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INTRABEAM SCATTERING

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We calculate the contribution to emittance growth rates due to Coulomb scattering of particles within relativistic beams such as those found in colliders and accumulator rings. We allow for the variation of lattice parameters around the ring, which is the case for typical strong-focusing lattices presently being studied. We find that the emittance growth corresponds to a tendency of the beam rest-frame momentum space to relax to a spherical shape. Finally, we apply our results to the Antiproton Accumulator and Energy Saver lattices currently being built at Fermilab.

I. INTRODUCTION

In anticipation of the existence of intense stored relativistic antiproton and proton beams in colliders and accumulator rings, there is a need to understand fully the effects of intrabeam scattering. This is already a rather well-understood subject. The most definitive published work appears to be that of Piwinski,[1] which treats the couplings of the various degrees of freedom in the phase space. However, it does not include the most general situation encountered in a strong-focusing lattice. This generalization was considered subsequently by Sacherer and Möhl, and a computer code which embodies those modifications does exist at CERN. After this work was completed, we were informed by Piwinski that he also worked out the general formulae in 1979. Meanwhile, we also worked out these results using somewhat different techniques. Throughout these derivations we set $\hbar = c = 1$. Our approach, which we believe to be reasonably simple and easy to apply, provides some insights into the nature of the old results and how the addition of a strong-focusing lattice affects them. The main conclusions are as follows:

1. In general, the total six-dimensional beam emittance will grow. For bunched beams, the growth rate is given by the simple formula

$$\frac{1}{\tau} = \frac{\pi^2\alpha^2 MN\,(\log)}{\gamma\Gamma}\,\langle H(\lambda_1, \lambda_2, \lambda_3)\rangle, \tag{1.1}$$

where α is the fine structure constant, M is the particle mass, N is the number of particles per bunch (for an unbunched beam, N is the number of particles in the beam), log is a Coulomb logarithm which we take to be 20 throughout these calculations, γ is the Lorentz contraction factor, Γ is the total 6-dimensional rms invariant phase volume (for an unbunched beam replace Γ by $\Gamma/\sqrt{2}$), and $H(\lambda_1, \lambda_2, \lambda_3)$ is a "shape factor" which has the following properties:

a. H is dimensionless
b. H is homogeneous, i.e.

$$H(\alpha\lambda_1, \alpha\lambda_2, \alpha\lambda_3) = H(\lambda_1, \lambda_2, \lambda_3); \tag{1.2}$$

it therefore depends upon only two independent variables.
c. H has a one-dimensional integral representation; it is essentially the integral introduced by Piwinski.[1]

2. In the rest frame of the beam, in general, the beam momentum space (assuming Gaussian distributions) will be ellipsoidal in shape. The parameters $1/\sqrt{\lambda_1}$, $1/\sqrt{\lambda_2}$, $1/\sqrt{\lambda_3}$ measure the dimensions of the ellipsoid along its principal axes.
3. $H > 0$ unless $\lambda_1 = \lambda_2 = \lambda_3$, in which case the rest-frame distribution is spherical. In that case $H = 0$; the small Coulomb scatterings leave the distribution—and the total phase volume—invariant. In all other cases, there is emittance growth, along with a tendency for the beam momentum space to relax to a spherical shape.

The symbol $\langle\ \rangle$ denotes an average over positions of the beam around the ring.

It is easy to show that whenever the combination of lattice functions $\phi \equiv \eta' - \eta\beta_x'/2\beta_x$ does not vanish, the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ cannot be all equal; hence H will not vanish. In general, the condition $\phi = 0$ will not be met at almost all points around a strong-focusing lattice; hence the emittance will always grow. One can only cut the losses, not eliminate them. This is true both below and above the transition energy.

Formulae, only sightly more complicated, exist for growth rates for the individual horizontal, vertical, and longitudinal emittances. They are given below.

II. CALCULATION OF FORMULAE

Following Piwinski, we consider a Gaussian phase-space distribution for the beam. We find it convenient to use the canonical phase-space distribution

$$\rho(x, p) = \frac{N}{\Gamma} e^{-S(x,p)}, \tag{2.1}$$

with

$$S(x, p) = \frac{1}{2} A_{ij}\, \delta p_i\, \delta p_j + B_{ij}\, \delta p_i\, \delta x_j + \frac{1}{2} C_{ij}\, \delta x_i\, \delta x_j \tag{2.1a}$$

$$\Gamma = \int d^3x\, d^3p\, e^{-S(x,p)}, \tag{2.1b}$$

where $\delta\mathbf{p}$ and $\delta\mathbf{x}$ are the momentum and coordinate deviations from the reference values $\mathbf{p}$ and $\mathbf{x}$ and N is the number of particles in the beam. For bunched beams, we consider only one bunch in the ring.

To set our notation, the expression for S is

$$S(x, p) = S^{(h)} + S^{(v)} + S^{(l)}, \tag{2.2}$$

where

$$S^{(h)} = \frac{\beta_x}{2\epsilon_x} x_\beta'^2 - \frac{\beta_x'}{2\epsilon_x} x_\beta x_\beta' + \frac{1}{2\epsilon_x\beta_x}\left(1 + \frac{\beta_x'^2}{4}\right) x_\beta^2, \tag{2.2a}$$

$$S^{(v)} = \frac{\beta_z}{2\epsilon_z} z'^2 - \frac{\beta_z' z z'}{2\epsilon_z} + \frac{z^2}{2\epsilon_z\beta_z}, \tag{2.2b}$$

$$S^{(l)} = \begin{cases} \dfrac{\delta^2}{2\sigma_\eta^2} & \text{(for an unbunched beam),} \\ \dfrac{\delta^2}{\sigma_\eta^2} + \dfrac{(s - \bar{s})^2}{2\sigma_s^2} & \text{(for a bunched beam),} \end{cases} \tag{2.2c}$$

where β_x and β_z are the horizontal and vertical betatron functions, ϵ_x and ϵ_z the rms beam emittances, σ_x, σ_z, σ_s, and σ_p the rms beam width, height, bunched-beam length, and momentum spread, and

$$\epsilon_x = \frac{\sigma_x^2}{\beta_x}, \tag{2.2d}$$

$$\epsilon_z = \frac{\sigma_z^2}{\beta_z}, \tag{2.2e}$$

$$\sigma_\eta = \frac{\sigma_p}{\bar{p}}, \tag{2.2f}$$

$$x_\beta = x - \eta(s)\delta, \tag{2.2g}$$

$$x_\beta' = x' - \eta'(s)\delta, \tag{2.2h}$$

where $\eta(s)$ is the momentum dispersion function and

$$x' \equiv \frac{\delta p_x}{\bar{p}}, \tag{2.2i}$$

$$z' \equiv \frac{\delta p_z}{\bar{p}}, \tag{2.2j}$$

$$\delta \equiv \frac{\delta p_s}{\bar{p}} \tag{2.2k}$$

The relativistic "Golden Rule" for the transition rate due to a two-body scattering process $p_1 + p_2 \to p_1' + p_2'$ can be written[2] in the form:

$$\frac{d\mathcal{P}}{dt} = \frac{1}{2}\int d^3x \frac{d^3p_1}{\gamma_1}\frac{d^3p_2}{\gamma_2}\,\rho(x, p_1)\,\rho(x, p_2)\,|\mathcal{M}|^2 \cdot \frac{d^3p_1'}{\gamma_1'}\frac{d^3p_2'}{\gamma_2'}\frac{\delta^{(4)}(p_1' + p_2' - p_1 - p_2)}{(2\pi)^2}, \tag{2.3}$$

where $\mathcal{M}$ is the invariant Coulomb scattering amplitude given by

$$\mathcal{M} = \frac{4\pi\alpha}{q^2}, \tag{2.3a}$$

with the 4-momentum transfer

$$q_u = (p_1' - p_1)_\mu, \tag{2.3b}$$

and

$$q^2 \equiv -Q^2 < 0. \tag{2.3c}$$

In Eq. (2.3), the factor 1/2 is what is left over from several factors of 2 associated with particles 1 and 2 being identical both before and after collision. We have assumed that small-angle scattering is dominant and omitted the exchange (interference) term in $|\mathcal{M}|^2$.

We will be interested in the rate of change of emittances or other functions $f(\bar{\mathbf{p}} + \delta\mathbf{p})$ of momentum deviations. The rate equation is given by

$$\begin{aligned}\frac{d\langle f(p)\rangle}{dt} &= \frac{N}{2\Gamma^2}\int d^3x \frac{d^3p_1}{\gamma_1}\frac{d^3p_2}{\gamma_2}\, e^{-S(x,p_1)-S(x,p_2)} \\ &\quad\times |\mathcal{M}|^2\,[f(p_1') - f(p_1) + f(p_2') - f(p_2)] \\ &\quad\times \frac{d^3p_1'}{\gamma_1'}\frac{d^3p_2'}{\gamma_2'}\frac{\delta^{(4)}(p_1' + p_2' - p_1 - p_2)}{(2\pi)^2}.\end{aligned} \tag{2.4}$$

But note that

$$\langle f(p_1') - f(p_1)\rangle = \langle f(p_2') - f(p_2)\rangle, \tag{2.5}$$

so that we can rewrite Eq. (2.4) as

$$\begin{aligned}\frac{d\langle f(p)\rangle}{dt} &= \frac{N}{\Gamma^2}\int d^3x \frac{d^3p_1}{\gamma_1}\frac{d^3p_2}{\gamma_2}\, e^{-S(x,p_1)-S(x,p_2)}\,|\mathcal{M}|^2\,[f(p_1') - f(p_1)] \\ &\quad\times \frac{d^3p_1'}{\gamma_1'}\frac{d^3p_2'}{\gamma_2'}\cdot\frac{\delta^{(4)}(p_1' + p_2' - p_1 - p_2)}{(2\pi)^2}.\end{aligned} \tag{2.6}$$

We are interested in the behavior of small momentum fluctuations about some mean value. In the last integral, we write the 4-vectors

$$p_1' = p_1 + q \tag{2.7}$$

and expand f to second order

$$f(\mathbf{p}_1') - f(\mathbf{p}_1) = q_i \frac{\partial f}{\partial p_i}\bigg|_{p=p_1} + \frac{q_i q_j}{2} \frac{\partial^2 f}{\partial p_i \partial p_j}\bigg|_{p=p_1} + \cdots. \tag{2.8}$$

The momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ can be expanded about the central value $\bar{\mathbf{p}}$ (we will often ignore the bar):

$$\mathbf{p}_1 = \bar{\mathbf{p}} + \frac{\boldsymbol{\xi}}{2} + \boldsymbol{\Delta} = \bar{\mathbf{p}} + \boldsymbol{\delta}_1, \tag{2.9}$$

$$\mathbf{p}_2 = \bar{\mathbf{p}} + \frac{\boldsymbol{\xi}}{2} - \boldsymbol{\Delta} = \bar{\mathbf{p}} + \boldsymbol{\delta}_2, \tag{2.10}$$

where $\boldsymbol{\xi} = \mathbf{p}_1 + \mathbf{p}_2 - 2\bar{p}$ and $\boldsymbol{\Delta} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)$.
To second order in all small quantities, we have

$$f(\mathbf{p}_1') - f(\mathbf{p}_1) = q_i\left(\frac{\xi_j}{2} + \Delta_j + \frac{q_j}{2}\right)\frac{\partial^2 f}{\partial p_i \partial p_j}\bigg|_{\bar{p}} + \cdots. \tag{2.11}$$

