Introduction to solving intractable problems

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UNSW

Outline

1. Algorithms for NP-hard problems
2. Exponential Time Algorithms
3. Parameterized Complexity
   - FPT Algorithm for Vertex Cover
   - Algorithms for Vertex Cover
4. Further Reading

Outline

Central question

P vs. NP

NP-hard problems

- no known polynomial time algorithm for any NP-hard problem
- belief: P $\neq$ NP
- What to do when facing an NP-hard problem?

Example problem

Monitoring a power grid

Tammy is responsible for fault detection on the power grid of an energy company.

She has access to k monitoring devices. Each one can be placed on a node of the electrical grid and can monitor the power lines that are connected to this node.

Tammy's objective is to place the monitoring devices in such a way that each power line is monitored by at least one monitoring device.

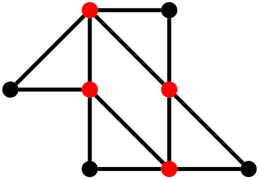
Let us first give an abstraction of this problem and formulate it as a decision problem for graphs.

Example problem: Vertex Cover

A vertex cover in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every edge of G has an endpoint in S.

Vertex Cover	
Input:	Graph G, integer k
Question:	Does G have a vertex cover of size k?

Note: Vertex Cover is NP-complete.



Coping with NP-hardness

- Approximation algorithms
  - There is a polynomial-time algorithm, which, given a graph G, finds a vertex cover of G of size at most $2 \cdot \mathsf{OPT}$, where OPT is the size of a smallest vertex cover of G.
- Exact exponential time algorithms
  - There is an algorithm solving Vertex Cover in time $O(1.1970^n)$, where $n = |V|$ (Xiao and Nagamochi, 2017).
- Fixed parameter algorithms
  - There is an algorithm solving Vertex Cover in time $O(1.2738^k + kn)$ (Chen, Kanj, and Xia, 2010).
- Heuristics
  - The COVER heuristic (COVer Edges Randomly) finds a smaller vertex cover than state-of-the-art heuristics on a suite of hard benchmark instances (Richter, Helmert, and Gretton, 2007).
- Restricting the inputs
  - Vertex Cover can be solved in polynomial time on bipartite graphs, trees, interval graphs, etc. (Golumbic, 2004).
- Quantum algorithms?
  - Not believed to solve NP-hard problems in polynomial time (Aaronson, 2005). Quadratic speedup possible in some cases.

Aims of this course

Design and analyze algorithms for NP-hard problems.

We focus on algorithms that solve NP-hard problems exactly and analyze their worst case running time.

Outline

Running times

Worst case running time of an algorithm.

- An algorithm is polynomial if $\exists c \in \mathbb{N}$ such that the algorithm solves every instance in time $O(n^c)$, where n is the size of the instance.
  Also: $n^{O(1)}$ or $\mathsf{poly}(n)$.
- quasi-polynomial: $2^{O(\log^c n)}$, $c \in O(1)$
- sub-exponential: $2^{o(n)}$
- exponential: $2^{\mathsf{poly}(n)}$
- double-exponential: $2^{2^{\mathsf{poly}(n)}}$

O^*-notation ignores polynomial factors in the input size:

$$O^*(f(n)) \equiv O(f(n) \cdot \mathsf{poly}(n))$$
$$O^*(f(k)) \equiv O(f(k) \cdot \mathsf{poly}(n))$$

Brute-force algorithms for NP-hard problems

Theorem 1

Every problem in NP can be solved in exponential time.

Brute-force algorithms for NP-hard problems

Theorem 1

Every problem in NP can be solved in exponential time.

For a proof, see the lecture on NP-completeness.

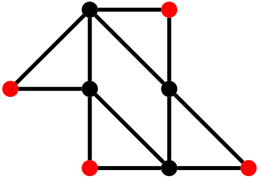
Three main categories for NP-complete problems

- Subset problems
- Permutation problems
- Partition problems

Subset Problem: Independent Set

An independent set in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that the vertices in S are pairwise non-adjacent in G.

Independent Set	
Input:	Graph G, integer k
Question:	Does G have an independent set of size k?



Brute-force:

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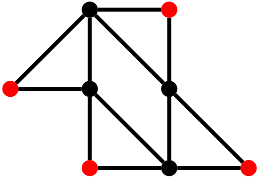
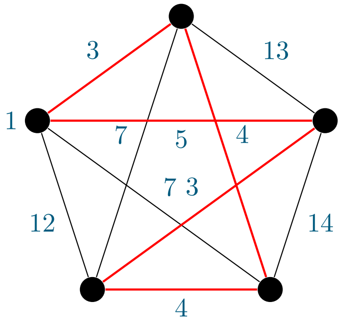


Brute-force: $O^*(2^n)$, where $n = |V(G)|$

Permutation Problem: Traveling SalesPerson

Traveling SalesPerson (TSP)

Input:	a set of n cities, the distance $d(i,j) \in \mathbb{N}$ between every two cities i and j, integer k
Question:	Is there a permutation of the cities (a tour) such that the total distance when traveling from city to city in the specified order, and returning back to the origin, is at most k?



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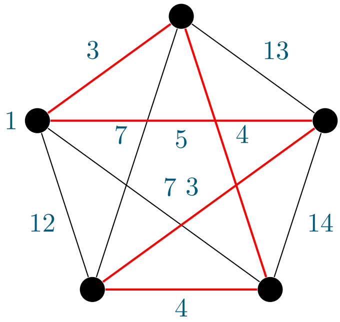
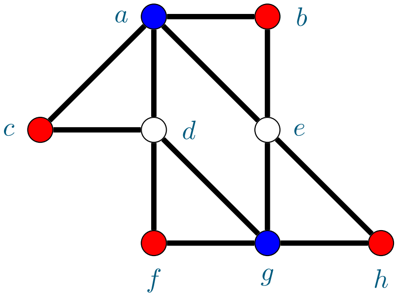


Brute-force: $O^*(n!) \subseteq 2^{O(n \log n)}$

Partition Problem: Coloring

A k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, \ldots, k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

Coloring	
Input:	Graph G, integer k
Question:	Does G have a k-coloring?



