MICROWAVE INSTABILITY ACROSS THE TRANSITION ENERGY*

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Abstract

It is well known that during the acceleration of hadrons in a storage ring, the beam always goes above the microwave instability threshold near the transition energy γ_t. The reason is that the longitudinal revolution frequency spread of the beam which otherwise provides Landau damping vanishes at the transition energy. The amount of the beam dilution near the transition energy is determined by τ_{th}, the length of time when the beam stays unstable, and the growth rate of the instability. It is pointed out in this paper that τ_{th} is proportional to the fourth power of γ_t, and thus the choice of a large γ_t is not desirable from this point of view. An analysis is also given of the microwave instability induced beam dilution for the proposed Relativistic Heavy Ion Collider at BNL.

Introduction

In the design of a large accelerator, the microwave instability is an important consideration at the transition energy, where the Landau damping becomes ineffective. Unfortunately, the heavy ions (from deuteron up to gold or uranium) have to be accelerated through the transition energy, γ_t, of the proposed Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory. The heavy ion beam has a large q^2/A ratio, where q and A are the charge and mass number of the heavy ions respectively, therefore the intrabeam scattering becomes very important. As a consequence, higher focusing strength is needed for a smaller beam size, and this gives rise to a larger γ_t. Thus from the point of view of the intrabeam Coulomb scattering larger γ_t is preferable.

On the other hand, large γ_t causes another problem, the growth rate of the microwave instability across the transition energy increases rapidly with γ_t, since the duration of time when η, $\eta = \gamma_t^{-2}-\gamma^{-2}$ is so small that the beam stay under the microwave threshold is roughly proportional to γ_t^4/a^2, where "a" is the bunch area. A compromise between microwave instability and intrabeam scattering has to be considered in the choice of γ_t.

In this paper, we intended to study the systematics of the total growth of the microwave instability across the transition energy. For an allowable total growth, we can set a limit on the impedance of the ring, the number of particles per bunch, the phase space area before the transition, and/or the acceleration rate in the design requirement. In the following we review the longitudinal phase space equation of motion and derive the Liouville-Poincare invariant. Then we apply the dispersion integral to calculate the growth rate of the microwave instability and the total integrated growth. The conclusion is given in the end.

Invariant of the Longitudinal Phase Space at the Transition Region

The equation of synchrotron motion for longitudinal phase space is given by

$$\dot{\theta} = \frac{h\eta\omega_o^2}{E\beta^2} W \tag{1}$$

$$\dot{W} = \frac{qeV}{2\pi}(\sin\phi - \sin\phi_s) \simeq \left(\frac{qeV}{2\pi}\cos\phi_s\right)\theta \tag{2}$$

where ω_o is the revolution frequency; βc, the speed of the particles; E, the energy of the particle; h, the harmonic number; qe, the charge of particles; V, the magnitude of r.f. voltage and $\theta = \phi-\phi_s$; and $W = \Delta E/\omega_o$ are conjugate variables of the synchrotron motion. Linearizing Eq. (2), we obtain

$$\frac{d}{dt}\left(\frac{1}{\Omega_s^2(t)}\dot{\theta}\right) + \theta = 0 \tag{3}$$

where

$$\Omega_s^2(t) = \frac{h\omega_\infty^2}{E(t)}\,\frac{qeV}{2\pi}\,|\eta(t)\cos\phi_s|$$
$$\omega_\infty = \frac{c}{R} = \frac{\omega_o}{\beta}\ . \tag{4}$$

Now let us assume that the particles are accelerated uniformly through the transition energy, i.e. $\gamma(t)$ is a linear function of t as

$$\gamma(t) = \gamma_t + \dot{\gamma}t \tag{5}$$

with constant $\dot{\gamma}$. Then $\eta(t)$ at the transition region may be written as

$$\eta(t) = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \simeq \frac{2\dot{\gamma}}{\gamma_t^3}\, t\ . \tag{6}$$

