# On the feasibility for the system of quadratic equations. Minimizing $z(c)$ when $|c_{\text{bad}}| = \infty$

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## 1 The framework

In this section, the framework for finding convex cuts is described. We start from the definition of a quadratic map, then the task is set formally and then is formulated as a constrainted optimization problem.

Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a quadratic map: $f_i(x) = x^T A_i x + 2b_i^T x$, $A_i^T = A_i$. We want to explore the image $F = f(\mathbb{R}^n)$ and find convex parts of $F$.

1. Consider a problem

$$g(c) = \min_{x \in \mathbb{R}^n} (c, f(x)) \tag{1}$$

It has a finite solution iff 3.2

$$\begin{cases} c \cdot A \geqslant 0 \\ (c \cdot b)^T \operatorname{Ker}(c \cdot A) = 0 \end{cases}$$

The $x$, minimizing 1 are ($Q^g$ is a pseudoinverse of $Q$)

$$x = \operatorname{Ker}(c \cdot A) - (c \cdot A)^g (c \cdot b)$$

This way, we found pre-images $X(c)$ of boundary points of $F$ lying on the supporting hyperplane with a normal vector $c$:

$$X(c) = \arg\min(c, f(x)) = \operatorname{Ker}(c \cdot A) - (c \cdot A)^g (c \cdot b)$$

2. Let $c_+$ be a vector from $\mathbb{R}^m$ s.t. $c_+ \cdot A > 0$. Then the supporting hyperplane with the normal vector $c_+$ is touching $F$ at a single point $y_0 = f(x_0)$, where

$$x_0 = -(c_+ \cdot A)^{-1} (c_+ \cdot b)$$

We want to find the maximum value $z_{\max}$ such that $Z \subseteq F$ is convex:

$$Z = \{y \big| (c_+, y - y_0) \in [0, z_{\max}]\}$$

To do that, we find c such that set $f(X(c))$ is nonconvex, project points of $f(X(c))$ to $c_+$ and find a minimum of the projection:

$$z_{\max} = \inf_c \tilde{z}(c),\ \tilde{z}(c) = \inf \big(c_+, f(X(c)) - f(x_0)\big) \tag{2}$$

3. To find $\tilde{z}(c) = \inf\limits_{x \in \operatorname{Ker}(c \cdot A) - (c \cdot A)^g (c \cdot b)} (c_+, f(x) - f(x_0))$, we introduce the plus norm in the $q \in \mathbb{R}^n$ space

$$||q||_+^2 = q^T (c_+ \cdot A) q$$

Note that 3.4 $(c_+, f(x) - f(x_0)) = ||x - x_0||_+^2$. Therefore, $\tilde{z}(c) = \rho_+^2(\operatorname{Ker}(c \cdot A) - (c \cdot A)^g (c \cdot b), x_0)$

4. Next, we change variables to find $\tilde{z}(c)$ explicitly. We use the following change $\begin{cases} x = S\hat{x} + x^0 \\ y = \hat{y} + y^0 \end{cases}$ $y = f(x)$, $\hat{y} = \hat{f}(\hat{x})$ and demand $\begin{cases} c_+ \cdot A = I \\ c_+ \cdot b = 0 \end{cases}$. It leads 3.5 to

$$x^0 = x_0,\ y^0 = y_0,\ S = S_1 S_2,\ S_1^T (c_+ \cdot A) S_1 = \Lambda,\ S_1^T S_1 = I,\ S_2 = \Lambda^{-1/2},\ \begin{cases} \hat{A}_i = S^T A_i S \\ \hat{b}_i = S^T (b_i + A_i x^0) \end{cases}$$

In new variables $||\hat{q}||_+^2 = \hat{q}^T \hat{q}$, i.e. the new basis is orthonormal w.r.t. $||\cdot||_+$, because $c_+ \cdot A = I$.

Notice that $\tilde{z}(c) = \rho_+^2(x_0, \mathrm{Ker}(c\cdot A) - (c\cdot A)^g (c\cdot b)) = \mathrm{diam}(\mathrm{Ker}(c\cdot\hat{A}) - (c\cdot\hat{A})^g (c\cdot\hat{b})) = ||(c\cdot A)^g (c\cdot b)||^2$ since $\mathrm{Ker}(c\cdot\hat{A})^T \big((c\cdot A)^g (c\cdot b)\big) = 0$ 3.2

5. We associate the condition $\mathrm{Rg}(c\cdot A) \leqslant n-1$, $c\cdot A \geqslant 0$ with nonconvexity of $f(X(c))$. The real sufficient condition for nonconvexity is given by the Theorem 3.4, which requires some additional constraints on $c$:
$$\begin{cases} Rg(c\cdot A) = n-1 \\ f^1 \nparallel f^2 \end{cases}$$
But we assume that other cases, in particular, $Rg(c\cdot A) < n-1$ or $f^1 \parallel f^2$ are rare.

6. Let us note that if $f(X(c))$ is nonconvex, then $\lambda_{\min}(c\cdot A) = 0$. Consider $\tilde{z}(c + \alpha c_+) = ||(c\cdot A + \alpha I)^g (c\cdot b)||^2$, since $c_+ \cdot \hat{A} = I$, $c_+ \cdot \hat{b} = 0$. Therefore, adding $\alpha c_+$ to $c$ adjusts the spectrum of $c\cdot A$. If and only of we choose $\alpha = -\lambda_{\min}(c\cdot A)$, $\tilde{z}$ will be associated with a $\tilde{c} = c + \alpha c_+$ for which $\lambda_{\min}(\tilde{c}\cdot A) = 0$. This way, for every given $c \in \mathbb{R}^m$ we can find one and only one $\tilde{c} = c + \alpha c_+$ such that $\lambda_{\min}(\tilde{c}\cdot A) = 0$

We define $z(c) = ||\big((c\cdot\hat{A}) - \lambda_{\min}(c\cdot\hat{A})\big)^g (c\cdot\hat{b})||^2$, which is the same as $\tilde{z}(c - \lambda_{\min} c_+)$. This way, we automatically choose the adjustment to the spectrum to ensure that $\lambda_{\min}(c\cdot A - \lambda_{\min}(c\cdot A)) = 0$

Notice that $z(c + \mu c_+) = z(c)$.

7. Next, we define $Q(c) = c\cdot\hat{A} - \lambda_{\min}(c\cdot\hat{A})$, $\hat{b}_c = c\cdot\hat{b}$, $v(c) = Q(c)^g b_c$. Then
$$z(c) = ||v(c)||^2 = v^T v$$
Note that $z(c)$ has its geometrical meaning 2 only if $c \in c_{\text{bad}} = \{c \big| \,\mathrm{Ker}\, Q(c) \perp b_c\}$.

