Power flow feasibility problem

Sergei Volodin
under supervision of Y. Maximov

Skolkovo Insitute of Science and Technology

The problem

1. Large-scale power grids
2. Need to know if a regime of the grid is «normal», «safe»
3. Ohm's law $\Rightarrow$ quadratic equations:

   $$y_i = f_i(x) = x^T A_i x + 2b_i^T x$$

   y (regime) known, x is not
4. Need to determine if $\exists x \colon y = f(x)$ (means «safe»)

This problem is known as Power Flow Feasibility problem.
To solve it, the image $f(\mathbb{R}^n)$ must be examined

State of the art

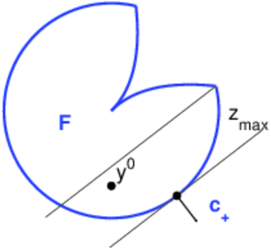
1. A. Dymarsky, Convexity of a Small Ball Under Quadratic Map, arXiv:1410.1553
2. A. Dymarsky, On the Convexity of Image of a Multidimensional Quadratic Map, arXiv:1410.2254

The solution

Given: the map $f : \mathbb{R}^n \to \mathbb{R}^m$, $f_i(x) = x^T A_i x + 2b_i^T x$, $A_i^T = A_i$
Proposed algorithm for examining $F = f(\mathbb{R}^n)$:

- **Input:** $y^0 \in F$, direction $c_+ : c_+ \cdot A > 0$
- **Output:** value $z_{\max}$ s.t. the cut $Q(c_+, z_{\max}, F)$ is convex

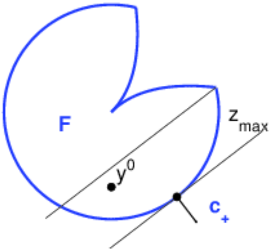
$$Q(c_+, z, F) = \{y \big| (y - y^*, c_+) \in [0, z]\} \cap F$$



The solution

The idea of the algorithm:

1. y^* — touching point of hyperplane c_+
2. Discovering boundary nonconvexities $\{F_i\}$ close to y^0
3. Projecting to c_+: (F_i, c_+)
4. Calculating $z_{\max} = \inf\limits_i \inf\limits_{y \in F_i} (c_+, y)$



The solution

Boundary points of F on supporting hyperplane c:

$$\partial F_c = f(\arg\min_{x \in \mathbb{R}^n}(c, f(x))) = f(\mathrm{Ker}(c \cdot A) - (c \cdot A)^g (c \cdot b))$$

$$\text{if } \begin{cases} c \cdot A \geqslant 0 \\ (c \cdot b)^T \mathrm{Ker}(c \cdot A) = 0 \end{cases} \text{, otherwise } \partial F_c = \emptyset$$

- ∂F_c is nonconvex $\Rightarrow$ F has nonconvexity
- ∂F_c is nonconvex $\Leftrightarrow^{(*)}$ $\mathrm{Rank}(c \cdot A) = n - 1$, $c \cdot A \geqslant 0$

Therefore,

$$z_{\max} = \inf_c \inf_{y \in \partial F_c} (c_+, \partial F_c)$$

(*) *We assume* $\mathrm{Rank}(c \cdot A) < n - 1$ *to be a rare case. Condition* $\mathrm{Rank}(c \cdot A) = n - 1$ *is associated with nonconvexity of* ∂F_c

The solution

Linear change of basis s.t. $\begin{cases} c_+ \cdot A = I, \; c_+ \cdot b = 0 \end{cases} \quad \Rightarrow$

- $\inf\limits_{y \in \partial F_c} (c_+, \partial F_c) = \|(c \cdot A)^g (c \cdot b)\|^2$ for $c\colon$ Rank$(c \cdot A) = n - 1$
- Adding γc_+ to c to ensure $\lambda_{\text{min}}\big((c + \gamma c_+) \cdot A\big) = 0$
  $c_+ \cdot A = I$ by our choice of variables
- Define $z(c) = \|(c \cdot A - \lambda_{\text{min}}(c \cdot A))^g (c \cdot b)\|^2$
- Define $c_{\mathrm{bad}} = \{c \,|\, \mathrm{Ker}(c \cdot A) \perp (c \cdot b)\}$
  $c \in c_{\mathrm{bad}} \Leftrightarrow z(c)$ *has its original meaning, useless otherwise*
- Then

$$\boxed{z_{\text{max}} = \inf_{c \in c_{\mathrm{bad}}} z(c)}$$

- In general case $|c_{\mathrm{bad}}|$ is continuum

The solution

We use gradient projection method to find

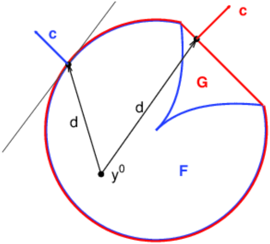
$$z_{\max} = \inf_{c \in \mathcal{C}_{\mathrm{bad}}} z(c)$$