Using the approximations of Eqs. (5) and (6), we obtain

$$\Omega_s^2(t) \simeq \frac{|t|}{T^3}$$

where

$$T = \left[\frac{E_o}{\omega_\infty^2 h}\,\frac{\gamma_t^4}{2\dot{\gamma}}\left(\frac{2\pi}{qeV|\cos\phi_s|}\right)\right]^{1/3}. \tag{7}$$

Defining $x = |t|/T$ and

$$y = \int_0^x dx'(x'^{1/2}) = \frac{2}{3}x^{3/2}\ ,$$

equation (3) becomes,

$$\frac{d^2\Phi}{dy^2} + \frac{1}{y}\frac{d\Phi}{dy} + \left(1 - \frac{(\frac{2}{3})^2}{y^2}\right)\Phi = 0 \tag{8}$$

with

$$\Phi = y^{-2/3}\,\theta\ .$$

Equation (8) is the Bessel equation of order 2/3. Thus the solution of θ and W are given by linear combination of $J_{2/3}(y)$ and $N_{2/3}(y)$, i.e.

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$$\theta = bx\ \left(\cos\psi\ J_{2/3}(y) + \sin\psi\ N_{2/3}(y)\right) \qquad (9)$$

$$\frac{h\eta\omega_o^2}{E\beta^2} W = \dot{\theta} = \frac{\theta}{Tx} + \frac{x^{2/3}b}{T}\ \left(\cos\psi\left(\frac{2J_{2/3}}{3y} - J_{5/3}\right)\right.$$

$$\left. + \sin\psi\ \left(\frac{2N_{2/3}}{3y} - N_{5/3}\right)\right) \ . \qquad (10)$$

The invariant phase space curve[1] can easily be derived from eqs. (9) and (10) as,

$$\alpha_{\theta\theta}\theta^2 + 2\alpha_{\theta_W}\theta W + \alpha_{ww}W^2 = 1, \qquad (11)$$

where

$$\alpha_{\theta\theta} = \left[\left(\tfrac{3}{2}\ yN_{5/3} - 2N_{2/3}\right)^2 + \left(2J_{2/3} - \tfrac{3}{2}\ yJ_{5/3}\right)^2\right]/b^2D^2x^5$$

$$\alpha_{\theta_W} = \frac{hT\eta\omega_o^2}{b^2D^2x^4E\beta^2}\ \left(N_{2/3}\left(\tfrac{3}{2}\ yN_{5/3} - 2N_{2/3}\right) - J_{2/3}\left(2J_{2/3} - \tfrac{3}{2}yJ_{5/3}\right)\right)$$

$$\alpha_{ww} = \frac{h^2T^2\eta^2\omega_o^4}{b^2D^2x^3E^2\beta^2}\ \left(J_{2/3}{}^2 + N_{2/3}{}^2\right)$$

with

$$D = J_{2/3}N_{5/3} - J_{5/3}N_{2/3} \ .$$

The ellipse defined by eq. (11) has a conserved phase space area, a. Using the asympototic expansion of Bessel and Newmann functions, we obtain at large y

$$\alpha_{\theta\theta} = 1/\hat{\theta}^2$$

$$\alpha_{ww} = 1/\hat{w}^2$$

$$\alpha_{\theta_W} = 0$$

with

$$\hat{\theta} = \left(\frac{2ah^2\omega_o^2\dot{\gamma}T^2}{\pi E_o\beta^2\gamma_t^4}\right)^{1/2}\left(\frac{|t|}{T}\right)^{1/4} \qquad (12)$$

$$\hat{\omega} = \frac{1}{h}\left(\frac{aE_o\ \beta^2\gamma_t}{2\pi h^2\omega_o^2\dot{\gamma}T^2}\right)^{4\ 1/2}\left(\frac{T}{|t|}\right)^{1/4} \qquad (13)$$

where the phase space area, a, has been used to replace the constant b

$$b^2 = \frac{2ah^2\omega_o^2\ \dot{\gamma}T^2}{3E_o\beta^2\gamma_t^4} \qquad (14)$$

