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DERIVATION OF THE FIRST-ORDER TRANSFORMATION MATRIX FOR A SIMPLE WIEN FILTER AND COMPARISON TO RESULTS OF NUMERICAL INTEGRATION

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The first-order transformation matrix is derived for a simple Wien filter. The Wien filter is approximated by square-edged, homogeneous, transverse E and B fields. The results are compared to results of numerical integration through a Wien filter with and without fringe fields. The derived transformation matrix is presently used in the first-order optics code TRACE to tune the 750 keV polarized proton injection transport at LAMPF. The Wien filter is used to precess the spin of the polarized proton beam.

1. Introduction

One of the elements of the 750 keV polarized H^- injection transport at LAMPF is a Wien ($E \times B$) filter. The Wien filter is used to precess the spin of the polarized H^- ions to the appropriate orientation needed by the experimenters. The proton-spin orientation is changed several times during a production cycle. At each new setting of the Wien filter, the beam is focused differently and the beam transport must be retuned for an optimum match into the linac. The transport is tuned interactively using the first-order optics code TRACE. The first-order transformation matrix for a simple Wien filter is developed for use in TRACE, and the transformation is compared to results of numerical integration to determine the validity of the first-order approximation.

2. Derivation of first-order transformation matrix

The fields of the simple Wien filter are assumed to be uniform, square-edged fields, with the z component of the E field implicitly represented by a delta function at the edges of the filter. The fields inside the filter are transverse electric and magnetic fields and are perpendicular to each other. The magnitude and direction of the E field is such that is cancels the effect of the B field on the synchronous particle. The synchronous particle's path is the center axis of the filter; particles either off-axis or off-momentum are deflected by the Wien filter. The equations of motion relative to the synchronous particle are derived, and the first-order solutions are found. These solutions are in a format compatible with particle optics codes such as TRACE.

2.1. Coordinate system and units

The beam axis is along the z-axis of a right-handed Cartesian coordinate system with the origin at the entrance of the filter. Gaussian units are used. The variables used to describe a particle are relative to the central trajectory along the z-axis of the synchronous particle. The general particle coordinates are

x, the x position,

$x' = \mathrm{d}x/\mathrm{d}z$, the angle in x relative to the z-axis,

y, the y position,

$y' = \mathrm{d}y/\mathrm{d}z$, the angle in y relative to the z-axis, and $\Delta p/p_s = (p_0 - p_s)/p_s$, the momentum difference, where p_0 is the initial momentum of the particle and p_s is the momentum of the synchronous particle.

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2.2. Synchronous particle

The synchronous particle has momentum p_s, energy ϵ_s, and charge q. It enters the Wien filter on axis, i.e., $x = 0$, $x' = 0$, $y = 0$, and $y' = 0$. The fields of the filter are set such that the forces on the synchronous particle are zero, therefore

$$F = 0 = q(E + v/c \times B).$$

The B field is defined as $\boldsymbol{B} = B_0 \hat{y}$, therefore the E field is given by $\boldsymbol{E} = E_0 \hat{x}$, where

$$E_0 = (p_s c/\epsilon_s) B_0. \tag{1}$$

2.3. General incoming particle

The momentum and energy of the incoming particle before it reaches the filter ($z < 0$) are

$$p_0 c = p_s c(1 + \Delta p/p_s) \quad \text{and} \quad \epsilon_0 = \left[p_0^2 c^2 + m^2 c^4\right]^{1/2}. \tag{2}$$

At the entrance of the filter ($z = 0$), the particle receives an impulse that is dependent on its distance from the E-field plates. Taking the potential at the origin as zero, the energy gain is

$$\epsilon' = \epsilon_0 + qE_0 x_0, \tag{3}$$

with the particle entering the filter with initial coordinates

$$x = x_0, \quad x' = x_0', \quad y = y_0, \quad \text{and } y' = y_0'. \tag{4}$$

The momentum is given by

$$p'c = \left[\epsilon'^2 - m^2 c^4\right]^{1/2} = p_z' c\left[1 + x'^2 + y'^2\right]^{1/2}. \tag{5}$$

