On the feasibility for the system of quadratic equations

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November 12, 2016

Abstract

We consider several problems related to quadratic equations that arise in power flow analysis, discrete optimization, uncertainty analysis and physical applications. we provide deep analysis for the image of the space of variables under quadratic map (the feasibility domain). There are several classes of quadratic maps representing the image convexity, while in general the image is nonconvex, nevertheless, demonstrates hidden convexity structure.

We propose numerical procedure to check the feasibility for the individual quadratic transformation. On this way we provide sufficient condition for infeasibility and investigate the numerical algorithms exploiting convex relaxation of quadratic mappings for discovering nonconvexity. Identifying convex parts of the image we verify feasibility of certain point. Finally, we address such problems as membership oracle and boundary oracle for the quadratic image as well as local convexity analysis.

AC power flow equations seem to be the natural application. $N-1$ contingency assessment, maximum loadability regime search as well as other problem of power systems security and stability analysis can be addressed via the proposed methodology.

Keywords Quadratic Maps, Feasibility, Hidden Convexity, Convex Relaxations, Power Flow analysis

1 Introduction

[EXPECTED...]

The paper is organized as follows. In Section 2 we formulate the problem, emphasize the role of convexity of the image under quadratic transformations and refer to known results on convexity for particular classes of quadratic functions. We formulate the main results in Section 3. It contains the analysis of the image in terms of supporting hyperplane, certificates of convexity/ nonconvexity. We also develop efficient algorithms to obtain the certificates. The algorithms exploit convex relaxation technique and so called “boundary oracle” for convex domains. Section 4 is devoted for the application of the proposed routine for the problem of checking attainability of operation regimes. Section 5 contains results on numerical simulation. Conclusions and directions for the future work can be found in final Section 6.

Notations

We consider multidimensional quadratic mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ or $f : \mathbb{C}^n \to \mathbb{R}^m$ of the form

$$\begin{aligned} & f(x) = (f_1(x), f_2(x), \dots, f_m(x))^T \\ & f_i(x) = x^* A_i x + b_i^* x + x^* b_i, \quad i = 1, \dots, m \le n. \end{aligned} \tag{1}$$

and its image

$$F = \{f(x) : x \in \mathbb{R}^n \text{ or } x \in \mathbb{C}^n\} \subseteq \mathbb{R}^m,$$

where * denotes transposition for real variable mapping or hermitian conjugation for complex variable mapping. F is the image of the whole space under the mapping f, we call it also *feasibility region*. Denote by $G = \mathrm{conv}(F)$ its convex hull.

∂F_c — the boundary points of F touched by the supporting hyperplane with normal vector $c \in \mathbb{R}^m$. We use notation C_+ for the set of c such that $A_c = \sum_{i=1}^m c_i A_i \succ 0$. For $c \in C_+$ it is clear that ∂F_c consists of a single point. Otherwise, we say $c \in C_-$.

For symmetric (hermitian) matrices $\langle X, Y \rangle = \mathrm{trace}(XY)$, and $X \succeq 0$ denotes nonnegative definite matrix X.

2 Problem formulation and the role of convexity

Quadratic transformations arise in discrete optimization, uncertainty analysis, physical applications. A variety of problems from combinatorial and continuous mathematical programming can be recast as quadratic minimization under quadratic constraints. These are binary integer programs (linear programs with binary variables), polynomial programs, scheduling problems of minimizing total length of the schedule under certain precedence constraints and machine availability constraints. Besides, a number of problems for power flow analysis involves quadratic maps. In general all the problems mentioned above are nonconvex, nevertheless, demonstrate hidden convexity structure and admit extremely efficient convex relaxations. The idea of convex relaxations for quadratic problems goes back to [19]; recent results and references can be found in [11] and [10]. Similar ideas and technique give the convex hull for the image set F, see also [20].

We examine several problems regarding the image of quadratic map F:

1. Feasibility or Membership Oracle.

   $$\text{For the given } y^0 \in \mathbb{R}^m \text{ determine whether } y^0 \in F.$$

   This problem is shown to be NP-hard [2] but it is extremely vital for numerous applications.

2. Boundary Oracle.

   $$\text{For } y^0 \in F \text{ and arbitrary direction } d \text{ find the maximal } t$$
   $$\text{such that for all } \tau \in [0,t]\ y^0 + \tau d \in F.$$

   Any iterative method for this problem requires the solution of the feasibility problem in every step. In contrast, the boundary points as well as their pre-images described by supporting hyperplane with normal $c \in C_+$ can be easily calculated. Nevertheless, boundary oracle is needed when the boundary point in a certain direction d is of interest. Besides, boundary oracle procedure allows us to implement Markov-Chain Monte Carlo techniques for efficient sampling even in nonconvex domains.