Brute-force:

Partition Problem: Coloring

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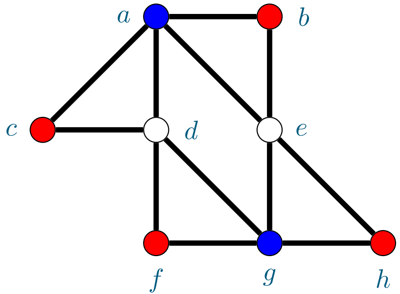


Brute-force: $O^*(k^n)$, where $n = |V(G)|$

Exponential Time Algorithms

- natural question in Algorithms:
  design faster (worst-case analysis) algorithms for problems
- might lead to practical algorithms
  - for small instances
    - you don't want to design software where your client/boss can find with better solutions *by hand* than your software
  - subroutines for
    - (sub)exponential time approximation algorithms
    - randomized algorithms with expected polynomial run time

Solve an NP-hard problem

- exhaustive search
  - trivial method
  - try all candidate solutions (certificates) for a ground set on n elements
  - running times for problems in NP
    - Subset Problems: $O^*(2^n)$
    - Permutation Problems: $O^*(n!)$
    - Partition Problems: $O^*(c^{n \log n})$
- faster exact algorithms
  - for some problems, it is possible to obtain provably faster algorithms
  - running times $O(1.0836^n), O(1.4689^n), O(1.9977^n)$

Exponential Time Algorithms in Practice

- How large are the instances one can solve in practice?

Available time nb. of operations	1 s 2^{38}	1 min $\sim 2^{44}$	1 hour $\sim 2^{50}$	3 days $\sim 2^{56}$	6 months $\sim 2^{62}$
n^5	194	446	1,024	2,352	5,404
n^{10}	14	21	32	49	74
1.05^n	540	625	711	796	881
1.1^n	276	320	364	407	451
1.5^n	65	75	85	96	106
2^n	38	44	50	56	62
5^n	16	19	22	24	27
$n!$	14	16	17	19	20

Note: Intel Core i7-8086K executes $\sim 2^{38}$ instructions per second at 5 GHz.

"For every polynomial-time algorithm you have, there is an exponential algorithm that I would rather run."

– Alan Perlis (1922-1990, programming languages, 1st recipient of Turing Award)

Hardware vs. Algorithms

- Suppose a 2^n algorithm enables us to solve instances up to size x
- Faster processors
  - processor speed doubles after 18–24 months (Moore's law)
  - can solve instances up to size $x + 1$
- Faster algorithm
  - design an $O^*(2^{n/2}) \subseteq O(1.4143^n)$ time algorithm
  - can solve instances up to size $2 \cdot x$

Outline

A story

A computer scientist meets a biologist . . .

Eliminating conflicts from experiments

$n = 1000$ experiments,
$k = 20$ experiments failed

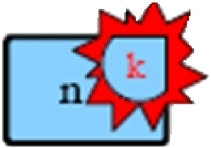
Theoretical	Running Time Number of Instructions	Real
2^n	$1.07 \cdot 10^{301}$	$4.941 \cdot 10^{282}$ years
n^k	10^{60}	$4.611 \cdot 10^{41}$ years
$2^k \cdot n$	$1.05 \cdot 10^9$	0.01526 seconds

Notes

- We assume that 2^{36} instructions are carried out per second.
- The Big Bang happened roughly $13.5 \cdot 10^9$ years ago.

Goal of Parameterized Complexity

Confine the combinatorial explosion to a parameter k.



For which problem–parameter combinations can we find algorithms with running times of the form

$$f(k) \cdot n^{O(1)},$$

where the f is a computable function independent of the input size n?

Examples of Parameters

A Parameterized Problem

Input:	an instance of the problem
Parameter:	a parameter k
Question:	a Yes/No question about the instance and the parameter

- A parameter can be
  - input size (trivial parameterization)
  - solution size
  - related to the structure of the input (maximum degree, treewidth, branchwidth, genus, ...)
  - etc.

Main Complexity Classes

P: class of problems that can be solved in time $n^{O(1)}$
FPT: class of problems that can be solved in time $f(k) \cdot n^{O(1)}$
W[·]: parameterized intractability classes
XP: class of problems that can be solved in time $f(k) \cdot n^{g(k)}$

$$\mathsf{P} \subseteq \mathsf{FPT} \subseteq \mathsf{W}[1] \subseteq \mathsf{W}[2] \cdots \subseteq \mathsf{W}[P] \subseteq \mathsf{XP}$$

Known: If FPT $=$ W[1], then the Exponential Time Hypothesis fails, i.e. 3-SAT can be solved in time $2^{o(n)}$.

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1. Algorithms for NP-hard problems

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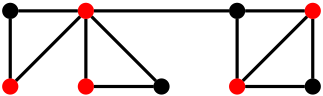
3. Parameterized Complexity
   - FPT Algorithm for Vertex Cover
   - Algorithms for Vertex Cover

4. Further Reading

Vertex Cover

Vertex Cover (VC)

Input:	A graph $G = (V, E)$ on n vertices, an integer k
Parameter:	k
Question:	Is there a set of vertices $C \subseteq V$ of size at most k such that every edge has at least one endpoint in C?



Outline

Brute Force Algorithms

- $2^n \cdot n^{O(1)}$ not FPT
- $n^k \cdot n^{O(1)}$ not FPT

An FPT Algorithm

```
Algorithm vc1(G, k);
if E = ∅ then                                          // all edges are covered
    return Yes
else if k ≤ 0 then                                     // we cannot select any vertex
    return No
else
    Select an edge uv ∈ E;
    return vc1(G − u, k − 1) ∨ vc1(G − v, k − 1)
```

Running Time Analysis

- Let us look at an arbitrary execution of the algorithm.
- Recursive calls form a search tree T
  - with depth $\leq k$
  - where each node has ≤ 2 children
- $\Rightarrow T$ has $\leq 2^k$ leaves and $\leq 2^k - 1$ internal nodes
- at each node the algorithm spends time $n^{O(1)}$
- The running time is $O^*(2^k)$

A faster FPT Algorithm

A faster FPT Algorithm

Algorithm vc2(G, k);

```
if E = ∅ then                                  // all edges are covered
    return Yes
else if k ≤ 0 then                             // we used too many vertices
    return No
else if Δ(G) ≤ 2 then                          // G has maximum degree ≤ 2
    Solve the problem in polynomial time;
else
    Select a vertex v of maximum degree;
    return vc2(G − v, k − 1) ∨ vc2(G − N[v], k − d(v))
```

Running time analysis of vc2

- Number of leaves of the search tree:

$$\begin{aligned} T(k) &\leq T(k-1) + T(k-3) \\ x^k &\leq x^{k-1} + x^{k-3} \\ x^3 - x^2 - 1 &\leq 0 \end{aligned}$$

