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Theory of "Resonant" Lattices for Synchrotrons with Negative Momentum Compaction Factor

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Abstract—In synchrotrons the transition energy $W_{tr} = m_0c^2(\gamma_{tr} - 1)$ is fundamentally important because it determines the maximum attainable accelerated currents. From this point of view it is desirable that the momentum compaction factor $\alpha = 1/\gamma_{tr}^2$ be as small as possible or even negative, which makes γ_{tr} imaginary and accordingly rules out transition energy crossing by particles under acceleration. In this connection a theory of "resonant" lattices with negative momentum compaction factor based on the correlation principle of simultaneous superperiodic modulation of the orbit curvature and lens gradient functions is worked out.

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1. INTRODUCTION

Alternating gradient synchrotrons feature a transition energy $\gamma = \gamma_{tr}$ at which the derivative of the particle revolution period T in the orbit with respect to the momentum p vanishes,

$$\frac{dT}{T} = \left(\frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma^2}\right)\frac{dp}{p}.$$

The quantities $\alpha = 1/\gamma_{tr}^2$ and $\eta = \alpha - 1/\gamma^2$ are known in acceleration physics as the momentum compaction factor and the slip factor. Since the frequency of longitudinal oscillations is proportional to the square root of the slip factor [1], the longitudinal frequency becomes zero as the transition energy is crossed, which means loss of longitudinal motion stability. In accelerator physics, transition energy crossing is classed among the most important problems and the momentum compaction factor is the most important parameter of the synchrotron lattice.

In this connection, many methods for transition energy crossing with minimum particle loss have been developed. In high-intensity accelerators, however, transition energy crossing should be utterly eliminated because of severe loss limitation. Moreover, since it follows from the theory of collective instabilities that the microwave instability limit defined by the Keil–Schnell criterion is proportional to the slip factor η and the required minimum spread of incoherent frequencies $\delta\omega/\omega$ in longitudinal motion has, according to the Landau theory [2], the form

$$\left(\frac{\delta\omega}{\omega}\right)^2 = \eta^2\left(\frac{\delta p}{p}\right)^2 \geq \frac{eI_{peak}|\eta|}{2\pi m_0 c^2\gamma\beta^2}\left|\frac{Z_{II}(n)}{n}\right|, \quad (1)$$

the slip factor $|\eta|$ should be as large as possible in absolute value at the momentum spread $\delta p/p$ in a beam of current I_{peak} and at the longitudinal impedance $|Z_{II}(n)/n|$.

A way to cope with the above problem is to make a lattice with the negative momentum compaction factor defined as follows:

$$\alpha = \frac{1}{C}\int^{C}\frac{D(s)}{\rho(s)}ds \leq 0. \quad (2)$$

Here, C is the length of the closed equilibrium orbit, s is the longitudinal coordinate along the equilibrium orbit, $D(s)$ is the horizontal dispersion function, and $\rho(s)$ is the curvature radius of the equilibrium orbit. In this case the transition energy takes on the complex value

$$\gamma_{tr} = -\frac{i}{\sqrt{|\alpha|}}$$

and it is obvious that transition energy crossing never occurs. In this case the slip factor

$$|\eta| = |\alpha| + \frac{1}{\gamma^2}$$

takes on its maximum possible value in magnitude.

The idea of a lattice with the negative momentum compaction factor was first put forward by Vladimirskiĭ and Tarasov in 1955 [3]. They proposed to use reversed field magnets with negative curvature $\rho(s) < 0$. Later, in 1958, Courant and Snyder mathematically described the idea of the negative momentum compaction factor [4]. They solved the dispersion equation using a special formalism involving the variable

$$d\theta = \frac{ds}{\nu\beta},$$

which allowed averaging over fast oscillations

$$\begin{aligned} \alpha &= \frac{\nu^3}{\bar{R}}\sum_{k=0}^{\infty}\frac{a_k^2}{\nu^2 - k^2}, \\ a_k &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\beta^{3/2}}{\rho}e^{-ik\theta}d\theta. \end{aligned} \tag{3}$$

Here, ν is the betatron frequency of horizontal oscillations, β is the horizontal beta-function, and $\bar{R}$ is the average radius of the orbit.

Based on expressions (3), Courant and Snyder formulated a simple conclusion: "If the betatron frequency ν is slightly less than an integer k and if a_k is large enough, the dominant role of the zero term a_0 can be canceled." Vladimirskiĭ and Tarasov proposed to do this by inserting "compensating magnets" with reversed fields.

However, the idea of thus eliminating transition energy remained overlooked for a long time. Beginning in the late 1980s, many authors tried to design a lattice with complex transition energy. For example, in [5] they proposed a "modular" method where quadrupoles spaced 180° apart modulated the β-function in a special way to get a negative momentum compaction factor. The same method was used by the Fermi National Laboratory team for their main injector [6]. The idea based on modulation of the orbit curvature by inserting empty spaces without bending magnets in the design of the Saturne synchrotron (France) was developed in [7]. At the GSI accelerator center (Germany) the required harmonic was excited by a group of quadrupole lenses inserted in the SIS-18 synchrotron [8]. At the initial designing stage of the TRIUMF KAON Factory (Canada) it was proposed to use a lattice with modulation of drift spaces [9]. All these methods employed either quadrupole modulation of the β-function or modulation of the orbit curvature, which directly followed from (3).