3. Convexity Oracle.

   $$\text{Check the convexity of } F \text{ for particular } f.$$

   In the general setting we do not assume that matrices A_i are sign-definite; nevertheless F sometimes occur to be convex ("hidden convexity" property). For a few low-dimensional classes of f the image convexity is guaranteed:

   - $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$, $m = 2$, $b_i \equiv 0$; [12], [17].
   - $x \in \mathbb{R}^n$, $m = 2$, $b_i \neq 0$ and $c_1 A_1 + c_2 A_2 \succ 0$; [14]. For $x \in \mathbb{C}^n$ see Lemma **??** below.
   - $x \in \mathbb{C}^n$, $m = 3$, $b_i \equiv 0$; [3]
   - $x \in \mathbb{R}^n$, $m = 3$ and there exists a positive-definite linear combination of matrices A_1, A_2, A_3 and $b_i \equiv 0$; see [13], [14].

   Besides, if $b_i \equiv 0$ and matrices A_i commute, then F is convex [15]. In general, checking the convexity of F for particular f, $m > 2$ is NP-hard.

4. Convex subparts of F.

   If A_i have positive off-diagonal entries, then "positive part of F" is convex (i.e. $F + R^m_+$ is convex) [16]. In the general case, instead of analyzing the whole image F some local information can be useful. One approach for local analysis exploits the fact that the image of the small ball $\|x - x^0\| \le \varepsilon$ remains convex for $\varepsilon \le \varepsilon_{\max}$ [18], [25]. But there is no regular routine for the tight estimation of $\varepsilon_{\max}$ and we desire to describe convex subpart in $\mathbb{R}^m$ rather then in the pre-image space.

There are links between the problems mentioned above. The convexity is the crucial property. Once the image of f is recognized as convex Membership and Boundary Oracles appear immediately. The convex relaxation is tight and it allows us to implement powerful numerical procedures of convex optimization (Linear Matrix Inequalities, duality theory). Namely, we can answer if a particular y^0 is feasible or not without solving equations and searching for the pre-image $x^0 : f(x^0) = y^0$. On the other hand, the infeasibility of certain y^0 can be checked easily if $y^0 \notin G$. When the image F is nonconvex but its part F_c is known to be convex we obtain *limited* Membership and Boundary Oracles that provide feasibility and boundary points efficiently for $y \in F_c$.

To solve the bunch of problems described above we provide several routines, their application for particular f and y^0 is schematically demonstrated in Fig. **??**.

The whole routine contains three main blocks.

1. Sufficient condition for infeasibility that checks if given y^0 is in the convex hull of F. Obviously, for convex images the sufficient condition is also necessary.

2. Nonconvexity certificate. Randomized numerical algorithm that discovers nonconvexity with probability one. The output of the algorithm is the set of vectors C_- such that supporting hyperplane with normal $c \in C_-$ touches the image F in more than one point and the subset of $C_= \subseteq C_-$ characterizing nonconvex boundary.

3. Theory for cutting convex subpart of F. Indeed, for the boundary points of F where the supporting hyperplane with normal $c \in C_+$ touches the single point with pre-image x^0, the convexity is preserved for the image of a small ball $\|x - x^0\| \le \varepsilon$. This in turn can be reformulated as the convexity of the intersection of F and $(c, f) \le z_{\max}$. Equipped with the set C_- from nonconvexity certificate we efficiently calculate the appropriate $z_{\max}$.

For membership oracle one should start with the sufficient condition for infeasibility for y^0. If y^0 is not infeasible the convexity issue arises and nonconvexity certificate is applied. If we obtain $C_- = \oslash$ it is the strong support of the convexity assumption and efficient Membership and Boundary Oracles are straightforward. When the procedure discovers nonempty C_-, under certain conditions we are able to cut convex subpart F_c by the inequality $(c, f) \leq z_{\max}$. This provides limited Membership and Boundary Oracle and we are able to check if y^0 belongs to this convex subpart. Finally, the proposed routine does not solve all the completely. There still remain blind zones for membership oracle since we can't certify feasibility for $y^0 \notin F_c$.

3 Feasibility certificates for convex and nonconvex images

In the general case we do not know in advance if the image of the quadratic map is convex or not. We provide the computational routine to address the problems from Section 2. In this section we describe all the steps above and formulate the final algorithm.

3.1 Sufficient condition for infeasibility

We remind some known facts on convex hull for quadratic image and exploit it to formulate the sufficient condition for infeasibility.

