25 Spring 439/639 TSA: Lecture 20

Dr Sergey Kushnarev

Table of contents

EWMA

EWMA stands for exponentially weighted moving average, which is a quick way to generate "forecasts". It is useful under some specific settings.

EWMA basically do the following

$$\widehat{Y}_t(1) = (1-\theta)Y_t + \theta\,\widehat{Y}_{t-1}(1)$$

which is linear combination of the observed Y_t and the predicted value at the previous time $\widehat{Y}_{t-1}(1)$. The parameter θ in this method is often chosen ad hoc.

EWMA can be useful in predicting IMA$(1,1)$. Consider a IMA$(1,1)$ (i.e. ARIMA$(0,1,1)$)

$$\begin{aligned} &Y_t - Y_{t-1} = W_t, \quad W_t = e_t - \theta\, e_{t-1}, \\ &\text{i.e.,} \quad Y_t = Y_{t-1} + e_t - \theta\, e_{t-1}. \end{aligned}$$

For invertible model, suppose it has an AR(∞) invertible representation

$$e_t = \pi_0 Y_t + \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \cdots$$

then Y_{t+1} can be written as

$$Y_{t+1} = e_{t+1} - \pi_1 Y_t - \pi_2 Y_{t-1} - \cdots.$$

Using the similar truncation/approximation idea we used for MA/ARMA model (see examples in lecture 18,19), and taking the conditional expectation $\mathbb{E}[\cdot \mid Y_{1,\dots,t}]$, we get

$$\widehat{Y}_t(1) = \mathbb{E}[Y_{t+1} \mid Y_{1,\dots,t}] = -\pi_1 Y_t - \pi_2 Y_{t-1} - \cdots - \pi_t Y_1.$$

So if we can find the coefficients π_j, then we can get $\widehat{Y}_t(1)$ by this method.

Sidenote: for ARIMA(p,d,q), the coefficients π_j in the invertible representation satisfy the following recursive formula

$$\pi_j = \begin{cases} \displaystyle\sum_{i=1}^{\min(j,q)} \theta_i\,\pi_{j-i} - \tilde{\phi}_j, & \text{if } 1 \le j \le p+d \\ \displaystyle\sum_{i=1}^{\min(j,q)} \theta_i\,\pi_{j-i}, & \text{if } j > p+d \end{cases}$$

where $1-\tilde{\phi}_1 x - \cdots - \tilde{\phi}_{p+d} x^{p+d} = (1-\phi_1 x - \cdots - \phi_p x^p)(1-x)^d$ is the AR polynomial of the ARMA$(p+d,q)$ corresponding to the original ARIMA(p,d,q). And $\pi_0 = 1$.

For our IMA$(1,1)$ (i.e. ARIMA$(0,1,1)$) setting,

$$Y_t - Y_{t-1} = e_t - \theta\, e_{t-1},$$

we can either use the previous general recursion formula, or just plug $e_t = \sum_{j=0}^{\infty} \pi_j Y_{t-j}$ into the IMA$(1,1)$, to get the following

$$\pi_0 = 1, \text{and } \pi_j = (\theta - 1)\theta^{j-1} \text{ for } j \geq 1.$$

Then use the earlier result, we get

$$\begin{aligned}
\widehat{Y}_t(1) &= -\pi_1 Y_t - \pi_2 Y_{t-1} - \cdots - \pi_t Y_1 \\
&= (1-\theta)Y_t + (1-\theta)\theta\, Y_{t-1} + (1-\theta)\theta^2 Y_{t-2} + \cdots + (1-\theta)\theta^{t-1} Y_1 \\
&= (1-\theta)Y_t + \theta\, \underbrace{\left[(1-\theta)Y_{t-1} + (1-\theta)\theta Y_{t-2} + \cdots + (1-\theta)\theta^{t-2} Y_1\right]}_{\widehat{Y}_{t-1}(1)}.
\end{aligned}$$

So we reached the EWMA formula we introduced at the beginning:

$$\widehat{Y}_t(1) = (1-\theta)Y_t + \theta\, \widehat{Y}_{t-1}(1).$$

We can also rewrite it as

$$\widehat{Y}_t(1) = \underbrace{\widehat{Y}_{t-1}(1)}_{\text{forecast}} + (1-\theta)\underbrace{\left(Y_t - \widehat{Y}_{t-1}(1)\right)}_{\text{forecast error}},$$

which can be seen as the forecast for Y_t at time $t-1$, plus the forecast error (after we observed the actual Y_t) multiplied by a smoothing factor $(1-\theta)$.

FYI: there are some other smoothing forecast methods, like Holt and Holt-Winters exponential moving average (double and triple exponential weighted moving average).

Seasonal ARIMA (SARIMA)

We combined AR and MA into the mixed model ARMA, and generalized to ARIMA. SARIMA is a further generalization of ARIMA.

Example 1: seasonal MA. Consider the model

$$Y_t = e_t - \Theta e_{t-12}.$$

This model is seasonal MA of order 1 with seasonal period 12, denoted by MA$(1)_{12}$.

Note: the equation for this model can also be seen as an MA(12) with $\theta_{12} = \Theta$ and $\theta_1 = \cdots = \theta_{11} = 0$. But MA(12) allows the parameters $\theta_1, ..., \theta_{11}$ to be nonzero. So MA(12) is "too large" for this model. Instead, MA$(1)_{12}$ is the correct model to characterize it.

