Category Theory for Scientists (Solutions)

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Contents

1 Introduction

The book *(2013, Spivak) "Category theory for scientists"* [1].

Chapter 2. The Category of Sets

2.3.3.1

Create an olog for human nuclear biological families that includes the concept of person, man, woman, parent, father, mother, and child. Make sure to label all the arrows, and make sure each arrow indicates a valid aspect in the sense of Section 2.3.2.1. Indicate with check-marks the diagrams that are intended to commute. If the 2-dimensionality of the page prevents a check-mark from being unambiguous, indicate the intended commutativity with an equation.

Answer:

$$\begin{tikzcd}
 & person & \\
man \arrow[ur, "is"] & & woman \arrow[ul, "is"'] \\
 & parent \arrow[uu, "is"] \arrow[dd, "has"] & \\
father \arrow[uu, "is"] \arrow[ur, "is"] \arrow[dr, "has"'] & & mother \arrow[uu, "is"'] \arrow[ul, "is"'] \arrow[dl, "has"] \\
 & child \arrow[uuuu, bend right=90, "is"'] &
\end{tikzcd}$$

Commutative square examples:

- "parent is person" ∘ "mother is parent" = "woman is person" ∘ "mother is woman"
- "mother has child" = "parent has child" ∘ "mother is parent"
- "child is person" ∘ "parent has child" ≠ "parent is person"

2.4 Products and coproducts

2.4.1.4

How many elements does the $\{a, b, c, d\} * \{1, 2, 3\}$ have?

Answer:

12

2.4.1.8

Answer:

(a) No, because $a(b + c) \neq (a + b)c$.

(b) No, because $x * 0 \neq x$.

(c) Yes.

2.4.1.15

(a) Let X and Y be sets.. construct the "swap map" $s : (X \times Y) \to (Y \times X)$

Answer:

$s : (X \times Y) \to (Y \times X) = (,) \circ \langle \pi_2, \pi_1 \rangle$
Note: we used angle brackets. Is it really correct?

(b) Can you prove that s is a isomorphism using only the universal property for product?
Note: $(f : X \to Y)$ is an isomorphism if $\exists (g : Y \to X) : g \circ f = id_X \wedge f \circ g = id_Y$
Note: diagram (f, g, h) commutes if $f \circ g = h$

Answer:

In universal property of products, put A equal to $Y \times X$ and get $\exists! g : (Y \times X) \to (X \times Y)$. In a similar way, we have $\exists! s : (X \times Y) \to (Y \times X)$. We need to show that $g \circ s = id_{(X \times Y)}$ and $s \circ g = id_{(Y \times X)}$. On the following diagram,

$$
\begin{tikzcd}
& A \times B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
A & & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, "\exists! id"'] &
\end{tikzcd}
$$

$id = id_{(A \times B)}$ - is the unique identity function. By combining diagrams for f and g we reduce the $g \circ f$ to the similar case.

2.4.2.4

Would you say that a phone is the coproduct of a `cellphone` and a `landline phone`?

Answer:

Yes, until we consider other types of phones besides cell- and landline ones.

2.4.2.10

Write the universal property for coproduct in terms of a relationship between $Hom_{Set}(X, A)$, $Hom_{Set}(Y, A)$ and $Hom_{Set}(X \sqcup Y, A)$.

Answer:

$Hom_{Set}(X, A) \times Hom_{Set}(Y, A) \cong Hom_{Set}(X \sqcup Y, A)$

2.4.2.13

TODO

2.4.2.14

TODO

2.5 Finite limits in Set

2.5.1.2

Answer:

$X \times_Z Y = \{(x_1, z_1, y_1), (x_2, z_2, y_2), (x_2, z_2, y_4), (x_3, z_2, y_2), (x_3, z_2, y_4)\}$

2.5.1.3

(a)

Answer:

Let $X = \{1, 2, 3, 4, 5\}; Y = \{a, b, c\}$; where $C = \{R, B, Y\}$

We have: $X \times_C Y = \{1a, 4a, 2c, 5c, 3b\}$
(b) TODO (obvious).

2.5.1.5

(a) Suppose that $Y = \emptyset$; what can you say about $X \times_Z Y$?

Answer:

$X \times_Z Y = \emptyset$

(b) $Z = 1$; what can you say about $X \times_Z Y$?