Theorem 3.1. (Convex hull)

The convex hull for the feasibility set F is

$$G = conv(F) = \{\mathcal{H}(X) : X \succeq 0, X_{n+1,n+1} = 1\},$$

where $X = X^* \in \mathbb{C}^{(n+1)\times(n+1)}$, $\mathcal{H}(X) = (\langle H_1, X\rangle, \langle H_2, X\rangle, \dots, \langle H_m, X\rangle)^T$, $H_i = \begin{pmatrix} A_i & b_i \\ b_i^* & 0 \end{pmatrix}$.

Hence we can provide simple sufficient conditions for checking if a particular point $y^0 \in \mathbb{R}^m$ is infeasible (does not belong to F). Indeed, for $y^0 \in F$ it is necessary to have $y^0 \in G$, that is to solve corresponding linear matrix inequality (LMI):

$$\mathcal{H}(X) = y^0, \quad X \succeq 0, \quad X_{n+1,n+1} = 1. \tag{2}$$

Alternatively, if the point does not belong to the convex domain it can be separated by a certain hyperplane. We introduce the variable $c \in \mathbb{R}^m$ and construct matrix $A_c = \sum c_i A_i$, vector $b_c = \sum c_i b_i$ and block matrix

$$H(c) = \begin{pmatrix} A_c & b_c \\ b_c^* & -(c, y^0) \end{pmatrix}. \tag{3}$$

Vector c is the normal for separating hyperplane (Fig. 1) and the sufficient condition for $y^0 \notin F$ has the form:

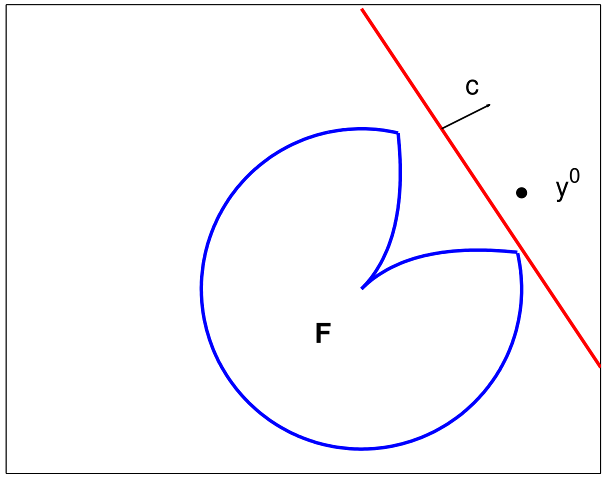

Figure 1: Infeasibility certificate via separating hyperplane.

Theorem 3.2. (Sufficient condition)

If for given y^0 there exists c such that $H(c) \succ 0$ (3), then y^0 is infeasible.

Proof. Via Schur complement $H(c) \succ 0 \Leftrightarrow A_c \succ 0$ and $-(c, y^0) - b_c^* A_c^{-1} b_c > 0$. But the latter inequality means

$$(c, y^0) < -b_c^* A_c^{-1} b_c = \min x^* A_c x + 2x^* b_c = \min_{y \in G}(c, y).$$

It means that there exists the separating hyperplane, defined by its normal c that strictly separates y^0 and $G = conv(F)$, hence y^0 does not belong to F. $\square$

Corollary 3.3. (Convex case)

If the image F is convex the sufficient condition given by Theorem 3.2 is also necessary.

Holding $H(c) \preceq 0$ for all c implies feasibility of y^0 just for convex F. Therefore we examine convexity of F.

3.2 Nonconvexity certificate

We examine convexity/nonconvexity of F in terms of the intersection with the supporting hyperplane with normal vector c. The geometry of ∂F_c is different depending on the spectrum of $A = \sum c_i A_i$. First, if A is sign-definite the corresponding supporting hyperplane intersects F at the unique point ∂F_c. Further we denote these c as c_+. Second, if A has both positive and negative eigenvalues then there is no corresponding supporting hyperplane in this case because F stretches to infinity in both directions along c. Finally, when A is semi-definite and singular we discover nonconvexity checking a few extra conditions.

Theorem 3.4. (Nonconvexity certificate)

Let $m \geq 3$, $n \geq 3$, $b_i \neq 0$, and for some $c = (c_1, c_2, \ldots, c_m)^T \neq 0$, the matrix $A = \sum c_i A_i \succeq 0$ has a simple zero eigenvalue $Ae = 0$, and for $b = \sum c_i b_i$ we have $b^* e + e^* b = 0$. Let also exist e^0: $Ax^0 = -b$. Denote $x^\alpha = \alpha e + e^0$, $f^\alpha = f(x^\alpha) = f^0 + f^1\alpha + f^2\alpha^2$. If $f^1 \not\parallel f^2$ then F is nonconvex.

