

# 1 Introduction to Math 120

## Course Setup and Expecta- tions

Start off by greeting the class and introducing yourself.

Welcome to Math 120, Introduction to College Mathematics.

This class is designed to be a practical math course where we learn how to get better at life.

We will answer questions such as:

- How does money get used in daily life, and how do we use money wisely?
- What strategies are optimal when gambling at a casino?
- How can we use data to make predictions about outcomes that we can't test on a small scale?

The main goal for today is to lay out the course structure and expectations so that everybody is on the same page.

The first thing you will need to know is how to access the course on Canvas, our course management system. To get to Canvas go to [www.nsc.instructure.com](http://www.nsc.instructure.com) and sign in using your student id and password.

Actually go to [www.nsc.instructure.com](http://www.nsc.instructure.com) at this point to demonstrate what it looks like.

We will be using calculators and computers in this class. Please bring your laptops and tablets to class each day.

laptop > tablet > smartphone > calculator

You are encouraged to install the WolframAlpha app and the Desmos app on your device. Make sure that your device is connected to the NSC wifi.

The homework will also rely heavily on web resources.

For each day of class there are two assignments to be completed.

- Prior to coming to class each student must complete a pre-class assignment that will prepare you for the group activities you will work on during class. The pre-class assignment is designed to baby-step you through some of the key ideas that will be covered during class.

Show them how to log into [student.desmos.com](http://student.desmos.com).

- After class you will complete a homework assignment that is to be handed in at the beginning of the next class period.

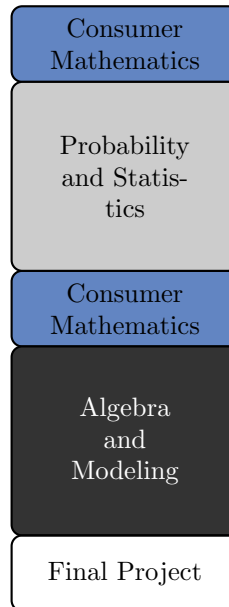
Show them how to access the class activities and homework file.

Class time will be devoted to group work and discussion with a small bit of lecture between problems. It is essential that students come prepared to work hard during class. There is no course textbook, so the only way to learn the material is to participate in class and to complete the pre-class assignments.

The course is broken up into 3 topics:

- Consumer Mathematics
- Probability and Statistics
- Algebra and Modeling

A rough timeline is presented below. Explain that the consumer mathematics content is broken up across the semester.



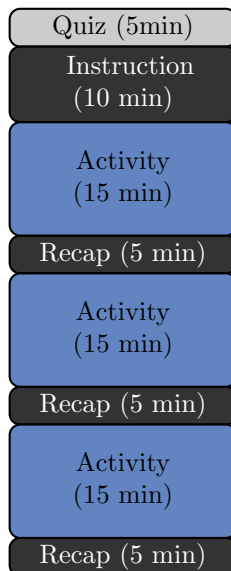
**Grades** Grades will be determined by your performance across several categories:

- Daily Quizzes (5-min at the start of class)
- Pre-Class Work (Completed in Desmos before class)
- In-Class Group Work (a daily write-up, completed during class)
- Homework (Due at the beginning of class, must be neat)
- Probability Exam and Modeling Exam (In-class exams)
- Consumer Math Portfolio (Completed as small investigations across the semester)
- Final Project (a group activity completed in the last few weeks of the course, culminating in a portfolio and video presentation)

A typical class period will be structured as we have displayed to the right. The quiz will be at the start of class, so you can't afford to be late. Homework from the previous class period will be handed in along with the quiz.

Most of the time will be devoted to exploring problems in small groups. Some days and topics will require more instruction than others, and some activities will be shorter than others.

At the end of the class period each group will hand in a write-up of their activities for the day. We will show an example of an activity write-up later in this class period.



How are grades determined?

Go to `student.desmos.com` and enter the code from the instructor.

The rest of today's class will be devoted to clarifying expectations for the homework and group activities.

Homework must be completed on time. It will be due at the beginning of class. No late work will be accepted (without prior permission).

- Write in full sentences—even when dealing with mathematical expressions.
- If you have scratch work, then box it off to the side so that it doesn't mix with the main work and conclusions.
- Staple all pages together.

Each day in class students will break into groups of 2 or 3 to work on problems as a group for 10–15 minutes at a time. Each group will complete a write-up of the group activities for the day that includes the names of each member of the group.

These write-ups will be graded very generously, but it is important to do neat work because the group activities are meant to prepare us for homework. If we don't practice doing neat work, then we will definitely fail to do neat work when exam time comes.

We will use the rest of class time to complete a group activity.

## Monty Hall Problem

### Monty Hall Problem

On a game show, the host shows the contestant three doors. Behind one of the doors is a prize, while nothing is behind the other two doors. The contestant chooses door 3.



Instead of opening up door 3, the host opens door 1 and shows the contestant that nothing is behind it.



He then gives the contestant the chance to switch his choice to door 2. Should the contestant switch?

Have them write down their answer and a short sentence describing their reasoning. Then take a class vote.

Do a few sample simulations to show them how it can be done. Warn the students that they will have only 10 minutes to complete the simulation and write up their conclusion. They don't need to do a great deal of writing in collecting the data. Just have the record whether the contestant won or lost.

#### Monty Hall Problem Simulation

15 min

Get in groups of 2. Write both of your names on a new sheet of paper, and follow the instructions below.

1. Simulate the game by playing 15 rounds. One player will be the host and will secretly pick a door. The other player will be the contestant, who will make a guess. The host will then cross off a door that doesn't have the prize. The player should stick with the original guess. Record the number of times the player won the prize.
2. Simulate the game for 15 more rounds where the player switches their guess each time. Record the number of times the player won the prize.
3. Decide whether the player should switch or not (or if it matters at all). Write a brief explanation of your conclusion.

Discuss the results and conclusions as a class. Carry on a debate if they are so inclined. Don't move on to the next slide before discussing the answer: You only have a  $1/3$  chance of winning if you stick with your initial guess, so you should always switch.

Here is a sample solution.

Bill S. Preston Esquire, Ted Theodore Logan

Monty Hall Problem Simulation

1. Outcomes for original guess: WLLLWLLLLWLLLLW  
The contestant won 5 times when sticking with the initial guess.
2. Outcomes for switching: LWWWWLWLLWLWLW  
The contestant won 9 times when switching from the initial guess.
3. It seems from the simulation that switching is the better choice. However, this seems unlikely, so we probably need to do more simulations. The difference is probably due to luck.

You can run several trials to collect data at <http://www.grand-illusions.com/simulator/montysim.htm>

Observe how the sample solution had the name of the students, and the name of the group activity. The work was written in complete sentences wherever possible.

The answer to part 3 was only semi-correct, but that doesn't matter at all. The answer did support the results from parts 1 and 2, and it showed that the students were thinking critically about the problem.

Class time is limited, so group activity time is time to get down to business with no delay. Most class periods will involve multiple group activities as well as some discussion and new content, so it is important that we use our time well.

You can use the same page for all of the group activities in one class period a given day. We encourage you to switch groups often to get to know your classmates and learn from them.

## Homework

## Homework

There is no homework to hand in at the beginning of class next time. However, you must complete each item below before the start of the next class.

1. Go to the class discussion board on Piazza (in Canvas) and introduce yourself on the thread titled "Introductions".
2. Bookmark the website <http://desmos.com> and play around with the calculator a little bit to get a feel for it.
3. Be sure to complete the pre-class activities before the beginning of the next class period.

Send out an announcement to remind students about the pre-class homework.

## 2 Percentages

### Consumer Math Portfolio Assignments

As part of the unit on Consumer Mathematics you will create a portfolio of written assignments by investigating various financial scenarios. The homework assignment at the end of each class period in the Consumer Math portion of the course is designated a Consumer Math Portfolio Assignment.

Unlike the other homework assignments, a Consumer Math Portfolio Assignment must be nicely typed, and it must be submitted to Canvas before the start of the next class period. The Consumer Math Portfolio will actually consist of 5 smaller assignments that will be graded separately, but you are encouraged to use the same theme and layout for each of the Consumer Math Portfolio Assignments.

### Reference Amounts

In the preview work, we reviewed the concept of percentages. The key formula for calculating with percentages is

$$\text{Percentage} = \frac{\text{Part}}{\text{Whole}} \times 100$$

This equation will continue to play an important role in our future calculations with percentages.

In 2012, a rain gauge in Boulder City recorded a total annual rainfall of 3.17 inches. In 2013, the same rain gauge recorded a total annual rainfall of 2.45 inches.

Channel 4 news reports that rainfall in 2013 was 23% lower than in 2012.

Channel 6 news reports that rainfall in 2012 was 29% higher than in 2013.

Can both reports be accurate? [Discuss!](#)

Discuss/debate this as a class before moving on.

When calculating a percentage, it is important to understand which amount is being [referenced](#) because this amount will be the [whole](#) used in calculating the percentage.

The actual difference in rainfall between the two years was 0.72 inches. If the rainfall from 2012 (3.17 inches) is the amount being referred to for comparison, then

$$\frac{0.72}{3.17} \times 100 \approx 23\%$$

and so 2013 did have 23% less rainfall *than* 2012.

But if the rainfall from 2013 (2.45 inches) is the amount being referred to for comparison, then

$$\frac{0.72}{2.45} \times 100 \approx 29\%$$

and so 2012 did have 29% more rainfall *than* 2013.

**Consecutive  
Percent-  
ages**

**Down and then Up**

**5 min**

Suppose your employer tells you that, due to a downturn in sales, your pay will be cut by 10% this month. If sales pick up next month, however, you will then receive a 10% raise. Discuss whether this scenario would result in your eventual salary being higher, lower, or the same as your original salary.

**Up and then Down**

Suppose your employer tells you that you've earned a 10% raise this month due to strong sales. If sales drop next month, however, your pay will then be cut by 10%. Discuss whether this scenario would result in your eventual salary being higher, lower, or the same as your original salary.

**Increase  
and  
Decrease**

In everyday conversation, you might refer to a pay increase or a pay reduction as simply a change in pay. The word **change** in common use does not indicate a direction for the change.

Mathematical language has a strong emphasis on precision. The ambiguity of a “directionless” change is not precise. For this reason, the words **increase** and **decrease** are often used instead to indicate which type of change took place. No one will confuse a 10% increase in pay with a 10% decrease in pay. If the word **change** is used, the direction of the change should be indicated by using positive and negative numbers to indicate the direction of the change.

For example, if Maury made \$100 last week and \$90 this week, we can say that his weekly earnings decreased by 10% or we can say that his weekly earnings changed by  $-10\%$ .

For the second activity, keep in mind that some students will likely not realize the reference amount changes each year due to the increase from the previous year.

**Changes in Pricing**

**10 min**

The cost of an unlocked iPhone 4s at the end of 2011 was \$649. At the end of 2012, the same iPhone 4s could be bought for \$549. Find the percent change in the cost of the iPhone 4s over that time period.

**Shifting References**

The average American spent \$8,508 on health care in 2011. The cost of health care is projected to rise about 5.8% per year for the years 2012–2022. Calculate the projected amount the average American will spend on health care in 2014.

*Hint: Note that the cost of health care in 2011, 2012, and 2013 all serve as reference amounts in this problem.*

**Homework**

**Consumer Math Portfolio Assignment 1**

1. Write up a 1-page personal bio-sketch that explains who you are and what your strengths are. Think of your audience as the class members, but be sure to write it in a tone that would impress a prospective employer. It might help to think about how someone might introduce you if you were the keynote speaker at an event. (If that doesn't help, think about what your friends might say at your funeral.) Be sure to include at least some details on your personal mathematics history.

Submit this assignment in Canvas. Remember to follow the guidelines on screen 1. (No hollow platitudes. By the end of the bio-sketch I should have a good feel for who you are.)

### 3 Interest

#### Simple Interest

#### Simple Interest

In the preview work, we introduced the idea of simple interest. This is interest that is calculated as a straightforward percentage of the principal annually. We can use the idea of simple interest to get a feel for how good a deal actually is.

A local rent-to-own store offers a 6-piece bedroom set for \$1,773.91. If a person is not able or does not wish to pay the full price upfront, the store offers a financing plan where you instead pay \$29.99 per week for 91 weeks. What is the annual simple interest rate this store is charging?

#### Simple Interest

$$\$29.99 \times 91 = \$2,729.09$$

This is the total cost of the set using the installment plan.

$$\$2,729.09 - \$1,773.91 = \$955.18$$

This is the amount of interest being charged.

$$\$955.18 = \$1,773.91 \times r \times \frac{91}{52}$$

This is the formula for simple interest, using 52 weeks in a year.

$$r \approx 0.308 = 30.8\%$$

The store is charging almost 31% annual interest!

#### Compound Interest

More commonly used than simple interest is **compound interest**. When interest is *compounded*, it is added to the principal. Then, the next time interest is calculated the principal is larger and so even more interest is charged. Compound interest means you earn (or are charged) interest on interest!

Example: Mary loans Phillip \$100, with the stipulation that he will owe 10% interest for each week he waits to pay it back. This is an example of simple interest. If he paid the loan back after 1 week, he'd owe  $\$100 + 10\% \times \$100 = \$100 + \$10 = \$110$ . If he paid the loan back after 2 weeks he'd owe,  $\$100 + 10\% \times \$100 + 10\% \times \$100 = \$100 + \$10 + \$10 = \$120$ .

If the interest were compounded, Phillip would still owe  $\$100 + 10\% \times \$100 = \$100 + \$10 = \$110$  after one week, but then the \$10 is added to the principal for a new principal of \$110. Therefore, after two weeks, Phillip would owe  $\$110 + 10\% \times \$110 = \$110 + \$11 = \$121$ . This \$121 would then be the principal for the 3<sup>rd</sup> week.



## Computing Compound Interest

5–7 min

Find how much Phillip would have to pay back if he didn't pay until after the following number of weeks using each type of interest.

	Compound Interest (10%)	Simple Interest (10%)
After 1 week	\$110	\$110
After 2 weeks	\$121	\$120
After 3 weeks		
After 4 weeks		
After 5 weeks		

Don't move to the next slide until they have worked on this problem.

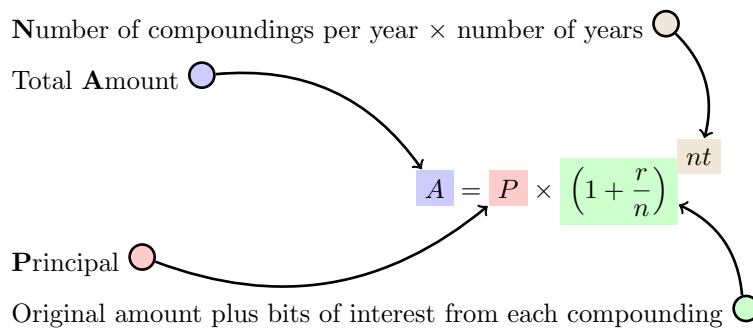
## Computing Compound Interest

Find how much Phillip would have to pay back if he didn't pay until after the following number of weeks using each type of interest.

	Compound Interest (10%)	Simple Interest (10%)
After 1 week	\$110	\$110
After 2 weeks	\$121	\$120
After 3 weeks	\$133.10	\$130
After 4 weeks	\$146.41	\$140
After 5 weeks	\$161.05	\$150

Warn them that the next formula is a bit intimidating. They don't have to memorize it, but they do have to be able to use it.

There *is* a specific formula for computing compound interest all at once rather than in multiple steps. Because calculations involving compound interest are typically done with years, rather than weeks or months, as the basic unit of time, the explanation for the equation is in terms of years.



Go to `desmos.com` and enter  $A = P(1 + \frac{r}{n})^{nt}$  and create sliders. Show the students how to enter values. They don't have to use `desmos.com`, but it's more fun that way.

Have the students make some guesses as a class before turning them loose to work it out.

### The Power of Compound Interest

10 min

When Benjamin Franklin died in 1790, his will left £1000 (British currency known as pounds) each to the cities of Boston and Philadelphia. Each city was to establish an account that would generate interest for 200 years, at which time the cities could withdraw the money to use for public works projects. If the Boston account compounded monthly at 4% and the Philadelphia account compounded monthly at 3.5%, how much more (in pounds) would Boston have at the end of the 200 years than Philadelphia? Write down a guess before calculating an exact result.

$$A = P \times \left(1 + \frac{r}{n}\right)^{nt}$$

**APY** One type of simple interest that is used widely is something called the annual percentage yield (APY).

An annual percentage yield is a hypothetical interest rate that *would have earned* the same amount as a lesser compound interest rate.

Show an example of how to set the slider to go up to 365 in desmos.

### APY

### Computing APY

5 min

A bank offers a CD (a type of investment account) that pays 2.27% interest compounded daily. The bank also lists this account has having an annual percentage yield of 2.30%.

Compute the compound interest earned on a one-year \$100 investment with the rate 2.27%, compounded daily, using the formula  $A = P \times \left(1 + \frac{r}{n}\right)^{nt}$ .

Compute the simple interest earned on a one-year \$100 investment with the annual rate 2.30% using the formula  $I = P \times r \times t$ .

*Hint: Remember that  $A$  and  $I$  represent different things.*

## Homework

### Consumer Math Portfolio Assignment 2

1. Visit the site [http://www.census.gov/govs/local/historical\\_data.html](http://www.census.gov/govs/local/historical_data.html) and select a year of historical finance data. Within the data file, choose a government (federal or a specific state) and locate the amount of money spent by that government on debt interest for that year.
  - (a) Use [bankrate.com](http://www.bankrate.com) to locate a long-term investment account (such as a CD, savings account, money market account, or bonds) paying a high rate of interest and calculate how much money the state or federal government would have earned to date if the money paid for debt interest was instead invested to earn interest in that account.
  - (b) Write a report of your work and your findings as if you were going to present the results to government officials. Include a paragraph on something you learned or thought about because of this exercise.
2. Planning for retirement can be daunting. In 2012, Fidelity Investments suggested<sup>1</sup> a person will need to save  $8\times$  their [final](#) salary before retirement to live comfortably post-retirement!
  - (a) Use the retirement calculator at <http://www.bankrate.com/calculators/retirement/roi-calculator.aspx> to explore several investment scenarios. Find a table of actual historical return rates on various types of investment, such as [http://pages.stern.nyu.edu/~adamodar/New\\_Home\\_Page/datafile/histretSP.html](http://pages.stern.nyu.edu/~adamodar/New_Home_Page/datafile/histretSP.html) and compute three scenarios for investing that seem reasonable with your own current and projected future finances.
  - (b) Keep a log of your calculations and write a summary of the results as well as a paragraph about what you learned or thought about as a result of this exercise.

Quiz could be a simple interest calculation for 2 years and the corresponding compound interest calculation for 2 years

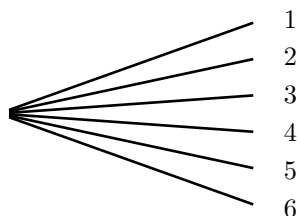
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<sup>1</sup><http://usatoday30.usatoday.com/money/business/story/2012/09/12/fidelity-issues-new-retirement-savings-guidelines/57756922/1>

## 4 Introduction to Probability and Statistics

### Probability Trees

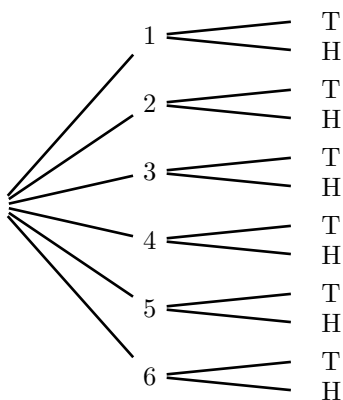
A *probability tree* is a visual representation of the events of an experiment. The following picture is a probability tree for rolling a standard 6-sided die:



When used with a single event, probability trees seem rather useless. We might as well just think of the sample space (sample space = the set of possible outcomes) as a list.

However, they can help to organize more complex information such as sequential events.

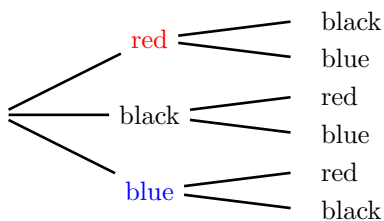
The following probability tree shows the outcomes when rolling a die followed by flipping a coin.



Notice that the tree gives an intuitive flow from left to right for sequential events and that the tree can help us list the entire sample space by listing the branch/sub-branch combinations.

The power of the probability tree is that not all events at deeper levels need to be identical.