Answer:

$\forall X, Y : X \times_Z Y \cong X \times Y$

2.5.1.6

.. Aristotelian space and time ..
$S = R^3$; $T = R$; $Y = S \times T$; $g1 : Y \to S$; $g2 : Y \to S$ where $g1, g2$ projects space-time to its components. $X = \{1\}$; $f_1 : X \to S$; $f_2 : X \to T$ is a set of one element and its space-time projections.

(a) What is the meaning of

$$\begin{tikzcd} W_1 \arrow[r] \arrow[d] & Y \arrow[d, "g_1"] \\ X \arrow[r, "f_2"] & S \end{tikzcd} \qquad \begin{tikzcd} W_2 \arrow[r] \arrow[d] & Y \arrow[d, "g_2"] \\ X \arrow[r, "f_2"] & S \end{tikzcd}$$

Answer:

1 is associated with its time and space. W_1 yields time points of Y corresponding to 1's position. W_2 yields the space points corresponding to 1's time.

(b) Interpret the sets in terms of the center of mass of MIT at the time of its founding.

Answer:

TODO: (**unsure**) Is it just the MIT-relative space and time points?

2.5.1.10

.. Appropriate or misleading olog labels ..

Answer:

(a) a person whose favorite color is blue - OK

(b) a dog whose owner is a woman - OK

(c) a good fit - Nope. We would say that a good fit requires less or equal width.

2.5.1.11

(a) Consider your olog from Exercise 2.3.3.1. Are any of the commutative squares there actually pullback squares?

Answer:

Yes, for example: "father" = "man" $\times_{person}$ "parent"

$$\begin{tikzcd} \boxed{\text{father}} \arrow[r, "is"] \arrow[d, "is"] & \boxed{\text{man}} \arrow[d, "is"] \\ \boxed{\text{parent}} \arrow[r, "is"] & \boxed{\text{person}} \end{tikzcd}$$

(b) Now use ologs with products and pullbacks to define what a brother is and what a sister is:

Answer:

$$\begin{tikzcd} brother \arrow[r, "is"] \arrow[d, "is"] & child \arrow[d, "p"] & \\ man \arrow[r, "p"] & pom \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \{T\} \arrow[d, "id"] \\ & person \arrow[r, "n_{ch}"] & \{T, F\} \end{tikzcd}$$

where: *pom* stands for "parent of many", p is "has as parent" and n_{ch} is "number of children"

2.5.1.13

Pullback diagram in which the fiber product is isomorphic to the preimage of $y \in Y$.

Answer:

$$\begin{tikzcd} Y^{-1} \times \{y\} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \{y\} \arrow[d, "id"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

2.5.1.15

Create an olog whose underlying shape is a commutative square. Now add the fiber product so that the shape is the same as that of Diagram (2.32). Assign English labels to the projections and to the dotted map A, such that these labels are as canonical as possible.

Answer:

TODO: (**unsure**)
The general impression is the following:

- The commutative square can be visualized as a pyramid in a 3D space.
- The pullback corresponds to a 2D plane with two axes.
- The pullback over one-element set corresponds to a 2D plane with two orthogonal axes.
- The projections π_1 and π_2 play the role of parallel projections from a point in the plane to its axes.
- The map $A \to (D \times_P W)$ acts as a projection from a point in 3D space to the plane.

$$\begin{tikzcd}
 & \boxed{(D \times_P W) \text{ woman dog owner}} \arrow[ddl, bend right, "\pi_1"'] \arrow[ddr, bend left, "\pi_2"] & \\
 & \boxed{(A)\begin{array}{l}\text{blond woman}\\ \text{dog owner}\end{array}} \arrow[u, "\langle\text{has, is}\rangle"] \arrow[dl, "has"'] \arrow[dr, "is"] & \\
\boxed{(D) \text{ dog}} \arrow[dr] & & \boxed{(W) \text{ woman}} \arrow[dl] \\
 & \boxed{(P) \text{ person}} &
\end{tikzcd}$$

2.5.1.18

(a) Create an olog that defines two people to be 'of approximately the same height' if and only if their height difference is less than half an inch, using a pullback.

Answer:

$$\begin{tikzcd}
\{((p_1, p_2), \Delta) | p_1, p_2 \in \text{people}, \Delta \in \mathbb{R}\} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \{\Delta | -0.5 < \Delta < 0.5\} \arrow[d, "is"] \\
\boxed{\text{pair of people}} \arrow[r, "h_2 - h_1"] & \mathbb{R}
\end{tikzcd}$$

(b) In the same olog, make a box for those people whose height is approximately the same as a person named 'The Virgin Mary'. You may need to use images.

