On the feasibility for the system of quadratic equations, explanations

1. Theorem 3.2 (Sufficient condition)

Consider $f\colon \mathbb{R}^n \to \mathbb{R}^m$, s.t. $f_i(x) = x^T A_i x + 2b_i^T x$, $A_i = A_i^T$. Define $F = f(\mathbb{R}^n)$.
Then why $A = \inf\limits_{y\in F}(c, y) = \inf\limits_{y\in \operatorname{conv} F}(c, y) = B$?

1. First, $F \subseteq \operatorname{conv} F$, therefore, $B \leqslant A$.

2. Secondly, let $y_k \in \operatorname{conv} F$ be a sequence s.t. $g_k = (c, y_k) \underset{k\to\infty}{\longrightarrow} B$. $y_k = \sum\limits_{i=1}^{n_k} \alpha_i^k y_i^k$.

   $g_k(c, y_k) = \sum\limits_{i=1}^{n_k} \alpha_i^k (c, y_i^k) = \sum\limits_{i=1}^{n_k} \alpha_i^k g_i^k$. Define $g_0^k = \min\limits_{i\in \overline{1,n_k}} g_i^k$. Then $B \leqslant g_0^k \leqslant g^k$. Therefore, $g_0^k \to B$ also. This way, we have constructed a sequence $y_0^k \in F$ s.t. $(c, y_0^k) \to B$, therefore, $A \leqslant B$.

2. Minimum of $f(x)$

Consider $f\colon \mathbb{R}^n \to \mathbb{R}^m$. $f_i(x) = x^T A_i x + 2b_i^T x$. $A_i^T = A_i$. Let $c \in \mathbb{R}^m$
We want to find $g(c) = \inf\limits_{x\in\mathbb{R}^n}(c, f(x))$.
Define $A_c \equiv c \cdot A = \sum\limits_{i=1}^m c_i A_i$, $b_c = c \cdot b = \sum\limits_{i=1}^m c_i b_i$.
$(c, f(x)) = \sum\limits_{i=1}^m c_i f_i(x) = \sum c_i (x^T A_i x + 2b_i^T x) = x^T A_c x + 2b_c^T x$.
If $\exists v\colon -\alpha = v^T A_c v < 0$ then $g(c) = -\infty$: $g(\beta v) = -\beta^2\alpha + \beta 2 b_c^T v \to -\infty$, $\beta \to +\infty$.
From this point on, we assume $A_c \geqslant 0$. Let R_0 be a zero eigenspace (=kernel) of A_c: $R_0 = \{v\colon A_c v = 0\} = \ker A_c$
If $\exists v \in R_0\colon v^T b_c \neq 0$ then $g(c) = -\infty$: Consider $f(\beta v) = \beta^2 v^T \cancelto{0}{(A_c v)} + 2\beta \underbrace{b_c^T v}_{\neq 0} \to -\infty$, $\beta \to \infty$
Then $R_0 \subseteq \{b_c\}^\perp$
Consider $A = \sum\limits_{i=1}^n \lambda_i s_i s_i^T = S\Lambda S^T$, $S = ||s_1 ... s_n||$, $S^T S = E$, $s_i^T s_j = \delta_{ij}$.
f is differentiable, then for finding $g(c)$ the gradiend $\nabla(c, f(x)) = 2A_c x + 2b_c = 0$.

$$S\Lambda S^T x = -b_c \Leftrightarrow \Lambda S^T x = -S^T b_c$$

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Let x be $x = x^\parallel + x^\perp$, $x^\parallel \in R_0$, $x^\perp \perp R_0$.
Then neither $f(x)$ nor $\Lambda S^T x$ depend on $x^\parallel$. This means that the x minimizing $g(c)$ is defined in terms of $x^\perp$ and $x^\parallel$ is arbitrary.
Define $\lambda_i^g = \begin{cases} 0, & \lambda_i = 0 \\ 1/\lambda_i, & \lambda_i \neq 0 \end{cases}$. Define $\Lambda^g = \operatorname{diag}(\lambda_1^g, ..., \lambda_n^g)$. Then $\Lambda\Lambda^g = \delta_{ij}[\lambda_i \neq 0]$. Then $S\Lambda^g \Lambda S^T$ is a projector on $R_0^\perp$.
Consider $\Lambda^g S^T(x^\parallel + x^\perp) = -\Lambda^g S^T b_c$. But $\Lambda S^T x^\parallel = 0$, therefore, $S\Lambda^g S^T x^\perp = -S\Lambda^g S^T b_c$. But $x^\perp$ is already in $R_0^\perp$, therefore, $x^\perp = -\underbrace{S\Lambda^g S^T}_{A^g} b_c$. Here A_c^g is a pseudoinverse of A_c.

Therefore, $\boxed{x = -A_c^g b_c + x^\parallel}$, where $x^\parallel \in R_0$.
Let us notice that since $A_c^{gT} = A_c^g$

Consider $f(x) = f(x^{\perp}) = b_c^T A_c^g A_c A_c^g b_c - 2b_c^T A_c^g b_c$. Consider $A_c A_c^g b_c = S\Lambda \cancelto{E}{S^T S} \Lambda^g S^T b_c$. Because $R_0 \in \{b_c\}^{\perp}$, $\Lambda\Lambda^g S^T b_c = S^T b_c$. Therefore, $A_c A_c^g b_c = b_c$. Then $f(x) = b_c^T A_c^g b_c - 2b_c^T A_c^g b_c = \boxed{-b_c^T A_c^g b_c}$

3. Finding c provided d

Let $H\colon \mathbb{R}^{n+1,n+1} \to \mathbb{R}^n$ be a map s.t. $H_i(X) = \mathrm{Tr}(H_i X)$,

$$H_i = \left\| \begin{matrix} A_i & b_i \\ b_i^T & 0 \end{matrix} \right\|^{\square}$$

.

