

# Solutions for Category Theory for Scientists by David Spivak

Sergei Mironov

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## 1 Introduction

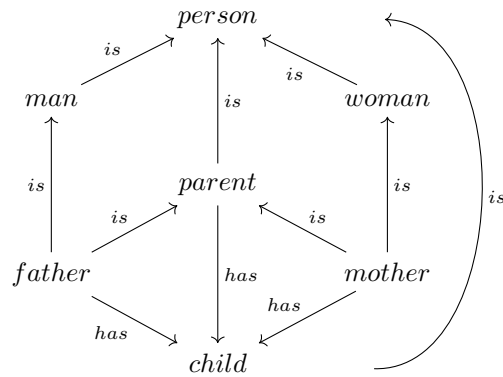
The book (2013, Spivak) “*Category theory for scientists*” [1].

## 2 The Category of Sets

### 2.3.3.1

Create an olog for human nuclear biological families that includes the concept of person, man, woman, parent, father, mother, and child. Make sure to label all the arrows, and make sure each arrow indicates a valid aspect in the sense of Section 2.3.2.1. Indicate with check-marks the diagrams that are intended to commute. If the 2-dimensionality of the page prevents a check-mark from being unambiguous, indicate the intended commutativity with an equation.

**Answer:**



Commutative square examples:

- "parent is person"  $\circ$  "mother is parent" = "woman is person"  $\circ$  "mother is woman"
- "mother has child" = "parent has child"  $\circ$  "mother is parent"
- "child is person"  $\circ$  "parent has child"  $\neq$  "parent is person"

## 2.4 Products and coproducts

### 2.4.1.4

How many elements does the  $\{a, b, c, d\} * \{1, 2, 3\}$  have?

**Answer:**

12

### 2.4.1.8

**Answer:**

(a) No, because  $a(b + c) \neq (a + b)c$ .

(b) No, because  $x * 0 \neq x$ .

(c) Yes.

### 2.4.1.15

(a) Let  $X$  and  $Y$  be sets.. construct the "swap map"  $s : (X \times Y) \rightarrow (Y \times X)$

**Answer:**

$s : (X \times Y) \rightarrow (Y \times X) = (, ) \circ \langle \pi_2, \pi_1 \rangle$

Note: we used angle brackets. Is it really correct?

(b) Can you prove that  $s$  is a isomorphism using only the universal property for product?

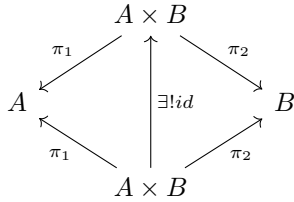
Note:  $(f : X \rightarrow Y)$  is an isomorphism if  $\exists(g : Y \rightarrow X) : g \circ f = id_X \wedge f \circ g = id_Y$

Note: diagram  $(f, g, h)$  commutes if  $f \circ g = h$

**Answer:**

In universal property of products, put  $A$  equal to  $Y \times X$  and get  $\exists!g : (Y \times X) \rightarrow (X \times Y)$ . In a similar way, we have  $\exists!s : (X \times Y) \rightarrow (Y \times X)$ . We need to show that  $g \circ s = id_{(X \times Y)}$  and  $s \circ g = id_{(Y \times X)}$ .

On the following diagram,



$id = id_{(A \times B)}$  - is the unique identity function. By combining diagrams for  $f$  and  $g$  we reduce the  $g \circ f$  to the similar case.

#### 2.4.2.4

Would you say that a phone is the coproduct of a **cellphone** and a **landline phone**?

**Answer:**

Yes, until we consider other types of phones besides cell- and landline ones.

#### 2.4.2.10

Write the universal property for coproduct in terms of a relationship between  $Hom_{Set}(X, A)$ ,  $Hom_{Set}(Y, A)$  and  $Hom_{Set}(X \sqcup Y, A)$ .

**Answer:**

$$Hom_{Set}(X, A) \times Hom_{Set}(Y, A) \cong Hom_{Set}(X \sqcup Y, A)$$

#### 2.4.2.13

TODO

#### 2.4.2.14

TODO

### 2.5 Finite limits in Set

#### 2.5.1.2

**Answer:**

$$X \times_Z Y = \{(x_1, z_1, y_1), (x_2, z_2, y_2), (x_2, z_2, y_4), (x_3, z_2, y_2), (x_3, z_2, y_4)\}$$

### 2.5.1.3

(a)

**Answer:**

Let  $X = \{1, 2, 3, 4, 5\}; Y = \{a, b, c\}$ ; where  $C = \{R, B, Y\}$

We have:  $X \times_C Y = \{1a, 4a, 2c, 5c, 3b\}$

(b) TODO (obvious).

### 2.5.1.5

(a) Suppose that  $Y = \emptyset$ ; what can you say about  $X \times_Z Y$  ?

**Answer:**

$$X \times_Z Y = \emptyset$$

(b)  $Z = 1$ ; what can you say about  $X \times_Z Y$  ?

**Answer:**

$$\forall X, Y : X \times_Z Y \cong X \times Y$$

### 2.5.1.6

.. Aristotelian space and time ..

$S = R^3$ ;  $T = R$ ;  $Y = S \times T$ ;  $g1 : Y \rightarrow S$ ;  $g2 : Y \rightarrow T$  where  $g1, g2$  projects space-time to its components.  $X = \{1\}$ ;  $f1 : X \rightarrow S$ ;  $f2 : X \rightarrow T$  is a set of one element and its space-time projections.

