25 Spring 439/639 TSA: Lecture 14

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Parameter estimation

Using all the tools we have seen (sample ACF/PACF/EACF, transformations, ADF test, ARMA subsets, etc.) we arrive at a few candidate models.

Next goal: Estimate the parameters ϕ_i $(i = 1, ..., p)$, θ_j $(j = 1, ..., p)$. And maybe the variance of the noise σ_e^2, the mean of the time series μ.

Method of Moments (MoM)

Recall that the task is estimating the parameters given the observed samples. The idea of **Method of Moments (MoM)** is to solve the parameters from the equation(s)

$$\text{theoretical moment} = \text{sample moment}$$

where the theoretical k-th moment $\mu_k = \mathbb{E}[Y^k]$ is a function of the parameters, and the sample k-th moment $m_k = \frac{1}{n}\sum_{i=1}^n Y_i^k$ is a function of the observed data.

Example 0. Suppose $Y_1, \cdots, Y_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$. Then we know that $\mu_1 = \mathbb{E}[Y] = \mu$, and $\mu_2 = \mathbb{E}[Y^2] = \mu^2 + \sigma^2$. The MoM method considers

$$\begin{cases} \mu_1 = m_1 \\ \mu_2 = m_2 \end{cases} \implies \begin{cases} \mathbb{E}[Y] = \dfrac{1}{n}\sum_{i=1}^n Y_i \\ \mathbb{E}[Y^2] = \dfrac{1}{n}\sum_{i=1}^n Y_i^2 \end{cases} \implies \begin{cases} \mu = \frac{1}{n}\sum_{i=1}^n Y_i = \overline{Y} \\ \mu^2 + \sigma^2 = \frac{1}{n}\sum_{i=1}^n Y_i^2 \end{cases}$$

Solving the system gives

$$\begin{cases} \hat{\mu}_{\text{MOM}} = \overline{Y} \\ \hat{\sigma}^2_{\text{MOM}} = \frac{1}{n}\sum_{i=1}^n Y_i^2 - \overline{Y}^2 = \frac{1}{n}\sum_{i=1}^n (Y_i - \overline{Y})^2 \end{cases}$$

Exercise: verify the last step above.

There is a variant method of MoM, called **generalized method of moments (GMoM)**. (For simplicity, we may also call it MoM.) The basic idea is, if we want to estimate some quantity $g(Y)$, then we can directly use $\frac{1}{n}\sum_{i=1}^n g(Y_i)$. This idea is useful in time series parameter estimation, since we can utilize the sequential structure of the observed data.

Example 1. Suppose $Y_1, \ldots, Y_n$ are from an AR(1) model with mean zero:

$$Y_t - \phi Y_{t-1} = e_t, \quad e_t \sim \text{iid}(0, \sigma_e^2).$$

We can apply the generalized MoM here by solving the equation(s)

$$\text{theoretical ACF} = \text{sample ACF}.$$

Note that the theoretical ACF is $\rho_k = \phi^k$, and in particular $\rho_1 = \phi$. To estimate ϕ, we can solve the equation $\rho_1 = r_1$ where r_1 is the sample ACF at lag 1:

$$r_1 = \frac{\sum_{t=1}^{n-1}(Y_{t+1} - \overline{Y})(Y_t - \overline{Y})}{\sum_{t=1}^{n}(Y_t - \overline{Y})^2}.$$

Solving the equation $\rho_1 = r_1$ gives the MoM (GMoM) estimate

$$\hat{\phi}_{\text{MOM}} = \frac{\sum_{t=1}^{n-1}(Y_{t+1} - \overline{Y})(Y_t - \overline{Y})}{\sum_{t=1}^{n}(Y_t - \overline{Y})^2}.$$

