

Branching algorithms

Serge Gaspers

UNSW

Outline

- 1 Branching algorithms
- 2 Running time analysis
- 3 Feedback Vertex Set
- 4 Maximum Leaf Spanning Tree
- 5 Further Reading

Outline

- 1 Branching algorithms
- 2 Running time analysis
- 3 Feedback Vertex Set
- 4 Maximum Leaf Spanning Tree
- 5 Further Reading

Branching Algorithm

Branching Algorithm

- **Selection**: Select a local configuration of the problem instance
 - **Recursion**: Recursively solve subinstances
 - **Combination**: Compute a solution of the instance based on the solutions of the subinstances
-
- **Halting** rule: 0 recursive calls
 - **Simplification** rule: 1 recursive call
 - **Branching** rule: ≥ 2 recursive calls

Example: Our first VERTEX COVER algorithm

Algorithm $\text{vc1}(G, k);$

```
1 if  $E = \emptyset$  then                // all edges are covered
2   | return Yes
3 else if  $k \leq 0$  then              // we cannot select any vertex
4   | return No
5 else
6   | Select an edge  $uv \in E;$ 
7   | return  $\text{vc1}(G - u, k - 1) \vee \text{vc1}(G - v, k - 1)$ 
```

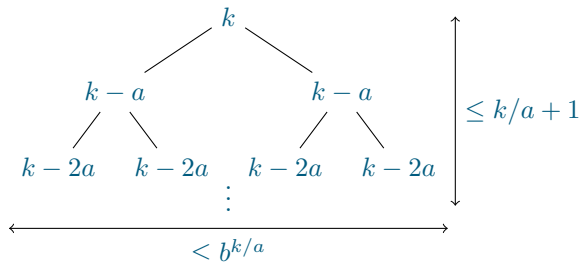
Outline

- 1 Branching algorithms
- 2 Running time analysis**
- 3 Feedback Vertex Set
- 4 Maximum Leaf Spanning Tree
- 5 Further Reading

Search trees

Recall: A **search tree** models the recursive calls of an algorithm.

For a b -way branching where the parameter k decreases by a at each recursive call, the number of nodes is at most $b^{k/a} \cdot (k/a + 1)$.



If k/a and b are upper bounded by a function of k , and the time spent at each node is **FPT** (typically, polynomial), then we get an **FPT** running time.

Outline

- 1 Branching algorithms
- 2 Running time analysis
- 3 Feedback Vertex Set**
- 4 Maximum Leaf Spanning Tree
- 5 Further Reading

Feedback Vertex Set

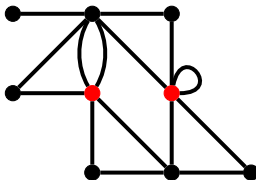
A **feedback vertex set** of a multigraph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $G - S$ is acyclic.

FEEDBACK VERTEX SET

Input: Multigraph $G = (V, E)$, integer k

Parameter: k

Question: Does G have a feedback vertex set of size at most k ?



Simplification Rules

We apply the first **applicable**¹ simplification/halting rule.

(Finished)

If G is acyclic and $k \geq 0$, then return **YES**.

(Budget-exceeded)

If $k < 0$, then return **No**.

¹A rule is **applicable** if it modifies the instance.

Simplification Rules

We apply the first **applicable**¹ simplification/halting rule.

(Finished)

If G is acyclic and $k \geq 0$, then return **YES**.

(Budget-exceeded)

If $k < 0$, then return **No**.

(Loop)

If G has a loop $vv \in E$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

¹A rule is **applicable** if it modifies the instance.

Simplification Rules

We apply the first **applicable**¹ simplification/halting rule.

(Finished)

If G is acyclic and $k \geq 0$, then return **YES**.

(Budget-exceeded)

If $k < 0$, then return **No**.

(Loop)

If G has a loop $vv \in E$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

(Multiedge)

If E contains an edge uv more than twice, remove all but two copies of uv .

¹A rule is **applicable** if it modifies the instance.

Simplification Rules II

(Degree-1)

If $\exists v \in V$ with $d_G(v) \leq 1$, then set $G \leftarrow G - v$.