The factors S can be simplified as well. Note that

$$\begin{aligned} S(\mathbf{x}, \boldsymbol{\delta}_1) + S(\mathbf{x}, \boldsymbol{\delta}_2) &= \frac{1}{2} A_{ij} \frac{\xi_i \xi_j}{2} + B_{ij} x_i \xi_j + C_{ij} x_i x_j + A_{ij}\, \Delta_i \Delta_j \\ &= S\left(\sqrt{2}\mathbf{x}, \frac{\boldsymbol{\xi}}{\sqrt{2}}\right) + \tilde{S}(\boldsymbol{\Delta}), \end{aligned} \tag{2.12}$$

with

$$\tilde{S}(\boldsymbol{\Delta}) = A_{ij}\, \Delta_i \Delta_j. \tag{2.13}$$

Thus we can rewrite Eq. (2.6) as follows

$$\begin{aligned} \frac{d\langle f\rangle}{dt} = \frac{4\alpha^2 N}{\gamma^2 \Gamma^2} \int d^3x\, d^3\xi\, d^3\Delta e^{-S(\sqrt{2}x, \xi/\sqrt{2}) - \tilde{S}(\Delta)} \cdot \int \frac{d^3 p_1'}{\gamma_1'} \frac{d^3 p_2'}{\gamma_2'} \frac{q_i\left(\frac{\xi_j}{2} + \Delta_j + \frac{q_j}{2}\right)}{(q^2)^2} \\ \times \frac{\partial^2 f}{\partial p_i \partial p_j}\bigg|_{\bar{p}} \cdot \delta^{(4)}(p_1' + p_2' - W), \end{aligned} \tag{2.14}$$

where we have introduced the 4-vector

$$W_\mu = (p_1 + p_2)_\mu, \tag{2.14a}$$

which with

$$\Delta_\mu = \frac{1}{2}(p_1 - p_2)_\mu \tag{2.14b}$$

[cf Eqs. (2.9) and (2.10)], has the properties

$$\Delta \cdot W = 0, \tag{2.14c}$$

$$\Delta^2 = -|\Delta|^2 < 0, \tag{2.14d}$$

$$W^2 + 4\Delta^2 = 4M^2. \tag{2.14e}$$

The x and ξ integrations can be done immediately, leaving

$$\frac{d\langle f\rangle}{dt} = \frac{4\alpha^2 N}{\gamma^2 \tilde{\Gamma}} \int d^3\,\Delta\, e^{-\mathfrak{S}(\Delta)} \frac{\partial^2 f}{\partial p_i \partial p_j} \left\{ I_i\, \Delta_j + \frac{I_{ij}}{2} \right\}, \tag{2.15}$$

with

$$\tilde{\Gamma} = \begin{cases} \Gamma & \text{for bunched beams,} \\ \dfrac{\Gamma}{\sqrt{2}} & \text{for unbunched beams.} \end{cases} \tag{2.16}$$

where we must do the integrals

$$I_\mu = \int \frac{d^3 p_1{}'}{\gamma_1{}'} \frac{d^3 p_2{}'}{\gamma_2{}'} \frac{q_\mu}{q^4} \delta^{(4)}(p_1{}' + p_2{}' - p_1 - p_2), \tag{2.17a}$$

$$I_{\mu\nu} = \int \frac{d^3 p_1{}'}{\gamma_1{}'} \frac{d^3 p_2{}'}{\gamma_2{}'} \frac{q_\mu q_\nu}{q^4} \delta^{(4)}(p_1{}' + p_2{}' - p_1 - p_2). \tag{2.17b}$$

The Lorentz tensor structure of the integrals allows for a reasonably effortless evalution. The integrals depend upon the two 4-vectors W_μ and Δ_μ. In the center-of-mass system (cms) of the collision shown in Fig. 1, we have

$$\Delta_0 = 0, \tag{2.18a}$$

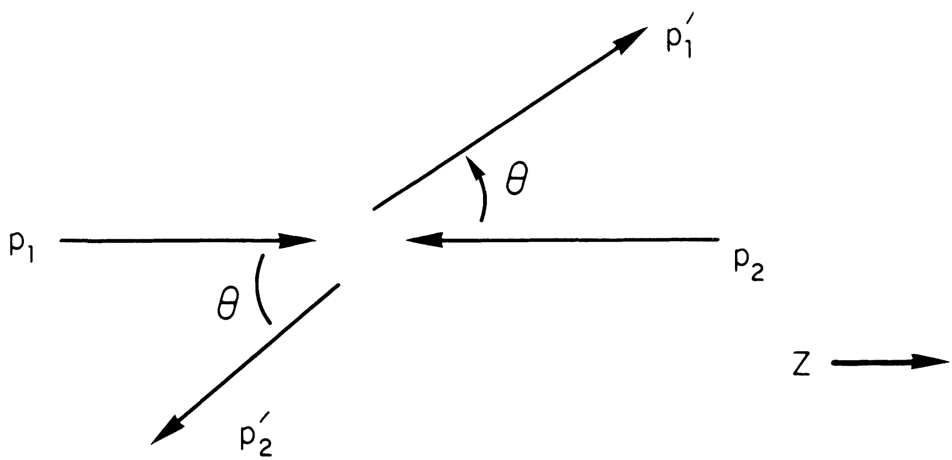


FIGURE 1. Scattering between two beam particles in their center-of-mass system.

$$I_0 = 0, \tag{2.18b}$$

$$I_{00} = 0, \tag{2.18c}$$

$$I_{0i} = 0. \tag{2.18d}$$

Thus

$$I_\mu = \Delta_\mu I(\Delta^2), \tag{2.19a}$$

$$I_{\mu\nu} = \left(-g_{\mu\nu} + \frac{W_\mu W_\nu}{W^2} - \frac{\lambda(\Delta^2)\Delta_\mu\Delta_\nu}{|\Delta^2|}\right)\tilde{I}(\Delta^2), \tag{2.19b}$$

and we need only to obtain I, λ, $\tilde{I}$. Using nonrelativistic kinematics in the cms, we get

$$I_3 = \Delta_3 I(\Delta^2) = \int \frac{d^3p_1' \, p(\cos\theta - 1)\delta(2E_1' - W)}{[2p^2(1 - \cos\theta)]^2}, \tag{2.20}$$

where p is just the magnitude of the 3-momenta in the cms. Using

$$d^3p_1' = 2\pi p_1' E_1' \, dE_1' \, d(\cos\theta),$$

we find

$$pI(\Delta^2) \simeq -\pi M \int d(\cos\theta) \frac{1}{4p^2(1 - \cos\theta)}, \tag{2.21}$$

and finally

$$I(\Delta^2) \simeq \frac{-\pi M}{4|\Delta^2|^{3/2}} \ln \frac{4}{\theta_{\min}^2}. \tag{2.22}$$

Now

$$I_{33} \sim \int d(\cos\theta) \frac{(1 - \cos\theta)^2}{(1 - \cos\theta)^2} \tag{2.23}$$

has no logarithm. Let us approximate it by zero. This means

$$\lambda = 1. \tag{2.24}$$

Finally,

$$\begin{aligned} I_{11} &= \tilde{I}(\Delta^2) \\ &= \int d^3p_1' \frac{\delta(2E_1' - W)\, p^2 \sin^2\theta \cos^2\phi}{[2p^2(1 - \cos\theta)]^2} \\ &\simeq \frac{\pi M}{4|\Delta^2|^{1/2}} \ln \frac{4}{\theta_{\min}^2}. \end{aligned} \tag{2.25}$$

The minimum θ occurs at maximum impact parameter $b_{\max}$. From the old argument

$$\Delta p_\perp = e\int E_\perp\, dt = \frac{e}{\beta}\int E_\perp\, dx = \frac{e^2}{2\pi\beta b} = \frac{2\alpha}{b\beta}, \tag{2.26}$$

or,

$$\theta_{\min} \simeq \frac{2\alpha}{b_{\max}\beta p} \simeq \frac{2M\alpha}{b_{\max}p^2}. \tag{2.27}$$

We define the Coulomb logarithm

$$\log \equiv \ln\frac{b_{\max}|\Delta^2|}{M\alpha} = \ln\frac{b_{\max}}{b_{\min}} = \ln\frac{2}{\theta_{\min}}. \tag{2.28}$$

It is a rather large number and we ignore the variation with $|\Delta^2|$. We further note that if one wants to go beyond the leading log approximation, one needs to keep the exchange term in the elastic scattering amplitude $\mathscr{M}$. Thus we have in any frame

$$I_\mu = \frac{-\pi M}{2|\Delta^2|^{3/2}}\Delta_\mu \log, \tag{2.29a}$$

$$I_{\mu\nu} = \left(-g_{\mu\nu} + \frac{W_\mu W_\nu}{W^2} - \frac{\Delta_\mu\Delta_\nu}{|\Delta^2|}\right)\frac{\pi M}{2|\Delta^2|^{1/2}}\log. \tag{2.29b}$$

Next we must evaluate I_μ and $I_{\mu\nu}$ in the laboratory frame. But first we note that to the requisite order in small quantities,

$$\frac{\mathbf{W}}{\sqrt{W^2}} = \boldsymbol{\beta}\gamma, \tag{2.30}$$

where $\boldsymbol{\beta}$ is the mean particle velocity in the laboratory frame. Therefore,

$$\begin{aligned} I_i\Delta_j + \frac{1}{2}I_{ij} &= \frac{\pi M\log}{4|\Delta^2|^{1/2}}\left\{-\frac{2\Delta_i\Delta_j}{|\Delta^2|} + \left(\delta_{ij} + \gamma^2\beta_i\beta_j - \frac{\Delta_i\Delta_j}{|\Delta^2|}\right)\right\} \\ &= \frac{\pi M\log}{4|\Delta^2|^{1/2}}\left\{\delta_{ij} - \frac{3\Delta_i\Delta_j}{|\Delta^2|} + \gamma^2\beta_i\beta_j\right\}. \end{aligned} \tag{2.31}$$

To evaluate $|\Delta^2|$ in the laboratory frame, we have

$$|\Delta^2| = |\boldsymbol{\Delta}|^2 - \Delta_0{}^2. \tag{2.32}$$

However,

$$\Delta\cdot W = \Delta_0 W_0 - \boldsymbol{\Delta}\cdot\mathbf{W} = 0$$

implies

$$\Delta_0 = \frac{\boldsymbol{\Delta}\cdot\mathbf{W}}{W_0} = \boldsymbol{\Delta}\cdot\boldsymbol{\beta}, \tag{2.33}$$

and therefore

$$|\Delta^2| = |\mathbf{\Delta}|^2 - |\mathbf{\Delta}\cdot\boldsymbol{\beta}|^2. \tag{2.34}$$

Thus, we arrive at

$$\frac{d\langle f\rangle}{dt} = \frac{\pi M\alpha^2 N(\log)}{\gamma^2\tilde{\Gamma}}\int \frac{d^3\Delta\, e^{-\tilde{S}(\Delta)}}{\sqrt{|\mathbf{\Delta}|^2 - |\mathbf{\Delta}\cdot\boldsymbol{\beta}|^2}}\cdot\frac{\partial^2 f}{\partial p_i \partial p_j}\left\{\delta_{ij} + \gamma^2\beta_i\beta_j - \frac{3\Delta_i\Delta_j}{[|\mathbf{\Delta}|^2 - |\mathbf{\Delta}\cdot\boldsymbol{\beta}|^2]}\right\}. \tag{2.35}$$

Now we change variables to

$$\Delta_x = \frac{1}{2}p\theta_x, \tag{2.36a}$$

$$\Delta_y = \frac{\gamma}{2}p\theta_y, \tag{2.36b}$$

$$\Delta_z = \frac{1}{2}p\theta_z. \tag{2.36c}$$

This accounts for the longitudinal Lorentz contraction associated with the transformation to the beam rest frame and helps establish correspondence with the Piwinski calculation.[1] Now we write

$$\tilde{S}(\mathbf{\Delta}) = A_{ij}\,\Delta_i\,\Delta_j = \frac{\theta_i\theta_j}{4}L_{ij}, \tag{2.37}$$

with

$$L_{ij} = L_{ij}^{(h)} + L_{ij}^{(l)} + L_{ij}^{(v)}, \tag{2.37a}$$

and

$$L^{(h)} = \frac{\beta_x}{\epsilon_x}\begin{pmatrix} 1 & -\gamma\phi & 0 \\ -\gamma\phi & \dfrac{\gamma^2\eta^2}{\beta_x{}^2} + \gamma^2\phi^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.37b}$$

$$L^{(l)} = \begin{cases} \dfrac{\gamma^2}{\sigma_\eta{}^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{(unbunched)}, \\[2ex] \dfrac{2\gamma^2}{\sigma_\eta{}^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{(bunched)}, \end{cases} \tag{2.37c}$$

$$L^{(v)} = \frac{\beta_z}{\epsilon_z}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.37d}$$

where

$$\phi = \eta' - \frac{\beta_x'\eta}{2\beta_x}. \tag{2.37e}$$

These matrices are all that is needed to characterize the emittance growth rates.