Consider an urn containing a red ball, a black ball, and a blue ball. The following probability tree describes the possibilities when drawing two balls without replacing the first one.



Probability trees can handle even more complex situations.

## Probability Trees

10 min

Draw a probability tree that fits the following situation:

- First, flip a coin.
- If the coin comes up heads, draw a ball from an urn containing a white ball and a black ball.
- If the coin comes up tails, roll a standard 6-sided die.

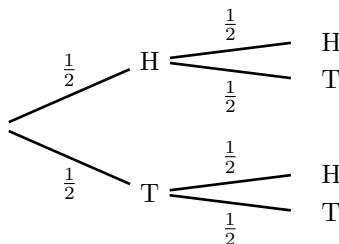
When all the groups have completed this activity ask one of them to display their solution.

### The Product Rule and Addition Rule

You might be wondering, “Where is the probability?”

The probability is hidden in the lines connecting the various events. What we normally do is put the probability of the event above or below the line.

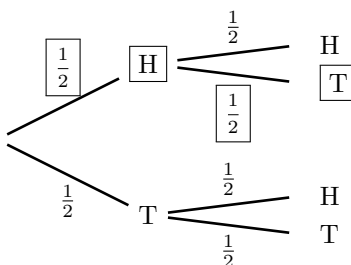
For example, the following is the probability tree for back-to-back coin flips.



Notice the probabilities at each level of branches adds to 1. This must always happen.

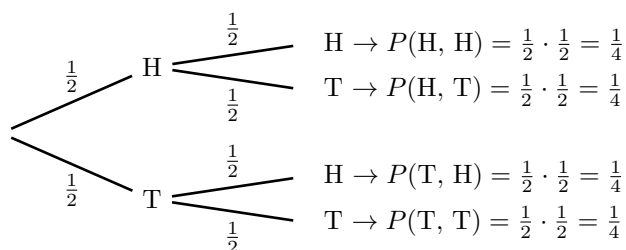
What’s the probability of flipping a head followed by a tail?

To determine the probability of a specific set of sequential events, you multiply the probabilities listed on the path taken in the probability tree. This is a special case of a more general rule known as the *product rule* for probabilities.



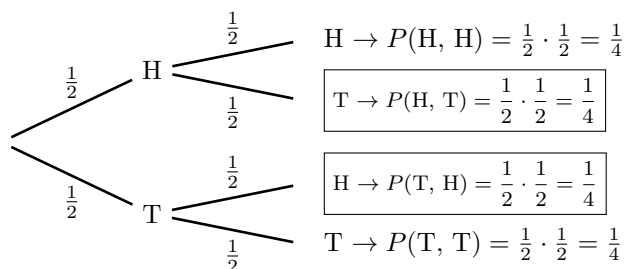
$$P(H, T) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

We can do this for each possibility and list the probabilities at the end of each path.



What is the probability of flipping one head and one tail?

Notice that this question is different from the previous one. There is only one path that is a head followed by a tail, but there are two paths that result in one head and one tail.



To combine the probability of different paths, we add the probabilities together. This is a special case of the *addition rule* for probabilities.

$$P(\text{one head and one tail}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Review the product rule and addition rule for probability. Multiply probabilities along branches of a tree (for sequential events). Add the probabilities of mutually exclusive outcomes to get the total probability.

**The Product Rule and Addition Rule**

**10 min**

Coins *A* and *B* are biased in the following manner:

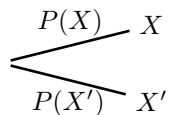
- Coin A:  $P(H) = \frac{2}{3}$ ,  $P(T) = \frac{1}{3}$
- Coin B:  $P(H) = \frac{1}{4}$ ,  $P(T) = \frac{3}{4}$

Do the following:

- Draw the probability tree for flipping coin *A* followed by coin *B*, including the probabilities at each step.
- Determine the probability of each branch.
- Determine the probability of flipping one head and one tail.
- Determine the probability of flipping the same side twice.

**Complementary Events** Any event  $X$  can either **happen** or it can **not happen**. The event that is  $X$  **not happening** is called the *complement* of  $X$  and we write it as  $X'$  (read as “ $X$  prime”).

We can draw a probability tree for this.



Since we know something happens 100% of the time, we have the following relationship:

$$P(X) + P(X') = 1.$$

This can be rewritten as

$$P(X') = 1 - P(X).$$

The complement rule only applies to either-or situations. The following are some examples of this:

- If a team won  $\frac{3}{4}$  of their games and there were no ties, then they lost  $\frac{1}{4}$  of their games.
- If a student got 90% of the multiple choice problems on a test correct, then the student got 10% of the problems wrong.

#### Complementary Events

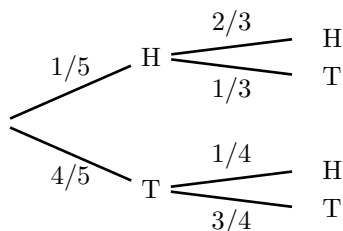
10 min

Write down 6 sets of complementary events and make up possible probabilities for the event and the complement. Use fractions for 2 of them, decimals for 2 of them, and percents for 2 of them.

## Homework

### Homework 1

1. For a certain biased coin, the probability of flipping a head is  $\frac{2}{7}$ . What is the probability of flipping a tail?
2. A ball is drawn from an urn containing 3 white balls, 6 yellow balls, and 1 blue ball. What is the probability of drawing each color of ball?
3. The following experiment is performed: First, a standard 6-sided die is rolled. After that, a ball is drawn from an urn containing 2 red balls and 3 green balls.
  - Draw a probability tree for this experiment (including the probabilities).
  - Determine the sample space for this experiment.
  - Determine the probability of each possible outcome.
  - Determine the probability that an even number will be rolled, followed by a red ball being drawn.
4. Urn 1 contains 4 red balls and 2 green balls. Urn 2 contains 5 red balls and 3 green balls. An experiment is performed in which a person draws a ball from urn 1 followed by a ball from urn 2.
  - Draw a probability tree for the experiment.
  - Determine the probability of each outcome.
  - Determine the probability of drawing two balls of the same color.
  - Determine the probability of drawing two balls of different colors. Use two different methods to get this value.
5. Consider the probability tree below for flipping some biased coins:



- Determine the probability of each outcome.
- Determine the probability of getting one head and one tail.
- Determine the probability of getting *at least* one tail.



## 5 Independent Events and the Multiplication Rule

### With or Without Replacement

A common framework for thinking about independent and dependent events is to draw balls from an urn with or without replacement. Putting the ball back into the urn means that every new draw will have the same probabilities as the original one. But if you do not replace the ball, the probabilities change.

For example, consider an urn with one red ball and one black ball. If you draw two balls without replacement, you will always have one red ball and one black ball. But if you draw with replacement, it's possible to draw the red ball both times.



### Balls and Urns - Part 1

10 min

Suppose you have an urn that contains 3 red balls and 7 black balls. Using your intuition, record your answers to the following questions.

- Are you more likely to draw two red balls if you draw with replacement or if you draw without replacement?
- Are you more likely to draw two black balls if you draw with replacement or if you draw without replacement?
- Are you more likely to draw one red ball and one black ball if you draw with replacement or if you draw without replacement?



Warn them that they will find the exact answers in a minute, but for now they will just make an intuitive guess.

### Balls and Urns - Part 2

10 min

Suppose you have an urn that contains 3 red balls and 7 black balls.

1. Create a probability tree for drawing two balls with replacement and a tree for drawing two balls without replacement.
2. Determine the probability of drawing two red balls in each situation. Was your guess correct?
3. Determine the probability of drawing two black balls in each situation. Was your guess correct?
4. Determine the probability of drawing one red ball and one black ball in each situation. Was your guess correct?



## The Birthday Paradox

If you pick a random person somewhere on the planet, how likely is it that this person will have the same birthday (month and day) as you do?

In order to answer this question, we will frame the question using probabilistic language. We can think about this as an experiment in which we're drawing balls from an urn, and the balls are labeled with all possible birthdays. Drawing a ball at random is similar to the randomness of the person's birthday.

For now, we will ignore February 29 as a possibility, but we'll come back to it later.

FYI: "Based on Harvard University research of birth records in the United States between 1973 and 1999, September 16 is the most common birthday in the United States and December 25 the least common birthday (other than February 29, because of leap years)." <http://en.wikipedia.org/wiki/Birthday>

Since there are 365 balls in the urn and only one of the balls has your birthday on it, we have that

$$P(\text{drawing your birthday}) = \frac{1}{365} \approx 0.27\%$$

This is a very unlikely!

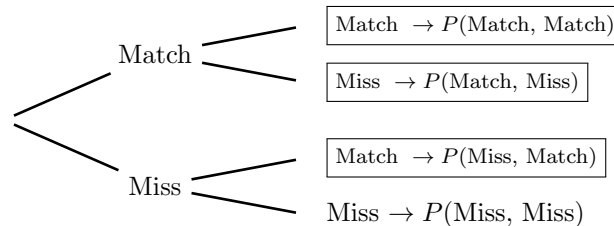
If you picked two people, how likely is it that at least one of them will have the same birthday as you?

- Frame this question in terms of balls and urns. What would you do if you actually wanted to conduct an experiment to answer this question?

Emphasize that these are drawn with replacement.

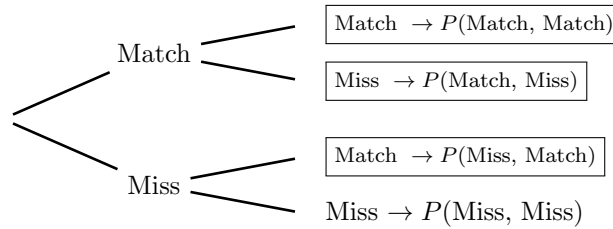
The experiment is that you would be drawing two balls with replacement, and checking whether either ball has your birthday on it.

We can put this into a probability tree and look to see which branches result in a match:



If we compute the probability directly, we would need to add up all three outcomes that have at least one match.

However, notice that there is only one path that doesn't have a match. We can use complements to make the calculation easier.



$$P(\text{At least one match}) = 1 - P(\text{Miss, Miss})$$

Since we are doing this experiment with replacement, the probability of missing doesn't change from one draw to the next. This means that we can use the multiplication rule to get

$$\begin{aligned} P(\text{Miss, Miss}) &= P(\text{Miss}) \cdot P(\text{Miss}) \\ &= P(\text{Miss})^2 \end{aligned}$$

Since we know that  $P(\text{Miss}) = \frac{364}{365}$ , we can very easily do this calculation:

$$\begin{aligned} P(\text{At least one match}) &= 1 - P(\text{Miss, Miss}) \\ &= 1 - P(\text{Miss})^2 \\ &= 1 - \left(\frac{364}{365}\right)^2 \\ &= 0.55\% \end{aligned}$$

This is still very unlikely!

What happens if we pick three people?

### Sharing Your Birthday

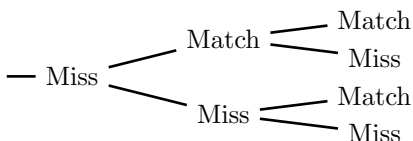
10 min

1. Create a tree that shows all of the possibilities for picking three random people and determine which branches lead to at least one person sharing your birthday.
2. Use the property of complements and the multiplication rule to compute the probability that at least one person out of three will share your birthday.
3. Look for a pattern that you can use to compute the probability that at least one person out of four will share your birthday.
4. How many people would you need to pick to have a 50% chance of at least one of them sharing your birthday?

We will now take the last question and see what happens if we change the language slightly: How many people would you need to pick to have a 50% chance that someone will have a shared birthday?

As before, we will frame this problem in terms of balls and urns. But this time, “miss” and “match” will mean missing or matching *any* of the previous draws, not some specific draw.

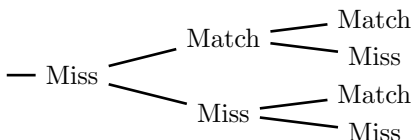
The start of the probability tree is different this time, because after one draw there’s only one outcome because there’s nothing to match. The following is the probability tree for three draws:



### The Birthday Paradox

10 min

1. Fill in the probabilities for the probability tree below. Note that “Miss” and “Match” mean that it misses *all* previously drawn birthdays or matches *some* previously drawn birthday.
  2. Determine the probability of at least two people sharing a birthday out of three people.
  3. Based on the change from the previous example, take a guess as to the number of people it would take to have a 50% chance that two people would share a birthday.
- The answer to #3 is surprisingly small, which is why this problem is known as the birthday paradox. You will do this calculation for homework. The key branch is the Miss-Miss-Miss branch!



**Coincidences** A *coincidence* is when two or more unusual events happen at the same time when there’s no connection between them. Because our minds tend to draw relevance between events when this happens, we sometimes read more facts into coincidences than are really there.

For example, a man might meet a woman who happens to have the same birthday. Starting from that commonality, they build a relationship and eventually marry. Some people might say that “fate” brought them together.

But even though the chances of two people sharing the same birthday is about 0.27%, the fact that there are billions of married people in the world means the chances that some couple would have the same birthday is very high.

The difficulty in interpreting events like this is that this calculation does not show that “fate” did not have a role bring this *specific* couple together. All it says is that if everything happened randomly, we would expect *some* couples to share a birthday.

The distinction between a *specific* couple and *some* couple is similar to the distinction in the birthday problems. In one case, we have a *specific* person whose birthday we’re trying to match. In the other, we simply said that *some* person’s birthday matches another’s in the collection of people.

Failing to specify certain details in advance, like *some* instead of *specific*, can dramatically change the probability calculations and can lead people to believe certain events are less likely to occur than their intuition tells them, and because of that they might incorrectly attribute an underlying reason to the relationship between the two events rather than a chance relationship between them.

### Coincidences

10 min

1. Determine the probability of flipping heads 10 times in a row in 10 flips.
2. Suppose every NSC student flipped a coin 10 times in a row. How many of them would you expect to get all heads? (There are approximately 3300 NSC students.)

1. In class, you estimated the number of people it would take to have a 50% chance of someone having a shared birthday. This was called the birthday paradox problem because the number of people is surprisingly small. We will do this calculation explicitly by taking advantage of complements. The complement of “somebody shares a birthday” is “nobody shares a birthday.” This is useful for simplifying calculations.
  - (a) If the probability that “nobody shares a birthday” is 20%, what is the probability that “somebody shares a birthday”? If the probability that “nobody shares a birthday” is  $x$ , what is the probability that “somebody shares a birthday”?

For the next set of problems, ignore February 29.

- (a) Given two people, determine the probability that nobody shares a birthday. (Hint: The first person can have any birthday, but the second person can only have one of 364 birthdays. Why?)
  - (b) Given three people, determine the probability that nobody shares a birthday.
  - (c) Given four people, determine the probability that nobody shares a birthday.
  - (d) Extend the pattern and determine the number of people there must be in order for there to be less than a 50% chance that nobody shares a birthday.
  - (e) Explain why this corresponds to there being a greater than a 50% chance that somebody shares a birthday.
2. (Optional) For the birthday calculations we did in class, we ignored February 29.
  - (a) At first, it may seem that to include February 29, we would simply increase the number of days to 366. This turns out to be incorrect. Why? (Hint: How often does February 29 occur?)
  - (b) Instead of working in one-year cycles, leap years compel us to work in 4-year cycles. In a 4-year cycle, how many balls would be labeled “January 1”? How many would be labeled “February 29”? How many balls would we need in the urn?
  - (c) Determine the number of draws it would take to have a 50% chance of drawing February 29.
  - (d) This calculation still isn’t quite right because every 100 years we skip a leap year. Would including this fact increase or decrease the number of draws it would take to have a 50% chance of drawing February 29? (Do not calculate this. Just think!)
    - Note: This still isn’t completely accurate because every 400 years, we don’t skip the 100 year leap year!
3. (Optional) Watch the video at <http://youtu.be/jY6F49oRiJM>. For the following problems, assume you have a fair coin.
  - (a) What is the probability of flipping 5 tails in a row in 5 flips?
  - (b) If two people flip a coin 5 times in a row, what is the probability that at least one of them will get 5 tails in a row?
  - (c) How many people would it take to have a 75% chance that someone would flip 5 tails in a row?
  - (d) How many people would it take to have a 75% chance that two people would have the same sequence (same results in same order) of 5 coin flips?

## 6 Conditional Probability

### Conditional Probability

In the pre-class work, we used conditional probability as a probability calculation in which there is a restriction on the sample space. This is the formula.

$$P(X|Y) = \frac{\# \text{ of successes for } X \text{ where } Y \text{ also happens}}{\# \text{ of possibilities where } Y \text{ happens}}$$

In the video, we saw that if we know a man has two children and at least one of them is a boy, then the probability the other one is a boy is  $\frac{1}{3}$  instead of  $\frac{1}{2}$ .

This was computed by looking at the sample space for two children and removing the combination that had no boys.

Boy-Boy	Boy-Girl
Girl-Boy	Girl-Girl

Knowing that there is at least one boy leads to two combinations where the other sibling is a girl and only one where the other sibling is a boy.

We will practice some calculations involving conditional probabilities using sample spaces.

Point out that if we are told that the **oldest** child is a boy, then the probability that the other one is a boy is  $\frac{1}{2}$ .

#### Conditional Probabilities 1

10 min

For each of the situations below, write out the entire sample space, remove portions based on the condition, then compute the probability.

1. When rolling two 6-sided dice, determine the probability of rolling a 5 if the total is 9.
2. When rolling two 6-sided dice, determine the probability of rolling a 1 or a 6 if the total is at least 7.
3. When rolling a 4-sided die and a 6-sided die together, determine the probability of rolling a 5 given that one of the dice is a 3.

The method of counting the elements in a sample space works when the events are equiprobable. But when this is not the case, we need to be a little more sophisticated in our approach.

For these situations, we use a more general formula for computing conditional probabilities:

$$P(X|Y) = \frac{P(X \text{ when } Y \text{ also happens})}{P(Y)}$$

We will be using probability trees to organize our information. Just as before, we may need to add up multiple paths to get the full probability.

**Conditional Probabilities 2****10 min**

For the following problem, assume that you have a biased coin and  $P(H) = \frac{1}{3}$ .

1. Draw the probability tree for two coin tosses and compute the probability of each outcome.
  2. Determine the probability of flipping at least one head.
  3. Determine the probability of flipping one head and one tail.
  4. Determine the probability of flipping one tail given that at least one head was flipped.
- Note: This is  $P(X|Y)$  where  $X$  = “Flip one tail” and  $Y$  = “Flip at least one head.”

**The False Positive Paradox**

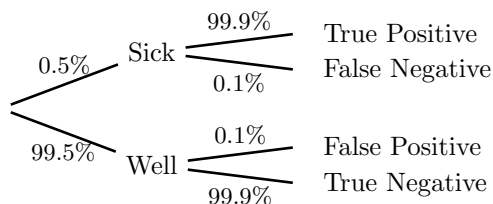
Suppose you show up to a walk-in clinic to be tested for a very rare disease (only 0.5% of the population has it) and the test is 99.9% accurate. This means that if you have the disease the test will correctly diagnose you 99.9% of the time, and if you don't have the disease the test will correctly diagnose you 99.9% of the time.

The test comes back positive. What is the probability that you have the disease?

- 99%
- 99.9%
- 99.99%
- Better than 99.99%
- Worse than 99%

This problem can be answered using conditional probability.

We begin with a probability tree. The first branch will correspond to whether the person has the illness (sick or well) and the second branch will correspond to the test result (positive or negative).



Notice that some people who are sick will come back with a negative test. This is known as a **false negative**. There are also some people who are well whose test will come back positive. This is called a **false positive**.



**False Positive Paradox****10 min**

Using the probability tree below, what is the probability that you have the disease if you test positive?

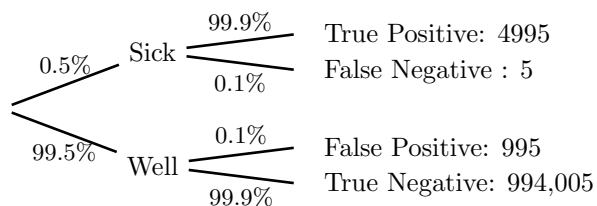
- Determine the probability of receiving a positive diagnosis.
- Determine the probability that you are sick and that you get a positive diagnosis.
- Determine the probability of being sick given that you received a positive diagnosis.

$$P(X|Y) = \frac{P(X \text{ when } Y \text{ also happens})}{P(Y)}$$



The underlying intuition of the false positive paradox is that if the disease is very rare, then the test will result in a large number of false positives. Instead of using percents, we will look at concrete numbers.