Geometrically the condition implies that the linear function (c, f) attains its minimum on F at points f^α only. But parabola f^α is nonconvex, thus the supporting hyperplane touches F on a nonconvex set. Further we denote by C_- vectors satisfying the condition of Theorem 3.4.

Now the main problem is to find C_- (if exists) and hence discovers nonconvexity of the feasible set. At the first glance, sampling in c can be applied. Take $c \in \mathbb{R}^m$ and minimize (c, f) on G if such minimum exists. If $A_c = \sum c_i A_i \succ 0$ the minimum is unique and obtained at rank-1 matrix xx^*, $x = -A_c^{-1} b_c$, $b_c = \sum c_i b_i$, and x gives a pre-image for the boundary point of F. However, to identify nonconvexity we should find c such that A is singular. The probability of this event is zero if we sample c randomly.

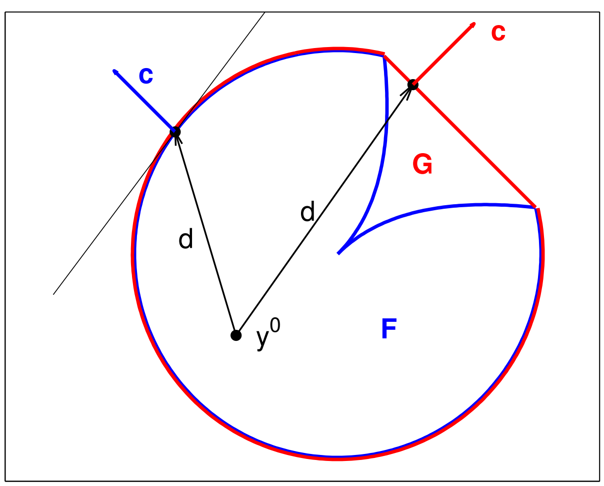

Figure 2: Idea of the nonconvexity certificate.

For efficient search of vectors C_- satisfying the condition of Theorem 3.4 we construct so called *boundary oracle* for G and exploit its dual problem. The idea of the nonconvexity certificate is demonstrated in Fig. 2. Boundary oracle is the procedure that returns the boundary point for given $y^0 \in G$ and the arbitrary direction $d \in \mathbb{R}^m$. For a convex hull of F the following Semidefinite Program (SDP) [21] with variables $t \in \mathbb{R}$, $X = X^* \in \mathbb{R}^{(n+1)\times(n+1)}$ specifies the boundary point $y^0 + td$ of the convex hull:

$$\begin{aligned} &\max\ t \\ \mathcal{H}(X) &= y^0 + td \\ X &\succeq 0 \\ X_{n+1,n+1} &= 1. \end{aligned} \tag{4}$$

If we obtain $\operatorname{rank}(X) = 1$ for the solution of (4) we claim that the obtained boundary point is on the boundary of F. Otherwise, the boundary point of the convex hull does not belong to F and lies on the "flat part" of the boundary of G.

On the other hand the dual problem to (4) gives us normal vector c for the boundary point:

$$\begin{aligned} &\min\ \gamma + (c, y^0) \\ &(c, d) = -1 \\ &H = \begin{pmatrix} \sum c_i A_i & \sum c_i b_i \\ \sum c_i b_i^* & \gamma \end{pmatrix} \succeq 0 \end{aligned} \tag{5}$$

This is again SDP in variables c and γ.

Equipped with boundary oracle technique (which provides both a boundary point of G and the normal vector c in this point) we are able to discover vectors c to identify nonconvexity as in Theorem 3.4. In our approach we sample random directions d (instead of c) and the probability of finding a boundary point on a "flat part" of the boundary of G (which correspond to nonconvex F) is positive.

Theorem 3.5. (Efficiency of the nonconvexity certificate)

Let $d \in U(S^{m-1})$ be uniformly distributed on the unit sphere and the random variable

$$\varphi(d) = \begin{cases} 1, & \text{if the solution } c \text{ of the problem (5) satisfies Theorem 3.4} \\ 0, & \text{otherwise.} \end{cases}$$

Then if the image F is nonconvex the expectation $\mathbb{E}(\varphi) > 0$.

To conclude it is the strong support of the convexity assumption if we do not discover nonconvexity after checking large number of directions d for various $y^0 \in G$.

3.3 Certifying convex parts of the image

Discovering nonconvexity in terms of C_- does not help directly to check feasibility for the given y^0. Nevertheless, exploiting local convexity [26] of F gives us opportunity to cut convex part of the image.

