lot when compared to all the trials represented.



250
200
150
100
50
0
Frequency
7
8
9
10
11
12
Average # of flips

While we can't simulate $100,000$ months of bicycle thefts directly, it turns out that many real-world phenomena follow very predictable distributions, including the normal distribution!



Recall that Phoebe estimated that $\frac{1}{100}$ of locked bicycles would be stolen in a given month. She can run simulations asking questions like:

- If the actual long-term proportion of bicycles stolen is $\frac{1}{50}$, how likely would I be to get a result of $\frac{1}{100}$ using one month's data?
- If the actual long-term proportion of bicycles stolen is $\frac{1}{200}$, how likely would I be to get a result of $\frac{1}{100}$ using one month's data?

If it's reasonable to assume that the proportion of bicycles stolen is normally distributed (and there are deep mathematical reasons for why might be), then Phoebe can use normal distributions centered on a conjectured proportion, μ, to estimate how likely it is that her historical data of $\frac{1}{100}$ is significantly wrong, and then she can adjust her premium accordingly.

In actual insurance practice, the calculations can be much more complicated to account for non-normally distributed phenomena and additional variables, but enormous amounts of data are collected to make the predictions as accurate as possible. We will focus on a simplified example.

Riches or Bust!

Suppose the actual proportion of bicycles stolen per month in Phoebe's neighborhood is $\frac{1}{40} = 0.025$. If the standard deviation of the monthly proportions of stolen bicycles is $\frac{1}{200} = 0.005$, how likely is Phoebe to get a month where no more than $\frac{1}{100} = 0.01$ locked bicycles is stolen? Assume that the proportion of bicycles stolen per month is normally distributed.

Hint: Remember the $68 - 95 - 99.7$ rule.

Coming Out Ahead

Phoebe consults a police officer, Christy, who believes the actual long-term proportion of locked bicycles stolen is not the $\frac{1}{100} = 0.01$ Phoebe got from examining recent data, but instead the much lower proportion of $\frac{1}{200} = 0.005$. Phoebe cuts her premium for locked bicycles based on this information, and insures $1,000$ locked bicycles the following month. If Christy is right, how likely is it that there will be 15 or more thefts of locked bicycles the next month, which would force Phoebe to pay out more in claims than she received in premiums? Assume the standard deviation of the monthly proportion of locked bicycle thefts is $\frac{1}{200} = 0.005$.

Homework 12 I

1. Make a list of 10 pieces of information that a car insurer collects from each applicant and conjecture how that piece of information might inform their calculations of premiums. For example, insurers collect zip codes because the location of an insured driver has a big impact on the likelihood and possible expense of an accident. Other factors being equal, a rural driver will probably have a lower premium than an urban driver.
2. Look up the names of at least four probability distributions besides the normal distribution. For each, sketch, by hand, a graph of the distribution and give one example of a phenomenon that follows that distribution.

Taxes

- Fermi Estimation
- Sin Tax
- Income Tax
- Sales Tax

The dollar amounts when discussing taxes are sometimes staggering. It can also be difficult to get all the needed information in order to come to a reasoned conclusion. Because of these issues, it will be helpful to use a process known as Fermi estimation in our calculations.

Fermi estimation, named after the physicist Enrico Fermi, is the process of using a sequence of estimates and assumptions of quantities to calculate an unknown.

Wild Guessing

How many smog check stations are in Clark County? Why did you guess that number?

Fermi Estimation

Let's assume there are about 2 million vehicles in Clark County. If a single smog station can perform

$$\frac{1 \text{ smog } \checkmark}{20 \text{ min.}} \times \frac{60 \text{ min.}}{1 \text{ hour}} \times \frac{8 \text{ hours}}{1 \text{ workday}} \times \frac{250 \text{ workdays}}{1 \text{ year}} = \frac{6,000 \text{ smog } \checkmark}{1 \text{ year}}$$

that means there need to be

$2,000,000$ smog $\checkmark$ per year $\times \dfrac{1 \text{ station}}{6,000 \text{ smog } \checkmark \text{ per year}} \approx 333$ stations in the county.

Fermi Estimation

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that means there need to be
$2,000,000$ smog $\checkmark$ per year $\times \dfrac{1 \text{ station}}{6,000 \text{ smog } \checkmark \text{ per year}} \approx 333$ stations in the county. A look at the Nevada DMV page shows listings for 318 smog check stations in Clark County, very close to our estimate!

Fermi estimation works on the idea that you can use nice round numbers during intermediate estimations because some will be too high and others too low; as long as you keep track of your units the final estimation won't be too bad. One Fermi estimation problem attributed[2] to the man himself is an estimate of the number of piano tuners in Chicago.

1. Estimate $5,000,000$ people in Chicago.
2. Estimate 2 people per household.
3. Estimate 1 in 20 households have a piano that gets tuned regularly.
4. Estimate pianos get tuned once a year.
5. Estimate it takes 2 hours to tune a piano.
6. Estimate a piano tuner works 8 hours a day, 5 days a week, for 50 weeks a year.

$$\frac{5,000,000 \text{ ppl}}{2 \text{ ppl/house}} \times \frac{1 \text{ piano}}{20 \text{ houses}} \times \frac{1 \text{ tuning per yr}}{1 \text{ piano}} = 125,000 \text{ tunings per year}$$

$$\frac{50 \text{ wks}}{1 \text{ yr}} \times \frac{5 \text{ days}}{1 \text{ wk}} \times \frac{8 \text{ hrs}}{1 \text{ day}} \times \frac{1 \text{ tuning per tuner}}{2 \text{ hrs}} = \frac{1000 \text{ tunings per year}}{1 \text{ tuner}}$$

$$125,000 \text{ tunings per year} \div \frac{1000 \text{ tunings per year}}{1 \text{ tuner}} = 125 \text{ tuners}$$

[2] `http://en.wikipedia.org/wiki/Fermi_problem` This page says there are 81.

As we saw in the preview work, tariffs are a type of indirect tax that impacts consumer (and producer) behavior to achieve a desired outcome. Sometimes taxes are used directly for this purpose, such as the so-called "sin (or vice) taxes," which are taxes levied on a product or service with the goal of reducing the consumption of that product or service and raising revenue to address societal costs brought on by the consumption of that good or service.

According to an article[3] from the *Los Angeles Times*, Nevada received 6% of its 2011 state revenue from sin taxes on things like gambling, and tobacco & liquor sales.

[3] `http://articles.latimes.com/2013/aug/19/business/la-fi-mo-states-sin-taxes-20130819`

Sin Tax

The Family Smoking Prevention and Tobacco Control Act (FSPTCA) states:

> *Reducing the use of tobacco by minors by* 50 *percent would prevent well over* $10,000,000$ *of today's children from becoming regular, daily smokers, saving over* $3,000,000$ *of them from premature death due to tobacco-induced disease. Such a reduction in youth smoking would also result in approximately* $\$75,000,000,000$ *in savings attributable to reduced health care costs.*

Use Fermi estimation to calculate a sin tax that would result in $\$75,000,000,000$ in revenue over the next 20 years to cover some of the future tobacco related health care expenses of tobacco-using youth. Your final answer should be in a unit like \$ per pack of cigarettes.