(a) What is the meaning of

$$\begin{array}{ccc} W_1 & \longrightarrow & Y \\ \downarrow & & \downarrow g_1 \\ X & \xrightarrow{f_2} & S \end{array}$$

$$\begin{array}{ccc} W_2 & \longrightarrow & Y \\ \downarrow & & \downarrow g_2 \\ X & \xrightarrow{f_2} & S \end{array}$$

**Answer:**

1 is associated with its time and space.  $W_1$  yields time points of  $Y$  corresponding to 1's position.  $W_2$  yields the space points corresponding to 1's time.

(b) Interpret the sets in terms of the center of mass of MIT at the time of its founding.

**Answer:**

TODO: (unsure) Is it just the MIT-relative space and time points?

### 2.5.1.10

.. Appropriate or misleading olog labels ..

**Answer:**

(a) a person whose favorite color is blue - OK

(b) a dog whose owner is a woman - OK

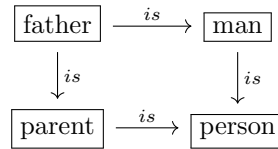
(c) a good fit - Nope. We would say that a good fit requires less or equal width.

### 2.5.1.11

(a) Consider your olog from Exercise 2.3.3.1. Are any of the commutative squares there actually pullback squares?

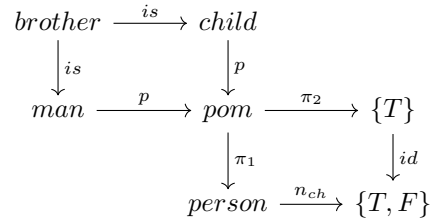
**Answer:**

Yes, for example: "father" = "man"  $\times_{person}$  "parent"



(b) Now use ologs with products and pullbacks to define what a brother is and what a sister is:

**Answer:**

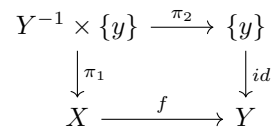


where: *pom* stands for "parent of many", *p* is "has as parent" and *n<sub>ch</sub>* is "number of children"

### 2.5.1.13

Pullback diagram in which the fiber product is isomorphic to the preimage of  $y \in Y$ .

**Answer:**



### 2.5.1.15

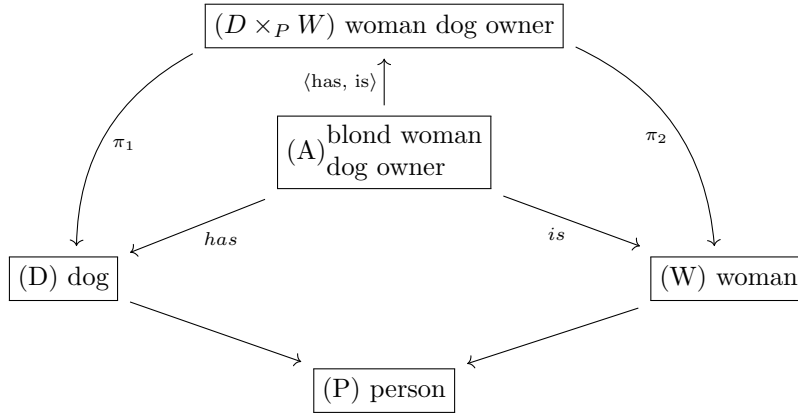
Create an olog whose underlying shape is a commutative square. Now add the fiber product so that the shape is the same as that of Diagram (2.32). Assign English labels to the projections and to the dotted map A, such that these labels are as canonical as possible.

**Answer:**

TODO: (**unsure**)

The general impression is the following:

- The commutative square can be visualized as a pyramid in a 3D space.
- The pullback corresponds to a 2D plane with two axes.
- The pullback over one-element set corresponds to a 2D plane with two orthogonal axes.
- The projections  $\pi_1$  and  $\pi_2$  play the role of parallel projections from a point in the plane to its axes.
- The map  $A \rightarrow (D \times_P W)$  acts as a projection from a point in 3D space to the plane.



### 2.5.1.18

(a) Create an olog that defines two people to be 'of approximately the same height' if and only if their height difference is less than half an inch, using a pullback.

**Answer:**

$$\begin{array}{ccc}
 \{((p_1, p_2), \Delta) | p_1, p_2 \in \text{people}, \Delta \in \mathbb{R}\} & \xrightarrow{\pi_2} & \{|\Delta| - 0.5 < \Delta < 0.5\} \\
 \downarrow \pi_1 & & \downarrow is \\
 \boxed{\text{pair of people}} & \xrightarrow{h_2 - h_1} & \mathbb{R}
 \end{array}$$

(b) In the same olog, make a box for those people whose height is approximately the same as a person named 'The Virgin Mary'. You may need to use images.

**Answer:**

$$\begin{array}{ccccc}
 \boxed{\text{people of approx Virgin Mary height}} & \xrightarrow{\pi_2} & \{((p_1, p_2), \Delta) | \dots\} & \xrightarrow{\pi_2} & \{|\Delta| - 0.5 < \Delta < 0.5\} \\
 \downarrow \pi_1 & & \downarrow \pi_1 & & \downarrow is \\
 \{(\boxed{\text{The Virgin Mary}}, p_2) | p_2 \in \text{people}\} & \xrightarrow{is} & \boxed{\text{pair of people}} & \xrightarrow{h_2 - h_1} & \mathbb{R}
 \end{array}$$

TODO: (**unsure**) We did not use images, despite the textbook suggestion.