Suppose there are 1,000,000 people in the population and all get tested. The probability tree below shows the number of people in each category.



As you can see, nearly 1,000 people will receive false positives. These false positives make it so that a random person testing positive is less likely to have the disease than the accuracy rate of the test would lead us to believe.

This calculation is part of the reason why doctors don't regularly test everyone for rare diseases all the time. If we blindly test people, we create large numbers of false positives that can lower the overall effectiveness of the test and create unnecessary stress for patients and their families.

Instead of testing blindly, doctors look for other risk factors before administering a test for a rare disease. In a sense, they run multiple tests and only conclude that a person has the disease if all or most of the tests come back positive.

1. Suppose a family has three children and at least one of them is a boy. Hint: Three children give rise to eight possibilities,  $\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$ .
  - What is the probability that at least one of the other children is a girl?
  - What is the probability that at least one of the other children is a boy?
  - What is the probability that both other children are girls?
  - What is the probability that both other children are boys?
2. Suppose there are two urns. One urn contains two black balls and the other contains one white ball and one black ball, but you don't know which is which. You walk over to an urn and pull out one ball. It's black. What is the probability that the other urn contains two black balls?
3. To mitigate the false positive paradox for a rare disease, a doctor implements a cheap pre-screening test that is 90% accurate (for both positive and negative results). The doctor only runs the second test if the first test comes back positive. The second test is 99.9% accurate (for both positive and negative results). Only 0.5% of the population has the disease.
  - What is the probability that someone who tests positive for the disease with the pre-screening test actually has the disease?
  - What is the probability that someone has the disease if the pre-screening comes back positive but the follow-up test is negative?
  - What is the probability that someone has the disease if both tests come back positive?

Hint: Construct a probability tree with three levels: one level for sick/well, the next level for pre-screening test positive/negative, and the final level for second test positive/negative.

## 7 Expected Value

In the pre-class work, we used the expected value formula to determine how much we would expect to win or lose on average when playing a probabilistic game:

$$E[X] = \sum P(X_i) \cdot V(X_i)$$

where

- $X_i$  is a notation that represents each possible outcome of the game  $X$ .
- $P(X_i)$  is the probability of the event  $X_i$ .
- The  $V(X_i)$  is the *value* of the event  $X_i$ .

We used this formula with probability trees to compute the expected value of some simple games.

**Roulette** In casino games, the player needs to pay money up front in order to play. This can sometimes cause people to misinterpret events.

For example, in the game of roulette the players have an opportunity to bet on red or black. There are 18 red numbers, 18 black numbers, and (usually) 2 green numbers. If you bet on red or black, you put a chip in the appropriate area. If you win, you get two chips pushed back to you. However, you have only won one chip because you paid one chip up front and it's just getting pushed back to you.

When doing expected value calculations, you need to pay attention to this because you care about your *net* winnings (winnings after accounting for anything you paid up front to play). The money that is pushed towards you is called your *gross* winnings. Both concepts are useful, so you will need to pay attention to the distinctions.

The words net and gross are also used in finance with the same meaning as usual. Your net pay is the actual amount of money you have after taxes, deductions, and other expenses. Your gross pay is the total amount of money you earned.

Payout odds in a casino are usually stated in the form  $A:B$  (read “ $A$  to  $B$ ”), which means that you win  $A$  dollars net when you wager  $B$  dollars.

In roulette, the payout odds for a bet on red or black is 1:1. This means that if you wager \$1 on red and a red number is spun, then you win \$1 (net). These values scale, so that if you wagered \$50 on red and red is spun, then you win \$50 (net). If you wager on a specific number, the payout is 35:1, which means that if you wager \$1, you would win \$35 (net).

The payouts are not always relative to a \$1 bet. In blackjack, casinos may pay 3:2 or 6:5, depending on the specific game you're playing.

- Which is a better payout, 3:2 or 6:5?

## Roulette Calculations

10 min

A roulette wheel consists of 18 red numbers, 18 black numbers, and 2 green numbers. Each number is equiprobable.

1. Betting on red pays 1:1. Determine the expected value of betting \$1 on red.
2. Betting on green pays 17:1. Determine the expected value of betting \$5 on green.
3. Betting on any specific number pays 35:1. Determine the expected value of betting \$25 on a specific number.

	00	3	6	9	12	15	18	21	24	27	30	33	36
	2	5	8	11	14	17	20	23	26	29	32	35	
0	1	4	7	10	13	16	19	22	25	28	31	34	

## Multi-Step Games

Some games involve multiple rounds of events. For example, a modern slot machine will sometimes have a regular payout, but it will sometimes also go into a bonus game in which you have an additional round of payouts.

We can keep track of these multiple rounds by using a probability tree. The probabilities are computed in exactly the same manner as before, and we simply use the payouts at the end of each branch to determine the value of each outcome.

## Multi-Step Games

10 min

A gambling game is constructed in the following manner. One ball is chosen at random from an urn that contains one red ball, one white ball, and one blue ball. If it's the red ball, the player loses \$20. If it's the white ball, the player flips a coin; heads wins \$10 and tails loses \$10. If it's the blue ball, the player rolls a 4-sided die and wins \$5 times the number of the die (so rolling a 2 wins \$10).

1. Draw the probability tree for this game and compute the probabilities for each outcome.
2. Determine the value of each outcome.
3. Determine the expected value of this game.

## House Edge

Casino games virtually always have a negative expected value. An important quantity to know is the *house edge* for a game. The house edge is stated as a percent and represents the percent of the wager that the house wins on average.

Because the house edge is relative to the initial wager, it is useful to think of the game in multiple steps, where the first step is the wager and each step afterwards is treated as a gross value instead of a net value. For example, when betting on red, it's best to think of the first step as losing \$1, and then winning \$2 if it comes up red and winning \$0 otherwise. The probability tree below displays this.

$$\begin{array}{rcl}
 \begin{array}{l} -\$1 \\ \hline \end{array} \text{Wager on Red} & \begin{array}{l} +\$2 \\ \hline +\$0 \end{array} & \begin{array}{l} \text{Red: Net \$1} \\ \text{Black/Green: Net } -\$1 \end{array}
 \end{array}$$

This setup is also useful for games that are more complex because it allows you to break it down into smaller steps.

We computed earlier that the expected value of a \$1 bet on red is about -\$0.053. Since the house wins all the money you lose, the house wins \$0.053. To get the percent, we divide the house winnings by the original wager.

$$\text{House Edge} \approx \frac{0.053}{1} = 0.053 = 5.3\%$$

We also computed that the expected value of a \$5 bet on green is about -\$0.263. This would give

$$\text{House Edge} \approx \frac{0.263}{5} \approx 0.053 = 5.3\%$$

It turns out that betting on red, black, or green all have the exact same house edge. If you bet on red, you win more often, but you win less when you win. If you bet on green, you don't win very often, but when you win you win a lot more. But in the end, it averages out to be the same.

Not all games are like this. For example, the house edge on craps varies dramatically depending on the type of bet you place.

The size of the house edge is used to determine how good or bad a game is for the player. Players prefer small house edges, but casinos prefer large ones.

- The house edge for slot machines usually run between 2% and 15%, depending on the game.
- The house edge for Keno is between 25% and 29%, depending on the wager.
- The house edge for blackjack is about 0.5%, as long as you make good decisions.

Computing the house edge for blackjack is extremely complicated because it depends on the decisions that the player makes, the type of blackjack payout (3:2 or 6:5), the number of decks, and several other factors. For very complex games like this, computers are used to simulate player decisions and estimate the house edge.

### The House Edge

10 min

Craps is played with two six-sided dice. It is possible to make wagers on the outcome of a single roll (called proposition bets).

- Betting on any 7 pays 4:1. Determine the house edge on this wager.
- Betting on craps (any 2, 3, or 12) pays 7:1. Determine the house edge on this wager.
- What would the house edge be if the payout odds for craps were 7.5:1 instead of 7:1? (As they are in some places.)

1. Determine the amount of money won given the initial wager and the payout odds.
  - (a) \$5 wager paying out 2 : 1
  - (b) \$10 wager paying out 4 : 1
  - (c) \$20 wager paying out 5 : 2
  - (d) \$50 wager paying out 6 : 5
  - (e) \$90 wager paying out 8 : 3
2. A game is played with a 6-sided die. If the numbers 1 or 2 are rolled, the player loses \$10. If the numbers 3 or 4 are rolled, the player wins \$5. If the numbers 5 or 6 are rolled, the player wins \$7. What is the expected value of this game?
3. A coin is flipped three times. If the player gets at least two heads in a row, he wins \$25. If the player gets three tails in a row, he wins \$50. Otherwise, he loses \$20. Determine the expected value of this game.
4. A player pays \$5 to play a game. In this game, he has a 50% chance of losing all of his money, a 20% chance of a *gross* winning of \$1, a 20% chance of a *gross* winning of \$5, and a 10% chance of a *gross* winning of \$25. Determine the expected value of this game.
5. A roulette wheel has the numbers 1–36, plus a 0 and a 00 for a total of 38 numbers. There are many bets a player may make.
  - (a) Betting on 1–12 wins whenever the number is 1–12. This bet pays 2 : 1.
    - i. What is the expected value of a \$50 wager?
    - ii. What is the house edge?
  - (b) A line bet means betting that any one of three specific numbers will come up. This pays 11 : 1.
    - i. What is the expected value of a \$75 wager?
    - ii. What is the house edge?
6. A simplified slot machine is set with the following rules:
  - 90% of the time, the player loses
  - 9% of the time, the machine pays 1 : 1
  - 1% of the time, the player enters a bonus round. In the bonus round, 1% of the time it pays 1000 : 1, and the rest of the time it pays 5 : 1.

Determine the house edge.

## 8 Introduction to Statistics

**Samples** In the pre-class materials, we talked about three special values, known as measures of central tendency.

- Mean: Add the values and divide by the number of data points.
- Median: This is the middle value when the numbers are arranged in order
- Mode: This is the most common value.

Each of these values describes some feature about the distribution of the data.

We used a model of drawing numbered balls from an urn without knowing how many balls are in the urn and what numbers are on the balls as a way to introduce the idea of statistical sampling.

You probably wondered why we couldn't just dump out all of the balls from the urn and count them up to see what was inside. The reason is that in applications of statistics, it's rarely possible to get complete information on the population. For example, to find the average height of a person, we would need to measure the height of every person on the planet, and then take the average. This is a logistical impossibility.

(The urn would represent the world, the balls would represent people, and the numbers on the balls would represent their heights.)

Instead, we need to take a *sample* of the population and use the information we gathered there to try to draw conclusions about the larger population from that. We always need to be careful when trying to *infer* information about a population from a sample because there are many types of errors that can be made.

This is the chart that was used in the pre-class materials:

Number	Frequency
7	2
8	5
9	2
10	1

We clearly see that 8 is the most common number in the sample. But does this mean that 8 is the most common number in the urn? Not necessarily.

- Name some reasons why 8 might not be the most common number in the urn.
- If you were to take a new sample from the urn, what actions could you take to account for the issues just discussed?

These simple ideas may appear obvious, but in the 1970 draft lottery (for the Vietnam War), people born in the month of December were more likely to be drafted because of a failure to mix the balls.

In more advanced courses in statistics, you can learn about different techniques that are used to reduce the effects of these types of biases. The goal is to have a *representative* sample, meaning that the distribution in the sample reflects the distribution of the larger population.

### Statistical Sampling 1

10 min

1. Gather data on the age of all the people in the classroom.
2. Determine the mean, median, and mode of the data.
3. Explain why this is probably not a representative sample of the ages of people living in the state of Nevada.

## Coin Flip Frequencies

Instead of making lists, we can also use a frequency table to keep track of the data. This is especially useful when the same numbers appear repeatedly. This tends to be much more compact and easier to work with. The following is the frequency table from the pre-class materials.

Number	Frequency
7	2
8	5
9	2
10	1

This table says that the number 7 was drawn 2 times, the number 8 was drawn 5 times, and so on.

Flipping a coin is another example of taking a statistical sample. The entire population would be the collection of all results from all coin flips, perhaps even all future coin flips.

### Statistical Sampling 2

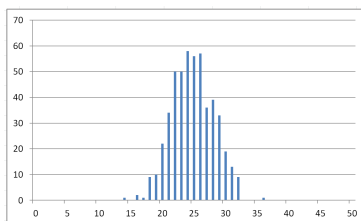
10 min

Everyone will contribute to the data by flipping a coin 10 times. (Catch the coin in the air!) You will report the number of heads that are flipped.

1. Create a frequency table for the coin-flipping data collected by the class.
2. Determine the mean, median, and mode of the data.

As the amount of data gets larger, frequency tables can become hard to read. For such data sets we often use a graphical representation to help us understand the data.

The following chart was constructed using 500 sets of 50 simulated coin tosses. The horizontal axis shows the number of heads flipped, out of 50, and the height of each bar is the frequency.

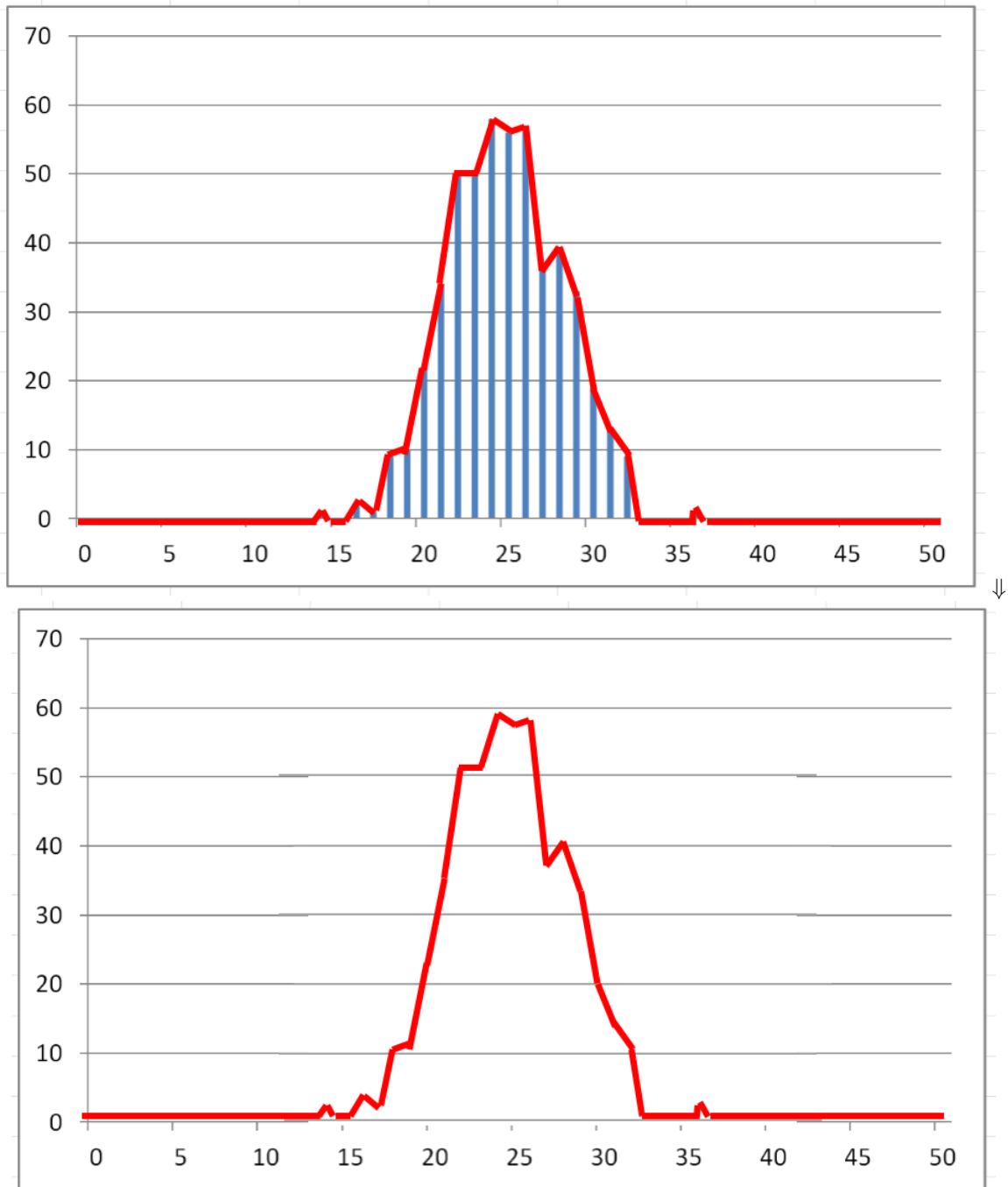


We can clearly see that the data peaks in the middle and tapers off on both sides. Shapes like this are referred to as normal distributions.



## Distributions

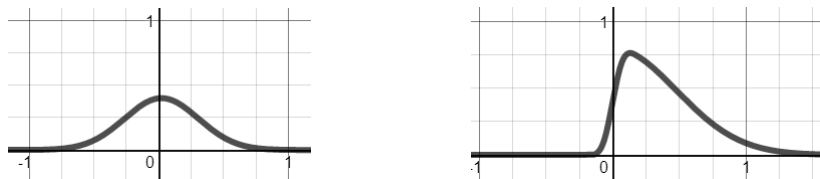
Often, we represent distributions by drawing just the outline of the bars, rather than the bars themselves. Thus, our simulation of 500 sets of 50 coin flips could be represented like this:



**Distribution Shapes** The normal distribution (also called the bell-shaped curve) is one of the most important statistical distributions because it describes so many natural phenomena.

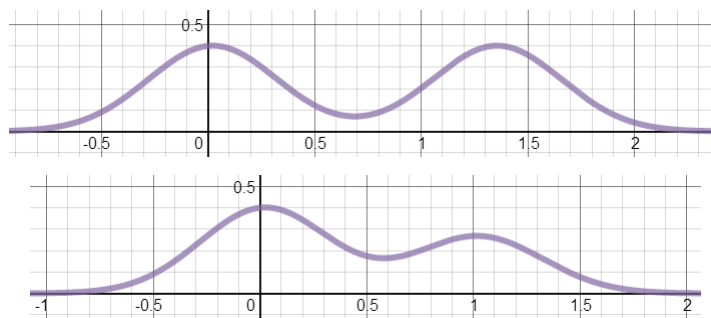
The normal curve is characterized by having a peak in the middle with the data tapering off symmetrically on both sides.

If the tapering is not symmetric, then we have a skewed distribution.



The number of children in families would form a skewed distribution because many families have small numbers of children while a few families have large numbers of children.

So far, the distributions have been centered around a single value. But sometimes the data is clustered around multiple values. Both of the following images show bimodal distributions (two peaks). An example of a bimodal data set would be the heights of people in an elementary school building. The adults and children would be clustered separately.



Sometimes the peaks are completely separate, sometimes they blend into each other, and sometimes they're not quite the same height. There are many shapes that distributions can take.

The graphs were created using an online graphing program called Desmos. It's free software at [desmos.com](https://desmos.com) that has a number of useful features that we will be using a few times throughout the course.

1. For the following data set

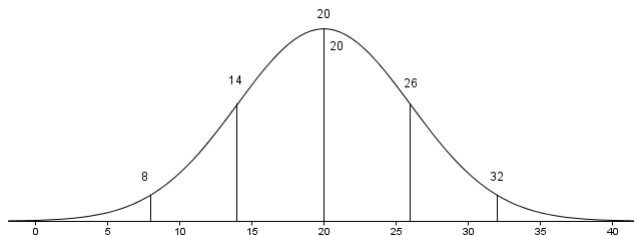
$\{3, 4, 4, 5, 5, 5, 5, 6, 6, 6, 7, 8, 8, 9, 10\}$

- Create a frequency table for the data.
  - Create a graph of the frequency of the data.
  - Is the data normally distributed? Skewed? Bimodal?
  - Determine the mean, median, and mode of the data. (Round to the nearest 0.1.)
2. Create a set of data with the following conditions. Use between 5 and 10 data points.
- The mean is 5 and the median is 7.
  - The mode is 9 and the mean is 8.
  - The median is 4 and the mode is 7.
  - A bimodal distribution where the mean and median are both 5.
3. Using a web browser, go to <https://www.desmos.com/calculator/6g19griuu>. You will see a graph with five sliders. Describe how the following sliders change the shape of the curve.
- The  $a$  slider
  - The  $b$  slider
  - The  $c$  slider
  - The  $d$  slider
  - The  $l$  slider

## 9 Normal Curves

### Computing Standard Deviations

In the pre-class materials, we looked at normal curves obtained by picking  $\mu$  and  $\sigma$  to be different values. The picture below is an example of a normal curve created by the GeoGebra program.