Answer:

$$\begin{array}{ccccc}
\boxed{\text{people of approx Virgin Mary height}} & \xrightarrow{\pi_2} & \{((p_1,p_2),\Delta)|...\} & \xrightarrow{\pi_2} & \{\Delta| -0.5<\Delta<0.5\} \\
\downarrow \pi_1 & & \downarrow \pi_1 & & \downarrow is \\
\{(\boxed{\text{The Virgin Mary}}, p_2)|p_2 \in \text{people}\} & \xrightarrow{is} & \boxed{\text{pair of people}} & \xrightarrow{h_2 - h_1} & \mathbb{R}
\end{array}$$

TODO: (**unsure**) We did not use images, despite the textbook suggestion.

2.5.1.19

Let $W = X \times_Z Y$ and $W' = X' \times_{Z'} Y'$, and form the diagram to the right. Use the universal property of fiber products to construct a map $W \to W'$ such that all squares commute

Answer:

Let $WX = \pi_1$, $WY = \pi_2$, $W'X' = \pi_1'$, $W'Y' = \pi_2'$. Define $f : W \to W' = (x,y)_z \mapsto (XX'(x), YY'(y))_{ZZ'(z)}$. We want to show that it makes $WW'Y'Y$ to commute. Indeed, $\pi_2' \circ f = YY' \circ \pi_2 = y \mapsto YY'(y)$. By analogy, it also commutes $WW'X'X$. The triangles $WW'Y'$ and $WW'X'$ also commute. The universal property for fiber products says that we can only have one such mapping, thus this is the answer.

2.5.2.6

(a) TODO: Copy the picture from the book notes
(b) TODO: Copy the picture from the book notes
(c) TODO: Find the correct answer

2.5.3.3

Come up with an olog that uses equalizers in a reasonably interesting way. Alternatively, use an equalizer to specify those published authors who have published exactly one paper. Hint: find a function from authors to papers; then find another.

Answer:

We follow the second way.

$$\text{Authors} \underset{\text{const}_1}{\overset{\text{num of papers}}{\rightrightarrows}} \mathbb{R}$$

where $const_1$: authors $\to \mathbb{R}$ maps all authors to 1.

2.5.3.4

Find a universal property enjoyed by the equalizer of two arrows, and present it in the style of Lemmas 2.4.1.10, 2.4.2.7, and 2.5.1.14.

Answer:

For any set A, function f and g there exists unique u mapping A to $Eq(f,g)$ such that everything commute, in particular, $\pi \circ u = p$.

$$\begin{tikzcd}
Eq(f,g) \arrow[dd, bend right=60, "\pi"'] \\
A \arrow[u, dashed, "\exists! u"] \arrow[d, "p"] \\
X \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\
Y
\end{tikzcd}$$

2.5.3.5

(a) A terminal set is a set S such that for every set X, there exists a unique function $X \to S$. Find a terminal set.

Answer:

Any set S with one element. There exist exactly one function $X \to S$ so it is unique.

(b) Do you think that the notion *terminal set* belongs in this section (Section 2.5)? How so? If products, pullbacks, and equalizers are all limits, what do limits have in common?

Answer:

A terminal set S is terminal because it has canonical projections (inclusions) back to X (and Y, if applicable). We suppose that the way they work does not change from category to category.

2.6 Finite colimits in Set

2.6.1.3

Let X be the set of people on earth; define a binary relation $R \subseteq (X \times X)$ on X as follows. For a pair (x, y) of people, say $(x, y) \in R$ if x spends a lot of time thinking about y.

Answer:

(a) Is this relation reflexive? No.
(b) Is it symmetric? No.
(c) Is it transitive? No.

2.6.1.5

Take a set I of sets; i.e. suppose that for each element $i \in I$ you are given a set X_i. For every two elements $i, j \in I$ say that $i \sim j$ if X_i and X_j are isomorphic. Is this relation an equivalence relation on I?

Answer:

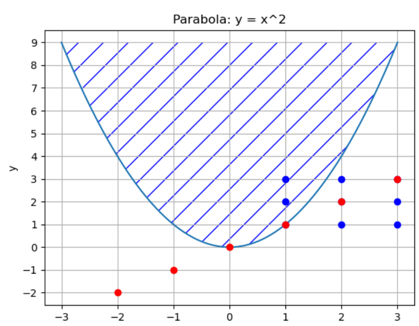
Any set is isomorphic to itself, so $\sim$ is reflexive; isomorphism is also symmetric and transitive. So, yes, $\sim$ is an equivalence relation.