1. **Input:** start point obtained via *nonconvexity certificate*

$$c^0 \in \mathcal{C}_{\mathrm{bad}}$$

2. Gradient $\frac{dz}{dc}$ is calculated explicitly
3. Normal vector n for $\mathcal{C}_{\mathrm{bad}}$
4. Projection of c' onto $\mathcal{C}_{\mathrm{bad}}$ is done by adjusting λ in $c' + \lambda n$
5. Repeat until $\frac{dz}{dc} \parallel n$
6. **Output:** minimal value $z_{\max} = z(c^*)$

Nonconvexity certificate:

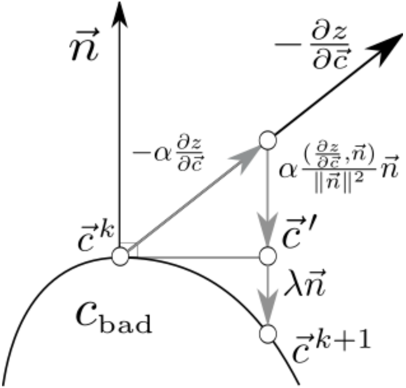


1. **Input:** y^0
2. Generating random directions d
3. Find t: $y^0 + td \in \partial \operatorname{conv} F$
4. $y^0 + td \in F$?
5. If not, obtain c via dual problem
6. **Output:** «nonconvex» $c \in c_{\mathrm{bad}}$

$\Rightarrow$ Obtained start point for gradient descent

The solution

Gradient step + projection



1. **Input:** current point $c = c^k \in c_{\mathrm{bad}}$
2. Calculate $n(c)$, $\frac{dz}{dc}$, $c' = c - \alpha(\hat{1} - \frac{(\cdot,n)}{(n,n)} n) \frac{dz}{dc}$
3. Project $c' \notin c_{\mathrm{bad}}$ onto c_{bad}
4. **Ouput:** $c^{k+1} \leftarrow$ result of projection

The solution

Projection

Given: point $c' \notin c_{\mathrm{bad}}$, normal vector n

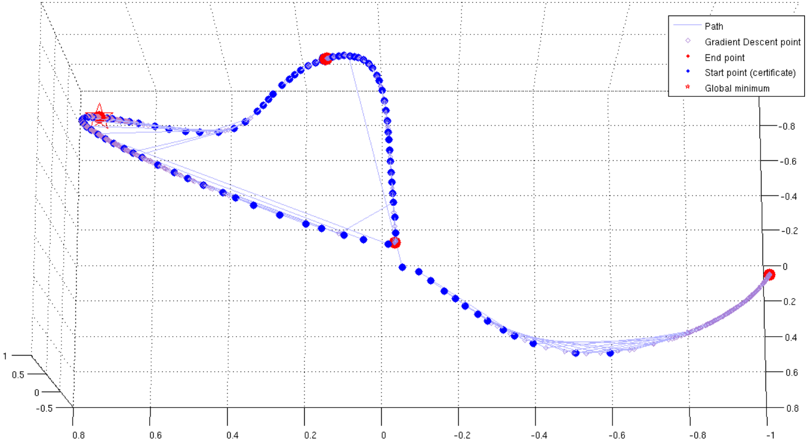
Find: point $c \in c_{\mathrm{bad}}$ close to c

1. $c(\lambda) = c' + \lambda n$
2. For some λ, $c(\lambda) \in c_{\mathrm{bad}}$
3. $c(\lambda) \in c_{\mathrm{bad}} \Leftrightarrow \mathrm{Ker}(c(\lambda) \cdot A) \perp c(\lambda) \cdot b$
4. $m(\lambda) = (c(\lambda) \cdot b)^T x_0(\lambda)$, $x_0(\lambda) \in \mathrm{Ker}(c(\lambda) \cdot A)$
5. Projection is done by finding root of $m(\lambda)$
6. Bisection method is used

Numerical experiment

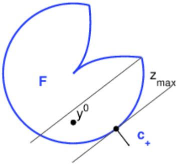
$f : \mathbb{R}^4 \to \mathbb{R}^4$

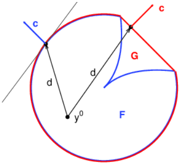
- 4 local minima
- Global minimum found

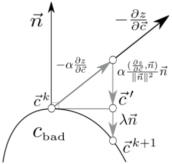


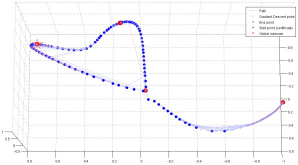
Algorithm for examining the set of «safe» regimes was proposed:

1. Can determine if the whole region is «safe» at one run
2. Cuts convex parts of the image F
3. General case when the set of nonconvexities is a continuum was considered
4. Algorithm was tested on a number of maps f









Thank you!
Questions?