- The equation $x^3 - x^2 - 1 = 0$ has a unique positive real solution: $x \approx 1.4655\ldots$
- Running time: $1.4656^k \cdot n^{O(1)}$

Outline

Further Reading

- Exponential-time algorithms
  - Chapter 1, *Introduction*, in (Fomin and Kratsch, 2010).
  - Survey on exponential-time algorithms (Woeginger, 2001).
  - Chapter 1, *Introduction*, in (Gaspers, 2010).
- Parameterized Complexity
  - Chapter 1, *Introduction*, in (Cygan et al., 2015)
  - Chapter 2, *The Basic Definitions*, in (Downey and Fellows, 2013)
  - Chapter I, *Foundations*, in (Niedermeier, 2006)
  - *Preface* in (Flum and Grohe, 2006)

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NP-completeness

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Polynomial time

Polynomial-time algorithm

Polynomial-time algorithm:
There exists a constant $c \in \mathbb{N}$ such that the algorithm has (worst-case) running-time $O(n^c)$, where n is the size of the input.

Polynomial time

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Example

Polynomial: n; $n^2 \log_2 n$; n^3; n^{20}
Super-polynomial: $n^{\log_2 n}$; $2^{\sqrt{n}}$; 1.001^n; 2^n; $n!$

Tractable problems

Central Question

Which computational problems have polynomial-time algorithms?

Million-dollar question

Intriguing class of problems: NP-complete problems.

NP-complete problems

It is unknown whether NP-complete problems have polynomial-time algorithms.

- A polynomial-time algorithm for one NP-complete problem would imply polynomial-time algorithms for all problems in NP.

Gerhard Woeginger's P vs NP page:
http://www.win.tue.nl/~gwoegi/P-versus-NP.htm

Polynomial vs. NP-complete

Polynomial

- Shortest Path: Given a graph G, two vertices a and b of G, and an integer k, does G have a simple a–b-path of length at most k?
- Euler Tour: Given a graph G, does G have a cycle that traverses each edge of G exactly once?
- 2-CNF SAT: Given a propositional formula F in 2-CNF, is F satisfiable?
  A k-CNF formula is a conjunction (AND) of clauses, and each clause is a disjunction (OR) of at most k literals, which are negated or unnegated Boolean variables.

NP-complete

- Longest Path: Given a graph G and an integer k, does G have a simple path of length at least k?
- Hamiltonian Cycle: Given a graph G, does G have a simple cycle that visits each vertex of G?
- 3-CNF SAT: Given a propositional formula F in 3-CNF, is F satisfiable?
  Example:
  $(x \vee \neg y \vee z) \wedge (\neg x \vee z) \wedge (\neg y \vee \neg z)$.

Overview

What's next?

- Formally define P, NP, and NP-complete (NPC)
- (New) skill: show that a problem is NP-complete

Outline

Decision problems and Encodings

<Name of Decision Problem>
Input: <What constitutes an instance>
Question: <Yes/No question>

Decision problems and Encodings

<Name of Decision Problem>
Input: <What constitutes an instance>
Question: <Yes/No question>

We want to know which decision problems can be solved in polynomial time – polynomial in the size of the input n.

- Assume a “reasonable” encoding of the input
- Many encodings are polynomial-time equivalent; i.e., one encoding can be computed from another in polynomial time.
- Important exception: unary versus binary encoding of integers.
  - An integer x takes $\lceil \log_2 x \rceil$ bits in binary and $x = 2^{\log_2 x}$ bits in unary.

Formal-language framework

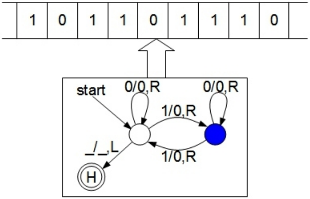
We can view decision problems as languages.

- Alphabet Σ: finite set of symbols. W.l.o.g., $\Sigma = \{0, 1\}$
- Language L over Σ: set of strings made with symbols from Σ: $L \subseteq \Sigma^*$
- Fix an encoding of instances of a decision problem Π into Σ
- Define the language $L_\Pi \subseteq \Sigma^*$ such that

$$x \in L_\Pi \Leftrightarrow x \text{ is a Yes-instance for } \Pi$$

Non-deterministic Turing Machine (NTM)

- input word $x \in \Sigma^*$ placed on an infinite tape (memory)
- read-write head initially placed on the first symbol of x
- computation step: if the machine is in state s and reads a, it can move into state s', writing b, and moving the head into direction $D \in \{L, R\}$ if $((s,a),(s',b,D)) \in \delta$.



- Q: finite, non-empty set of states
- Γ: finite, non-empty set of tape symbols
- $\_ \in \Gamma$: blank symbol (the only symbol allowed to occur on the tape infinitely often)
- $\Sigma \subseteq \Gamma \setminus \{b\}$: set of input symbols
- $q_0 \in Q$: start state
- $A \subseteq Q$: set of accepting (final) states
- $\delta \subseteq (Q \setminus A \times \Gamma) \times (Q \times \Gamma \times \{L, R\})$: transition relation, where L stands for a move to the left and R for a move to the right.

Accepted Language

Definition 1

A NTM accepts a word $x \in \Sigma^*$ if there exists a sequence of computation steps starting in the start state and ending in an accept state.

Definition 2

The language accepted by an NTM is the set of words it accepts.

Acceptance in polynomial time

Definition 3

A language L is accepted in polynomial time by an NTM M if

- L is accepted by M, and
- there is a constant k such that for any word $x \in L$, the NTM M accepts x in $O(|x|^k)$ computation steps.

Deterministic Turing Machine

Definition 4

A Deterministic Turing Machine (DTM) is a Non-deterministic Turing Machine where the transition relation contains at most one tuple $((s,a),(\cdot,\cdot,\cdot))$ for each $s \in Q \setminus A$ and $a \in \Gamma$.

The transition relation δ can be viewed as a function $\delta : Q \setminus A \times \Gamma \to Q \times \Gamma \times \{L, R\}$.
$\Rightarrow$ For a given input word $x \in \Sigma^*$, there is exactly one sequence of computation steps starting in the start state.

DTM equivalents

Many computational models are polynomial-time equivalent to DTMs:

- Random Access Machine (RAM, used for algorithms in the textbook)
- variants of Turing machines (multiple tapes, infinite only in one direction, ...)
- ...

