

# Power flow feasibility problem

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# The problem

- ① Large-scale power grids
- ② Need to know if a regime of the grid is «normal», «safe»
- ③ Ohm's law  $\Rightarrow$  quadratic equations:

$$y_i = f_i(x) = x^T A_i x + 2b_i^T x$$

$y$  (regime) known,  $x$  is not

- ④ Need to determine if  $\exists x: y = f(x)$  (means «safe»)

This problem is known as Power Flow Feasibility problem.  
To solve it, the image  $f(\mathbb{R}^n)$  must be examined

- ① A. Dymarsky, Convexity of a Small Ball Under Quadratic Map, arXiv:1410.1553
- ② A. Dymarsky, On the Convexity of Image of a Multidimensional Quadratic Map, arXiv:1410.2254

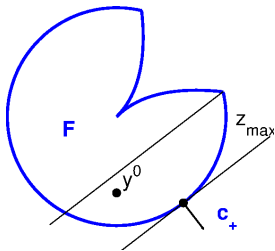
# The solution

Given: the map  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $f_i(x) = x^T A_i x + 2b_i^T x$ ,  $A_i^T = A_i$

Proposed algorithm for examining  $F = f(\mathbb{R}^n)$ :

- **Input:**  $y^0 \in F$ , direction  $c_+$ :  $c_+ \cdot A > 0$
- **Output:** value  $z_{\max}$  s.t. the cut  $Q(c_+, z_{\max}, F)$  is convex

$$Q(c_+, z, F) = \{y \mid (y - y^*, c_+) \in [0, z]\} \cap F$$

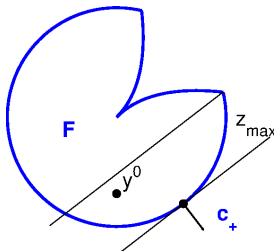


$y^*$  — touching point of hyperplane  $c_+$

# The solution

The idea of the algorithm:

- 1 Discovering boundary nonconvexities  $\{F_i\}$  close to  $y^0$
- 2 Projecting to  $c_+$ :  $(F_i, c_+)$
- 3 Calculating  $z_{\max} = \inf_i \inf_{y \in F_i} (c_+, y)$



Boundary points of  $F$  on supporting hyperplane  $c$ :

$$\partial F_c = f(\arg \min_{x \in \mathbb{R}^n} (c, f(x))) = f(\text{Ker}(c \cdot A) - (c \cdot A)^g(c \cdot b))$$

$$\text{if } \begin{cases} c \cdot A \geq 0 \\ (c \cdot b)^T \text{Ker}(c \cdot A) = 0 \end{cases}, \text{ otherwise } \partial F_c = \emptyset$$

- $\partial F_c$  is nonconvex  $\Rightarrow F$  has nonconvexity
- $\partial F_c$  is nonconvex  $\Leftrightarrow^{(*)}$  Rank( $c \cdot A$ ) =  $n - 1$ ,  $c \cdot A \geq 0$

Therefore,

$$z_{\max} = \inf_c \inf_{y \in \partial F_c} (c_+, \partial F_c)$$

(\*) We assume Rank( $c \cdot A$ ) <  $n - 1$  to be a rare case. Condition Rank( $c \cdot A$ ) =  $n - 1$  is associated with nonconvexity of  $\partial F_c$

Linear change of basis s.t.  $\{c_+ \cdot A = I, c_+ \cdot b = 0 \Rightarrow$

- $\inf_{y \in \partial F_c} (c_+, \partial F_c) = \|(c \cdot A)^g(c \cdot b)\|^2$  for  $c$ :  $\text{Rank}(c \cdot A) = n - 1$
- Adding  $\gamma c_+$  to  $c$  to ensure  $\lambda_{\min}((c + \gamma c_+) \cdot A) = 0$   
 $c_+ \cdot A = I$  by our choice of variables
- Define  $z(c) = \|(c \cdot A - \lambda_{\min}(c \cdot A))^g(c \cdot b)\|^2$
- Define  $c_{\text{bad}} = \{c \mid \text{Ker}(c \cdot A) \perp (c \cdot b)\}$   
 $c \in c_{\text{bad}} \Leftrightarrow z(c)$  has its original meaning, useless otherwise
- Then

$$z_{\max} = \inf_{c \in c_{\text{bad}}} z(c)$$

- In general case  $|c_{\text{bad}}|$  is continuum

We use gradient projection method to find

$$z_{\max} = \inf_{c \in c_{\text{bad}}} z(c)$$

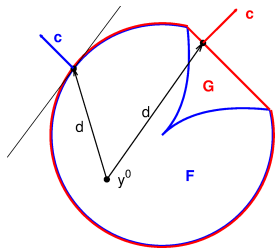
- ➊ **Input:** start point obtained via *nonconvexity certificate*

$$c^0 \in c_{\text{bad}}$$

- ➋ Gradient  $\frac{dz}{dc}$  is calculated explicitly
- ➌ Normal vector  $n$  for  $c_{\text{bad}}$
- ➍ Projection of  $c'$  onto  $c_{\text{bad}}$  is done by adjusting  $\lambda$  in  $c' + \lambda n$
- ➎ Repeat until  $\frac{dz}{dc} \parallel n$
- ➏ **Output:** minimal value  $z_{\max} = z(c^*)$