The income tax system in the United States is very complex and the subject of much criticism. Proposals for replacing it include:

- A national sales tax
- A flat tax (strict % of income)
- A value-added tax (tax charged on goods at various points in production based on how much the good's value has increased)
- More esoteric options like an automatically depreciating dollar[4] or an expenditures tax that takes into account the amount of income a person saves instead of spends[5]

Some tax proposals are *progressive*, meaning they take a larger portion when the amount taxed is greater, while others are *regressive*, meaning they take a larger portion when the amount taxed is lower.

[4] http://www.forbes.com/sites/kellyphillipserb/2011/09/02/guest-post-income-tax-alternative/

[5] https://www.kansascityfed.org/PUBLICAT/EconRev/EconRevArchive/1984/3q84mill.pdf

Sales taxes are sometimes considered regressive because of their effect on economic staples. For example, a person with a $\$30,000$ income in a state with 8% sales tax might pay about $\$200$ a year in sales tax on food, representing 0.67% of their income. A person with a $\$50,000$ income in the same state might buy more expensive food, but still only end up paying about $\$300$ a year in sales tax for that food, representing only 0.6% of their income.

Some states offer sales tax holidays, days where no sales tax is charged on certain items, in an effort to offset the potentially regressive nature of sales tax. For example, Georgia charges no sales tax on school supplies and a limited amount of clothing for two days in August prior to the start of the new school year.

Sales Tax Holiday

Suppose Nevada is considering a sales tax holiday to mitigate the potentially regressive nature of the tax. Choose 2 items you think should be considered eligible for the sales tax holiday. Use Fermi estimation to calculate how much tax revenue might be lost by the state in providing a sales tax holiday for your items.

Consumer Math Portfolio Assignment 4

Use Fermi estimation to consider the effect of the following alternative tax proposals on (a) a single person with an income of $\$100,000$ per year; (b) a family of 4 with a combined income of $\$60,000$ per year; and (c) a retired couple with an income of $\$25,000$ per year. You may want to watch the explanation at `http://www.showme.com/sh/?h=80c1L9c` first. Be sure to carefully write out your assumptions and show your calculations.

A. A national sales tax of 21%.

B. A flat tax rate of 17% of income after housing and food expenses.

C. A graduated income tax using the following rates:

Your first	**Taxed at rate**
$\$0$–$\$15,000$	0%
$\$15,000.01$–$\$50,000$	5%
$\$50,000.01$–$\$75,000$	10%
$\$75,000+$	15%

Mortgages

- Affordability Guidelines
- The Total Cost of a Mortgage
- Buying Points

There are many methods in use for calculating how much home a person can afford. During the housing bubble, lenders often approved borrowers for more than the borrowers could reasonably be able to repay. With this history, it is important to not rely solely on a loan officer's willingness to approve a mortgage to assess its affordability. What follows are three rules of thumb for assessing the affordability of a mortgage. None of them is perfect and each might give differing answers depending on available interest rates, hidden expenses, and other factors.

Guideline #1: The amount of the mortgage loan should not exceed three times the borrower's annual gross income.

Guideline #2: If a family has other *significant* monthly debt obligations, possibly including car payments, credit cards, or student loans, the family's monthly housing expenses, including mortgage payment, property taxes, and private mortgage insurance, should be limited to no more than 25% of their monthly gross income (income prior to deductions).

Guideline #3: If the family has no other significant monthly debt obligations, the monthly housing costs could range as high as 38% of their gross monthly income.

Affordability Guidelines

Use one or more of the affordability guidelines to suggest maximum mortgage amounts for the following individuals:

Hunter has approximately $\$25,000$ in student loan debt, makes $\$45,000$/year, and is single with no dependents.

Amy & Ben have 2 young children, one of whom requires frequent medical care. Ben makes $\$42,000$/year while Amy makes $\$37,000$/year. The couple pay $\$600$/month in childcare and have two modest car payments.

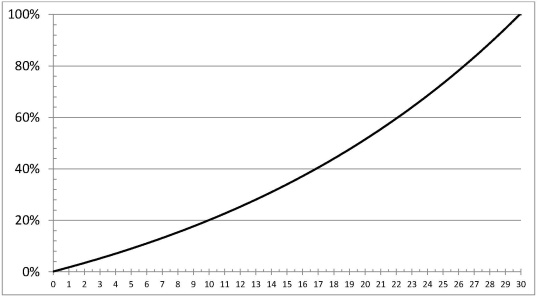
Stephanie is a single-mom of one school-aged child who makes $\$50,000$/year at her regular job and $\$1,000$/month in consulting fees. She spends $\$150$/month on student-loan payments.

Joanna is a recent college graduate with $\$20,000$ in outstanding student loans, $\$3,000$ credit card debt, and is interviewing for full-time jobs while supporting herself on tips from her job as a waitress.

Additional mortgage expenses, such as private mortgage insurance (PMI), homeowner's insurance, and property taxes can be significant.

PMI

A homeowner who cannot provide a 20% down payment is usually required to pay several hundred dollars each month in PMI. This insurance protects the lender against early default on the mortgage which might result in the lender reselling the property for less than is still owed. After a homeowner has paid 20% of the mortgage they can usually discontinue the PMI, though it can take about 10 years to build that level of equity on a traditional 30-year loan. The graph below shows the level of equity earned each year by paying the minimum monthly payment.



Homeowner's Insurance

Homeowner's insurance protects the homeowner against damage to or loss of the home and is usually required by lenders. The price can vary substantially, but it's not uncommon for a homeowner to pay $\$50$, $\$100$, or more each month in homeowner's insurance.

Property Taxes

Property taxes are collected on the appraised value of the home, which may differ from its market value. In the United States, these taxes are collected locally and are spent on local government services, such as public education and infrastructure. These rates can vary from less than a quarter of a percent to nearly two percent of appraised value annually. Nevada's 2014 property tax rate averaged about 1.1% of the appraised home value annually.

The Total Cost of a Mortgage

Cynthia & Simon are interested in purchasing a home in a city where the property tax rate is 1.25%. The home they are considering is appraised at $\$200,000$ and would have an annual insurance bill of $\$660$.
Additionally, because they cannot afford a large down payment, PMI will cost them $\$200$ per month.
Calculate the total monthly cost of the mortgage for Cynthia & Simon if the mortgage payment itself is $\$965.05$.

As we've seen, mortgages can be expensive. A person can easily pay more than the original cost of the house *just in interest* over the life of the loan, and this isn't even counting things like taxes and insurance!

To reduce the amount of interest s/he pays, a borrower can

1. borrow less money,
2. pay the mortgage back faster, or
3. borrow at a lower interest rate.

The first two options are fairly straightforward, if somewhat difficult to do in practice. The third option, however, is one that borrowers sometimes overlook but can result in significant long-term savings. In addition to simply shopping around for a lender offering a low rate, sometimes borrowers can engage in a process known as buying points.

A point, in the context of mortgages, refers to a permanent reduction in the interest rate of the mortgage in exchange for an upfront cash payment. The reduced rate results in smaller monthly payments and can add up to significant savings over the life of a mortgage.