Simplification Rules III

(Degree-2)

If $\exists v \in V$ with $d_G(v) = 2$, then denote vu, vw its two incident edges and set $G \leftarrow G' = (V \setminus \{v\}, (E \setminus \{vu, vw\}) \cup \{uw\})$.

Simplification Rules III

(Degree-2)

If $\exists v \in V$ with $d_G(v) = 2$, then denote vu, vw its two incident edges and set $G \leftarrow G' = (V \setminus \{v\}, (E \setminus \{vu, vw\}) \cup \{uw\})$.

Lemma 1

(Degree-2) is sound.

Proof.

Suppose S is a feedback vertex set of G of size at most k . Let

$$S' = \begin{cases} S & \text{if } v \notin S \\ (S \setminus \{v\}) \cup \{u\} & \text{if } v \in S. \end{cases}$$

Now, $|S'| \leq k$ and S' is a feedback vertex set of G' since every cycle in G' corresponds to a cycle in G , with, possibly, the edge uw replaced by the walk (u, v, w) .

Suppose S' is a feedback vertex set of G' of size at most k . Then, S' is also a feedback vertex set of G . □

Remaining issues

- A select–discard branching decreases k in only one branch
- One could branch on all the vertices of a cycle, but the length of a shortest cycle might not be bounded by any function of k

Remaining issues

- A select–discard branching decreases k in only one branch
- One could branch on all the vertices of a cycle, but the length of a shortest cycle might not be bounded by any function of k

Idea:

- An acyclic graph has average degree < 2
- After applying simplification rules, G has average degree ≥ 3
- The selected feedback vertex set needs to be incident to many edges
- Does a feedback vertex set of size at most k contain at least one vertex among the $f(k)$ vertices of highest degree?

The fvs needs to be incident to many edges

Lemma 2

If S is a feedback vertex set of $G = (V, E)$, then

$$\sum_{v \in S} (d_G(v) - 1) \geq |E| - |V| + 1$$

The fvs needs to be incident to many edges

Lemma 2

If S is a feedback vertex set of $G = (V, E)$, then

$$\sum_{v \in S} (d_G(v) - 1) \geq |E| - |V| + 1$$

Proof.

Since $F = G - S$ is acyclic, $|E(F)| \leq |V| - |S| - 1$.

Since every edge in $E \setminus E(F)$ is incident with a vertex of S , we have

$$\begin{aligned} |E| &= |E| - |E(F)| + |E(F)| \\ &\leq \left(\sum_{v \in S} d_G(v) \right) + (|V| - |S| - 1) \\ &= \left(\sum_{v \in S} (d_G(v) - 1) \right) + |V| - 1. \end{aligned}$$



The fvs needs to contain a high-degree vertex

Lemma 3

Let G be a graph with minimum degree at least 3 and let H denote a set of $3k$ vertices of highest degree in G .

Every feedback vertex set of G of size at most k contains at least one vertex of H .

The fvs needs to contain a high-degree vertex

Lemma 3

Let G be a graph with minimum degree at least 3 and let H denote a set of $3k$ vertices of highest degree in G .

Every feedback vertex set of G of size at most k contains at least one vertex of H .

Proof.

Suppose not. Let S be a feedback vertex set with $|S| \leq k$ and $S \cap H = \emptyset$. Then,

$$\begin{aligned} 2|E| - |V| &= \sum_{v \in V} (d_G(v) - 1) \\ &= \sum_{v \in H} (d_G(v) - 1) + \sum_{v \in V \setminus H} (d_G(v) - 1) \\ &\geq 3 \cdot \left(\sum_{v \in S} (d_G(v) - 1) \right) + \sum_{v \in S} (d_G(v) - 1) \\ &\geq 4 \cdot (|E| - |V| + 1) \\ \Leftrightarrow \quad 3|V| &\geq 2|E| + 4. \end{aligned}$$

But this contradicts the fact that every vertex of G has degree at least 3. □

Algorithm for Feedback Vertex Set

Theorem 4

FEEDBACK VERTEX SET *can be solved in $O^*((3k)^k)$ time.*

Proof (sketch).

- Exhaustively apply the simplification rules.
- The branching rule computes H of size $3k$, and branches into subproblems $(G - v, k - 1)$ for each $v \in H$.



