Solutions for Category Theory for Scientists by David Spivak

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Contents

1 Introduction

The book *(2013, Spivak) "Category theory for scientists"* [1].

2 The Category of Sets

2.3.3.1

Create an olog for human nuclear biological families that includes the concept of person, man, woman, parent, father, mother, and child. Make sure to label all the arrows, and make sure each arrow indicates a valid aspect in the sense of Section 2.3.2.1. Indicate with check-marks the diagrams that are intended to commute. If the 2-dimensionality of the page prevents a check-mark from being unambiguous, indicate the intended commutativity with an equation.

Answer:

$$
\begin{tikzcd}
 & person & \\
man \arrow[ur, "is"] & & woman \arrow[ul, "is"'] \\
 & parent \arrow[uu, "is"] \arrow[dd, "has"] & \\
father \arrow[uu, "is"] \arrow[ur, "is"] \arrow[dr, "has"'] & & mother \arrow[uu, "is"'] \arrow[ul, "is"'] \arrow[dl, "has"] \\
 & child \arrow[uuuu, bend right=60, "is"'] &
\end{tikzcd}
$$

Commutative square examples:

- "parent is person" $\circ$ "mother is parent" = "woman is person" $\circ$ "mother is woman"
- "mother has child" = "parent has child" $\circ$ "mother is parent"
- "child is person" $\circ$ "parent has child" $\neq$ "parent is person"

2.4 Products and coproducts

2.4.1.4

How many elements does the $\{a, b, c, d\} * \{1, 2, 3\}$ have?

Answer:

12

2.4.1.8

Answer:

(a) No, because $a(b + c) \neq (a + b)c$.

(b) No, because $x * 0 \neq x$.

(c) Yes.

2.4.1.15

(a) Let X and Y be sets.. construct the "swap map" $s : (X \times Y) \to (Y \times X)$

Answer:

$s : (X \times Y) \to (Y \times X) = (,) \circ \langle \pi_2, \pi_1 \rangle$
Note: we used angle brackets. Is it really correct?

(b) Can you prove that s is a isomorphism using only the universal property for product?

Note: $(f : X \to Y)$ is an isomorphism if $\exists (g : Y \to X) : g \circ f = id_X \land f \circ g = id_Y$
Note: diagram (f, g, h) commutes if $f \circ g = h$

Answer:

In universal property of products, put A equal to $Y \times X$ and get $\exists! g : (Y \times X) \to (X \times Y)$. In a similar way, we have $\exists! s : (X \times Y) \to (Y \times X)$. We need to show that $g \circ s = id_{(X \times Y)}$ and $s \circ g = id_{(Y \times X)}$. On the following diagram,

$$\begin{tikzcd}
& A \times B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
A & & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, "\exists! id"] &
\end{tikzcd}$$

$id = id_{(A \times B)}$ - is the unique identity function. By combining diagrams for f and g we reduce the $g \circ f$ to the similar case.

2.4.2.4

Would you say that a phone is the coproduct of a `cellphone` and a `landline phone`?

Answer:

Yes, until we consider other types of phones besides cell- and landline ones.

2.4.2.10

Write the universal property for coproduct in terms of a relationship between $Hom_{Set}(X, A)$, $Hom_{Set}(Y, A)$ and $Hom_{Set}(X \sqcup Y, A)$.

Answer:

$Hom_{Set}(X, A) \times Hom_{Set}(Y, A) \cong Hom_{Set}(X \sqcup Y, A)$

2.4.2.13

TODO

2.4.2.14

TODO

2.5 Finite limits in Set

2.5.1.2

Answer:

$X \times_Z Y = \{(x_1, z_1, y_1), (x_2, z_2, y_2), (x_2, z_2, y_4), (x_3, z_2, y_2), (x_3, z_2, y_4)\}$

2.5.1.3

(a)

Answer:

Let $X = \{1, 2, 3, 4, 5\}; Y = \{a, b, c\}$; where $C = \{R, B, Y\}$

We have: $X \times_C Y = \{1a, 4a, 2c, 5c, 3b\}$
(b) TODO (obvious).

2.5.1.5

(a) Suppose that $Y = \emptyset$; what can you say about $X \times_Z Y$?

Answer:

$X \times_Z Y = \emptyset$

(b) $Z = 1$; what can you say about $X \times_Z Y$?

Answer:

$\forall X, Y : X \times_Z Y \cong X \times Y$

2.5.1.6

.. Aristotelian space and time ..
$S = R^3$; $T = R$; $Y = S \times T$; $g1 : Y \to S$; $g2 : Y \to S$ where $g1, g2$ projects space-time to its components. $X = \{1\}$; $f_1 : X \to S$; $f_2 : X \to T$ is a set of one element and its space-time projections.

(a) What is the meaning of

$$\begin{tikzcd} W_1 \arrow[r] \arrow[d] & Y \arrow[d, "g_1"] \\ X \arrow[r, "f_2"] & S \end{tikzcd} \qquad \begin{tikzcd} W_2 \arrow[r] \arrow[d] & Y \arrow[d, "g_2"] \\ X \arrow[r, "f_2"] & S \end{tikzcd}$$

Answer:

1 is associated with its time and space. W_1 yields time points of Y corresponding to 1's position. W_2 yields the space points corresponding to 1's time.

(b) Interpret the sets in terms of the center of mass of MIT at the time of its founding.

Answer:

TODO: (**unsure**) Is it just the MIT-relative space and time points?

2.5.1.10

.. Appropriate or misleading olog labels ..

Answer:

(a) a person whose favorite color is blue - OK

(b) a dog whose owner is a woman - OK

(c) a good fit - Nope. We would say that a good fit requires less or equal width.

2.5.1.11

(a) Consider your olog from Exercise 2.3.3.1. Are any of the commutative squares there actually pullback squares?

Answer:

Yes, for example: "father" = "man" $\times_{person}$ "parent"

$$\begin{tikzcd}
\boxed{\text{father}} \arrow[r, "is"] \arrow[d, "is"] & \boxed{\text{man}} \arrow[d, "is"] \\
\boxed{\text{parent}} \arrow[r, "is"] & \boxed{\text{person}}
\end{tikzcd}$$

(b) Now use ologs with products and pullbacks to define what a brother is and what a sister is:

Answer:

$$\begin{tikzcd}
brother \arrow[r, "is"] \arrow[d, "is"] & child \arrow[d, "p"] & \\
man \arrow[r, "p"] & pom \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \{T\} \arrow[d, "id"] \\
& person \arrow[r, "n_{ch}"] & \{T, F\}
\end{tikzcd}$$

where: *pom* stands for "parent of many", p is "has as parent" and n_{ch} is "number of children"

2.5.1.13

Pullback diagram in which the fiber product is isomorphic to the preimage of $y \in Y$.

Answer:

$$\begin{tikzcd}
Y^{-1} \times \{y\} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \{y\} \arrow[d, "id"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

2.5.1.15

Create an olog whose underlying shape is a commutative square. Now add the fiber product so that the shape is the same as that of Diagram (2.32). Assign English labels to the projections and to the dotted map A, such that these labels are as canonical as possible.

Answer:

TODO: (**unsure**)
The general impression is the following:

- The commutative square can be visualized as a pyramid in a 3D space.
- The pullback corresponds to a 2D plane with two axes.
- The pullback over one-element set corresponds to a 2D plane with two orthogonal axes.
- The projections π_1 and π_2 play the role of parallel projections from a point in the plane to its axes.
- The map $A \to (D \times_P W)$ acts as a projection from a point in 3D space to the plane.

$$\begin{tikzcd}
 & & \boxed{(D \times_P W) \text{ woman dog owner}} \arrow[dll, bend right, "\pi_1"'] \arrow[drr, bend left, "\pi_2"] & & \\
 & & \boxed{\text{(A) blond woman dog owner}} \arrow[u, "\langle \text{has, is} \rangle"] \arrow[dll, "has"] \arrow[drr, "is"'] & & \\
\boxed{\text{(D) dog}} \arrow[drr] & & & & \boxed{\text{(W) woman}} \arrow[dll] \\
 & & \boxed{\text{(P) person}} & &
\end{tikzcd}$$

2.5.1.18

(a) Create an olog that defines two people to be 'of approximately the same height' if and only if their height difference is less than half an inch, using a pullback.

Answer:

$$\begin{tikzcd}
\{((p_1, p_2), \Delta) | p_1, p_2 \in \text{people}, \Delta \in \mathbb{R}\} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \{\Delta | -0.5 < \Delta < 0.5\} \arrow[d, "is"] \\
\boxed{\text{pair of people}} \arrow[r, "h_2 - h_1"] & \mathbb{R}
\end{tikzcd}$$

(b) In the same olog, make a box for those people whose height is approximately the same as a person named 'The Virgin Mary'. You may need to use images.