However, the most successful approach, which found wide application in many international projects, was the lattice of the Moscow Kaon Factory [10, 11]. It was based on resonantly correlated functions of the orbit curvature and lens gradient modulation. This lattice was then adapted for the TRIUMF KAON Factory [12, 13]. Later it was considered the best for the SSC Booster [14], then was adopted for the Neutrino Factory at CERN (Switzerland) [15], and ultimately was implemented in the J-PARC accelerator complex (Japan) [16, 17]. Now this lattice has also been adopted for the high-energy antiproton storage ring HESR within the framework of the international project FAIR (Germany) [18, 19].

The lattice in question has the following features:

(i) a negative momentum compaction factor produced by simultaneous modulation of the orbit curvature and quadrupole lens gradient;

(ii) variation of transition energy in a wide range from $\gamma_{tr} \approx \nu$ to $\gamma_{tr} \approx i\nu$ by quadrupole lens gradient modulation alone;

(iii) independent tuning of the momentum compaction factor and betatron frequencies of arcs by three families (two focusing and one defocusing) of quadrupole lenses;

(iv) the possibility of having zero dispersion in straight sections;

(v) functional division of the accelerator lattice into arcs and straight sections and their independent tuning;

(vi) effective system for correction of chromaticity by two families of sextupoles;

(vii) mutual compensation for the nonlinear effect of chromatic sextupoles in the first order of the perturbation theory;

(viii) low sensitivity to multipole errors in quadrupoles and magnets due to a special structure of the arcs.

In this paper a theory of resonant lattices allowing an optimum approach to their construction is developed.

2. THE GENERAL VIEW OF THE DISPERSION EQUATION FOR THE LATTICE WITH SUPERPERIODICITY INTRODUCED

In [5–9], the authors, relying on the Courant–Snyder principle (see (3)), try to use modulations of the mixed function $\beta^{3/2}/\rho$ for controlling the momentum compaction factor. This is not quite correct because the momentum compaction factor is actually governed by two functions, the orbit curvature function $\rho(s)$ and the lens gradient function $G(s)$, and the modulation of the function $\beta(s)$ is a consequence of the gradient modulation. Moreover, in all procedures to be carried out with the above functions, we should seek the minimum possible perturbation of the β-function, whose modulation ultimately affects the dynamic aperture of the accelerator. Since, as is shown below, the momentum compaction factor depends on the modulation of gradients and the modulation of curvature in different way, these functions should be treated separately.

The general form of the dispersion equation for the periodic alternating focusing lattice is

$$\frac{d^2D}{ds^2} + [K(s) + \varepsilon k(s)]D = \frac{1}{\rho(s)}, \tag{4}$$

where the gradients $G(s)$ and the orbit curvature $\rho(s)$ are related to each other through the functions

$$K(s) = \frac{eG(s)}{p}, \quad \varepsilon k(s) = \frac{e\Delta G(s)}{p},$$

where $p = m_0\gamma v$ is the particle momentum. The latter of the functions is a superperiodic modulation of the gradients $\Delta G(s)$. Obviously, in a periodic lattice the function $K(s)$ has a focusing cell periodicity L_c and the function $k(s)$ has a superperiod periodicity $L_s = nL_c$ with multiplicity $n = 1, 2, \ldots$. In other words, the lattice has a double periodicity governed by the functions $K(s + L_c) = K(s)$ and $k(s + L_s) = k(s)$. The orbit curvature has a single periodicity coinciding with the length of the superperiod, $\rho(s + L_s) = \rho(s)$.

Let us turn to another longitudinal coordinate,

$$d\phi = \frac{2\pi}{L_s}ds,$$

measured in radians and varying by 2π over one superperiod. Considering that the length of the superperiod can be expressed in terms of the average orbit radius $\bar{R}$

$$L_s = \frac{2\pi\bar{R}}{S},$$

where S is the number of superperiods, Eq. (4) can be written in the system with the new longitudinal coordinate

$$\begin{gathered}\frac{d^2D}{d\phi^2} + \left(\frac{\bar{R}}{S}\right)^2[K(\phi) + \varepsilon k(\phi)]D = \left(\frac{\bar{R}}{S}\right)^2\frac{1}{\rho(\phi)}, \\ d\phi = \frac{S}{\bar{R}}ds.\end{gathered} \tag{5}$$

Equation (5) is an inhomogeneous linear differential equation of the second order. To solve it, we should first find fundamental solutions φ_1 and φ_2 of the homogeneous equation

$$\frac{d^2X}{d\phi^2} + \left(\frac{\bar{R}}{S}\right)^2[K(\phi) + \varepsilon k(\phi)]X = 0, \tag{6}$$

and then, using the method of constant variation [20], obtain the partial solution of the dispersion equation with allowance for variation in gradients and orbit curvature,

$$D(\phi) = \left(\frac{\bar{R}}{S}\right)^2\left[\varphi_2\int\frac{\varphi_1}{\rho W}d\phi - \varphi_1\int\frac{\varphi_2}{\rho W}d\phi\right], \tag{7}$$

where W is the Wronskian of the fundamental solution of the homogeneous equation.

3. THE FUNDAMENTAL SOLUTION OF THE DISPERSION EQUATION FOR A LATTICE WITH A PERIODICALLY ALTERNATING GRADIENT

Since, by physical definition, dispersion is the partial solution of the inhomogeneous equation of betatron oscillations with its right-hand part proportional to the momentum spread, i.e.,

$$\propto \frac{1}{\rho}\frac{\Delta p}{p},$$

we replace the variable D in (6) by the variable X, thus stressing that the solution of the homogeneous equation is only an intermediate solution which is not dispersion.