2.6.1.9

Consider the set R of real numbers. Draw the coordinate plane $\mathbb{R} \times \mathbb{R}$, give it coordinates x and y. A binary relation on $\mathbb{R}$ is a subset $S \subseteq \mathbb{R} \times \mathbb{R}$, which can be drawn as a set of points in the plane.

(a) Draw the relation $\{(x, y)|y = x^2\}$.
(b) Draw the relation $\{(x, y)|y \geq x^2\}$.
(c) Let S_0 be the equivalence relation on R generated (in the sense of Lemma 2.6.1.7) by the empty set. Draw S as a subset of the plane.
(d) Consider the equivalence relation S_1 generated by $\{(1, 2), (1, 3)\}$. Draw S_1 in the plane. Highlight the equivalence class containing $(1, 2)$.
(e) The reflexivity property and the symmetry property have pleasing visualizations in $\mathbb{R} \times \mathbb{R}$; what are they?
(f) Is there a nice heuristic for visualizing the transitivity property?

Answer:



(a) the line

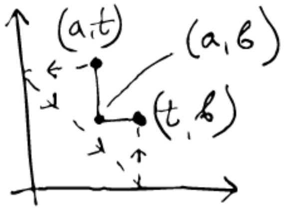
(b) the filled area

(c) red dots

(d) square of blue dots, including the diagonal

(e) The representation of a reflexivity relation in rational numbers is the diagonal line on a 2D plane. The representation of a symmetry of a set on a 2D plane is the set itself and its mirror image across the diagonal

(f) TODO: Something like this?



2.6.1.10

Consider the binary relation $R = \{(n, n+1) | n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}$.

(a) What is the equivalence relation generated by R?

(b) How many equivalence classes are there?

Answer:

(a) Whole $\mathbb{Z}$.

(b) One.

2.6.1.11

Suppose N is a network (or graph). Let X be the nodes of the network, and let $R \subseteq X \times X$ denote the relation such that $(x, y) \in R$ iff there exists an arrow connecting x to y.

(a) What is the equivalence relation $\sim$ generated by R?

(b) What is the quotient $X/\sim$?

Answer:

(a) For each disjoint subgraph $g \subseteq N$, the relation R represents a fully connected graph g' formed by the vertices of g.

(b) A set of disjoint graphs.

2.6.2.4

(The picture) Write down the cardinality of $P \cong \underline{n}$ as a natural number $n \in \mathbb{N}$.

Answer:

1, because the described $W \to X$ and $W \to Y$ make all elements of X and Y equivalent.

2.6.2.5

Suppose that $W = \emptyset$; what can you say about $X \sqcup_W Z$?

Answer:

$X \sqcup_W Z$ becomes $X \sqcup Z$ because $(X \sqcup W \sqcup Z)/\sim$ is filled with distinct elements of both X and Z.

2.6.2.6

Let $W := \mathbb{N} = \{0, 1, 2, ...\}$ denote the set of natural numbers, let $X = \mathbb{Z}$ denote the set of integers, and let $Y = \{☺\}$ denote a one-element set. Define $f : W \to X$ by $f(w) = -(w + 1)$, and define $g : W \to Y$ to be the unique map. Describe the set $X \sqcup_W Y$.

Answer:

$X \sqcup_W Y = \{😄, 1, 2, 3\}$ where 😄 is equivalent to all the items of $\{..., -2, -1, 0, 😄\}$.

2.6.2.7

Let $i : R \subseteq X \times X$ be an equivalence relation. Composing with the projections $\pi_1, \pi_2 : X \times X \to X$, we have two maps $\pi_1 \circ i : R \to X$ and $\pi_2 \circ i : R \to X$.

(a) What is the pushout $X - R - X$

(b) If $i : R \subseteq X \times X$ is not assumed to be an equivalence relation, we can still define the pushout above. Is there a relationship between the pushout $X - R - X$ and the equivalence relation generated by $R \subseteq X \times X$

Answer:

(a) For any equivalence class in R, the pushout $X \sqcup_i X$ merges all the points belonging to it into a single element.