Consider c_+ such that $A_+ = \sum (c_+)_i A_i \succ 0$ the supporting hyperplane orthogonal to c_+ touches F in a single point ∂F_{c_+} with pre-image $x^0 = -A_+^{-1} b_+$, $b_+ = \sum (c_+)_i b_i$. The image of the ellipsoid centered in x^0

$$(x - x^0)^* A_+ (x - x^0) \le \varepsilon^2$$

preserves convexity for $\varepsilon \le \varepsilon_{\max}$. Moreover, every ellipsoid $(x - x^0)^* A_+ (x - x^0) \le \varepsilon^2$ is mapped into its own hyperplane $(c_+, f) = (c_+, f(x^0)) + \varepsilon^2$ and we can certify convexity of

$$F \cap \{(c_+, f) \le z_{\max}\}, \quad z_{\max} = (c_+, f(x^0)) + \varepsilon_{\max}^2. \tag{6}$$

We call the inequality in (6) *convex cut* and illustrate it in Fig. 3.

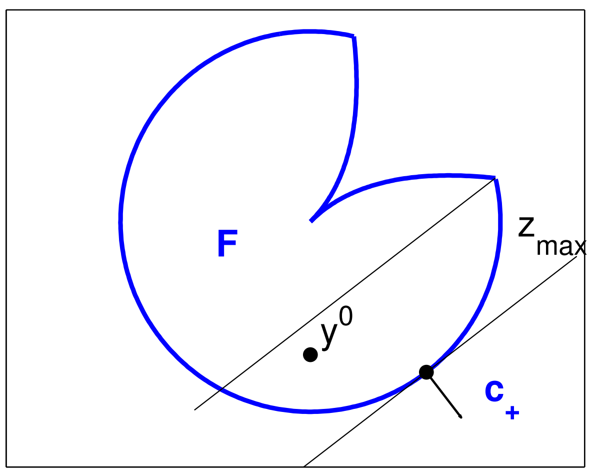

Figure 3: Cutting convex part.

There are several estimates for $\varepsilon_{\max}$ (and $z_{\max}$) requiring various computational effort. But the knowledge of C_- – the set of vectors c such that supporting hyperplane orthogonal to c touches the image F in "flat edge" and discovers nonconvexity – brings the clue to find $z_{\max}$ exactly.

Indeed, the nonconvex boundary points lie at the parabola $f(\alpha e + e^0)$ and nearest point to ∂F_{c_+} can be found straightforward as $z_c = \min_\alpha (c_+, f(\alpha e + e^0))$. Finally, $z_{\max}$ is obtained minimizing z_c among all discovered nonconvexities

$$z_{\max} = \min_{c \in C_-} z_c. \tag{7}$$

3.4 Numerical algorithm

Now we summarize all the reasoning above in the algorithmic form.

Algorithm **input:** the mapping $f : \mathbb{R}^n(\text{ or } \mathbb{C}^n) \to \mathbb{R}^m$ and $y^0 \in \mathbb{R}^m$.

1. For the given y^0 check its infeasibility via LMI $H(c) \succ 0$ (3). If LMI holds then **output:** y^0 is infeasible $\Rightarrow$ STOP.

2. Generate N random directions d^i on the unit sphere in $\mathbb{R}^m$. For every random direction d^i solve SDP (5). If the obtained c satisfies Theorem 3.4, collect c in C_- as well as corresponding e and e^0.

3. Specify cutting hyperplane normal vector c_+ such that

$$A_+ = \sum (c_+)_i A_i \succ 0, \quad b_+ = \sum (c_+)_i b_i.$$

4. For every $c \in C_-$ calculate $z_c = \min_{\alpha}(c_+, f(\alpha e + e^0))$ and take $z_{\max} = \min_{c \in C_-} z_c$.

5. If $(c_+, y^0) < z_{\max}$ then y^0 is in convex part of F. **output:** y^0 is feasible $\Rightarrow$ STOP.

6. If other c_+ could be found repeat steps 3-5.

7. **output:** y^0 may be feasible or infeasible $\Rightarrow$ STOP.

Discussion issues

- The choice of N and the efficiency of Step 2 depending on y^0.

  Actually, we are not forced to exploit y^0 we are checking feasibility for, any other $y \in G$ could be used for the problem (5).

- The cardinality or co-dimension of C_-.

  For small size examples with $m = 3, 4$ we encounter with the final number of $c \in C_-$. How does the set C_- depends on m?

- Appropriate choice of c_+.

  In general the set of c_+ is rich enough and can be described in term of LMI

$$\sum c_i A_i \succ 0$$
$$\sum c_i = 1 \text{(or any other normalization)}$$

  Practical recommendations for the number of c_+ to be repeat steps 3-5 are of interest.

- When the algorithm terminated with uncertain output (step 7) deeper attainability analysis is needed.