Example 2. Suppose $Y_1, \dots, Y_n$ are from an AR(2) model with mean zero:

$$Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} = e_t.$$

The first two YW equations are

$$\begin{cases} \gamma_1 - \phi_1\gamma_0 - \phi_2\gamma_1 = 0 \\ \gamma_2 - \phi_1\gamma_1 - \phi_2\gamma_0 = 0 \end{cases} \implies \begin{cases} \rho_1 = \phi_1 + \phi_2\rho_1 \\ \rho_2 = \phi_1\rho_1 + \phi_2 \end{cases} \implies \begin{cases} \rho_1 = \frac{\phi_1}{1-\phi_2} \\ \rho_2 = \frac{\phi_1^2 - \phi_2^2 + \phi_2}{1-\phi_2} \end{cases}$$

Using MoM, we need to solve ϕ_1, ϕ_2 from the equations

$$\begin{cases} \rho_1(\phi_1, \phi_2) = r_1 \\ \rho_2(\phi_1, \phi_2) = r_2 \end{cases} \quad \text{i.e.,} \quad \begin{cases} \frac{\phi_1}{1-\phi_2} = r_1 \\ \frac{\phi_1^2 - \phi_2^2 + \phi_2}{1-\phi_2} = r_2 \end{cases}$$

Alternatively, we can also replace the theoretical ACF with sample ACF in the YW equations, and then solve for ϕ_1, ϕ_2: (this is equivalent to the procedure above)

$$\begin{cases} r_1 = \phi_1 + \phi_2 r_1 \\ r_2 = \phi_1 r_1 + \phi_2 \end{cases}$$

Exercise: verify that

$$\begin{cases} \hat{\phi}_1^{\text{MOM}} = \frac{r_1(1-r_2)}{1-r_1^2} \\ \hat{\phi}_2^{\text{MOM}} = \frac{r_2 - r_1^2}{1-r_1^2} \end{cases}$$

Example 3. Suppose $Y_1, \dots, Y_n$ are from an AR(p) model with mean zero:

$$Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \cdots - \phi_p Y_{t-p} = e_t.$$

Similar to the AR(2) example, we start from the first p YW equations

$$\begin{cases} \rho_1 = \phi_1 + \phi_2\rho_1 + \cdots + \phi_p\rho_{p-1} \\ \rho_2 = \phi_1\rho_1 + \phi_2 + \cdots + \phi_p\rho_{p-2} \\ \quad\vdots \\ \rho_p = \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \cdots + \phi_p \end{cases}$$

which can be written in the matrix form

$$\begin{bmatrix} \rho_0 & \rho_1 & \cdots & \rho_{p-1} \\ \rho_1 & \rho_0 & \cdots & \rho_{p-2} \\ \vdots & \vdots & & \vdots \\ \rho_{p-1} & \rho_{p-2} & \cdots & \rho_0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{bmatrix}.$$

Replace the theoretical ACF ρ_k by the sample ACF r_k, and solve the MoM estimate for $(\phi_1, ..., \phi_p)$. So the MoM estimates satisfy

$$\widehat{R}_p \; \hat{\vec{\phi}} = \hat{\vec{\rho}}_p, \quad \text{where} \quad \widehat{R}_p = \begin{bmatrix} r_0 & r_1 & \cdots & r_{p-1} \\ r_1 & r_0 & \cdots & r_{p-2} \\ \vdots & \vdots & & \vdots \\ r_{p-1} & r_{p-2} & \cdots & r_0 \end{bmatrix}, \quad \hat{\vec{\phi}} = \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \vdots \\ \hat{\phi}_p \end{bmatrix}, \quad \hat{\vec{\rho}}_p = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_p \end{bmatrix}.$$

Then we get the MoM estimate

$$\hat{\vec{\phi}}_{\text{MOM}} = \widehat{R}_p^{-1} \; \hat{\vec{\rho}}_p.$$

Remark: The matrix $\widehat{R}_p$ defined above is always invertible, which is guaranteed by the particular way (and details) we used to construct the sample ACF r_k in lecture 9. In fact, we remarked in lecture 9 that our construction of r_k makes the "sample ACF matrix" invertible, and this matrix is nothing but the $\widehat{R}_p$ we have just seen.