(b) If $i' : R' \subseteq X \times X$ is not an equivalence relation, then its pushout $|X \sqcup_{i'} X| > |R|$ where R is the equivalence relation generated by R'.

2.6.3.2

Let $X = R$ be the set of real numbers. What is the coequalizer of the two maps $X \to X$ given by $f = x \mapsto x$ and $g = x \mapsto (x + 1)$ respectively?

Answer:

The coequalizer $Coeq(f, g) \sim [0, 1) \subset \mathbb{R}$.

2.6.3.3

Find a universal property enjoyed by the coequalizer of two arrows.

Answer:

For any set A, function f and g, there exists unique u mapping A to $Coeq(f,g)$ such that everything commute, in particular, $u \circ i = q$.

$$
\begin{tikzcd}
X \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\
Y \arrow[d, "q"] \arrow[dd, bend left=60, "i"] \\
A \\
Coeq(f,g) \arrow[u, dashed, "\exists! u"]
\end{tikzcd}
$$

2.6.3.4

(Initial object). An initial set is a set S such that for every set A, there exists a unique function $S \to A$

(a) Find an initial set

(b) Do you think that the notion initial set belongs in this section (Section 2.6)? How so? If coproducts, pushouts, and coequalizers are all colimits, what do colimits have in common?

Answer:

(a) Initial object for sets is the empty set.

(b) Colimits are isomorphic up to the unique isomorphism (???)

2.7 Other notions in Set

2.7.1.2

Create an olog that includes sets X and Y , and functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = id_X$ but such that $f \circ g \neq id_Y$; that is, such that f is a retract section but not an isomorphism.

Answer:

Any X and Y meet this condition if $|X| < |Y|$. For example, mothers and their children. We may say that $f : X \to Y$ maps mothers to their first children, while $g : Y \to X$ maps children to their mothers.

2.7.2.2

For a finite set A, let $|A| \in \mathbb{N}$ denote the cardinality of (number of elements in) A. If A and B are both finite (including the possibility that one or both are empty), is it always true that $|B^A| = |B|^{|A|}$?

Answer:

The statement is true when both sets are non-empty and if only one set is empty. When both sets are empty, it is true if we put $0^0 = 1$.

2.7.2.4

Let $X = \{1, 2\}$, $A = \{a, b\}$, and $Y = \{x, y\}$

(a) Write down three distinct elements of $L := Hom_{Set}(X \times A, Y)$

(b) Write down all the elements of $M := Hom_{Set}(A, Y)$

(c) For each of the three elements $l \in L$ you chose in part (a), write down the corresponding function $\phi(l) : X \to M$ guaranteed by Proposition 2.7.2.3.

Answer:

(a) E.g. $l_1 = \{(1, a) \mapsto x, (1, b) \mapsto x, (2, a) \mapsto x, (2, b) \mapsto x\}$, $l_2 = \{(1, a) \mapsto x, (1, b) \mapsto x, (2, a) \mapsto x, (2, b) \mapsto y\}$, $l_3 = \{(1, a) \mapsto x, (1, b) \mapsto x, (2, a) \mapsto y, (2, b) \mapsto x\}$

(b) There are: $f_{xx} = \{a \mapsto x, b \mapsto x\}$, $f_{xy} = \{a \mapsto x, b \mapsto y\}$, $f_{yx} = \{a \mapsto y, b \mapsto x\}$, $f_{yy} = \{a \mapsto x, b \mapsto y\}$

(c) $\phi(l_1) = \{1 \mapsto f_{xx}, 2 \mapsto f_{xx}\}$, $\phi(l_2) = \{1 \mapsto f_{xx}, 2 \mapsto f_{xy}\}$, $\phi(l_3) = \{1 \mapsto f_{xx}, 2 \mapsto f_{yx}\}$

2.7.2.5

Let A and B be sets. We know that $Hom_{Set}(A, B) = B^A$, so we have a function $id_{B^A} : Hom_{Set}(A, B) \to B^A$. Look at Proposition 2.7.2.3, making the substitutions $X = Hom_{Set}(A, B)$, $Y = B$, and $A = A$. Consider the function

$\phi^{-1} : HomSet(Hom_{Set}(A, B), B^A) \to Hom_{Set}(Hom_{Set}(A, B) \times A, B)$

obtained as the inverse of (2.42). We have a canonical element id_{B^A} in the domain of ϕ^{-1}. We can apply the function ϕ^{-1} and obtain an element $ev = \phi^{-1}(id_{B^A}) \in Hom_{Set}(Hom_{Set}(A, B) \times A, B)$, which is itself a function:

$ev : Hom_{Set}(A, B) \times A \to B$

(a) Describe the function ev in terms of how it operates on elements in its domain.