Answer:

$$\begin{tikzcd}
\boxed{\text{people of approx Virgin Mary height}} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \{((p_1,p_2),\Delta)|...\} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \{\Delta| -0.5 < \Delta < 0.5\} \arrow[d, "is"] \\
\{(\boxed{\text{The Virgin Mary}}, p_2)|p_2 \in \text{people}\} \arrow[r, "is"] & \boxed{\text{pair of people}} \arrow[r, "h_2 - h_1"] & \mathbb{R}
\end{tikzcd}$$

TODO: (**unsure**) We did not use images, despite the textbook suggestion.

2.5.1.19

Let $W = X \times_Z Y$ and $W' = X' \times_{Z'} Y'$, and form the diagram to the right. Use the universal property of fiber products to construct a map $W \to W'$ such that all squares commute

Answer:

Let $WX = \pi_1$, $WY = \pi_2$, $W'X' = \pi'_1$, $W'Y' = \pi'_2$. Define $f : W \to W' = (x, y)_z \mapsto (XX'(x), YY'(y))_{ZZ'(z)}$. We want to show that it makes $WW'Y'Y$ to commute. Indeed, $\pi'_2 \circ f = YY' \circ \pi_2 = y \mapsto YY'(y)$. By analogy, it also commutes $WW'X'X$. The triangles $WW'Y'$ and $WW'X'$ also commute. The universal property for fiber products says that we can only have one such mapping, thus this is the answer.

2.5.2.6

(a) TODO: Copy the picture from the book notes
(b) TODO: Copy the picture from the book notes
(c) TODO: Find the correct answer

2.5.3.3

Come up with an olog that uses equalizers in a reasonably interesting way. Alternatively, use an equalizer to specify those published authors who have published exactly one paper. Hint: find a function from authors to papers; then find another.

Answer:

We follow the second way.

$$\begin{tikzcd}
\text{Authors} \arrow[r, shift left, "\text{num of papers}"] \arrow[r, shift right, "\text{const}_1"'] & \mathbb{R}
\end{tikzcd}$$

where $const_1$: authors $\to \mathbb{R}$ maps all authors to 1.

2.5.3.4

Find a universal property enjoyed by the equalizer of two arrows, and present it in the style of Lemmas 2.4.1.10, 2.4.2.7, and 2.5.1.14.

Answer:

For any set A, function f and g there exists unique u mapping A to $Eq(f, g)$ such that everything commute, in particular, $\pi \circ u = p$.

$$\begin{tikzcd}
Eq(f,g) \arrow[dd, bend right, "\pi"'] \\
A \arrow[u, dashed, "\exists! u"] \arrow[d, "p"] \\
X \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\
Y
\end{tikzcd}$$

2.5.3.5

(a) A terminal set is a set S such that for every set X, there exists a unique function $X \to S$. Find a terminal set.

Answer:

Any set S with one element. There exist exactly one function $X \to S$ so it is unique.

(b) Do you think that the notion *terminal set* belongs in this section (Section 2.5)? How so? If products, pullbacks, and equalizers are all limits, what do limits have in common?

Answer:

A terminal set S is terminal because it has canonical projections (inclusions) back to X (and Y, if applicable). We suppose that the way they work does not change from category to category.

2.6 Finite colimits in Set

2.6.1.3

Let X be the set of people on earth; define a binary relation $R \subseteq (X \times X)$ on X as follows. For a pair (x, y) of people, say $(x, y) \in R$ if x spends a lot of time thinking about y.

Answer:

(a) Is this relation reflexive? No.
(b) Is it symmetric? No.
(c) Is it transitive? No.

2.6.1.5

Take a set I of sets; i.e. suppose that for each element $i \in I$ you are given a set X_i. For every two elements $i, j \in I$ say that $i \sim j$ if X_i and X_j are isomorphic. Is this relation an equivalence relation on I?

Answer:

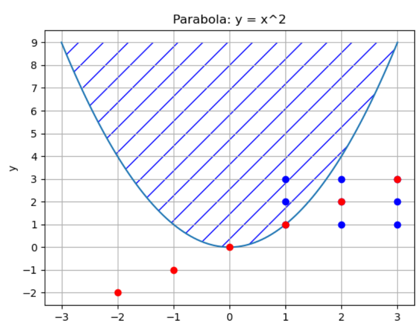
Any set is isomorphic to itself, so $\sim$ is reflexive; isomorphism is also symmetric and transitive. So, yes, $\sim$ is an equivalence relation.

2.6.1.9

Consider the set R of real numbers. Draw the coordinate plane $\mathbb{R} \times \mathbb{R}$, give it coordinates x and y. A binary relation on $\mathbb{R}$ is a subset $S \subseteq \mathbb{R} \times \mathbb{R}$, which can be drawn as a set of points in the plane.

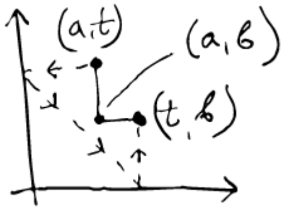
(a) Draw the relation $\{(x, y) | y = x^2\}$.
(b) Draw the relation $\{(x, y) | y \geq x^2\}$.
(c) Let S_0 be the equivalence relation on R generated (in the sense of Lemma 2.6.1.7) by the empty set. Draw S as a subset of the plane.
(d) Consider the equivalence relation S_1 generated by $\{(1, 2), (1, 3)\}$. Draw S_1 in the plane. Highlight the equivalence class containing $(1, 2)$.
(e) The reflexivity property and the symmetry property have pleasing visualizations in $\mathbb{R} \times \mathbb{R}$; what are they?
(f) Is there a nice heuristic for visualizing the transitivity property?

Answer:



(a) the line
(b) the filled area
(c) red dots
(d) square of blue dots, including the diagonal
(e) The representation of a reflexivity relation in rational numbers is the diagonal line on a 2D plane. The representation of a symmetry of a set on a 2D plane is the set itself and its mirror image across the diagonal

(f) TODO: Something like this?



2.6.1.10

Consider the binary relation $R = \{(n, n+1) | n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}$.

(a) What is the equivalence relation generated by R?

(b) How many equivalence classes are there?

Answer:

(a) Whole $\mathbb{Z}$.

(b) One.

2.6.1.11

Suppose N is a network (or graph). Let X be the nodes of the network, and let $R \subseteq X \times X$ denote the relation such that $(x, y) \in R$ iff there exists an arrow connecting x to y.

(a) What is the equivalence relation $\sim$ generated by R?

(b) What is the quotient $X/\sim$?

Answer:

(a) For each disjoint subgraph $g \subseteq N$, the relation R represents a fully connected graph g' formed by the vertices of g.

(b) A set of disjoint graphs.

2.6.2.4

(The picture) Write down the cardinality of $P \cong \underline{n}$ as a natural number $n \in \mathbb{N}$.

Answer:

1, because the described $W \to X$ and $W \to Y$ make all elements of X and Y equivalent.

2.6.2.5

Suppose that $W = \emptyset$; what can you say about $X \sqcup_W Z$?

Answer:

$X \sqcup_W Z$ becomes $X \sqcup Z$ because $(X \sqcup W \sqcup Z)/\sim$ is filled with distinct elements of both X and Z.

2.6.2.6

Let $W := \mathbb{N} = \{0, 1, 2, ...\}$ denote the set of natural numbers, let $X = \mathbb{Z}$ denote the set of integers, and let $Y = \{$☺$\}$ denote a one-element set. Define $f : W \to X$ by $f(w) = -(w + 1)$, and define $g : W \to Y$ to be the unique map. Describe the set $X \sqcup_W Y$.

Answer:

$X \sqcup_W Y = \{$😄$, 1, 2, 3\}$ where 😄 is equivalent to all the items of $\{..., -2, -1, 0,$ 😄$\}$.

2.6.2.7

Let $i : R \subseteq X \times X$ be an equivalence relation. Composing with the projections $\pi_1, \pi_2 : X \times X \to X$, we have two maps $\pi_1 \circ i : R \to X$ and $\pi_2 \circ i : R \to X$.

(a) What is the pushout $X - R - X$

(b) If $i : R \subseteq X \times X$ is not assumed to be an equivalence relation, we can still define the pushout above. Is there a relationship between the pushout $X - R - X$ and the equivalence relation generated by $R \subseteq X \times X$

Answer:

(a) For any equivalence class in R, the pushout $X \sqcup_i X$ merges all the points belonging to it into a single element.

(b) If $i' : R' \subseteq X \times X$ is not an equivalence relation, then its pushout $|X \sqcup_{i'} X| > |R|$ where R is the equivalence relation generated by R'.

2.6.3.2

Let $X = R$ be the set of real numbers. What is the coequalizer of the two maps $X \to X$ given by $f = x \mapsto x$ and $g = x \mapsto (x + 1)$ respectively?

Answer:

The coequalizer $Coeq(f, g) \sim [0, 1) \subset \mathbb{R}$.

2.6.3.3

Find a universal property enjoyed by the coequalizer of two arrows.

