On the feasibility for the system of quadratic equations, explanations

Theorem 3.2 (Sufficient condition)

Consider $f\colon \mathbb{R}^n \to \mathbb{R}^m$, s.t. $f_i(x) = x^T A_i x + 2b_i^T x$, $A_i = A_i^T$. Define $F = f(\mathbb{R}^n)$.
Then why $A = \inf\limits_{y \in F} (c, y) = \inf\limits_{y \in \operatorname{conv} F} (c, y) = B$?

1. First, $F \subseteq \operatorname{conv} F$, therefore, $B \leqslant A$.

2. Secondly, let $y_k \in \operatorname{conv} F$ be a sequence s.t. $g_k = (c, y_k) \underset{k\to\infty}{\longrightarrow} B$. $y_k = \sum\limits_{i=1}^{n_k} \alpha_i^k y_i^k$.

   $g_k(c, y_k) = \sum\limits_{i=1}^{n_k} \alpha_i^k (c, y_i^k) = \sum\limits_{i=1}^{n_k} \alpha_i^k g_i^k$. Define $g_0^k = \min\limits_{i \in \overline{1, n_k}} g_i^k$. Then $B \leqslant g_0^k \leqslant g^k$. Therefore, $g_0^k \to B$ also. This way, we have constructed a sequence $y_0^k \in F$ s.t. $(c, y_0^k) \to B$, therefore, $A \leqslant B$.

Finding c provided d

Let $H\colon \mathbb{R}^{n+1,n+1} \to \mathbb{R}^n$ be a map s.t. $H_i(X) = \operatorname{Tr}(H_i X)$,

$$H_i = \left\| \begin{matrix} A_i & b_i \\ b_i^T & 0 \end{matrix} \right\|^{\square}$$

.

Consider a boundary point X, which is a solution of:

$$\begin{aligned} &\sup \qquad t \\ &\begin{cases} H(X) = y^0 + td \\ X \geqslant 0 \\ X_{n+1,n+1} = 1 \end{cases} \end{aligned}$$

.

Define $f(t, X) = t$, $D_0 = \{(t, X) \big| X \geqslant 0,\, X_{n+1,n+1} = 1\}$, $D_1 = \{(t, X) \big| H(X) = y^0 + td\}$.
Then supremum is equivalent to

$$\sup_{(t,X) \in D_0 \cap D_1} f(t, X)$$

Define a Lagrange function $L(c, t, X) = \underbrace{t}_{f(t,X)} + \sum\limits_{i=1}^{m} c_i (y_i^0 + td_i - H_i(X))$

Then the dual function is $g(c) = \sup\limits_{(t,X) \in D_0} L(c, t, X)$.

Because $L = t(1 + \sum\limits_{i=1}^{m} c_i d_i) + \sum\limits_{i=1}^{n} c_i (y_i^0 - H_i(X))$, $g = +\infty$ when $(c, d) \neq -1$. From this point we assume that $\boxed{(c, d) = -1}$.

Now, $g(c) = \sup\limits_{X_{n+1,n+1}=1, X \geqslant 0} (c, y^0 - H(X)) = (c, y^0) + \sup\limits_{y \in \operatorname{conv} F} -(c, y) = (c, y^0) - \inf\limits_{y \in \operatorname{conv} F} (c, y)$.

Then the dual problem is

$$g(c) \to \inf_{(c,d) = -1}$$