- Determine the values of  $\mu$  and  $\sigma$  from the picture.

We used WolframAlpha to compute the standard deviation of small data sets, but didn't talk about the actual calculation. The calculation is not difficult, but it can get long if there are lots of data points.

When computing the standard deviation of a data set derived from taking a sample of a population, we use the following formula:

$$\sigma = \sqrt{\frac{1}{N-1} \sum (x_i - \mu)^2}$$

This formula looks more complicated than it really is, but it will require a bit of explanation.

$$\sigma = \sqrt{\frac{1}{N-1} \sum (x_i - \mu)^2}$$

- $N$  represents the total number of data points in the data set.
- $x_i$  represent the actual data points.
- $\sum$  means that you're going to add together several terms. In this case, you will add together the  $(x_i - \mu)^2$  terms.

Mathematical formulas are very often written like this, and it takes time to learn to read them just like learning to read any other language.

To begin to understand how to use this formula, we will work through an example.

Example: Compute the standard deviation of the data set  $\{3, 6, 10, 8, 13\}$ .

We can see right away that  $N = 5$  for this data set because there are 5 data points.

Next, we compute  $\mu$ :

$$\mu = \frac{3 + 6 + 10 + 8 + 13}{5} = 8$$

We need to compute the differences  $x_i - \mu$  and square the results. In order to facilitate the organization of this, we will put the information into a simple table.

$i$	$x_i$	$\mu$	$x_i - \mu$	$(x_i - \mu)^2$
1	3	8	-5	25
2	6	8	-2	4
3	10	8	2	4
4	8	8	0	0
5	13	8	5	25

To get  $\sum (x_i - \mu)^2$ , we simply add the values in the last column to get 58. From here, we can simply plug it into the formula:

$$\begin{aligned}
 \sigma &= \sqrt{\frac{1}{N-1} \sum (x_i - \mu)^2} \\
 &= \sqrt{\frac{1}{4} \cdot 58} \\
 &= \sqrt{14.5} \\
 &\approx 3.81
 \end{aligned}$$

### Computing Standard Deviations

10 min

Consider the following sample data set:

$$\{22, 18, 19, 23, 23, 15\}$$

1. Determine  $N$  and  $\mu$  for this data set.
2. Create a chart with the columns  $i$ ,  $x_i$ ,  $\mu$ ,  $x_i - \mu$ , and  $(x_i - \mu)^2$ .
3. Compute  $\sigma$ .
4. Construct a frequency table for the data.
5. Plot the distribution. Is the distribution normal or skewed?

We often use the standard deviation when measuring data even if the distribution is slightly skewed. As the skewing becomes more and more prevalent, we need to choose more sophisticated tools to describe the shape of the curve.

At some point, you may see a formula for standard deviation that uses  $N$  instead of  $N-1$  in the denominator. The reasons are technical, but if you are finding the standard deviation of a sample (as we are) you use  $N-1$ . If you are finding the standard deviation of the entire population, then you use  $N$ .

**Percentiles** *Percentiles* are a way of describing the location of a data point in a distribution. You often hear this term used when talking about data points on the higher end of the distribution.

To say that a data point is “in the 90<sup>th</sup> percentile” means that data point *is larger than* 90% of the data. In general, for a data point to be “in the  $n^{\text{th}}$  percentile” means that the data point is larger than  $n\%$  of the data.

We will find percentiles of some data points in a small list of data in order to build some intuition.

**Percentile Calculations****10 min**

Example: Consider the following list of data, listed from smallest to largest for convenience:

$$\{2, 6, 8, 10, 14, 16, 19, 19, 20, 100\}$$

1. How many data points is the number 10 greater than?
2. 10 is in what percentile?
3. What percentile is 19 in? (Hint: 19 is not greater than 19.)
4. What numbers are in the 100<sup>th</sup> percentile?

For the normal curve, there are tables or formulas that can be used to work out the percentiles. The tables are known as  $z$ -tables.

The reason they're called  $z$ -tables relates back to the idea of a  $z$ -score, which is a number that describes the location of a data point using the mean and the standard deviation. Each data point has a  $z$ -score associated with it and that  $z$ -score is the number of standard deviations from the mean the data point is located.

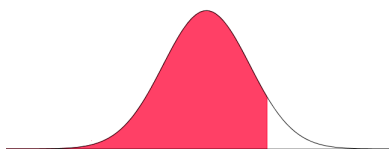
To find the  $z$ -score of the data value  $x$ , you would use the formula:

$$z = \frac{x - \mu}{\sigma}$$

The formula for locating a data point,  $x$ , with a particular  $z$ -score is:

$$x = \mu + \sigma \cdot z$$

Since percentiles represent the percent of data that is below a certain value, the following picture is commonly associated with a  $z$ -table. A  $z$ -table is a chart of values that give percentiles for normally distributed data based on their  $z$ -scores.



The shaded portion represents the data that is below a particular value (the value right at the boundary of the red and white regions), so the numbers obtained from the  $z$ -table represent the percent of data to the left of that data point.

The table below is a  $z$ -table. The percents are listed as decimals, so that .500 is the 50<sup>th</sup> percentile.

$z$	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
-2	.023	.018	.014	.012	.008	.006	.005	.003	.003	.002
-1	.159	.136	.115	.097	.081	.067	.055	.045	.036	.029
-0	.500	.460	.420	.372	.344	.308	.274	.242	.212	.184
0	.500	.540	.580	.618	.656	.692	.726	.758	.788	.816
1	.841	.864	.885	.903	.919	.933	.945	.955	.964	.971
2	.977	.982	.986	.988	.992	.994	.995	.997	.997	.998

For example, a data point with a  $z$ -score of 1.2 is in the 88.5<sup>th</sup> percentile, and if a data point has a  $z$ -score of -0.8 it is in the 21.2<sup>nd</sup> percentile.

- Determine the percentile of data points with  $z$ -scores of -1.5, 1.9, and -0.3.

$z$	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
-2	.023	.018	.014	.012	.008	.006	.005	.003	.003	.002
-1	.159	.136	.115	.097	.081	.067	.055	.045	.036	.029
-0	.500	.460	.420	.372	.344	.308	.274	.242	.212	.184
0	.500	.540	.580	.618	.656	.692	.726	.758	.788	.816
1	.841	.864	.885	.903	.919	.933	.945	.955	.964	.971
2	.977	.982	.986	.988	.992	.994	.995	.997	.997	.998

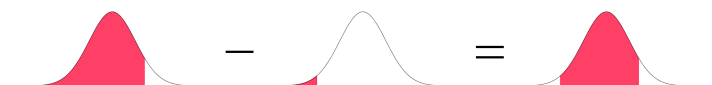
We can also use the table to work backwards from a percentile to the approximate  $z$ -score.

To find the  $z$ -score of the 70th percentile, we find the value closest to 0.700 in the chart and pick the corresponding  $z$ -score. The closest value is .692, which corresponds to the  $z$ -score 0.5, so we know that the 70<sup>th</sup> percentile has a  $z$ -score of about 0.5 (slightly above that).

If the value is exactly in the middle, take the average of the two  $z$ -scores.

- Determine the  $z$ -score of the 65th percentile, the 10th percentile, and the 99th percentile.

Another use of the  $z$ -table is to approximate the percent of data that falls between two points on the normal curve. This process is best explained by a picture.



In other words, to determine the percent of data between two points, we take the percent of data to the left of the larger value and subtract off the percent of data to the left of the lower value.

$z$	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
-2	.023	.018	.014	.012	.008	.006	.005	.003	.003	.002
-1	.159	.136	.115	.097	.081	.067	.055	.045	.036	.029
-0	.500	.460	.420	.372	.344	.308	.274	.242	.212	.184
0	.500	.540	.580	.618	.656	.692	.726	.758	.788	.816
1	.841	.864	.885	.903	.919	.933	.945	.955	.964	.971
2	.977	.982	.986	.988	.992	.994	.995	.997	.997	.998

### Percent of Data Between Two Values

10 min

1. Determine the percent of data that falls between  $z = -1$  and  $z = 1$ .
2. Determine the percent of data that falls between  $z = -2$  and  $z = 2$ .
3. Determine the percent of data that falls between  $z = 0$  and  $z = 1$ .

The first two values obtained in the previous exercise reflect what is known as the 68-95-99.7 rule. For normally distributed data:

- 68% of the data falls within 1 standard deviation of the mean.
- 95% of the data falls within 2 standard deviations of the mean.
- 99.7% of the data falls within 3 standard deviations of the mean.

Practically speaking, this means that in most circumstances, finding a data point outside of 3 standard deviations of the mean warrants further investigation. It could be just chance, but it could also be a sign of something more sinister like fraud. We will look into this possibility in the next section.

1. Consider the following set of data:

$\{21, 29, 33, 25, 28\}$

- (a) Determine the mean, median, and mode of the data set.
  - (b) Determine the standard deviation of the data set by hand.
  - (c) Use **WolframAlpha.com** to determine the standard deviation of the data set. (Your answers should match up.)
  - (d) The data point 29 is in which percentile?
  - (e) Approximately what is the  $z$ -score for the data point 29?
  - (f) According to the  $z$ -table, approximately what is the percentile of the data point 29?
    - Note: These two values should be somewhat close. The gap between the values is related to the fact that we have a small data set.
2. The results of IQ tests are normally distributed with  $\mu = 100$  and  $\sigma = 15$ .
- (a) Determine the IQs corresponding to the following  $z$ -scores:  $z = -2$ ,  $z = -0.8$ ,  $z = 0.3$ , and  $z = 1.7$ .
  - (b) Determine the  $z$ -scores for the following IQs: 90, 105, 135, 72.
  - (c) Determine the percentiles corresponding to the following IQs: 94, 85, 100, 142.
  - (d) Determine the percent of the population that has an IQ in the following ranges: between 70 and 100, between 94 and 115, below 88, above 130.
3. In the next class, we are going to talk about detecting fraud. Your assignment is to make up two lists of 75 random coin tosses. The first list will be made up off the top of your head. Try to make it as random as possible. After making your first list obtain a second list by flipping a coin 75 times in a row and recording the results. Flip the coin higher than your head and catch it in the air.



## 10 Detecting Frauds

### 75 Coin Flips

For the homework, you were asked to create two lists of 75 coin flips, one fake and one real. We are going to run a series of analyses in order to try to find different ways of picking out the fake.

This is part math and part psychology. Remember that while we can draw some mathematical conclusions about the data, the math does not give us a definitive answer. Rare events happen when there are lots of samples drawn, so we need to be careful to not overstate our conclusions.

When trying to fake data, most people try to give things at least a superficial appearance of being legitimate, so it doesn't make sense for us to do a straight heads/tails count and look for one to be out of the ordinary. However, people tend not to analyze their lists for more complex distributions.

- Consider the following sets of three-coin sequences. In a set of three flips, which should be the most common?

HHH	HHT	HTH	THH
TTT	TTH	THT	HTT

In fact, these are all equally likely to occur in any particular set of three flips. If we draw the probability tree, we will see that each three-coin outcome has a  $\frac{1}{8}$  chance of appearing. If we take the 75 coin flips and break them into 25 three-coin sequences, we expect to see each one appearing about 3 times.

#### Three-Coin Sequences

10 min

1. Analyze both lists of 75 tosses by breaking them each into 25 three-coin sequences and creating a frequency table for each possibility. Keep the two lists separate.
2. If every outcome is equally likely, what is the probability that a random list of 25 three-coin sequences will not have any HHH or TTT combinations? (Hint: Think about a hit-miss probability tree.)
3. Does one of the two distributions appear to be more even than the other? If so, which one looks more even?
4. Compare your results with others. Is there a general pattern that is emerging?

### Swap Analysis

Another way to measure sequences is to count the number of swaps in the sequence. A swap in a sequence is when a streak changes from head to tail or tail to head. For example, the sequence HTTTTHHHHTT contains three swaps.

Initially, it may not seem obvious how to estimate swaps. But it turns out that it's very similar to just flipping coins.

- Suppose the previous coin flip was a head. What must the next coin flip be in order to get a swap? What is the probability of this happening?
- Suppose the previous coin flip was a tail. What must the next coin flip be in order to get a swap? What is the probability of this happening?

Since we have 75 coin tosses and the first flip can't count as a swap (why not?), there are only 74 chances to get a swap. Since each one has a 50% chance of giving a swap, we expect to have 37 swaps on average.

**Swap Analysis****10 min**

1. Analyze both lists of 75 tosses by counting the number of swaps.
  2. For this number of coin tosses, the number of swaps is normally distributed with  $\bar{x} = 37$  and  $s \approx 4.3$ . Determine the  $z$ -score for the number of swaps in both lists.
  3. Does either list fall outside of 2 standard deviations from the mean on this measure?
  4. Only 5% of the data should fall outside of 2 standard deviations from the mean. Determine the percent of data in the class that falls outside of 2 standard deviations from the mean. If this is significantly above 5%, then this is an indication that fraud exists in the data.
- Note: Being a suspect is not the same as being guilty. In a classroom of 20 students, we expect to have some student's legitimate data fall into the list of suspects. This is just another warning to not draw conclusions too quickly.

**Runs Analysis**

Our last analysis will be an analysis of runs. For any given list of 75 flips, we can count up the total number of runs of each length. For example, HTTTHHHTT has 1 run of 1, 1 run of 2, and 2 runs of 3.

This gives us two pieces of data on the lists:

- A distribution of runs
- The size of the longest run

We will now analyze the lists based on these measures.

**Runs Analysis****10 min**

1. Analyze both lists of 75 tosses by counting the number of runs of each length and determining the longest run.
2. Below are the runs results of 10,000 simulated lists of 75 coin tosses each with the average number of runs of a given length falling into each category. How closely does your distribution match the simulation?

Length	1	2	3	4	5	6	7	8	9
Runs	18.7	9.4	4.8	2.3	1.1	0.6	0.3	0.1	0.1

3. Below is the percent of lists that had different length longest runs. Determine the percentile for the length of the longest run for both of your lists.

Length	$\leq 3$	4	5	6	7	8
Percent	0.2	6.8	22.6	26.8	19.6	11.6
Length	9	10	11	12	$\geq 13$	
Percent	6.4	3.2	1.3	0.8	0.7	

1. Write a short (one paragraph) summary of the analysis of the two lists you generated. Be sure to answer the following questions:
  - Which was determined to be the fraudulent list?
  - What types of analysis did you use?
  - Did the analysis lead to the correct conclusion?
2. Write a paragraph describing how you might test whether a 6-sided die is a fair die.
3. A 6-sided die is rolled 20 consecutive times without the number 1 showing up. What is the probability of this happening? (Remember that if the events are independent, then you can use the product property.)

## 11 Credit

As we saw in the preview work, consumer credit plays a large role in the modern economy. Those who use credit wisely and prove themselves creditworthy can save thousands of dollars on future purchases that must be financed.

Consumer credit generally comes in two types:

**Installment credit** provides for one-time purchases and usually includes fixed repayment terms.

**Revolving credit** allows for the on-going issuance of (usually smaller) amounts of credit that are repaid over time according to the amount borrowed and the specific terms of the credit.

Ask the students to provide examples of each type of credit.

### Installment Credit

#### Monthly Car Payments

10 min

A car has a price of \$12,000. It may be purchased on installment plans of several lengths. The longer the plan, the more total interest the borrower pays. For each of the loan terms shown below, calculate the simple interest rate the borrower will be charged.

Auto Loan		
Loan term	Total Interest	Interest Rate
48 <i>months</i>	\$3,500	7.3%
60 <i>months</i>	\$5,000	8.3%
72 <i>months</i>	\$6,500	9.0%

Usually, a car loan comes with a set compound interest rate instead of a preset amount of interest to be charged. This can make computing the monthly payment a little more involved. This is because the interest is compounded on the remaining balance each month. Thus, you are charged more in interest on the first month of your loan than you are on the last month of your loan.

This also means that at the beginning of the loan, more of your payment goes toward interest and less towards the principal of the loan. An [amortization table](#) is a table showing the details of each payment over the life of the loan and can help illustrate this concept.

Go over the first 3 months in the table in detail. Make sure the students know where the numbers are coming from.

### A small amortization table

\$1,000 loan at 4% over 12 months

Month	Payment	Principal paid	Interest paid	Balance
Jan.	\$85.15	\$81.82	\$3.33	\$918.18
Feb.	\$85.15	\$82.09	\$3.06	\$836.09
Mar.	\$85.15	\$82.36	\$2.79	\$753.73
Apr.	\$85.15	\$82.64	\$2.51	\$671.09
May	\$85.15	\$82.91	\$2.24	\$588.18
June	\$85.15	\$83.19	\$1.96	\$504.99
July	\$85.15	\$83.47	\$1.68	\$421.52
Aug.	\$85.15	\$83.74	\$1.41	\$337.78
Sep.	\$85.15	\$84.02	\$1.13	\$253.76
Oct.	\$85.15	\$84.30	\$0.85	\$169.45
Nov.	\$85.15	\$84.59	\$0.56	\$84.87
Dec.	\$85.15	\$84.87	\$0.28	\$0.00
—	\$1,021.80	\$1,000.00	\$21.80	—

In practice, the amount of principal paid with early payments can be dwarfed by the amount of interest paid. This means it is advantageous to make additional principal payments (if possible) to reduce the total amount of interest you will be charged over the life of the loan.

### Amortization Tables

10 min

Fill out the remainder of the first 5 entries in this amortization table showing a home loan of \$165,000 over 360 months (30 years) at 4.5%.

Month	Payment	Interest paid	Principal paid	Balance
Month 1	\$836.03	\$618.75	\$217.28	\$164,782.72
Month 2	\$836.03	\$617.94	\$218.09	\$164,564.63
Month 3	\$836.03	\$617.12	\$218.91	\$164,345.71
Month 4	\$836.03	\$616.30	\$219.73	\$164,125.98
Month 5	\$836.03	\$615.47	\$220.56	\$163,905.42

*Hint: The interest paid in month 2 is the amount found when the annual rate of 4.5% is applied to \$164,782.72 for one month.*

### Using simple interest

The second kind of credit is revolving credit, which means the term is open-ended and additional amounts may be borrowed or repaid according to specific terms over time. The most common form of consumer revolving credit is that provided by a credit card.

Credit cards often charge an *annual* percentage rate (APR) on debt accumulated on the card, but assess those charges at the end of each *monthly* billing cycle. To do this accurately, the APR is divided by 365 to get a daily interest rate, and this daily rate is multiplied by the number of days in the billing cycle to compute the interest rate.

In fact, the formula for computing the amount of interest charged by a credit card company bears a striking resemblance to that used for calculating simple interest.

### Credit card calculations

$$I = P \times r \times t$$

$$\text{FC} = \text{ADB} \times \frac{\text{APR}}{365} \times \text{BC}$$

- **Finance Charge** — The amount of interest charged by the credit card company.
- **Average Daily Balance** — The average of how much debt was carried on the credit card each day.
- **Daily interest rate** — The APR divided by the number of days in a year.
- **Billing Cycle** — The number of days in the billing cycle.

Help them fill in the first few days. Then tell them that they only have a few minutes to finish the whole table.

### Credit Cards

#### A credit card example

**10 min**

Suppose a card's billing cycle began Aug. 8 and carried a balance of \$57.54. The card user purchases \$63.44 of gasoline on Aug. 12, \$32.30 for entertainment on Aug. 19, and \$66.00 for jeans on Sep. 1. The card user also makes a \$100 partial payment to the card company on Aug. 25.

Find the card's balance for each day of the BC. Then find the ADB.

Aug. 8	Aug. 9	Aug. 10	Aug. 11	Aug. 12	Aug. 13	Aug. 14
\$57.54	\$57.54	\$57.54	\$57.54	\$120.98	\$120.98	\$120.98
Aug. 15	Aug. 16	Aug. 17	Aug. 18	Aug. 19	Aug. 20	Aug. 21
\$120.98	\$120.98	\$120.98	\$120.98	\$153.28	\$153.28	\$153.28
Aug. 22	Aug. 23	Aug. 24	Aug. 25	Aug. 26	Aug. 27	Aug. 28
\$153.28	\$153.28	\$153.28	\$53.28	\$53.28	\$53.28	\$53.28
Aug. 29	Aug. 30	Aug. 31	Sep. 1	Sep. 2	Sep. 3	Sep. 4
\$53.28	\$53.28	\$53.28	\$119.28	\$119.28	\$119.28	\$119.28
Sep. 5	Sep. 6	Sep. 7				
\$119.28	\$119.28	\$119.28				

**Computing a Finance Charge****10 min**

Using the formula, compute the finance charge for the ADB and BC you just found with an APR of 27.99%.  
**ADB = \$103.37, FC = \$2.46**

$$FC = ADB \times \frac{APR}{365} \times BC$$

What will the balance be on the credit card on Sep. 8? **\$121.74**

Answer this one as a class. Just have them write it down as part of the group work so that they have the calculation recorded all in one spot.