At this point, we have a function $z(c)$ defined on $c_{\text{bad}}$ and we want to minimize it. We assume that it is continuous and differentiable, and also that $c_{\text{bad}}$ contains connected parts (manifolds)

# 2 Minimization

In this section we address the problem of finding minimum of $z(c)$ stated in the previous section. Again, we assume that $c_{\text{bad}}$ contains a manifold and we have a point $c$ in it already. We use well-known techique, Gradient projection method to solve it. We use not the Euclidean projection, but a very special one based on the geometry of $c_{\text{bad}}$. The method's two steps, gradient step and projection, are discussed in details below.

Formally, we want to find
$$z_{\max} = \inf_{c \in c_{\text{bad}}} z(c) \tag{3}$$
Let $\pi_{c_{\text{bad}}}$ be an operator which projects a point onto $c_{\text{bad}}$. Then the method is the following:
$$c^{(k+1)} = \underbrace{\pi_{c_{\text{bad}}}}_{\text{proj.}} \underbrace{(c^{(k)} - \beta^k \nabla z(c^{(k)}))}_{\text{gradient step}}$$

Nonconvexity certificate gives a single point $c \in c_{\text{bad}}$, which is used as a start point $c^{(1)}$. We assume $\mathrm{Rg}\, Q(c) = n-1$

Define $k(c) \in \mathbb{R}^n$ s.t. $k(c) \in \mathrm{Ker}\, Q(c)$, $||k(c)|| = 1$. We assume that for any $c'$ the dot product $k(c')^T k(c) > 0$ (change sign if false). Each iteration has two key elements: gradient step and projection (see image below)

Assume that we have done $k-1$ steps already and obtained a point $c = c^{(k)}$. The next two sections describe how to obtain the next point $\tilde{c} = c^{(k+1)}$.

## 2.1 Gradient step

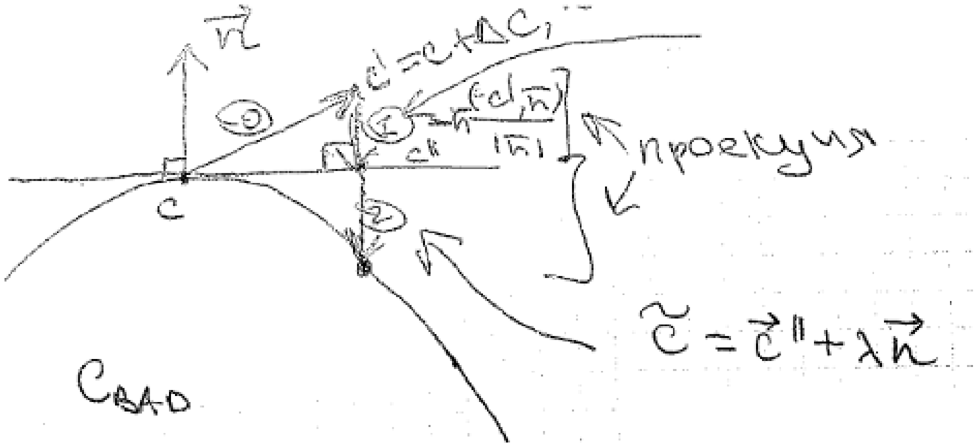
Define $R_i = \hat{A}_i - k^T \hat{A}_i k$, $q_i = \hat{b}_i - R_i v$. Then 3.9

$$\nabla_i z \equiv \frac{\partial z}{\partial c_i} = 2v^T Q^g q_i$$

Define $n_i = k^T q_i$. Then $n$ is a normal vector to $c_{\text{bad}}$ at a point $c$ (see picture). We eliminate normal part of the gradient to decrease the distance to $c_{\text{bad}}$ (see picture):

$$c' = c - \beta \left( I - n\frac{(\cdot, n)}{|n|^2} \right) \nabla z$$

## 2.2 Projection



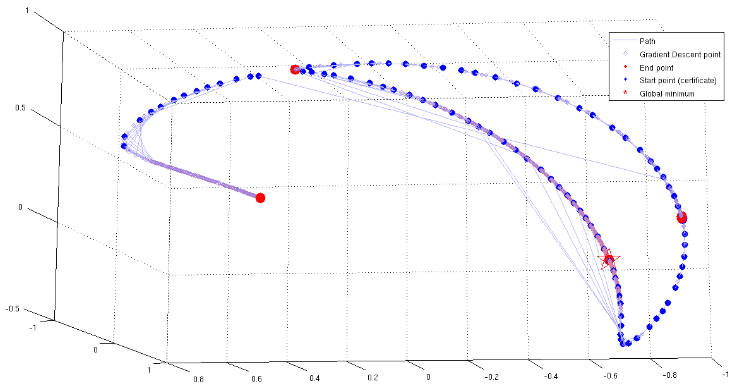

We next assume that for some $\lambda \in \mathbb{R}$ vector $\tilde{c}(\lambda) = c' + \lambda n \in c_{\text{bad}}$. To find $\lambda$, we write the condition $\tilde{c} \in c_{\text{bad}} \Leftrightarrow (\tilde{c} \cdot \tilde{b})^T k(\tilde{c}(\lambda)) = 0$. We define a function

$$m(\lambda) = (\tilde{c}(\lambda) \cdot \tilde{b})^T k(\tilde{c}(\lambda))$$

This function is continuous if $\operatorname{Rg} Q(\lambda) = n - 1$ in the neighbourhood of $\lambda$. Next, we find its root using bisection method on $[-\lambda_0, \lambda_0]$, $\lambda_0 = ||c - c'||$.

For some $\lambda$, $m(\lambda) = 0$ means that $\tilde{c} = c' + \lambda n \in c_{\text{bad}}$, and that the projection step was a success.

If the method does not converge on the interval given, or rank problem ($\operatorname{Rg} Q \neq n - 1$) occurs, we reduce the gradient step $\beta \to \theta\beta$, $\theta < 1$, recalculate $c'$ and try the projection again.




This way, we can construct a new point $\tilde{c}$ from the previous one $c$, and $\tilde{c}$ has lower value of $z(\cdot)$. We continue until the Gradient projection method condition holds:

$$\nabla z || n$$

If this condition holds, then the resulting $\tilde{c}$ is the same as $c$ on previous iteration, and the iterations stop.