Answer:

For any set A, function f and g, there exists unique u mapping $Coeq(f, g)$ to A such that everything commute, in particular, $u \circ i = q$.

$$\begin{tikzcd}
X \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\
Y \arrow[d, "q"] \arrow[dd, bend left=80, "i"] \\
A \\
Coeq(f,g) \arrow[u, dashed, "\exists! u"]
\end{tikzcd}$$

2.6.3.4

(Initial object). An initial set is a set S such that for every set A, there exists a unique function $S \to A$

(a) Find an initial set

(b) Do you think that the notion initial set belongs in this section (Section 2.6)? How so? If coproducts, pushouts, and coequalizers are all colimits, what do colimits have in common?

Answer:

(a) Initial object for sets is the empty set.

(b) Colimits are isomorphic up to the unique isomorphism (???)

2.7 Other notions in Set

2.7.1.2

Create an olog that includes sets X and Y , and functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = id_X$ but such that $f \circ g \neq id_Y$; that is, such that f is a retract section but not an isomorphism.

Answer:

Any X and Y meet this condition if $|X| < |Y|$. For example, mothers and their children. We may say that $f : X \to Y$ maps mothers to their first children, while $g : Y \to X$ maps children to their mothers.

2.7.2.2

For a finite set A, let $|A| \in \mathbb{N}$ denote the cardinality of (number of elements in) A. If A and B are both finite (including the possibility that one or both are empty), is it always true that $|B^A| = |B|^{|A|}$?

Answer:

The statement is true when both sets are non-empty and if only one set is empty. When both sets are empty, it is true if we put $0^0 = 1$.

2.7.2.4

Let $X = \{1, 2\}$, $A = \{a, b\}$, and $Y = \{x, y\}$

(a) Write down three distinct elements of $L := Hom_{Set}(X \times A, Y)$

(b) Write down all the elements of $M := Hom_{Set}(A, Y)$

(c) For each of the three elements $l \in L$ you chose in part (a), write down the corresponding function $\phi(l) : X \to M$ guaranteed by Proposition 2.7.2.3.

Answer:

(a) E.g. $l_1 = \{(1, a) \mapsto x, (1, b) \mapsto x, (2, a) \mapsto x, (2, b) \mapsto x\}$, $l_2 = \{(1, a) \mapsto x, (1, b) \mapsto x, (2, a) \mapsto x, (2, b) \mapsto y\}$, $l_3 = \{(1, a) \mapsto x, (1, b) \mapsto x, (2, a) \mapsto y, (2, b) \mapsto x\}$

(b) There are: $f_{xx} = \{a \mapsto x, b \mapsto x\}$, $f_{xy} = \{a \mapsto x, b \mapsto y\}$, $f_{yx} = \{a \mapsto y, b \mapsto x\}$, $f_{yy} = \{a \mapsto x, b \mapsto y\}$

(c) $\phi(l_1) = \{1 \mapsto f_{xx}, 2 \mapsto f_{xx}\}$, $\phi(l_2) = \{1 \mapsto f_{xx}, 2 \mapsto f_{xy}\}$, $\phi(l_3) = \{1 \mapsto f_{xx}, 2 \mapsto f_{yx}\}$

2.7.2.5

Let A and B be sets. We know that $Hom_{Set}(A, B) = B^A$, so we have a function $id_{B^A} : Hom_{Set}(A, B) \to B^A$. Look at Proposition 2.7.2.3, making the substitutions $X = Hom_{Set}(A, B)$, $Y = B$, and $A = A$. Consider the function

$\phi^{-1} : HomSet(Hom_{Set}(A, B), B^A) \to Hom_{Set}(Hom_{Set}(A, B) \times A, B)$

obtained as the inverse of (2.42). We have a canonical element id_{B^A} in the domain of ϕ^{-1}. We can apply the function ϕ^{-1} and obtain an element $ev = \phi^{-1}(id_{B^A}) \in Hom_{Set}(Hom_{Set}(A, B) \times A, B)$, which is itself a function:

$ev : Hom_{Set}(A, B) \times A \to B$

(a) Describe the function ev in terms of how it operates on elements in its domain.

(b) Why might one be tempted to denote this function by ev?

Answer:

(a) According to (2.4), ϕ^{-1} is

$$\psi : [(A \to B) \to (A \to B)] \to [(A \to B) \times A \to B]$$

defined as

$$\psi = (g : G) \mapsto ((f, a) : (A \to B) \times A) \mapsto g(f)(a)$$

where we put $G = [(A \to B) \to (A \to B)]$. Indeed, G has an identity element $id_{B^A} = f \mapsto f$. Now,

$$ev = \psi(id_{B^A}) = (f, a) \mapsto id_{B^A}(f)(a) = (f, a) \mapsto b$$

where $f : A \to B = a \mapsto b$

(b) ev must be a shortening of "evaluate". It maps f and its input argument to the result.

2.7.2.6

In Example 2.4.1.7 we said that $\mathbb{R}^2$ is an abbreviation for $\mathbb{R} \times \mathbb{R}$, but in (2.43) we say that $\mathbb{R}^2$ is an abbreviation for $\mathbb{R}^{\underline{2}}$. Use Exercise 2.1.2.14, Proposition 2.7.2.3, Exercise 2.4.2.10, and the fact that $1 + 1 = 2$, to prove that these are isomorphic, $\mathbb{R}^{\underline{2}} = \mathbb{R} \times \mathbb{R}$

Answer:

Reminder:
2.1.2.14: Find a set A such that for any set X there is a isomorphism of sets $X = Hom_{Set}(A, X)$ (Answer - {☺}).
2.7.2.3: Currying
2.4.2.10: Universal property of coproducts: $Hom_{Set}(X, A) \times Hom_{Set}(Y, A) = Hom_{Set}(X \sqcup Y, A)$.
So we go:

$$\mathbb{R}^{\underline{2}} = \{1, 2\} \to \mathbb{R} \cong \{1\} \sqcup \{2\} \to \mathbb{R} \cong (\{1\} \to \mathbb{R}) \times (\{2\} \to \mathbb{R}) \cong \mathbb{R} \times \mathbb{R}$$

where we write $Hom_{Set}(A, B)$ as just $A \to B$.
TODO: (**unsure**) The problem in the proof is that we didn't use currying and the $1 + 1 = 2$ fact. Should we actually continue by saying that $\mathbb{R} \times \mathbb{R} = \mathbb{R}^{1+1}$? But then again, what about currying?

2.7.3.2

Everything in Proposition 2.7.3.1 is true except in one case, namely that of

$$\underline{0}^{\underline{0}}$$

In this case, we get conflicting answers, because for any set A, including $A = \emptyset = \underline{0}$, we have claimed both that $A^0 = \underline{1}$ and that $0^A = \underline{0}$. What is the correct answer for $\underline{0}^{\underline{0}}$, based on the definitions of $\underline{0}$ and $\underline{1}$, given in (2.6), and of A^B, given in (2.41)?

Answer:

Reminder:

2.41: $B^A := Hom_{Set}(A, B)$, which we sometimes write just as $A \to B$.
2.6: The definitions are: $\underline{1} := \{1\}$, $\underline{0} := \emptyset$.

$Hom_{Set}(A, B)$, by definition, consists of sets of the form $\{(a_1, b_1), (a_2, b_2), ...\}$ where $\{a_1, a_2, ...\} = A$ and $\forall i : b_i \in B$. In contrast to $Hom_{Set}(A, \{\})$, for which we can not define any such functions, in $Hom_{Set}(\{\}, \{\})$ we can define a single empty function, so $Hom_{Set}(\{\}, \{\}) = \{\{\}\}$. Thus:

$$\underline{0}^{\underline{0}} \cong \underline{1}$$

2.7.3.3

It is also true of natural numbers that if $a, b \in \mathbb{N}$ and $ab = 0$ then either $a = 0$ or $b = 0$. Is the analogous statement true of all sets?

Answer:

The proposition is: if $A \times B = \{\}$ then either $A = \{\}$ or $B = \{\}$.
TODO: check the proof of this fact for numbers.

2.7.3.4

Explain why there is a canonical function $\underline{5}^{\underline{3}} \times \underline{3} \to \underline{5}$ but not a canonical function $\underline{575} \to \underline{5}$.

Answer:

Following the definition of Currying (2.7.2.3), we can transform $\underline{5}^{\underline{3}} \times \underline{3} \to \underline{5}$ back to $Hom_{Set}(\underline{3}, \underline{5}) \to Hom_{Set}(\underline{3}, \underline{5})$. There is an canonical function $id_{Hom_{Set}(\underline{3}, \underline{5})}$ mapping every $\underline{3} \to \underline{5}$ function to itself.
In contrast, for $\underline{575} \to \underline{5}$ which is $Hom_{Set}(\underline{575}, \underline{5})$ we can't name any outstanding functions.

2.7.4.2

(a) How many elements does $\mathbb{P}(\emptyset)$ have?

(b) How many elements does $\mathbb{P}(\{☺\})$ have?

(c) How many elements does $\mathbb{P}(\{1, 2, 3, 4, 5, 6\})$ have?

(d) Any idea why they may have named it "power set"?