Outline

- 1 Branching algorithms
- 2 Running time analysis
- 3 Feedback Vertex Set
- 4 Maximum Leaf Spanning Tree**
- 5 Further Reading

Maximum Leaf Spanning Tree

A **leaf** of a tree is a vertex with degree 1. A **spanning tree** in a graph $G = (V, E)$ is a subgraph of G that is a tree and has $|V|$ vertices.

MAXIMUM LEAF SPANNING TREE

Input: connected graph G , integer k

Parameter: k

Question: Does G have a spanning tree with at least k leaves?

Property

A k -leaf tree in G is a subgraph of G that is a tree with at least k leaves.

A k -leaf spanning tree in G is a spanning tree in G with at least k leaves.

Lemma 5

Let $G = (V, E)$ be a connected graph.

G has a k -leaf tree $\Leftrightarrow G$ has a k -leaf spanning tree.

Proof.

(\Leftarrow): trivial

(\Rightarrow): Let T be a k -leaf tree in G . By induction on $x := |V| - |V(T)|$, we will show that T can be extended to a k -leaf spanning tree in G .

Base case: $x = 0$ ✓.

Induction: $x > 0$, and assume the claim is true for all $x' < x$. Choose $uv \in E$ such that $u \in V(T)$ and $v \notin V(T)$. Since $T' := (V(T) \cup \{v\}, E(T) \cup \{uv\})$ has $\geq k$ leaves and $< x$ external vertices, it can be extended to a k -leaf spanning tree in G by the induction hypothesis. \square

- The branching algorithm will check whether G has a k -leaf tree.
- A tree with ≥ 3 vertices has at least one **internal** (= non-leaf) vertex.
- “Guess” an internal vertex r , i.e., do a $|V|$ -way branching fixing an initial internal vertex r .

- The branching algorithm will check whether G has a k -leaf tree.
- A tree with ≥ 3 vertices has at least one **internal** (= non-leaf) vertex.
- “Guess” an internal vertex r , i.e., do a $|V|$ -way branching fixing an initial internal vertex r .
- In any branch, the algorithm has computed
 - T – a tree in G
 - I – the internal vertices of T , with $r \in I$
 - B – a subset of the leaves of T where T may be extended: the boundary set
 - L – the remaining leaves of T
 - X – the external vertices $V \setminus V(T)$

- The branching algorithm will check whether G has a k -leaf tree.
- A tree with ≥ 3 vertices has at least one **internal** (= non-leaf) vertex.
- “Guess” an internal vertex r , i.e., do a $|V|$ -way branching fixing an initial internal vertex r .
- In any branch, the algorithm has computed
 - T – a tree in G
 - I – the internal vertices of T , with $r \in I$
 - B – a subset of the leaves of T where T may be extended: the boundary set
 - L – the remaining leaves of T
 - X – the external vertices $V \setminus V(T)$
- The question is whether T can be extended to a k -leaf tree where all the vertices in L are leaves.

Simplification Rules

Apply the first applicable simplification rule:

(Halt-Yes)

If $|L| + |B| \geq k$, then return **YES**.

(Halt-No)

If $|B| = 0$, then return **No**.

(Non-extendable)

If $\exists v \in B$ with $N_G(v) \cap X = \emptyset$, then move v to L .

Lemma 6 (Branching Lemma)

Suppose $u \in B$ and there exists a k -leaf tree T' extending T where u is an internal vertex.

Then, there exists a k -leaf tree T'' extending $(V(T) \cup N_G(u), E(T) \cup \{uv : v \in N_G(u) \cap X\})$.

Branching Lemma

Lemma 6 (Branching Lemma)

Suppose $u \in B$ and there exists a k -leaf tree T' extending T where u is an internal vertex.

Then, there exists a k -leaf tree T'' extending $(V(T) \cup N_G(u), E(T) \cup \{uv : v \in N_G(u) \cap X\})$.

Proof.

Start from $T'' \leftarrow T'$ and perform the following operation for each $v \in N_G(u) \cap X$. If $v \notin V(T')$, then add the vertex v and the edge uv .