# 3 Explanations

This section contains explanations and proofs for the main article and draft.pdf.

## 3.1 $F$ and $\operatorname{conv} F$

Consider $f\colon \mathbb{R}^n \to \mathbb{R}^m$, s.t. $f_i(x) = x^T A_i x + 2b_i^T x$, $A_i = A_i^T$. Define $F = f(\mathbb{R}^n)$.
Then why $A = \inf\limits_{y\in F}(c, y) = \inf\limits_{y\in \operatorname{conv} F}(c, y) = B$?

1. First, $F \subseteq \operatorname{conv} F$, therefore, $B \leqslant A$.

2. Secondly, let $y_k \in \operatorname{conv} F$ be a sequence s.t. $g_k = (c, y_k) \underset{k\to\infty}{\longrightarrow} B$. $y_k = \sum\limits_{i=1}^{n_k} \alpha_i^k y_i^k$.

   $g_k(c, y_k) = \sum\limits_{i=1}^{n_k} \alpha_i^k (c, y_i^k) = \sum\limits_{i=1}^{n_k} \alpha_i^k g_i^k$. Define $g_0^k = \min\limits_{i\in\overline{1,n_k}} g_i^k$. Then $B \leqslant g_0^k \leqslant g^k$. Therefore, $g_0^k \to B$ also. This way, we have constructed a sequence $y_0^k \in F$ s.t. $(c, y_0^k) \to B$, therefore, $A \leqslant B$.

## 3.2 Minimum of $f(x)$

Consider $f\colon \mathbb{R}^n \to \mathbb{R}^m$. $f_i(x) = x^T A_i x + 2b_i^T x$. $A_i^T = A_i$. Let $c \in \mathbb{R}^m$
We want to find $g(c) = \inf\limits_{x\in\mathbb{R}^n} (c, f(x))$.
Define $A_c \equiv c \cdot A = \sum\limits_{i=1}^{m} c_i A_i$, $b_c = c \cdot b = \sum\limits_{i=1}^{m} c_i b_i$.
$(c, f(x)) = \sum\limits_{i=1}^{m} c_i f_i(x) = \sum c_i (x^T A_i x + 2b_i^T x) = x^T A_c x + 2b_c^T x$.
If $\exists v\colon -\alpha = v^T A_c v < 0$ then $g(c) = -\infty$: $g(\beta v) = -\beta^2\alpha + \beta 2b_c^T v \to -\infty$, $\beta \to +\infty$.
From this point on, we assume $A_c \geqslant 0$. Let $R_0$ be a zero eigenspace (=kernel) of $A_c$: $R_0 = \{v\colon A_c v = 0\} = \ker A_c$

If $\exists v \in R_0\colon v^T b_c \neq 0$ then $g(c) = -\infty$: Consider $f(\beta v) = \beta^2 v^T \cancelto{0}{(A_c v)} + 2\beta \underbrace{b_c^T v}_{\neq 0} \to -\infty$, $\beta \to \infty$

Then $R_0 \subseteq \{b_c\}^\perp$
Consider $A = \sum\limits_{i=1}^{n} \lambda_i s_i s_i^T = S\Lambda S^T$, $S = ||s_1 ... s_n||$, $S^T S = E$, $s_i^T s_j = \delta_{ij}$.
$f$ is differentiable, then for finding $g(c)$ the gradiend $\nabla(c, f(x)) = 2A_c x + 2b_c = 0$.

$$S\Lambda S^T x = -b_c \Leftrightarrow \Lambda S^T x = -S^T b_c \,(*)$$

.
Let $x$ be $x = x^\parallel + x^\perp$, $x^\parallel \in R_0$, $x^\perp \perp R_0$.
Then neither $f(x)$ nor $\Lambda S^T x$ depend on $x^\parallel$. This means that the $x$ minimizing $g(c)$ is defined in terms of $x^\perp$ and $x^\parallel$ is arbitrary.

Define $\lambda_i^g = \begin{cases} 0, & \lambda_i = 0 \\ 1/\lambda_i, & \lambda_i \neq 0 \end{cases}$. Define $\Lambda^g = \operatorname{diag}(\lambda_1^g, ..., \lambda_n^g)$. Then $\Lambda\Lambda^g = \delta_{ij}[\lambda_i \neq 0]$. Then $S\Lambda^g\Lambda S^T$ is a projector on $R_0^\perp$.

Consider $(*) \Leftrightarrow \Lambda^g \Lambda S^T (x^\parallel + x^\perp) = -\Lambda^g S^T b_c$. But $\Lambda S^T x^\parallel = 0$, therefore, $\underbrace{S\Lambda^g\Lambda S^T}_{\text{projector}} x^\perp = -S\Lambda^g S^T b_c$.
But $x^\perp$ is already in $R_0^\perp$, therefore, $x^\perp = -\underbrace{S\Lambda^g S^T}_{A_c^g} b_c$. Here $A_c^g$ is a pseudoinverse of $A_c$.

Therefore, $\boxed{x = -A_c^g b_c + x^\parallel}$, where $x^\parallel \in R_0$.

Consider $(c, f(x)) = (c, f(x^\perp)) = b_c^T A_c^g A_c A_c^g b_c - 2b_c^T A_c^g b_c$. Consider $A_c A_c^g b_c = S\Lambda \cancelto{E}{S^T S} \Lambda^g S^T b_c$. Because $R_0 \subseteq \{b_c\}^\perp$, $\Lambda\Lambda^g S^T b_c = S^T b_c$. Therefore, $A_c A_c^g b_c = b_c$. Then $(c, f(x)) = b_c^T A_c^g b_c - 2b_c^T A_c^g b_c = \boxed{-b_c^T A_c^g b_c}$

### 3.3 Finding $c$ provided $d$

Let $H\colon \mathbb{R}^{n+1,n+1} \to \mathbb{R}^n$ be a map s.t. $H_i(X) = \mathrm{Tr}(H_i X)$,

$$H_i = \left\| \begin{matrix} A_i & b_i \\ b_i^T & 0 \end{matrix} \right\|^{\square}$$

.

Consider a boundary point $X$, which is a solution of (main article, (4)):

$$\boxed{\begin{matrix} \sup & t \\ \begin{cases} H(X) = y^0 + td \\ X \geqslant 0 \\ X_{n+1,n+1} = 1 \end{cases} & \end{matrix}}$$

Define $f(t, X) = t$, $D_0 = \{(t, X)\big| X \geqslant 0,\ X_{n+1,n+1} = 1\}$, $D_1 = \{(t, X)\big| H(X) = y^0 + td\}$.
Then supremum is equivalent to

$$\sup_{(t,X)\in D_0\cap D_1} f(t, X)$$

Define a Lagrange function $L(c, t, X) = \underbrace{t}_{f(t,X)} + \sum\limits_{i=1}^{m} c_i(y_i^0 + td_i - H_i(X))$.