Answer:

(a) $\mathbb{P}(\emptyset) = \{\emptyset\}$ - one element.

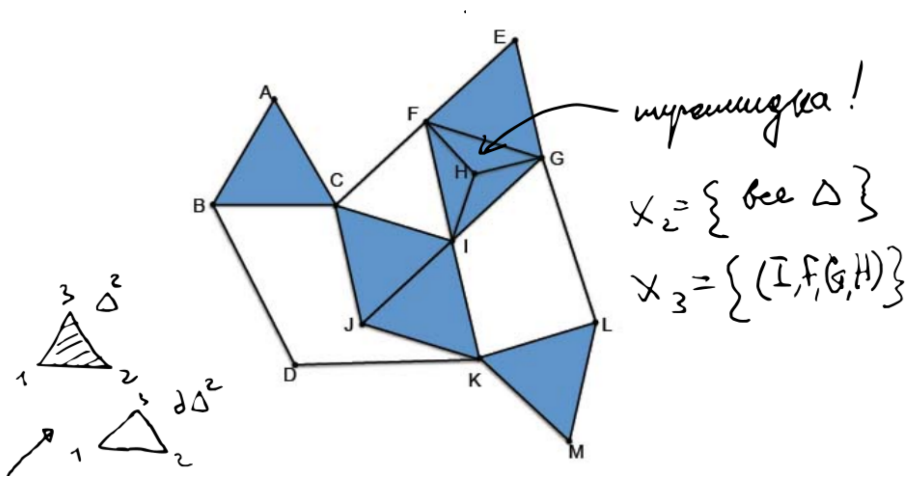
(b) $\mathbb{P}(\{☺\}) = \{\{\}, \{☺\}\}$ - two elements.

(c) $\mathbb{P}(\{1, 2, 3, 4, 5, 6\})$ has 64 elements.

(d) The name probably originates from the fact that $|\mathbb{P}(A)| = 2^{|A|}$.

2.7.4.6

Let X be the following simplicial complex, so that $X_0 = \{A, B, ..., M\}$



TODO: insert a clean picture
In this case X_1 consists of elements like $\{A, B\}$ and $\{D, K\}$ but not $\{D, J\}$. Write out X_2 and X_3 (hint: the drawing of X indicates that X_3 should have one element).

Answer:

$X_2 = \{ABC, CIJ, IJK, KLM, EFG, FGH, GHI, FHI, IFG\}$ (all the blue trinagles).
$X_3 = \{IHGF\}$ (the pyramid).

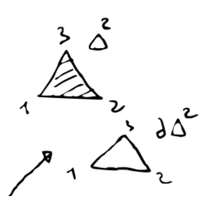
2.7.4.7

The 2-simplex Δ^2 is drawn as a filled-in triangle with vertices $V = \{1, 2, 3\}$. There is a simplicial complex $X = \delta\Delta^2$ that would be drawn as an empty triangle with the same set of vertices.

(a) Draw Δ^2 and X side by side and make clear the difference.

(b) Write down the data for X as a simplicial complex. In other words what are the sets $\{X_0, X_1, X_2, X_3, ...\}$?

Answer:



(a) (TODO: insert a clean picture).

(b) $X_0 = \{1, 2, 3\}$, $X_1 = \{(1, 2), (1, 3), (2, 3)\}$, $X_2 = X_3 = ... = \emptyset$.

2.7.4.12

Let $f : A \to \Omega$ denote the characteristic function of some $A' \subseteq A$, and define $A'' \subseteq A$ to be its complement, $A'' = A - A'$ (i.e. $a \in A''$ if and only if $a \notin A'$).

(a) What is the characteristic function of $A'' \subseteq A$?

(b) Can you phrase it in terms of some function $\Omega \to \Omega$?

Answer:

$g : A'' \to \Omega = not \circ f$
where
$not : \Omega \to \Omega = \{True \mapsto False, False \mapsto True\}$.

2.7.5.6

Show, in analogy to Proposition 2.7.5.5, that pushouts preserve epimorphisms.

Answer:

Let $f : Y \to X$ be an epimorphism. For any function g, the bottom map $f' : A \to X \sqcup_Y A$ is again an epimorphism.

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\forall g"] & X \arrow[d, "g'"] \arrow[ddr, bend left, shift left, "r"] \arrow[ddr, bend left, shift right, "q"'] & \\
A \arrow[r, "f'"] \arrow[drr, bend right, "p"'] & X \sqcup_Y A \arrow[dr, shift left, "n"] \arrow[dr, shift right, "m"'] & \\
& & B
\end{tikzcd}$$

The proof goes as follows: (1) We add some B and two functions $m, n : X \sqcup_Y A \to B$ such that $m \circ f' = n \circ f'$. We denote this function by p. (2) We look at the functions $q = m \circ g'$ and $r = n \circ g'$: $q \circ f = m \circ g' \circ f = m \circ f' \circ g = n \circ f' \circ g = n \circ g' \circ f = r \circ f$. So we can speak of only one function q and ignore r. (3) From the universal property of pushouts, there exist exactly one function $X \sqcup_Y A \to B$, so we have $m = n$. (4) We conclude that f' is an epimorphism by the definition.

2.7.5.9

Consider the subobject classifier Ω, the singleton $\{☺\}$ and the map $\{☺\} \to_{True} \Omega$ from Definition 2.7.4.9. Look at diagram 2.44 (below) and in the spirit of Exercise(sic!) 2.7.5.7, come up with a label for Ω, a label for $\{☺\}$, and a label for $True$. Given a label for X and a label for f, come up with a label for A, a label for i and a label for f', such that the English smoothly fits the mathematics.

$$\begin{tikzcd}
A \arrow[r, "f'"] \arrow[d, "i"] & \{\text{☺}\} \arrow[d, "True"] \\
X \arrow[r, "f"] & \Omega
\end{tikzcd}$$

Answer:

$$\begin{tikzcd} \boxed{\text{X which matches a Pattern}} \arrow[r, "matches"] \arrow[d, "is"] & \boxed{\text{Pattern}} \arrow[d, "is"] \\ \boxed{\text{X}} \arrow[r, "matches"] & \boxed{\text{Example}} \end{tikzcd}$$

2.7.6.2

(a) Come up with some notion of mapping for multisets that generalizes functions when the notion is restricted to sets.

(b) Suppose that $X = (1, 1, 2, 3)$ and $Y = (a, b, b, b)$, i.e. $X = \{1, 2, 3\}$ with 1 having multiplicity 2, and $Y = \{a, b\}$ with b having multiplicity 3. What are all the maps $X \to Y$ in your notion?

Answer:

(a) The right definition is given in the textbook below this exercise. Here we try a brute-force approach. Let Multiset be a set of pairs $\{(e_i, n_i)\}$ where $\{e_i\} = E$ is the set of elements and $n_i \in \mathbb{N} : n_i \geq 1$ the multiplicity of each element.

With this definition, $X = \{(1, 2), (2, 1), (3, 1)\}$ and $Y = \{(a, 1), (b, 3)\}$.

(b) We are going to have some troubles with the definition of what the map is in our multiset. We want to preserve the totality, so we define map $A \to B$ as a set of pairs $\{((a_i, n_i), B_i)\}$ where $B_i \subseteq B$ such that (1) every (a_i, n_i) is unique among the map, (2) $\{(a_i, n_i)\} = A$, (3) sum of multiplicities of B_i is equal to the n_i.

Some examples of $X \to Y$ are:

(a) $\{((1, 2), \{(a, 1), (b, 1)\}), ((2, 1), \{(a, 1)\}), ((3, 1), \{(a, 1)\})\}$
(b) $\{((1, 2), \{(b, 2)\}), ((2, 1), \{(a, 1)\}), ((3, 1), \{(a, 1)\})\}$
(c) $\{((1, 2), \{(b, 2)\}), ((2, 1), \{(b, 1)\}), ((3, 1), \{(b, 1)\})\}$
(d) ...

TODO: should we really list all the maps?

2.7.6.4

Suppose that a pseudo-multiset is defined to be almost the same as a multiset, except that π is not required to be surjective.

(a) Write down a pseudo-multiset that is not a multi-set.

(b) Describe the difference between the two notions in terms of multiplicities.

(c) The multiset commutative diagram might no longer commute for the pseudo-multisets.

Answer:

Recap: We say that $f : X \to Y$ is surjective if, for all $y \in Y$ there exists some $x \in X$ such that $f(x) = y$. In other words, uncovered elements in Y are not allowed, but X can be larger than Y. Being "non-surjective" means that the multiset is allowed to contain unused names in it.

(a) Example of a pseudo-multiset which is not a multiset: $(E = \{1_1, 1_2, 2, 3\}, B = \{1, 2, 3, ☺\}, \pi = \{1_1 \mapsto 1, 1_2 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3\})$. π doesn't map anything to ☺ thus it is not surjective.

(b) Pseudo-multisets are allowed to contain elements with undefined multiplicities.