P and NP

Definition 5 (P)

$\mathsf{P} = \{L \subseteq \Sigma^* : \text{ there is a DTM accepting } L \text{ in polynomial time}\}$

Definition 6 (NP)

$\mathsf{NP} = \{L \subseteq \Sigma^* : \text{ there is a NTM accepting } L \text{ in polynomial time}\}$

Definition 7 (coNP)

$\mathsf{coNP} = \{L \subseteq \Sigma^* : \Sigma^* \setminus L \in \mathsf{NP}\}$

coP?

Theorem 8

If $L \in \mathsf{P}$, then there is a polynomial-time DTM that halts in an accepting state on every word in L and it halts in a non-accepting state on every word not in L.

coP?

Theorem 8

If $L \in \mathsf{P}$, then there is a polynomial-time DTM that halts in an accepting state on every word in L and it halts in a non-accepting state on every word not in L.

Proof sketch.

Suppose $L \in \mathsf{P}$. By the definition of P, there is a DTM M that accepts L in polynomial time.
Idea: design a DTM M' that simulates M for $c \cdot n^k$ steps, where $c \cdot n^k$ is the running time of M and transitions to a non-accepting state if M does not halt in an accepting state.
(Note that this proof is nonconstructive: we might not know the running time of M.) $\square$

NP and certificates

Non-deterministic choices

A NTM for an NP-language L makes a polynomial number of non-deterministic choices on input $x \in L$.
We can encode these non-deterministic choices into a certificate c, which is a polynomial-length word.
Now, there exists a DTM, which, given x and c, verifies that $x \in L$ in polynomial time.

Thus, $L \in \mathrm{NP}$ iff there is a DTM V and for each $x \in L$ there exists a polynomial-length certificate c such that $V(x, c) = 1$, but $V(y, \cdot) = 0$ for each $y \notin L$.

CNF-SAT is in NP

- A CNF formula is a propositional formula in conjunctive normal form: a conjunction (AND) of clauses; each clause is a disjunction (OR) of literals; each literal is a negated or unnegated Boolean variable.
- An assignment $\alpha : \text{var}(F) \to \{0, 1\}$ satisfies a clause C if it sets a literal of C to true, and it satisfies F if it satisfies all clauses in F.

CNF-SAT

Input:	CNF formula F
Question:	Does F have a satisfying assignment?

Example: $(x \vee \neg y \vee z) \wedge (\neg x \vee z) \wedge (\neg y \vee \neg z)$.

Lemma 9

CNF-SAT $\in$ NP.

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Lemma 9

CNF-SAT $\in$ NP.

Proof.

Certificate: assignment α to the variables.
Given a certificate, it can be checked in polynomial time whether all clauses are satisfied. $\square$

Brute-force algorithms for problems in NP

Theorem 10

Every problem in NP can be solved in exponential time.

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Every problem in NP can be solved in exponential time.

Proof.

Let Π be an arbitrary problem in NP. [Use certificate-based definition of NP]
We know that $\exists$ a polynomial p and a polynomial-time verification algorithm V such that:

- for every $x \in \Pi$ (i.e., every YES-instance for Π) $\exists$ string $c \in \{0,1\}^*$, $|c| \le p(|x|)$, such that $V(x,c) = 1$, and
- for every $x \notin \Pi$ (i.e., every NO-instance for Π) and every string $c \in \{0,1\}^*$, $V(x,c) = 0$.

Brute-force algorithms for problems in NP

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Proof.

Let Π be an arbitrary problem in NP. [Use certificate-based definition of NP]
We know that $\exists$ a polynomial p and a polynomial-time verification algorithm V such that:

- for every $x \in \Pi$ (i.e., every YES-instance for Π) $\exists$ string $c \in \{0,1\}^*$, $|c| \le p(|x|)$, such that $V(x,c) = 1$, and
- for every $x \notin \Pi$ (i.e., every NO-instance for Π) and every string $c \in \{0,1\}^*$, $V(x,c) = 0$.

Now, we can prove there exists an exponential-time algorithm for Π with input x:

- For each string $c \in \{0,1\}^*$ with $|c| \le p(|x|)$, evaluate $V(x,c)$ and return YES if $V(x,c) = 1$.
- Return NO.

Running time: $2^{p(|x|)} \cdot n^{O(1)} \subseteq 2^{O(2 \cdot p(|x|))} = 2^{O(p(|x|))}$, but non-constructive. $\square$

Outline

1 Overview

2 Turing Machines, P, and NP

3 Reductions and NP-completeness

4 NP-complete problems

5 Further Reading

Polynomial-time reduction

Definition 11

A language L_1 is polynomial-time reducible to a language L_2, written $L_1 \leq_P L_2$, if there exists a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ such that for all $x \in \Sigma^*$,

$$x \in L_1 \Leftrightarrow f(x) \in L_2.$$

A polynomial time algorithm computing f is a reduction algorithm.

New polynomial-time algorithms via reductions

Lemma 12

If $L_1, L_2 \in \Sigma^*$ are languages such that $L_1 \leq_P L_2$, then $L_2 \in$ P implies $L_1 \in$ P.

Definition 13 (NP-hard)

A language $L \subseteq \Sigma^*$ is NP-hard if

$$L' \leq_P L \text{ for every } L' \in \text{NP}.$$

Definition 14 (NP-complete)

A language $L \subseteq \Sigma^*$ is NP-complete (in NPC) if

1. $L \in$ NP, and
2. L is NP-hard.

A first NP-complete problem

Theorem 15

CNF-SAT is NP-complete.

Proved by encoding NTMs into SAT (Cook, 1971; Levin, 1973) and then CNF-SAT (Karp, 1972).

Proving NP-completeness

Lemma 16

If L is a language such that $L' \leq_P L$ for some $L' \in$ NPC, then L is NP-hard. If, in addition, $L \in$ NP, then $L \in$ NPC.

Proving NP-completeness

Lemma 16

If L is a language such that $L' \leq_P L$ for some $L' \in$ NPC, then L is NP-hard. If, in addition, $L \in$ NP, then $L \in$ NPC.

Proof.

For all $L'' \in$ NP, we have $L'' \leq_P L' \leq_P L$.
By transitivity, we have $L'' \leq_P L$.
Thus, L is NP-hard. □

Proving NP-completeness (2)

Method to prove that a language L is NP-complete:

1. Prove $L \in$ NP
2. Prove L is NP-hard.
   - Select a known NP-complete language L'.
   - Describe an algorithm that computes a function f mapping every instance $x \in \Sigma^*$ of L' to an instance $f(x)$ of L.
   - Prove that $x \in L' \Leftrightarrow f(x) \in L$ for all $x \in \Sigma^*$.
   - Prove that the algorithm computing f runs in polynomial time.