To simplify the calculation, we can use the Gaussian bunch. The particle density distribution in the bunch is given by

$$\rho(\theta,w) = N_b\ \frac{3\left(\alpha_{\theta\theta}\alpha_{ww}-\alpha_{\theta_W}^2\right)}{\pi}\ e^{-3\left(\alpha_{\theta\theta}\theta^2 + 2\alpha_{\theta_W}\theta w+\alpha_{ww}W^2\right)} \qquad (15)$$

$$= N_b\ G(\theta)\ g_\theta(w)$$

where

$$G(\theta) = \sqrt{\frac{3}{\pi}}\left(\frac{\alpha_{\theta\theta}\alpha_{ww}-\alpha_{\theta_W}^2}{\alpha_{ww}}\right)^{1/2} e^{-3\left(\frac{\alpha_{\theta\theta}\alpha_{ww}-\alpha_{\theta_W}^2}{\alpha_{ww}}\right)\theta^2} \qquad (16)$$

$$g_\theta(w) = \sqrt{\frac{3\alpha_{ww}}{\pi}}\ e^{-3\alpha_{ww}\ \left(W + \frac{\alpha_{\theta_W}}{\alpha_{ww}}\ \theta\right)^2} \ . \qquad (17)$$

The factors 3 above come about because we choose the bunch area "a" to contain 95% of the beam particles. The factors in eq. (15), $G(\theta)$ and $g_\theta(w)$, are normalized separately.

<u>Growth Rate and the Total Growth of the Microwave Instability at Transition Region</u>

Without the energy spread, the longitudinal coherent frequency shift $\Delta\Omega$, $\Delta\Omega = \Omega - n\omega_o$, is given by

$$\Delta\Omega/n\ ^2 = i\ \frac{qe\eta\omega_o^2\ I_o}{2\pi\beta^2E}\ \frac{Z}{n} = i\ \frac{Z}{n}\ \frac{q^2e^2N_b\eta\omega_o^2}{A2\pi\beta\gamma mc} \qquad (18)$$

where n is the mode number, I_o is the current of the beam, and Z is the total impedence of the chamber of accelerator. Note here that when $\mathrm{Im}(\Delta\Omega) > 0$, the beam suffers microwave instability. In the presence of a frequency spread described by a normalized frequency distribution $F(\omega)$, the coherent frequency, Ω, of the collective mode is given by the solution of the dispersion relation[2]

$$1 = -\left(\frac{\Delta\Omega}{n}\right)^2\ \int \frac{\frac{dF_\omega(\omega)}{d\omega}}{\left(\frac{\Omega}{n} - \omega\right)}\ d\omega \qquad (19)$$

with

$$\int F(\omega)d\omega = 1 \ . \qquad (20)$$

The revolution frequency, ω, $\omega = \omega_o + \Delta\omega$ is related to W by

$$\Delta\omega = -\eta\ \frac{\omega^2}{\beta E^2}\ \left(\frac{\Delta E}{\omega}\right) = -\eta\ \frac{\omega^2}{\beta E^2}\ W = -J^{-1}W \ , \qquad (21)$$

where J is the Jacobian of the transformation. Using the density distribution of Section 2, we obtain

$$\rho_\omega(\theta,\omega) = J\rho_W = \frac{\sqrt{3}}{\pi}\ \left(\alpha_{\theta\theta}\alpha_{\omega\omega}-\alpha_{\theta\omega}^2\right)^{1/2} e^{-3\left(\alpha_{\theta\theta}\theta^2+2\alpha_{\theta\omega}\theta\omega+\alpha_{\omega\omega}\omega^2\right)} \qquad (22)$$

where

$$\alpha_{\theta\omega} = -J\alpha_{\theta W} \qquad (23)$$

$$\alpha_{\omega\omega} = -J^2\alpha_{WW} \ . \qquad (24)$$

The dispersion integral is then used to solve $\delta\Omega$ of the coherent shift in the beam. Figure 1 shows $\mathrm{Im}(\delta\Omega)$ as a function of time around the transition energy for gold ion. At time t = 0, the particle crosses the transition energy at $\gamma_t = 26.4$. The parameters are