Here we divided the constraints into two parts: $D_1$ goes to the Lagrange function, $D_0$ goes to the inner supremum. *Stephen Boyd, Lieven Vandenberghe. Convex Optimization. Page ???. Cambridge University Press*

Then the dual function is $g(c) = \sup\limits_{(t,X)\in D_0} L(c, t, X)$.

Because $L = t(1 + \sum\limits_{i=1}^{m} c_i d_i) + \sum\limits_{i=1}^{n} c_i(y_i^0 - H_i(X))$, $g = +\infty$ when $(c, d) \neq -1$. From this point we assume that $\boxed{(c, d) = -1}$.

Now, $g(c) = \sup\limits_{X_{n+1,n+1}=1, X\geqslant 0} (c, y^0 - H(X)) = (c, y^0) + \sup\limits_{y\in \mathrm{conv}\, F} -(c, y) = (c, y^0) - \inf\limits_{y\in \mathrm{conv}\, F} (c, y)$.

Then the dual problem is

$$g(c) \to \inf_{(c,d)=-1}$$

Let us prove that $\inf\limits_{y\in \mathrm{conv}\, F} (c, y) = \inf\limits_{H=\left\| \begin{smallmatrix} A_c & b_c \\ b_c^T & \gamma \end{smallmatrix} \right\| \geqslant 0} (-\gamma)$

Via Schur complement $H \geqslant 0 \Leftrightarrow \begin{cases} A_c \geqslant 0 \\ \gamma - b_c^T A_c^g b_c \geqslant 0 \\ (E - A_c A_c^g) b_c = 0 \end{cases}$.

$A_c \geqslant 0$ is a necessary condition for $\exists g(c) \in \mathbb{R}$ (see part 2).

$(E - A_c A_c^g) b_c = 0$ is another necessary condition for $\exists g(c) \in \mathbb{R}$.

Statement $\gamma \geqslant b_c^T A_c b_c$ means $-\gamma \leqslant -b_c^T A_c b_c = \inf\limits_{y\in \mathrm{conv}\, F} (c, y)$, which means that $-\gamma$ is a lower bound for $\inf\limits_{y\in \mathrm{conv}\, F} (c, y)$.

Then $H \geqslant 0 \Leftrightarrow -\gamma \leqslant \inf\limits_{y\in \mathrm{conv}\, F} (c, y)$.

Then $g(c) = (c, y^0) - \inf\limits_{H\geqslant 0} -\gamma$.

Then the dual problem is:

$$\inf_{(c,d)=-1} g(c) \Leftrightarrow \inf_{(c,d)=-1} \left[(c, y^0) - \inf_{H\geqslant 0}(-\gamma)\right] \Leftrightarrow \inf_{(c,d)=-1}\inf_{H\geqslant 0} (c, y^0) + \gamma = \boxed{\begin{matrix} \inf & \gamma + (c, y^0) \\ \begin{cases} H \geqslant 0 \\ (c, d) = -1 \end{cases} & \end{matrix}}.$$

This problem is exactly (5) from main article ∎.

## 3.4 What is $z_{\max}$?

Consider $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $f_i(x) = x^T A_i x + 2b_i^T x$, $A_i = A_i^T$.

Let $c_+ \in \mathbb{R}^m$ s.t. $A_+ = \sum c_i A_i > 0$. Then the minimum $\inf\limits_x (c, f(x))$ is obtained at a single point $x_0 = -A_+^{-1} b_+$, $b_+ = \sum c_i b_i$.

Consider $S_\varepsilon^+ = \{x \in \mathbb{R}^n \big| (x - x_0)^T A_+ (x - x_0) = \varepsilon^2\}$. Then $f(S_\varepsilon^+) = \{y \in \mathbb{R}^m \big| (c_+, y) = (c_+, f(x_0)) + \varepsilon^2\}$.

Indeed, if $\begin{cases} P = (c_+, f(x) - f(x_0)) \\ Q = (x - x_0)^T A_+ (x - x_0) \end{cases}$ then $P - Q = \underline{x^T A_+ x} - x_0^T A_+ x_0 + 2b_+^T x - 2b_+^T x_0 - \underline{x^T A_+ x} - x_0^T A_+ x_0 + 2x^T A_+ x_0 = /x_0 = -A_+^{-1} b_+/ = 2x_0^T b_+ + 2b_+^T x - 2b_+^T x_0 - 2x^T b_+ = 0$.

$\boxed{\textbf{Therefore, the image of } B_\varepsilon^+ \textbf{ is a } \textbf{\textit{convex cut}}\ \{y \big| (c_+, y) \in (c_+, f(x_0)) + [0, z_{\max}]\}}$

## 3.5 Variables s.t. $c \cdot A = I$, $c \cdot b = 0$

Given: the map $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $f_i(x) = x^T A_i x + 2b_i^T x$, $A_i = A_i^T$, vector $c \in \mathbb{R}^m\colon c \cdot A > 0$

We need to find a new pair of bases s.t. $\begin{cases} \tilde{c} \cdot \hat{A} = I & (1) \\ \tilde{c} \cdot \hat{b} = 0 & (2) \end{cases}$

We choose $x = S\hat{x} + x^0$, $y = \hat{y} + y^0$, $f(x) = y$, $\hat{f}(\hat{x}) = \hat{y}$. Consider $f_i(x) = x^{0T} A_i x^0 + \hat{x}^T S^T A_i S \hat{x} + 2x^{0T} A_i S \hat{x} + 2b_i^T S \hat{x} + 2b_i^T x^0 = \hat{x}^T \underbrace{S^T A_i S}_{\hat{A}_i} \hat{x} + 2\hat{x}^T \underbrace{S^T (b_i + A_i x^0)}_{\hat{b}_i} + \underbrace{x^{0T} A_i x^0 + 2b_i^T x^0}_{y_i^0}$

$c \cdot \hat{A} = S^T c \cdot A S = I$, then $S = S_1 S_2$, $S_1^T A_i S_1 = \Lambda = \text{diag}$, $S_1^T S_1 = I$, $S_2 = \Lambda^{-1/2}$.

$c \cdot \hat{b} = S^T (c \cdot b + (c \cdot A) x^0)$. Then $x^0 = -(c \cdot A)^{-1} (c \cdot b)$.