For example, suppose Gina is considering a $\$150,000$ mortgage at 4.75% over 30 years. Her monthly mortgage payment would be $\$782.47$ (not including homeowner's insurance, PMI, or any taxes). Gina's bank offers her a point in the form of a 0.25% reduction in the interest rate in exchange for 1% of the mortgage amount. In other words, Gina could pay the bank $0.01 \times \$150,000 = \$1,500$ now but her interest rate will drop to 4.50%, resulting in a new monthly mortgage payment of $\$760.03$. Considering that the mortgage is for 30 years, this results in a savings of $360 \times (\$782.47 - \$760.03) - \$1,500 = \$6,578.40$ in interest over the 30-year loan.

The exact cost of points and the interest rate reduction received through buying points can change from day-to-day based on national fluctuations in interest rates, but it's possible to get a sense of their potential value by performing some simplified calculations.

Buying Points

Lola is interested in buying points for her mortgage. Her 30-year mortgage amount is $\$190,000$ with a 6.00% interest rate. Her bank will let her buy up to two points worth 0.25% points off her rate each, at a cost of 1% of the mortgage per point.

Use a mortgage payment calculator like that found at `http://www.bankrate.com/calculators/mortgages/mortgage-calculator.aspx` to help you compute how many months it will take Lola to save as much in interest as she would originally spend on the points.

Homework 14 I

1. Find the average 30-year mortgage rate and price of a house in your hometown for your birth year, if possible (otherwise, use 1980 Las Vegas). Using `http://www.bankrate.com/calculators/mortgages/mortgage-calculator.aspx`, create an amortization table showing the progress of a loan repayment using that information. Then, compute the percentage and $\$$ value of equity that would have been earned by the time of important events in your life (age 6 - learned to ride a bike, age 13 - first kiss, etc.). Compile your findings for submission.

2. Buying points comes with an opportunity cost. The money you spend on points could have instead been used towards a down payment, reducing your mortgage principal and therefore the interest charged over the life of the loan. The calculator at `http://www.bankrate.com/calculators/mortgages/mortgage-loan-points-calculator.aspx` can help you see how

Homework 14 II

buying points compares to putting the extra money towards the down payment. The calculator has sliders that let you set the amount of the mortgage, the length of the loan, the number of points you buy, and the interest rate *after* buying the points. It simultaneously calculates what would happen if you instead spent the money on a down payment and so had a higher interest rate on the loan. Use the calculator to examine the following scenarios a person could face in buying a home. In your report, describe for each scenario whether it is advantageous to buy points. Include in your report an overview of your impression of buying points after using this calculator:

a. A $\$200,000$ 30-year mortgage at 5% where each point reduces the rate by 0.25 percentage points. (Calculate for a variety of points.)

b. A $\$150,000$ 20-year mortgage at 6% where each point reduces the rate by 0.15 percentage points. (Calculate for a variety of points.)

Homework 14 III

c. The median price of a Las Vegas home as listed at `http://www.zillow.com/las-vegas-nv/home-values/`, the current average mortgage rate as listed at `http://www.bankrate.com/finance/mortgages/current-interest-rates.aspx`, and the current reduction offered by buying points as listed at `https://www.suntrust.com/Mortgage/Rates`. (Calculate for a variety of points for both 15-year and 30-year loan terms.)

Inflation

As we saw in the preview work, inflation reduces the value of currency by lowering its ability to purchase goods and services as a result of higher prices. Even low levels of inflation can have a serious impact on the everyday transactions of consumers.

Interest vs. Inflation

If Stefano invested $\$1,000$ in a CD in 2008 at 1.8%, how much would the account have after 5 years? Recall: $A = P\left(1+\frac{r}{n}\right)^{nt}$
The average rates of inflation for the 5 years of Stefano's investment are shown below.[a]

Inflation (%)				
2008	2009	2010	2011	2012
3.8	−0.4	3.2	2.1	1.5

1. What is the dollar amount of Stefano's investment at the end of 5 years?
2. Using the inflation rates, determine how much money in 2012 is equivalent to $\$1,000$ in 2008.

[a] `http://www.usinflationcalculator.com/inflation/historical-inflation-rates/`

Unfortunately, traditional savings mechanisms such as savings accounts and investment CDs offer rates of return lower than the usual rate of inflation, meaning that the money will continually worth less and less even though there is more of it.

As a result, most investors look for opportunities to receive returns at a rate higher than inflation, thus increasing the overall buying power of their investment. Higher rates, however, usually come with higher risks. For example, real estate was long seen as a solid investment, offering returns of approximately 9% annually,[6] far above inflation. The housing crash of 2006 (driven in part by large numbers of investors hoping for large returns) caused home values to fall significantly, dropping nearly 50% in some places, leading to many billions of dollars in lost investment.

[6] http://www.money.cnn.com/galleries/2007/real_estate/0704/gallery.stocks_v_realestate.moneymag/index.html

The decreased buying power caused by inflation is especially important to consider when contemplating long-term effects, such as the standard of living that you can afford after retirement.

Inflation vs. Retirement

David and Victoria are each 5 years from retirement with a combined income of $\$100,000$. If inflation is projected to continue at 2.5% annually, how much will they need to withdraw from savings during their first year of retirement to match their current buying power?

Many retirement plans strive to provide returns that outstrip inflation by investing in a mixture of safer low-yield investments, and riskier high-yield investments. One common investment option is stocks.

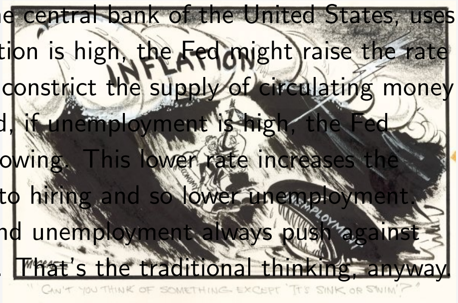
Stocks are shares (or ownership interests) in companies that can be bought, sold, or traded. When a company wants an influx of cash for expansion, for example, it may sell stocks to interested investors. Those investors then own pieces of the company and receive a portion of its profits.

Inflation vs. Savings

Carissa is ready to retire and has accumulated savings of $\$500,000$ to do so. She currently spends $\$50,000$ a year.

1. If inflation occurs at 3% per year, how much will Carissa have to spend each year, for the next 10 years, to maintain her buying power?
2. If Carissa withdraws from savings the amount she needs each year on Jan. 1, and her remaining savings earns 2% simple interest each year, what will be her savings balance after a 3 years? b 5 years? c 10 years?
3. Give an estimate of how much Carissa would need in her savings at retirement to maintain her buying power for 20 years, taking into consideration the effects of inflation. Give reasons for your estimate.