### 2.5.1.19

Let  $W = X \times_Z Y$  and  $W' = X' \times_{Z'} Y'$ , and form the diagram to the right. Use the universal property of fiber products to construct a map  $W \rightarrow W'$  such that all squares commute

**Answer:**

Let  $WX = \pi_1$ ,  $WY = \pi_2$ ,  $W'X' = \pi'_1$ ,  $W'Y' = \pi'_2$ . Define  $f : W \rightarrow W' = (x, y)_z \mapsto (XX'(x), YY'(y))_{ZZ'(z)}$ . We want to show that it makes  $WW'Y'Y$  to commute. Indeed,  $\pi'_2 \circ f = YY' \circ \pi_2 = y \mapsto YY'(y)$ . By analogy, it also commutes  $WW'X'X$ . The triangles  $WW'Y'$  and  $WW'X'$  also commute. The universal property for fiber products says that we can only have one such mapping, thus this is the answer.

### 2.5.2.6

- (a) TODO: Copy the picture from the book notes
- (b) TODO: Copy the picture from the book notes
- (c) TODO: Find the correct answer

### 2.5.3.3

Come up with an olog that uses equalizers in a reasonably interesting way. Alternatively, use an equalizer to specify those published authors who have published exactly one paper. Hint: find a function from authors to papers; then find another.

**Answer:**

We follow the second way.

$$\text{Authors} \xrightleftharpoons[\text{const}_1]{\text{num of papers}} \mathbb{R}$$

where  $\text{const}_1 : \text{authors} \rightarrow \mathbb{R}$  maps all authors to 1.

### 2.5.3.4

Find a universal property enjoyed by the equalizer of two arrows, and present it in the style of Lemmas 2.4.1.10, 2.4.2.7, and 2.5.1.14.

**Answer:**

For any set  $A$ , function  $f$  and  $g$  there exists unique  $u$  mapping  $A$  to  $Eq(f, g)$  such that everything commute, in particular,  $\pi \circ u = p$ .

$$\begin{array}{c}
 Eq(f, g) \\
 \uparrow \exists! u \\
 A \\
 \downarrow p \\
 X \\
 \downarrow f \parallel g \\
 Y
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow \pi \\
 \searrow
 \end{array}$$

### 2.5.3.5

(a) A terminal set is a set  $S$  such that for every set  $X$ , there exists a unique function  $X \rightarrow S$ . Find a terminal set.

**Answer:**

Any set  $S$  with one element. There exist exactly one function  $X \rightarrow S$  so it is unique.

(b) Do you think that the notion *terminal set* belongs in this section (Section 2.5)? How so? If products, pullbacks, and equalizers are all limits, what do limits have in common?

**Answer:**

A terminal set  $S$  is terminal because it has canonical projections (inclusions) back to  $X$  (and  $Y$ , if applicable). We suppose that the way they work does not change from category to category.

## 2.6 Finite colimits in Set

### 2.6.1.3

Let  $X$  be the set of people on earth; define a binary relation  $R \subseteq (X \times X)$  on  $X$  as follows. For a pair  $(x, y)$  of people, say  $(x, y) \in R$  if  $x$  spends a lot of time thinking about  $y$ .

**Answer:**

- (a) Is this relation reflexive? No.
- (b) Is it symmetric? No.
- (c) Is it transitive? No.



### 2.6.1.5

Take a set  $I$  of sets; i.e. suppose that for each element  $i \in I$  you are given a set  $X_i$ . For every two elements  $i, j \in I$  say that  $i \sim j$  if  $X_i$  and  $X_j$  are isomorphic. Is this relation an equivalence relation on  $I$ ?

**Answer:**

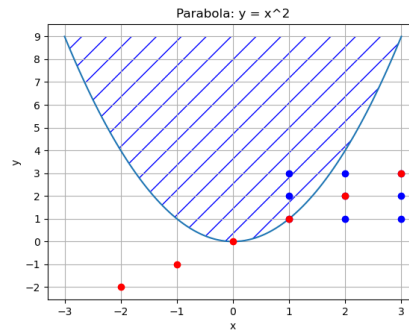
Any set is isomorphic to itself, so  $\sim$  is reflexive; isomorphism is also symmetric and transitive. So, yes,  $\sim$  is an equivalence relation.

### 2.6.1.9

Consider the set  $\mathbb{R}$  of real numbers. Draw the coordinate plane  $\mathbb{R} \times \mathbb{R}$ , give it coordinates  $x$  and  $y$ . A binary relation on  $\mathbb{R}$  is a subset  $S \subseteq \mathbb{R} \times \mathbb{R}$ , which can be drawn as a set of points in the plane.

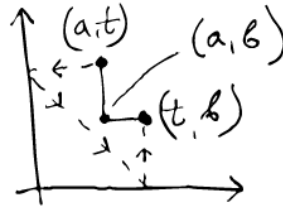
- Draw the relation  $\{(x, y) | y = x^2\}$ .
- Draw the relation  $\{(x, y) | y \geq x^2\}$ .
- Let  $S_0$  be the equivalence relation on  $\mathbb{R}$  generated (in the sense of Lemma 2.6.1.7) by the empty set. Draw  $S_0$  as a subset of the plane.
- Consider the equivalence relation  $S_1$  generated by  $\{(1, 2), (1, 3)\}$ . Draw  $S_1$  in the plane. Highlight the equivalence class containing  $(1, 2)$ .
- The reflexivity property and the symmetry property have pleasing visualizations in  $\mathbb{R} \times \mathbb{R}$ ; what are they?
- Is there a nice heuristic for visualizing the transitivity property?