According to a Floquet theorem, the fundamental solution of (6) when $\varepsilon \longrightarrow 0$ can be represented as

$$x_{1,2} = a_{1,2}f(\phi)e^{\pm i\psi(\phi)}, \tag{8}$$

where $f(\phi)$ is a Floquet function modulus that is periodic, $f(\phi) = f(\phi + 2\pi)$; $\psi(\phi)$ is the Floquet function phase; $a_{1,2}$ are arbitrary constants. Note that the Floquet function modulus coincides with the square root of the $\beta(\phi)$ function.

Substituting (8) into (6) and dividing the resulting expression into real and imaginary parts, we get a system of equations

$$\begin{gathered}\frac{d^2f}{d\phi^2} + \left(\frac{\bar{R}}{S}\right)^2K(\phi)f - \frac{1}{f^3} = 0, \\ \frac{d\psi}{d\phi} = \frac{1}{f^2}.\end{gathered} \tag{9}$$

The second equation in (9) is easily integrated, and over an arbitrary length $\phi = 2\pi j$ consisting of $1, \ldots, j$ superperiods the solution has the form

$$\psi(2\pi j) = \int_0^{2\pi j}\frac{d\xi}{f^2(\xi)} + \psi(0). \tag{10}$$

In what follows we put $\psi(0) = 0$ because in the class of lattices under consideration the superperiod has mirror symmetry about its center and the ϕ coordinate is measured from this point. Since the function $f(\phi)$ is periodic with a period 2π, we represent integral (10) as a sum of j integrals over the integration length 2π

$$\psi(2\pi j) = j\int_0^{2\pi}\frac{d\xi}{f^2(\xi)}$$

or

$$\psi(\phi) = \phi\frac{1}{2\pi}\int_0^{2\pi}\frac{d\xi}{f^2(\xi)}. \tag{11}$$

Let us find the phase advance μ per superperiod normalized to 2π as the average of $1/f^2$:

$$\mu = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\xi}{f^2(\xi)}. \tag{12}$$

Then the Floquet function phase can be represented as

$$\psi(\phi) = \mu\phi. \tag{13}$$

In addition, we know that the Floquet function may be represented as a product of the average f_0 by the function describing oscillation in the alternating-focusing lattice:

$$f(\phi) = f_0[1+q(\phi)]. \tag{14}$$

The function $q(\phi)$, introduced by Kapchinskiĭ in [21], is periodic with the cell period, $q(\phi) = q(\phi + 2\pi/n)$, and its average over the period is zero,

$$\overline{q(\phi)} = \int_0^{2\pi/n} q(\phi)d\phi = 0.$$

The function

$$\frac{f}{f_0} = 1 + q(\phi)$$

is a normalized function describing oscillation of the beam envelope in the focusing channel. We denote it by

$$\hat{f} = \frac{f}{f_0}.$$

Considering the smallness of the function $q(\phi) \ll 1$ and its periodicity and substituting (14) into (12), we get an expression that relates the Floquet function to the normalized phase advance:

$$\begin{aligned}\mu &= \frac{1}{f_0^2},\\ f(\phi) &= \frac{1}{\sqrt{\mu}}[1+q(\phi)].\end{aligned} \tag{15}$$

Thus, knowing the function $K(\phi)$, we can obtain the solution of (6) for $\varepsilon = 0$, when μ and f are defined by (9), (12), and (14).

Now let us consider the case where superperiodicity is introduced by the function $\varepsilon k(\phi)$. The Bogolyubov–Mitropolsky method [22] is used to find fundamental solutions of (6) for $\varepsilon \neq 0$ in a biperiodic structure with gradient modulation. Since the superperiod is usually mirror symmetrical, the only terms remaining in the expansion of the perturbation function in a Fourier series are those with cosines,

$$\varepsilon k(\phi) = \sum_{k=0}^{\infty} g_k \cos k\phi, \tag{16}$$

where

$$g_k = \frac{1}{\bar{R}B}\frac{1}{\pi}\int_{-\pi}^{\pi}\Delta G\cos k\phi d\phi. \tag{17}$$

The latter involves the well-known relation

$$\frac{e}{p} = \frac{1}{\bar{R}B},$$

where B is the maximum magnetic field in the bending magnets.

Substituting (16) into (6) and rearranging the term responsible for perturbation to the right-hand side of the equation, we get

$$\begin{aligned}&\frac{d^2X}{d\phi^2} + \left(\frac{\bar{R}}{S}\right)^2 K(\phi)X\\ &= -X\left(\frac{\bar{R}}{S}\right)^2\sum_{k=0}^{\infty} g_k\cos k\phi.\end{aligned} \tag{18}$$

The right-hand side of (18) depends on X and ϕ. The solution of this equation should be sought for by the Bogolyubov–Mitropolsky asymptotic method [22] in the form

$$X = a_1(\phi)f(\phi)\cos\psi + a_2(\phi)f(\phi)\sin\psi, \tag{19}$$

where the coefficients a_1 and a_2 are the functions of the longitudinal ϕ coordinate and the phase $\psi = \mu\phi$.