Similar to the MA model, we can derive the ACVF and ACF for MA$(1)_{12}$:

$$\begin{cases} \gamma_0 = (1+\Theta^2)\sigma_e^2 \\ \gamma_{12} = -\Theta\sigma_e^2 \\ \gamma_k = 0, \text{ if } k \neq 0, 12 \end{cases}$$

$$\begin{cases} \rho_0 = 1 \\ \rho_{12} = \frac{-\Theta}{1+\Theta^2} \\ \rho = 0, \text{ if } k \neq 0, 12 \end{cases}$$

Exercise: verify the ACVF and ACF above.

In general, the seasonal MA model $\mathrm{MA}(Q)_s$ with order Q and seasonal period s, has the equation

$$Y_t = e_t - \Theta_1 e_{t-s} - \Theta_2 e_{t-2s} - \cdots - \Theta_Q e_{t-Qs}.$$

The MA polynomial is of order Qs:

$$\Theta(x) = 1 - \Theta_1 x^s - \Theta_2 x^{2s} - \cdots - \Theta_Q x^{Qs}.$$

Example 2: seasonal AR. Consider the $\mathrm{AR}(1)_{12}$ model (seasonal AR of order 1 with seasonal period 12)

$$Y_t = \Phi Y_{t-12} + e_t.$$

The causality condition for $\mathrm{AR}(1)_{12}$ is $|\Phi| < 1$. *Note:* The AR polynomial is $1 - \Phi x^{12}$ which has 12 roots. All the roots have the same modulus $|\Phi|^{-\frac{1}{12}}$. So the roots are outside the unit disk if (and only if) $|\Phi| < 1$.

We can find that $\rho_0 = 1, \rho_{12} = \Phi, \rho_{24} = \Phi^2, \ldots$. The ACF for $\mathrm{AR}(1)_{12}$ is

$$\begin{cases} \rho_{12\cdot k} = \Phi^k, & \text{for integer } k \geq 0 \\ \rho_n = 0, & \text{if } n \neq 12k \end{cases}$$

Exercise: verify the ACF above.

In general, the seasonal AR model $\mathrm{AR}(P)_s$ with order P and seasonal period s, has the equation

$$Y_t = \Phi_1 Y_{t-s} + \Phi_2 Y_{t-2s} + \cdots + \Phi_P Y_{t-Ps} + e_t.$$

The AR polynomial is of order Ps:

$$\Phi(x) = 1 - \Phi_1 x^s - \cdots - \Phi_P x^{Ps}.$$

Seasonal ARMA. Similar to $\mathrm{ARMA}(p,q)$, we can also combine $\mathrm{AR}(P)_s$ and $\mathrm{MA}(Q)_s$ into $\mathrm{ARMA}(P,Q)_s$. (The seasonal period s for AR part and MA part are the same.)

Multiplicative seasonal ARMA model

We can combine a nonseasonal $\mathrm{ARMA}(p,q)$ and a seasonal $\mathrm{ARMA}(P,Q)_s$ together by multiplying the AR/MA polynomials. This multiplicative seasonal ARMA model is denoted as $\mathrm{ARMA}(p,q) \times (P,Q)_s$:

$$\mathrm{ARMA} \underbrace{(p,q)}_{\text{nonseasonal}} \times \underbrace{(P,Q)_s}_{\text{seasonal}}.$$

It is still in the ARMA form, with the following AR polynomial and MA polynomial

$$\begin{aligned} &\text{AR polynomial:} \quad \underbrace{\phi(x)}_{\text{order } p} \cdot \underbrace{\Phi(x)}_{\text{order } Ps} \\ &\text{MA polynomial:} \quad \underbrace{\theta(x)}_{\text{order } q} \cdot \underbrace{\Theta(x)}_{\text{order } Qs} \end{aligned}$$

where $\phi(x)$, $\Phi(x)$ are the AR polynomials of $\mathrm{ARMA}(p,q)$ and $\mathrm{ARMA}(P,Q)_s$ respectively, and similarly $\theta(x)$, $\Theta(x)$ are MA polynomials.

Example. Consider an $\mathrm{ARMA}(0,1) \times (1,0)_{12}$. Then the AR polynomial and MA polynomial are

$$\begin{aligned} &\text{AR polynomial:} \quad 1 \cdot (1 - \Phi x^{12}) \\ &\text{MA polynomial:} \quad (1 - \theta x) \cdot 1 \end{aligned}$$

So the equation for $\mathrm{ARMA}(0,1) \times (1,0)_{12}$ is

$$\begin{aligned} &(1 - \Phi B^{12})\, Y_t = (1 - \theta B)\, e_t, \\ &\text{i.e.,} \quad Y_t - \Phi Y_{t-12} = e_t - \theta e_{t-1}. \end{aligned}$$

For this model, (assume it is causal,) we can find $\gamma_0 = \frac{1+\theta^2}{1-\Phi^2}\sigma_e^2$, $\gamma_1 = \frac{-\theta}{1-\Phi^2}\sigma_e^2$, $\gamma_{12} = \Phi\gamma_0$, $\gamma_{11} = \gamma_{13} = \Phi\gamma_1$,....
The ACVF is

$$\begin{cases} \gamma_{12\cdot k} = \frac{1+\theta^2}{1-\Phi^2}\Phi^k\sigma_e^2, & \text{for integer } k \geq 0 \\ \gamma_{12\cdot k\pm 1} = \frac{-\theta}{1-\Phi^2}\Phi^k\sigma_e^2, & \text{for integer } k \geq 0 \\ \gamma_n = 0, & \text{if } n \neq 12k, 12k \pm 1 \end{cases}$$

Exercise: derive the ACF above. (Hint: you can use YW method; or write it as a GLP.)