**Answer:**



- the line
- the filled area
- red dots
- square of blue dots, including the diagonal
- The representation of a reflexivity relation in rational numbers is the diagonal line on a 2D plane. The representation of a symmetry of a set on a 2D plane is the set itself and its mirror image across the diagonal

(f) TODO: Something like this?



#### 2.6.1.10

Consider the binary relation  $R = \{(n, n+1) | n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}$ .

- (a) What is the equivalence relation generated by  $R$ ?
- (b) How many equivalence classes are there?

**Answer:**

- (a) Whole  $\mathbb{Z}$ .
- (b) One.

#### 2.6.1.11

Suppose  $N$  is a network (or graph). Let  $X$  be the nodes of the network, and let  $R \subseteq X \times X$  denote the relation such that  $(x, y) \in R$  iff there exists an arrow connecting  $x$  to  $y$ .

- (a) What is the equivalence relation  $\sim$  generated by  $R$ ?
- (b) What is the quotient  $X/\sim$ ?

**Answer:**

- (a) For each disjoint subgraph  $g \subseteq N$ , the relation  $R$  represents a fully connected graph  $g'$  formed by the vertices of  $g$ .
- (b) A set of disjoint graphs.

#### 2.6.2.4

(The picture) Write down the cardinality of  $P \cong \underline{n}$  as a natural number  $n \in \mathbb{N}$ .

**Answer:**

1, because the described  $W \rightarrow X$  and  $W \rightarrow Y$  make all elements of  $X$  and  $Y$  equivalent.

### 2.6.2.5

Suppose that  $W = \emptyset$ ; what can you say about  $X \sqcup_W Z$ ?

**Answer:**

$X \sqcup_W Z$  becomes  $X \sqcup Z$  because  $(X \sqcup W \sqcup Z)/\sim$  is filled with distinct elements of both  $X$  and  $Z$ .

### 2.6.2.6

Let  $W := \mathbb{N} = \{0, 1, 2, \dots\}$  denote the set of natural numbers, let  $X = \mathbb{Z}$  denote the set of integers, and let  $Y = \{\ominus\}$  denote a one-element set. Define  $f : W \rightarrow X$  by  $f(w) = -(w + 1)$ , and define  $g : W \rightarrow Y$  to be the unique map. Describe the set  $X \sqcup_W Y$ .

**Answer:**

$X \sqcup_W Y = \{\ominus, 1, 2, 3\}$  where  $\ominus$  is equivalent to all the items of  $\{\dots, -2, -1, 0, \ominus\}$ .

### 2.6.2.7

Let  $i : R \subseteq X \times X$  be an equivalence relation. Composing with the projections  $\pi_1, \pi_2 : X \times X \rightarrow X$ , we have two maps  $\pi_1 \circ i : R \rightarrow X$  and  $\pi_2 \circ i : R \rightarrow X$ .

- (a) What is the pushout  $X - R - X$
- (b) If  $i : R \subseteq X \times X$  is not assumed to be an equivalence relation, we can still define the pushout above. Is there a relationship between the pushout  $X - R - X$  and the equivalence relation generated by  $R \subseteq X \times X$

**Answer:**

- (a) For any equivalence class in  $R$ , the pushout  $X \sqcup_i X$  merges all the points belonging to it into a single element.
- (b) If  $i' : R' \subseteq X \times X$  is not an equivalence relation, then its pushout  $|X \sqcup_{i'} X| > |R|$  where  $R$  is the equivalence relation generated by  $R'$ .

### 2.6.3.2

Let  $X = \mathbb{R}$  be the set of real numbers. What is the coequalizer of the two maps  $X \rightarrow X$  given by  $f = x \mapsto x$  and  $g = x \mapsto (x + 1)$  respectively?

**Answer:**

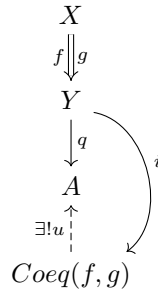
The coequalizer  $\text{Coeq}(f, g) \sim [0, 1) \subset \mathbb{R}$ .

### 2.6.3.3

Find a universal property enjoyed by the coequalizer of two arrows.

**Answer:**

For any set  $A$ , function  $f$  and  $g$ , there exists unique  $u$  mapping  $A$  to  $\text{Coeq}(f, g)$  such that everything commute, in particular,  $u \circ i = q$ .



### 2.6.3.4

(Initial object). An initial set is a set  $S$  such that for every set  $A$ , there exists a unique function  $S \rightarrow A$

- Find an initial set
- Do you think that the notion initial set belongs in this section (Section 2.6)? How so? If coproducts, pushouts, and coequalizers are all colimits, what do colimits have in common?

**Answer:**

- Initial object for sets is the empty set.
- Colimits are isomorphic up to the unique isomorphism (???)

## 2.7 Other notions in Set

### 2.7.1.2

Create an olog that includes sets  $X$  and  $Y$ , and functions  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that  $g \circ f = id_X$  but such that  $f \circ g \neq id_Y$ ; that is, such that  $f$  is a retract section but not an isomorphism.