2.4. Equations of motion

As the particle travels through the filter, ($0 < z < L$), the energy varies as x:

$$\epsilon = \epsilon_0 + qE_0 x = \epsilon_0 + q(p_s c/\epsilon_s) B_0 x. \tag{6}$$

The momentum is given by

$$pc = \left[\epsilon^2 - m^2 c^4\right]^{1/2} = p_z c\left[1 + (\mathrm{d}x/\mathrm{d}z)^2 + (\mathrm{d}y/\mathrm{d}z)^2\right]^{1/2}. \tag{7}$$

The z component of force on the particle is

$$F_z = qB_0 v_x/c.$$

The force can be written

$$F_z = \frac{\mathrm{d}}{\mathrm{d}t} p_z = \frac{p_x c}{\epsilon} \frac{\mathrm{d}}{\mathrm{d}x} p_z c = \frac{p_x c}{\epsilon} qB_0.$$

Solving the differential equation gives

$$p_z c = qB_0 x + C,$$

where C is a constant. Using the initial conditions to solve for C, this becomes

$$p_z c = p_z' c + qB_0(x - x_0). \tag{8}$$

Eqs. (1), (5), (7), and (8) are used to derive

$$\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 = x_0'^2 + \frac{(\epsilon_0 + \beta_s qB_0 x)^2 - m^2 c^4}{(p_z' c + qB_0(x - x_0))^2} - \left(\frac{p'c}{p_z' c}\right)^2.$$

This equation can be expanded in a Taylor's series about $x = x_0$ giving

$$\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 = x_0'^2 - 2\frac{(p'c)^2 - \beta_s\epsilon' p_z'c}{(p_z'c)^2}\left(\frac{qB_0}{p_z'c}\right)(x - x_0)$$
$$+\left[\frac{\beta_s^2(p_z'c)^2 - 4\beta'\epsilon_s p_z'c + 3(p'c)^2}{(p_z'c)^2}\right]\left(\frac{qB_0}{p_z'c}\right)^2(x - x_0)^2 + \sum_{m=3}^{\infty}(-1)^m\left[(m-1)\beta_s^2 - 2m\frac{\beta_s\epsilon'}{p_z'c}\right.$$
$$\left. + (m+1)\left(\frac{p'c}{p_z'c}\right)^2\right]\left(\frac{qB_0}{p_z'c}\right)^m(x - x_0)^m. \tag{9}$$

Then for $p'c \sim p_z'c$, $\beta_s\epsilon' \sim p'c'$, and $qB_0/(p'c) \ll 1$ (at LAMPF $qB_0/(p'c) = 0.0067$) eq. (9) can be approximated by truncation to second order in x:

$$\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 \simeq x_0'^2 - 2\left[\frac{(p'c)^2 - \beta_s\epsilon' p_z'c}{(p_z'c)^2}\right]\left(\frac{qB_0}{p_z'c}\right)(x - x_0)$$
$$-\left[\frac{4\beta_s\epsilon' p_z'c - 3(p'c)^2 - \beta_s^2(p_z'c)^2}{(p_z'c)^2}\right]\left(\frac{qB_0}{p_z'c}\right)^2(x - x_0)^2.$$

The solution to the above differential equation, satisfying the boundary condition at $z = 0$, is

$$x = x_0 - \left[\frac{(p'c)^2 - \beta_s\epsilon' p_z'c}{4\beta_s\epsilon' p_z'c - 3(p'c)^2 - \beta_s^2(p_z'c)^2}\right]\left(\frac{p_z'c}{qB_0}\right)(1 - \cos(\kappa z)) + \frac{x_0'}{\kappa}\sin(\kappa z), \tag{10}$$

where κ is given by

$$\kappa^2 = \left[\frac{4\beta_s\epsilon' p_z'c - 3(p'c)^2 - \beta_s^2(p_z'c)^2}{(p_z'c)^2}\right]\left(\frac{qB_0}{p_z'c}\right).$$