(c) Pseudo-multisets are less useful because it is hard to choose what is better: to include an element with an undefined multiplicity or to not include it at all.

2.7.6.5

Consider the multisets described in Exercise 2.7.6.2.

(a) Write each of them in the form (E, B, π), as in Definition 2.7.6.3.

(b) In terms of the same definition, what are the mappings $X \to Y$?

(c) If we remove the restriction that diagram 2.45 must commute, how many mappings $X \to Y$ are there?

Answer:

(a) $X = (E = \{1_1, 1_2, 2, 3\}, B = \{1, 2, 3\}, \pi = \{1_1 \mapsto 1, 1_2 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3\})$.
$Y = (E = \{a, b_1, b_2, b_3\}, B = \{a, b\}, \pi = \{a \mapsto a, b_1 \mapsto b, b_2 \mapsto b, b_3 \mapsto b\})$.

(b) The set of mappings is $\{(f_1, f_2) | f_2(\pi_X(x)) = \pi_Y(f_1(x))\}$. It is the set containing all $f_1 : E_X \to E_Y$ except those which leads to different instances of B_X to have different names in B_Y. So, for example, $\{1_1 \mapsto a, 1_2 \mapsto b_1\}$ is forbidden.

(c) Without the commutativity restriction, we must have $|E_X \to E_Y| * |B_X \to B_Y| = 4^4 * 3^2 = 2304$ different mappings.

2.7.6.8

Given sets X, Y, Z and functions $f : X \to Y$ and $g : Y \to Z$, we can compose them to get a function $X \to Z$. If B is a set, if (X, p), (Y, q) and (Z, r) are relative sets over B, and if $f : (X, p) \to (Y, q)$ and $g : (Y, q) \to (Z, r)$ are mappings, is there a reasonable notion of composition such that we get a mapping of relative sets $(X, p) \to (Z, r)$? Hint: draw diagrams.

Answer:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[drr, "h"] \arrow[dd, "p"] & Y \arrow[dr, "g"] \arrow[ddl, "q"] & \\
& & Z \arrow[dll, "r"] \\
B & &
\end{tikzcd}
$$

We can define $h : X \to Z = g \circ f$ and the since all the p, q, r triangles do commute by the definition of mappings of relative sets. So if we have $f_{rel} : (X, p) \to (Y, q)$ and $g_{rel} : (Y, q) \to (Z, r)$ then $g_{rel} \circ f_{rel}$ is just $g \circ f$.

2.7.6.9

(a) Let {☺} denote a set with one element. What is the difference between sets over {☺} and simply sets?

(b) Describe the sets relative to $\emptyset$. How many are there?

Answer:

(a) We need to explore the $Hom_{RelSet}((A, a), (B, b))$ where both A and B are relative to $E =$ {☺} and a, b are unique mappings from A, B to E.

- A and B are non-empty: trivial. The behaviour is similar to $Hom_{Set}(A, B)$.
- A is empty: The only possible $a : A \to E$ is $\{\}$. The problem might come from the fact that a must be a part of commutative triangle with b. TODO: check if we can say that empty function commutes with anything or not.
- B is empty: The only possible $f : A \to B$ is $\{\}$ when A is also empty. Again, need to check the triangle commutement statement for empty functions.

(b) In order for (S, π) to be relative to $\{\}$, the $\pi : S \to \{\}$ must exist which is possible only if S is also empty. Thus, we have only one set relative to the empty set.

2.7.6.13

Let {☺} denote a one element set. What are {☺}-indexed sets and mappings between them?

Answer:

The {☺}-indexed sets are equivalent to just sets and the mappings between them are equivalent to functions over sets.

2.7.6.14

There is a strong relationship between A-indexed sets and relative sets over A. What is it?

Answer:

For any I-relative set A_I, where I is also an index set of some $(S_a)_{a \in I}$, A_I could be used as an alternative index set. TODO: not sure.

3 Categories and functors, without admitting it

3.1 Monoids

3.1.1.6

Let $M = \mathbb{N}$ be the set of natural numbers. Taking $e = 1$, come up with a formula for $*$ that gives $\mathbb{N}$ the structure of a monoid.

Answer:

We set $*$ is multiplication. Then we have both identitites $a * e = a$, $e * a = a$ and the associativity.

3.1.1.7

Come up with an operation on the set $M = \{1, 2, 3, 4\}$, i.e. a legitimate function $f : M \times M \to M$, such that f cannot be the multiplication formula for a monoid on M. That is, either it is not associative, or no element of M can serve as a unit.

Answer:

(a) Has unit, but is not associative: $f = (a, b) \mapsto max(div(a, b), 1)$ modulo 4, where div denotes integer division.

(b) Associative, but has no unit: any function returning a constant element, say 1.

3.1.1.8

In both Example 3.1.1.3 and Exercise 3.1.1.6, the monoids $(M, e, *)$ satisfied an additional rule called commutativity, namely $m * n = n * m$ for every m, $n \in M$. There is a monoid $(M, e, *)$ lurking in linear algebra textbooks that is not commutative; if you have background in linear algebra try to answer this: what M, e, and $*$ might I be referring to?

Answer:

$M = \{1, a, b\}$ link.

3.1.1.9

Recall the notion of commutativity for monoids from Exercise 3.1.1.8.

(a) What is the smallest set M that you can give the structure of a non-commutative monoid?

(b) What is the smallest set M that you can give the structure of a monoid?

Answer:

(a) Three-element set. It will not work for two-element set $\{a, b\}$ because the non-associativity condition $a * b \neq b * a$ is not compatible with the monoid identity laws.

(b) One-element set is the smallest set which can have a monoid structure, its only element must be the monoid's identity.

3.1.1.16

Let $\{☺\}$ denote a one-element set.

(a) What is the free monoid generated by $\{☺\}$?

(b) What is the free monoid generated by $\emptyset$?

Answer:

(a) The $\{☺\}$ generates the monoid $(M, [], ++)$ where M is the set of lists

$$\{(0, \{\}), ..., (n, \{i \mapsto ☺ | i \in 1..n\}), ...\}$$

In other words, it is a set of all possible lists of ☺ of different lengths.

(b) $\emptyset$ generates a monoid consisting of the empty list alone: $(\{[]\}, [], ++)$.

3.1.1.21

Let's consider the buffer concept again (see Application 3.1.1.20), but this time only having size 3 rather than size 32. Show using Definition 3.1.1.17 that with relations given by $\sim_1$ we indeed have $[a, b, c, d, e, f] = [a, b, f]$ and that with relations given by $\sim_2$ we indeed have $[a, b, c, d, e, f] = [a, b, c]$.

Answer:

We can to state that in order to use $\sim_1$ and $\sim_2$ we need to extend the task by defining buffers of size 4 and 5. The answer seems obvious assuming that we accept this assumption.
TODO

3.1.1.22

Let $K := \{BS, a, b, c, ..., z\}$ a set having 27 elements. Suppose you want to think of $BS \in K$ as the "backspace key" and the elements $\{a, b, ...z\} \in K$ as the letter keys on a keyboard. Then the free monoid $List(K)$ is not quite appropriate as a model because we want $[a, b, d, BS] = [a, b]$.

(a) Choose a set of relations for which the monoid presented by generators K and the chosen relations is appropriate to this application.

(b) Under your relations, how does $[BS]$ compare with $[]$? Is that suitable?

Answer:

(a) The set of relations is $\{[i, BS] \sim [] | i \in \{a, ..., z\}\}$.

(b) Adding $[BS]$ makes list smaller rather than larger. With respect to the relations, we can say that $[BS]$ has negative length.

3.1.1.27

(Classify the cyclic monoids). Classify all the cyclic monoids up to isomorphism. That is, come up with a naming system such that every cyclic monoid can be given a name in your system, such that no two non-isomorphic cyclic monoids have the same name, and such that no name exists in the system unless it refers to a cyclic monoid.
Hint: one might see a pattern in which the three monoids in Example 3.1.1.25 correspond respectively to ∞, 1, and 12, and then think "Cyclic monoids can be classified by (i.e. systematically named by elements of) the set $\mathbb{N} \cup \{\infty\}$". That idea is on the right track, but is not correct.

Answer:

Since the cyclic monoids are built on just one generator, it does not matter how many relationships we define for it. Only the relationship with the smallest maximum argument is important, e.g. $[Q, Q], [Q, Q, Q]$ is more important then $[Q], [Q, Q, Q, Q]$.
So, any cyclic monoid can be classified by two numbers: the length of "tail" and the length of "loop".

3.1.2.4

(a) Realize the set $T := [0, 12) \subseteq \mathbb{R}$ as the coequalizer of a pair of arrows $\mathbb{R} \rightrightarrows \mathbb{R}$.

(b) For any $x \in \mathbb{R}$, realize the mapping $x \cdot - : T \to T$, implied by Example 3.1.2.3, using the universal property of coequalizers.

(c) Prove that it is an action.