Defining a kernel K_{ij} as

$$K_{ij} = \frac{\pi\alpha^2 MN\,(\log)}{4\gamma\tilde{\Gamma}}\int \frac{d^3\theta\, e^{-\theta_i\theta_j L_{ij}/4}}{(\theta_x{}^2 + \theta_y{}^2 + \theta_z{}^2)^{3/2}}(\delta_{ij}\theta^2 - 3\theta_i\theta_j), \tag{2.38}$$

we derive the diffusion equation

$$\frac{d\langle f\rangle}{dt} = K_{ij}[D_{ij}f], \tag{2.39}$$

with

$$[Df] = p^2\begin{pmatrix} \dfrac{\partial^2 f}{\partial p_x{}^2} & \gamma\dfrac{\partial^2 f}{\partial p_x \partial p_y} & \dfrac{\partial^2 f}{\partial p_x \partial p_z} \\ \gamma\dfrac{\partial^2 f}{\partial p_x \partial p_y} & \gamma^2\dfrac{\partial^2 f}{\partial p_y^2} & \gamma\dfrac{\partial^2 f}{\partial p_y \partial p_z} \\ \dfrac{\partial^2 f}{\partial p_x \partial p_z} & \gamma\dfrac{\partial^2 f}{\partial p_y \partial p_z} & \dfrac{\partial^2 f}{\partial p_z{}^2} \end{pmatrix}_{\bar{p}} \tag{2.40}$$

Again the powers of γ reflect the effect of the transformation into the rest frame of the beam, where the dynamics is simplest.

To further simplify the integral we perform the following steps:

(1) Write

$$\frac{1}{[\theta^2]^{3/2}} = \int_0^\infty \frac{d\lambda\,\lambda^{1/2}}{4\sqrt{\pi}} e^{-\lambda\theta^2/4}. \tag{2.41}$$

(2) Treat L_{ij} temporarily as a general matrix and write

$$\theta_i\theta_j e^{-(\theta_k L_{kl}\theta_l)/4} = -4\frac{\partial e^{-(\theta_k L_{kl}\theta_l)/4}}{\partial L_{ij}} \tag{2.42}$$

(3) Do the Gaussian integration

(4) Carry out the differentiation with the help of the identity

$$\frac{\partial}{\partial L_{ij}}\det(L + \lambda I) = \left(\frac{1}{L + \lambda I}\right)_{ij}\det(L + \lambda I). \tag{2.43}$$

Thus we are led to the basic formula for K_{ij}

$$K_{ij} = \frac{\pi^2\alpha^2 MN\,(\log)}{\gamma\tilde{\Gamma}} \int_0^\infty \frac{d\lambda\,\lambda^{1/2}}{[\det(L+\lambda I)]^{1/2}} \cdot \left\{\delta_{ij}\, T_r\left(\frac{1}{L+\lambda I}\right) - 3\left(\frac{1}{L+\lambda I}\right)_{ij}\right\}. \tag{2.44}$$

The diffusion coefficients K_{ij} depend only upon the shape of the momentum distribution as expressed by the tensor L_{ij}.

III. GROWTH RATES

Let us now consider various choices for the function f. An interesting one is energy. The original equation, the "Golden Rule," shows directly that $(d/dt)/\langle\sqrt{p^2+m^2}\rangle = 0$, leading to the traceless condition for K. This expresses the fact that kinetic energy is conserved in the Coulomb scattering process. However, in a strong-focusing lattice there is exchange of kinetic and potential energy, and thus this does not lead to an invariant of the motion.

Our main concern will be the emittance growths themselves. Since

$$S^{(a)}_{(x,p)} = \frac{1}{\epsilon_a}\left\{\frac{1}{2}\,\sigma^{(a)}_{ij}\,\delta p_i \delta p_j + \text{terms in } \delta x \delta p \text{ and in } (\delta x)^2\right\}, \tag{3.1}$$

where $a = h, v, l$ and the $\sigma^{(a)}_{ij}$ depend only upon lattice parameters and γ, then evidently

$$\begin{aligned}
\langle \epsilon_a S^{(a)}\rangle &= \sigma^{(a)}_{ij}\left\langle \frac{1}{2}\,\delta p_i \delta p_j + \text{terms in } \delta x\delta p \text{ and in } (\delta x)^2\right\rangle \\
&= \epsilon_a \frac{\displaystyle\int d^3x d^3p\, S^{(a)}\, e^{-S}}{\displaystyle\int d^3x d^3p\, e^{-S}} \\
&= -\epsilon_a\left[\frac{\partial}{\partial\lambda}\ln\int dx_a dp_a\, e^{-\lambda S^{(a)}}\right]_{\lambda=1} \\
&= \epsilon_a .
\end{aligned} \tag{3.2}$$

It follows that the emittance growth rate is

$$\frac{1}{\tau_a} = \frac{1}{\epsilon_a}\frac{d\epsilon_a}{dt} = \frac{1}{2}A^{(a)}_{ij}\frac{d}{dt}\langle \delta p_i \delta p_j\rangle. \tag{3.3}$$

We may now (i) rescale longitudinal momenta by the factor γ, (ii) replace the matrix $A^{(a)}_{ij}$ by $L^{(a)}_{ij}$, and finally, (iii) evaluate $(d/dt)/\langle\delta p_i \delta p_j\rangle$ in terms of the kernel K_{ij}, observing that all Lorentz contraction factors γ neatly cancel. We then find[3]

$$\frac{1}{\tau_a} = \sum_{ij} K_{ij} L_{ij}^{(a)} = \frac{\pi^2 \alpha^2 MN \,(\log)}{\gamma \tilde{\Gamma}} \left\langle \int_0^\infty \frac{d\lambda\, \lambda^{1/2}}{[\det(L + \lambda I)]^{1/2}} \right.$$
$$\left. \times \left\{ TrL^{(a)} Tr\left(\frac{1}{L + \lambda I}\right) - 3TrL^{(a)}\left(\frac{1}{L + \lambda I}\right)\right\}\right\rangle. \tag{3.4}$$

The brackets $\langle\cdots\rangle$ indicate that this must be averaged around the ring.

A simpler expression exists for the total growth rate, obtained by summation over $a = h, v, l$:[3]

$$\frac{1}{\tau} = \sum_a \frac{1}{\tau_a} = \frac{\pi^2 \alpha^2 MN \,(\log)}{\gamma \tilde{\Gamma}} \left\langle \int_0^\infty \frac{d\lambda\, \lambda^{1/2}}{[\det(L + \lambda I)]^{1/2}} \right.$$
$$\left. \times \left\{ Tr(L + \lambda I)\, Tr\left(\frac{1}{L + \lambda I}\right) - 9\right\}\right\rangle. \tag{3.5}$$

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ be the eigenvalues of the matrix L. Then

$$\frac{1}{\tau} = \frac{\pi^2 \alpha^2 MN \,(\log)}{\gamma \tilde{\Gamma}}$$
$$\times \left\langle (\lambda_1 - \lambda_2)^2 \int_0^\infty \frac{d\lambda\, \lambda^{1/2}}{(\lambda_1 + \lambda)^{3/2} (\lambda_2 + \lambda)^{3/2} (\lambda_3 + \lambda)^{1/2}} \right.$$
$$\left. + \text{ two cyclic permutations} \right\rangle. \tag{3.6}$$

As mentioned in the introduction, whenever $\phi \neq 0$ the eigenvalues λ_i cannot be all equal. This happens at almost all points around a strong-focusing lattice; thus the emittance will always grow. However, in many instances we have $\lambda_1 > \lambda_2 \simeq \lambda_3$. In such cases, we can analytically evaluate (see Appendix)

$$H \equiv \left[(\lambda_1 - \lambda_2)^2 \int_0^\infty \frac{d\lambda\, \lambda^{1/2}}{(\lambda_1 + \lambda)^{3/2} (\lambda_2 + \lambda)^{3/2} (\lambda_3 + \lambda)^{1/2}} \right.$$
$$\left. + \text{ two cyclic permutations} \right]$$
$$= \left[\frac{2(\lambda_1 + 2\lambda_2)}{\sqrt{\lambda_2(\lambda_1 - \lambda_2)}} \sin^{-1} \sqrt{\frac{\lambda_1 - \lambda_2}{\lambda_1}} - 6 \right]. \tag{3.7}$$

For the case $\lambda_1 > \lambda_2 > \lambda_3$, we can express the integrals in H in terms of elliptic integrals of the first and second kinds:[4]

$$F(\psi, k) = \int_0^\psi \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}, \tag{3.8a}$$

$$E(\psi, k) = \int_0^\psi \sqrt{1 - k^2 \sin^2 \alpha}\, d\alpha, \tag{3.8b}$$

where in our case

$$\psi = \arc \sin \sqrt{\frac{\lambda_1 - \lambda_3}{\lambda_1}}, \tag{3.9a}$$

$$k = \sqrt{\frac{\lambda_1(\lambda_2 - \lambda_3)}{\lambda_2(\lambda_1 - \lambda_3)}}. \tag{3.9b}$$

Through somewhat lengthy manipulations, we arrive at

$$\begin{aligned}
&\int_0^\infty \frac{d\lambda\, \lambda^{1/2}}{(\lambda_1 + \lambda)^{3/2}(\lambda_2 + \lambda)^{3/2}(\lambda_3 + \lambda)^{1/2}} \\
&\quad = \frac{2}{(\lambda_1 - \lambda_2)} \left\{ \frac{2}{(\lambda_1 - \lambda_2)} \left[\sqrt{\frac{\lambda_2}{\lambda_1 - \lambda_3}}\, E(\psi, k) - 1 \right] \right. \\
&\quad \left. + \frac{1}{\sqrt{\lambda_2(\lambda_1 - \lambda_3)}} \left[E(\psi, k) + \frac{\lambda_3}{\lambda_2 - \lambda_3} (E(\psi, k) - F(\psi, k)) \right] \right\},
\end{aligned} \tag{3.10a}$$

$$\begin{aligned}
&\int_0^\infty \frac{d\lambda\, \lambda^{1/2}}{(\lambda_1 + \lambda)^{3/2}(\lambda_3 + \lambda)^{3/2}(\lambda_2 + \lambda)^{1/2}} \\
&\quad = \frac{2}{(\lambda_1 - \lambda_2)} \sqrt{\frac{\lambda_2}{(\lambda_1 - \lambda_3)^3}} \left\{ E(\psi, k) \right. \\
&\quad \left. + \left(\frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_3} \right) [E(\psi, k) - F(\psi, k)] - \sqrt{\frac{\lambda_1 - \lambda_3}{\lambda_2}} \right\},
\end{aligned} \tag{3.10b}$$

$$\begin{aligned}
&\int_0^\infty \frac{d\lambda\, \lambda^{1/2}}{(\lambda_2 + \lambda)^{3/2}(\lambda_3 + \lambda)^{3/2}(\lambda_1 + \lambda)^{1/2}} \\
&\quad = \frac{2}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} \left\{ 1 + \left[\sqrt{\frac{\lambda_2}{\lambda_1 - \lambda_3}} - \frac{2\sqrt{\lambda_2(\lambda_1 - \lambda_3)}}{(\lambda_2 - \lambda_3)} \right] \cdot E(\psi, k) \right. \\
&\quad \left. + \frac{(\lambda_2 + \lambda_3)(\lambda_1 - \lambda_2)}{(\lambda_2 - \lambda_3)\sqrt{\lambda_2(\lambda_1 - \lambda_3)}} F(\psi, k) \right\}.
\end{aligned} \tag{3.10c}$$

These expressions are useful if one wants to evaluate $H(\lambda_1, \lambda_2, \lambda_3)$ without going to the computer, since many tables of elliptic integrals are available (e.g., see Ref. 4). We find that Eq. (3.10c) is not as useful as Eqs. (3.10a and b) because in Eq. (3.10c) the elliptic integrals must be known to high enough accuracy for certain careful cancellations to occur and give the correct result.