Governments have attempted to curb inflation in a variety of ways. Currently, the Federal Reserve Bank, which serves as the central bank of the United States, uses interest rates to impact inflation. If inflation is high, the Fed might raise the rate banks pay to borrow money. This would constrict the supply of circulating money and dampen inflation. On the other hand, if unemployment is high, the Fed might reduce the rate banks pay for borrowing. This lower rate increases the amount of money in circulation, leading to hiring and so lower unemployment. The complementary beasts of inflation and unemployment always push against the economy in a bid to outdo the other. That's the traditional thinking, anyway.



http://uws.edu.au/__data/assets/image/0009/479052/Economy_cartoon.jpg

Consumer Math Portfolio Assignment 5 I

1. For this problem you are going to have an evil twin who makes different financial decisions than you. You will buy a house while your evil twin buries money in the yard like a pirate. First, let's project what will happen to you.

   - Go to `www.zillow.com` and find the estimated value of a house you like.
   - Find a current 30-year mortgage rate at `www.zillow.com/mortgage-rates/`.
   - With those two pieces of information, use the calculator at `http://www.bankrate.com/calculators/mortgages/mortgage-calculator.aspx` to create a home mortgage amortization table and record the monthly mortgage payment and the total amount paid over the life of the mortgage.

Consumer Math Portfolio Assignment 5 II

Now, let's take a look at the results of your evil twin's decisions. Instead of buying a house like yours, each month he or she takes the amount of your monthly mortgage payment and bury it in a jar in the yard. At the end of 30 years you own a house and your twin has 360 jars of monthly payments totaling the amount you paid for the house, including interest. With this buried money, your evil twin finally wants to buy a house and wants it to be even nicer than your house. With all the buried money, your twin won't need a loan and so won't have to pay interest. However, because of inflation, the money in the jars doesn't buy as much as it did when it was first buried. How much is your twin's money worth, in terms of buying power, compared to what you spent? To find out, you are going to use an inflation calculator.

The calculator at `http://www.bankrate.com/calculators/savings/price-inflation-calculator.aspx` allows you to calculate how much buying power your twin's jar money will have in

Consumer Math Portfolio Assignment 5 III

terms of today's dollars. To compare your decisions with his/hers, enter the following values into the inflation calculator.

- Years: 30 (This is the length of the mortgage.)
- Monthly contributions: Your monthly mortgage payment (This is how much your evil twin buried in the yard each month.)
- Amount currently invested: $\$0$
- Expected rate of return: 0% (Money buried in the yard earns no interest.)
- Expected inflation rate: 3.22% (This is the average annual inflation rate over the past 100 years.)
- Federal tax rate: 0% (There's no tax penalty for digging up your yard.)
- State tax rate: 0% (For the same reason as above.)

Consumer Math Portfolio Assignment 5 IV

Click "VIEW REPORT" to see the results of the calculation. The number next to "Ending balance with taxes and inflation" is how much your twin's savings will be worth in current dollars. Who made the better decision?

2. Rerun the same calculations as before with two other rates of inflation (one higher and one lower) and two other houses. You can find historical inflation rates at `http://www.usinflationcalculator.com/inflation/historical-inflation-rates/` or you can just choose a value.
3. Prepare a report of your findings, detailing the consequences of your and your twin's financial decisions in each of the three sets of calculations.

Functions

- Dependent and Independent Variables
- Graphics vs Tables vs Formulas
- Desmos Water Problem

Recall that a function is a relationship where each value of the input variable (independent variable) is assigned exactly one value of the output variable (dependent variable).

Thus the number of hours of daylight is a function of the day of the year.

- Is the day of the year also a function of the number of hours of daylight?

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Thus the number of hours of daylight is a function of the day of the year.

- Is the day of the year also a function of the number of hours of daylight?

The formula $y = x^2$ defines y as a function of x.

- Is x also a function of y?

Is it a Function?

For each item below determine whether one quantity is a function of the other quantity. If yes, then also do the following:

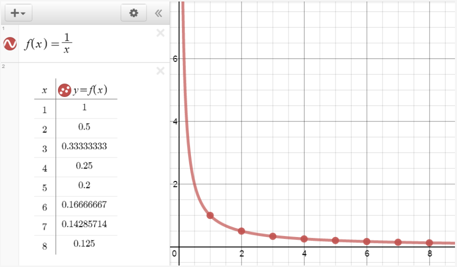
(a) Determine which variable is independent and which is dependent.
(b) Sketch a reasonable graph to illustrate the relationship.
(c) Write one or two sentences justifying the shape and behavior of your sketch.

1. The temperature in the LAS parking lot and the time of day—
2. The cost of pizza and the diameter of the pizza—
3. Your course grade and your shoe size—
4. Education level of a person and the amount of that person's salary—
5. Credit card interest owed and the balance due on the account—
6. The day of the year and the number of hours of daylight—
7. A male's optimal weight and a male's height—

In the preview assignment we encountered three methods of displaying functions.

- in a graph,
- in a table, and
- in symbols.

What are some pros and cons of each method?



Napolean's Keyrings

Napolean runs a home business selling boondoggle keyrings. His business license costs $\$10$/month. He makes a profit of $\$0.80$ for each keyring he sells. Define a function whose independent variable is the number of keyrings he sells in a given month and whose dependent variable is the profit he makes that month. Provide

- a graph of the function,
- a table of function values, and
- a symbolic form of the function.

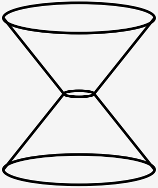
Water Filling a Container

Go to `https://student.desmos.com/` and enter the code your instructor gives you.
Be sure to enter both names. Follow the instructions of the activity.
There is no write-up for this activity.

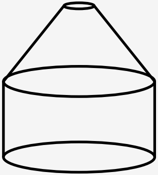
Homework 8 I

1 Suppose water is poured at a uniform rate into a container. Draw a function that models the height of the water over time for each container below.

(a)



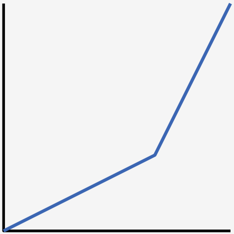
(b)



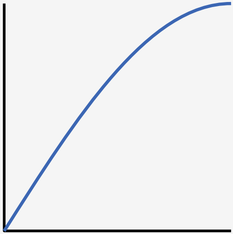
Homework 8 II

2 Suppose water is poured at a uniform rate into a container. For each graph below draw a container whose height over time is modeled by the graph.

(a)



(b)



Homework 8 III

3. Represent the function $f\left(x\right) = \dfrac{x}{x^2+1}$ as both a table and as a graph.

4. Represent the function below as a graph. (As an optional challenge, see if you can find a possible formula for $f(x)$.)

x	0	1	2	3	4	5	6
$f(x)$	0	-3	-4	-3	0	5	12

5. Present a function whose input is the 10 U.S. cities with the greatest population, and the output is the population. Decide which of the three methods for displaying a function is most appropriate, and then display the function. (Yes, you will need to do a search to find this data. Yes, you must cite your source.)

Homework 8 IV

6. For each item below determine whether one quantity is a function of the other quantity. If yes, then also do the following:
   1. Determine which variable is independent and which is dependent.
   2. Sketch a reasonable graph to illustrate the relationship.
   3. Write one or two sentences justifying the shape and behavior of your sketch.
      1. The number of rings in a tree trunk and the age of the tree—
      2. The cost of a television and the size of the screen—
      3. The perimeter of a square and its area—
      4. The cost to make a movie and the amount of money the movie makes—
      5. The height of a wife and the height of her husband—
      6. A student's letter grade in a class and their total score for the class—

Algebra of Functions

- Profit Function
- Dating Ages
- Coupons and Discounts

Profits From Boondoggle

Napolean wants to start a business selling boondoggle keyrings. Let x be the price he charges for each keychain.