Answer:

Note: $T = \{x \in \mathbb{R} | 0 \leq x < 12\}$.
Note: The coequalizer of f and g is $Coeq(f, g) := Y/f(x) \sim g(x)$ i.e. the coequalizer of f and g is the quotient of Y by the equivalence relation generated by $\{(f(x), g(x)) | x \in X\} \subseteq Y \times Y$

(a) Indeed, consider an identity $f : \mathbb{R} \to \mathbb{R} = x \mapsto x$ and $g : \mathbb{R} \to \mathbb{R} = x \mapsto x + 12$. Their coequalizer is in 1-1 correspondence with the $[0, 12)$ range on $\mathbb{R}$.

(b) Consider the diagram for coequalizer universal property. Assume that there exists $q : \mathbb{R} \to (\mathbb{R} \to T)$ such that $q \circ f = q \circ g$.

$$\begin{tikzcd}
\mathbb{R} \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\
\mathbb{R} \arrow[d, "q"] \arrow[dd, bend left, "i"] \\
(\mathbb{R} \to T) \\
T := Coeq(f, g) \arrow[u, dashed, "\exists! u"]
\end{tikzcd}$$

The universal property of coequalizer states that there exists a unique $u : T \to (\mathbb{R} \to T)$ where $T = Coeq(f, g)$. Now we can define $u = t \mapsto r \mapsto (t + r)`mod`12$.

(c) (1) $t \mapsto 0 \mapsto (t`div`12 = t)$. (2) $t \mapsto (a+b) \mapsto (t+a+b)`div`12 = (((t+a)`div`12) + b)`div`12$.

3.1.2.5

Let B denote the set of buttons (or positions) of a video game controller (other than, say 'start' and 'select'), and consider the free monoid $List(B)$ on B.

(a) What would it mean for $List(B)$ to act on the set of states of some game? Imagine a video game G' that uses the controller, but for which $List(B)$ would not be said to act on the states of G' . Now imagine a simple game G for which $List(B)$ would be said to act.

(b) Can you think of a state s of G, and two distinct elements $l, l' \in List(B)$ such that $l \circlearrowleft s = l' \circlearrowleft s$? In video game parlance, what would you call an element $b \in B$ such that, for every state $s \in G$, one has $b \circlearrowleft s = s$?

(c) In video game parlance, what would you call a state $s \in S$ such that, for every sequence of buttons $l \in List(B)$, one has $l \circlearrowleft s = s$?

Answer:

(a) The G could be a simple maze game. Can G' be a non-deterministic game, where state changes happen as a result of both user actions $List(B)$ and the outcome of some random number generator?

(b) (1) For maze game: the maze with more than one solution. (2) An unused button.

(c) A terminal state? A hunged (buggy) game?

3.1.2.13

Notes: Let $A = Hom_{Set}(List(\sum) \times S, S)$ and $B = Hom_{Set}(\sum \times S, S)$, let $\phi : A \to B$ and $\psi : B \to A$, also let $f \in A$ and $g \in B$.

Consider the functions ϕ and ψ defined above.

(a) Show that for any $g : \sum \times S \to S$, the map $\psi(g) : List(\sum) \times S \to S$ constitutes an action.

(b) Show that ϕ and ψ are mutually inverse functions (i.e. $\phi \circ \psi = id_{Hom(\sum \times S,S)}$ and $\psi \circ \phi = id_A$)

Answer:

(a) $\forall g, \psi(g) = (l, s) \mapsto \psi(g)(l_{0..-1}, g(l_{-1}, s))$ with a special case of $\psi(g) = ([], s) \mapsto s$. Now, $\psi(g)$ is an action, because: (1) $\psi(g)([], s) \mapsto s$ by definition; (2) $\psi(g)(l + l') = g(l_0, ...g(l'_{-2}, g(l'_{-1}, s))) = \psi(g)(l, \psi(g)(l', s))$

(b) (1) $\phi(\psi(g)) = (\epsilon, s) \mapsto \psi(g)([\epsilon], s) = g(\epsilon, s)$ (2) $\psi(\phi(f)) = (l, s) \mapsto \psi(\phi(f))(l_{0..-1}, \phi(f)(l_{-1}, s)) = f([l_0], ...f([l_{-2}], f([l_{-1}], s))) =_1 f(l, s)$, where $=_1$ holds because f is an action.

3.1.3.3

Let $\mathbb{N}$ be the additive monoid of natural numbers, let $S = \{0, 1, 2, ..., 11\}$, and let $\cdot : \mathbb{N} \times S \to S$ be the action given in Example 3.1.2.3. Using a nice small generating set for the monoid, write out the corresponding action table.

Answer:

IPython

```
def _act(a,b):
  return (a + b) % 12

for n in {0, 2, 3, 5, 7, 11}:
  print(f"{n}", end='')
  for s in {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}:
    print(f"& {_act(s,n)}", end='')
  print("\\\\")
  print("\\hline")
```

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
5	5	6	7	8	9	10	11	0	1	2	3	4
7	7	8	9	10	11	0	1	2	3	4	5	6
11	11	0	1	2	3	4	5	6	7	8	9	10

TODO: Not sure I get the problem correctly, need to check.

3.1.4.6

For any $m \in \mathbb{N}$ let $i_m : \mathbb{N} \to \mathbb{Z}$ be the function $i_m(x) = m * x$. All such functions are monoid homomorphisms $(\mathbb{N}, 0, +) \to (\mathbb{Z}, 0, +)$. Do any monoid homomorphisms $(\mathbb{N}, 0, +) \to (\mathbb{Z}, 0, +)$ not come in this way? For example, what about using $n \mapsto 5 * n + 1$ or $n \mapsto n^2$, or some other function?

Answer:

Lets assume that $\exists f : (\forall m : f \neq i_m)$. Then there should be at least one point $d \neq 0$ such that $f(d) \neq i_m(d)$. Lets break down $f(d) = f(1 + 1 + ... + 1) = f(1) * d \neq m * d$. So we have $\forall m : f(1) \neq m$ which is a contradiction.

3.1.4.7

Let $\mathcal{M} := (\mathbb{N}, 0, +)$ be the additive monoid of natural numbers, let $\mathcal{N} = (\mathbb{R}_{\geq 0}, 0, +)$ be the additive monoid of nonnegative real numbers, and let $\mathcal{P} := (\mathbb{R}_{>0}, 1, *)$ be the multiplicitive monoid of positive real numbers. Can you think of any nontrivial monoid homomorphisms of the following sorts:
$\mathcal{M} \to \mathcal{N}$; $\mathcal{M} \to \mathcal{P}$; $\mathcal{N} \to \mathcal{P}$; $\mathcal{N} \to \mathcal{M}$; $\mathcal{P} \to \mathcal{N}$?

Answer:

(a) $\mathcal{M} \to \mathcal{N}$: $f = x_{\mathcal{M}} \mapsto x_{\mathcal{N}}$;

(b) $\mathcal{M} \to \mathcal{P}$: $f = x_{\mathcal{M}} \mapsto e^x$;

(c) $\mathcal{N} \to \mathcal{P}$: $f = x_{\mathcal{N}} \mapsto e^x$;

(d) $\mathcal{N} \to \mathcal{M}$: There are no such homomorphisms. The reason is $|\mathcal{N}| > |\mathcal{M}|$ so f can not be injective which is required in order for $+_{\mathcal{M}}$ to work;

(e) $\mathcal{P} \to \mathcal{N}$: $f = x \mapsto log(x)$.

3.1.4.10

Let $G = \{a, b\}$, let $M := (M, e, *)$ be any monoid, and let $f : G \to M$ be given by $f(a) = m$ and $f(b) = n$, where $m, n \in M$. If $\psi : Hom_{Set}(G, M) \to Hom_{Mon}(F(G), M)$ is the function from the proof of Proposition 3.1.4.9 and $L = [a, a, b, a, b]$, what is $\psi(f)(L)$?

Answer:

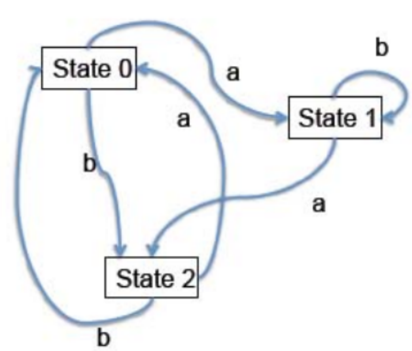
$\psi(f)(L) = m * m * n * m * n$

3.1.4.15

Let N be the free monoid on one generator, let $\sum = \{a, b\}$, and let $S = \{State0, State1, State2\}$. Consider the map of monoids $f : \mathbb{N} \to List(\sum)$ given by sending $1 \mapsto [a, b, b]$. The monoid action $\alpha : List(\sum) \times S \to S$ given in Example 3.1.3.1 can be transformed by restriction of scalars along f to an action $\delta_f(\alpha)$ of $\mathbb{N}$ on S. Write down its action table

Answer:

The referenced Example contains the following picture:



By monoid map we probably mean a monoid homomorphism which preserves the monoid properties, like $f(a * b) \mapsto f(a) *' f(b)$ where $*$ is a monoid operation. So e.g. $2 \mapsto [a, b, b, a, b, b]$.