IV. APPLICATIONS

Now that we have expressions for $1/\tau$ and $1/\tau_a$, we can apply these results to accelerator rings, including those with varying lattice parameters such as the Antiproton Accumulator ($\bar{p}A$) and the Energy Saver at Fermilab.

A. Antiproton Accumulator

Two lattices denoted $\bar{p}A$ Lattice 1 and 2 which have been suggested for Fermilab's $\bar{p}A$ (shown in Fig. 2) are given in Tables I and II. The $\bar{p}A$ has a radius of 75.45 m. As usual, we take

$$\beta_x' = -2\alpha_x. \tag{4.1}$$

The sets of lattice parameter values shown are symmetrically repeated around the ring.

For the antiprotons in the $\bar{p}A$, we take $\gamma = 9.53$, corresponding to an energy of 8 GeV. If we take an average of the parameter values for $\bar{p}A$ Lattice 1 we have

$$\bar{\beta}_x = 23 \text{ m}, \tag{4.2a}$$

$$\bar{\beta}_z = 20 \text{ m}, \tag{4.2b}$$

$$\bar{\eta}_x = 4 \text{ m}, \tag{4.2c}$$

$$\bar{\beta}_x' = -0.03, \tag{4.2d}$$

$$\bar{\eta}_x' = -0.2, \tag{4.2e}$$

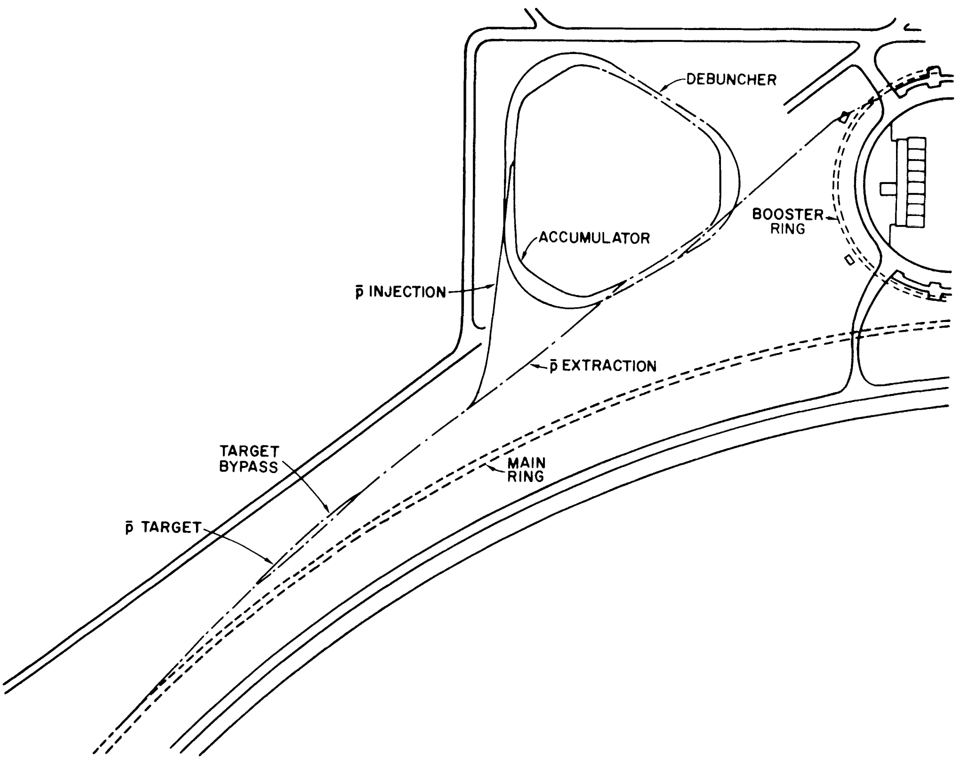


FIGURE 2. Fermilab antiproton accumulator.

TABLE I
($\bar{p}A$ LATTICE 1)

n	$\beta_x(n)$	$\beta_z(n)$	$\eta_x(n)$	$\alpha_x(n)$	$\eta_x'(n)$
1	16.06408	16.06449	8.38079	-.91442	0.00000
2	21.23496	15.68017	9.05335	−3.82147	1.18619
3	23.59398	14.95611	9.40920	−4.04191	1.18619
4	26.48412	14.30821	9.79566	−3.04559	.71863
5	42.39209	12.78990	11.43442	−3.93037	.71863
6	35.12411	20.70422	9.71488	6.57736	−2.20918
7	31.29110	23.50540	9.05213	6.19931	−2.20918
8	22.71608	30.64482	7.10634	.59576	−.69190
9	21.58421	30.51809	6.41445	.53611	−.69190
10	18.57151	29.84594	4.27707	.32821	−.53449
11	18.37995	29.76338	4.11672	.31031	−.53449
12	16.94134	28.53499	2.52800	.10241	−.37709
13	16.76557	27.89769	1.86809	−.00197	−.37709
14	17.07768	27.27316	1.69537	−.62588	−.31482
15	19.51782	23.69816	1.14443	−.76849	−.31482
16	25.86528	17.52494	.32141	−1.05255	−.15742
17	26.50414	17.05877	.27418	−1.07700	−.15742
18	35.00226	12.50054	−.00018	−1.36105	−.00001
19	35.82623	12.19578	−.00019	−1.38550	−.00001
20	34.08160	12.81345	−.00019	4.15174	.00002
21	18.52780	20.76027	−.00015	2.98566	.00002
22	16.81931	21.55618	−.00014	.72711	0.00000
23	11.00240	18.00078	−.00014	0.00000	0.00000
24	8.74872	8.75124	8.38079	0.00000	0.00000

n denotes lattice location.

Also, we take

$$\epsilon_x = \epsilon_z = 0.42 \times 10^{-6}\ \text{m}, \tag{4.3a}$$

$$\sigma_\eta = 1.2 \times 10^{-4}, \tag{4.3b}$$

$$I \equiv \text{Current} = 0.041\ \text{Amperes}, \tag{4.3c}$$

giving

$$\frac{\bar{\beta}_x}{\epsilon_x} = 5.48 \times 10^7, \tag{4.4a}$$

$$\frac{\bar{\beta}_z}{\epsilon_z} = 4.76 \times 10^7, \tag{4.4b}$$

$$\frac{\gamma^2\bar{\eta}^2}{\epsilon_x\bar{\beta}_x} = 1.50 \times 10^8, \tag{4.4c}$$

$$\frac{\bar{\beta}_x}{\epsilon_x}\gamma^2\bar{\phi}^2 = 1.94 \times 10^8, \tag{4.4d}$$

$$\frac{\gamma^2}{\sigma_\eta^{\ 2}} = 6.31 \times 10^9. \tag{4.4e}$$

TABLE II
($\bar{p}A$ LATTICE 2)

n	$\beta_x(n)$	$\beta_z(n)$	$\eta_x(n)$	$\alpha_x(n)$	$\eta_x'(n)$
1	7.51071	7.55360	.02120	0.00000	.00000
2	16.03401	16.02850	.02120	−1.06528	.00000
3	15.53793	19.17085	.02007	1.90064	−.00393
4	13.68349	24.17460	.01807	1.74984	−.00393
5	16.76594	23.99550	.01766	−4.58600	.00326
6	27.44113	14.41993	.02098	−5.92108	.00326
7	31.26892	11.15060	.02152	.76887	−.00173
8	29.92010	9.79484	.01996	.72286	−.00173
9	27.83752	7.99675	.08378	.64540	.08559
10	21.66828	8.03922	.63055	.32034	.08559
11	19.42273	9.41216	.64624	3.79226	−.02674
12	3.14183	29.95906	.55938	1.21987	−.02674
13	2.15912	30.19702	.59697	.09827	.12271
14	25.97882	1.88764	1.49889	−3.33891	.12271
15	26.11610	2.61163	1.45152	3.16693	−.24735
16	7.13551	24.09558	.42778	1.41897	−.24735
17	6.38438	25.88834	.27920	−.33727	−.16284
18	12.79658	12.57896	−.44220	−1.11011	−.16284
19	21.13383	7.18600	−.67070	−1.63914	.01214
20	22.84421	6.59317	−.66453	−1.72776	.01214
21	22.84815	6.58917	−.63626	1.71935	.11084
22	18.91308	8.15721	−.50112	1.50824	.11084
23	8.81615	18.37091	.59492	.72561	.37414
24	8.12362	19.83055	.78498	.63765	.37414
25	8.12377	19.81626	.98524	−.63799	.50692
26	9.93691	16.38485	1.60328	−.84916	.50692
27	21.15170	7.19325	4.48961	−1.63202	.77023
28	22.85453	6.59882	4.88089	−1.72000	.77023
29	23.22550	6.48369	4.98959	−.42352	.48906
30	23.66890	6.38794	5.23803	−.44932	.48906
31	28.76809	8.75719	8.04362	−.67884	.75236
32	31.30352	11.16025	9.36221	−.76783	.75236
33	27.46905	14.42967	9.12488	5.92904	−1.41913
34	16.77984	24.00716	7.68304	4.59183	−1.41913
35	13.68958	24.18218	7.86105	−1.74848	1.71086
36	15.54252	19.17523	8.73017	−1.89903	1.71086
37	16.03626	16.03021	9.22419	1.06761	.00000
38	7.49431	7.53979	9.22419	−.00000	.00000
39	7.51071	7.55360	.02120	.00000	.00000

n denotes lattice location.