Outline

3-CNF SAT is NP-hard

Theorem 17

3-CNF SAT is NP-complete.

Proof.

3-CNF SAT is NP-hard

Theorem 17

3-CNF SAT is NP*-complete.*

Proof.

3-CNF SAT is in NP, since it is a special case of CNF-SAT.

3-CNF SAT is NP-hard

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To show that 3-CNF SAT is NP-hard, we give a polynomial reduction from CNF-SAT.

3-CNF SAT is NP-hard

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Proof.

3-CNF SAT is in NP, since it is a special case of CNF-SAT.
To show that 3-CNF SAT is NP-hard, we give a polynomial reduction from CNF-SAT.
Let F be a CNF formula. The reduction algorithm constructs a 3-CNF formula F' as follows. For each clause C in F:

- If C has at most 3 literals, then copy C into F'.
- Otherwise, denote $C = (\ell_1 \vee \ell_2 \vee \cdots \vee \ell_k)$.

3-CNF SAT is NP-hard

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- If C has at most 3 literals, then copy C into F'.
- Otherwise, denote $C = (\ell_1 \vee \ell_2 \vee \cdots \vee \ell_k)$. Create $k-3$ new variables $y_1, \dots, y_{k-3}$, and add the clauses $(\ell_1 \vee \ell_2 \vee y_1), (\neg y_1 \vee \ell_3 \vee y_2), (\neg y_2 \vee \ell_4 \vee y_3), \dots, (\neg y_{k-3} \vee \ell_{k-1} \vee \ell_k)$.

3-CNF SAT is NP-hard

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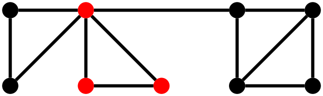
Show that F is satisfiable $\Leftrightarrow$ F' is satisfiable.
Show that F' can be computed in polynomial time (trivial; use a RAM). $\square$

Clique

A clique in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every two vertices of S are adjacent in G.

CLIQUE
Input: Graph G, integer k
Question: Does G have a clique of size k?



Theorem 18

CLIQUE *is* NP*-complete.*

Clique (2)

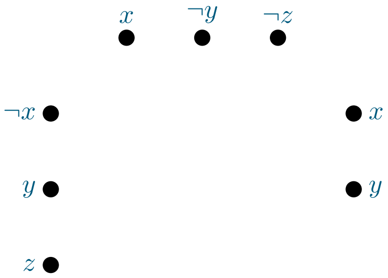
- CLIQUE is in NP

Clique (2)

- CLIQUE is in NP
- Let $F = C_1 \wedge C_2 \wedge \ldots C_k$ be a 3-CNF formula
- Construct a graph G that has a clique of size k iff F is satisfiable

$(\neg x \vee y \vee z) \wedge (x \vee \neg y \vee \neg z) \wedge (x \vee y)$

Clique (2)



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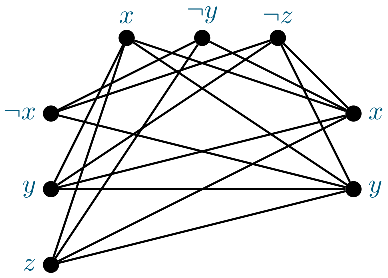
- For each clause $C_r = (\ell_1^r \vee \cdots \vee \ell_w^r)$, $1 \leq r \leq k$, create w new vertices $v_1^r, \ldots, v_w^r$

Clique (2)



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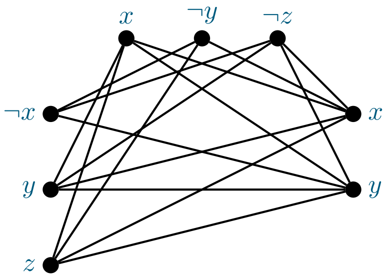
- Construct a graph G that has a clique of size k iff F is satisfiable
- For each clause $C_r = (\ell_1^r \vee \cdots \vee \ell_w^r)$, $1 \leq r \leq k$, create w new vertices $v_1^r, \dots, v_w^r$
- Add an edge between v_i^r and v_j^s if

$$\begin{aligned} r &\neq s && \text{and} \\ \ell_i^r &\neq \neg \ell_j^s && \text{where } \neg\neg x = x. \end{aligned}$$

- Check correctness and polynomial running time

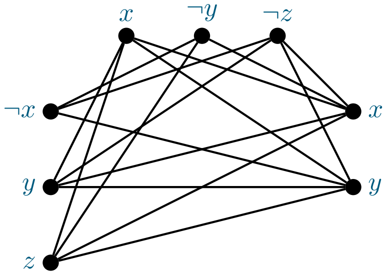
Clique (2)



$(\neg x \vee y \vee z) \wedge (x \vee \neg y \vee \neg z) \wedge (x \vee y)$

- Correctness: F has a satisfying assignment iff G has a clique of size k.

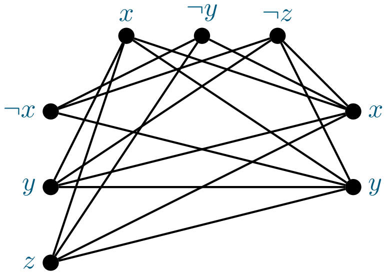
Clique (2)



$(\neg x \vee y \vee z) \wedge (x \vee \neg y \vee \neg z) \wedge (x \vee y)$

- Correctness: F has a satisfying assignment iff G has a clique of size k.
- $(\Rightarrow)$: Let α be a sat. assignment for F. For each clause C_r, choose a literal ℓ_i^r with $\alpha(\ell_i^r) = 1$, and denote by s^r the corresponding vertex in G. Now, $\{s^r : 1 \leq r \leq k\}$ is a clique of size k in G since $\alpha(x) \neq \alpha(\neg x)$.

Clique (2)



$(\neg x \vee y \vee z) \wedge (x \vee \neg y \vee \neg z) \wedge (x \vee y)$

- Correctness: F has a satisfying assignment iff G has a clique of size k.
- $(\Rightarrow)$: Let α be a sat. assignment for F. For each clause C_r, choose a literal ℓ_i^r with $\alpha(\ell_i^r) = 1$, and denote by s^r the corresponding vertex in G. Now, $\{s^r : 1 \leq r \leq k\}$ is a clique of size k in G since $\alpha(x) \neq \alpha(\neg x)$.
- $(\Leftarrow)$: Let S be a clique of size k in G. Then, S contains exactly one vertex $s_r \in \{v_1^r, \ldots, v_w^r\}$ for each $r \in \{1, \ldots, k\}$. Denote by l^r the corresponding literal. Now, for any r, r', it is not the case that $l_r = \neg l_{r'}$. Therefore, there is an assignment α to $\mathrm{var}(F)$ such that $\alpha(l_r) = 1$ for each $r \in \{1, \ldots, k\}$ and α satisfies F.