Demand With the price of each keychain at x dollars, Napoleon expects to sell $500 - 100x$ keychains.

Costs The rent for the facilities will be $\$200$ while the material costs $\$0.20$ per keychain.

1. Find the cost function $C(x)$.
2. Find the revenue function $R(x)$.
3. Find the profit function $P(x)$.
4. At what price should Napolean sell the keychains to maximize his profit? (Try plotting $P(x)$ on `desmos.com`.)

Bounds for Dating Ages

It has been suggested that a person of age x should only date a person whose age is at least $f(x) = \frac{1}{2}x + 7$.

1. According to this formula, what is the minimum age of a person that a 24-year old should date? And for a 36-year old?
2. Find the inverse function of $f(x)$ and call it $g(x)$. What is the intepretation of the output of $g(x)$?
3. According to this formula what is the maximum age of a person that a 24-year old should date? And for a 36-year old?
4. In `desmos.com` plot $f(x)$, $g(x)$ and $f(x) < y < g(x)$. What does the shaded region represent?
5. Plot a few points at `desmos.com` in the form (x, y) to represent couples you know. Does the model fit?

Coupons and Discounts

You clip a $\$5$ off coupon from the newspaper for a pair of shoes. At the store you find that all the shoes are 10% off.

1. Write a function $f(x)$ that shows the effect of the coupon.
2. Write another function $g(x)$ that shows the effect of the store discount.
3. Make a new function $h(x) = f(g(x))$ and determine whether this function applies the coupon before or after the discount.
4. Make a new function $k(x) = g(f(x))$ and determine whether this function applies the coupon before or after the discount.
5. Which pricing model do you think the store is most likely to choose? (Hint: compare $h(50)$ and $k(50)$.)

Homework 9 I

1. Recall that $F = \frac{9}{5}C + 32$ is the formula for converting degrees Celsius to degrees Fahrenheit. At what temperature is the Fahrenheit temperature equal to the Celsius Temperature? (i.e., where is $F = C$?) Either Desmos or WolframAlpha could be helpful.

2. If a person of age x should not date a person younger than $y = \frac{1}{2}x + 7$, then what is the lowest allowable dating age for any individual?

Homework 9 II

3. John is trying to find the ideal number of dates he should go on each week. He notes that his dating life satisfaction (on a scale of 1 to 10) is modeled by $S(d) = -\frac{1}{2}d^2 + 4d + 2$ where $0 \leq d \leq 8$ is the number of dates he goes on each week.

   1. How satisfied is John if he goes on 0 dates in a week?
   2. How satisfied is John if he goes on 2 dates per week?
   3. How many dates should John go on each week to maximize his dating life satisfaction? (Hint: Graph John's satisfaction function.)

Homework 9 III

4. John notes that the number of dates he has each week depends on the number of women whom he asks. He goes on approximately $d(w) = \sqrt{w}$ dates per week when he asks out w women that week $(0 \leq w \leq 64)$.
    1. About how many dates will John go on if he asks out 9 women?
    2. Approximately how many women did John ask out if he went on 5 dates?
    3. Using the function $S(d)$ from the previous problem, explain the input and output of the composite function $S(d(w))$.
    4. How many women should John ask out to maximize his dating satisfaction?
    5. Is John trying too hard? Should he try online dating?

Homework 9 IV

5. Suppose $f(x) = x + 1$. Find a formula for $f(f(x))$, $f(f(f(x)))$, and $f(f(f(f(x))))$. In general, what is the formula for n compositions of f: $f^{(n)}(x) = \underbrace{f(f(\cdots f(f(x))\cdots))}_{n \text{ times}}$?

6. Suppose $g(x) = 2x$. Find a formula for $g(g(x))$, $g(g(g(x)))$, and $g(g(g(g(x))))$. In general, what is the formula for n compositions of g: $g^{(n)}(x) = \underbrace{g(g(\cdots g(g(x))\cdots))}_{n \text{ times}}$?

Parent Functions 1

- The Shapes of Parent Functions
- Circles
- Finding a Model From Data

Warm-Up

In the pre-lecture assignment you learned some of the key characteristics of several functions. Take a minute to sketch the graph of each of the following functions.

1. $y = x$
2. $y = \sqrt{x}$
3. $y = x^2$
4. $y = \frac{1}{x}$
5. $y = x^3$
6. $y = |x|$

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We can check our answers by looking at `https://www.desmos.com/calculator/mtayxqpik8`.

Parent Functions in Models

Discuss each question below with a partner. Identify a function and determine what the x and y-variables might represent.

1. Which of the parent functions would be the best function to model an infant's growth from birth?
2. Which of the parent functions could model the distance that an object has fallen over time?
3. Which of the parent functions would give us the volume of a cube with side length x?
4. Which of the parent functions might model the amount of money you earn at your job over time?

The standard form of the equation of a circle is

$$(x-h)^2 + (y-k)^2 = r^2.$$

Here we view x and y as variables, while h, k, and r take specific values for particular circles. In this activity we will use Desmos to explore the role played by the constants h, k, and r.

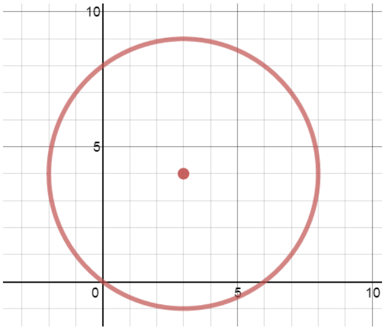
Transforming a Circle

Open up `desmos.com` and enter the code `(x-h)^2+(y-k)^2=r^2`. Then create a slider for each of the variables. With the help of the sliders describe the effect on the circle for each scenario below.

1. When h increases ______________________________.
2. When k increases ______________________________.
3. When r increases ______________________________.

Finding a Formula of a Circle

Find a formula for the circle in the image. (It is centered at the point $(3, 4)$ and it goes through the origin.)



$\in$

Finding a Formula for Distance Fallen

An object is thrown off a building. The function $f(t)$ gives the distance (in ft) that the object has fallen after t seconds. We observe the following values.

t	0	0.5	1	1.5	2	2.5	3
$f(t)$	0	4	16	36	64	100	144

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2. How far has the object fallen after 10 seconds?
3. How far does the object fall between seconds 6 and 7?

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1. Find a formula for $f(t)$.
2. How far has the object fallen after 10 seconds?
3. How far does the object fall between seconds 6 and 7?

Hint: Create a table in Desmos using x in place of t and $y = f(x)$. Then change the window setting so that $-1 < x < 5$ and $-5 < y < 150$. Decide which of the parent functions looks the most promising and try plotting it. Then start tweaking it.

Homework 10 I

1. Find an equation of a circle of radius 6 centered at the point $(1, -2)$. Identify the coordinates of at least one point on the circle.