**Answer:**

Any  $X$  and  $Y$  meet this condition if  $|X| < |Y|$ . For example, mothers and their children. We may say that  $f : X \rightarrow Y$  maps mothers to their first children, while  $g : Y \rightarrow X$  maps children to their mothers.

### 2.7.2.2

For a finite set  $A$ , let  $|A| \in \mathbb{N}$  denote the cardinality of (number of elements in)  $A$ . If  $A$  and  $B$  are both finite (including the possibility that one or both are empty), is it always true that  $|B^A| = |B|^{|A|}$ ?

**Answer:**

The statement is true when both sets are non-empty and if only one set is empty. When both sets are empty, it is true if we put  $0^0 = 1$ .

### 2.7.2.4

Let  $X = \{1, 2\}$ ,  $A = \{a, b\}$ , and  $Y = \{x, y\}$

- (a) Write down three distinct elements of  $L := \text{Hom}_{\text{Set}}(X \times A, Y)$
- (b) Write down all the elements of  $M := \text{Hom}_{\text{Set}}(A, Y)$
- (c) For each of the three elements  $l \in L$  you chose in part (a), write down the corresponding function  $\phi(l) : X \rightarrow M$  guaranteed by Proposition 2.7.2.3.

**Answer:**

- (a) E.g.  $l_1 = \{(1, a) \mapsto x, (1, b) \mapsto x, (2, a) \mapsto x, (2, b) \mapsto x\}$ ,  $l_2 = \{(1, a) \mapsto x, (1, b) \mapsto x, (2, a) \mapsto x, (2, b) \mapsto y\}$ ,  $l_3 = \{(1, a) \mapsto x, (1, b) \mapsto x, (2, a) \mapsto y, (2, b) \mapsto x\}$
- (b) There are:  $f_{xx} = \{a \mapsto x, b \mapsto x\}$ ,  $f_{xy} = \{a \mapsto x, b \mapsto y\}$ ,  $f_{yx} = \{a \mapsto y, b \mapsto x\}$ ,  $f_{yy} = \{a \mapsto y, b \mapsto y\}$
- (c)  $\phi(l_1) = \{1 \mapsto f_{xx}, 2 \mapsto f_{xx}\}$ ,  $\phi(l_2) = \{1 \mapsto f_{xx}, 2 \mapsto f_{xy}\}$ ,  $\phi(l_3) = \{1 \mapsto f_{xx}, 2 \mapsto f_{yx}\}$

### 2.7.2.5

Let  $A$  and  $B$  be sets. We know that  $\text{Hom}_{\text{Set}}(A, B) = B^A$ , so we have a function  $\text{id}_{B^A} : \text{Hom}_{\text{Set}}(A, B) \rightarrow B^A$ . Look at Proposition 2.7.2.3, making the substitutions  $X = \text{Hom}_{\text{Set}}(A, B)$ ,  $Y = B$ , and  $A = A$ . Consider the function

$$\phi^{-1} : \text{Hom}_{\text{Set}}(\text{Hom}_{\text{Set}}(A, B), B^A) \rightarrow \text{Hom}_{\text{Set}}(\text{Hom}_{\text{Set}}(A, B) \times A, B)$$

obtained as the inverse of (2.42). We have a canonical element  $\text{id}_{B^A}$  in the domain of  $\phi^{-1}$ . We can apply the function  $\phi^{-1}$  and obtain an element  $ev = \phi^{-1}(\text{id}_{B^A}) \in \text{Hom}_{\text{Set}}(\text{Hom}_{\text{Set}}(A, B) \times A, B)$ , which is itself a function:

$$ev : \text{Hom}_{\text{Set}}(A, B) \times A \rightarrow B$$

- (a) Describe the function  $ev$  in terms of how it operates on elements in its domain.
- (b) Why might one be tempted to denote this function by  $ev$ ?

**Answer:**

(a) According to (2.4),  $\phi^{-1}$  is

$$\psi : [(A \rightarrow B) \rightarrow (A \rightarrow B)] \rightarrow [(A \rightarrow B) \times A \rightarrow B]$$

defined as

$$\psi = (g : G) \mapsto ((f, a) : (A \rightarrow B) \times A) \mapsto g(f)(a)$$

where we put  $G = [(A \rightarrow B) \rightarrow (A \rightarrow B)]$ . Indeed,  $G$  has an identity element  $id_{B^A} = f \mapsto f$ . Now,

$$ev = \psi(id_{B^A}) = (f, a) \mapsto id_{B^A}(f)(a) = (f, a) \mapsto b$$

where  $f : A \rightarrow B = a \mapsto b$

(b)  $ev$  must be a shortening of "evaluate". It maps  $f$  and its input argument to the result.

### 2.7.2.6

In Example 2.4.1.7 we said that  $\mathbb{R}^2$  is an abbreviation for  $\mathbb{R} \times \mathbb{R}$ , but in (2.43) we say that  $\mathbb{R}^2$  is an abbreviation for  $\mathbb{R}^2$ . Use Exercise 2.1.2.14, Proposition 2.7.2.3, Exercise 2.4.2.10, and the fact that  $1 + 1 = 2$ , to prove that these are isomorphic,  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

**Answer:**

Reminder:

2.1.2.14: Find a set  $A$  such that for any set  $X$  there is a isomorphism of sets  $X = Hom_{Set}(A, X)$  (Answer -  $\{\odot\}$ ).

2.7.2.3: Currying

2.4.2.10: Universal property of coproducts:  $Hom_{Set}(X, A) \times Hom_{Set}(Y, A) = Hom_{Set}(X \sqcup Y, A)$ .