For both the Accumulator and the Energy Saver, $1/\tau_z \ll 1/\tau_x$ and $1/\tau_z \ll 1/\tau_l$, so that we are mainly interested in understanding the ratio τ_x/τ_l of horizontal and longitudinal intrabeam-scattering diffusion times. In addition for both the Accumulator and the Energy Saver, a useful approximation is obtained by expanding Eq. (3.4) and neglecting β_x/ϵ_x and β_z/ϵ_z relative to

$$\frac{\gamma^2\eta^2}{\epsilon_x\beta_x}, \frac{\beta_x}{\epsilon_x}\gamma^2\phi^2, \quad \text{and} \quad \frac{\gamma^2}{\sigma_\eta{}^2}.$$

After tedious manipulations of Eq. (3.4), we are led to

$$\frac{1}{\tau_x} = \frac{\pi^2\alpha^2 MN\,(\log)}{\gamma\tilde{\Gamma}}\left[\frac{\gamma^2\eta^2}{\epsilon_x\beta_x} + \frac{\beta_x}{\epsilon_x}\gamma^2\phi^2\right]\cdot\int_0^\infty \frac{d\lambda\,\sqrt{\lambda}\,[2a\lambda + b]}{\{\lambda^3 + a\lambda^2 + b\lambda + c\}^{3/2}}, \tag{4.5}$$

$$\frac{1}{\tau_l} = \frac{\pi^2\alpha^2 MN\,(\log)}{\lambda\tilde{\Gamma}}\left[\frac{m\gamma^2}{\sigma_\eta{}^2}\right]\int_0^\infty \frac{d\lambda\,\sqrt{\lambda}\,[2a\lambda + b]}{\{\lambda^3 + a\lambda^2 + b\lambda + c\}^{3/2}}, \tag{4.6}$$

$$\frac{1}{\tau_z} = \frac{\pi^2\alpha^2 MN\,(\log)}{\gamma\tilde{\Gamma}}\left[\frac{\beta_z}{\epsilon_z}\right]\cdot\int_0^\infty \frac{d\lambda\,\sqrt{\lambda}\left[-a\lambda + \left(b - \dfrac{3\epsilon_z}{\beta_z}c\right)\right]}{\{\lambda^3 + a\lambda^2 + b\lambda + c\}^{3/2}}, \tag{4.7}$$

where $m = 1(2)$ for an unbunched (bunched) beam and

$$a = \frac{\gamma^2\eta^2}{\epsilon_x\beta_x} + \frac{\beta_x}{\epsilon_x}\gamma^2\phi^2 + \frac{m\gamma^2}{\sigma_\eta{}^2}, \tag{4.8a}$$

$$b = \left[\left(\frac{\beta_x}{\epsilon_x} + \frac{\beta_z}{\epsilon_z}\right)\left(\frac{\gamma^2\eta^2}{\epsilon_x\beta_x} + \frac{m\gamma^2}{\sigma_\eta{}^2}\right) + \frac{\beta_x}{\epsilon_x}\frac{\beta_z}{\epsilon_z}\gamma^2\phi^2\right], \tag{4.8b}$$

$$c = \frac{\beta_x}{\epsilon_x}\frac{\beta_z}{\epsilon_z}\left(\frac{\gamma^2\eta^2}{\epsilon_x\beta_x} + \frac{m\gamma^2}{\sigma_\eta{}^2}\right). \tag{4.8c}$$

Thus we obtain

$$\frac{\tau_x}{\tau_l} = \frac{\left\langle \dfrac{m}{\sigma_\eta{}^2} \right\rangle}{\left\langle \left[\dfrac{\eta^2}{\epsilon_x\beta_x} + \dfrac{\beta_x}{\epsilon_x}\phi^2\right] \right\rangle}. \tag{4.9}$$

In Eq. (4.7), we see the tendency for $1/\tau_z$ to be negative and several orders of magnitude less than $1/\tau_x$ and $1/\tau_l$. As long as dropping β_x/ϵ_x and β_z/ϵ_z relative to the quantities in Eqs. (4.4c–e) is justified, we have

$$\frac{1}{\tau_l} > \frac{1}{\tau_x}, \tag{4.10a}$$

whenever

$$\frac{1}{\sigma_\eta{}^2} \gg \frac{\eta^2}{\epsilon_x\beta_x} + \frac{\beta_x\phi^2}{\epsilon_x}. \tag{4.10b}$$

We numerically evaluate Eq. (3.4) for $1/\tau_x$, $1/\tau_z$, $1/\tau_l$ for each of the lattice locations shown in Table I for $\bar{p}A$ Lattice 1. For an unbunched beam, we have

$$\Gamma_{\text{unbunched}} = \frac{(2\pi)^{5/2}}{\sqrt{2}}\,\beta^3\gamma^3 M^3\epsilon_x\epsilon_z\sigma_\eta C, \tag{4.11}$$

where C is the ring circumference, so that from Eq. (3.4) we have[3]

$$\frac{\pi^2\alpha^2 MN\,(\log)}{\gamma\Gamma_{\text{unbunched}}} = \frac{r_0{}^2\, I(\log)(6.25\times 10^{18})}{4\sqrt{\pi}\,\beta^3\gamma^4\epsilon_x\epsilon_z\sigma_\eta}, \tag{4.12}$$

where I is measured in amperes and

$$r_0 = \frac{\alpha}{M} = 1.4 \times 10^{-18}\ \text{m} \tag{4.13}$$

is the classical proton radius. Recall that throughout these calculations we take the Coulomb log = 20. For ϵ_x, ϵ_z, σ_η, and I we use the values given in Eqs. (4.3a–c). The results are given in Table III. A summary of the averaged diffusion rates is given in

TABLE III

(Diffusion Rates for $\bar{p}A$ Lattice 1)

n	$\frac{1}{\tau_l}$ (hrs^{-1})	$\frac{1}{\tau_x}$ (hrs^{-1})	$\frac{1}{\tau_z}$ (hrs^{-1})
1	0.920	0.252	−0.00267
2	0.857	0.235	−0.00275
3	0.842	0.231	−0.00273
4	0.823	0.226	−0.00271
5	0.731	0.202	−0.00261
6	0.683	0.188	−0.00314
7	0.683	0.188	−0.00322
8	0.695	0.191	−0.00316
9	0.708	0.194	−0.00310
10	0.758	0.126	−0.00311
11	0.762	0.127	−0.00309
12	0.801	0.0692	−0.00314
13	0.810	0.0701	−0.00312
14	0.812	0.0703	−0.00313
15	0.818	0.0709	−0.00320
16	0.828	0.0187	−0.00320
17	0.828	0.0187	−0.00318
18	0.825	−0.00287	−0.00287
19	0.823	−0.00281	−0.00284
20	0.827	−0.00293	−0.00290
21	0.874	−0.00322	−0.00325
22	0.889	−0.00315	−0.00321
23	1.060	−0.00269	−0.00282
24	1.270	0.347	−0.00209

$$\epsilon_x = \epsilon_z = 0.42 \times 10^{-6}\ \text{m}$$

$$\sigma_\eta = 1.2 \times 10^{-4}$$

$$I = 0.041\ \text{A}$$

$$\gamma = 9.53$$

$$\left(\frac{1}{\tau_l}\right)_{\text{avg}} = 0.830\ \text{hrs}^{-1}$$

$$\left(\frac{1}{\tau_x}\right)_{\text{avg}} = 0.117\ \text{hrs}^{-1}$$

$$\left(\frac{1}{\tau_z}\right)_{\text{avg}} = -0.00297\ \text{hrs}^{-1}$$

TABLE IV

(Averaged Diffusion Rates in hrs^{-1} for $\bar{p}A$ Lattice 1)

	$I = 0.041\text{A}$	$\gamma = 9.53$	
σ_η	$\left(\frac{1}{\tau_l}\right)_{\text{avg}}$	$\left(\frac{1}{\tau_x}\right)_{\text{avg}}$	$\left(\frac{1}{\tau_z}\right)_{\text{avg}}$
	$(\epsilon_x = \epsilon_z = 0.21 \times 10^{-6}\text{ m})$		
1.8×10^{-4}	0.789	0.4950	−0.01150
	$(\epsilon_x = \epsilon_z = 0.42 \times 10^{-6}\text{ m})$		
0.10×10^{-4}	151.0	0.149	−0.00390
0.50×10^{-4}	5.56	0.137	−0.00357
0.90×10^{-4}	1.58	0.125	−0.00322
1.20×10^{-4}	0.830	0.117	−0.00297
1.80×10^{-4}	0.325	0.102	−0.00250
2.80×10^{-4}	0.110	0.0834	−0.00189
	$(\epsilon_x = \epsilon_z = 0.65 \times 10^{-6}\text{ m})$		
1.8×10^{-4}	0.182	0.0371	−0.000929

TABLE V

(Averaged Diffusion Rates in hrs^{-1} for $\bar{p}A$ Lattice 2)

	$I = 0.041$	$\gamma = 9.53$	
σ_η	$\left(\frac{1}{\tau_l}\right)_{\text{avg}}$	$\left(\frac{1}{\tau_x}\right)_{\text{avg}}$	$\left(\frac{1}{\tau_z}\right)_{\text{avg}}$
	$(\epsilon_x = \epsilon_z = 0.21 \times 10^{-6}\text{ m})$		
1.8×10^{-4}	1.08	0.420	−0.00923
	$(\epsilon_x = \epsilon_z = 0.42 \times 10^{-6}\text{ m})$		
0.10×10^{-4}	186.0	0.124	−0.00289
0.50×10^{-4}	6.96	0.115	−0.00266
0.90×10^{-4}	2.01	0.106	−0.00243
1.20×10^{-4}	1.07	0.0987	−0.00227
1.80×10^{-4}	0.431	0.0865	−0.00196
2.80×10^{-4}	0.151	0.0706	−0.00154
	$(\epsilon_x = \epsilon_z = 0.65 \times 10^{-6}\text{ m})$		
1.8×10^{-4}	0.237	0.0313	−0.000716

Tables IV and V for $\bar{p}A$ Lattices 1 and 2, respectively. In Figs. 3a–h, we plot $(1/\tau_l)_{\text{avg}}$ and $(1/\tau_x)_{\text{avg}}$ vs. emittance and σ_η for $\bar{p}A$ Lattices 1 and 2, with $I = 0.041\text{A}$ and $\gamma = 9.53$.

B. Energy Saver

The Fermilab Energy Saver has far too many lattice parameter values to present in a tables here; $n_{\max} = 567$ and the radius is 1 km. The average lattice parameter values are