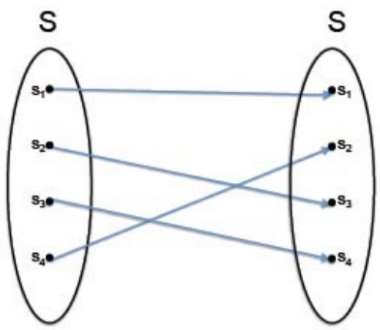
Actions	
$\mathbb{N}$	1
State 0	State 1
State 1	State 2
State 2	State 0

For other number $n \in \mathbb{N}$ just apply the above tables recursively for $1 + (n - 1)$ until we reach $0 \mapsto []$.

3.2 Groups

3.2.1.7

Let S be a finite set. A permutation of S is an isomorphism $f : S \to S$.



(a) Come up with an identity, and a multiplication formula, such that the set of permutations of S forms a monoid.

(b) Is it a group?

Answer:

(a) The identity is the identity function $f = id_S$. The multiplication is the function composition operator $\circ$.

(b) Yes, it is a group, because for every permutation we can define a reverse permutation, so $f^{-1} \circ f = id_S$.

3.2.1.8

In Exercise 3.1.1.27 you classified the cyclic monoids. Which of them are groups?

Answer:

Cyclic monoids are characterized by two parameters: the length of their "tail" and the length of their "cycle." A monoid is a group if its "tail" length is zero.

3.2.1.11

Let X be a set and consider the group of permutations of X (see Exercise 3.2.1.7), which we will denote Σ_X. Find a canonical action of Σ_X on X.

Answer:

The canonical action is just application of the permutation to the set. The action is canonical because it follows directly from the definition of the Group.

3.2.1.14

(a) Consider the $U(1)$ action on $\mathbb{R}^3$ given in Example 3.2.1.10. Describe the set of orbits of this action.

(b) What are the orbits of the action of the permutation group $\Sigma_{\{1,2,3\}}$ on the set $\{1, 2, 3\}$? (See Exercise 3.2.1.11.)

Answer:

(a) The $U(1)$ action rotates points around the Z axis in $\mathbb{R}^3$. So the set of orbits consists of the circles centered at the Z axis.

(b) The permutation group action orbits are permutations of the set S because we can set S to any of its permutations using the action.

3.2.1.15

Let G be a group and X a set on which G acts by $\circlearrowleft: G \times X \to X$. Is "being in the same orbit" an equivalence relation on X?

Answer:

Yes, it is. It is a reflexive, symmetrical and transitive relation.

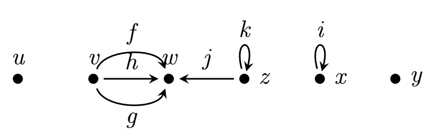
3.3 Graphs

3.3.1.4

(a) Draw the graph corresponding to a graph tables. (b) Write down graph tables corresponding to a graph.

Answer:

(a)

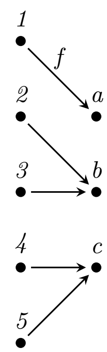


(b) (Obvious)

3.3.1.5

Let $A = \{1,2,3,4,5\}$ and $B = \{a,b,c\}$. Draw them and choose an arbitrary function $f : A \to B$ and draw it. Let $A \sqcup B$ be the coproduct of A and B (Definition 2.4.2.1) and let $A \to A \sqcup B \leftarrow B$ be the two inclusions. Consider the two functions src, tgt : $A \to A \sqcup B$, where src $= i_1$ and tgt is the composition $A \to B \to A \sqcup B$. Draw the associated graph $(A \sqcup B, A, \text{src}, \text{tgt})$.

Answer:



3.3.1.6

(a) Let V be a set. Suppose we just draw the elements of V as vertices and have no arrows between them. Is this a graph?

(b) Given V, is there any other "canonical" or somehow automatic non-random procedure for generating a graph with those vertices?

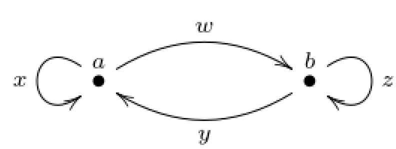
Answer:

(a) Yes, it is. Both its *src* and *dst* functions are empty sets.

(b) No, there isn't. If such a procedure existed, it would mean that a random set has predetermined relationships, but we know it does not.

3.3.1.10

Let G be the graph depicted below:



Draw (using ellipses "…" if necessary) the graph $T(G)$ defined in Example 3.3.1.9.

Answer:

The answer is something like the following:

$$
\begin{tikzcd}
a0 \arrow[r] \arrow[dr] & a1 \arrow[r] \arrow[dr] & a2 \\
b0 \arrow[r] \arrow[ur] & b1 \arrow[r] \arrow[ur] & b2
\end{tikzcd}
$$

3.3.1.11

Consider the infinite graph $G = (V, A, \mathrm{src}, \mathrm{tgt})$ depicted below,

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
(0,2) \arrow[r] \arrow[u] & (1,2) \arrow[r] \arrow[u] & (2,2) \arrow[r] \arrow[u] & \cdots \\
(0,1) \arrow[r] \arrow[u] & (1,1) \arrow[r] \arrow[u] & (2,1) \arrow[r] \arrow[u] & \cdots \\
(0,0) \arrow[r] \arrow[u] & (1,0) \arrow[r] \arrow[u] & (2,0) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}
$$

(a) Write down the sets A and V.

(b) What are the source and target functions $A \to V$?

Answer:

(a) $V = \mathbb{N} \times \mathbb{N}$; $A = (\mathbb{N} \times \mathbb{N}) \sqcap \{N, E\}$

(b) $src = \pi_1$ so $((x, y), ..) \mapsto (x, y)$; $dst = \{((x, y), N) \mapsto (x, y + 1)\} \cup \{((x, y), E) \mapsto (x + 1, y)\}$ where $x, y \in \mathbb{N}$.

3.3.1.12

A graph is a pair of functions $A \rightrightarrows V$. This sets up the notion of equalizer and coequalizer (see Definitions 2.5.3.1 and 2.6.3.1).

(a) What feature of a graph is captured by the equalizer of its source and target functions?

(b) What feature of a graph is captured by the coequalizer of its source and target functions?

Answer:

Notes:

- Let $f, g : X \to Y$.
- The equalizer of f and g is $Eq(f, g) := \{x \in X | f(x) = g(x)\} \subseteq X$.
- The coequalizer of f and g is $Coeq(f, g) := Y/f(x) \sim g(x)$ i.e. the coequalizer of f and g is the quotient of Y by the equivalence relation generated by $\{(f(x), g(x))|x \in X\} \subseteq Y \times Y$

(a) $Eq(src, dst)$ are edges that are loops (the source vertex matches the destination vertex).

(b) $Coeq(src, dst)$ is the relation that represents connected vertices.

3.3.2.3

How many paths are there in the following graph? ($V = \{1, 2, 3\}$, $A = \{f, g\}$. f goes from 1 to 2, g goes from 2 to 3).

Answer:

There are 6 paths: 3 of length 0, 2 of length 1 and 1 of length 2.

3.3.2.4

Let G be a graph and consider the set $Path_G$ of paths in G. Suppose someone claimed that there is a monoid structure on the set $Path_G$, where the multiplication formula is given by concatenation of paths. Are they correct? Why or why not? Hint: what should be the identity element?

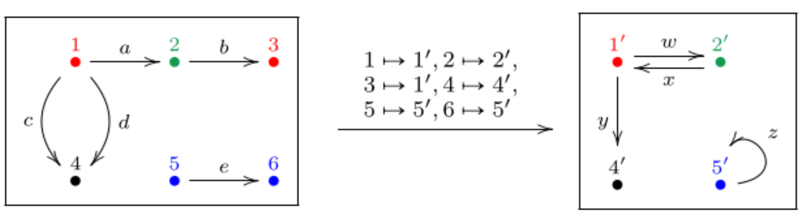
Answer:

No, because the monoid $(M, e, *)$ should have $* : M \times M \to M$ defined for any element. It is not the case for path concatenation.

3.3.3.4

(a) Where are a, b, c, d, e sent under $f_1 : A \to A'$ in Diagram (3.7)?

(b) Choose a couple of elements of A and check that they behave as specified by Diagram (3.6).



(3.7)

Answer:

(a) To w, x, y, y, z correspondingly.

(b) We need to check that $f_0 : V \to V'$ and $f_1 : A \to A'$ commute with src and dst functions of both graphs. Indeed, $f_0(src(e)) = 5' = src'(f_1(e))$.

3.3.3.5

Let G be a graph, let $n \in N$ be a natural number, and let $[n]$ be the chain graph of length n, as in Example 3.3.1.8. Is a path of length n in G the same thing as a graph homomorphism $[n] \to G$, or are there subtle differences? More precisely, is there always an isomorphism between the set of graph homomorphisms $[n] \to G$ and the set $Path_G^{(n)}$ of length-n paths in G?