4 Application: power flow analysis

We consider AC power flow model. The network is characterized by complex admittance matrix

$$Y \in \mathbb{C}^{\mathcal{N}\times\mathcal{N}},$$

where $\mathcal{N}$ is the total number of buses (including slack bus with fixed voltage magnitude and phase). Power injections defined by Kirchhoff's laws can be written through matrix Y and complex voltages V_i:

$$s_i = V_i(YV)_i^*, \quad i = 1, \ldots, \mathcal{N}.$$

We treat all $V \in \mathbb{C}^{\mathcal{N}}$ feasible while the constraints are stated in the space of quadratic image of V. Introducing real vector $x = [\mathrm{Re}(V)^T,\ \mathrm{Im}(V)^T]^T$ active and reactive powers are real-valued quadratic functions of x.

The problem of checking attainability of operations regimes if highly important. For instance, in $N-1$ contingency assessment the high number of regimes needs to be checked. Solving power flow system of quadratic equations for every regime looks as an overkill since the only question of interest is the existence of solution rather than the particular solution itself. Maximum loadability search as well as other problem of power systems security and stability analysis can be addressed via the methodology described in Section 3.

5 Numerical results

In this section we apply the proposed routine for several test maps. The first one is artificially constructed while the others describe power flow feasibility region for 3-bus networks. Starting form a specified feasible y^0 and $N = 10^4$ we run the algorithm to obtain vector c

such that the supporting hyperplane (c, y) touches the image $y(x)$ in more than one point and thus certifies its nonconvexity. We distinguish nonconvexities discovered by different vectors c and examine the portion of random directions d resulted in every c. The results are summarized in Table 1.

Table 1: Numerical results for discovering nonconvexity of test mappings

Source	Map	Number of nonconvexities	Portion of d's per nonconvexity
Artificial	$\mathbb{R}^3 \to \mathbb{R}^3$	3	0.04 0.13 0.03
[22]	$\mathbb{R}^3 \to \mathbb{R}^3$	1	0.02
[23]	$\mathbb{C}^2 \to \mathbb{R}^4$	1	0.06
[24]	$\mathbb{C}^2 \to \mathbb{R}^4$	2	< 0.001 < 0.001

Artificial system We start with the artificially constructed quadratic mapping $\mathbb{R}^3 \to \mathbb{R}^3$ with

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad A_3 = I$$

$$b_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T, \qquad b_2 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T, \qquad b_3 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T.$$

Note that A_1 is positive semidefinite thus $c^1 = (1, 0, 0)^T$ specifies the supporting hyperplane (c, y) touching the image in more than one point. For this map we obtain two other critical $c^2 = (1, 0.5, -1)^T$ and $c^3 = (1, 4.75, 1.38)^T$. For $y^0 = (0, 1, 1)^T$ the portion for every c is given in Table 1. We analytically justify that there is no other c specifying nonconvexity for this map and plot several 2-D sections of the image in $\mathbb{R}^3$ (Fig. 4). For fixed $0 \leq y_3 \leq 1/3$ the section appears to be convex and for $y_3 = 4$ we visualize all three nonconvexities.

Constant power loads [22] This example is borrowed form [22], where feasible points of F are addressed as equilibria of system with constant power loads. The map of interest has the form:

$$\begin{aligned} P_1(x) &= x_1^2 - 0.5x_1x_2 + x_1x_3 - 1.5x_1 \\ P_2(x) &= x_2^2 - 0.5x_1x_2 - x_2x_3 + 0.5x_2 \\ P_3(x) &= x_3^2 - 2\epsilon x_3(x_1 + x_2) - x_3, \quad \epsilon = 0.01. \end{aligned}$$

