Via likelihood:

$$\log p(x) = \log \int p(x|z)\, p(z)\, dz \qquad \text{add } q(z|x)$$

$$\log p(x) = \log \int \frac{\boxed{q(z|x)}}{q(z|x)}\, p(x|z)\, p(z)\, dz =$$

$$= \log \int \underbrace{q(z|x)\; p(x|z)}_{?}\; \frac{p(z)}{q(z|x)}\, dz =$$

$$= \log \mathbb{E}_{z \sim q(z|x)}\left[p(x|z)\, \frac{p(z)}{q(z|x)} \right]$$

For convex function φ

$\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)]$ Jensen's inequality

$$\log \mathbb{E}_{z \sim q(z|x)}\left[p(x|z)\, \frac{p(z)}{q(z|x)} \right]$$

concave

$$\log p(x) \geq \mathbb{E}_z\left[\log p(x|z) + \log \frac{p(z)}{q(z|x)}\right] =$$

D_{KL}

$$\mathbb{E}_z\left[\log p(x|z)\right] + \mathbb{E}_z\left[\log \frac{p(z)}{q(z|x)}\right] =$$

$$\mathbb{E}_z\left[\log p(x|z)\right] - D_{KL}\left(q(z|x) \,\|\, p(z)\right) \blacksquare$$

Via KL

$$D_{KL}(q(z|x)\,|\,p(z|x)) = -\mathbb{E}_{z\sim q(z|x)}\left[\log \frac{p(z|x)}{q(z|x)}\right] =$$

$$= -\mathbb{E}_{z\sim q(z|x)}\left[\log p(z|x) - \log q(z|x)\right] =$$

$$= -\mathbb{E}_{z}\left[\log \frac{p(x|z)\,p(z)}{p(x)} - \log q(z|x)\right] =$$

$$= -\mathbb{E}_{z}\left[\log p(x|z) + \log p(z) - \log p(x) - \log q(z|x)\right] =$$

$$= -\mathbb{E}_{z}\left[\log p(x|z)\right] - \mathbb{E}_{z}\left[\log p(z) - \log q(z|x)\right] - \log p(x)$$

$$D_{KL}(P\|Q) = \sum_{x} P(x)\log\frac{P(x)}{Q(x)} = -\sum_{x} P(x)\log\frac{Q(x)}{P(x)}$$

$$D_{KL}(Q\|P) = -\sum_{x} Q(x)\log\frac{P(x)}{Q(x)}$$

$$D_{KL}(q(z|x)\,|\,p(z|x)) = -\mathbb{E}_{z}\left[\log p(z|x)\right] +$$

$$+ D_{KL}(q(z|x)\,\|\,p(z)) + \log p(x)$$

$$\log p(x) = +\underbrace{D_{KL}(q(z|x)\,|\,p(z|x))}_{\geq 0} + E_z\left[\log p(z|x)\right] - D_{KL}(q(z|x)\,\|\,p(z))$$

$$\log p(x) \geq \mathbb{E}_z\left[\log p(z|x)\right] - D_{KL}(q(z|x)\,\|\,p(z))$$