$$\bar{\beta}_x = 73\text{ m}, \tag{4.14a}$$

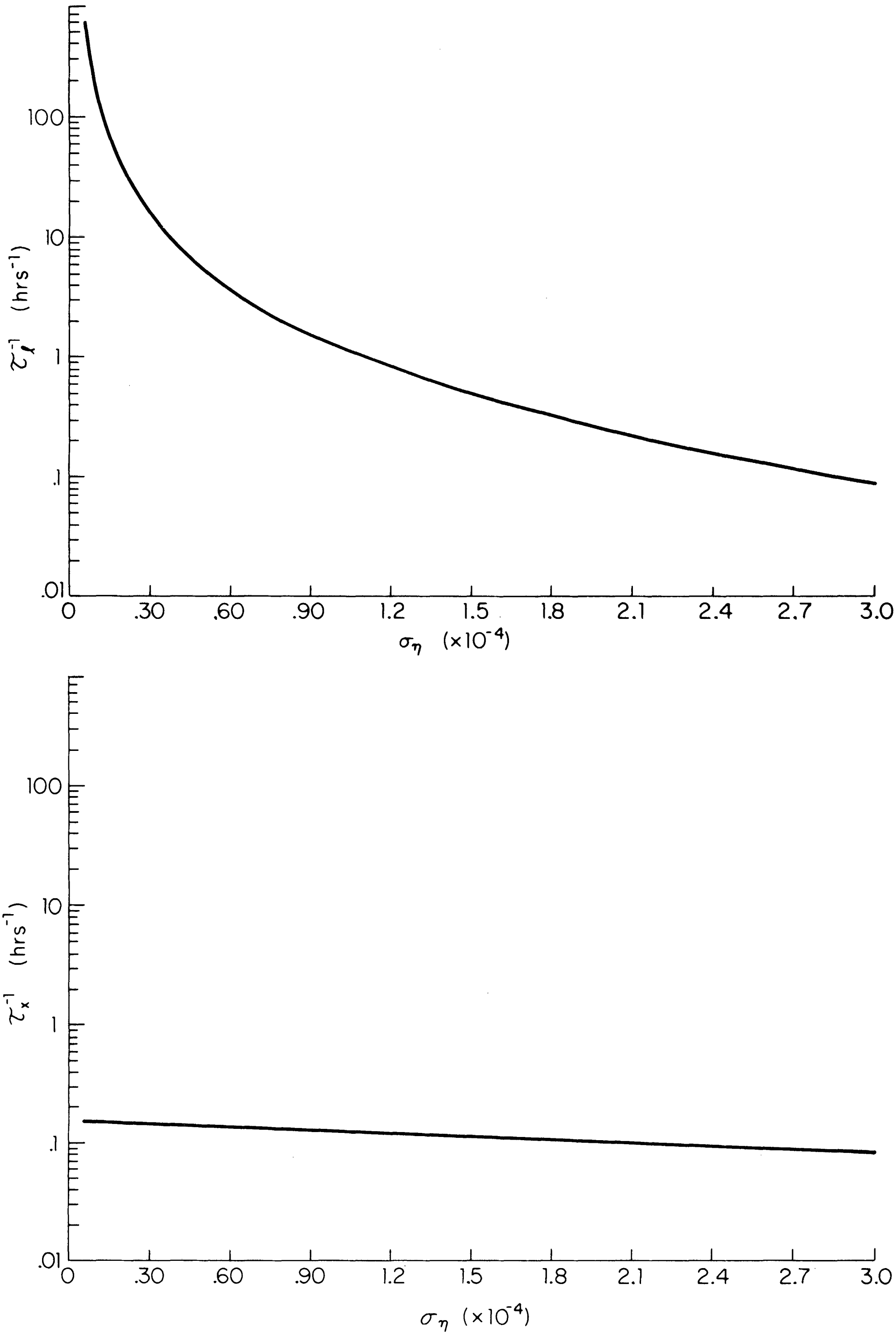


FIGURE 3[a, b]. $(1/\tau_l)_{avg}$ $[(1/\tau_x)_{avg}]$ vs. σ_η for $\bar{p}A$ Lattice 1 with $I = 0.041$A and $\gamma = 9.53$.

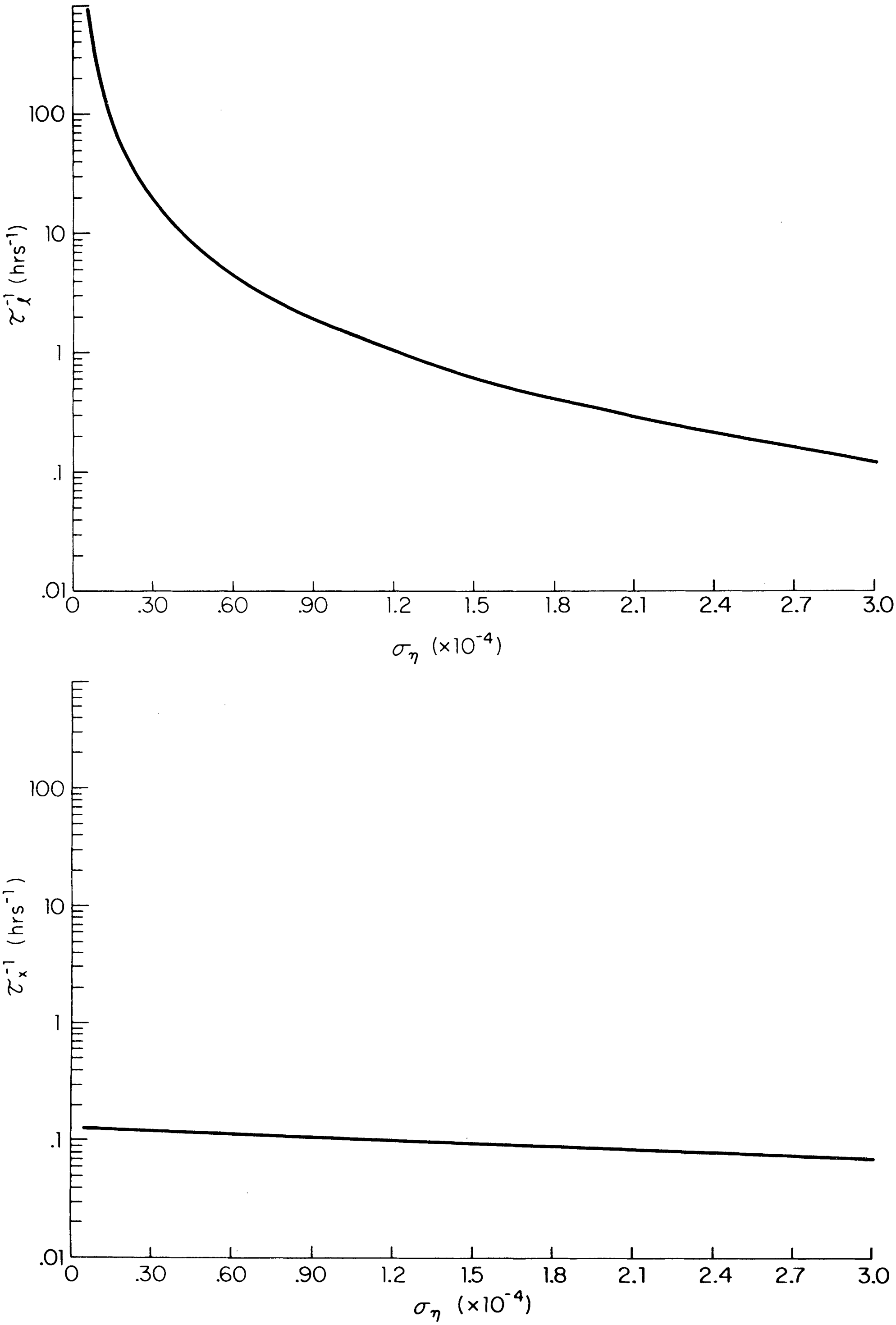
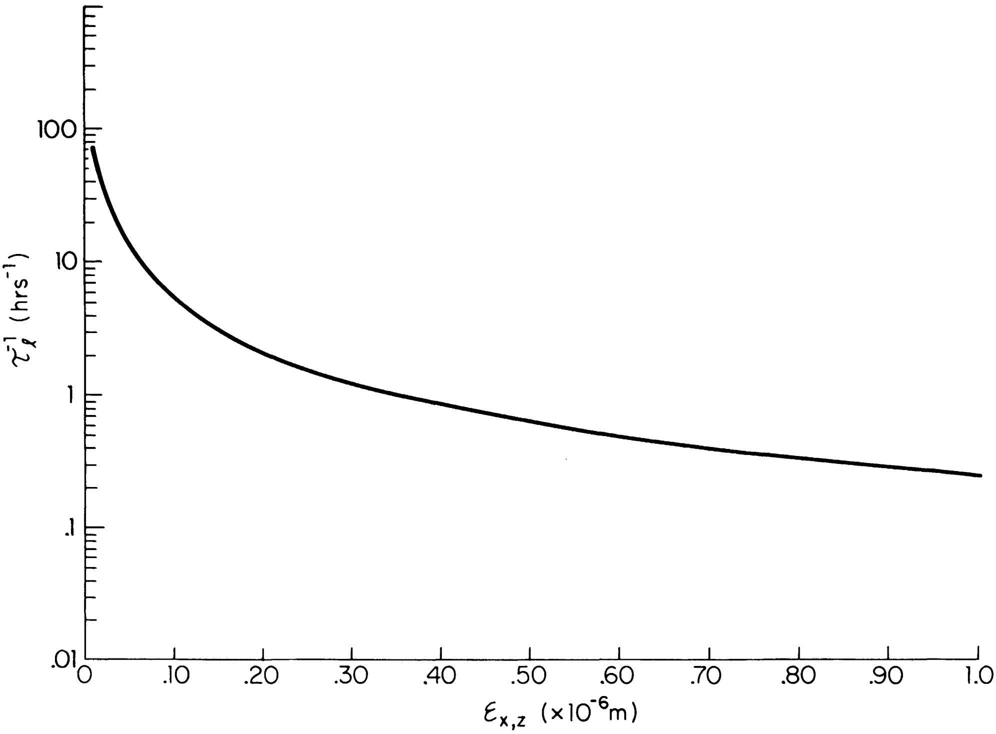


FIGURE 3[c, d]. $(1/\tau_l)_{\text{avg}}$ $[(1/\tau_x)_{\text{avg}}]$ vs. σ_η for $\bar{p}A$ Lattice 2 with $I = 0.041$A and $\gamma = 9.53$.



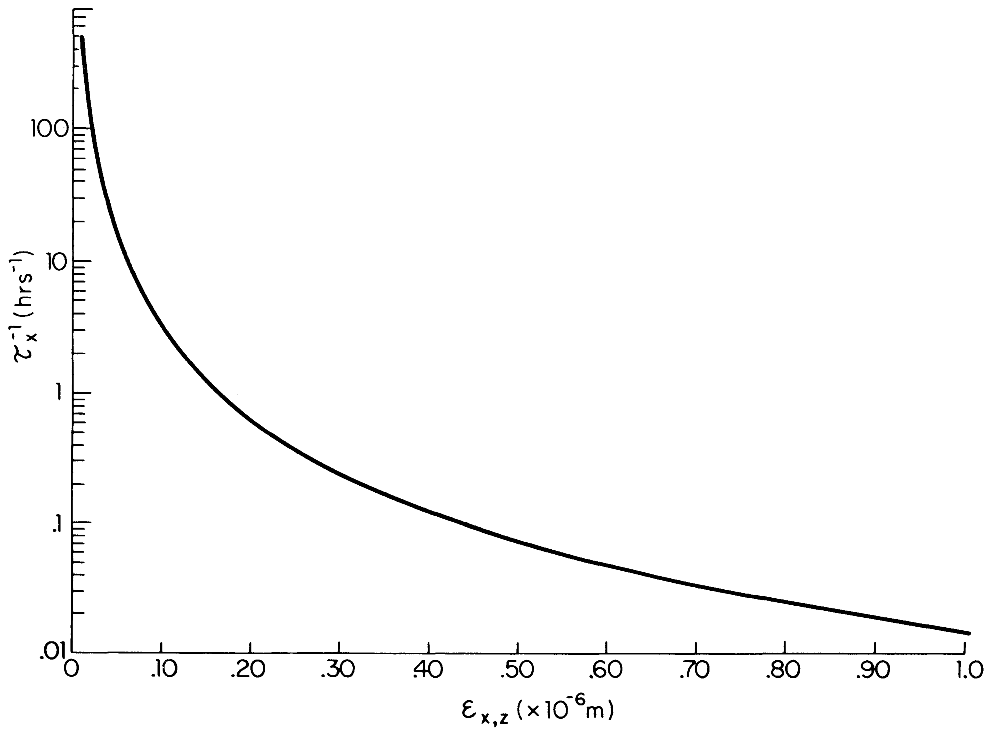
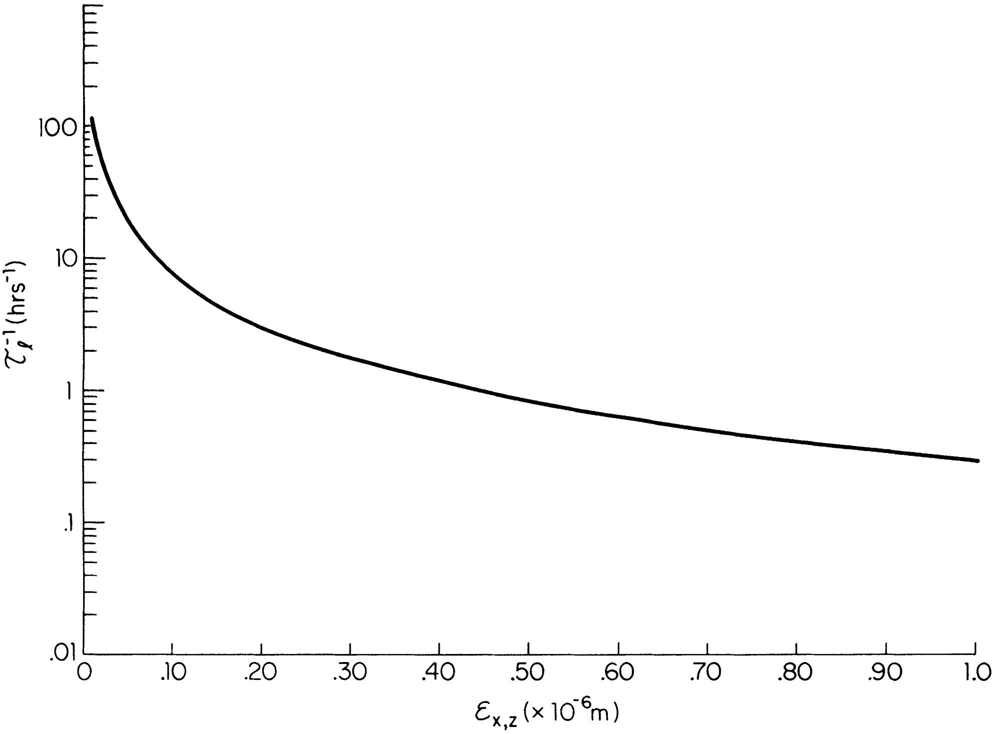


