Second Order Slip-factor

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Objectives

- At transition crossing the slip factor, $\eta(p)=\alpha-1/\gamma^2$, becomes zero.
- Consequently, the spread of particle revolution frequencies approaches zero near transition

$$\frac{\Delta f}{f}=\eta(p)\frac{\Delta p}{p}+\eta_p(p)\left(\frac{\Delta p}{p}\right)^2+\ldots$$

⇨ At transition $\sigma_f\equiv\sqrt{\overline{\left(\frac{\Delta f}{f}\right)^2}}=\eta_p(p_{tr})\sqrt{\overline{\left(\frac{\Delta p}{p}\right)^4}}\xrightarrow[\text{Distribution}]{\text{For Gaussian}}\sqrt{3}\eta_p(p_{tr})\sigma_p{}^2$

- The second order slip-factor plays important role at the transition and good understanding of its behavior on machine parameters are important for both simulations and transition crossing tuning
- Initial computations using MAD showed that it does not deliver a reliable number
  - As result a study was initiated
    - Other participants: F. Schmidt - CERN, A. Valishev – FNAL
  - Also compared to two earlier publications

Smooth Lattice Approximation

- It is straightforward to get the exact analytical result in the smooth lattice approximation

$$B_z = B_0\left(1+g\frac{r}{R_0}+s\left(\frac{r}{R_0}\right)^2+\dots\right)$$ where the reference orbit is determined as $\frac{1}{R_0}=\frac{eB_0}{p_0c}$

$$\Rightarrow \quad p_0c\left(1+\frac{\delta p}{p_0}\right)=eR_0\left(1+\frac{r}{R_0}\right)B_0\left(1+g\frac{r}{R_0}+s\left(\frac{r}{R_0}\right)^2+\dots\right) \Rightarrow \frac{\delta p}{p_0}=\frac{r}{R_0}(1+g)+(g+s)\left(\frac{r}{R_0}\right)^2+\dots$$

- On other hand the betatron frequency is determined by

$$\nu_x^2=1+\frac{R}{B}\frac{dB}{dr}=1+g+\frac{r}{R_0}\left(g-g^2+2s\right)+\dots$$

$\Rightarrow$ Betatron tune: $\boxed{\nu_{x0}=\sqrt{1+g}}$ and Chromaticity $\boxed{\xi\equiv p\frac{d\nu_x}{dp}=\frac{g-g^2+2s}{2\nu_x^{\ 3}}}$

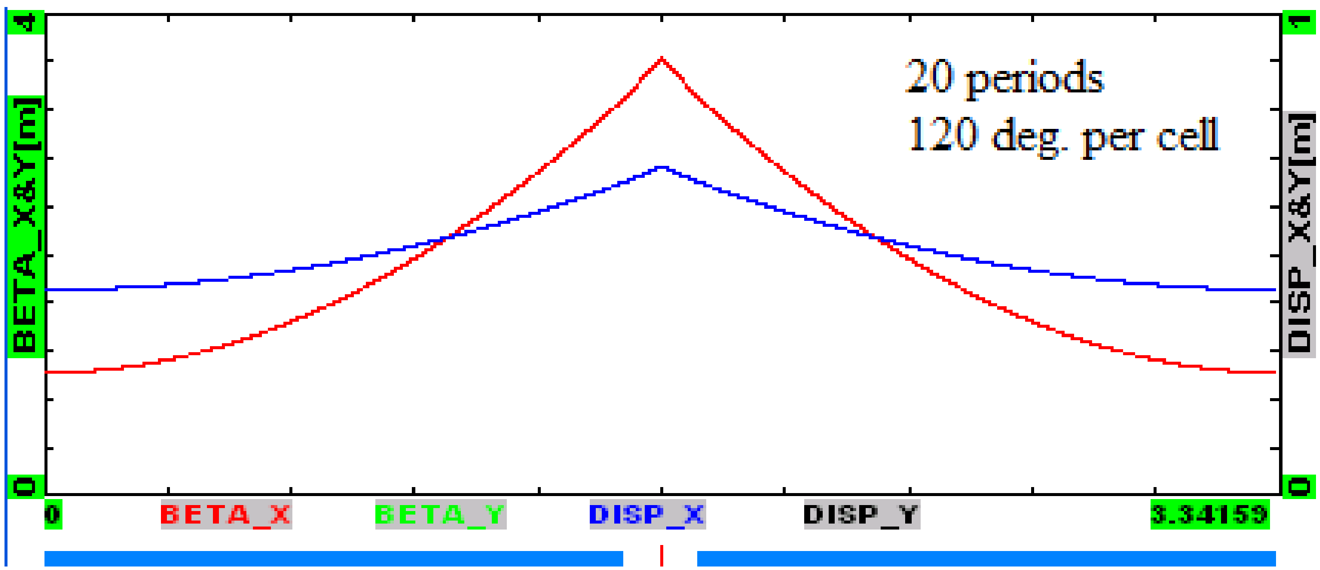
- For the slip-factors we have: $\boxed{\alpha=\frac{p_0}{R_0}\frac{dr}{dp}=\frac{1}{1+g}=\frac{1}{\nu_x^{\ 2}}}$, $\boxed{\alpha_p\equiv\frac{p_0^{\ 2}}{2R_0}\frac{d^2r}{dp^2}=-\frac{g+s}{(1+g)^3}=-\frac{g+s}{\nu_x^{\ 2}}}$
- That allows to bind up the chromaticity and the 2nd order slip-factor