So we go:

$$\mathbb{R}^2 = \{1, 2\} \rightarrow \mathbb{R} \cong \{1\} \sqcup \{2\} \rightarrow \mathbb{R} \cong (\{1\} \rightarrow \mathbb{R}) \times (\{2\} \rightarrow \mathbb{R}) \cong \mathbb{R} \times \mathbb{R}$$

where we write  $Hom_{Set}(A, B)$  as just  $A \rightarrow B$ .

TODO: (**unsure**) The problem in the proof is that we didn't use currying and the  $1 + 1 = 2$  fact. Should we actually continue by saying that  $\mathbb{R} \times \mathbb{R} = \mathbb{R}^{1+1}$ ? But then again, what about currying?

### 2.7.3.2

Everything in Proposition 2.7.3.1 is true except in one case, namely that of

$$\underline{0}^0$$

In this case, we get conflicting answers, because for any set  $A$ , including  $A = \emptyset = \underline{0}$ , we have claimed both that  $A^0 = \underline{1}$  and that  $0^A = \underline{0}$ . What is the correct answer for  $\underline{0}^0$ , based on the definitions of  $\underline{0}$  and  $\underline{1}$ , given in (2.6), and of  $A^B$ , given in (2.41)?

**Answer:**

Reminder:

2.41:  $B^A := \text{Hom}_{\text{Set}}(A, B)$ , which we sometimes write just as  $A \rightarrow B$ .

2.6: The definitions are:  $\underline{1} := \{1\}$ ,  $\underline{0} := \emptyset$ .

$\text{Hom}_{\text{Set}}(A, B)$ , by definition, consists of sets of the form  $\{(a_1, b_1), (a_2, b_2), \dots\}$  where  $\{a_1, a_2, \dots\} = A$  and  $\forall i : b_i \in B$ . In contrast to  $\text{Hom}_{\text{Set}}(A, \{\})$ , for which we can not define any such functions, in  $\text{Hom}_{\text{Set}}(\{\}, \{\})$  we can define a single empty function, so  $\text{Hom}_{\text{Set}}(\{\}, \{\}) = \{\{\}\}$ . Thus:

$$\underline{0}^{\underline{0}} \cong \underline{1}$$

### 2.7.3.3

It is also true of natural numbers that if  $a, b \in \mathbb{N}$  and  $ab = 0$  then either  $a = 0$  or  $b = 0$ . Is the analogous statement true of all sets?

**Answer:**

The proposition is: if  $A \times B = \{\}$  then either  $A = \{\}$  or  $B = \{\}$ .

TODO: check the proof of this fact for numbers.

### 2.7.3.4

Explain why there is a canonical function  $\underline{5}^{\underline{3}} \times \underline{3} \rightarrow \underline{5}$  but not a canonical function  $\underline{575} \rightarrow \underline{5}$ .

**Answer:**

Following the definition of Currying (2.7.2.3), we can transform  $\underline{5}^{\underline{3}} \times \underline{3} \rightarrow \underline{5}$  back to  $\text{Hom}_{\text{Set}}(\underline{3}, \underline{5}) \rightarrow \text{Hom}_{\text{Set}}(\underline{3}, \underline{5})$ . There is an canonical function  $\text{id}_{\text{Hom}_{\text{Set}}(\underline{3}, \underline{5})}$  mapping every  $\underline{3} \rightarrow \underline{5}$  function to itself. In contrast, for  $\underline{575} \rightarrow \underline{5}$  which is  $\text{Hom}_{\text{Set}}(\underline{575}, \underline{5})$  we can't name any outstanding functions.

### 2.7.4.2

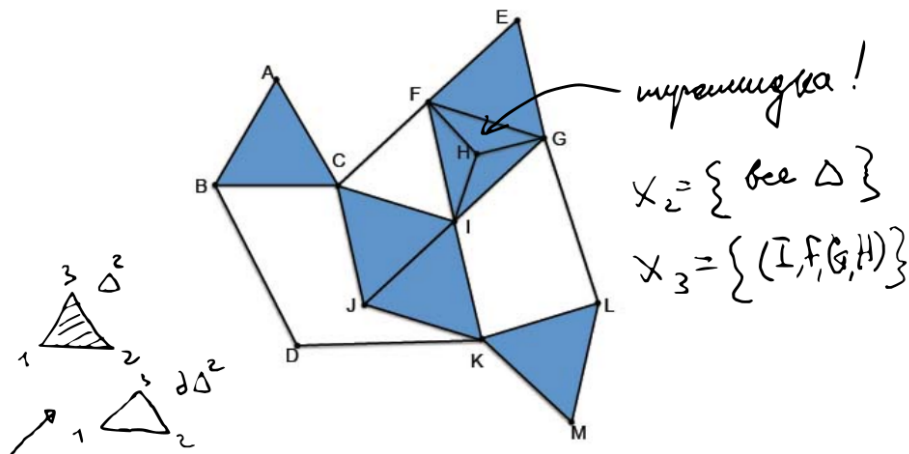
- (a) How many elements does  $\mathbb{P}(\emptyset)$  have?
- (b) How many elements does  $\mathbb{P}(\{\odot\})$  have?
- (c) How many elements does  $\mathbb{P}(\{1, 2, 3, 4, 5, 6\})$  have?
- (d) Any idea why they may have named it "power set"?