Amortization tables can help you know how long it will take to pay off a debt and how much interest you will pay in doing so. Paying a larger amount each month can drastically reduce the amount of interest over the term of the loan.

For example, a \$3,000 credit card balance with a 17.0% APR will take 40 months to pay off and cost \$934 in interest with a \$100 monthly payment. However it will take only 24 months to pay off and cost \$553 in interest with a \$150 monthly payment.

**Homework****Consumer Math Portfolio Assignment 3**

1. Explore the credit card calculators at <http://www.creditcards.com/calculators/>. For a specific loan amount and interest rate of your choosing, use the calculators to find out how much interest you could save by paying more than the minimum payment. (You will probably need to use more than one of the calculators for this problem.) Write a report of your findings.
2. Find three cars for sale in the surrounding area (you should search dealership ads, used car websites, etc.) that you would like to own. For each of these, use the car loan calculator at <http://www.edmunds.com/calculators/car-loan.html> to compute financing. Write an in-depth report listing the features of each vehicle and financing details as if you were presenting options for purchase to your significant other. Make a recommendation based on your findings and provide logical reasoning to support your conclusion.

## 12 Mortgages

**Affordability Guidelines** There are many methods in use for calculating how much home a person can afford. During the housing bubble, lenders often approved borrowers for more than the borrowers could reasonably be able to repay. With this history, it is important to not rely solely on a loan officer's willingness to approve a mortgage to assess its affordability. What follows are three rules of thumb for assessing the affordability of a mortgage. None of them is perfect and each might give differing answers depending on available interest rates, hidden expenses, and other factors.

**Guideline #1:** The amount of the mortgage loan should not exceed three times the borrower's annual gross income.

**Guideline #2:** If a family has other *significant* monthly debt obligations, possibly including car payments, credit cards, or student loans, the family's monthly housing expenses, including mortgage payment, property taxes, and private mortgage insurance, should be limited to no more than 25% of their monthly gross income (income prior to deductions).

**Guideline #3:** If the family has no other significant monthly debt obligations, the monthly housing costs could range as high as 38% of their gross monthly income.

### Affordability Guidelines

10 min

Use one or more of the affordability guidelines to suggest maximum mortgage amounts for the following individuals:

**Hunter** has approximately \$25,000 in student loan debt, makes \$45,000/year, and is single with no dependents.

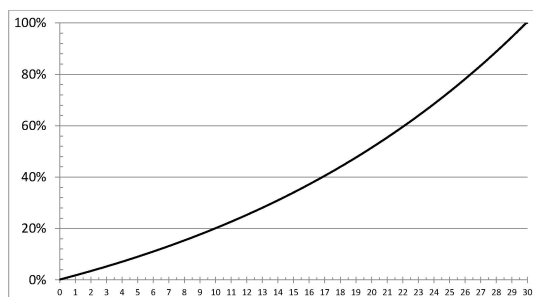
**Amy & Ben** have 2 young children, one of whom requires frequent medical care. Ben makes \$42,000/year while Amy makes \$37,000/year. The couple pay \$600/month in childcare and have two modest car payments.

**Stephanie** is a single-mom of one school-aged child who makes \$50,000/year at her regular job and \$1,000/month in consulting fees. She spends \$150/month on student-loan payments.

**Joanna** is a recent college graduate with \$20,000 in outstanding student loans, \$3,000 credit card debt, and is interviewing for full-time jobs while supporting herself on tips from her job as a waitress.

**The Total Cost of a Mortgage** Additional mortgage expenses, such as private mortgage insurance (PMI), homeowner's insurance, and property taxes can be significant.

**PMI** A homeowner who cannot provide a 20% down payment is usually required to pay several hundred dollars each month in PMI. This insurance protects the lender against early default on the mortgage which might result in the lender reselling the property for less than is still owed. After a homeowner has paid 20% of the mortgage they can usually discontinue the PMI, though it can take about 10 years to build that level of equity on a traditional 30-year loan. The graph below shows the level of equity earned each year by paying the minimum monthly payment.





**Homeowner's Insurance** Homeowner's insurance protects the homeowner against damage to or loss of the home and is usually required by lenders. The price can vary substantially, but it's not uncommon for a homeowner to pay \$50, \$100, or more each month in homeowner's insurance.

**Property Taxes** Property taxes are collected on the appraised value of the home, which may differ from its market value. In the United States, these taxes are collected locally and are spent on local government services, such as public education and infrastructure. These rates can vary from less than a quarter of a percent to nearly two percent of appraised value annually. Nevada's 2014 property tax rate averaged about 1.1% of the appraised home value annually.

### The Total Cost of a Mortgage

10 min

Cynthia & Simon are interested in purchasing a home in a city where the property tax rate is 1.25%. The home they are considering is appraised at \$200,000 and would have an annual insurance bill of \$660. Additionally, because they cannot afford a large down payment, PMI will cost them \$200 per month.

Calculate the total monthly cost of the mortgage for Cynthia & Simon if the mortgage payment itself is \$965.05.

### Buying Points

As we've seen, mortgages can be expensive. A person can easily pay more than the original cost of the house *just in interest* over the life of the loan, and this isn't even counting things like taxes and insurance!

To reduce the amount of interest s/he pays, a borrower can

1. borrow less money,
2. pay the mortgage back faster, or
3. borrow at a lower interest rate.

The first two options are fairly straightforward, if somewhat difficult to do in practice. The third option, however, is one that borrowers sometimes overlook but can result in significant long-term savings. In addition to simply shopping around for a lender offering a low rate, sometimes borrowers can engage in a process known as **buying points**.

A point, in the context of mortgages, refers to a permanent reduction in the interest rate of the mortgage in exchange for an upfront cash payment. The reduced rate results in smaller monthly payments and can add up to significant savings over the life of a mortgage.

For example, suppose Gina is considering a \$150,000 mortgage at 4.75% over 30 years. Her monthly mortgage payment would be \$782.47 (not including homeowner's insurance, PMI, or any taxes). Gina's bank offers her a point in the form of a 0.25% reduction in the interest rate in exchange for 1% of the mortgage amount. In other words, Gina could pay the bank  $0.01 \times \$150,000 = \$1,500$  now but her interest rate will drop to 4.50%, resulting in a new monthly mortgage payment of \$760.03. Considering that the mortgage is for 30 years, this results in a savings of  $360 \times (\$782.47 - \$760.03) - \$1,500 = \$6,578.40$  in interest over the 30-year loan.

The exact cost of points and the interest rate reduction received through buying points can change from day-to-day based on national fluctuations in interest rates, but it's possible to get a sense of their potential value by performing some simplified calculations.

### Buying Points

10 min

Lola is interested in buying points for her mortgage. Her 30-year mortgage amount is \$190,000 with a 6.00% interest rate. Her bank will let her buy up to two points worth 0.25% points off her rate each, at a cost of 1% of the mortgage per point.

Use a mortgage payment calculator like that found at <http://www.bankrate.com/calculators/mortgages/mortgage-calculator.aspx> to help you compute how many months it will take Lola to save as much in interest as she would originally spend on the points.

## Homework

### Homework 12

1. Find the average 30-year mortgage rate and price of a house in your hometown for your birth year, if possible (otherwise, use 1980 Las Vegas). Using <http://www.bankrate.com/calculators/mortgages/mortgage-calculator.aspx>, create an amortization table showing the progress of a loan repayment using that information. Then, compute the percentage and \$ value of equity that would have been earned by the time of important events in your life (age 6 - learned to ride a bike, age 13 - first kiss, etc.). Compile your findings for submission.
2. Buying points comes with an opportunity cost. The money you spend on points could have instead been used towards a down payment, reducing your mortgage principal and therefore the interest charged over the life of the loan. The calculator at <http://www.bankrate.com/calculators/mortgages/mortgage-loan-points-calculator.aspx> can help you see how buying points compares to putting the extra money towards the down payment. The calculator has sliders that let you set the amount of the mortgage, the length of the loan, the number of points you buy, and the interest rate *after* buying the points. It simultaneously calculates what would happen if you instead spent the money on a down payment and so had a higher interest rate on the loan. Use the calculator to examine the following scenarios a person could face in buying a home. In your report, describe for each scenario whether it is advantageous to buy points. Include in your report an overview of your impression of buying points after using this calculator:
  - (a) A \$200,000 30-year mortgage at 5% where each point reduces the rate by 0.25 percentage points. (Calculate for a variety of points.)
  - (b) A \$150,000 20-year mortgage at 6% where each point reduces the rate by 0.15 percentage points. (Calculate for a variety of points.)
  - (c) The median price of a Las Vegas home as listed at <http://www.zillow.com/las-vegas-nv/home-values/>, the current average mortgage rate as listed at <http://www.bankrate.com/finance/mortgages/current-interest-rates.aspx>, and the current reduction offered by buying points as listed at <https://www.suntrust.com/Mortgage/Rates>. (Calculate for a variety of points for both 15-year and 30-year loan terms.)

## 13 Inflation

As we saw in the preview work, inflation reduces the value of currency by lowering its ability to purchase goods and services as a result of higher prices. Even low levels of inflation can have a serious impact on the everyday transactions of consumers.

The inflation values are year-to-year. Explain to students that they need to calculate it in steps and cannot simply add across to get the total. In 2008, the average rate of return on an investment CD was about 1.8% annual interest, compounded daily. <http://www.bankrate.com/finance/cd-rates-history-0112.aspx>

### Interest vs. Inflation

10 min

If Stefano invested \$1,000 in a CD in 2008 at 1.8%, how much would the account have after 5 years? Recall:  $A = P \left(1 + \frac{r}{n}\right)^{nt}$

The average rates of inflation for the 5 years of Stefano's investment are shown below.<sup>2</sup>

Inflation (%)				
2008	2009	2010	2011	2012
3.8	-0.4	3.2	2.1	1.5

1. What is the dollar amount of Stefano's investment at the end of 5 years?
2. Using the inflation rates, determine how much money in 2012 is equivalent to \$1,000 in 2008.

Unfortunately, traditional savings mechanisms such as savings accounts and investment CDs offer rates of return lower than the usual rate of inflation, meaning that the money will continually worth less and less even though there is more of it.

As a result, most investors look for opportunities to receive returns at a rate higher than inflation, thus increasing the overall buying power of their investment. Higher rates, however, usually come with higher risks. For example, real estate was long seen as a solid investment, offering returns of approximately 9% annually,<sup>3</sup> far above inflation. The housing crash of 2006 (driven in part by large numbers of investors hoping for large returns) caused home values to fall significantly, dropping nearly 50% in some places, leading to many billions of dollars in lost investment.

The decreased buying power caused by inflation is especially important to consider when contemplating long-term effects, such as the standard of living that you can afford after retirement.

### Inflation vs. Retirement

10 min

David and Victoria are each 5 years from retirement with a combined income of \$100,000. If inflation is projected to continue at 2.5% annually, how much will they need to withdraw from savings during their first year of retirement to match their current buying power?

<sup>2</sup><http://www.usinflationcalculator.com/inflation/historical-inflation-rates/>

<sup>3</sup>[http://www.money.cnn.com/galleries/2007/real\\_estate/0704/gallery.stocks\\_v\\_realestate.moneymag/index.html](http://www.money.cnn.com/galleries/2007/real_estate/0704/gallery.stocks_v_realestate.moneymag/index.html)

Many retirement plans strive to provide returns that outstrip inflation by investing in a mixture of safer low-yield investments, and riskier high-yield investments. One common investment option is stocks.

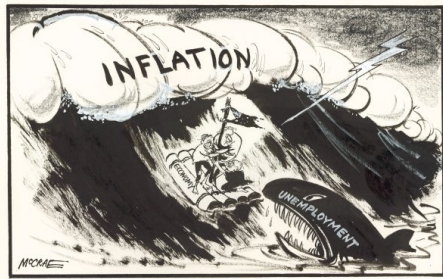
Stocks are shares (or ownership interests) in companies that can be bought, sold, or traded. When a company wants an influx of cash for expansion, for example, it may sell stocks to interested investors. Those investors then own pieces of the company and receive a portion of its profits.

Do a couple years' calculations with them. Help them see that the answers to part 1 are used in part 2.

**Inflation vs. Savings** **10 min**

Carissa is ready to retire and has accumulated savings of \$500,000 to do so. She currently spends \$50,000 a year.

1. If inflation occurs at 3% per year, how much will Carissa have to spend each year, for the next 10 years, to maintain her buying power?
2. If Carissa withdraws from savings the amount she needs each year on Jan. 1, and her remaining savings earns 2% simple interest each year, what will be her savings balance after 3 years? 5 years? 10 years?
3. Give an estimate of how much Carissa would need in her savings at retirement to maintain her buying power for 20 years, taking into consideration the effects of inflation. Give reasons for your estimate.



[http://uws.edu.au/\\_\\_data/assets/image/0009/479052/Economy\\_cartoon.jpg](http://uws.edu.au/__data/assets/image/0009/479052/Economy_cartoon.jpg) Governments have attempted to curb inflation in a variety of ways. Currently, the Federal Reserve Bank, which serves as the central bank of the United States, uses interest rates to impact inflation. If inflation is high, the Fed might raise the rate banks pay to borrow money. This would constrict the supply of circulating money and dampen inflation. On the other hand, if unemployment is high, the Fed might reduce the rate banks pay for borrowing. This lower rate increases the amount of money in circulation, leading to hiring and so lower unemployment. The complementary beasts of inflation and unemployment always push against the economy in a bid to outdo the other. That's the traditional thinking, anyway.

## Homework

### Consumer Math Portfolio Assignment 5

1. For this problem you are going to have an evil twin who makes different financial decisions than you. You will buy a house while your evil twin buries money in the yard like a pirate. First, let's project what will happen to you.
  - Go to [www.zillow.com](http://www.zillow.com) and find the estimated value of a house you like.
  - Find a current 30-year mortgage rate at [www.zillow.com/mortgage-rates/](http://www.zillow.com/mortgage-rates/).
  - With those two pieces of information, use the calculator at <http://www.bankrate.com/calculators/mortgages/mortgage-calculator.aspx> to create a home mortgage amortization table and record the monthly mortgage payment and the total amount paid over the life of the mortgage.

Now, let's take a look at the results of your evil twin's decisions. Instead of buying a house like yours, each month he or she takes the amount of your monthly mortgage payment and bury it in a jar in the yard. At the end of 30 years you own a house and your twin has 360 jars of monthly payments totaling the amount you paid for the house, including interest. With this buried money, your evil twin finally wants to buy a house and wants it to be even nicer than your house. With all the buried money, your twin won't need a loan and so won't have to pay interest. However, because of inflation, the money in the jars doesn't buy as much as it did when it was first buried. How much is your twin's money worth, in terms of buying power, compared to what you spent? To find out, you are going to use an inflation calculator.

The calculator at <http://www.bankrate.com/calculators/savings/price-inflation-calculator.aspx> allows you to calculate how much buying power your twin's jar money will have in terms of today's dollars. To compare your decisions with his/hers, enter the following values into the inflation calculator.

- Years: 30 (This is the length of the mortgage.)
- Monthly contributions: Your monthly mortgage payment (This is how much your evil twin buried in the yard each month.)
- Amount currently invested: \$0
- Expected rate of return: 0% (Money buried in the yard earns no interest.)
- Expected inflation rate: 3.22% (This is the average annual inflation rate over the past 100 years.)
- Federal tax rate: 0% (There's no tax penalty for digging up your yard.)
- State tax rate: 0% (For the same reason as above.)

Click "VIEW REPORT" to see the results of the calculation. The number next to "Ending balance with taxes and inflation" is how much your twin's savings will be worth in current dollars. Who made the better decision?

2. Rerun the same calculations as before with two other rates of inflation (one higher and one lower) and two other houses. You can find historical inflation rates at <http://www.usinflationcalculator.com/inflation/historical-inflation-rates/> or you can just choose a value.
3. Prepare a report of your findings, detailing the consequences of your and your twin's financial decisions in each of the three sets of calculations.

## 14 Functions

### Dependent and Independent Variables

Recall that a **function** is a relationship where each value of the input variable (independent variable) is assigned exactly one value of the output variable (dependent variable).

Thus the number of hours of daylight is a function of the day of the year.

- Is the day of the year also a function of the number of hours of daylight?

The formula  $y = x^2$  defines  $y$  as a function of  $x$ .

- Is  $x$  also a function of  $y$ ?

**Is it a Function?**

**10 min**

For each item below determine whether one quantity is a function of the other quantity. If yes, then also do the following:

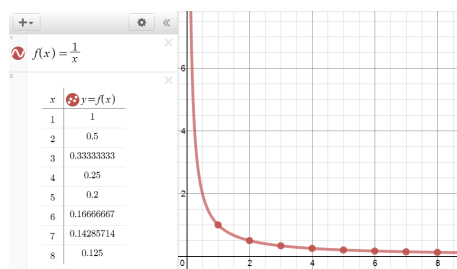
- Determine which variable is independent and which is dependent.
  - Sketch a reasonable graph to illustrate the relationship.
  - Write one or two sentences justifying the shape and behavior of your sketch.
- The temperature in the LAS parking lot and the time of day—
  - The cost of pizza and the diameter of the pizza—
  - Your course grade and your shoe size—
  - Education level of a person and the amount of that person's salary—
  - Credit card interest owed and the balance due on the account—
  - The day of the year and the number of hours of daylight—
  - A male's optimal weight and a male's height—

### Graphics vs Tables vs Formulas

In the preview assignment we encountered three methods of displaying functions.

- in a graph,
- in a table, and
- in symbols.

What are some pros and cons of each method?



**Napolean's Keyrings****10 min**

Napolean runs a home business selling boondoggle keyrings. His business license costs \$10/month. He makes a profit of \$0.80 for each keyring he sells. Define a function whose independent variable is the number of keyrings he sells in a given month and whose dependent variable is the profit he makes that month. Provide

- a graph of the function,
- a table of function values, and
- a symbolic form of the function.

**Desmos  
Water  
Problem**

Go to <https://teacher.desmos.com/dashboard> and log in. Then create a Water Line activity and have the class spend 15 minutes working on it. Warn them that there is no time to waste.

**Water Filling a Container****15 min**

Go to <https://student.desmos.com/> and enter the code your instructor gives you.

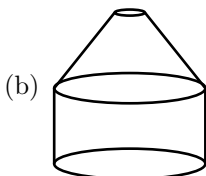
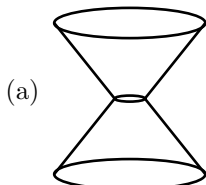
Be sure to enter both names. Follow the instructions of the activity. There is no write-up for this activity.

## Homework

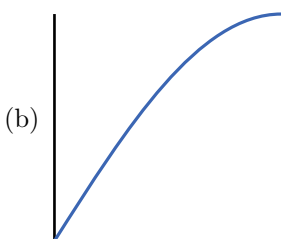
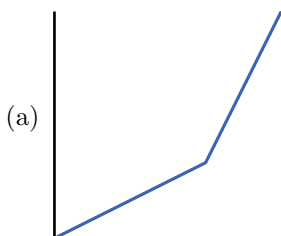
## Homework 8

Be sure to show your work. If/when you use a tool such as [desmos.com](https://www.desmos.com) or [wolframalpha.com](https://www.wolframalpha.com) you should indicate exactly how you used the tool to obtain your answer.

- Suppose water is poured at a uniform rate into a container. Draw a function that models the height of the water over time for each container below.



- Suppose water is poured at a uniform rate into a container. For each graph below draw a container whose height over time is modeled by the graph.



- Represent the function  $f(x) = \frac{x}{x^2 + 1}$  as both a table and as a graph.
- Represent the function below as a graph. (As an optional challenge, see if you can find a possible formula for  $f(x)$ .)

$x$	0	1	2	3	4	5	6
$f(x)$	0	-3	-4	-3	0	5	12

- Present a function whose input is the 10 U.S. cities with the greatest population, and the output is the population. Decide which of the three methods for displaying a function is most appropriate, and then display the function. (Yes, you will need to do a search to find this data. Yes, you must cite your source.)
- For each item below determine whether one quantity is a function of the other quantity. If yes, then also do the following:
  - Determine which variable is independent and which is dependent.