$$\boxed{\alpha_p=\frac{1-\nu_x^{\ 2}-2\nu_x\xi}{2\nu_x^{\ 4}}}$$

  - The 2nd order slip-factor is uniquely determined by the measurable values: the betatron tune and the tune chromaticity

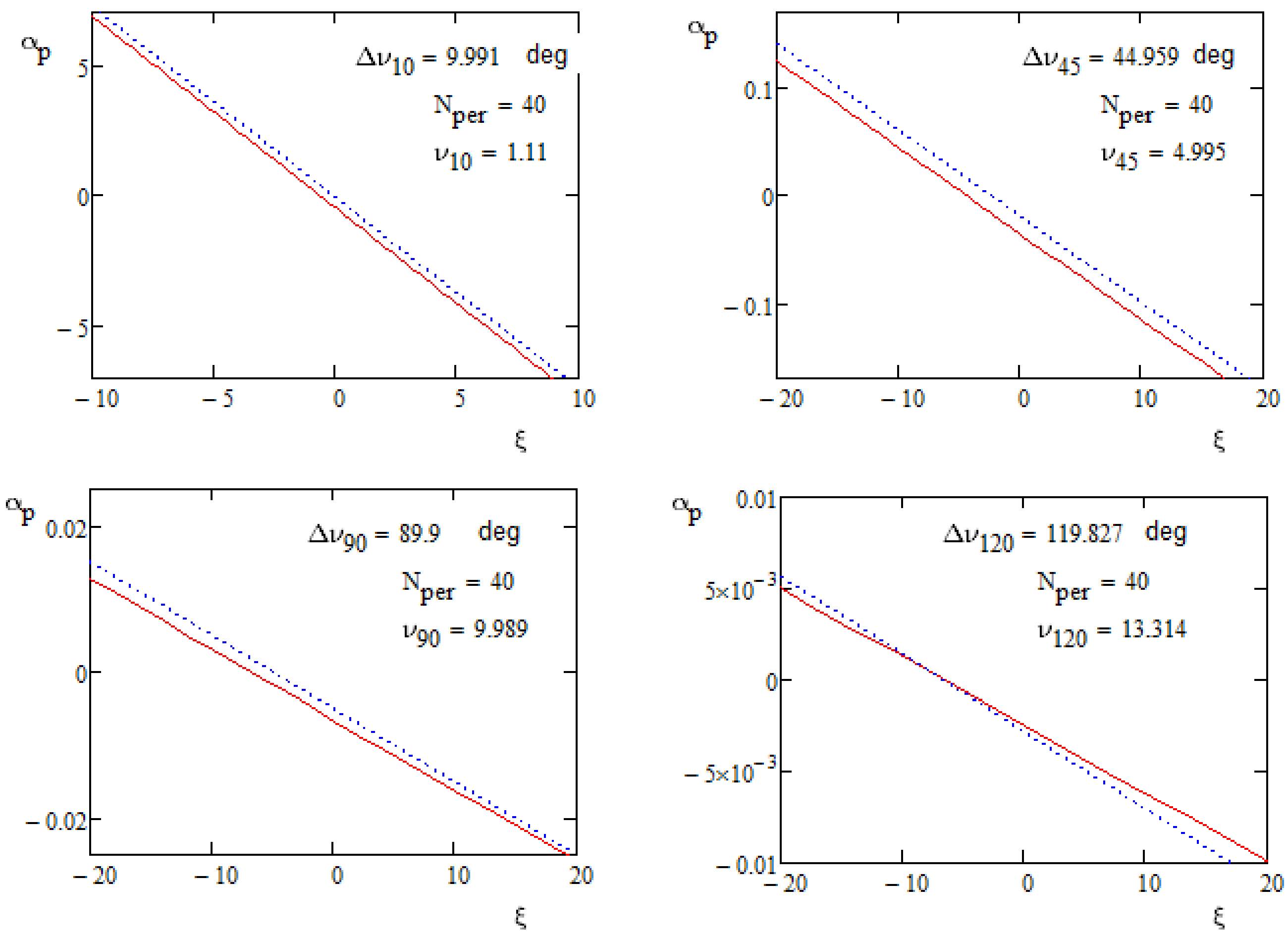
FO Lattice (with rectangular dipoles and thin lenses)