Otherwise, add the edge uv , creating a cycle C in T and remove the other edge of C incident to v . This does not decrease the number of leaves, since it only increases the number of edges incident to u , and u was already internal. \square

Follow Path Lemma

Lemma 7 (Follow Path Lemma)

Suppose $u \in B$ and $|N_G(u) \cap X| = 1$. Let $N_G(u) \cap X = \{v\}$.

If there exists a k -leaf tree extending T where u is internal, but no k -leaf tree extending T where u is a leaf, then there exists a k -leaf tree extending T where both u and v are internal.

Follow Path Lemma

Lemma 7 (Follow Path Lemma)

Suppose $u \in B$ and $|N_G(u) \cap X| = 1$. Let $N_G(u) \cap X = \{v\}$.

If there exists a k -leaf tree extending T where u is internal, but no k -leaf tree extending T where u is a leaf, then there exists a k -leaf tree extending T where both u and v are internal.

Proof.

Suppose not, and let T' be a k -leaf tree extending T where u is internal and v is a leaf. But then, $T - v$ is a k -leaf tree as well. \square

Algorithm

- Apply halting & simplification rules
- Select $u \in B$. Branch into
 - $u \in L$
 - $u \in I$. In this case, add $X \cap N_G(u)$ to B (Branching Lemma).
 - In the special case where $|X \cap N_G(u)| = 1$, denote $\{v\} = X \cap N_G(u)$, make v internal, and add $N_G(v) \cap X$ to B , continuing the same way until reaching a vertex with at least 2 neighbors in X (Follow Path Lemma).
 - In the special case where $|X \cap N_G(u)| = 0$, return **No**.

Algorithm

- Apply halting & simplification rules
- Select $u \in B$. Branch into
 - $u \in L$
 - $u \in I$. In this case, add $X \cap N_G(u)$ to B (Branching Lemma).
 - In the special case where $|X \cap N_G(u)| = 1$, denote $\{v\} = X \cap N_G(u)$, make v internal, and add $N_G(v) \cap X$ to B , continuing the same way until reaching a vertex with at least 2 neighbors in X (Follow Path Lemma).
 - In the special case where $|X \cap N_G(u)| = 0$, return **No**.
- In one branch, a vertex moves from B to L ; in the other branch, $|B|$ increases by at least 1.

Running time analysis

- Consider the “measure” $\mu := 2k - 2|L| - |B|$
- We have that $0 \leq \mu \leq 2k$
- Branch where $u \in L$:
 - $|B|$ decreases by 1, $|L|$ increases by 1
 - μ decreases by 1
- Branch where $u \in I$.
 - u moves from B to I
 - ≥ 2 vertices move from X to B
 - μ decreases by at least 1
- Binary search tree of height $\leq \mu \leq 2k$

Result for Maximum Leaf Spanning Tree

Theorem 8 ((Kneis, Langer, and Rossmanith, 2011))

MAXIMUM LEAF SPANNING TREE *can be solved in $O^*(4^k)$ time.*

Outline

- 1 Branching algorithms
- 2 Running time analysis
- 3 Feedback Vertex Set
- 4 Maximum Leaf Spanning Tree
- 5 Further Reading

Further Reading

- Chapter 3, *Bounded Search Trees* in (Cygan et al., 2015)
- Chapter 3, *Bounded Search Trees* in (Downey and Fellows, 2013)
- Chapter 8, *Depth-Bounded Search Trees* in (Niedermeier, 2006)

References I

- Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh (2015). *Parameterized Algorithms*. Springer. DOI: [10.1007/978-3-319-21275-3](https://doi.org/10.1007/978-3-319-21275-3).
- Rodney G. Downey and Michael R. Fellows (2013). *Fundamentals of Parameterized Complexity*. Springer. DOI: [10.1007/978-1-4471-5559-1](https://doi.org/10.1007/978-1-4471-5559-1).
- Joachim Kneis, Alexander Langer, and Peter Rossmanith (2011). “A New Algorithm for Finding Trees with Many Leaves”. In: *Algorithmica* 61.4, pp. 882–897.
- Rolf Niedermeier (2006). *Invitation to Fixed Parameter Algorithms*. Oxford University Press. DOI: [10.1093/ACPROF:OS0/9780198566076.001.0001](https://doi.org/10.1093/ACPROF:OS0/9780198566076.001.0001).