**Answer:**

- (a)  $\mathbb{P}(\emptyset) = \{\emptyset\}$  - one element.
- (b)  $\mathbb{P}(\{\odot\}) = \{\{\}, \{\odot\}\}$  - two elements.
- (c)  $\mathbb{P}(\{1, 2, 3, 4, 5, 6\})$  has 64 elements.
- (d) The name probably originates from the fact that  $|\mathbb{P}(A)| = 2^{|A|}$ .

### 2.7.4.6

Let  $X$  be the following simplicial complex, so that  $X_0 = \{A, B, \dots, M\}$



TODO: insert a clean picture

In this case  $X_1$  consists of elements like  $\{A, B\}$  and  $\{D, K\}$  but not  $\{D, J\}$ . Write out  $X_2$  and  $X_3$  (hint: the drawing of  $X$  indicates that  $X_3$  should have one element).

**Answer:**

$X_2 = \{ABC, CIJ, IJK, KLM, EFG, FGH, GHI, FHI, IFG\}$  (all the blue triangles).

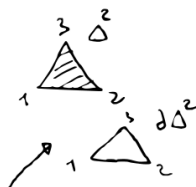
$X_3 = \{IHGF\}$  (the pyramid).

### 2.7.4.7

The 2-simplex  $\Delta^2$  is drawn as a filled-in triangle with vertices  $V = \{1, 2, 3\}$ . There is a simplicial complex  $X = \delta\Delta^2$  that would be drawn as an empty triangle with the same set of vertices.

- Draw  $\Delta^2$  and  $X$  side by side and make clear the difference.
- Write down the data for  $X$  as a simplicial complex. In other words what are the sets  $\{X_0, X_1, X_2, X_3, \dots\}$ ?

**Answer:**



- (TODO: insert a clean picture).
- $X_0 = \{1, 2, 3\}$ ,  $X_1 = \{(1, 2), (1, 3), (2, 3)\}$ ,  $X_2 = X_3 = \dots = \emptyset$ .



### 2.7.4.12

Let  $f : A \rightarrow \Omega$  denote the characteristic function of some  $A' \subseteq A$ , and define  $A'' \subseteq A$  to be its complement,  $A'' = A - A'$  (i.e.  $a \in A''$  if and only if  $a \notin A'$ ).

- (a) What is the characteristic function of  $A'' \subseteq A$ ?
- (b) Can you phrase it in terms of some function  $\Omega \rightarrow \Omega$ ?

**Answer:**

$$g : A'' \rightarrow \Omega = \text{not} \circ f$$

where

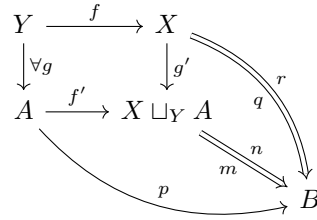
$$\text{not} : \Omega \rightarrow \Omega = \{True \mapsto False, False \mapsto True\}.$$

### 2.7.5.6

Show, in analogy to Proposition 2.7.5.5, that pushouts preserve epimorphisms.

**Answer:**

Let  $f : Y \rightarrow X$  be an epimorphism. For any function  $g$ , the bottom map  $f' : A \rightarrow X \sqcup_Y A$  is again an epimorphism.



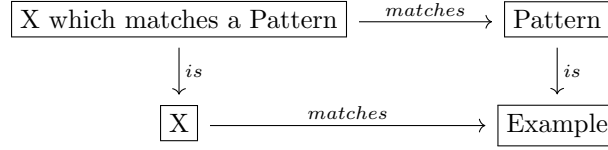
The proof goes as follows: (1) We add some  $B$  and two functions  $m, n : X \sqcup_Y A \rightarrow B$  such that  $m \circ f' = n \circ f'$ . We denote this function by  $p$ . (2) We look at the functions  $q = m \circ g'$  and  $r = n \circ g'$ :  $q \circ f = m \circ g' \circ f = m \circ f' \circ g = n \circ f' \circ g = n \circ g' \circ f = r \circ f$ . So we can speak of only one function  $q$  and ignore  $r$ . (3) From the universal property of pushouts, there exist exactly one function  $X \sqcup_Y A \rightarrow B$ , so we have  $m = n$ . (4) We conclude that  $f'$  is an epimorphism by the definition.

### 2.7.5.9

Consider the subobject classifier  $\Omega$ , the singleton  $\{\odot\}$  and the map  $\{\odot\} \rightarrow_{True} \Omega$  from Definition 2.7.4.9. Look at diagram 2.44 (below) and in the spirit of Exercise(sic!) 2.7.5.7, come up with a label for  $\Omega$ , a label for  $\{\odot\}$ , and a label for  $True$ . Given a label for  $X$  and a label for  $f$ , come up with a label for  $A$ , a label for  $i$  and a label for  $f'$ , such that the English smoothly fits the mathematics.

$$\begin{array}{ccc} A & \xrightarrow{f'} & \{\odot\} \\ \downarrow i & & \downarrow True \\ X & \xrightarrow{f} & \Omega \end{array}$$

**Answer:**



#### 2.7.6.2

- Come up with some notion of mapping for multisets that generalizes functions when the notion is restricted to sets.
- Suppose that  $X = (1, 1, 2, 3)$  and  $Y = (a, b, b, b)$ , i.e.  $X = \{1, 2, 3\}$  with 1 having multiplicity 2, and  $Y = \{a, b\}$  with  $b$  having multiplicity 3. What are all the maps  $X \rightarrow Y$  in your notion?