FIGURE 3[e, f]. $(1/\tau_l)_{avg}$ $[(1/\tau_x)_{avg}]$ vs. $\epsilon_{x,z}$ for $\bar{p}A$ Lattice 1 with $I = 0.041$A and $\gamma = 9.53$.



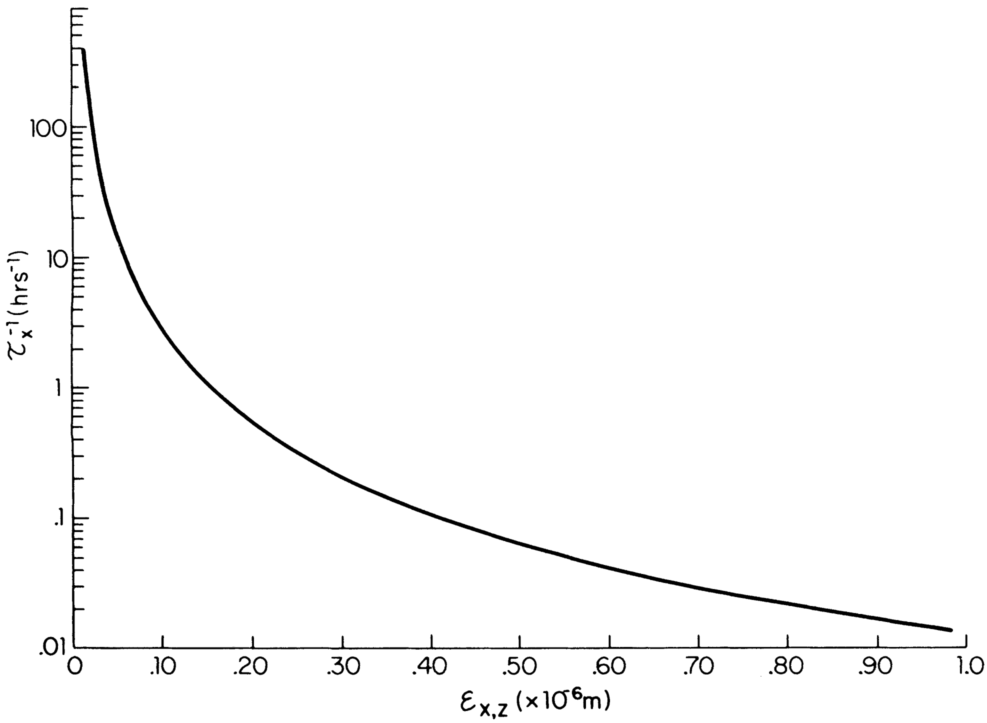


FIGURE 3[g, h]. $(1/\tau_l)_{avg}$ $[(1/\tau_x)_{avg}]$ vs. $\epsilon_{x,z}$ for $\bar{p}A$ Lattice 2 with $I = 0.041$A and $\gamma = 9.53$.

$$\bar{\beta}_z = 75 \text{ m}, \tag{4.14b}$$

$$\bar{\eta}_x = 2.7 \text{ m}, \tag{4.14c}$$

$$\bar{\beta}_x' = -1.6, \tag{4.14d}$$

$$\bar{\eta}_x' = -0.01. \tag{4.14e}$$

For a bunched beam, we have

$$\gamma = 10^3, \tag{4.15a}$$

$$\epsilon_x = \epsilon_z = 4.17 \times 10^{-9} \text{ m}, \tag{4.15b}$$

$$N = 10^{11}, \tag{4.15c}$$

$$\sigma_\eta = 10^{-4}, \tag{4.15d}$$

$$\sigma_s = \text{rms bunch length} = 0.4 \text{ m}, \tag{4.15e}$$

giving

$$\frac{\bar{\beta}_x}{\epsilon_x} = 1.75 \times 10^{10}, \tag{4.16a}$$

$$\frac{\bar{\beta}_z}{\epsilon_z} = 1.80 \times 10^{10}, \tag{4.16b}$$

$$\frac{\gamma^2 \bar{\eta}^2}{\epsilon_x \bar{\beta}_x} = 2.39 \times 10^{13}, \tag{4.16c}$$

$$\frac{\bar{\beta}}{\epsilon_x} \gamma^2 \bar{\phi}^2 = 6.72 \times 10^{12}, \tag{4.16d}$$

$$\frac{2\gamma^2}{\sigma_\eta^2} = 2.0 \times 10^{14}. \tag{4.16e}$$

Thus, the quantities in Eqs. (4.16a–b) are considerably smaller than those in Eqs. (4.16c–e), so that our approximations leading to Eq. (4.9) are justified.

We now evaluate Eq. (3.4) numerically for $1/\tau_x$, $1/\tau_z$, $1/\tau_l$ for the 567 lattice locations and compute the averages.[3] For a bunched beam, we have

$$\Gamma_{\text{bunched}} = (2\pi)^3 \, \beta^3 \gamma^3 M^3 \epsilon_x \epsilon_z \sigma_\eta \sigma_s, \tag{4.17}$$

and use the values for γ, N, and σ_s, given in Eqs. (4.15). We summarize the averaged diffusion rates for the Energy Saver in Table VI. In Figs. 4a–d, we plot $1/\tau_l$ and $1/\tau_x$ vs. emittance and σ_η, using the average lattice parameters given in Eqs. (4.14a–e) and the same γ, N, and σ_s given above.

TABLE VI
(Averaged Diffusion Rates in hrs^{-1} for the Energy Saver)

σ_η	$\left(\frac{1}{\tau_l}\right)_{avg}$	$\left(\frac{1}{\tau_x}\right)_{avg}$	$\left(\frac{1}{\tau_z}\right)_{avg}$
	$N = 10^{11}$, $\gamma = 10^3$, $\sigma_s = 0.4$ m		
	$(\epsilon_x = \epsilon_z = 0.104 \times 10^{-8}$ m)		
0.1×10^{-3}	0.492	0.489	−0.0000579
0.2×10^{-3}	0.0904	0.306	−0.0000403
0.3×10^{-3}	0.0316	0.220	−0.0000302
0.4×10^{-3}	0.0147	0.171	−0.0000239
0.5×10^{-3}	0.00803	0.140	−0.0000197
0.6×10^{-3}	0.00487	0.118	−0.0000166
0.7×10^{-3}	0.00318	0.102	−0.0000143
0.8×10^{-3}	0.00219	0.0899	−0.0000126
0.9×10^{-3}	0.00157	0.0803	−0.0000112
1.0×10^{-3}	0.00117	0.0726	−0.0000100
σ_η	$\left(\frac{1}{\tau_l}\right)_{avg}$	$\left(\frac{1}{\tau_x}\right)_{avg}$	$\left(\frac{1}{\tau_z}\right)_{avg}$
	$(\epsilon_x = \epsilon_z = 0.2085 \times 10^{-8}$ m)		
0.1×10^{-3}	0.198	0.1050	−0.0000120
0.2×10^{-3}	0.0389	0.0713	−0.00000906
0.3×10^{-3}	0.0141	0.0529	−0.00000710
0.4×10^{-3}	0.00667	0.0418	−0.00000578
0.5×10^{-3}	0.00369	0.0344	−0.00000485
0.6×10^{-3}	0.00226	0.0293	−0.00000415
0.7×10^{-3}	0.00149	0.0254	−0.00000362
0.8×10^{-3}	0.00103	0.0225	−0.00000320
0.9×10^{-3}	0.000744	0.201	−0.00000287
1.0×10^{-3}	0.000555	0.0182	−0.00000259
σ_η	$\left(\frac{1}{\tau_l}\right)_{avg}$	$\left(\frac{1}{\tau_x}\right)_{avg}$	$\left(\frac{1}{\tau_z}\right)_{avg}$
	$(\epsilon_x = \epsilon_z = 0.4165 \times 10^{-8}$ m)		
0.1×10^{-3}	0.0767	0.0218	−0.00000238
0.2×10^{-3}	0.0161	0.0160	−0.00000194
0.3×10^{-3}	0.00606	0.0124	−0.00000160
0.4×10^{-3}	0.00294	0.00998	−0.00000134
0.5×10^{-3}	0.00165	0.00835	−0.00000115
0.6×10^{-3}	0.00103	0.00716	−0.00000100
0.7×10^{-3}	0.000680	0.00626	−0.000000886
0.8×10^{-3}	0.000475	0.00556	−0.000000791
0.9×10^{-3}	0.000345	0.00500	−0.000000714
1.0×10^{-3}	0.000259	0.00454	−0.000000649

V. DISCUSSION AND SUMMARY

In this paper we have extended existing published calculations on intrabeam scattering to include the case of strong-focusing lattices. We have found that whenever the lattice function $\phi = \eta' - \beta_x'\eta/2\beta_x$ is nonvanishing, the overall 6-dimensional emittance grows. The formula for this growth rate, Eq. (3.6), is found to be especially simple. We have also obtained approximate analytic formulae for various limiting situations. In