2.5. Expansion to first order

Eq. (10) is expanded in a Maclaurin's series about $x_0 = 0$, $x_0' = 0$, and $\Delta p/p_s = 0$ giving

$$x = x_0\cos(\kappa_0 z) + x_0'\frac{1}{\kappa_0}\sin(\kappa_0 z) - \frac{\Delta p}{p_s}\frac{1}{\gamma_s\kappa_0}(1 - \cos(\kappa_0 z)), \tag{11}$$

where κ_0 is given by

$$\kappa_0^2 = \frac{1}{\gamma_s^2}\left(\frac{qB_0}{p_s c}\right)^2.$$

Differentiating the equation for x with respect to z gives the expression for x':

$$x' = -x_0\kappa_0\sin(\kappa_0 z) + x_0'\cos(\kappa_0 z) - \frac{\Delta p}{p_s}\frac{1}{\gamma_s}\sin(\kappa_0 z). \tag{12}$$

Since the force in the y direction is zero, the equations of motion are simply

$$y = y_0 + y_0' z, \quad \text{and} \quad y' = y_0'. \tag{13}$$

The final energy of the particle (for $z > L$) is the same as the initial energy of the particle (for $z < 0$); therefore, the total momentum is a constant and

$$\Delta p/p_s = \Delta p_0/p_s. \tag{14}$$

The first order transformation matrix given by eqs. (11)–(14) is

$$\mathbf{R} = \begin{bmatrix} \cos(\kappa_0 L) & \dfrac{1}{\kappa_0}\sin(\kappa_0 L) & 0 & 0 & -\dfrac{1}{\gamma_s \kappa_0}(1-\cos(\kappa_0 L)) \\ -\kappa_0 \sin(\kappa_0 L) & \cos(\kappa_0 L) & 0 & 0 & -\dfrac{1}{\gamma_s}\sin(\kappa_0 L) \\ 0 & 0 & 1 & L & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\mathbf{R}$ transforms the particle initial coordinates at $z = 0$ to the final coordinates at $z = L$ by

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ \dfrac{\Delta p}{p_s} \end{bmatrix} = \mathbf{R} \begin{bmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ \dfrac{\Delta p_0}{p_s} \end{bmatrix}.$$

3. Comparison with results of numerical integration

The particle equations of motion are integrated numerically and the results are compared to the derived first-order transformation matrix. The calculations are done for a Wien filter with square-edged fields and a Wien filter with fringe fields.

The integration used 1000 input particles with a Gaussian distribution. Fig. 1 shows the input particles plotted in one of the phase-space planes. The input particles are parametrized by the Courant–Snyder parameters α, β, and γ, where $\beta\gamma - \alpha^2 = 1$. E is the root-mean-square emittance of the beam. ξ is the root-mean-square width of the momentum spread. The phase-space distribution is given by:

$$f(\Delta p/p_s) = \frac{1}{\sqrt{2\pi}\,\xi} e^{-(\Delta p/p)^2/2\xi^2},$$

$$f(x, x') = \frac{1}{2\pi E_x} e^{-(\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2)/2E_x},$$

$$f(y, y') = \frac{1}{2\pi E_y} e^{-(\gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2)/2E_y}.$$

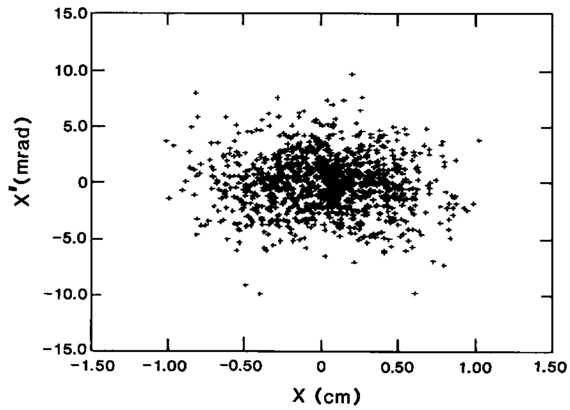

Fig. 1. Input particle distribution used for the integration of the equations of motion.