2. Find the center and radius of the circle $(x+5)^2 + y^2 = 9$. Identify the coordinates of at least one point on the circle.

3. Plot the points given in the table below. Then identify a parent function that best fits the data. Tweak your guess until you have found the exact formula for $f(x)$.

x	-3	-2	-1	0	1	2	3
$f(x)$	12	8	4	0	4	8	12

Homework 10 II

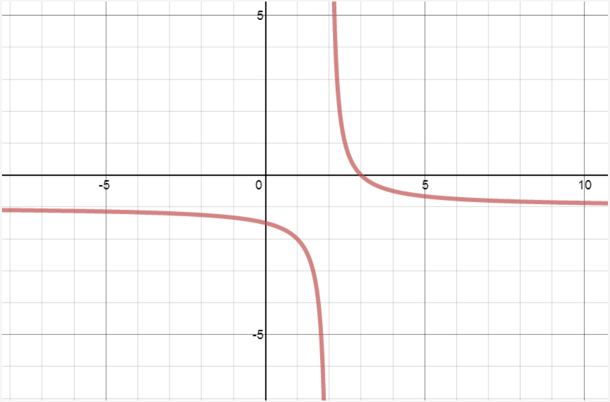
4 A stalactite is forming in an inverted cone shape. The volume and height of the stalactite are measured at various stages. You have been asked to model the volume, $V(x)$, (in ft^3) of the stalactite in terms of the height, x (in ft).

x	1	2	3	4	5	6
$V(x)$	0.25	2	6.75	16	31.25	54

Plot the points and decide on a parent function that best fits the data. Then tweak the function to obtain an exact value of $V(x)$.

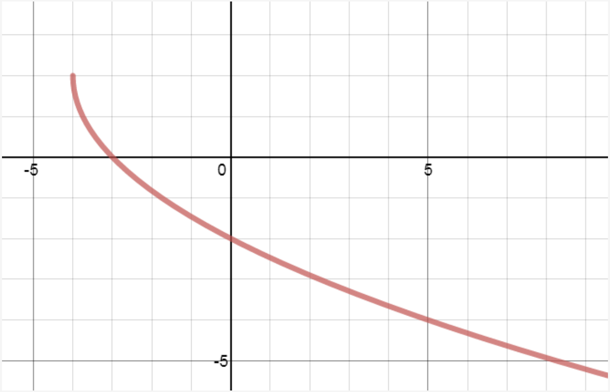
5 Determine the values of a, h, and k for the function $f(x) = a \cdot \frac{1}{x-h} + k$ whose graph is below. (The values of a, h, and k are integers.)

Homework 10 III



6 Determine the values of a, h, and k for the function $g\left(x\right) = a\sqrt{x-h} + k$ whose graph is below. (The values of a, h, and k are integers.)

Homework 10 IV



Parent Functions 2

- Wave Functions
- Exponential Functions
- Logarithms

In the pre-class assignment we learned about the sine and cosine functions. We don't need to have a deep understanding of these wave functions, but we need to be familiar with the following bits of information about the parent functions $y = \sin(x)$ and $y = \cos(x)$.

1. The ouput values repeat every 2π units.
2. The minimum output is -1 and the maximum output is 1.
3. The graphs of both functions are waves, so we use either function to model objects that oscillate.

Measuring the Tide

The tide level is measured every 5 hours. The table below gives the level of the tide in feet above the low tide level.

time (in h)	0	5	10	15	20	25	30	35
tide level (in ft)	0	1.48	3.73	3.41	1.00	0.07	2.00	3.93

1. Find a function of the form $f(x) = a\cos(bx) + k$ that outputs the tide level after x hours.
2. Estimate the level of the tide after 27 hours.
3. Does your model seem realistic? Explain.

We need to know the following things about exponential functions.

1. The graph of $y = a^x$ goes through the point $(0, 1)$, and only outputs positive values.
2. If $0 < a < 1$, then the graph is decreasing with a horizontal asymptote along the positive x-axis.
3. If $1 < a$, then the graph is increasing with a horizontal asymptote along the negative x-axis.
4. Exponential functions are useful in modeling the amount of some substance where the change in the amount of the substance depends only on the amount that you have now. Examples include compound interest, population, and radioactive decay.

A Population Model

A settlement starts out with a population of 1000. Each year the population increases by 10%. Let $P(t)$ be the function that gives the population in the settlement after t years.

1. Find the missing values in the table below.

t	0	1	2	3	4	5	6	7
$P(t)$	1000							

2. Find a formula for $P(t)$ using the values from the table.
3. Find a formula for $P(t)$ by reasoning it out directly.
4. What will the population be after 100 years?
5. Is this a realistic model?

Recall that each exponential equation $a^x = b$ has a corresponding logarithmic equation $\log_a(b) = x$. Both equations mean the same thing.

Logarithm Practice

Evaluate each logarithm. Try to do it without a calculator.

1. $\log_6(36)$
2. $\log_7\left(\frac{1}{7}\right)$
3. $\log_{10}(1000)$
4. $\ln(1)$
5. $\log_2(4)$
6. $\log_4(2)$

To solve an equation we can plot both halves of the equation and see where the curves intersect.

Solve the equation $2^x = 5$ using three different methods.

1. Plot $y = 2^x$ and $y = 5$ in `desmos.com`. Then click on the point where the curves intersect.
2. Convert the exponential equation $2^x = 5$ to logarithmic form.
3. Go to `wolframalpha.com` and enter:

```
2^x = 5
```

Homework 11 I

1. A scientist measures the percent, y, of the moon's face illuminated on day x. The values of her measurements are shown below (in decimal form) in a table.

x	1	8	15	22	28
y	0.000	0.502	0.999	0.550	0.006

   1. The moon phases repeat regularly, so we hope that a wave function will model it. Fit a curve of the form $y = a\sin(bx + c) + \frac{1}{2}$ to the data. Your answer will be an approximation. The values of a, b, and c might not be pretty.
   2. Use your model to estimate the portion of the moon that is visible on day 19.
   3. According to your model, how many days does it take for the moon to repeat its phases?

Homework 11 II

2. A bacteria colony is dying off. The colony loses half of its members each day. Let $N(t)$ be the number of bacteria remaining after t days.