- (b) Sketch a reasonable graph to illustrate the relationship.
- (c) Write one or two sentences justifying the shape and behavior of your sketch.
  - i. The number of rings in a tree trunk and the age of the tree—
  - ii. The cost of a television and the size of the screen—
  - iii. The perimeter of a square and its area—
  - iv. The cost to make a movie and the amount of money the movie makes—
  - v. The height of a wife and the height of her husband—
  - vi. A student's letter grade in a class and their total score for the class—

## 15 Algebra of Functions

### Profit Function

Preface this problem by asking what cost, revenue, and profits are. Warn them that they will need to come up with each function on their own, but that they are not hard to find.  $C(x) = 200 + 0.2(500 - 100x)$ ,  $R(x) = x(5 - 0.01x)$ ,  $P(x) = R(x) - C(x)$ . If they enter these functions into [desmos.com](https://www.desmos.com) they can click on the top of the profit function to find the coordinates of the vertex. Talk them through the meaning of cost, revenue, and profit before turning them loose. The max occurs at  $x = 2.6$ .

#### Profits From Boondoggle

10 min

Napoleon wants to start a business selling boondoggle keyrings. Let  $x$  be the price he charges for each keychain.

**Demand** With the price of each keychain at  $x$  dollars, Napoleon expects to sell  $500 - 100x$  keychains.

**Costs** The rent for the facilities will be \$200 while the material costs \$0.20 per keychain.

1. Find the cost function  $C(x)$ .
2. Find the revenue function  $R(x)$ .
3. Find the profit function  $P(x)$ .
4. At what price should Napoleon sell the keychains to maximize his profit? (Try plotting  $P(x)$  on [desmos.com](https://www.desmos.com).)

### Dating Ages

Be sensitive with this example to avoid causing offense. Be sure to review inverse functions before starting them on this example.

#### Bounds for Dating Ages

10 min

It has been suggested that a person of age  $x$  should only date a person whose age is at least  $f(x) = \frac{1}{2}x + 7$ .

1. According to this formula, what is the minimum age of a person that a 24-year old should date? And for a 36-year old?
2. Find the inverse function of  $f(x)$  and call it  $g(x)$ . (Recall that [wolframalpha.com](https://www.wolframalpha.com) can help you find an inverse function.) What is the interpretation of the output of  $g(x)$ ?
3. According to this formula what is the maximum age of a person that a 24-year old should date? And for a 36-year old?
4. In [desmos.com](https://www.desmos.com) plot  $f(x)$ ,  $g(x)$  and  $f(x) < y < g(x)$ . What does the shaded region represent?
5. Plot a few points at [desmos.com](https://www.desmos.com) in the form  $(x, y)$  to represent couples you know. Does the model fit?

**Coupons  
and  
Discounts**

Do a quick review of how to compose functions.

**Coupons and Discounts**

**10 min**

You clip a \$5 off coupon from the newspaper for a pair of shoes. At the store you find that all the shoes are 10% off.

1. Write a function  $f(x)$  that shows the effect of the coupon.
2. Write another function  $g(x)$  that shows the effect of the store discount.
3. Make a new function  $h(x) = f(g(x))$  and determine whether this function applies the coupon before or after the discount.
4. Make a new function  $k(x) = g(f(x))$  and determine whether this function applies the coupon before or after the discount.
5. Which pricing model do you think the store is most likely to choose? (Hint: compare  $h(50)$  and  $k(50)$ .)

## Homework

## Homework 9

Be sure to show your work. If/when you use a tool such as [desmos.com](https://www.desmos.com) or [wolframalpha.com](https://www.wolframalpha.com) you should indicate exactly how you used the tool to obtain your answer.

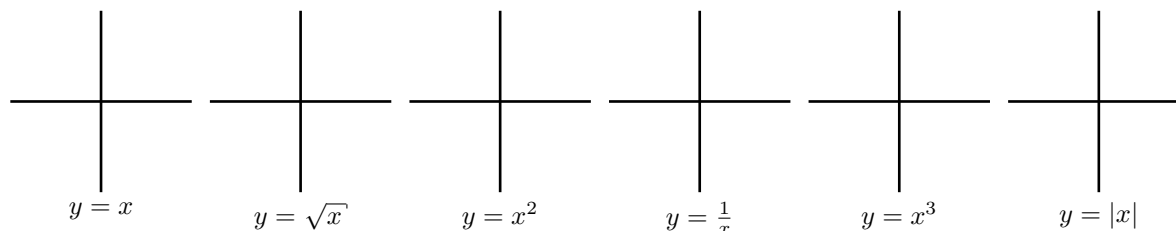
- Recall that  $F = \frac{9}{5}C + 32$  is the formula for converting degrees Celsius to degrees Fahrenheit. At what temperature is the Fahrenheit temperature equal to the Celsius Temperature? (i.e., where is  $F = C$ ?) Either Desmos or WolframAlpha could be helpful.
- If a person of age  $x$  should not date a person younger than  $y = \frac{1}{2}x + 7$ , then what is the lowest allowable dating age for any individual?
- John is trying to find the ideal number of dates he should go on each week. He notes that his dating life satisfaction (on a scale of 1 to 10) is modeled by  $S(d) = -\frac{1}{2}d^2 + 4d + 2$  where  $0 \leq d \leq 8$  is the number of dates he goes on each week.
  - How satisfied is John if he goes on 0 dates in a week?
  - How satisfied is John if he goes on 2 dates per week?
  - How many dates should John go on each week to maximize his dating life satisfaction? (Hint: Graph John's satisfaction function.)
- John notes that the number of dates he has each week depends on the number of women whom he asks. He goes on approximately  $d(w) = \sqrt{w}$  dates per week when he asks out  $w$  women that week ( $0 \leq w \leq 64$ ).
  - About how many dates will John go on if he asks out 9 women?
  - Approximately how many women did John ask out if he went on 5 dates?
  - Using the function  $S(d)$  from the previous problem, explain the input and output of the composite function  $S(d(w))$ .
  - How many women should John ask out to maximize his dating satisfaction? (If you enter these functions in Desmos you may want to use  $x$  as the input variable instead of  $d$  or  $w$ .)
  - Is John trying too hard? Should he try online dating?
- Suppose  $f(x) = x + 1$ . Find a formula for  $f(f(x))$ ,  $f(f(f(x)))$ , and  $f(f(f(f(x))))$ . In general, what is the formula for  $n$  compositions of  $f$ :  $f^{(n)}(x) = \underbrace{f(f(\cdots f(f(x))\cdots))}_{n \text{ times}}$ ?
- Suppose  $g(x) = 2x$ . Find a formula for  $g(g(x))$ ,  $g(g(g(x)))$ , and  $g(g(g(g(x))))$ . In general, what is the formula for  $n$  compositions of  $g$ :  $g^{(n)}(x) = \underbrace{g(g(\cdots g(g(x))\cdots))}_{n \text{ times}}$ ?

## 16 Parent Functions 1

### The Shapes of Parent Functions

### Warm-Up

In the pre-lecture assignment you learned some of the key characteristics of several functions. Take a minute to sketch the graph of each of the following functions.



We can check our answers by looking at <https://www.desmos.com/calculator/mtayxqpik8>.

### Parent Functions in Models

10 min

Discuss each question below with a partner. Identify a function and determine what the  $x$  and  $y$ -variables might represent.

1. Which of the parent functions would be the best function to model an infant's growth from birth?
2. Which of the parent functions could model the distance that an object has fallen over time?
3. Which of the parent functions would give us the volume of a cube with side length  $x$ ?
4. Which of the parent functions might model the amount of money you earn at your job over time?

**Circles** The standard form of the equation of a circle is

$$(x - h)^2 + (y - k)^2 = r^2.$$

Here we view  $x$  and  $y$  as variables, while  $h$ ,  $k$ , and  $r$  take specific values for particular circles. In this activity we will use Desmos to explore the role played by the constants  $h$ ,  $k$ , and  $r$ .

### Transforming a Circle

10 min

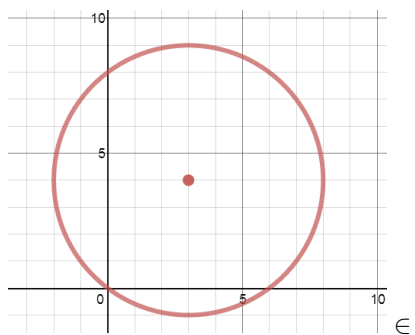
Open up [desmos.com](https://www.desmos.com) and enter the code  $(x-h)^2+(y-k)^2=r^2$ . Then create a slider for each of the variables. If you are lazy you can just click [here](#). With the help of the sliders describe the effect on the circle for each scenario below.

1. When  $h$  increases \_\_\_\_\_.
2. When  $k$  increases \_\_\_\_\_.
3. When  $r$  increases \_\_\_\_\_.

### Finding a Formula of a Circle

2 min

Find a formula for the circle in the image. (It is centered at the point  $(3, 4)$  and it goes through the origin.)



### Finding a Model From Data

### Finding a Formula for Distance Fallen

10 min

An object is thrown off a building. The function  $f(t)$  gives the distance (in ft) that the object has fallen after  $t$  seconds. We observe the following values.

$t$	0	0.5	1	1.5	2	2.5	3
$f(t)$	0	4	16	36	64	100	144

1. Find a formula for  $f(t)$ .
2. How far has the object fallen after 10 seconds?
3. How far does the object fall between seconds 6 and 7?

Hint: Create a table in Desmos using  $x$  in place of  $t$  and  $y = f(x)$ . Then change the window setting so that  $-1 < x < 5$  and  $-5 < y < 150$ . Decide which of the parent functions looks the most promising and try plotting it. Then start tweaking it.

Note: To get a graph that is twice as tall as the graph  $y = f(x)$  we would want  $y = 2f(x)$ .

## Homework

### Homework 10

- Find an equation of a circle of radius 6 centered at the point  $(1, -2)$ . Identify the coordinates of at least one point on the circle.
- Find the center and radius of the circle  $(x + 5)^2 + y^2 = 9$ . Identify the coordinates of at least one point on the circle.
- Plot the points given in the table below. Then identify a parent function that best fits the data. Tweak your guess until you have found the exact formula for  $f(x)$ .

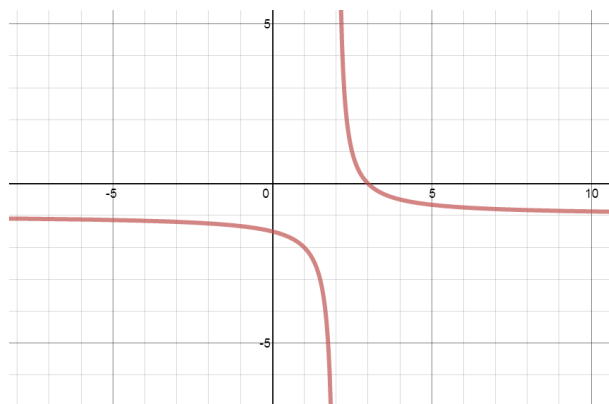
$x$	-3	-2	-1	0	1	2	3
$f(x)$	12	8	4	0	4	8	12

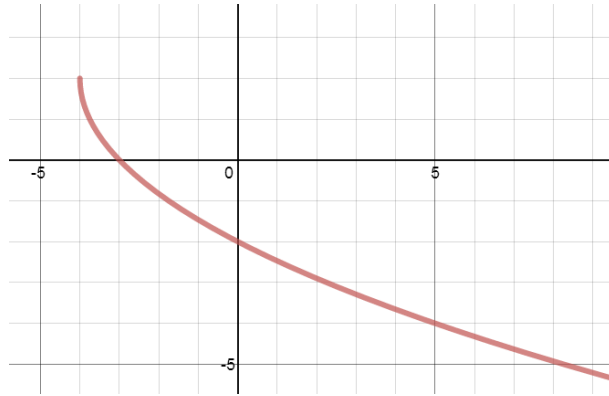
- A stalactite is forming in an inverted cone shape. The volume and height of the stalactite are measured at various stages. You have been asked to model the volume,  $V(x)$ , (in  $\text{ft}^3$ ) of the stalactite in terms of the height,  $x$  (in ft).

$x$	1	2	3	4	5	6
$V(x)$	0.25	2	6.75	16	31.25	54

Plot the points and decide on a parent function that best fits the data. Then tweak the function to obtain an exact value of  $V(x)$ .

- Determine the values of  $a$ ,  $h$ , and  $k$  for the function  $f(x) = a \cdot \frac{1}{x-h} + k$  whose graph is below. (The values of  $a$ ,  $h$ , and  $k$  are integers.)







## 17 Parent Functions 2

### Wave Functions

In the pre-class assignment we learned about the sine and cosine functions. We don't need to have a deep understanding of these wave functions, but we need to be familiar with the following bits of information about the parent functions  $y = \sin(x)$  and  $y = \cos(x)$ .

1. The output values repeat every  $2\pi$  units.
2. The minimum output is  $-1$  and the maximum output is  $1$ .
3. The graphs of both functions are waves, so we use either function to model objects that oscillate.

### Measuring the Tide

10 min

The tide level is measured every 5 hours. The table below gives the level of the tide in feet above the low tide level.

time (in h)	0	5	10	15	20	25	30	35
tide level (in ft)	0	1.48	3.73	3.41	1.00	0.07	2.00	3.93

1. Find a function of the form  $f(x) = a \cos(bx) + k$  that outputs the tide level after  $x$  hours.<sup>4</sup>
2. Estimate the level of the tide after 27 hours.
3. Does your model seem realistic? Explain.

### Exponential Functions

We need to know the following things about exponential functions.

1. The graph of  $y = a^x$  goes through the point  $(0, 1)$ , and only outputs positive values.
2. If  $0 < a < 1$ , then the graph is decreasing with a horizontal asymptote along the positive  $x$ -axis.
3. If  $1 < a$ , then the graph is increasing with a horizontal asymptote along the negative  $x$ -axis.
4. Exponential functions are useful in modeling the amount of some substance where the change in the amount of the substance depends only on the amount that you have now. Examples include compound interest, population, and radioactive decay.

### A Population Model

10 min

A settlement starts out with a population of 1000. Each year the population increases by 10%. Let  $P(t)$  be the function that gives the population in the settlement after  $t$  years.

1. Find the missing values in the table below.

$t$	0	1	2	3	4	5	6	7
$P(t)$	1000							

2. Find a formula for  $P(t)$  using the values from the table.
3. Find a formula for  $P(t)$  by reasoning it out directly.
4. What will the population be after 100 years?
5. Is this a realistic model?

<sup>4</sup>This is not the only possible form of the function, but it will work in this case. In general a wave function might have the form  $a \sin(b(x-h)) + k$  or  $a \cos(b(x-h)) + k$ . We will learn the specific details about shifting and scaling parameters in another section.

**Logarithms** Recall that each exponential equation  $a^x = b$  has a corresponding logarithmic equation  $\log_a(b) = x$ . Both equations mean the same thing.

**Logarithm Practice**

**10 min**

Evaluate each logarithm. Try to do it without a calculator.

1.  $\log_6(36)$
2.  $\log_7\left(\frac{1}{7}\right)$
3.  $\log_{10}(1000)$
4.  $\ln(1)$
5.  $\log_2(4)$
6.  $\log_4(2)$

Just do this one for the students unless you have extra time. This is the only opportunity to teach them how to solve exponential equations.

To solve an equation we can plot both halves of the equation and see where the curves intersect.

Solve the equation  $2^x = 5$  using three different methods.

1. Plot  $y = 2^x$  and  $y = 5$  in [desmos.com](https://www.desmos.com). Then click on the point where the curves intersect.
2. Convert the exponential equation  $2^x = 5$  to logarithmic form. (This is how the problem could be solved algebraically.)
3. Go to [wolframalpha.com](https://www.wolframalpha.com) and enter:

$$2^x = 5$$

## Homework

### Homework 11

Be sure to show your work. If/when you use a tool such as [desmos.com](https://www.desmos.com) or [wolframalpha.com](https://www.wolframalpha.com) you should indicate exactly how you used the tool to obtain your answer.

1. A scientist measures the percent,  $y$ , of the moon's face illuminated on day  $x$ . The values of her measurements are shown below (in decimal form) in a table.

$x$	1	8	15	22	28
$y$	0.000	0.502	0.999	0.550	0.006

- (a) The moon phases repeat regularly, so we hope that a wave function will model it. Fit a curve of the form  $y = a \sin(bx + c) + \frac{1}{2}$  to the data. (The constant  $1/2$  comes from the fact that on average we expect 50% of the moon's surface to be illuminated.) Your answer will be an approximation. The values of  $a$ ,  $b$ , and  $c$  might not be pretty.
  - (b) Use your model to estimate the portion of the moon that is visible on day 19.
  - (c) According to your model, how many days does it take for the moon to repeat its phases?
2. A bacteria colony is dying off. The colony loses half of its members each day. Let  $N(t)$  be the number of bacteria remaining after  $t$  days.

- (a) Complete the table below.

$t$	0	1	2	3	4	5	6
$N(t)$	4,000						

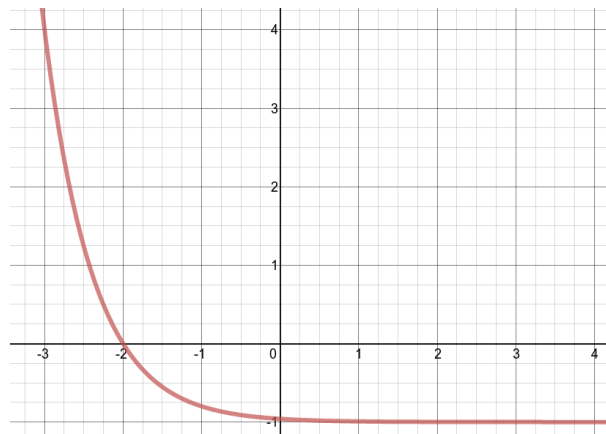
- (b) Find an exponential model for  $N(t)$ .
  - (c) According to your model, how many bacteria are present after 36 hours?
  - (d) How many days will it take for the bacteria population to reach 10?
3. Solve each equation.
    - (a)  $x^5 = x$
    - (b)  $4^x = x^4$  (Be sure to get all three solutions.)
    - (c)  $2x - x^3 = x^2$
    - (d)  $\sqrt{x} = \log_2(x)$
  4. Find the points of intersection of the curves

$$y = x^2 - 5 \quad \text{and} \quad x^2 + y^2 = 25.$$

5. Find the formula of a function,  $f(x)$  that approximates the data in the table.

$x$	1	2	3	4	5	6	7	8
$f(x)$	1.5	2.25	3.375	5.063	7.594	11.391	17.086	25.629

6. Determine the values of  $b$ ,  $h$ , and  $k$  for the function  $f(x) = b^{x-h} + k$  whose graph is below.



## 18 Transformations of Functions

### Rules for Transformations

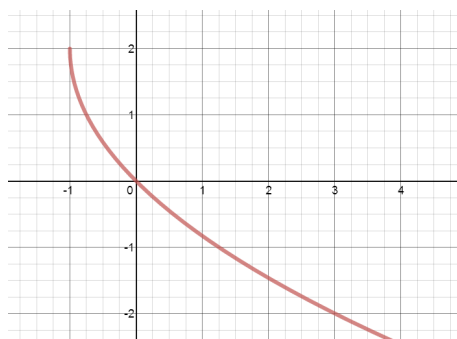
By this point we have mastered the graphs of over a dozen parent functions.

Recall the rules for shifting and scaling the graph of a function.

- To shift  $y = f(x)$  to the right by  $h$  units use  $y = f(x - h)$
- To shift  $y = f(x)$  to up by  $k$  units use  $y = f(x) + k$
- To scale  $y = f(x)$  horizontally by a factor of  $b$  use  $y = f\left(\frac{1}{b}x\right)$ .
- To scale  $y = f(x)$  vertically by a factor of  $a$  use  $y = af(x)$ .

The curve is  $y = -2\sqrt{x+1} + 2$ . In this example you need to emphasize that the order of scaling and shifting in the same direction matters. They could have shifted the graph vertically by 1 unit and then scaled vertically by a factor of 2. Alternatively, they could have scaled vertically by a factor of 2 and then shifted the graph vertically by 2 units.

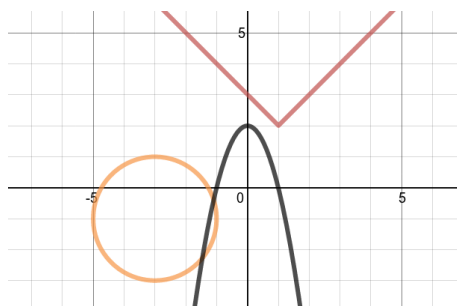
Below we see the graph of a curve. As a class we will identify which parent function this curve is based off of. Then we will identify how the parent function was shifted and/or scaled.