- It is a straightforward to obtain a semi-analytical solution in the case of one thin lens and one rectangular dipole per period
  - We expect that for a small betatron phase advance per cell the solution will coincide with smooth lattice approximation
  - We are interested to see the effect of beta-beating on the 2nd order slip-factor
- The following lattice was used in simulations: R_0=5 m, L_{dr}=0.3 m
  - Number of periods and betatron phase advance were varied



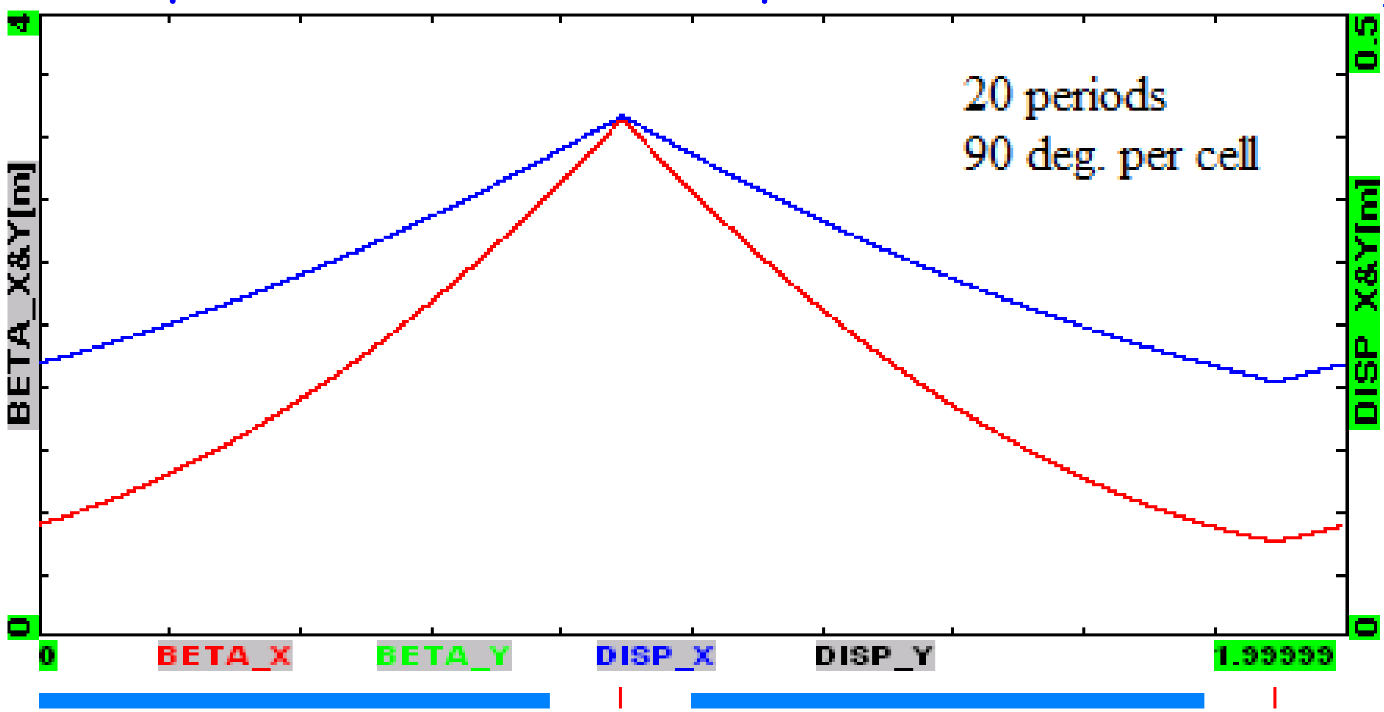
Results of Computations for the FO Lattice

- Results of numeric computations show quite good coincidence with the smooth lattice model
  - Computations were done for 5, 10, 20 and 40 periods in the ring
  - All computations show behavior which is very close to the smooth lattice approximation
    - The reasons are not clear yet
  - It has been used to verify that MADX delivers correct result
    - Looks like for a knowledgeable person it does
  - Computations for the Booster and Main Injector will follow