In the absence of perturbation, when $\varepsilon = 0$ and a_1 and a_2 do not depend on ϕ, the derivative $dX/d\phi$ can be determined by simple differentiation of (19)

$$\begin{aligned}\frac{dX}{d\phi} &= a_1\frac{df}{d\phi}\cos\psi + a_2\frac{df}{d\phi}\sin\psi\\ &- a_1 f\frac{d\psi}{d\phi}\sin\psi + a_2 f\frac{d\psi}{d\phi}\cos\psi.\end{aligned} \tag{20}$$

When $\varepsilon \neq 0$, Eqs. (19) and (20) still hold providing that a_1 and a_2 are functions of ϕ. Therefore, we will treat these equations as replacements of variables, and $a_1(\phi)$ and $a_2(\phi)$ are taken to be new functions of the variable ϕ, which, when found, will allow us to find the solution of (18). Differentiating (19) with all the aforesaid taken into account, we get

$$\begin{aligned}\frac{dX}{d\phi} &= \frac{da_1}{d\phi}f\cos\psi + a_1\frac{df}{d\phi}\cos\psi + \frac{da_2}{d\phi}f\sin\psi\\ &+ a_2\frac{df}{d\phi}\sin\psi - a_1 f\frac{d\psi}{d\phi}\sin\psi + a_2 f\frac{d\psi}{d\phi}\cos\psi.\end{aligned} \tag{21}$$

Now, equating the right-hand side of the above equation to the right-hand side of (20), we arrive at the first relation needed for determining of $a_1(\phi)$ and $a_2(\phi)$:

$$\frac{da_1}{d\phi}f\cos\psi + \frac{da_2}{d\phi}f\sin\psi = 0. \tag{22}$$

The second equation is found in a similar way. Let us differentiate (20) with the dependence of a_1 and a_2 on ϕ ignored,

$$\begin{aligned}\frac{d^2X}{d\phi^2} &= a_1\frac{d^2f}{d\phi^2}\cos\psi - 2a_1\frac{df}{d\phi}\frac{d\psi}{d\phi}\sin\psi \\ &+ a_2\frac{d^2f}{d\phi^2}\sin\psi + 2a_2\frac{df}{d\phi}\frac{d\psi}{d\phi}\cos\psi \\ &- a_1 f\left(\frac{d\psi}{d\phi}\right)^2\cos\psi - a_2 f\left(\frac{d\psi}{d\phi}\right)^2\sin\psi,\end{aligned} \tag{23}$$

and (21) with the dependence of a_1 and a_2 on ϕ taken into account,

$$\begin{aligned}\frac{d^2X}{d\phi^2} &= \frac{da_1}{d\phi}\frac{df}{d\phi}\cos\psi + a_1\frac{d^2f}{d\phi^2}\cos\psi \\ &- a_1\frac{df}{d\phi}\frac{d\psi}{d\phi}\sin\psi + \frac{da_2}{d\phi}\frac{df}{d\phi}\sin\psi \\ &+ a_2\frac{d^2f}{d\phi^2}\sin\psi + a_2\frac{df}{d\phi}\frac{d\psi}{d\phi}\cos\psi \\ &- \frac{da_1}{d\phi}f\frac{d\psi}{d\phi}\sin\psi - a_1\frac{df}{d\phi}\frac{d\psi}{d\phi}\sin\psi \\ &- a_1 f\left(\frac{d\psi}{d\phi}\right)^2\cos\psi + \frac{da_2}{d\phi}f\frac{d\psi}{d\phi}\cos\psi \\ &+ a_2\frac{df}{d\phi}\frac{d\psi}{d\phi}\cos\psi - a_2 f\left(\frac{d\psi}{d\phi}\right)^2\sin\psi.\end{aligned} \tag{24}$$

After substituting (23) and (24) into (18) and comparing them, we arrive at the second relation needed for determination of $a_1(\phi)$ and $a_2(\phi)$,

$$\begin{aligned}&\frac{da_1}{d\phi}\left[\frac{df}{d\phi}\cos\psi - f\frac{d\psi}{d\phi}\sin\psi\right] \\ &+ \frac{da_2}{d\phi}\left[\frac{df}{d\phi}\sin\psi + f\frac{d\psi}{d\phi}\cos\psi\right] \\ &= -\varepsilon k(\phi)\left(\frac{\bar{R}}{S}\right)^2[a_1 f\cos\psi + a_2 f\sin\psi].\end{aligned} \tag{25}$$

Thus, we have two equations, (22) and (25), which together with the second equation of system (9) make it possible to find $a_1(\phi)$ and $a_2(\phi)$ in the first-order approximation of the Bogolyubov–Mitropol'skii asymptotic theory,

$$\begin{aligned}\frac{da_1}{d\phi} &= f^2\left(\frac{\bar{R}}{S}\right)^2\sum_{k=0}^{\infty}g_k\cos k\phi \\ &\times(a_{10}\cos\mu\phi + a_{20}\sin\mu\phi)\sin\mu\phi, \\ \frac{da_2}{d\phi} &= -f^2\left(\frac{\bar{R}}{S}\right)^2\sum_{k=0}^{\infty}g_k\cos k\phi \\ &\times(a_{10}\cos\mu\phi + a_{20}\sin\mu\phi)\cos\mu\phi,\end{aligned} \tag{26}$$

where a_{10} and a_{20} are the zero approximation of solution (19). System of equations (26) involves Fourier representation (16) of the perturbation function $\varepsilon k(\phi)$.