The algorithm:

1. Compute $S_1$ via the eigenbasis of $c \cdot A$, $S_1^T (c \cdot A) S_1 = \Lambda$
2. Compute $S_2 = \Lambda^{-1/2}$, $S = S_1 S_2$.
3. Compute $x^0 = -(c \cdot A)^{-1} (c \cdot b)$
4. Compute $\tilde{A}_i = S^T A_i S$, $\tilde{b}_i = S^T (b_i + A_i x^0)$

## 3.6 $z(c) = ?$

Given: the map $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $f_i(x) = x^T A_i x + 2b_i^T x$, $A_i = A_i^T$, two vectors $c, c_+ \in \mathbb{R}^m$. $c_+ A = I$, $c_+ b = 0$. Find: $z(c) = \inf\limits_{y \in Y} (c_+, y)$ where $Y$ is an intersection of $f(\mathbb{R}^n)$ with a tangent hyperplane defined by its normal vector $c$.

Define $\sigma(Q) = \{\lambda \big| \dim \operatorname{Ker}(Q - \lambda E) > 0\}$. Define $\lambda_{\min}(Q) = \min \sigma(Q)$ — minimal eigenvalue of $Q$.

1. If $\lambda_{\min}(c \cdot A) < 0$ then the tangent hyperplane does not exist, and $z(c) = +\inf$
2. Then $\lambda_{\min}(c \cdot A) > 0$, then there is no nonconvexity, and $z(c) = \inf$
3. For $\lambda_{\min} = 0$ in part 2 $Y$ was found explicitly: $Y = \{f(x) \big| x = x^{\|} - A_c^g b_c, x^{\|} \in \operatorname{Ker} A_c\}$. Then for $y \in Y$ $(c_+, y) = x^T (\cancelto{I}{c_+ \cdot A}) x + 2(\cancelto{0}{c_+ \cdot b})^T x = \boxed{x^T x}$
4. $x^T x = ||x||^2 = ||x^{\|}||^2 + ||A_c^g b_c||^2$. We want to minimize $(c_+, y)$, therefore, we choose $x^{\|} = 0$. Then $z(c) = ||A_c^g b_c||^2 = \boxed{\left|(c \cdot A)^g (c \cdot b)\right|^2}$
5. Consider $\inf\limits_{x \in \mathbb{R}^n} (c_+, f(x)) = -(\cancelto{0}{c_+ \cdot b})(\cancelto{I}{A_{c_+}})^{-1} (c_+ \cdot b) = 0$. Therore, $z(c) = \inf\limits_{y \in Y} (c_+, y) - \inf\limits_{y \in F} (c_+, y)$

Now consider $z(c) = \begin{cases} |(c \cdot A)^g (c \cdot b)|^2, & \lambda_{\min}(c \cdot A) = 0 \\ +\infty, & \text{otherwise} \end{cases}$

Consider $z(c + \alpha c_+) = |(c \cdot A + \alpha \cancelto{I}{c_+ \cdot A})^g b_c|^2 = |(c \cdot A + \alpha I)^g b_c|^2$. We need $\alpha$ s.t. $\lambda_{\min}(c \cdot A + \alpha I) = 0$, therefore, $\alpha = -\lambda_{\min}(c \cdot A)$.

Define $\hat{z}(c) = |(c \cdot A - \lambda_{\min}(A))^g (c \cdot b)|^2$. Consider $\mathbb{R}^m \ni c = c^{\|} + c^{\perp}$, $c^{\|} \parallel c_+$, $(c_+, c^{\perp}) = 0$.

Then $\hat{z}(c) = \hat{z}(c^{\perp})$, i.e. $\hat{z}$ does not depend on $c^{\|}$. It depends only on $c^{\perp}$, and $c^{\|}$ is chosen in a way that $\lambda_{\min}(c \cdot A) = 0$.

## 3.7 Theorem 3.4 (Nonconvexity certificate)

Given.

1. The map $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $f_i(x) = x^T A_i x + 2b_i^T x$, $A_i = A_i^T$, $m, n \geqslant 3$. The vector $c \in \mathbb{R}^m$.
2. $A_c \geqslant 0$, $\dim \operatorname{Ker} A_c = 1$ (=simple zero eigenvalue), $\operatorname{Ker} A_c = \operatorname{Lin}\{e\}$
3. $b_c \perp \operatorname{Ker} A_c$
4. $b \perp \operatorname{Ker} A_c$, $e^0 = -A_c^g b_c$
5. $f(\alpha e + e^0) = f^0 + 2\alpha f^1 + \alpha^2 f^2$, $f^1 \nparallel f^2$

Then $F = \operatorname{Im} f$ is nonconvex.

Consider $\inf\limits_{y\in F}(c, y)$ (part 2). $x = \underbrace{x^{\|}}_{\alpha e} + (\underbrace{-A_c^g b_c}_{e^0})$. Then

$$f_i(\alpha e + e^0) = \alpha^2 \underbrace{e^T A_i e}_{f_i^2} + 2\alpha(\underbrace{b_i^T e + e^T A_i e^0}_{f_i^1}) + \underbrace{2b_i^T e^0 + e^{0T} A_i e^0}_{f_i^0}$$

If $f^1 \nparallel f^2$, then $\{f(\alpha e + e^0 \big| \alpha \in \mathbb{R})\} = F \cap \{y \big| (c, y) = \inf\limits_{y\in F}(c, y)\}$ is nonconvex. Then F is nonconvex.

## 3.8 Equations (0.18)-(0.21)

Consider $A(t)\colon n\times n$, $\exists \dot{A}$, $A^T = A$, $A \geqslant 0$, $A$ has a simple zero eigenvalue: $\forall t\, A(t)x_0(t) = 0$, $x_0^T x_0 = 0$.

Then $A = S\Lambda S^T$, $S^T S = E$, $A^g = S\Lambda^g S^T$. Define $\lambda_i = \Lambda_{ii}$. $\Lambda_{ii}^g = \begin{cases} \frac{1}{\lambda_i}, & \lambda_i \neq 0 \\ 0, & \lambda_i = 0 \end{cases}$.