Answer:

Note: We proof isomorphism between A and B by showing $(f : A \to B) \circ (g : B \to A) = id_B$ and $g \circ f = id_A$.
Consider a set of graph homomorphisms $H = [n] \to G$. It has the following structure: $(f_0, f_1) = (\{0_v \mapsto v_0, 1_V \mapsto v_1, ...\}, \{0_A \mapsto a_1, 1_A \mapsto a_2, ...\})$ where i_V,i_A - vertices and arrows of $[n]$ and v_i, a_i - vertices and arrows of G.
We define $f : Path^{(n)} \to H$ and $g : H \to Path^{(n)}$ as follows: $f = v(a_i)^{[n-1]} \mapsto (\{i_V \mapsto v(a_i)^{[i]}\}, \{i_A \mapsto a_i\})$, $g = (\{...\}, \{...\}) \mapsto v(a_i)^{[n-1]}$ where $(a_i)^{[n]} = a_0 a_1 ... a_{(n-1)}$, $(a_i)^0$ expands to nothing (no arrows), $n \in \{1, ...\}$ - number of path vertices.
(sounds pretty obvious that $f \circ g = id = g \circ f$)
TODO?

3.3.3.6

Given a morphism of graphs $f : G \to G'$, there is an induced function $\mathrm{Path}(f) : \mathrm{Path}(G) \to \mathrm{Path}(G')$.

(a) Is it the case that for every $n \in \mathbb{N}$, the function $\mathrm{Path}(f)$ carries $\mathrm{Path}^{(n)}(G)$ to $\mathrm{Path}^{(n)}(G')$, or can path lengths change in this process?

(b) Suppose that f_0 and f_1 are injective (meaning no two distinct vertices in G are sent to the same vertex, respectively for arrows, under f). Does this imply that $\mathrm{Path}(f)$ is also injective (meaning no two distinct paths are sent to the same path under f)?

(c) Suppose that f_0 and f_1 are surjective (meaning every vertex in G' and every arrow in G' is in the image of f). Does this imply that $\mathrm{Path}(f)$ is also surjective? *Hint*: at least one of the answers to these three questions is "no".

Answer:

(a) Yes.

(b) Yes.

(c) No. $\mathrm{Path}(f)$ Graph G can acquire a loop, so the set of paths in G' can become infinitely large.

3.3.3.7

Given a graph (V, A, src, tgt), let $i : A \to V \times V$ be the function guaranteed by the universal property for products, as applied to $src, tgt : A \to V$. One might hope to summarize Condition (3.6) for graph homomorphisms by the commutativity of the single square

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\
V \times V \arrow[r, "f_0 \times f_0"'] & V' \times V'.
\end{tikzcd}$$

Is the commutativity of the diagram in (above) indeed equivalent to the commutativity of the diagrams in (3.6)?

Answer:

I can not see reasons why the above diagram is not the equivalent of the two diagrams in (3.6). TODO: Need to double-check! Very interesting! Can it be related to the no-arrows graphs? Can it be related to loop-arrows in graphs? Note: Is it about problems with sibling arrows? Most likely yes!

3.3.3.10

A relation on $\mathbb{R}$ is a subset of $\mathbb{R} \times \mathbb{R}$, and one can indicate such a subset of the plane by shading. Choose an error bound $\epsilon > 0$ and draw the relation one might refer to as "ϵ-approximation". To say it another way, draw the relation "x is within ϵ of y".

Answer:

An ascending bar centered at the $y = x$ line, with $2 * \epsilon$ height.

3.3.3.11

(Binary relations to graphs).

(a) If $R \subseteq S \times S$ is a binary relation, find a natural way to make a graph out of it, having vertices S.

(b) What is the set A of arrows?

(c) What are the source and target functions $\mathrm{src}, \mathrm{tgt} : A \to S$?

(d) Take the left-hand table in (3.9) and consider its first 7 rows (i.e. forget the ..). Draw the corresponding graph (do you see a tetrahedron?).

(e) Do the same for the right-hand table.

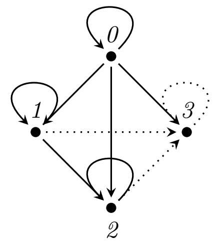
Answer:

(a) Points go to Vertices, Points in relation go to Arrows, *src* and *dst* mark left-hand side and right-hand side points in relation.

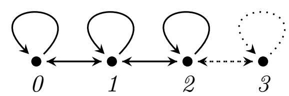
(b) Arrows indicate points in relation to each other.

(c) *src* selects left-hand side points in relation, *dst* selects right-hand side points.

(d) Graph corresponding to the left table in (3.9).



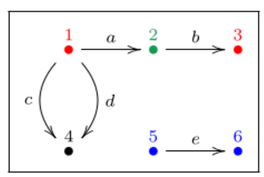
(e) Graph corresponding to the right-hand table in (3.9)



3.3.3.12

(Graphs to binary relations).

- If (V, A, src, tgt) is a graph, find a natural way to make a binary relation $R \subseteq V \times V$ out of it.
- Take the left-hand graph G from (3.7) and write out the corresponding binary relation in table form.



Answer:

- Let $R_G \subseteq V \times V = \{(src(a), dst(a)) | a \in A_G\}$.
- $R_G = \{(1,4), (1,2), (2,3), (5,6)\}$. Note: What about duplications, here $(1,4)$?

3.3.3.13

(Going around the loops)

- Given a binary relation $R \subseteq S \times S$, you know from Exercise 3.3.3.11 how to construct a graph out of it, and from Exercise 3.3.3.12 how to make a new binary relation out of that. How does the resulting relation compare with the original?
- Given a graph (V, A, src, tgt), you know from Exercise 3.3.3.12 how to make a new binary relation out of it, and from Exercise 3.3.3.11 how to construct a new graph out of that. How does the resulting graph compare with the original?

Answer:

- The new relation will be identical to the original
- The new graph will not be identical to the original. The sibling arrows will be joined into one arrow. The vertices without arrows will disappear.

3.4 Orders

3.4.1.2

- Decide whether the table to the left in Display (3.9) constitutes a linear order.
- Show that neither of the other tables are even preorders.

Answer:

- Yes it is a linear order. It is reflexive, transitive, antisymmetric and comparable.
- The $|n - m| \leq 1$ is not transitive, thus, it is not a preorder. The $n = 5m$ relation has gaps (not reflexive), so, again, not a preorder.

3.4.1.8

Let $S = \{1, 2, 3, 4\}$.

(a) Find a preorder $R \subseteq S \times S$ such that the set R is as small as possible. Is it a partial order? Is it a linear order?

(b) Find a preorder $R_1 \subseteq S \times S$ such that the set R_1 is as large as possible. Is it a partial order? Is it a linear order?

Answer:

(a) $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$. It is reflexive, and formally transitive. It is a partial order. It is not a linear order.

(b) R_1 should be a fully-connected graph. It is a reflexive and transitive by definition, but it contains non-trivial loops (symmetric). Thus, it is not a partial order. Also it is not a linear order.

3.4.1.9

(a) List all the preorder relations possible on the set $\{1, 2\}$.

(b) For any $n \in \mathbb{N}$, how many linear orders exist on the set $\{1, 2, 3, \ldots, n\}$.

(c) Does your formula work when $n = 0$?

Answer:

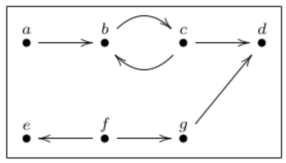
(a) There are $\{(1,1),(2,2)\}$, $\{(1,1),(2,2),(1,2)\}$, $\{(1,1),(2,2),(2,1)\}$, $\{(1,1),(2,2),(1,2),(2,1)\}$.

(b) Linear orders are permuations of set elements. Thus, for n-element set there are $n!$ linear orders.

(c) 0! is set to be 1, and there is one linear order $R \subseteq \{\} \times \{\} = \{\}$. So, the formula works.

3.4.1.12

Let $G = (V, A, src, tgt)$ be the graph below.



In the corresponding pre-order which of the following are true:
(a) $a \leq b$? (b) $a \leq c$? (c) $c \leq b$? (d) $b = c$? (e) $e \leq f$? (f) $f \leq d$?

Answer:

(a) True (b) True (c) True (d) False. The equality is a partial-order requirement. (e) False (f) True.

3.4.1.13

(a) Let $S = \{1, 2\}$. The subsets of S form a partial order; draw the associated graph.

(b) Repeat this for $Q = \emptyset$, $R = \{1\}$, and $T = \{1, 2, 3\}$.

(c) Do you see n - dimensional cubes?

Answer:

TODO

3.4.1.15

True or false: a partial order is a preorder that has no cliques. (If false, is there a ”nearby” true statement?)

Answer:

TODO

References

[1] David I. Spivak. "Category theory for scientists". In: (2013). URL: https://api.semanticscholar.org/CorpusID:126379073.