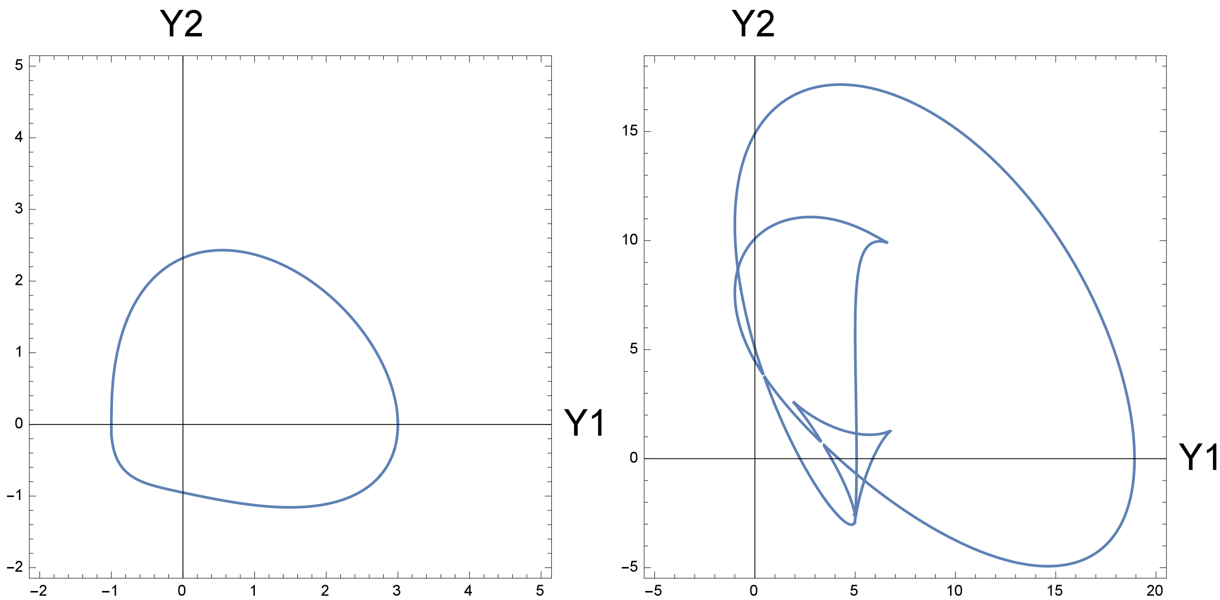

Figure 4: Two sections of the feasibility domain: first we fix $y_3 = 1/3$ and obtain convex section, then for $y_3 = 4$ the section is nonconvex.

For $P^0 = (0.5, 0.5, 0.25)^T$ we obtain a single $c = (1, 2.9021, 0.7329)^T$ identifying nonconvexity. It means that the supporting parabola

$$f^\alpha = \begin{pmatrix} -0.5442 \\ 0.0022 \\ -0.0339 \end{pmatrix} + \alpha \begin{pmatrix} 0.5043 \\ 0.0400 \\ -0.8463 \end{pmatrix} + \alpha^2 \begin{pmatrix} -0.0022 \\ -0.1991 \\ 0.7912 \end{pmatrix}$$

provides boundary points for the image $P(x)$, $x \in \mathbb{R}^3$ but the convex combination of two boundary points $\lambda f^{\alpha_1} + (1-\lambda) f^{\alpha_2}$ is infeasible for $0 < \alpha < 1$, $f^{\alpha_1} \neq f^{\alpha_2}$.

3-bus [23] Consider tree unbalanced 3-bus system (1 slack, 2 PQ-buses) with the admittance matrix

$$Y = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2-i & 1+i \\ 0 & 1+i & -1-i \end{pmatrix}.$$

The feasibility region in the space of P_2, Q_2, P_3, Q_3 is known to be nonconvex [23]. Here P_i and Q_i denote active and reactive power at the i-th bus, $y = (P_2, Q_2, P_3, Q_3)^T$. For $y^0 = (0, 0, 1, 1)^T$ our numerical routing obtains the single c generating nonconvexity for approximately 6% of random directions.

3-bus [24] This example of 3-bus cycle network with slack, PV and PQ-bus is borrowed from [24] (Fig. 5).