Note that since the perturbation in question is known not to be on resonance, it should be dropped and the first approximation of the asymptotic theory should be quite enough. On integrating (26), we get

$$\begin{aligned}a_1 &= a_{10} + \left(\frac{\bar{R}}{S}\right)^2\int_0^{\phi}f^2\sum_{k=0}^{\infty}g_k\cos kt \\ &\times\sin\mu t(a_{10}\cos\mu t + a_{20}\sin\mu t)dt, \\ a_2 &= a_{20} - \left(\frac{\bar{R}}{S}\right)^2\int_0^{\phi}f^2\sum_{k=0}^{\infty}g_k\cos kt \\ &\times\cos\mu t(a_{10}\cos\mu t + a_{20}\sin\mu t)dt.\end{aligned} \tag{27}$$

Substituting solutions (27) into initial solution (19), we get

$$\begin{aligned}X(\phi) &= a_{10}f\Bigg[\cos\mu\phi + \cos\mu\phi\left(\frac{\bar{R}}{S}\right)^2 \\ &\times\int_0^{\phi}f^2\sum_{k=1}^{\infty}g_k\cos kt\sin\mu t\cos\mu t dt \\ &- \sin\mu\phi\left(\frac{\bar{R}}{S}\right)^2\int_0^{\phi}f^2\sum_{k=1}^{\infty}g_k\cos kt\cos\mu t\cos\mu t dt\Bigg] \\ &+ a_{20}f\Bigg[\sin\mu\phi - \sin\mu\phi\left(\frac{\bar{R}}{S}\right)^2 \\ &\times\int_0^{\phi}f^2\sum_{k=1}^{\infty}g_k\cos kt\sin\mu t\cos\mu t dt + \cos\mu\phi\left(\frac{\bar{R}}{S}\right)^2 \\ &\times\int_0^{\phi}f^2\sum_{k=1}^{\infty}g_k\cos kt\sin\mu t\sin\mu t dt\Bigg].\end{aligned} \tag{28}$$

Since the Floquet function f is fast oscillating as compared with other functions in the integrand (see (15)), it can be factored outside the integral sign. Considering definition (12) and the fact that $\mu = \nu/S$, the first bracketed expression in (28) can be rearranged by simple manipulations to give

$$\cos\mu\phi\left(\frac{\bar{R}}{S}\right)^2\int_0^\phi f^2\sum_{k=1}^{\infty} g_k\cos kt\sin\mu t\cos\mu t dt$$
$$-\sin\mu\phi\left(\frac{\bar{R}}{S}\right)^2\int_0^\phi f^2\sum_{k=1}^{\infty} g_k\cos kt\cos\mu t\cos\mu t dt$$
$$= -\frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{k=1}^{\infty} g_k\left\{\frac{\cos(\mu\phi+k\phi)}{1-(1+kS/\nu)^2}+\frac{\cos(\mu\phi-k\phi)}{1-(1-kS/\nu)^2}\right\} \tag{29}$$
$$= -\frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} g_k\frac{\cos(\mu\phi-k\phi)}{1-(1-kS/\nu)^2}.$$

The same can be done with the second bracketed expression in (28):

$$-\sin\mu\phi\left(\frac{\bar{R}}{S}\right)^2\int_0^\phi\sum_{k=1}^{\infty} g_k\cos kt\sin\mu t\cos\mu t dt$$
$$+\cos\mu\phi\left(\frac{\bar{R}}{S}\right)^2\int_0^\phi\sum_{k=1}^{\infty} g_k\cos kt\sin\mu t\cos\mu t dt$$
$$= -\frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{k=1}^{\infty} g_k\left\{\frac{\sin(\mu\phi+k\phi)}{1-(1+kS/\nu)^2}+\frac{\sin(\mu\phi-k\phi)}{1-(1-kS/\nu)^2}\right\} \tag{30}$$
$$= -\frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} g_k\frac{\sin(\mu\phi-k\phi)}{1-(1-kS/\nu)^2}.$$

At the last step of the rearrangements, in view of the fact that $g_k = g_{-k}$, we went from the summation over $k = 1 - \infty$ to the summation over $k = -\infty$–∞ in (29) and (30).

Ultimately, the expression for the solution of equation (18) can be represented in a compact form:

$$X(\phi) = a_{10}\varphi_1 + a_{20}\varphi_2, \tag{31}$$

where φ_1 and φ_2 are fundamental solutions of Eq. (6) for a bioperiodic structure with modulation of gradients:

$$\varphi_1 = f(\phi)\left[\cos\mu\phi - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} g_k\frac{\cos(\mu\phi-k\phi)}{1-(1-kS/\nu)^2}\right],$$
$$\varphi_2 = f(\phi)\left[\sin\mu\phi - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} g_k\frac{\sin(\mu\phi-k\phi)}{1-(1-kS/\nu)^2}\right]. \tag{32}$$

In what follows we always keep in mind that summation is taken over all values $k = -\infty$–∞ except $k = 0$.