   1. Complete the table below.

t	0	1	2	3	4	5	6
$N(t)$	$4,000$						

   2. Find an exponential model for $N(t)$.

   3. According to your model, how many bacteria are present after 36 hours?

   4. How many days will it take for the bacteria population to reach 10?

Homework 11 III

3. Solve each equation.

   1. $x^5 = x$
   2. $4^x = x^4$ (Be sure to get all three solutions.)
   3. $2x - x^3 = x^2$
   4. $\sqrt{x} = \log_2(x)$

4. Find the points of intersection of the curves

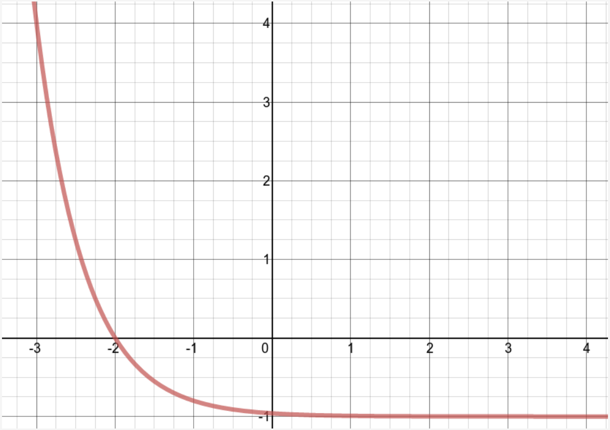
$$y = x^2 - 5 \qquad \text{and} \qquad x^2 + y^2 = 25.$$

5. Find the formula of a function, $f(x)$ that approximates the data in the table.

x	1	2	3	4	5	6	7	8
$f(x)$	1.5	2.25	3.375	5.063	7.594	11.391	17.086	25.629

Homework 11 IV

6 Determine the values of b, h, and k for the function $f(x) = b^{x-h} + k$ whose graph is below.



Transformations of Functions

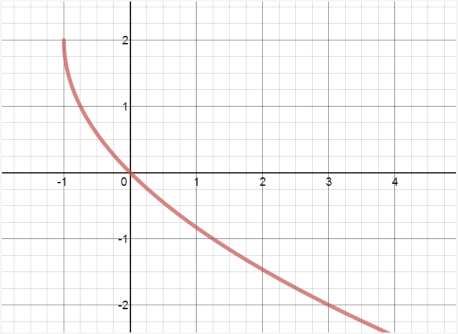
- Rules for Transformations

By this point we have mastered the graphs of over a dozen parent functions.

Recall the rules for shifting and scaling the graph of a function.

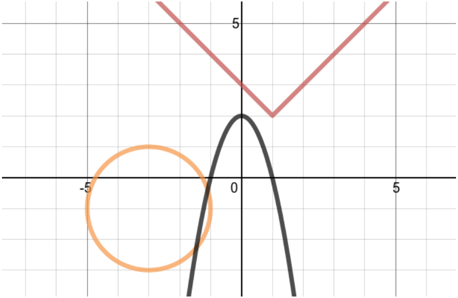
- To shift $y = f(x)$ to the right by h units use $y = f(x - h)$
- To shift $y = f(x)$ to up by k units use $y = f(x) + k$
- To scale $y = f(x)$ horizontally by a factor of b use $y = f\left(\frac{1}{b}x\right)$.
- To scale $y = f(x)$ vertically by a factor of a use $y = af(x)$.

Below we see the graph of a curve. As a class we will identify which parent function this curve is based off of. Then we will identify how the parent function was shifted and/or scaled.



Mystery Curves

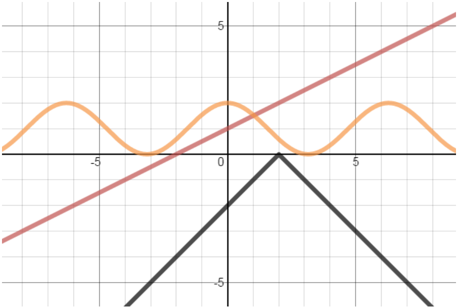
Below we see the graphs of three curves. Find the equation of each curve.



Remember that if $f(x)$ is the parent function then one possible way of accounting for scaling and shifting transformations is to consider the function $af\big(b(x-h)\big)+k$.

Mystery Curves II

Below we see the graphs of three more curves. Find the equation of each curve.



Remember that if $f(x)$ is the parent function then one possible way of accounting for scaling and shifting transformations is to consider the function $af\big(b(x-h)\big)+k$.

A Change of Heart

Go to https://www.desmos.com/calculator/h4p6yr39pi and create 5 transformations of the heart-shaped graph. Try to do it without peeking at the answer. Record the equations below.

Purple:

Blue:

Orange:

Green:

Black:

Homework 12 I

1. Go to `https://www.desmos.com/calculator/lpdp6e2cnk`. For this problem you should find equations for 6 different graphs. You will need to shift, scale, and/or reflect the graphs of parent functions to obtain a graph of each of the six functions. Write the equation for each function below. (Answers might not be unique.)

Red:

Orange:

Black:

Purple:

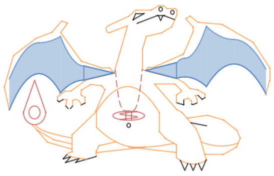
Blue:

Green:

Transformations of Functions II

- Monsters and Restricted Curves
- Desman

For your homework today you will demonstrate your mastery of the functions and function transformations by creating a monster out of functions.
The dragon to the right was made by a student at `http://desmos.com`.



You aren't required to make something so elaborate, but you will need to actively participate during the rest of class to be able to do the homework.

An instructor made sample monsters at
`https://www.desmos.com/calculator/b96obpwkcg` and at
`https://www.desmos.com/calculator/eowtedjmg7`.

Each of the examples took 30 minutes to make.

Go to `http://desmos.com` and enter the function $y = 1 - x^2$.

Sometimes we want to plot only a single part of a graph. Desmos has its own coding convention that allows us to restrict the x or y-values. At the end of the expression, simply put an inequality inside curly braces such as

$$\{a < x < b\} \quad \text{or} \quad \{c < y < d\}.$$

How would we display only the part of the graph above the x-axis?

A 2-minute video showing one strategy can be found at
`https://www.youtube.com/watch?v=Pd-JguTH8gw`

Creating a Desman

In this activity we will use Desmos to create a person.
Get with a partner and go to `http://student.desmos.com` and enter the class code the instructor displays.
You don't need to hand in anything for this activity because the instructor will have a record of it. Be sure to include your names after you sign in.
We will spend the rest of the class period working on our Desman.

Homework 13 I

1. Create a monster using Desmos. Go to `http://student.desmos.com` and enter the code: `bdan`. Instead of creating a person you should create a monster. This homework assignment will be graded out of 10 points according to the rubric below. Good luck.

   - 2 pts The monster is created using functions based off of several different parent functions.
   - 2 pts The color and style is consistent/appropriate for the whole picture.
   - 2 pts The monster consists only of curves from equations in Desmos.
   - 2 pts The project demonstrates that the student has a solid mastery of the shapes of graphs given by an equation.
   - 1 pts The final product compares favorably with the work of your classmates.
   - 1 pts Your monster looks cool.

Models and Predictions

- Braking Distance
- Squares in a Grid
- Brick Wall

Braking Distance

The table below gives the distance that a car will travel after applying the brakes at a given speed.

Speed (in mi/h**)**	**Distance to stop (in** ft**)**
10	5
20	19
30	43
40	76.5
50	120
60	172
70	234

1. Find a function that outputs stopping distance when you input speed.
2. Estimate the stopping distance for a car that is traveling 43 mi/h.

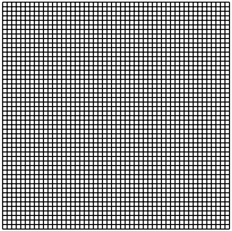
By making a model we are able to answer questions that seem insurmountable at first glance. For example, recall the problem where the tide level is measured every 5 hours. The table below gives the level of the tide in feet above the low tide level.