$$h = 342$$

$$V = 1.2 \text{ MV}$$

$$\theta_s = 2.268°$$

$$\dot{\gamma} = 1.6/\text{sec}$$

$$N_b = 1.2 * 10^9 \text{ particles} .$$

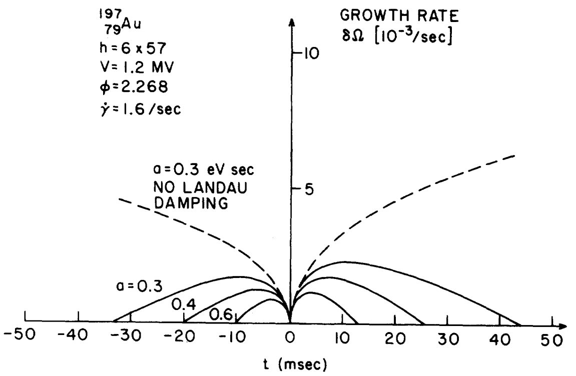

Fig. 1. Microwave growth rate at the transition region.

The initial phase space area a = .3 eV sec, .4 eV sec, and .6 eV sec are shown for comparison. The growth rate for a = .3 eV sec without Landau damping is also plotted for comparison. The impedance is assumed to be Z/n = 5 + $i\zeta_{\text{space charge}}$ ohms. We observe that the effect of Landau damping is quite important. The total growth of the amplitude is given by

$$G = e^{\int_{-\infty}^{\infty} \delta\Omega_i dt} \quad \delta\Omega_i > 0 = e^S \quad . \tag{26}$$

Figure 2 shows the total growth as a function of $|Z/n|N_b$. We observe that the total growth is very sensitive to the initial phase space area. We also show the total growth with $\dot{\gamma}$ = 3.2/sec of r.f. system. The dependence is less sensitive in this respect. The calculation shows a universal dependence of the growth exponent g as

$$S \propto (\frac{Z}{n} N_b q^2/A)^2 \quad , \tag{27}$$

when other ion species are considered.

Conclusion

We have studied the growth of microwave instability for a bunched beam. The width of ω distribution is proportional to $\sqrt{a}/\gamma_t^2$ (see eqs. 13 and 21), where a is the phase space area and γ_t is the transition gamma. For larger γ_t, the Landau damping becomes less effective.

We investigated the dependence of total integrated growth on the impendence, charges and masses of particles and number of particles per bunch. We found a systematic dependance of the integrated growth on $|Z/n| \cdot N_b \cdot q^2/A$. Note also that the total growth of the bunch area decreases faster than $1/a^2$ with increasing a.

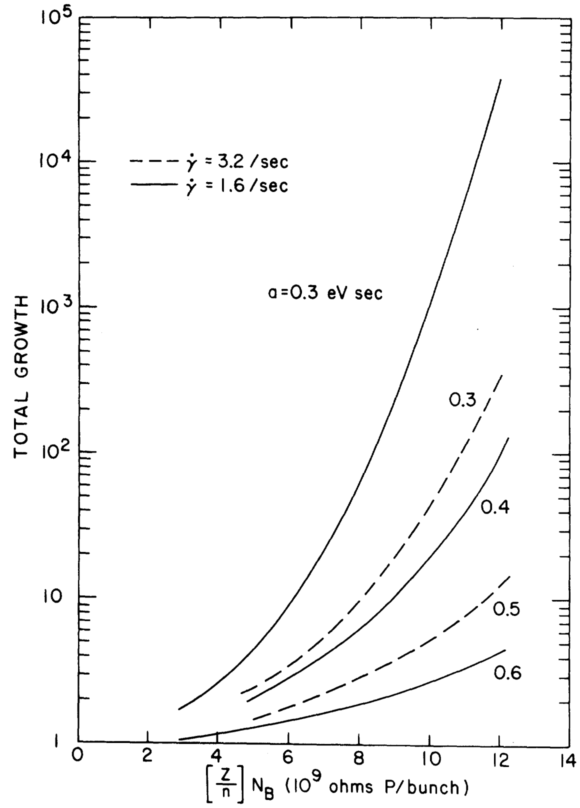

Fig. 2. Total growth of microwave amplitude.

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