Vertex Cover

A vertex cover in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every edge of G has an endpoint in S.

> VERTEX COVER
> Input: Graph G, integer k
> Question: Does G have a vertex cover of size k?

Theorem 19

VERTEX COVER *is* NP*-complete.*

The proof is left as an exercise.

Hamiltonian Cycle

A Hamiltonian Cycle in a graph $G = (V, E)$ is a cycle visiting each vertex exactly once.
(Alternatively, a permutation of V such that every two consecutive vertices are adjacent and the first and last vertex in the permutation are adjacent.)

HAMILTONIAN CYCLE
Input: Graph G
Question: Does G have a Hamiltonian Cycle?

Theorem 20

HAMILTONIAN CYCLE *is* NP*-complete.*

Proof sketch.

Hamiltonian Cycle

A Hamiltonian Cycle in a graph $G = (V, E)$ is a cycle visiting each vertex exactly once.
(Alternatively, a permutation of V such that every two consecutive vertices are adjacent and the first and last vertex in the permutation are adjacent.)

HAMILTONIAN CYCLE	
Input:	Graph G
Question:	Does G have a Hamiltonian Cycle?

Theorem 20

HAMILTONIAN CYCLE *is* NP*-complete.*

Proof sketch.

- HAMILTONIAN CYCLE is in NP: the certificate is a Hamiltonian Cycle of G.

Hamiltonian Cycle

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Input:	Graph G
Question:	Does G have a Hamiltonian Cycle?

Theorem 20

HAMILTONIAN CYCLE *is* NP*-complete.*

Proof sketch.

- HAMILTONIAN CYCLE is in NP: the certificate is a Hamiltonian Cycle of G.
- Let us show: VERTEX COVER $\leq_P$ HAMILTONIAN CYCLE

...

Hamiltonian Cycle (2)

Theorem 21

HAMILTONIAN CYCLE *is* NP*-complete.*

Proof sketch (continued).

- Let us show: VERTEX COVER $\leq_P$ HAMILTONIAN CYCLE

Hamiltonian Cycle (2)

Theorem 21

HAMILTONIAN CYCLE *is* NP*-complete.*

Proof sketch (continued).

- Let us show: VERTEX COVER $\leq_P$ HAMILTONIAN CYCLE
- Let $(G = (V, E), k)$ be an instance for VERTEX COVER (VC).
- We will construct an equivalent instance G' for HAMILTONIAN CYCLE (HC).

Hamiltonian Cycle (2)

Theorem 21

HAMILTONIAN CYCLE *is* NP*-complete.*

Proof sketch (continued).

- Let us show: VERTEX COVER $\leq_P$ HAMILTONIAN CYCLE
- Let $(G = (V, E), k)$ be an instance for VERTEX COVER (VC).
- We will construct an equivalent instance G' for HAMILTONIAN CYCLE (HC).
- Intuition: Non-deterministic choices
  - for VC: which vertices to select in the vertex cover
  - for HC: which route the cycle takes

. . . □

Hamiltonian Cycle (3)

Theorem 22

HAMILTONIAN CYCLE *is* NP*-complete.*

Proof sketch (continued).

- Add k vertices $s_1, \dots, s_k$ to G' (*selector vertices*)

Hamiltonian Cycle (3)

Theorem 22

HAMILTONIAN CYCLE *is* NP*-complete.*

Proof sketch (continued).

- Add k vertices $s_1, \ldots, s_k$ to G' (*selector vertices*)
- Each edge of G will be represented by a gadget (subgraph) of G'
- s.t. the set of edges covered by a vertex x in G corresponds to a partial cycle going through all gadgets of G' representing these edges.

Hamiltonian Cycle (3)

Theorem 22

HAMILTONIAN CYCLE *is* NP*-complete.*

Proof sketch (continued).

- Add k vertices $s_1, \dots, s_k$ to G' (*selector vertices*)
- Each edge of G will be represented by a gadget (subgraph) of G'
- s.t. the set of edges covered by a vertex x in G corresponds to a partial cycle going through all gadgets of G' representing these edges.
- Attention: we need to allow for an edge to be covered by both endpoints

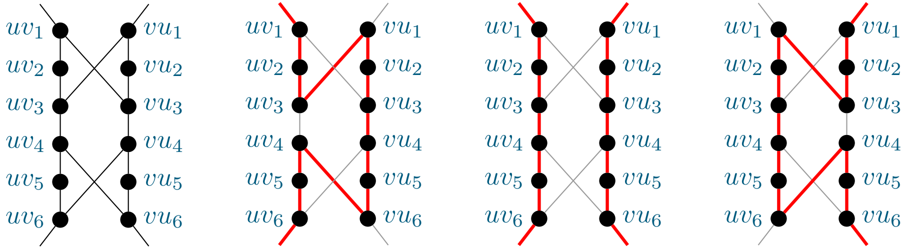
...

□

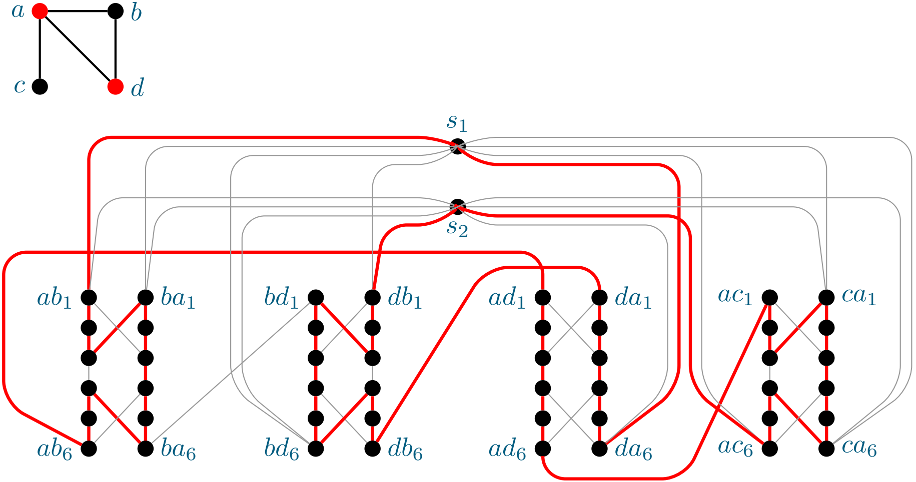
Hamiltonian Cycle (4)

Gadget representing the edge $\{u, v\} \in E$

Its states: 'covered by u', 'covered by u and v', 'covered by v'



Hamiltonian Cycle (5)



Outline

Further Reading

- Chapter 34, **NP-Completeness**, in (Cormen et al., 2009)
- Garey and Johnson's influential reference book (Garey and Johnson, 1979)

References I

Stephen A. Cook (1971). "The Complexity of Theorem-Proving Procedures". In: *Proceedings of the 3rd Annual ACM Symposium on Theory of Computing (STOC 1971)*, pp. 151–158.