Then $AA^g = A^g A = S\Lambda S^T S \Lambda^g S^T = S\Lambda\Lambda^g S^T = \sum\limits_{\lambda_i \neq 0} s_i s_i^T = 1 - x_0 x_0^T$ (0.20)

Consider $\frac{d}{dt}Ax_0 = \dot{A}x_0 + A\dot{x}_0$. Multiplying by $A^g$ from the left: $A^g\dot{A}x_0 + A^g A\dot{x}_0 = 0$. Consider $A^g A\dot{x}_0 = (1 - x_0 x_0^T)\dot{x}_0 = \dot{x}_0 - x_0 x_0^T \dot{x}_0$. Since $||x_0||^2 = x_0^T x_0 = 1$, $x_0^T\dot{x}_0 = 0$. Then $-A^g\dot{A}x_0 = \dot{x}_0$ (0.19).

Consider $\dot{A}x_0 + A\dot{x}_0 = 0$. Multiplying by $x_0^T$ from the left: $x_0^T\dot{A}x_0 + \cancelto{0}{x_0^T A}\,\dot{x}_0 = 0$. Then $x_0^T\dot{A}x_0 = 0$ (0.18)

Consider $AA^g = 1 - x_0 x_0^T$. Then $\dot{A}A^g + A\dot{A}^g = -\dot{x}_0 x_0^T - x_0\dot{x}_0^T = A^g\dot{A}x_0 x_0^T + x_0 x_0^T \dot{A}A^g$ (a)

Consider $A^g x_0 = S\Lambda^g S^T x_0 = 0$.

Multiplying (a) by $x_0$ from the right: $A\dot{A}^g x_0 = A^g\dot{A}x_0$. Multiplying by $A^g$ from the left: $AA^g\dot{A}^g x_0 = A^g A^g \dot{A}x_0$. Then $(1 - x_0 x_0^T)\dot{A}^g x_0 = A^g A^g \dot{A}x_0$. Then $\dot{A}^g x_0 = A^{-2}\dot{A}x_0 + x_0 x_0^T \dot{A}^g x_0$.

Let's multiply (a) by $A^g$ from the left: $A^g\dot{A}A^g + A^g A\dot{A}^g = A^{-2}\dot{A}x_0 x_0^T + \cancelto{0}{A^g x_0} x_0^T \dot{A}A^g$.

Consider $A^g A\dot{A}^g = \dot{A}^g - x_0 x_0^T \dot{A}^g = \dot{A}^g - x_0 x_0^T \dot{A}A^{-2} - x_0 x_0^T \dot{A}^g x_0 x_0^T$.

Consider $x_0^T\dot{A}^g x_0 = x_0^T(\dot{S}\Lambda^g S^T + S\dot{\Lambda}^g S^T + S\Lambda^g \dot{S}^T)x_0 = x_0^T\dot{S}\Lambda^g S^T x_0 = 0$.

Then $\frac{d}{dt}A^{-1} = -A^{-1}\dot{A}A^{-1} + x_0 x_0^T \dot{A}A^{-2} + A^{-2}\dot{A}x_0 x_0^T$ (0.21).

Case $\operatorname{Rg}A < n - 1$ is not considered since probability of such event is small.

## 3.9 Gradient descent

1. (0.5): $(\underbrace{c \cdot A - \lambda_{\min}(c \cdot A)}_{Q})x_0 = 0$, then $\dot{x}_0 = -Q^{-1}\dot{Q}x_0$. $\frac{d}{dt}\lambda_{\min} = \frac{d}{dt}x_0^T(c{\cdot}A)x_0 = 2\dot{x}_0\underbrace{(c \cdot A)x_0}_{\lambda_{\min}x_0} + x_0^T(\dot{c}\cdot A)x_0 = 2\lambda_{\min}\cancelto{0}{\dot{x}_0^T x_0} + x_0^T(\dot{c}\cdot A)x_0$.

   Then $\dot{x}_0 = -(A_c - \lambda_{\min}(Ac))^{-1}(\dot{c}\cdot A - x_0^T(\dot{c}\cdot A)x_0)x_0$ (correct).
2. (0.6). $x_0^T(c \cdot b) = 0$, use (0.5) (correct).
3. (0.7). $\frac{\partial}{\partial t}||v(c)||^2 = \frac{\partial}{\partial t}\sum\limits_j v_j^2(c) = 2\sum\limits_j v_j \frac{\partial}{\partial t}v_j = 2v^T(c)\frac{d}{dt}v(c(t))$ (correct). $v(c) = (\underbrace{c \cdot A - \lambda_{\min}(c \cdot A)}_{Q})^{-1}(c \cdot b) = Q^{-1}(c \cdot b)$.

4. (0.8) Define $Q = c\cdot A - \lambda_{\min}(c\cdot A)$. Define $v = Q^{-1}(c\cdot b)$. Then $z(c) = ||v||^2$ and $\dot z = 2v^T\dot v$.
   $\dot v = \dot Q^{-1}(c\cdot b) + Q^{-1}(\dot c\cdot b)$.
   Consider (0.21) $\dot Q^{-1} = -Q^{-1}\dot Q Q^{-1} + x_0x_0^T\dot Q Q^{-2} + Q^{-2}\dot Q x_0 x_0^T$.
   Then $\dot z = 2v^T\left(Q^{-1}(\dot c\cdot b) + (-Q^{-1}\dot Q Q^{-1} + x_0x_0^T\dot Q Q^{-2} + Q^{-2}\dot Q x_0x_0^T)(c\cdot b)\right)\boxed{=}$

   $\boxed{=}2v^TQ^{-1}(\dot c\cdot b) - 2v^TQ^{-1}\dot Q Q^{-1}(c\cdot b) + 2\cancelto{0}{v^Tx_0}\ x_0^T\dot Q Q^{-2}(c\cdot b) + 2v^TQ^{-2}\dot Q x_0 \cancelto{0}{x_0^T(c\cdot b)}\boxed{=}$
   Since $x_0\in\operatorname{Ker}Q\perp(c\cdot b)$, we have $x_0^T(c\cdot b) = 0$.
   Since $Qx_0 = 0$, $Q^{-1}x_0 = 0$: $Q^{-1}x_0 = S\Lambda^{-1}S^Tx_0 = S*0 = 0$. Since $v^T = (c\cdot b)^TQ^{-1}$. Then $v^Tx_0 = 0$

   $\boxed{=}2v^TQ^{-1}(\dot c\cdot b) - 2v^TQ^{-1}\dot Q\underbrace{Q^{-1}(c\cdot b)}_{v} = 2v^TQ^{-1}(\dot c\cdot b) - 2v^TQ^{-1}\dot Q v = \boxed{\dot z = 2v^TQ^{-1}(\dot c\cdot b - \dot Q v)}$,