Time (in h**)**	**Tide level (in** ft**)**
0	0
5	1.48
10	3.73
15	3.41
20	1.00
25	0.07
30	2.00
35	3.93

If we use the `fit` function in wolfram alpha we get a model that models the data poorly.

Squares in a Grid

Suppose that we want to know the number of squares inside a 50×50 grid. It doesn't seem practical to try to count them all. Notice that the squares come in many sizes.



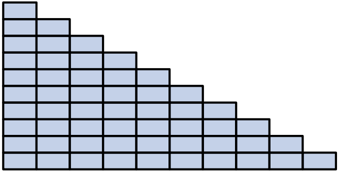
Let $f(x)$ be the function that gives the number of squares in an $x \times x$ grid. Complete the chart below.

x	1	2	3	4	5
$f(x)$					

Then use Desmos to find an exact formula for $f(x)$. How many squares are in a 50×50 grid? What about a 2000×2000 grid?

Brick Wall

You plan to make a diagonal brick wall as shown, but you don't know how many bricks to buy. You figure that the brick wall will have 100 rows.



Let $f(x)$ be the number of bricks used through x. Thus $f(1) = 1$, $f(2) = 3$, $f(3) = 6$, and so forth.

1. Find $f(4)$, $f(5)$, and $f(6)$.
2. Find a formula for $f(x)$.
3. How many bricks will be needed to complete 100 rows?
4. How many bricks would be needed to complete 1000 rows?

Homework 14 I

1 A required dose of a certain drug depends on weight as follows.

Weight (lbs)	100	150	200	250	300
Dosage (mg)	40	60	85	115	150

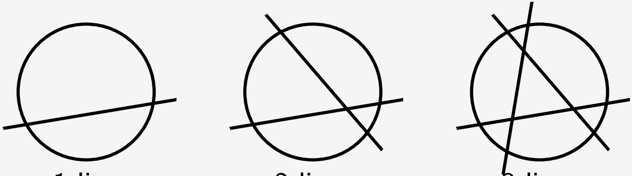
A doctor needs to know the dosage for a 450 lb patient. Find an equation that inputs the weight and outputs the dosage and use it to obtain the correct dosage. Be sure to write in full sentences. Hint: Enter the command

```
fit (100,40), (150,60), (200,85), (250,115), (300,150)
```

in WolframAlpha to get a curve that fits the data.

Homework 14 II

2. In this question we investigate the number of regions into which we can divide a circular disc using straight lines. We can see below that 1 line can divide the disc into 2 pieces, 2 lines can make 4 pieces, and 3 lines can make as many as 7 pieces.



1 line, 2 pieces | 2 lines, 4 pieces | 3 lines, 7 pieces

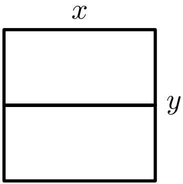
Optimization

- Bessy The Cow
- The Biggest Rectangle
- Minimizing A Perimeter

Consider the curve $y = 1 - x^4$. Which point on that curve is closest to the origin? Go to `http://desmos.com/calculator/71w9rpo46g` to investigate.

Bessy's Dream Pen

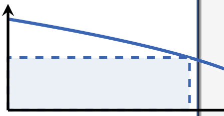
Bessy the cow wants to build her dream pen out of 600 ft of fencing. Her design calls for a rectangular pen that she will subdivide into two rooms as pictured. What are the values of x and y that will provide Bessy with the maximum enclosed area?



1. Write an equation that gives the total number of feet of fence in terms of x, y and 600 ft
2. Solve that equation for y.
3. Write an expression that gives the area of the pen, A, in terms of x and y.
4. Replace y with an appropriate function of x.
5. Find the value of x that maximizes the area function $A(x)$.

A Rectangle Under a Curve

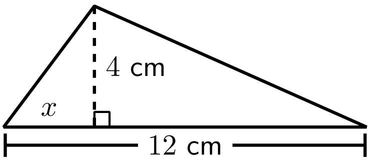
A rectangle in Quadrant I has one corner at the origin and the opposite corner on the curve $y = \sqrt{9-x}$. In this problem we want to find the length and width of the rectangle that has the maximum area.



1. Plot the curve and a sample rectangle.
2. Make a guess for the value x that gives a maximum area for the rectangle.
3. Write out a formula for the area of the rectangle.
4. Find the dimensions of the rectangle of maximum area under the curve.

The Minimum Perimeter of a Triangle

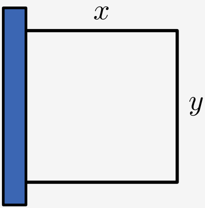
Suppose we have a triangle with a base of 12 cm and a height of 4 cm as pictured. There are many such triangles possible, one of which has smaller perimeter than all others. Our goal in this problem is to find the length, x, that gives the triangle with smallest perimeter.



1. Use the Pythagorean Theorem to find the length of the missing sides of the triangle.
2. Create a function $P(x)$ that gives the perimeter of the the big triangle.
3. What is the perimeter when $x = 0$?
4. Find the value of x that minimizes the perimeter function $P(x)$.
5. What is the minimum possible perimeter?

Homework 15 I

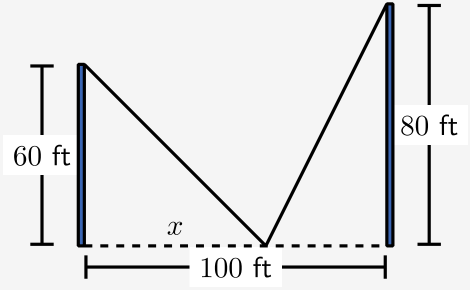
1. Bessy the cow wants to build her dream pen out of 300 feet of fencing. Her design calls for a rectangular pen built up against an existing wall, as pictured. Find the dimensions x and y that maximize the area of the enclosed pen.



2. Find the coordinates of the point(s) on the curve $y = \dfrac{1}{1+x^2}$ closest to the origin.

Homework 15 II

3 Two vertical towers of heights 60 ft and 80 ft stand on level ground with their bases 100 ft apart. A cable is to be strung from the top of one pole to some point on the ground between the poles, and then to the top of the other pole. What is the minimum possible length of cable required?

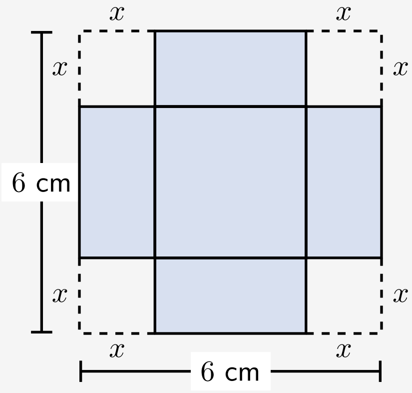


Homework 15 III

4. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis lying on the parabola $y = 8 - x^2$.

Homework 15 IV

5 A box is created by taking a $6\,\text{cm} \times 6\,\text{cm}$ sheet of metal and cutting out square corners, then folding up the sides as pictured below. Find the value of x that will result in a box with maximum possible volume.



Homework 15 V

(After you have solved the problem you can go to `http://geogebratube.org/student/m48703` to check your answer, but make sure you know how to solve it from scratch.)

Final Project