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Kernelization

Serge Gaspers

UNSW

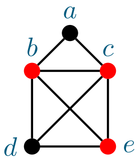
Outline

Outline

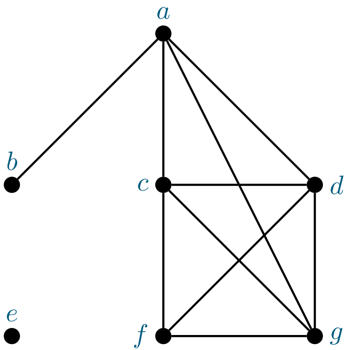
Vertex cover

A vertex cover of a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that for each edge $\{u, v\} \in E$, we have $u \in S$ or $v \in S$.

VERTEX COVER	
Input:	A graph $G = (V, E)$ and an integer k
Parameter:	k
Question:	Does G have a vertex cover of size at most k?



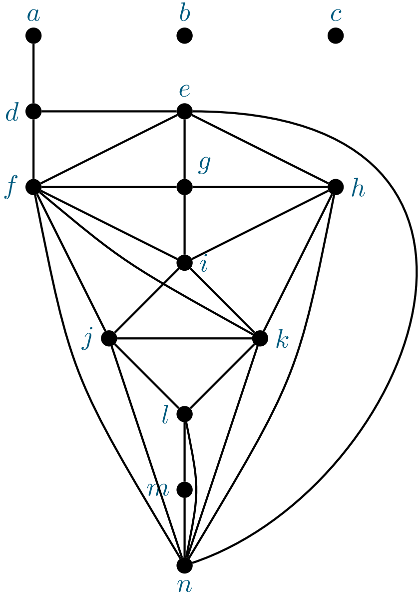
Exercise 1



$k = 4$

Is this a YES-instance for VERTEX COVER?
(Is there $S \subseteq V$ with $|S| \leq 4$, such that $\forall\, uv \in E$, $u \in S$ or $v \in S$?)

Exercise 2



$k = 7$

Outline

1. Vertex Cover
   - Simplification rules
   - Preprocessing algorithm
2. Kernelization algorithms
3. Kernel for Hamiltonian Cycle
4. Kernel for Edge Clique Cover
5. Kernels and Fixed-parameter tractability
6. Further Reading

Simplification rules for Vertex Cover

(Degree-0)

If $\exists v \in V$ such that $d_G(v) = 0$, then set $G \leftarrow G - v$.

Simplification rules for Vertex Cover

(Degree-0)

If $\exists v \in V$ such that $d_G(v) = 0$, then set $G \leftarrow G - v$.

Proving correctness. A simplification rule is sound if for every instance, it produces an equivalent instance. Two instances I, I' are equivalent if they are both Yes-instances or they are both No-instances.

Lemma 1

(Degree-0) is sound.

Simplification rules for VERTEX COVER

(Degree-0)

If $\exists v \in V$ such that $d_G(v) = 0$, then set $G \leftarrow G - v$.

Proving correctness. A simplification rule is sound if for every instance, it produces an equivalent instance. Two instances I, I' are equivalent if they are both YES-instances or they are both NO-instances.

Lemma 1

(Degree-0) is sound.

Proof.

First, suppose $(G - v, k)$ is a YES-instance. Let S be a vertex cover for $G - v$ of size at most k. Then, S is also a vertex cover for G since no edge of G is incident to v. Thus, (G, k) is a YES-instance.
Now, suppose $(G - v, k)$ is a NO-instance. For the sake of contradiction, assume (G, k) is a YES-instance. Let S be a vertex cover for G of size at most k. But then, $S \setminus \{v\}$ is a vertex cover of size at most k for $G - v$; a contradiction. $\square$

Simplification rules for Vertex Cover

(Degree-1)

If $\exists v \in V$ such that $d_G(v) = 1$, then set $G \leftarrow G - N_G[v]$ and $k \leftarrow k - 1$.

Simplification rules for VERTEX COVER

(Degree-1)

If $\exists v \in V$ such that $d_G(v) = 1$, then set $G \leftarrow G - N_G[v]$ and $k \leftarrow k - 1$.

Lemma 1

(Degree-1) is sound.

Proof.

Let u be the neighbor of v in G. Thus, $N_G[v] = \{u, v\}$.
If S is a vertex cover of G of size at most k, then $S \setminus \{u, v\}$ is a vertex cover of $G - N_G[v]$ of size at most $k - 1$, because $u \in S$ or $v \in S$.
If S' is a vertex cover of $G - N_G[v]$ of size at most $k - 1$, then $S' \cup \{u\}$ is a vertex cover of G of size at most k, since all edges that are in G but not in $G - N_G[v]$ are incident to u. $\square$

Simplification rules for VERTEX COVER

(Large Degree)

If $\exists v \in V$ such that $d_G(v) > k$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

Simplification rules for Vertex Cover

(Large Degree)

If $\exists v \in V$ such that $d_G(v) > k$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

Lemma 1

(Large Degree) is sound.

Proof.

Let S be a vertex cover of G of size at most k. If $v \notin S$, then $N_G(v) \subseteq S$, contradicting that $|S| \leq k$. $\square$

Simplification rules for Vertex Cover

(Number of Edges)

If $d_G(v) \leq k$ for each $v \in V$ and $|E| > k^2$ then return No

Simplification rules for Vertex Cover

(Number of Edges)

If $d_G(v) \leq k$ for each $v \in V$ and $|E| > k^2$ then return No

Lemma 1

(Number of Edges) is sound.

Proof.