   $\dot Q = \dot c\cdot A - x_0^T(\dot c\cdot A)x_0$
   Since $\dot z = \sum\frac{\partial z}{\partial c_i}\frac{\partial c_i}{\partial t}$, $\frac{\partial z}{\partial c_i}$ can be found as a coefficient at $\dot c_i$ in $\dot z$
   $\dot z = 2v^TQ^{-1}\sum\limits_i\left(\dot c_ib_i - \dot c_iA_iv + x_0^T\dot c_iA_ix_0v\right) = \sum\limits_i\dot c_i\left[2v^TQ^{-1}(b_i - (A_i - x_0^TA_ix_0)v)\right]$

   Thus, $\boxed{\dfrac{\partial z}{\partial c_i} = 2v^TQ^{-1}(b_i - (A_i - x_0^TA_ix_0)v)}$, $Q = c\cdot A - \lambda_{\min}(c\cdot A)$, $v = Q^{-1}(c\cdot b)$, $x_0\in\operatorname{Ker}Q$, $||x_0|| = 1$
   **not the same as (0.8) in draft.pdf (but numerically same)**:
   $\dot z_{(0.8)} = 2\underbrace{(c\cdot b)^TQ^{-2}}_{v^TQ^{-1}}(\dot c\cdot b) - v^T(Q^{-1}\dot Q + \dot Q Q^{-1})v = 2v^TQ^{-1}(\dot c\cdot b) - v^TQ^{-1}\dot Q v - v^T\dot Q Q^{-1}v$

5. (0.10). If $\dot c = \beta c_+$, then $\dot z = 2v^TQ^{-1}(\dot c\cdot b - \dot Q v) = \boxed{=}$. Since $c_+\cdot b = 0$, $c_+\cdot A = I$, $\dot Q = c_+\cdot A - x_0^T(c_+\cdot A)x_0 = I - x_0^Tx_0 = I - 1 = 0$. And $\boxed{=}0$ (correct with new $\dot z$).
6. (0.14) $n_i = \left(b_i^T - v^T(A_i - x_0^TA_ix_0)\right)x_0$
7. (0.16) $P(\lambda) = Q^{-1}Q = S\Lambda^gS^TS\Lambda S^T = S\Lambda\Lambda^gS^T = 1 - x_0x_0^T$ — projector on $(\operatorname{Ker}Q)^\perp$ (correct)
8. (0.15) $P(\lambda)(c(\lambda)\cdot b) = c(\lambda)\cdot b \Leftrightarrow \underbrace{(c(\lambda)\cdot b)\perp\operatorname{Ker}Q}_{\Leftrightarrow c(\lambda)\in c_{\text{bad}}} \Leftrightarrow c(\lambda)\cdot b\in\operatorname{Im}Q \Leftrightarrow \exists\hat x\colon\ Q\hat x = c(\lambda)\cdot b$ (0.17)

   (correct)

### 3.10 Gradient descent. Projection

We have $c\in\mathbb{R}^m$, $c\in c_{\text{bad}} = \{c\big|||c|| = 1, \operatorname{Ker}Q(c)\perp(c\cdot b)\}$. $Q = c\cdot A - \lambda_{\min}(c\cdot A)$, $v = Q^{-1}(c\cdot b)$, $x_0\in\operatorname{Ker}Q$, $||x_0|| = 1$

1. Calculate $\frac{\partial z}{\partial c_i} = 2v^TQ^{-1}(b_i - (A_i - x_0^TA_ix_0)v)$. Define $\Delta c = -\nabla z(c)$
2. Calculate $\hat n_i = \left(b_i^T - v^T(A_i - x_0^TA_ix_0)\right)x_0$, define $n_i = \frac{\hat n_i}{|\hat n|}$
3. Define $c' = c + \Delta c - n(\Delta c, n)$
4. Define $c(\lambda) = c' + \lambda n$. Define $x_0(\lambda)$ s.t. $x_0(\lambda)\in\operatorname{Ker}Q(\lambda)$, $x_0(\lambda)^Tx_0 > 0$, $||x_0(\lambda)|| = 1$. Define $m(\lambda) = (c(\lambda)\cdot b)^Tx_0(\lambda)$. Beware of $\mathrm{Rg}Q < n-1$.
5. Find root of $m(\lambda)$ using binary search on $[-\lambda^0, \lambda^0]$, $\lambda^0 = ||c - c'||$.
6. Next $c$: $c(\lambda)$

$m(\lambda) = 0 \Leftrightarrow (c(\lambda)\cdot b)\perp\{x_0\} = \operatorname{Ker}Q$ if $\mathrm{Rg}Q = n-1$.

### 3.11 Equation 0.19

Consider $A = S\Lambda S^T$, $S^T S = E$, $x_0$ is a simple zero eigenvector of $A$: $Ax_0 = 0$, $||x_0|| = 1$.
$A^{-1} = S\Lambda_1 S^T$, $\Lambda = (\lambda_1, ..., \lambda_{n-1}, 0)$, $\Lambda_1 = (\lambda_1^{-1}, ..., \lambda_{n-1}^{-1}, 0)$.
Equation (0.19):

$$\dot{x_0} = -A^{-1}\dot{A}x_0$$

Consider $A^{-1}\dot{A}x_0 = S\Lambda_1 S^T(\dot{S}\Lambda S^T + S\dot{\Lambda}S^T + S\Lambda\dot{S}^T)x_0\boxed{=}$.
Consider $x_0 = Sy_0$, where $y_0 = (0, 0, ..., 1)$. Therefore, $0 = \dot{y_0} = \dot{S}^T x_0 + S^T\dot{x}_0$
Going back to 0.19, the part $\Lambda S^T x_0 = \Lambda y_0 = 0$, another part $\dot{\Lambda}S^T x_0 = 0$. Consequently,
$\boxed{=}S\Lambda_1 S^T S\Lambda\dot{S}^T x_0 = S\Lambda_1\Lambda\dot{S}^T x_0 = -S\Lambda_1\Lambda S^T\dot{x}_0 = -\sum\limits_{i=1}^{n-1} s_i s_i^T\dot{x}_0 = -(E - x_0 x_0^T)\dot{x}_0 = -\dot{x}_0 + x_0 x_0^T\dot{x}_0\boxed{=}$.
Taking a derivative $||x_0|| = 1$, we get $x_0^T\dot{x_0} = 0$, therefore,
$\boxed{=} - \dot{x}_0$

### 3.12 Nonconvexity certificate in $b_i = 0$ case

Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a quadratic map: $f_i(x) = x^T A_i x$, $A_i = A_i^T$.
Consider $c \in \mathbb{R}^m$ and boundary points $\partial F_c$:

$$c \cdot f(x) \to \min_c$$

Where $A = \sum c_i A_i$. Assuming $A \geqslant 0$.
Minimization leads to $Ax = 0$. The following cases hold:

1. $RgA = n$. $x = 0$ is a unique solution
2. $RgA = n - 1$. $x = \alpha e$, $f(x) = \alpha^2 f(e)$
3. $RgA = n - 2$. In this case $x = \alpha_1 e^1 + \alpha_2 e^2$. Consider $f(x) = \alpha_1^2 f_{11} + 2\alpha_1\alpha_2 f_{12} + \alpha_2^2 f_{22}$.
   (a) $f_{11}$, $f_{22}$, $f_{12}$ are linearly independent. In this case $\partial F_c$ is nonconvex
   (b) $Rg||f_{11}f_{12}f_{22}|| = 1$. $\partial F_c$ convex.