4. THE GENERAL SOLUTION OF THE DISPERSION EQUATION WITH THE PERIODICALLY VARYING GRADIENT AND ORBIT CURVATURE

Having obtained fundamental solutions of homogeneous equation (6), we can solve general dispersion equation (5) with the biperiodically varying gradient $K(\phi) + \varepsilon k(\phi)$ and the periodically varying orbit curvature $1/\rho(\phi)$. Considering the mirror symmetry of the superperiod, the orbit curvature can be represented as a Fourier series:

$$\frac{1}{\rho(\phi)} = \frac{1}{\bar{R}}\left(1+\sum_{n=1}^{\infty} r_n\cos n\phi\right), \tag{33}$$

where r_n is the Fourier harmonic of orbit curvature function normalized to $\bar{R}$:

$$r_n = \frac{\bar{R}}{\pi}\int_{\pi}^{-\pi}\frac{\cos n\phi}{\rho(\phi)}d\phi. \tag{34}$$

In accordance with the method of constant variation [20], the general solution is defined by expression (7) with the Wronskian

$$W = \varphi_1\frac{d\varphi_2}{d\phi} - \varphi_2\frac{d\varphi_1}{d\phi}$$
$$= f^2\left[\cos^2\psi\frac{d\psi}{d\phi} + \sin^2\psi\frac{d\psi}{d\phi}\right] = f^2\frac{1}{f^2} = 1. \tag{35}$$

Substituting (32), (33), and (35) into (7), we get the following general expression for dispersion:

$$D(\phi) = \left(\frac{\bar{R}}{S}\right)^2\frac{1}{\bar{R}}\left\{f(\phi)\left[\sin\mu\phi - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2\right.\right.$$

$$\times \sum_{k=-\infty}^{\infty} g_k \frac{\sin(\mu-k)\phi}{1-(1-kS/\nu)^2}\Bigg]$$

$$\times \int_0^{\phi} f(t)\left[\cos\mu t - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2 \sum_{k=-\infty}^{\infty} g_k \frac{\cos(\mu-k)t}{1-(1-kS/\nu)^2}\right]$$

$$\times\left[1+\sum_{n=1}^{\infty} r_n \cos nt\right]dt \tag{36}$$

$$-f(\phi)\left[\cos\mu\phi - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2 \sum_{k=-\infty}^{\infty} g_k \frac{\cos(\mu-k)\phi}{1-(1-kS/\nu)^2}\right]$$

$$\times \int_0^{\phi} f(t)\left[\sin\mu t - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2 \sum_{k=-\infty}^{\infty} g_k \frac{\sin(\mu-k)t}{1-(1-kS/\nu)^2}\right]$$

$$\times\left[1+\sum_{n=1}^{\infty} r_n \cos nt\right]dt\Bigg\}.$$

A product of two sums can be replaced by double summation

$$\sum_{k=1}^{\infty} g_k \cos k\phi \sum_{n=1}^{\infty} r_n \cos n\phi = \frac{1}{2}\sum_{k=1}^{\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} g_k r_n \cos(k-n)\phi. \tag{37}$$

In what follows we also always keep in mind that summation is taken over all values $n = -\infty$–∞ except $n = 0$.

Using the same property of the function $f(\phi)$ as was employed above to obtain the pair of fundamental solutions (32), namely, that the period of this function is much smaller than the period of the harmonics appearing in the integrand in (36) and governing the ultimate solution, we may factor this function outside the integral sign in the form of its average $f_0^2 = 1/\mu$. Integration of (36) using property (37) and the periodicity condition $D(\phi) = D(\phi + 2\pi)$ yields the expression for the periodic dispersion

$$D(\phi) = \frac{\bar{R}}{\nu^2}\hat{f}\Bigg\{\Bigg[\sin\mu\phi - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2$$

$$\times \sum_{k=-\infty}^{\infty} g_k \frac{\sin(\mu-k)\phi}{1-(1-kS/\nu)^2}\Bigg]\Bigg[\sin\mu\phi - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2$$

$$\times \sum_{k=-\infty}^{\infty} g_k \frac{\sin(\mu-k)\phi}{(1-kS/\nu)[1-(1-kS/\nu)^2]}\Bigg]$$

$$+\left[\cos\mu\phi - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2 \sum_{k=-\infty}^{\infty} g_k \frac{\cos(\mu-k)\phi}{1-(1-kS/\nu)^2}\right]$$

$$\times\Bigg[\cos\mu\phi - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2$$

$$\times \sum_{k=-\infty}^{\infty} g_k \frac{\cos(\mu-k)\phi}{(1-kS/\nu)[1-(1-kS/\nu)^2]}\Bigg]$$

$$+\left[\sin\mu\phi - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2 \sum_{k=-\infty}^{\infty} g_k \frac{\sin(\mu-k)\phi}{1-(1-kS/\nu)^2}\right] \tag{38}$$

$$\times\Bigg[\frac{1}{2}\sum_{n=-\infty}^{\infty} r_n \frac{\sin(\mu-n)\phi}{1-nS/\nu} - \frac{1}{4}\left(\frac{\bar{R}}{\nu}\right)^2$$

$$\times \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g_k r_n \frac{\sin[\mu-(k-n)]\phi}{[1-(k-n)S/\nu][1-(1-kS/\nu)^2]}\Bigg]$$

$$+\left[\cos\mu\phi - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2 \sum_{k=-\infty}^{\infty} g_k \frac{\cos(\mu-k)\phi}{1-(1-kS/\nu)^2}\right]$$

$$\times\Bigg[\frac{1}{2}\sum_{n=-\infty}^{\infty} r_n \frac{\cos(\mu-n)\phi}{1-nS/\nu} - \frac{1}{4}\left(\frac{\bar{R}}{\nu}\right)^2$$

$$\times \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g_k r_n \frac{\cos[\mu-(k-n)]\phi}{[1-(k-n)S/\nu][1-(1-kS/\nu)^2]}\Bigg]\Bigg\}.$$