Mystery Curves

10 min

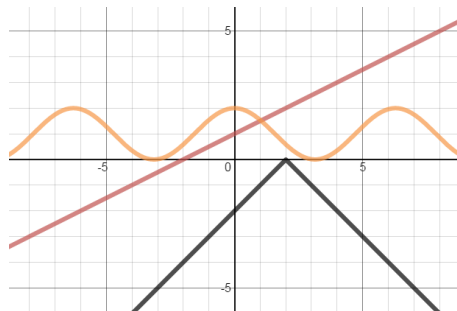
Below we see the graphs of three curves. Find the equation of each curve.



Remember that if  $f(x)$  is the parent function then one possible way of accounting for scaling and shifting transformations is to consider the function  $af(b(x - h)) + k$ .

**Mystery Curves II****10 min**

Below we see the graphs of three more curves. Find the equation of each curve.



Remember that if  $f(x)$  is the parent function then one possible way of accounting for scaling and shifting transformations is to consider the function  $af(b(x - h)) + k$ .

As an alternative to this activity, you can have students complete the Desmos activity Change of Heart — Transformations at <https://teacher.desmos.com/activitybuilder/custom/578e7bd8093fcd68077616d4>.

**A Change of Heart****10 min**

Go to <https://www.desmos.com/calculator/h4p6yr39pi> and create 5 transformations of the heart-shaped graph. Try to do it without peeking at the answer. Record the equations below.

**Purple:**

**Blue:**

**Orange:**

**Green:**

**Black:**

## Homework

### Homework 12

1. Go to <https://www.desmos.com/calculator/lpdp6e2cnk>. For this problem you should find equations for 6 different graphs. You will need to shift, scale, and/or reflect the graphs of parent functions to obtain a graph of each of the six functions. Write the equation for each function below. (Answers might not be unique.)

Red:

Orange:

Black:

Purple:

Blue:

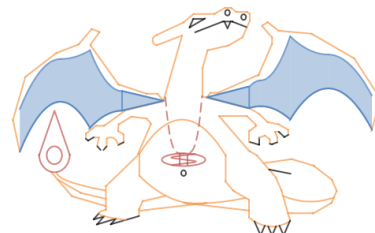
Green:

## 19 Transformations of Functions II

### Monsters and Restricted Curves

Take several minutes to show the class what the homework assignment will be like. There is nothing to hand in for this section.

For your homework today you will demonstrate your mastery of the functions and function transformations by creating a monster out of functions. The dragon to the right was made by a student at <http://desmos.com>.



You aren't required to make something so elaborate, but you will need to actively participate during the rest of class to be able to do the homework.

An instructor made sample monsters at <https://www.desmos.com/calculator/b96obpwkcg> and at <https://www.desmos.com/calculator/eowtedjmg7>.

Each of the examples took 30 minutes to make.

Go to <http://desmos.com> and enter the function  $y = 1 - x^2$ .

Sometimes we want to plot only a single part of a graph. Desmos has its own coding convention that allows us to restrict the  $x$  or  $y$ -values. At the end of the expression, simply put an inequality inside curly braces such as

$$\{a < x < b\} \quad \text{or} \quad \{c < y < d\}.$$

How would we display only the part of the graph above the  $x$ -axis?

Either use  $y = 1 - x^2\{-1 < x < 1\}$  or  $y = 1 - x^2\{y > 0\}$ .

### Desman

Go to <https://teacher.desmos.com/dashboard> and sign in. Then create a class activity for <https://teacher.desmos.com/activitybuilder/custom/57964af28bea723b4b917537>. Plan for this activity to take 35 minutes with the last few minutes for you to display some highlights. Only show the video after they have had a chance to sign in.

A 2-minute video showing one strategy can be found at <https://www.youtube.com/watch?v=Pd-JguTH8gw>

Creating a Desman

35 min

In this activity we will use Desmos to create a person.

Get with a partner and go to <http://student.desmos.com> and enter the class code the instructor displays.

You don't need to hand in anything for this activity because the instructor will have a record of it. Be sure to include your names after you sign in.

We will spend the rest of the class period working on our Desman.



## Homework

Create an activity at <https://teacher.desmos.com/activitybuilder/custom/5b75e2343a92af0ae16d07e5> and distribute the code to your students for their homework.

### Homework 13

1. Create a monster using Desmos. Go to <http://student.desmos.com> and enter the code from the instructor. Use Desmos to create a monster. This homework assignment will be graded out of 10 points according to the rubric below. Good luck.
  - 2 pts The monster is created using functions based off of several different parent functions.
  - 2 pts The color and style is consistent/appropriate for the whole picture.
  - 2 pts The monster consists only of curves from equations in Desmos.
  - 2 pts The project demonstrates that the student has a solid mastery of the shapes of graphs given by an equation.
  - 1 pts The final product compares favorably with the work of your classmates.
  - 1 pts Your monster looks cool.

If you are looking for inspiration or if you feel you are not an artist, you can always import an image into desmos and then try to fit curves to it. Keep it simple and have fun.

## 20 Models and Predictions

Consider showing off some of the best monsters from the homework assignment.

Begin the class by going to <https://student.desmos.com> with the code `vrt3`. Show the class some of the models and discuss the range of guesses. Talk about why the quadratic model was the right choice (the circle is two dimensional so the area increases in proportion to the radius squared). Ask for feedback and questions.

### Braking Distance

#### Braking Distance

15 min

The table below gives the distance that a car will travel after applying the brakes at a given speed.

Speed (in mi/h)	Distance to stop (in ft)
10	5
20	19
30	43
40	76.5
50	120
60	172
70	234

1. Find a function that outputs stopping distance when you input speed.
2. Estimate the stopping distance for a car that is traveling 43 mi/h.

After the students have completed the previous activity and you have discussed their solutions it is time to teach them about the WolframAlpha `fit` command. Open <http://wolframalpha.com> and enter the code:

`fit (10,5), (20,19), (30,43), (40,76.5), (50,120), (60,172), (70,234)`. Point out that the quadratic is probably the best model because of how small the coefficients are for the other terms in the other models. Then show how to fit a curve in desmos by typing  $y_1 \sim ax_1^2 + bx_1 + c$  into an empty cell.

By making a model we are able to answer questions that seem insurmountable at first glance. For example, recall the problem where the tide level is measured every 5 hours. The table below gives the level of the tide in feet above the low tide level.

Time (in h)	Tide level (in ft)
0	0
5	1.48
10	3.73
15	3.41
20	1.00
25	0.07
30	2.00
35	3.93

If we use the `fit` function in wolfram alpha we get a model that models the data poorly.

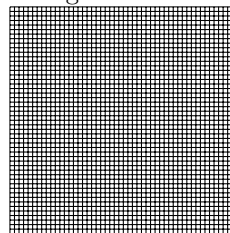
## Squares in a Grid

### Squares in a Grid

15 min

Suppose that we want to know the number of squares inside a  $50 \times 50$  grid. It doesn't seem practical to try

to count them all. Notice that the squares come in many sizes.



Let  $f(x)$  be the function that gives the number of squares in an  $x \times x$  grid. Complete the chart below.

$x$	1	2	3	4	5
$f(x)$					

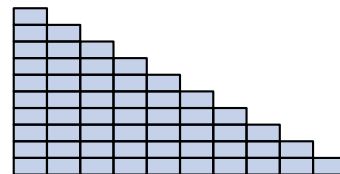
Then use Desmos to find an exact formula for  $f(x)$ . How many squares are in a  $50 \times 50$  grid? What about a  $2000 \times 2000$  grid?

## Brick Wall

### Brick Wall

15 min

You plan to make a diagonal brick wall as shown, but you don't know how many bricks to buy. You figure that the brick wall will have 100 rows. The top row is row 1; the next row is row 2; etc.



Let  $f(x)$  be the number of bricks used through  $x$ . Thus  $f(1) = 1$ ,  $f(2) = 3$ ,  $f(3) = 6$ , and so forth.

1. Find  $f(4)$ ,  $f(5)$ , and  $f(6)$ .
2. Find a formula for  $f(x)$ .
3. How many bricks will be needed to complete 100 rows?
4. How many bricks would be needed to complete 1000 rows?

## Homework

### Homework 14

Be sure to show your work. If/when you use a tool such as <http://desmos.com> or <http://wolframalpha.com> you should indicate exactly how you used the tool to obtain your answer.

1. A required dose of a certain drug depends on weight as follows.

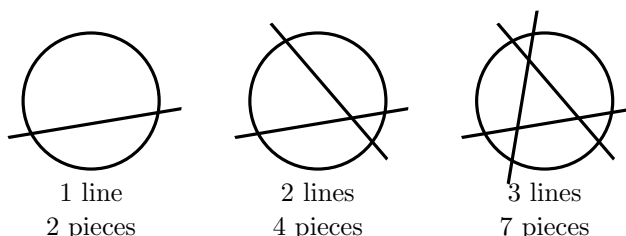
Weight (lbs)	100	150	200	250	300
Dosage (mg)	40	60	85	115	150

A doctor needs to know the dosage for a 450 lb patient. Find an equation that inputs the weight and outputs the dosage and use it to obtain the correct dosage. Be sure to write in full sentences. Hint: Enter the command

`fit (100,40), (150,60), (200,85), (250,115), (300,150)`

in WolframAlpha to get a curve that fits the data.

- In this question we investigate the number of regions into which we can divide a circular disc using straight lines. We can see below that 1 line can divide the disc into 2 pieces, 2 lines can make 4 pieces, and 3 lines can make as many as 7 pieces.



- (a) Find at least one more data point for this question.
  - (b) Find a formula for the function  $(x)$  that gives the maximum number of pieces into which we can divide a circular disc using  $x$  straight lines.
  - (c) Use the formula to find the maximum number of pieces that can be created using 100 lines. (Find the exact answer and not an estimate. Be sure you have the correct model.)
3. The table below shows the number of committees of size 3 that can be created from a group of a given size. For example, there are 10 committees of size 3 that can be created from a group of 5 people. If we label the five people  $A, B, C, D$ , and  $E$  then the committees are

	ABC	ABD	ABE	ACD	ACE				
	ADE	BCD	BCE	BDE	CDE				
Group size	3	4	5	6	7	8	9	10	100
Committees of size 3	1		10	20	35	56	84		

- (a) Find the number of committees of size 3 that can be chosen from among 4 people. Write out your reasoning without using the function that you will find in the next part.
  - (b) Find the function  $f(x)$  that will give the number of committees of size 3 that can be created from a group of  $x$  people.
  - (c) Fill in the missing entries in the table.
4. The Tylenol dosage recommendations for children are based on weight.

Weight (lbs)	8	14	20	29	41	54	65
Dosage (mg)	40	80	120	160	240	320	400

- (a) Find a function that approximates the dosage when you input the weight of the child.

(b) Estimate the proper dosage for a 25 lb child and for a 50 lb child.

## 21 Optimization

Start class by going over the homework problem involving the box formed by cutting out little  $x \times x$  squares from a  $6 \times 8$  sheet. This section will require a lot of hand holding. You can do a full demo at <http://geogebraTube.org/student/m48703>

Set it in projector mode and lead a discussion through this whole example. You can drag the black point around the curve to illustrate. Slowly and carefully derive the distance formula,  $d(x)$ , for them and talk them through how to find the minimum value of  $d(x)$ . As an afterthought, show them the circle method that is also at the same Desmos page.

Consider the curve  $y = 1 - x^4$ . Which point on that curve is closest to the origin? Go to <http://desmos.com/calculator/71w9rpo46g> to investigate.

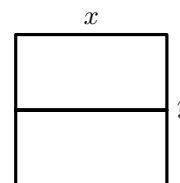
**Bessy The Cow**

A demonstration with resizable pens is available at <https://www.desmos.com/calculator/kmgv5l0wke>.

**Bessy's Dream Pen**

**15 min**

Bessy the cow wants to build her dream pen out of 600 ft of fencing. Her design calls for a rectangular pen that she will subdivide into two rooms as pictured. What are the values of  $x$  and  $y$  that will provide Bessy with the maximum enclosed area?



1. Write an equation that gives the total number of feet of fence in terms of  $x$ ,  $y$  and 600 ft
2. Solve that equation for  $y$ .
3. Write an expression that gives the area of the pen,  $A$ , in terms of  $x$  and  $y$ .
4. Replace  $y$  with an appropriate function of  $x$ .
5. Find the value of  $x$  that maximizes the area function  $A(x)$ .

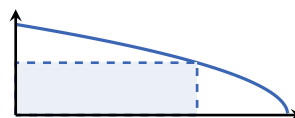
**The Biggest Rectangle**

A demonstration with a moveable rectangle is available at <https://www.desmos.com/calculator/uc1rqmq6ms>.

**A Rectangle Under a Curve**

**15 min**

A rectangle in Quadrant I has one corner at the origin and the opposite corner on the curve  $y = \sqrt{9 - x}$ . In this problem we want to find the length and width of the rectangle that has the maximum area.



1. Plot the curve and a sample rectangle.
2. Make a guess for the value  $x$  that gives a maximum area for the rectangle.
3. Write out a formula for the area of the rectangle.
4. Find the dimensions of the rectangle of maximum area under the curve.

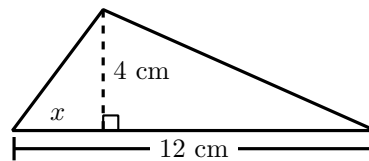
**Minimizing  
A  
Perimeter**

This problem is modeled at <http://desmos.com/calculator/kj1e70kjbw> in case you choose to show it to the class.

**The Minimum Perimeter of a Triangle**

**15 min**

Suppose we have a triangle with a base of 12 cm and a height of 4 cm as pictured. There are many such triangles possible, one of which has smaller perimeter than all others. Our goal in this problem is to find the length,  $x$ , that gives the triangle with smallest perimeter.



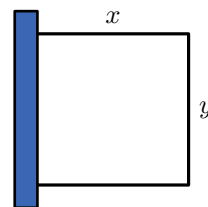
1. Use the Pythagorean Theorem to find the length of the missing sides of the triangle.
2. Create a function  $P(x)$  that gives the perimeter of the the big triangle.
3. What is the perimeter when  $x = 0$ ?
4. Find the value of  $x$  that minimizes the perimeter function  $P(x)$ .
5. What is the minimum possible perimeter?

## Homework

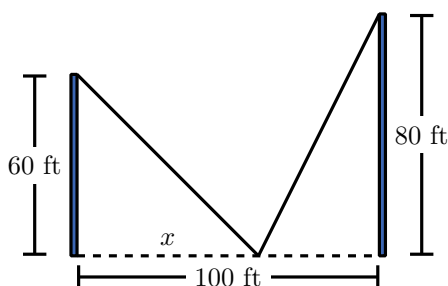
### Homework 15

Be sure to show your work. If/when you use a tool such as <http://desmos.com> or <http://wolframalpha.com> you should indicate exactly how you used the tool to obtain your answer.

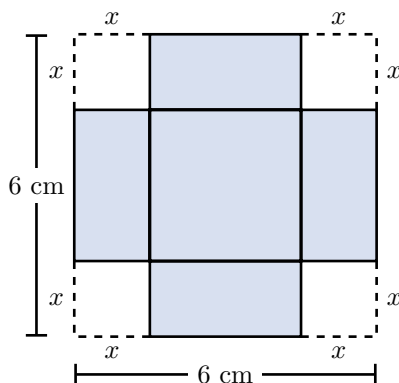
1. Bessy the cow wants to build her dream pen out of 300 feet of fencing. Her design calls for a rectangular pen built up against an existing wall, as pictured. Find the dimensions  $x$  and  $y$  that maximize the area of the enclosed pen.



2. Find the coordinates of the point(s) on the curve  $y = \frac{1}{1+x^2}$  closest to the origin.
3. Two vertical towers of heights 60 ft and 80 ft stand on level ground with their bases 100 ft apart. A cable is to be strung from the top of one pole to some point on the ground between the poles, and then to the top of the other pole. What is the minimum possible length of cable required?



4. Find the dimensions of the rectangle of largest area that has its base on the  $x$ -axis and its other two vertices above the  $x$ -axis lying on the parabola  $y = 8 - x^2$ .
5. A box is created by taking a  $6 \text{ cm} \times 6 \text{ cm}$  sheet of metal and cutting out square corners, then folding up the sides as pictured below. Find the value of  $x$  that will result in a box with maximum possible volume.



(After you have solved the problem you can go to <http://geogebraTube.org/student/m48703> to check your answer, but make sure you know how to solve it from scratch.)



## 22 Final Project — Portfolio and Video

The final project is a small group project where students will answer a research question by applying the statistics and/or modeling techniques learned in class to real-world data. The presentation should be fun and interesting because part of your grade is based on your ability to engage your classmates.

**Purpose** The purpose of this activity is to allow students to showcase the mathematical skills that they have developed this semester and to demonstrate their ability to successfully communicate solutions in an effective and fun manner.

**Tasks** Groups will be chosen in class. Each group will select one of the projects from the pages that follow and thoroughly solve the problem. Then students will

The project will consist of two components.

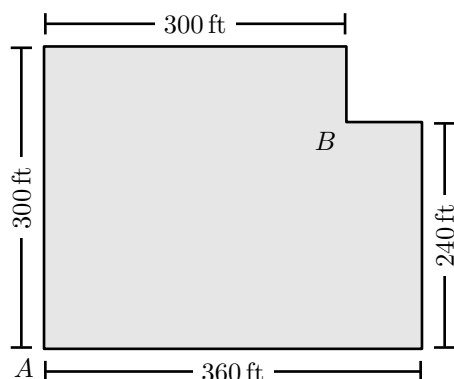
1. A neatly typed portfolio containing a full solution will be submitted in Canvas. The portfolio will contain the following components.
  - (a) Cover Page: creative and appropriate design by students
  - (b) Title Page: project name, date, instructor name, students' names
  - (c) Executive Summary: one page abstract of the problem, approach used, results obtained, and conclusions reached
  - (d) Supporting Data and Materials: computations, assumptions, collection methods, labeled drawings, labeled computer plots and/or printouts, references cited
  - (e) Evaluation Summary: (at most one page) of the group's learning experience in working on the project
2. a 4-7 minute video presentation that we will watch in class. The video will be uploaded to a video sharing site (like YouTube) and a link will be submitted in Canvas. The video should do the following.
  - (a) Introduce the viewers to the problem independently of the portfolio,
  - (b) Explain the solution method.
  - (c) Be fun and engaging. Do not make it boring!

Points are distributed as in the tables that follow.

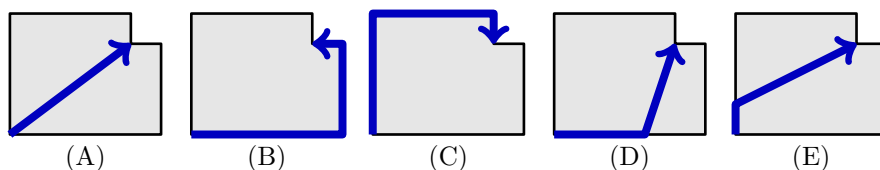
	Level/Criterion	Understanding	Reasoning/Critical Thinking	Communication
<b>Portfolio</b>	Novice (1-2 pts)	There is no solution, or the solution has no relationship to the task	No evidence of a strategy or procedure, or uses a strategy that does not help solve the problem. No evidence of mathematical reasoning.	There is no explanation of the solution. The explanation cannot be understood, or it is unrelated to the problem.
	Apprentice (3-5 pts)	The solution is not complete, indicating that parts of the problem are not understood.	Some evidence of mathematical reasoning. Some parts may be correct, but a correct answer is not achieved	There is an incomplete explanation; it may not be clearly presented.
	Practitioner (6-8 pts)	The solution shows that the student has a broad understanding of the problem and the major concepts necessary for its solution.	Uses effective mathematical reasoning. Mathematical procedures used. All parts are correct and a correct answer is achieved.	There is a clear explanation using mathematical notation.
	Expert (9-10 pts)	The solution shows a deep understanding of the problem including the ability to identify the appropriate mathematical concepts and the information necessary for its solution.	Employs refined and complex reasoning. Applies procedures accurately to correctly solve the problem and verify results. Verifies the reasonableness of the solution.	There is a clear, effective explanation detailing how the problem is solved. The important steps are included so that the reader does not need to infer how and why decisions were made. The problem is presented in a fun and cool manner.