**Result** If exist $c \in \mathbb{R}^m$:

1. $RgA = n - 2$
2. $A \geqslant 0$
3. $f_{ij}$ are linearly independent

Then $F = \mathrm{Im} f$ is nonconvex.

### 3.13 conv $F$

Пусть $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $f_i(x) = x^T A_i x + 2b_i^T x$, $A_i = A_i^T$.
Обозначим $F = f(\mathbb{R}^n)$, $G = \mathrm{conv}F$
Обозначим $H_i = \left\| \begin{matrix} A_i & b_i \\ b_i^T & 0 \end{matrix} \right\|$
Обозначим $X = \left\| \begin{matrix} x \\ 1 \end{matrix} \right\| \left\| \begin{matrix} x^T & 1 \end{matrix} \right\| = \left\| \begin{matrix} xx^T & x \\ x^T & 1 \end{matrix} \right\|$
Тогда $f_i(x) = \mathrm{tr}H_i X$, $f(x) = H(X)$
Обозначим $V = \{X \in \mathbb{R}^{(n+1)\times(n+1)} | X = X^T,\ X \geqslant 0,\ X_{n+1,n+1} = 1\}$
Обозначим $G_1 = H(V)$.
Доказать: $G_1 = G$ (On the feasibility for the system of quadratic equations, Theorem 3.1. (Convex hull))

1. $G \subseteq G_1$. Пусть $y \in G$. Тогда $\exists \{y_i\}_{i=1}^l \subset F$, $\{\lambda_i\}_{i=1}^l \colon y = \sum\limits_{i=1}^l \lambda_i y_i$, где $\lambda_i \geqslant 0$, $\sum \lambda_i = 1$. Поскольку $y_i \in F$, $\exists \{X_i\} \colon y_i = H(X_i)$, причем $X_i = \begin{Vmatrix} x_i x_i^T & x_i \\ x_i^T & 1 \end{Vmatrix} \in V$. Рассмотрим $j$-ю компоненту $y^j = \sum \lambda_i y_i^j = \sum \lambda_i \mathrm{tr} H_j X_i = \mathrm{tr} H_j \underbrace{\sum \lambda_i X_i}_{X}$. То есть, найден $X \in V \colon y = H(X)$. Значит, $y \in G_1$

2. $G_1 \subseteq G$. Пусть $y \in G_1$. Тогда $y = H(X)$, $X \in V$. Доказать: $y \in G = \mathrm{conv} F$. Представим $X$ в виде выпуклой комбинации $X = \sum \lambda_i X_i$, где $X_i \in V$, причем $X_i = \begin{Vmatrix} x_i x_i^T & x_i \\ x_i^T & 1 \end{Vmatrix} \in V$ для некоторого $x_i$. Это докажет $y \in G$.

   Рассмотрим $X = \sum\limits_{k=1}^n \lambda_k s_k s_k^T$ — спектральное разложение. Поскольку $X \in V$, $X \geqslant 0$, значит, $\lambda_k \geqslant 0$. Обозначим $\Lambda = \sum \lambda_k$. Пусть $s_{k,n+1} \neq \frac{1}{\Lambda}$. Тогда переопределим $s_{k,n+1} = \frac{1}{\Lambda}$. Это можно сделать, т.к. $H_i(s_k s_k^T)$ не изменится (прямая проверка, использовать $H_{i,n+1,n+1} = 0$). Рассмотрим $X = \sum \underbrace{\frac{\lambda_k}{\Lambda}}_{\alpha_k} \underbrace{\Lambda s_k s_k^T}_{X_k}$. Получили представление $X = \alpha_k X_k$, $X_k$ имеет нужный вид, $\alpha_k$ — выпуклая комбинация. Получаем $y \in G$

## 3.14 Small ball

Пусть $f \colon \mathbb{R}^n \to \mathbb{R}^m$, $f_i(x) = x^T A_i x - 2b_i^T x$, $A_i = A_i^T$. Пусть $c \in \mathbb{R}^m$.
Обозначим $c \cdot A = \sum\limits_{i=1}^n c_i A_i$, $c \cdot b = \sum\limits_{i=1}^n c_i b_i$, $F_c(x) = c^T f(x)$
**Хотим найти:**

$$\min_{||x||^2=1} F_c(x)$$

Функция Лагранжа: $L(x, \lambda) = x^T (c \cdot A) x - 2(c \cdot b)^T x - \lambda(||x||^2 - 1)$.
Находим $L_x = 2(c \cdot A)x - 2c \cdot b - 2\lambda x = 0$, $L_\lambda = ||x||^2 - 1 = 0$,
Получаем систему $\begin{cases} ||x|| = 1 \\ (c \cdot A - \lambda)x = c \cdot b \end{cases}$ Это совпадает с (2.3).
Далее переходим в базис из собственных векторов $\{x_i\}$ симметричной матрицы $c \cdot A$

$$S = ||x_1 ... x_n||,\ S^T S = E$$

$$x = Sy,\ \Lambda = S^T (c \cdot A) S,\ c \cdot b = S\alpha$$

Получаем

$$\begin{cases} ||y|| = 1 & (1) \\ (\Lambda - \lambda)y = \alpha & (2) \end{cases}$$

Выражаем из (2)

$$y_k = \frac{\alpha_k}{\lambda_k - \lambda}$$

Подставляем в (1) — это совпадает с (2.7)

$$\sum_{i=1}^n \left(\frac{\alpha_k}{\lambda_k - \lambda}\right)^2 = 1$$

Это соотношение определяет $\lambda$, по $\lambda$ находим $y_k$, затем находим $x$ в старом базисе.