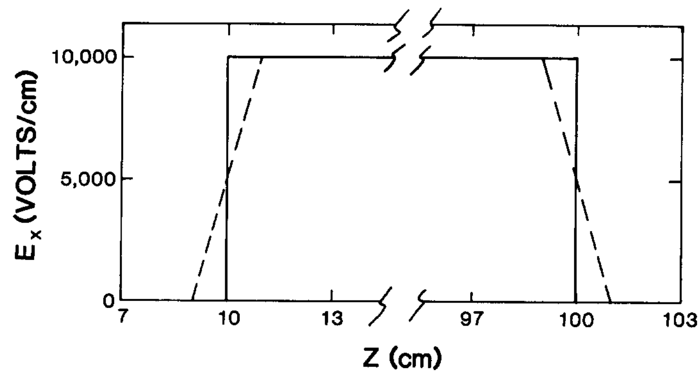

Fig. 2. The E_x fields used to model the Wien filter, square-edged (solid line) and with fringe fields (dashed line).

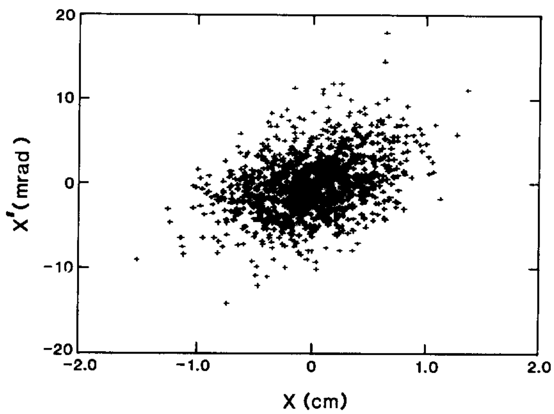

Fig. 3. Output particle distribution results from integration of the equations of motion.

Table 1
The Wien filter transformation matrices obtained from numerically integrated equations of motion and from the derived solution of the equations of motion (units are cm and mrad)

(a) Numerical integration with square-edged fields

0.787	0.101	0.000	0.000	−31.88
−3.766	0.789	0.000	0.000	−564.2
0.000	0.000	1.000	0.110	0.000
0.000	0.000	0.000	1.000	0.000
0.000	0.000	0.000	0.000	1.000

(b) Numerical integration with fringe fields

0.789	0.101	0.000	0.000	−31.87
−3.741	0.790	0.000	0.000	−564.3
0.000	0.000	1.000	0.110	0.000
0.000	0.000	0.001	1.000	0.000
0.000	0.000	0.000	0.000	1.000

(c) Evaluation of derived transformation matrix

0.788	0.101	0.000	0.000	−31.81
−3.761	0.788	0.000	0.000	−564.0
0.000	0.000	1.000	0.110	0.000
0.000	0.000	0.000	1.000	0.000
0.000	0.000	0.000	0.000	1.000

The fields used for the integration approximate the fields of the Wien filter used at LAMPF. The lengths of the fields are chosen such that $\int B_y \, dz$ is the same for both the case with fringe fields and the case without fringe fields. Fig. 2 shows the E_x field for both cases. For the fringe-field case, there are also E_z and B_z field components such that Maxwell's equations are satisfied.

An example of the particle phase-space output resulting from the integration is shown in fig. 3. To compare these results to the derived first-order transformation, the first-order transformation between the input and output particles is derived using a least-squares fitting method. This method can be extended to calculate higher-order transformation terms. Table 1 shows the numerical values for the derived matrix and the results of the fit to the integration for the two field cases. All three transformations are in good agreement.

4. Conclusion

A first-order transformation matrix for the transport of beam through a Wien filter is derived. The derivation assumes fields with square edges. The results of numerical integration are compared to the derived transformation. The comparison gives an estimate of the accuracy of the derived transformation.

Models for general optical elements in particle-beam transports can be derived and validated in similar fashion. First, the equations of motion are solved to first or higher order. The equations of motion can be numerically integrated through theoretical or measured fields of the element, and the integration results can be fit to a first- or higher-order transformation. The transformations from both the derivation and the integration are compared to give an indication of accuracy. The validated transformation can be incorporated into optics codes such as TRACE and used to design and tune particle-beam transports.