Finite Differences

Consider Poisson's Equation:

$$\partial_x^2 U(x,y) + \partial_y^2 U(x,y) = -f(x,y) \text{ for } (x,y) \in \Omega$$

$$U(x,y) = g(x,y) \text{ for } (x,y) \in \partial\Omega$$

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{\Delta y^2} = -f_{i,j}$$

If $\Delta x = \Delta y$:

$$U_{i,j} = \frac{1}{4}\left(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} + \Delta x^2 f_{i,j}\right)$$

Gauss-Seidel Iteration:

$$\Rightarrow U_{i,j}^{m+1} = \frac{1}{4}\left(U_{i+1,j}^{m} + U_{i-1,j}^{m+1} + U_{i,j+1}^{m} + U_{i,j-1}^{m+1} + \Delta x^2 f_{i,j}\right)$$

Multigrid 1

Uses a collection of grids beginning with the finest and getting coarser. Suppose we have the domain 0<x,y<L and N interior points in each direction.

$$\text{Let } N = 2^l + 2 \text{ and } \Delta x = \Delta y = h = \frac{1}{N-1}$$

Sequence of coarser grids with # of interior points:

$$2^l, 2^{l-1}, 2^{l-2}, \ldots, 2^0 = 1$$

When l=0, there is only one interior point:

$$U_{1,1} = \frac{1}{4}\left(U_{2,1} + U_{0,1} + U_{1,2} + U_{1,0} + h^2 f_{1,1}\right)$$

Main idea is to recursively estimate the correction to the solution by solving the problem on successively coarser grids and projecting back onto the fine grid.

Multigrid 2

Suppose $A_h U = b_h$, use several Gauss-Seidel iterations to get approximation U_h

We want the error $e_h = U_h - U$ but can get the residual:

$$r_h = A_h U_h - b_h = A_h(U_h - U) = A_h e_h$$

Project r_h to coarser grid: $r_H = A_H e_H$ and solve for e_H

Recursively obtain $e_H = A_H^{-1} r_H$

Through a sequence of coarser to fine grid transformations, obtain approximation to e_h and use to correct solution.

$$\Rightarrow r_h = \frac{1}{h^2}\left(U^m_{i+1,j} + U^{m+1}_{i-1,j} + U^m_{i,j+1} + U^{m+1}_{i,j-1} - 4U_{i,j}\right) - f_{i,j}$$

Multigrid 3

$$e_h = U_h - U \qquad\qquad r_h = \nabla^2 U_h + f$$

$$\nabla^2 e_h = \nabla^2 U_h + f - \nabla^2 U - f = r_h - (\nabla^2 U + f) = r_h$$

Same form as Poisson equation, e_h uknown, r_h known.

Multigrid Sequence:

1) Check the grid level, if l=0, return solution from

$$U_{1,1} = \frac{1}{4}\left(U_{2,1} + U_{0,1} + U_{1,2} + U_{1,0} + h^2 f_{1,1}\right)$$

2) Perform Gauss-Seidel pre-smoothing iterations

3) Calculate residual and restrict to coarser grid.

4) Estimate the correction term by calling multigrid routine recursively with residual vector as rhs.

Multigrid 4

5) Following the recursive call to compute e_H transfer it to fine grid to get approximation for e_h

6) Add correction to previously obtained approximate solution and do some post smoothing GS iterations.

Need coarse to fine and fine to coarse grid transfer functions to implement above ideas:

```
for(i=1; i<nc; i++){
    for(j=1; j<nc; j++){
        i_f = 2*(i-1);
        j_f = 2*(j-1);
        fine_vec[i_f][j_f] = coarse_vec[i][j];
        fine_vec[i_f+1][j_f] = coarse_vec[i][j];
        fine_vec[i_f][j_f+1] = coarse_vec[i][j];
        fine_vec[i_f+1][j_f+1] = coarse_vec[i][j];
    }
}
```

Multigrid 5

```
for(i=1; i<nc; i++){
    for(j=1; j<nc; j++){
        i_f = 2*(i-1);
        j_f = 2*(j-1);
        coarse_vec[i][j] = (1/4)*( fine_vec[i_f][j_f] + fine_vec[i_f+1][j_f] +
fine_vec[i_f][j_f+1] + fine_vec[i_f+1][j_f+1] );
    }
}
```

In the code, slightly more complicated functions are used for improved accuracy.

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = f(x,y) = -2\pi^2 sin(\pi x) sin(\pi y) \text{ on } \Omega$$

$$u(x,y) = g(x,y) = 0 \text{ on } \partial\Omega$$

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = f(x,y) = 4 \text{ on } \Omega$$

$$u(x,y) = g(x,y) = x^2 + y^2 \text{ on } \partial\Omega$$

$$\Omega = (x,y) : 0 < x < 1; 0 < y < 1$$