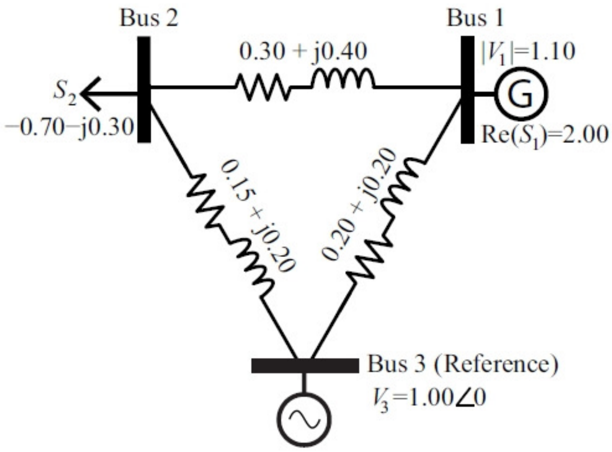

Figure 5: Three-bus system

Power flow equations take the form

$$
\begin{aligned}
P_1 &= x^T \begin{pmatrix} 3.7 & -0.6 & 0 & -0.8 \\ -0.6 & 0 & 0.8 & 0 \\ 0 & 0.8 & 3.7 & -0.6 \\ -0.8 & 0 & -0.6 & 0 \end{pmatrix} x + \\
&+ 2((-1.25, 0, 1.25, 0)^T, x), \\
V_1 &= x_1^2 + x_3^2, \\
P_2 &= x^T \begin{pmatrix} 0 & -0.6 & 0 & 0.8 \\ -0.6 & 3.6 & -0.8 & 0 \\ 0 & -0.8 & 0 & -0.6 \\ 0.8 & 0 & -0.6 & 3.6 \end{pmatrix} x + \\
&+ 2((0, -1.2, 0, 1.6)^T, x), \\
Q_2 &= x^T \begin{pmatrix} 0 & -0.8 & 0 & -0.6 \\ -0.8 & 4.8 & 0.6 & 0 \\ 0 & 0.6 & 0 & -0.8 \\ -0.6 & 0 & -0.8 & 4.8 \end{pmatrix} x + \\
&+ 2((0, -1.6, 0, -1.2)^T, x).
\end{aligned}
$$

We remind that $x = (\mathrm{Re}V_1,\ \mathrm{Re}V_2,\ \mathrm{Im}V_1,\ \mathrm{Im}V_2)^T$ and $V_3 = 1$ for slack bus. Starting at the given operation regime $P_1 = 2,\ V_1^2 = 1.21,\ P_2 = -0.7,\ Q_2 = -0.3$ we obtain at least two vectors c identifying nonconvexity but it requires more computational effort than in previous examples. Although our routine is capable to catch nonconvexity with probability one we are not sure that obtained vectors describe all the nonconvexities for this system.

The example may look artificial since the line resistances are as high as the reactances. We run our algorithm for the five times larger line reactances to model transmission grid and still discover nonconvexity of the image.

Remark We do not propose any routine to solve power flow equation, and we do not pretend to compete with any power flow solvers, neither DC equations nor iterative methods. Our method is focused on the certification and numerical description of nonconvexities of the image that is the reason for inexactness of convex relaxation for OPF.

6 Conclusions

We investigate the image of the quadratic mapping. For many applications convexity and "hidden convexity" plays crucial role for effectiveness of the numerical procedures based on convex relaxations.

We attack the problem of checking feasibility for a given y^0. First, we formulate the sufficient condition for infeasibility. Then we discover the particular structure of nonconvexity in terms of the intersection of the image with supporting hyperplane. We claim that nonconvexity can be characterized by the vector c of the supporting hyperplane when it satisfies certain conditions. Straightforward sampling in the space of c has no chances for success but we provide numerical algorithm based on dual representation of the boundary oracle. This algorithm discovers nonconvexity with probability one. Finally, having discovered nonconvexity we formulate additional cutting constraint that guarantees convexity of $F \cup \{(c_+, f) \leq z\}$ and feasibility of $y^0 \in \mathrm{conv}(F)$ and $(c_+, y^0) \leq z$.

Acknowledgements

The authors are thankful Prof. Janusz Bialek who gave us opportunity working together. The collaboration between Institute for Control Sciences RAS and Center for Energy Systems of Skolkovo Institute of Science and Technology is gratefully acknowledged.

A Lemma for $m = 2$, $x \in \mathbb{C}^n$

Lemma A.1. *For $m = 2$, $x \in \mathbb{C}^n$ the set F is convex.*

Proof. Introduce $z = (u, v)^T \in \mathbb{C}^{m+1}$, $v \in \mathbb{C}$ and three functions

$$\begin{aligned} g_i(z) &= u^* A_i u + v u^* b_i + v^* b_i^* u, \quad i = 1, 2, \\ g_3(z) &= (\text{Re } v)^2, \\ g(z) &= (g_1(z), g_2(z), g_3(z))^T. \end{aligned}$$

These functions are homogeneous quadratic forms, and the image

$$G = \{g(z) : \|z\| = 1, z \in \mathbb{C}^{m+1}\} \in \mathbb{R}^3$$

is convex due to the theorem on numerical range [3]. Thus its conic hull $Q = \{\lambda G, \lambda \geq 0\} \in \mathbb{R}^3$ is also convex. But

$$\begin{aligned} Q &= \{\lambda g(z) : \|z\| = 1, z \in \mathbb{C}^{m+1}, \lambda \geq 0\} \\ &= \{g(\sqrt{\lambda} z) : \|z\| = 1, z \in \mathbb{C}^{m+1}, \lambda \geq 0\} \\ &= \{g(z), z \in \mathbb{C}^{m+1}\}. \end{aligned}$$

Then its cross section by the hyperplane $g_3 = 1$ is convex as well: $H = \{(g_1(z), g_2(z))^T : g_3(z) = 1\}$ is convex. But H coincides with F. Indeed, $g_3 = 1$ implies either Re $v = 1$ and then $g_i(z) = f_i(u)$, $i = 1, 2$ or Re $v = -1$ and then $g_i(z) = f_i(-u)$, $i = 1, 2$. $\square$

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