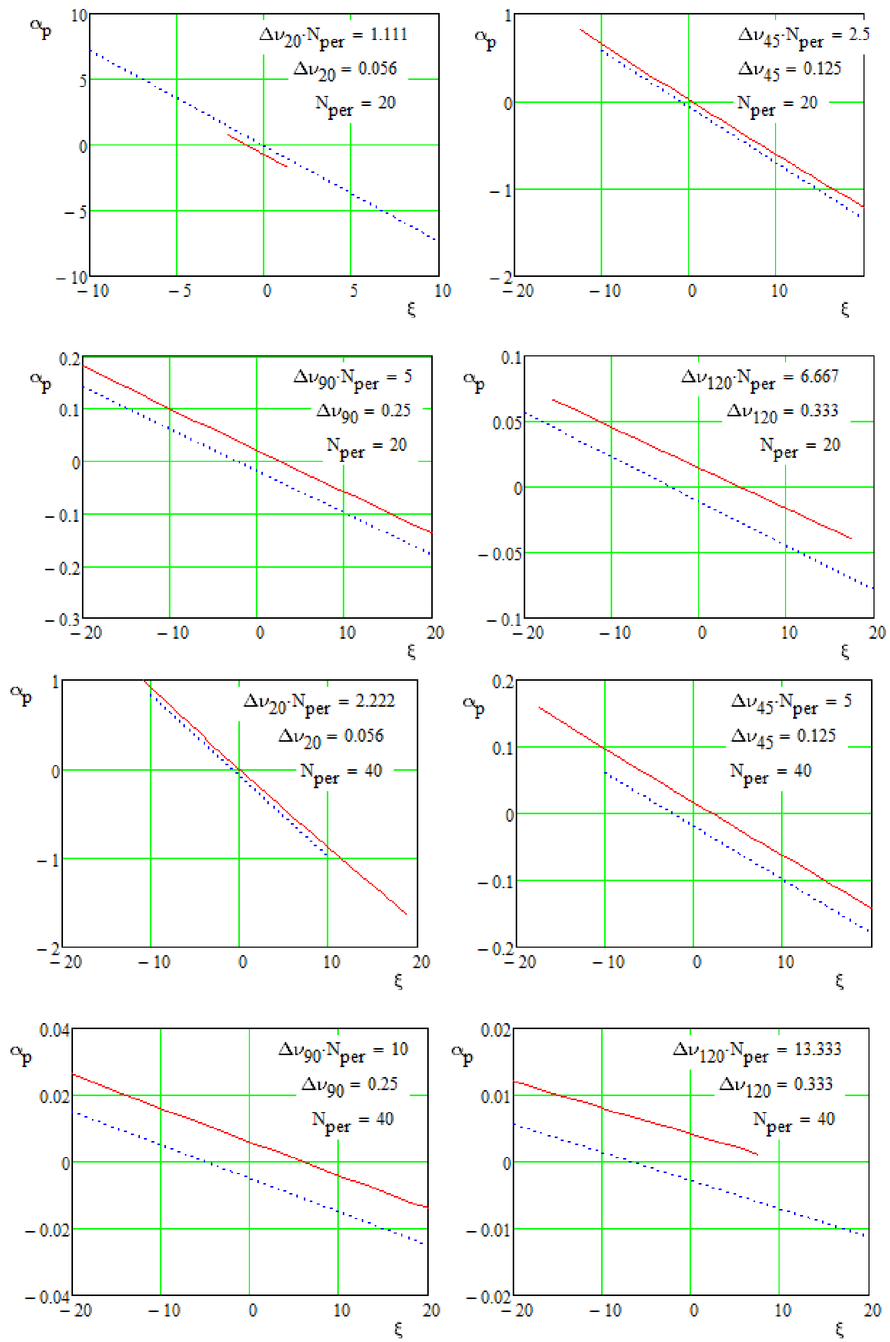
FODO Lattice (with sector dipoles and thin lenses)

- Another simple example is FODO lattice with sector dipoles and no straights
- Exact Solution is obtained numerically
  - Iterations are used to find the trajectory locations in quads
- The following lattice was used in simulations: R_0=5 m
  - Number of periods and betatron phase advance were varied



Results of Computations for the FODO Lattice

- The same as for the FO lattice the results of computations show quite good coincidence with the smooth lattice model
  - Computations were done for 20 and 40 periods in the ring
  - All computations show that the slope is very close to the smooth lattice approximation
    - The reasons of offsets are not clear
  - Typically, zeroing of 2nd order slip-factor happens for positive chromaticity in the range [0,10] for the phase advances of 45-90 deg. per cell