Consider a boundary point X, which is a solution of (main article, (4)):

$$\boxed{\begin{matrix} \sup\limits_{\begin{cases} H(X) = y^0 + td \\ X \geqslant 0 \\ X_{n+1,n+1} = 1 \end{cases}} & t \end{matrix}}$$

Define $f(t, X) = t$, $D_0 = \{(t, X) \big| X \geqslant 0,\, X_{n+1,n+1} = 1\}$, $D_1 = \{(t, X) \big| H(X) = y^0 + td\}$.
Then supremum is equivalent to

$$\sup_{(t,X) \in D_0 \cap D_1} f(t, X)$$

Define a Lagrange function $L(c, t, X) = \underbrace{t}_{f(t,X)} + \sum\limits_{i=1}^{m} c_i(y_i^0 + td_i - H_i(X))$.

Here we divided the constraints into two parts: D_1 goes to the Lagrange function, D_0 goes to the inner supremum. *Stephen Boyd, Lieven Vandenberghe. Convex Optimization. Page ???. Cambridge University Press*

Then the dual function is $g(c) = \sup\limits_{(t,X) \in D_0} L(c, t, X)$.

Because $L = t(1 + \sum\limits_{i=1}^{m} c_i d_i) + \sum\limits_{i=1}^{n} c_i(y_i^0 - H_i(X))$, $g = +\infty$ when $(c, d) \neq -1$. From this point we assume that $\boxed{(c, d) = -1}$.

Now, $g(c) = \sup\limits_{X_{n+1,n+1}=1, X \geqslant 0} (c, y^0 - H(X)) = (c, y^0) + \sup\limits_{y \in \mathrm{conv}\, F} -(c, y) = (c, y^0) - \inf\limits_{y \in \mathrm{conv}\, F} (c, y)$.

Then the dual problem is

$$g(c) \to \inf_{(c,d)=-1}$$

Let us prove that $\inf\limits_{y \in \mathrm{conv}\, F} (c, y) = \inf\limits_{H = \left\| \begin{smallmatrix} A_c & b_c \\ b_c^T & \gamma \end{smallmatrix} \right\| \geqslant 0} (-\gamma)$

Via Schur complement $H \geqslant 0 \Leftrightarrow \begin{cases} A_c \geqslant 0 \\ \gamma - b_c^T A_c^g b_c \geqslant 0 \\ (E - A_c A_c^g) b_c = 0 \end{cases}$.

$A_c \geqslant 0$ is a necessary condition for $\exists g(c) \in \mathbb{R}$ (see part 2).

$(E - A_c A_c^g) b_c = 0$ is another necessary condition for $\exists g(c) \in \mathbb{R}$.

Statement $\gamma \geqslant b_c^T A_c b_c$ means $-\gamma \leqslant -b_c^T A_c b_c = \inf\limits_{y \in \mathrm{conv}\, F} (c, y)$, which means that $-\gamma$ is a lower bound for $\inf\limits_{y \in \mathrm{conv}\, F} (c, y)$.

Then $H \geqslant 0 \Leftrightarrow -\gamma \leqslant \inf\limits_{y \in \mathrm{conv}\, F} (c, y)$.

Then $g(c) = (c, y^0) - \inf\limits_{H \geqslant 0} -\gamma$.

Then the dual problem is:

$$\inf_{(c,d)=-1} g(c) \Leftrightarrow \inf_{(c,d)=-1} \left[(c, y^0) - \inf_{H \geqslant 0} (-\gamma) \right] \Leftrightarrow \inf_{(c,d)=-1} \inf_{H \geqslant 0} (c, y^0) + \gamma = \boxed{\begin{matrix} \inf\limits_{\begin{cases} H \geqslant 0 \\ (c, d) = -1 \end{cases}} & \gamma + (c, y^0) \end{matrix}}.$$

This problem is exactly (5) from main article ■.

4. What is $z_{\max}$?

Consider $f\colon \mathbb{R}^n \to \mathbb{R}^m,\ f_i(x) = x^T A_i x + 2b_i^T x,\ A_i = A_i^T$.

Let $c_+ \in \mathbb{R}^m$ s.t. $A_+ = \sum c_i A_i > 0$. Then the minimum $\inf\limits_x (c, f(x))$ is obtained at a single point $x_0 = -A_+^{-1} b_+,\ b_+ = \sum c_i b_i$.

Consider $S_\varepsilon^+ = \{x \in \mathbb{R}^n \big| (x - x_0)^T A_+ (x - x_0) = \varepsilon^2\}$. Then $f(S_\varepsilon^+) = \{y \in \mathbb{R}^m \big| (c_+, y) = (c_+, f(x_0)) + \varepsilon^2\}$.

Indeed, if $\begin{cases} P = (c_+, f(x) - f(x_0)) \\ Q = (x - x_0)^T A_+ (x - x_0) \end{cases}$ then $P - Q = \underline{x^T A_+ x} - x_0^T A_+ x_0 + 2b_+^T x - 2b_+^T x_0 - \underline{x^T A_+ x} - x_0^T A_+ x_0 + 2x^T A_+ x_0 = /x_0 = -A_+^{-1} b_+/ = 2x_0^T b_+ + 2b_+^T x - 2b_+^T x_0 - 2x^T b_+ = 0.$

$\boxed{\textbf{Therefore, the image of } B_\varepsilon^+ \textbf{ is a } \textbf{\textit{convex cut}}\ \{y \big| (c_+, y) \in (c_+, f(x_0)) + [0, z_{\max}]\}}$

5. Finding minimum of $z(c)$ when c is in manifold

TODO