**Answer:**

- The right definition is given in the textbook below this exercise. Here we try a brute-force approach. Let Multiset be a set of pairs  $\{(e_i, n_i)\}$  where  $\{e_i\} = E$  is the set of elements and  $n_i \in \mathbb{N} : n_i \geq 1$  the multiplicity of each element.

With this definition,  $X = \{(1, 2), (2, 1), (3, 1)\}$  and  $Y = \{(a, 1), (b, 3)\}$ .

- We are going to have some troubles with the definition of what the map is in our multiset. We want to preserve the totality, so we define map  $A \rightarrow B$  as a set of pairs  $\{((a_i, n_i), B_i)\}$  where  $B_i \subseteq B$  such that (1) every  $(a_i, n_i)$  is unique among the map, (2)  $\{(a_i, n_i)\} = A$ , (3) sum of multiplicities of  $B_i$  is equal to the  $n_i$ .

Some examples of  $X \rightarrow Y$  are:

- $\{((1, 2), \{(a, 1), (b, 1)\}), ((2, 1), \{(a, 1)\}), ((3, 1), \{(a, 1)\})\}$
- $\{((1, 2), \{(b, 2)\}), ((2, 1), \{(a, 1)\}), ((3, 1), \{(a, 1)\})\}$
- $\{((1, 2), \{(b, 2)\}), ((2, 1), \{(b, 1)\}), ((3, 1), \{(b, 1)\})\}$
- ...

TODO: should we really list all the maps?

#### 2.7.6.4

Suppose that a pseudo-multiset is defined to be almost the same as a multiset, except that  $\pi$  is not required to be surjective.

- Write down a pseudo-multiset that is not a multi-set.
- Describe the difference between the two notions in terms of multiplicities.
- The multiset commutative diagram might no longer commute for the pseudo-multisets.

**Answer:**

Recap: We say that  $f : X \rightarrow Y$  is surjective if, for all  $y \in Y$  there exists some  $x \in X$  such that  $f(x) = y$ . In other words, uncovered elements in  $Y$  are not allowed, but  $X$  can be larger than  $Y$ . Being "non-surjective" means that the multiset is allowed to contain unused names in it.

- (a) Example of a pseudo-multiset which is not a multiset:  $(E = \{1_1, 1_2, 2, 3\}, B = \{1, 2, 3, \odot\}, \pi = \{1_1 \mapsto 1, 1_2 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3\})$ .  $\pi$  doesn't map anything to  $\odot$  thus it is not surjective.
- (b) Pseudo-multisets are allowed to contain elements with undefined multiplicities.
- (c) Pseudo-multisets are less useful because it is hard to choose what is better: to include an element with an undefined multiplicity or to not include it at all.

#### 2.7.6.5

Consider the multisets described in Exercise 2.7.6.2.

- (a) Write each of them in the form  $(E, B, \pi)$ , as in Definition 2.7.6.3.
- (b) In terms of the same definition, what are the mappings  $X \rightarrow Y$ ?
- (c) If we remove the restriction that diagram 2.45 must commute, how many mappings  $X \rightarrow Y$  are there?

**Answer:**

- (a)  $X = (E = \{1_1, 1_2, 2, 3\}, B = \{1, 2, 3\}, \pi = \{1_1 \mapsto 1, 1_2 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3\})$ .  
 $Y = (E = \{a, b_1, b_2, b_3\}, B = \{a, b\}, \pi = \{a \mapsto a, b_1 \mapsto b, b_2 \mapsto b, b_3 \mapsto b\})$ .
- (b) The set of mappings is  $\{(f_1, f_2) \mid f_2(\pi_X(x)) = \pi_Y(f_1(x))\}$ . It is the set containing all  $f_1 : E_X \rightarrow E_Y$  except those which leads to different instances of  $B_X$  to have different names in  $B_Y$ . So, for example,  $\{1_1 \mapsto a, 1_2 \mapsto b_1\}$  is forbidden.
- (c) Without the commutativity restriction, we must have  $|E_X \rightarrow E_Y| * |B_X \rightarrow B_Y| = 4^4 * 3^2 = 2304$  different mappings.

#### 2.7.6.8

Given sets  $X, Y, Z$  and functions  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , we can compose them to get a function  $X \rightarrow Z$ . If  $B$  is a set, if  $(X, p), (Y, q)$  and  $(Z, r)$  are relative sets over  $B$ , and if  $f : (X, p) \rightarrow (Y, q)$  and  $g : (Y, q) \rightarrow (Z, r)$  are mappings, is there a reasonable notion of composition such that we get a mapping of relative sets  $(X, p) \rightarrow (Z, r)$ ? Hint: draw diagrams.

**Answer:**

TODO

### 2.7.6.9

- (a) Let  $\{\odot\}$  denote a set with one element. What is the difference between sets over  $\{\odot\}$  and simply sets?
- (b) Describe the sets relative to  $\emptyset$ . How many are there?

**Answer:**

TODO

### 2.7.6.13

Let  $\{\odot\}$  denote a one element set. What are  $\{\odot\}$ -indexed sets and mappings between them?

**Answer:**

TODO

### 2.7.6.14

There is a strong relationship between  $A$ -indexed sets and relative sets over  $A$ . What is it?

**Answer:**

TODO

## References

- [1] David I. Spivak. “Category theory for scientists”. In: (2013). URL: <https://api.semanticscholar.org/CorpusID:126379073>.