Obviously, a rule similar to (37) is valid for the product of three sums. After rearrangements in (38) we get

$$D(\phi) = \frac{\bar{R}}{\nu^2}\hat{f}\Bigg\{1 - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2 \sum_{k=-\infty}^{\infty} \frac{g_k \cos k\phi}{(1-kS/\nu)[1-(1-kS/\nu)^2]} \tag{39}$$

$$-\frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2 \sum_{k=-\infty}^{\infty} \frac{g_k \cos k\phi}{1-(1-kS/\nu)^2} + \frac{1}{4}\left(\frac{\bar{R}}{\nu}\right)^4$$

$$\times \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{g_k g_n \cos(k-n)\phi}{(1-kS/\nu)[1-(1-kS/\nu^2)][1-(1-nS/\nu)^2]}$$
$$+\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{r_n\cos n\phi}{1-nS/\nu}-\frac{1}{4}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{g_k r_n\cos(k-n)\phi}{[1-(k-n)S/\nu][1-(1-kS/\nu)^2]}$$
$$-\frac{1}{4}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{g_k r_n\cos(k-n)\phi}{(1-nS/\nu)[1-(1-kS/\nu)^2]}$$
$$\left.+\frac{1}{8}\left(\frac{\bar{R}}{\nu}\right)^4\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{g_k r_n g_m\cos(k-n-m)\phi}{[1-(k-n)S/\nu][1-(1-kS/\nu)^2][1-(1-mS/\nu)^2]}\right\}.$$

It follows from this expression that in the case of no gradient modulation ($g_k = 0$) and curvature modulation ($r_k = 0$) the dispersion is defined by a simple expression

$$D(\phi) = \frac{\bar{R}}{\nu^2}\hat{f}(\phi),$$

coinciding with the known expression for an ordinary periodic structure.

The maximum value of the dispersion is defined by the following three terms:

$$D_{\max} = \frac{\bar{R}}{\nu^2}\hat{f}\left\{1-\frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2\right.$$
$$\times\sum_{k=-\infty}^{\infty}\frac{g_k}{(1-kS/\nu)[1-(1-kS/\nu)^2]}$$
$$-\frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{k=-\infty}^{\infty}\frac{g_k}{1-(1-kS/\nu)^2} \quad (40)$$
$$\left.+\frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{r_k}{1-kS/\nu}+O(g_k^i, r_k^i, i+j\geq 2)\right\}.$$

Thus, we see that the maximum dispersion value depends on the factor $1/(1 - kS/\nu)$, i.e., on how much the perturbation harmonic frequency differs from the horizontal betatron frequency.

5. DEFINITION OF THE RESONANT LATTICE AND ITS MAIN PROPERTIES

Now we know the functions $D(\phi)$ and $\rho(\phi)$ needed for calculating the momentum compaction factor α in a lattice with modulation of gradients and orbit curvature. According to (2), the momentum compaction factor is defined by the average of the function $D(\phi)/\rho(\phi)$. The dispersion and the orbit curvature may be represented in the general form in terms of their averages $\bar{D}$ and $\bar{R}$ and functions oscillating about these averages:

$$D(\phi) = \bar{D}+\tilde{D}(\phi), \quad \frac{1}{\rho(\phi)} = \frac{1}{\bar{R}}(1+\tilde{r}(\phi)).$$

Then the momentum compaction factor can be written as a sum:

$$\alpha = \frac{\bar{D}}{\bar{R}}+\frac{\overline{\tilde{D}(\phi)\tilde{r}(\phi)}}{\bar{R}}. \quad (41)$$

In an ordinary lattice without gradient and orbit curvature modulation the oscillating components are equal to zero, $\tilde{D}(\phi) = 0$, $\tilde{r}(\phi) = 0$, and the momentum compaction factor is governed by the first term in (41), namely, $\alpha = \bar{D}/\bar{R}$. On the other hand, the average dispersion is $\bar{D} = \bar{R}/\nu^2$ in this case (see (39)). Obviously, in lattices like this, the momentum compaction factor is fully governed by the betatron oscillation frequency, $\alpha = 1/\nu^2$, and its variation is limited.

The situation is essentially different in a lattice with gradient and orbit curvature modulation. Indeed, if both functions (gradient and orbit curvature) are modulated with an identical frequency ($k = n$ in (39)), the second term in (41) may make a considerable contribution to the momentum compaction factor provided that the value $1 - kS/\nu$ is small. We call this lattice based on the resonant perturbation of the magnetic optical channel parameters the resonant lattice. The closest harmonic producing the maximum effect on the momentum compaction factor is called the fundamental harmonic. This harmonic has kS oscillations over the entire magnetic optical structure. In most of the cases under consideration, this harmonic coincides with the number of superperiods, i.e., $kS = S$.

To have a resonant lattice, the following conditions should be met:

(i) the horizontal betatron frequency should be smaller than the fundamental modulation frequency of the superperiod parameters, $\nu < kS$, but as close to it as possible;

(ii) the orbit curvature modulation should be in antiphase with the lens gradient modulation, $g_k r_k < 0$;

(iii) the amplitudes of each fundamental harmonic should be as high as possible.

The exact equality of the frequencies $\nu = kS$ and $\nu = kS/2$, when the dispersion and the β-function increase without bound, should be ruled out.