(b) Why might one be tempted to denote this function by ev?

Answer:

(a) According to (2.4), ϕ^{-1} is

$$\psi : [(A \to B) \to (A \to B)] \to [(A \to B) \times A \to B]$$

defined as

$$\psi = (g : G) \mapsto ((f, a) : (A \to B) \times A) \mapsto g(f)(a)$$

where we put $G = [(A \to B) \to (A \to B)]$. Indeed, G has an identity element $id_{B^A} = f \mapsto f$. Now,

$$ev = \psi(id_{B^A}) = (f, a) \mapsto id_{B^A}(f)(a) = (f, a) \mapsto b$$

where $f : A \to B = a \mapsto b$

(b) ev must be a shortening of "evaluate". It maps f and its input argument to the result.

2.7.2.6

In Example 2.4.1.7 we said that $\mathbb{R}^2$ is an abbreviation for $\mathbb{R} \times \mathbb{R}$, but in (2.43) we say that $\mathbb{R}^2$ is an abbreviation for $\mathbb{R}^{\underline{2}}$. Use Exercise 2.1.2.14, Proposition 2.7.2.3, Exercise 2.4.2.10, and the fact that $1 + 1 = 2$, to prove that these are isomorphic, $\mathbb{R}^{\underline{2}} = \mathbb{R} \times \mathbb{R}$

Answer:

Reminder:
2.1.2.14: Find a set A such that for any set X there is a isomorphism of sets $X = Hom_{Set}(A, X)$ (Answer - {☺}).
2.7.2.3: Currying
2.4.2.10: Universal property of coproducts: $Hom_{Set}(X, A) \times Hom_{Set}(Y, A) = Hom_{Set}(X \sqcup Y, A)$.
So we go:

$$\mathbb{R}^{\underline{2}} = \{1, 2\} \to \mathbb{R} \cong \{1\} \sqcup \{2\} \to \mathbb{R} \cong (\{1\} \to \mathbb{R}) \times (\{2\} \to \mathbb{R}) \cong \mathbb{R} \times \mathbb{R}$$

where we write $Hom_{Set}(A, B)$ as just $A \to B$.
TODO: (**unsure**) The problem in the proof is that we didn't use currying and the $1 + 1 = 2$ fact. Should we actually continue by saying that $\mathbb{R} \times \mathbb{R} = \mathbb{R}^{1+1}$? But then again, what about currying?

2.7.3.2

Everything in Proposition 2.7.3.1 is true except in one case, namely that of

$$\underline{0}^{\underline{0}}$$

In this case, we get conflicting answers, because for any set A, including $A = \emptyset = \underline{0}$, we have claimed both that $A^0 = \underline{1}$ and that $0^A = \underline{0}$. What is the correct answer for $\underline{0}^{\underline{0}}$, based on the definitions of $\underline{0}$ and $\underline{1}$, given in (2.6), and of A^B, given in (2.41)?

Answer:

Reminder:

2.41: $B^A := Hom_{Set}(A, B)$, which we sometimes write just as $A \to B$.
2.6: The definitions are: $\underline{1} := \{1\}$, $\underline{0} := \emptyset$.

For $Hom_{Set}(A, B)$, by definition, consists sets of the form $\{(a_1, b_1), (a_2, b_2), ...\}$ where $\{a_1, a_2, ...\} = A$ and $\forall i : b_i \in B$. In contrast to $Hom_{Set}(A, \{\})$, for which we can not define any such functions, in $Hom_{Set}(\{\}, \{\})$ we can define a single empty function, so $Hom_{Set}(\{\}, \{\}) = \{\{\}\}$. Thus:

$$\underline{0}^{\underline{0}} \cong \underline{1}$$

2.7.3.3

It is also true of natural numbers that if $a, b \in \mathbb{N}$ and $ab = 0$ then either $a = 0$ or $b = 0$. Is the analogous statement true of all sets?

Answer:

The proposition is: if $A \times B = \{\}$ then either $A = \{\}$ or $B = \{\}$.

References

[1] David I. Spivak. "Category theory for scientists". In: (2013). URL: https://api.semanticscholar.org/CorpusID:126379073.