	Level/Criterion	Quality	Mathematics	Communication
<b>Video</b>	Novice (1-2 pts)	The video is poorly formatted. Sound issues abound.	No evidence of a strategy or procedure, or uses a strategy that does not help solve the problem. No evidence of mathematical reasoning.	There is no explanation of the problem. The solution cannot be understood, or it is unrelated to the problem.
	Apprentice (3-5 pts)	The video is unpleasant to watch. The narrative is not cohesive.	Some evidence of mathematical reasoning. Some parts may be correct, but a correct answer is not achieved	There is an incomplete explanation of the problem or the solution; it may not be clearly presented.
	Practitioner (6-8 pts)	The audio and video quality are good. The viewer can easily understand the dialog and images.	Uses effective mathematical reasoning. Mathematical procedures used. The important details are explained.	There is a clear explanation using mathematical notation. The problem is made to look fun and interesting.
	Expert (9-10 pts)	The audio and video quality are good. The set and other choices show forethought. The acting is enjoyable to watch.	The mathematical question is explained in a manner that any viewer could understand. All of the important details are explained.	There is a clear, effective explanation detailing how the problem is solved. The important steps are included so that the reader does not need to infer how and why decisions were made. The problem is presented in a fun and cool manner.

**Project:** Two buildings at opposite corners of a parking lot need to be connected by a cable that will be buried under ground. It costs \$11/ft to lay the cable across the parking lot and \$8/ft to lay the cable along the outer edge of the lot. We need to run cable from point  $A$  to point  $B$ .



You have been given the task of finding the path to lay the cable that will minimize the total cost. There are 5 types of paths that need to be considered.



Path (A) cuts straight across the parking lot. Paths (B) and (C) go around the parking lot. Paths (D) and (E) cut across the parking lot after following an edge for some distance.

You can see a restatement of the problem at <https://www.desmos.com/calculator/thbnkjppwh>.

1. Choose an appropriate variable (or variables) and label your picture. Then create a function (or functions) that will give you the cost for each path.
2. Find the path that results in the minimum total cost.
3. Write up your recommendation for your boss and prepare a presentation for the next board meeting so that the job can be carried out. Keep it professional.

**Project:**  
**The Value**  
**of Money**

The table below shows the cost of printing U.S. currency incurred by the Federal Reserve Board.

Denomination	Cost per note
\$1 & \$2	5.4¢
\$5	10.1¢
\$10	9.2¢
\$20 & \$50	10.2¢
\$100	13.1¢

The table below provides information on the size and composition of U.S. coinage.

Denomination	Cent	Nickel	Dime	Quarter	Half Dollar	Susan B. Anthony	\$1 Coin
<b>Composition</b>							
Copper	2.5%	75%	91.67%	91.67%	91.67%	87.5%	88.5%
Zinc	97.5%	—	—	—	—	—	6%
Manganese	—	—	—	—	—	—	3.5%
Nickel	—	25%	8.33%	8.33%	8.33%	12.5%	2%
<b>Weight (g)</b>	2.5	5	2.268	5.67	11.34	8.1	8.1

From July 2013 to July 2014, the price of copper varied from a low of \$6.44 per kg to a high of \$7.43 per kg. Over the same time period, zinc ranged from \$1.79 per kg to \$2.32 per kg, manganese from \$2.10 per kg to \$2.35 per kg, and nickel from \$13.27 per kg to \$21.21 per kg.<sup>5</sup>

Use the above data and Fermi estimation to calculate the answers to at least four interesting questions about U.S. money. For example, you might compute the maximum and minimum total real value (not face value) of all U.S. money in circulation during the time period spanned by the data, or you might compute the cost savings of replacing all pennies with nickels (in a 5 : 1 ratio, of course), or the added cost of making nickels out of pure nickel and pennies out of pure copper. There are many possible questions so be creative and craft questions you find genuinely intriguing.

<sup>5</sup><http://www.infomine.com/Investment/>

**Project:**  
**Spinning**  
**Pennies**

Most people are familiar with the concept of loaded dice. These are dice that have been weighted so that it increases the chances of certain numbers coming up. The effect is small enough so that it doesn't happen every single time, but it's large enough to swing the expected value from one side to the other. This subtle type of fraud can be very difficult to detect, but it can be done using the types of statistical techniques that we have developed (and even some more complex ones known as hypothesis testing that you can learn about in a true statistics class).

Let's say that you're going to play a very simple gambling game against a casino. You're going to spin a penny. If it's heads, the casino pays you \$1. If it's tails, you pay the casino \$1. Most people think this game is fair, but that's because they assume that spinning a coin is a 50-50 proposition. But is that true? In this project, we will test the question of whether spinning a penny is truly 50-50.

In order to test this, follow the procedure below:

- Step 1** (Collect data): We will be testing the question by looking at clusters of 10 spins. A valid spin is done on a hard, flat surface in such a way that the spin lasts at least three seconds and where the coin does not collide with any objects (such as a table leg if you're spinning on a hardwood floor) and does not leave the surface (such as spinning on a table and having the coin fall off the edge). Gather data on 1000 sets of 10 penny spins, for a total of 10,000 spins. (Hint: Find some open space and spin 10 coins at the same time instead of spinning one coin 10 times. You may want to enlist the help of family and friends and have a coin-spinning party to make it go faster because this can take 3 hours or more if one person is doing it alone.)
- Step 2** (Organize the data): Create a distribution table of the number of heads spun in each 10 spin cluster. Put this information into a graph.
- Step 3** (Analyze the data): Compute the mean, median, mode, and standard deviation. (Hint: For the standard deviation, WolframAlpha won't let you type in 1000 values, so you will need to do it in Excel or by hand. If you do it by hand, think about how we calculated this value by hand using a chart and devise a way of doing it using multiplication as repeated addition to make the calculation simpler.)
- Step 4** (Graphically compare with theoretical values): The theoretical distribution for the number of heads spun in 10 spins with a coin where it is a 50-50 proposition is:

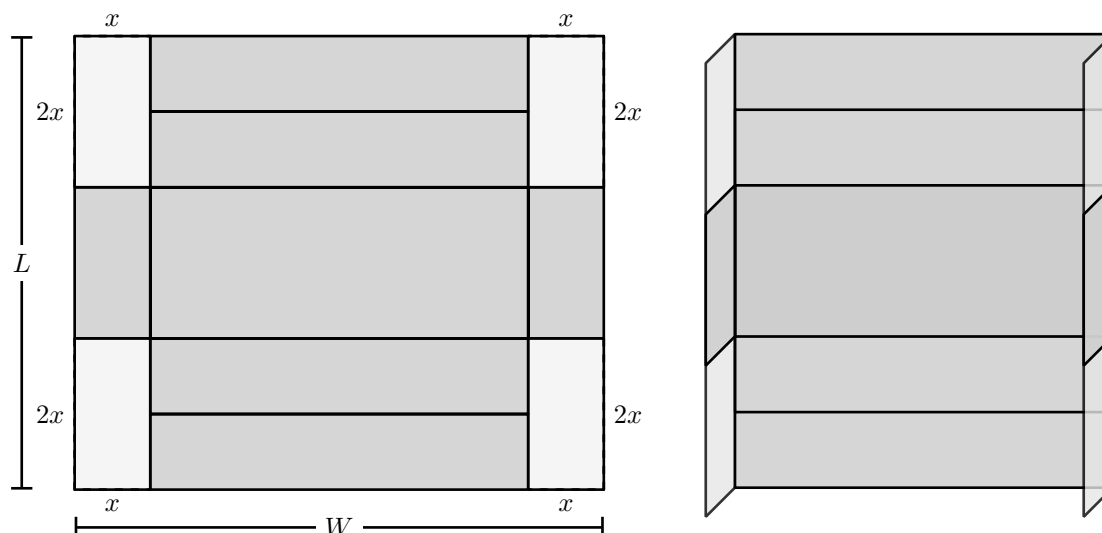
# of H	0	1	2	3	4	5	6	7	8	9	10
Freq.	0.1%	1.0%	4.4%	11.7%	20.5%	24.6%	20.5%	11.7%	4.4%	1.0%	0.1%

Compute the relative frequency of the number of times you got each number of heads and compare with the theoretical distribution. Graph both distributions on the same axes. How similar are the two graphs? What do you conclude from this observation?

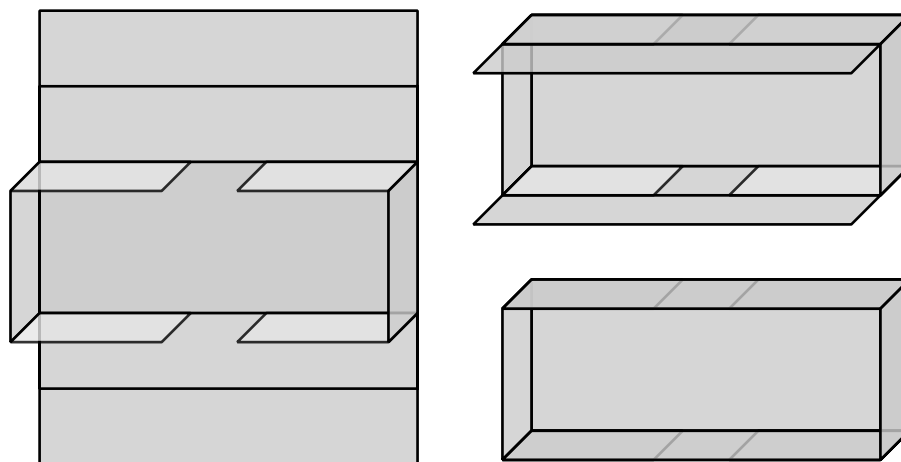
- Step 5** (Quantifiably compare with theoretical values): The theoretical mean for the average number of heads is  $\mu = 5$ . Calculate your sample-size adjusted  $z$ -score by computing  $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{1000}}$ , where  $\bar{x}$  is your average number of heads per cluster of 10 spins and  $\sigma$  is the standard deviation for your sample of the number of heads in 10 spins from your 1000 clusters of spins. What does this  $z$ -score say about the likelihood that a spin coming up H really is 50-50 and your results are simply due to chance? Write up your conclusion.

**Project:**  
**Folding a**  
**Box**

In this project we want to discover the maximum possible volume of an open-topped, tabbed box folded from an  $8.5'' \times 11''$  piece of paper. The construction goes as follows. Draw lines that are offset from each edge by length  $x$ . Then along one pair of sides draw another set of lines offset by  $2x$ . Next fold up the two edges with a single offset line as pictured below.



Then cut tabs where the offset rectangles meet and fold the tabs in, and fold up the remaining edges.



Note that we have labeled the box with  $L$  and  $W$  because there are two possible choices for deciding which edge gets the double fold.

Determine the length of  $x$  and the orientation ( $W = 8.5''$  or  $W = 11''$ ) that maximize the volume of the box.

**Project:** You are playing a basketball game against another team. You are ahead by two points with only a few seconds left. There is just enough time for the opposing team to get one shot off. You need to decide whether to foul the shooter or not. The shooting stats for each player are below.

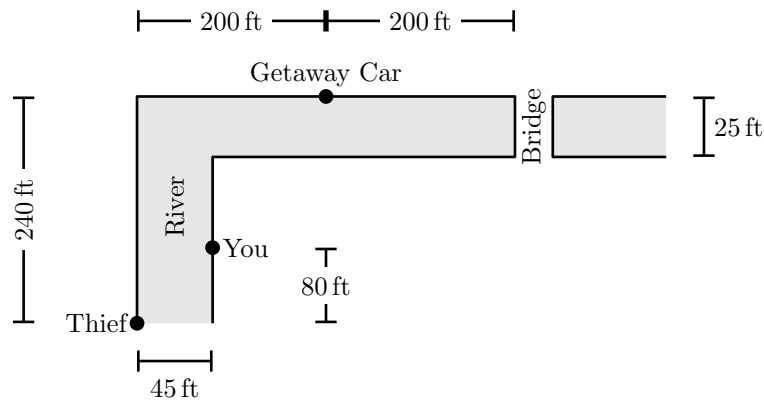
Player	3-pt %	3-pt % when fouled	2-pt %	2-pt % when fouled	free-throw %
point guard	41%	0%	51%	15%	82%
shooting guard	47%	0%	46%	14%	91%
small forward	31%	0%	52%	10%	74%
power forward	23%	0%	48%	9%	67%
center	21%	0%	53%	14%	51%

If the score is tied, then the game will go into overtime. In overtime, your team has a 65% chance of winning because the opponent's star player has fouled out of the game. The opposing player gets 2 free throws if fouled on a 2-pt shot and 3 free throws if fouled on a 3-pt shot. If the player makes the shot when fouled they get a single extra free throw.

1. Draw out a probability tree to represent all the possible outcomes if you foul the player when they attempt a 2-pt shot. Your tree should include all the possibilities of them making their shot, the subsequent free-throws, and overtime (if needed). (You might choose to label the branches with variable probabilities such as  $p$  and  $q$  so that the tree could represent any of the 5 players.)
2. For each of the five players determine the probability of your team winning if you foul that player on a 2-pt shot, and determine whether you should foul the player or not on a 2-pt shot.
3. Draw out a probability tree to represent all the possible outcomes if you foul the player when they attempt a 3-pt shot. Your tree should include all the possibilities of them making their shot, the subsequent free-throws, and overtime (if needed).
4. For each of the five players determine the probability of your team winning if you foul that player on a 3-pt shot, and determine whether you should foul the player or not on a 3-pt shot.

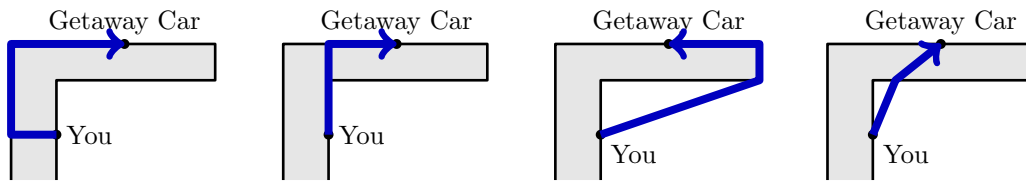
**Project:** You are a police officer in the foot pursuit of a thief. You are on the opposite side of a muddy river as the perpetrator with a 90 degree bend in it and you can see the getaway car waiting for him. A map of the area is provided below (not to scale).

**Police Pursuit**



You can run 24 feet per second on solid ground when you're dry, and slog through the muddy river at 2 feet per second. However, after coming out of the muck, you can only run 19 feet per second. The thief can run 18 feet per second on solid ground and follows the path around to the getaway car. (In other words, he does not get into the river.) Your goal is to get to catch the thief directly or at least beat him to the getaway car.

There are a few different ways you can try to capture the criminal:



1. What would happen if you tried to cut straight across the river to try to cut off the thief's path? Would you reach the opposite side before he does? If not, maybe you can catch up to him before he gets to the getaway car?
2. What if you ran straight ahead to where the river is a little bit more narrow? Could you catch him then?
3. Maybe if you just stayed out of the muddy river completely, you could catch him. What if you were to run to the bridge? Could you get to the getaway car before he gets there?
4. What would happen if you headed diagonally to the edge of the river, and then diagonally across the river? Is there a spot you can pick to cross over that will allow you to beat him there?

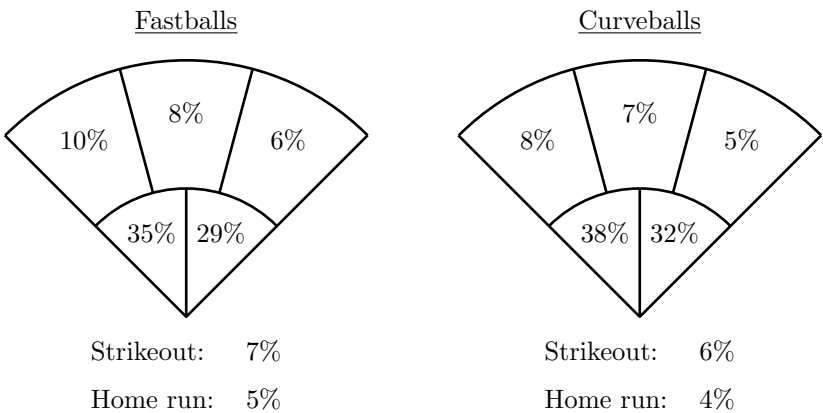


**Project:  
Bottom of  
the Ninth**

You're the statistician for a baseball team. The owners have asked you to provide an analysis of the manager's decision-making. They've provided you with a high pressure situation in which they're second-guessing the manager, and they want you to provide a mathematical analysis of the situation for review.

It's the bottom of the ninth inning, the bases are loaded, and you're trying to hold onto a one run lead. The momentum has been against you for the last couple innings, so the owners believe that giving up a run here would break the team's morale and the team would definitely lose. So they want you to look to maximize the chances of getting the out and not worry about extra innings.

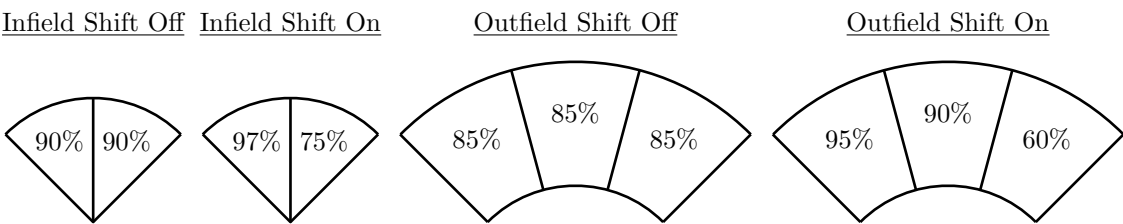
Here is the scouting for the upcoming batter. These are charts that display the chances that he will hit the ball into different regions of the field based on whether the pitcher throws fastballs or curve balls. It's broken down into two infield sections and three outfield sections, plus the chances of a home run or strikeout.



There are two different defensive adjustments that you can recommend.

- Infield shift (Off/On): By shifting your infield towards third base, you increase your chances of getting an out on a ball hit to the infield on the third base side at the risk of an increased chance that a ball on the first base side will get through.
- Outfield shift (Off/On): By shifting your outfield towards the third base side, you increase your chances of getting an out in both left and center field, but you also leave a large hole in right field.

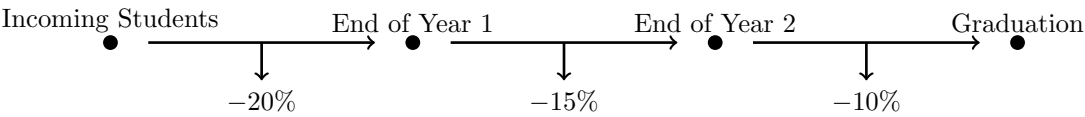
Here are the charts that tell you the probabilities of getting outs for both the infield and outfield based on the defensive moves.



Provide your owners with a report of what the manager should have done based on the mathematics of the situation. What should the manager have communicated to the pitcher? Should he have called for an infield shift or an outfield shift?

**Project:**  
**Student**  
**Success**  
**Initiatives**

You are hired as a consultant to help a school maximize the number of students that graduate from their three year specialized academic program. Here is their current retention pipeline:



This shows the percent of students that do not persist towards graduation for each year of the program. The percent losses are relative to the number of students that have reached the previous level of achievement. In other words, between the end of year 1 and the end of year 2, the school is losing 15% of the students that completed the first year, not 15% of the incoming class of students.

Based on your research, you’ve found a number of programs and opportunities that can be applied during each year of the program. Each program has a different cost and a different level of improvement. The improvement changes the drop rate by the listed amount. For example, if the school invests in General Education Tutoring, then the drop rate during the first year changes from  $-20\%$  to  $-11\%$

Year	Intervention	Cost	Improvement
Year 1	General Education Tutoring	\$4000	9%
Year 1	Study Skills Workshops	\$2500	6%
Year 2	Community Mentoring Programs	\$1500	2%
Year 2	Department-Based Tutoring Services	\$3000	4%
Year 2	Internship Programs	\$6000	8%
Year 3	Push to Graduation	\$3500	5%
Year 3	Career Counseling Services	\$2000	3%

The school has \$10,000 to spend on student success initiatives in order to improve their graduation rates. Any combination of these initiatives can be applied as long as the cost does not exceed \$10,000. In addition to these large scale initiatives, you believe that for every \$500 of unspent money, the school can leverage social events that would improve the first year retention by 0.5%. So if there is \$1500 in unspent money after these initiatives, they can improve their first year retention by another 1.5%.

Write up a report to the school leadership that tells them how to best spend the money if they want to maximize the number of students that graduate from their program.