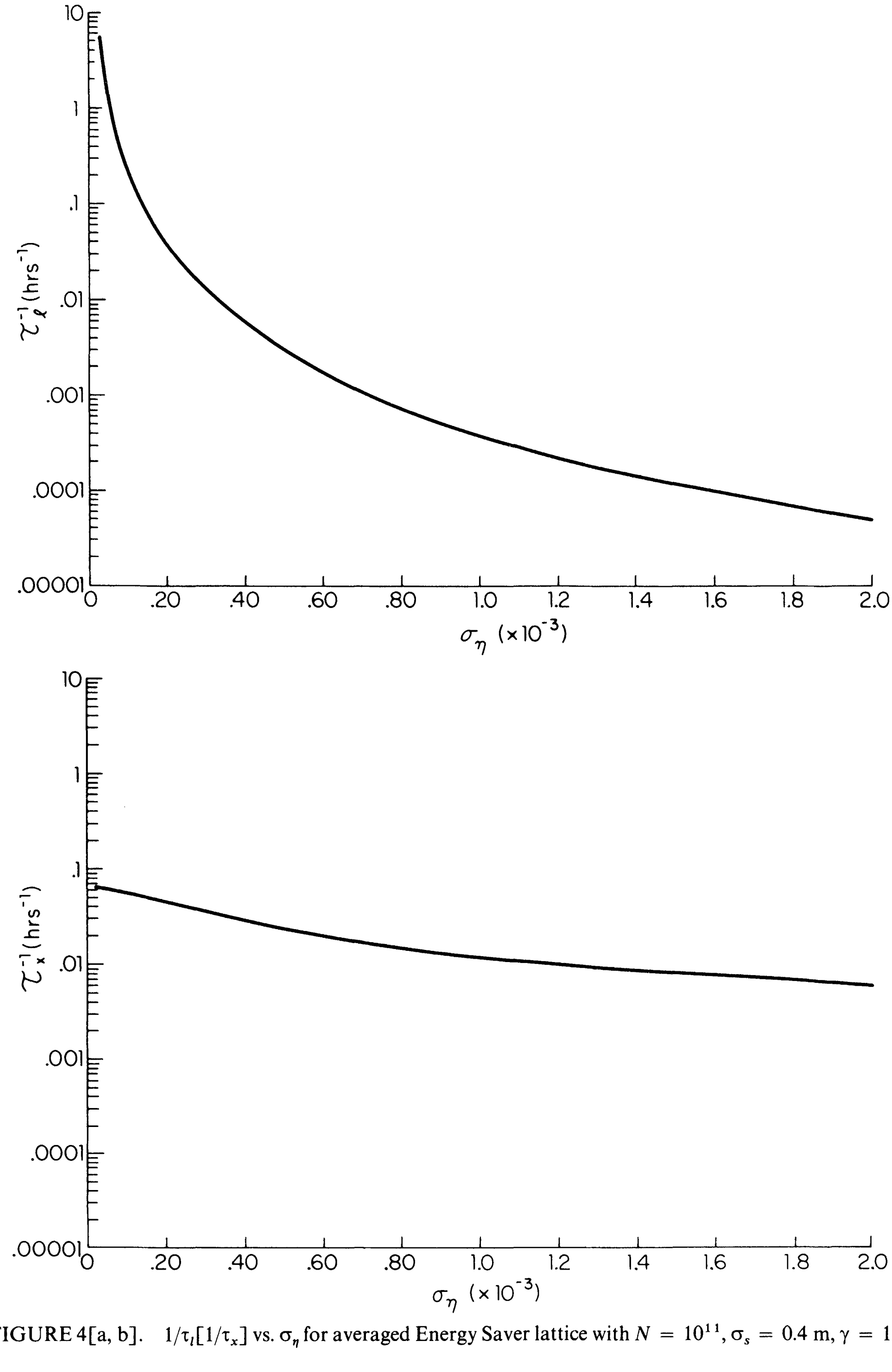


FIGURE 4[a, b]. $1/\tau_l[1/\tau_x]$ vs. σ_η for averaged Energy Saver lattice with $N = 10^{11}$, $\sigma_s = 0.4$ m, $\gamma = 10^3$

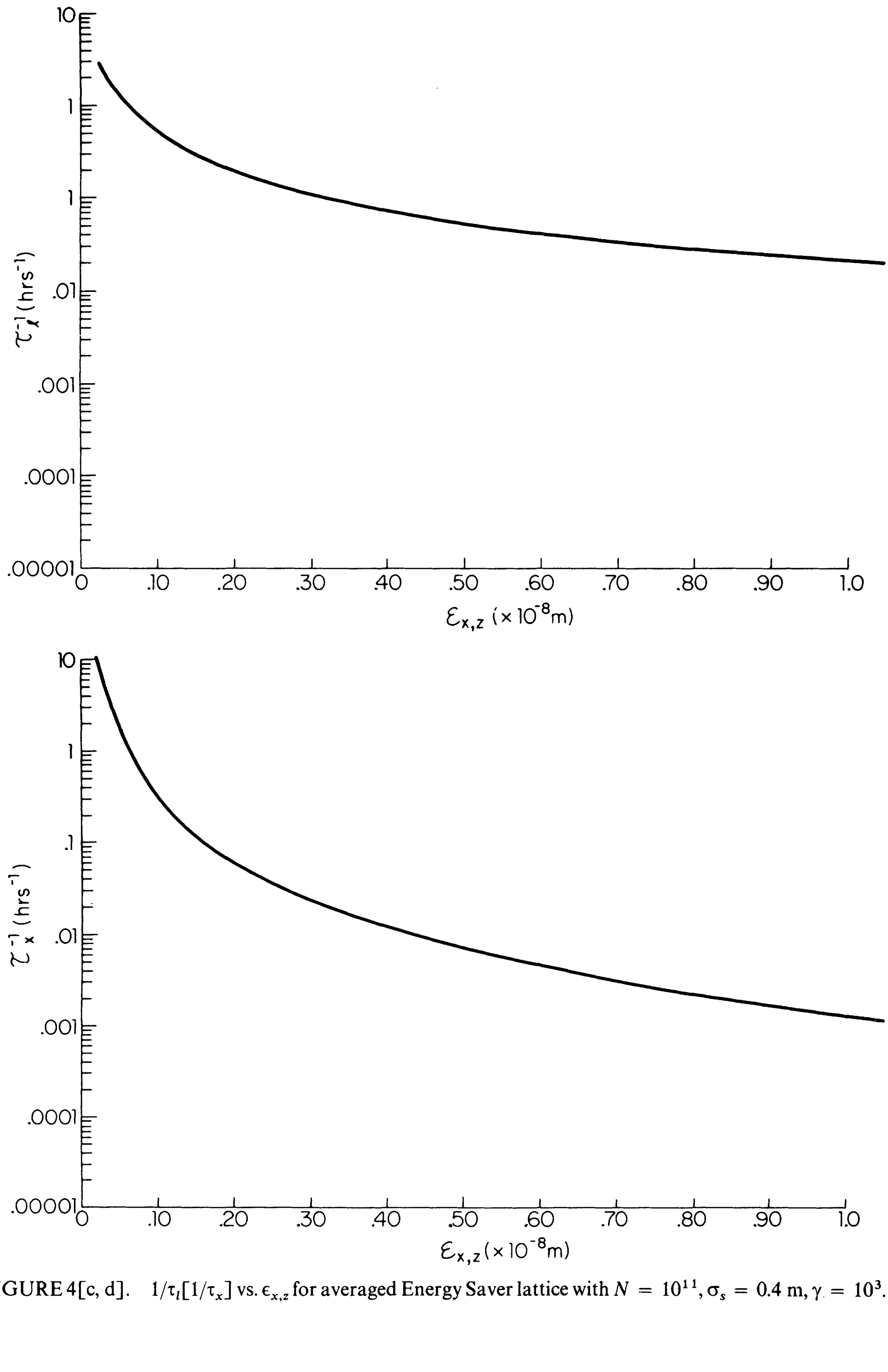


FIGURE 4[c, d]. $1/\tau_l[1/\tau_x]$ vs. $\epsilon_{x,z}$ for averaged Energy Saver lattice with $N = 10^{11}, \sigma_s = 0.4$ m, $\gamma = 10^3$.

particular, whenever γ^2/σ_η^2 dominates the other terms in L[cf., Eq. (2.37)], it is useful to write $1/\tau = 1/\tau_x + 1/\tau_z + 1/\tau_l \simeq 1/\tau_l$ and express $1/\tau_l$ in terms of elliptic integrals.

The results of this work have been applied to both preliminary designs for the Fermilab Antiproton Accumulator ring and to the Energy Saver and are presented in various tables and graphs. We find that for reasonable parameter choices the emittance growth rates are not rapid enough to be a severe design constraint.

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APPENDIX

We want to evaluate analytically the integral H[cf., Eq. (3.7)] for the case $\lambda_1 > \lambda_2 \simeq \lambda_3$. Write

$$H = 2(\lambda_1 - \lambda_2)^2 \int_0^\infty \frac{d\lambda\, \lambda^{1/2}}{(\lambda_1 + \lambda)^{3/2}(\lambda_2 + \lambda)^2}. \tag{A1}$$

But remember that $H(\lambda_1, \lambda_2, \lambda_3)$ is homogeneous, so that in Eqs. (1.2) and (A1) we can set

$$\alpha = \frac{1}{\lambda_1}, \tag{A2}$$

$$\epsilon = \frac{\lambda_2}{\lambda_1}, \tag{A3}$$

and write

$$H = 2(1 - \epsilon)^2 \int_0^\infty \frac{d\lambda\, \lambda^{1/2}}{(1 + \lambda)^{3/2}(\epsilon + \lambda)^2}. \tag{A4}$$

Then let

$$\lambda = \epsilon \tan^2 \theta, \tag{A5a}$$

$$d\lambda = 2\epsilon \tan\theta \sec^2\theta\, d\theta, \tag{A5b}$$

so that

$$H = \frac{4(1 - \epsilon)^2}{\sqrt{\epsilon}} \int_0^{\pi/2} \frac{d\theta \tan^2\theta \sec^2\theta}{\sec^3\theta\,(\cos^2\theta + \epsilon \sin^2\theta)^{3/2} \sec^4\theta}$$

$$= \frac{4(1-\epsilon)^2}{\sqrt{\epsilon}} \int_0^{\pi/2} \frac{d\theta \cos^3\theta \sin^2\theta}{[1-(1-\epsilon)\sin^2\theta]^{3/2}}$$

$$= \frac{4(1-\epsilon)^2}{\sqrt{\epsilon}} \int_0^{1} \frac{dv\, v^2(1-v^2)}{[1-(1-\epsilon)v^2]^{3/2}}. \tag{A6}$$

Letting

$$v = \frac{1}{\sqrt{1-\epsilon}} \sin\phi, \tag{A7a}$$

$$\sin\phi_0 = \sqrt{1-\epsilon}, \tag{A7b}$$

we have

$$H = \frac{4\sqrt{1-\epsilon}}{\sqrt{\epsilon}} \int_0^{\phi_0} \frac{d\phi \cos\phi \sin^2\phi}{\cos^3\phi} \left[1 - \frac{1}{(1-\epsilon)}\sin^2\phi\right]$$

$$= \frac{4}{\sqrt{\epsilon(1-\epsilon)}} \int_0^{\phi_0} d\phi \left(\frac{1}{\cos^2\phi} - 1\right)(\cos^2\phi - \epsilon)$$

$$= \frac{4}{\sqrt{\epsilon(1-\epsilon)}} \left\{\phi_0(1+\epsilon) - \frac{1}{2}(\phi_0 + \sin\phi_0\cos\phi_0 - \epsilon\tan\phi_0\right\}$$

$$= \frac{2(1+2\epsilon)}{\sqrt{\epsilon(1-\epsilon)}} \sin^{-1}\sqrt{1-\epsilon} - 6. \tag{A8}$$

Using Eq. (A3), we find

$$H = \frac{2(\lambda_1 + 2\lambda_2)}{\sqrt{\lambda_2(\lambda_1 - \lambda_2)}} \sin^{-1}\sqrt{\frac{\lambda_1 - \lambda_2}{\lambda_1}} - 6. \tag{A9}$$

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2. James D. Bjorken and Sidney D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), p. 113.
3. Note that in evaluating $1/\tau_a$, to establish the connection to conventional units simply replace α^2/M^2 by $cr_0{}^2$, where $c = 3 \times 10^8$ m/sec.
4. Paul. F. Byrd and Morris D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists* (Springer-Verlag, New York, 1971).