Results obtained in earlier studies

- Two earlier publication were also considered:

  [1] J. Shan, S. Peggs, S. Bogacz, "Analytical evaluation of the second order momentum compaction factor and comparison with MAD results", Fermilab-Pub-92/124 (1992)

  [2] K.-Y. Ng, "Higher order momentum compaction for a simplified FODO lattice and comparison with SYNCH", Fermilab-FN-578 (1991)

- Both papers
  - Written to understand transition crossing in the MI
  - use sector dipoles and zero length quadrupoles
  - For zero sextupole strength and quadrupole focusing much larger than focusing coming from the dipoles their results (e.g. Eq. 5.8 in Ref. 2) coincide well with calculations presented above
    - Effect of sextupoles (and chromaticity) is not presented in sufficient details in Ref. [1]
    - Part of the equation describing effect of sextupoles on α_p in Ref [2] has incorrect dimension

From 2nd order momentum compaction to 2nd order slip-factor

$$\frac{\Delta f}{f} = \eta(p)\frac{\Delta p}{p} + \eta_p(p)\left(\frac{\Delta p}{p}\right)^2 + \dots$$

$$\eta(p) = \alpha - \frac{1}{\gamma^2}$$

- Non-linear dependence of particle velocity on momentum should be also accounted

$$\eta_p(p) = \alpha_p + \frac{3}{2}\frac{\beta^2}{\gamma^2} - \frac{\eta(p)}{\gamma^2}$$

Relationship of 2nd Momentum Compaction & Chromaticity

- Kick from sextupole: $\Delta\theta = -\dfrac{e(SL)}{2pc}(D\delta)^2$
- Let's find orbit change:

$$\begin{bmatrix} x \\ \theta-\Delta\theta \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ \theta \end{bmatrix} = \frac{1}{2-M_{11}-M_{22}}\begin{bmatrix} 1-M_{22} & M_{12} \\ M_{21} & 1-M_{11} \end{bmatrix}\begin{bmatrix} 0 \\ \Delta\theta \end{bmatrix}$$

- Corresponding orbit lengthening: $\Delta s = M_{51}x + M_{52}\theta$

$$M_{11} = c_\nu + \alpha s_\nu, \quad M_{12} = \beta_x s_\nu, \quad M_{21} = -\left(1+\alpha^2\right)s_\nu / \beta_x, \quad M_{22} = c_\nu - \alpha s_\nu,$$

- Using $M_{51} = D'\left(1-c_\nu-\alpha s_\nu\right) - \dfrac{1+\alpha^2}{\beta_x}Ds_\nu, \quad M_{52} = D\left(c_\nu - \alpha s_\nu - 1\right) - D'\beta_x s_\nu,$

where $c_\mu = \cos(2\pi\nu), \quad s_\mu = \sin(2\pi\nu), \quad D' = \dfrac{dD}{ds}, \quad \alpha = -\dfrac{1}{2}\dfrac{d\beta_x}{ds}$

one obtains: $\Delta s = -\dfrac{e(SL)D^3\delta^2}{2pc} \quad \Rightarrow \quad \Delta\alpha_p \equiv \dfrac{1}{2C}\dfrac{d^2}{d\delta^2}\Delta s = -\dfrac{e(SL)D^3}{2pcC}$

- For many sextupoles we have: $\boxed{\Delta\alpha_p = -\dfrac{e}{2pcC}\sum_n (SL)_n D_n^{\,3}}$
- While for the chromaticity we have: $\boxed{\Delta\xi = \dfrac{e}{4\pi pc}\sum_n (SL)_n D_n \beta_{x_n}}$
- If all sextupoles are located at the same D's and β's one obtains: $\boxed{\dfrac{d\alpha_p}{d\xi} = -\dfrac{2\pi D^2}{C\beta_x} \xrightarrow[\textit{optics appr.}]{\textit{Smooth}} -\dfrac{1}{\nu_x^{\,3}}}$

Conclusions

- The main goal of this work is to verify that MADX delivers correct value for the second order slip-factor
- As a by-product, we found that the 2nd order slip-factor and the horizontal chromaticity are related by simple equation

$$\boxed{\frac{d\alpha_p}{d\xi} \approx -\frac{1}{{\nu_x}^3}}$$

- For typical cases the 2nd order slip-factor at zero chromaticity is

$$\boxed{\left|\alpha_p(\xi = 0)\right| \le \frac{1}{{\nu_x}^2}}$$