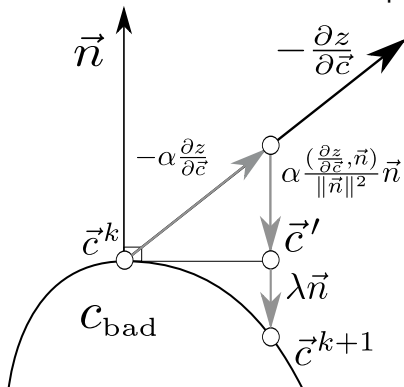
Nonconvexity certificate:



- 1 **Input:**  $y^0$
- 2 Generating random directions  $d$
- 3 Find  $t: y^0 + td \in \partial \text{conv } F$
- 4  $y^0 + td \in F?$
- 5 If not, obtain  $c$  via dual problem
- 6 **Output:** «nonconvex»  $c \in c_{\text{bad}}$

$\Rightarrow$  Obtained start point for gradient descent

Gradient step + projection



- 1 **Input:** current point  $c = c^k \in c_{\text{bad}}$
- 2 Calculate  $n(c)$ ,  $\frac{dz}{dc}$ ,  
 $c' = c - \alpha \left( \hat{1} - \frac{(\cdot, n)}{(n, n)} n \right) \frac{dz}{dc}$
- 3 Project  $c' \notin c_{\text{bad}}$  onto  $c_{\text{bad}}$
- 4 **Output:**  $c^{k+1} \leftarrow$  result of projection

## Projection

**Given:** point  $c' \notin c_{\text{bad}}$ , normal vector  $n$

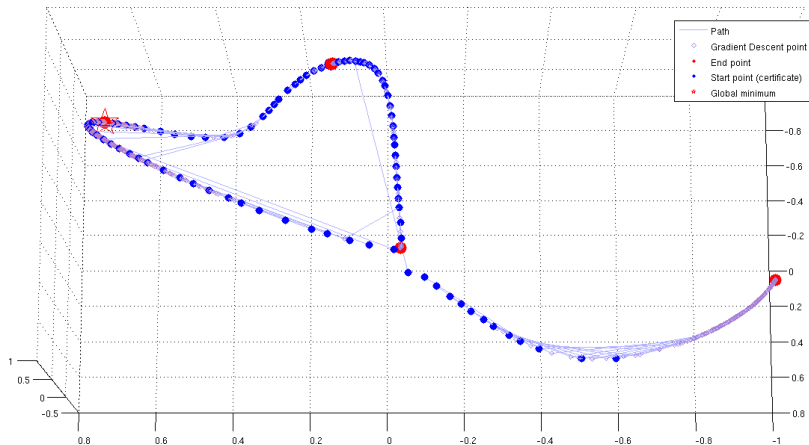
**Find:** point  $c \in c_{\text{bad}}$  close to  $c'$

- ①  $c(\lambda) = c' + \lambda n$
- ② For some  $\lambda$ ,  $c(\lambda) \in c_{\text{bad}}$
- ③  $c(\lambda) \in c_{\text{bad}} \Leftrightarrow \text{Ker}(c(\lambda) \cdot A) \perp c(\lambda) \cdot b$
- ④  $m(\lambda) = (c(\lambda) \cdot b)^T x_0(\lambda)$ ,  $x_0(\lambda) \in \text{Ker}(c(\lambda) \cdot A)$
- ⑤ Projection is done by finding root of  $m(\lambda)$
- ⑥ Bisection method is used

# Numerical experiment

$$f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$$

- 4 local minima
- Global minimum found

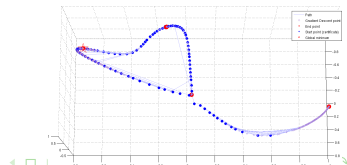
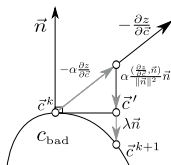
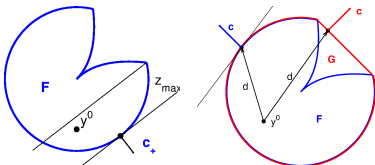


Algorithm for examining the set of «safe» regimes was proposed:

- ① Can determine if the whole subset is «safe» at one run
- ② Cuts convex parts of the image  $F$
- ③ General case when the set of nonconvexities is a continuum was considered
- ④ Algorithm was tested on a number of maps  $f$

Plan:

- ① What happens to  $c_{\text{bad}}$  if  $\text{Rank}(c \cdot A) = n - 2$ ?
- ②  $\mathbb{C}$  case
- ③ Testing for  $n, m > 4$
- ④ Testing for  $n, m \gg 1$



Thank you!  
Questions?