6. THE MOMENTUM COMPACTION FACTOR IN THE RESONANT LATTICE WITH SUPERPERIODIC MODULATION OF LENS GRADIENTS AND ORBIT CURVATURE

Let us assume that we have the possibility of modulating the lens gradient and the orbit curvature with an arbitrary frequency which is a multiple of the number of superperiods S. As a result, we get two additional factors allowing us to influence the momentum compaction factor.

One of these factors is the average dispersion

$$\begin{aligned}\bar{D} = \frac{\bar{R}}{\nu^2}\hat{f}\Bigg\{ & 1 + \frac{1}{4}\left(\frac{\bar{R}}{\nu}\right)^4 \\ & \times \sum_{k=-\infty}^{\infty}\frac{g_k^2}{(1-kS/\nu)[1-(1-kS/\nu)^2]^2} \\ & - \frac{1}{4}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{k=-\infty}^{\infty}\frac{g_k r_k}{1-(1-kS/\nu)^2} \\ & - \frac{1}{4}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{k=-\infty}^{\infty}\frac{g_k r_k}{(1-kS/\nu)[1-(1-kS/\nu)^2]} \\ & + O(g_k^i r_k^j, i+j \ge 3)\Bigg\}.\end{aligned} \tag{42}$$

We see that the average dispersion in the resonant lattice can be varied in a wide range from positive to negative values while the degree of its dependence on the modulation of the gradient g_k and the orbit curvature r_k is to a great extent governed by how close the horizontal betatron frequency ν is to the fundamental harmonic of superperiod modulation kS. Note that the gradient modulation considerably affects the average dispersion: when there is no gradient modulation, the average dispersion remains unchanged.

The other factor is phase correlation of resonant harmonics of the gradient and orbit curvature perturbation. It may be the most effective tool to control the momentum compaction factor. Let us first consider one superperiod where

$$\alpha_s = \frac{1}{2\pi\bar{R}}\int_0^{2\pi} D(\phi)\left(1 + \sum_{n=1}^{\infty} r_n \cos n\phi\right)d\phi. \tag{43}$$

In the so-called circular lattices consisting of S superperiods, the momentum compaction factor completely coincides with its value for one superperiod. In the lattices consisting of arks with a total of S superperiods separated by straight sections of length $L_{\rm str}$, the momentum compaction factor is defined by the expression

$$\alpha = \alpha_s\frac{SL_s}{SL_s + L_{s\rm tr}}.$$

Thus, knowing the momentum compaction factor for one superperiod, one can easily find its value for the entire accelerator. Substituting (39) into (43), integrating, and ultimately keeping only terms of not higher than the second order, we get the following expression for the momentum compaction factor:

$$\begin{aligned}\alpha_s = \frac{1}{\nu^2}\Bigg\{ & 1 + \frac{1}{4}\left(\frac{\bar{R}}{\nu}\right)^4 \\ & \times \sum_{k=-\infty}^{\infty}\frac{g_k^2}{(1-kS/\nu)[1-(1-kS/\nu)^2]^2} \\ & + \frac{1}{4}\sum_{k=-\infty}^{\infty}\frac{r_k^2}{1-kS/\nu} - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2 \\ & \times \sum_{k=-\infty}^{\infty}\frac{r_k g_k}{(1-kS/\nu)[1-(1-kS/\nu)^2]} \\ & - \frac{1}{2}\left(\frac{\bar{R}}{\nu}\right)^2\sum_{k=-\infty}^{\infty}\frac{r_k g_k}{1-(1-kS/\nu)^2} \\ & + O(g_k^i r_k^i, i+j \ge 3)\Bigg\}.\end{aligned} \tag{44}$$

Thus, in the resonant lattice the momentum compaction factor is mainly governed by four sums appearing in (44). The first and second sums demonstrate that the gradient modulation and the orbit curvature modulation may independently control the momentum compaction factor. The third and fourth sums correspond to their joint action on the momentum compaction factor. The first three sums have a resonant factor of $1/(1 - kS/\nu)$ enhancing the effect when the modulation frequency is close to the frequency of horizontal betatron oscillations. Obviously, if resonant action occurs, the fourth sum makes practically no contribution and can be omitted. Note that for the integral effect to be enhanced the gradient and orbit curvature modulations should be excited in antiphase, $g_k r_k < 0$. Considering that the major contribution comes only from the fundamental modulation harmonics, we can simplify (44) by representing it as

$$\alpha_s = \frac{1}{\nu^2}\left\{1 + \frac{1}{4(1-kS/\nu)}\right.$$
$$\left.\times\left[\left(\frac{\bar{R}}{\nu}\right)^2 \frac{g_k}{1-(1-kS/\nu)^2} - r_k\right]^2\right\}. \quad (45)$$

This expression makes it possible to determine the amount of modulation of the gradient and orbit curvature functions for a magnetic optical lattice and their relation to obtain the required momentum compaction factor.

7. CONCLUSIONS

The theory of resonant lattices for synchrotrons is developed in this paper. It allows requirements to the design of a lattice without transition energy crossing to be properly formulated and its optimum parameters to be determined. The theory is based on solution of the dispersion equation for a lattice with the introduced superperiodicity using modulation of the lens gradient and orbit curvature functions. On the basis of this solution, mathematical expressions are derived for the dispersion and the momentum compaction factor for both separate and joint resonantly controlled modulation of the above-mentioned functions. Simple formulas allowing evaluation for the case of one fundamental modulation harmonics are derived. This class of lattices has already found wide application in various international accelerator centers.

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