Assume $d_G(v) \leq k$ for each $v \in V$ and $|E| > k^2$.
Suppose $S \subseteq V$, $|S| \leq k$, is a vertex cover of G.
We have that S covers at most k^2 edges.
However, $|E| \geq k^2 + 1$.
Thus, S is not a vertex cover of G. $\square$

Outline

1. Vertex Cover
   - Simplification rules
   - Preprocessing algorithm
2. Kernelization algorithms
3. Kernel for HAMILTONIAN CYCLE
4. Kernel for EDGE CLIQUE COVER
5. Kernels and Fixed-parameter tractability
6. Further Reading

Preprocessing algorithm for VERTEX COVER

VC-preprocess
Input: A graph G and an integer k.
Output: A graph G' and an integer k' such that G has a vertex cover of size at most k if and only if G' has a vertex cover of size at most k'.

$G' \leftarrow G$
$k' \leftarrow k$
repeat
    Execute simplification rules (Degree-0), (Degree-1), (Large Degree), and (Number of Edges) for (G', k')
until *no simplification rule applies*
return (G', k')

Effectiveness of preprocessing algorithms

- How effective is VC-preprocess?
- We would like to study preprocessing algorithms mathematically and quantify their effectiveness.

First try

- Say that a preprocessing algorithm for a problem Π is nice if it runs in polynomial time and for each instance for Π, it returns an instance for Π that is strictly smaller.

First try

- Say that a preprocessing algorithm for a problem Π is nice if it runs in polynomial time and for each instance for Π, it returns an instance for Π that is strictly smaller.
- $\rightarrow$ executing it a linear number of times reduces the instance to a single bit
- $\rightarrow$ such an algorithm would solve Π in polynomial time
- For NP-hard problems this is not possible unless P $=$ NP
- We need a different measure of effectiveness

Measuring the effectiveness of preprocessing algorithms

- We will measure the effectiveness in terms of the parameter
- How large is the resulting instance in terms of the parameter?

Effectiveness of VC-preprocess

Lemma 2

For any instance (G, k) for Vertex Cover, *VC-preprocess produces an equivalent instance (G', k') of size $O(k^2)$.*

Proof.

Since all simplification rules are sound, $(G = (V, E), k)$ and $(G' = (V', E'), k')$ are equivalent.
By (Number of Edges), $|E'| \leq (k')^2 \leq k^2$.
By (Degree-0) and (Degree-1), each vertex in V' has degree at least 2 in G'.
Since $\sum_{v \in V'} d_{G'}(v) = 2|E'| \leq 2k^2$, this implies that $|V'| \leq k^2$.
Thus, $|V'| + |E'| \subseteq O(k^2)$. $\square$

Outline

Kernelization: definition

Definition 3

A kernelization for a parameterized problem Π is a **polynomial time** algorithm, which, for any instance I of Π with parameter k, produces an **equivalent** instance I' of Π with parameter k' such that $|I'| \leq f(k)$ and $k' \leq f(k)$ for a computable function f.
We refer to the function f as the size of the kernel.

Note: We do not formally require that $k' \leq k$, but this will be the case for many kernelizations.

VC-preprocess is a quadratic kernelization

Theorem 4

VC-preprocess is a $O(k^2)$ kernelization for VERTEX COVER.

Outline

Hamiltonian Cycle I

A Hamiltonian cycle of G is a subgraph of G that is a cycle on $|V(G)|$ vertices.

vc-Hamiltonian Cycle

Input:	A graph $G = (V, E)$.
Parameter:	$k = vc(G)$, the size of a smallest vertex cover of G.
Question:	Does G have a Hamiltonian cycle?

Thought experiment: Imagine a very large instance where the parameter is tiny. How can you simplify such an instance?

Hamiltonian Cycle II

Issue: We do not actually know a vertex cover of size k.
We do not even know the value of k (it is not part of the input).

Hamiltonian Cycle III

- Obtain a vertex cover using an approximation algorithm. We will use a 2-approximation algorithm, producing a vertex cover of size $\leq 2k$ in polynomial time.
- If C is a vertex cover of size $\leq 2k$, then $I = V \setminus C$ is an independent set of size $\geq |V| - 2k$.
- No two consecutive vertices in the Hamiltonian Cycle can be in I.
- A kernel with $\leq 4k$ vertices can now be obtained with the following simplification rule.

(Too-large)

Compute a vertex cover C of size $\leq 2k$ in polynomial time.
If $2|C| < |V|$, then return No

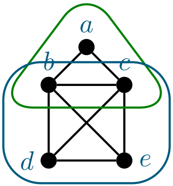
Outline

Edge Clique Cover

Definition 5

An edge clique cover of a graph $G = (V, E)$ is a set of cliques in G covering all its edges.
In other words, if $\mathcal{C} \subseteq 2^V$ is an edge clique cover then each $S \in \mathcal{C}$ is a clique in G and for each $\{u, v\} \in E$ there exists an $S \in \mathcal{C}$ such that $u, v \in S$.

Example: $\{\{a, b, c\}, \{b, c, d, e\}\}$ is an edge clique cover for this graph.



Edge Clique Cover

Edge Clique Cover

Input:	A graph $G = (V, E)$ and an integer k
Parameter:	k
Question:	Does G have an edge clique cover of size at most k?

The size of an edge clique cover $\mathcal{C}$ is the number of cliques contained in $\mathcal{C}$ and is denoted $|\mathcal{C}|$.

Helpful properties

Definition 5

A clique S in a graph G is a maximal clique if there is no other clique S' in G with $S \subset S'$.

Lemma 6

A graph G has an edge clique cover $\mathcal{C}$ of size at most k if and only if G has an edge clique cover $\mathcal{C}'$ of size at most k such that each $S \in \mathcal{C}'$ is a maximal clique.

Proof sketch.

$(\Rightarrow)$: Replace each clique $S \in \mathcal{C}$ by a maximal clique S' with $S \subseteq S'$.
$(\Leftarrow)$: Trivial, since $\mathcal{C}'$ is an edge clique cover of size at most k. □

Simplification rules for Edge Clique Cover

Thought experiment: Imagine a very large instance where the parameter is tiny. How can you simplify such an instance?

Simplification rules for Edge Clique Cover II

The instance could have many degree-0 vertices.

(Isolated)

If there exists a vertex $v \in V$ with $d_G(v) = 0$, then set $G \leftarrow G - v$.

Lemma 7

(Isolated) is sound.

Proof sketch.

Since no edge is incident to v, a smallest edge clique cover for $G - v$ is a smallest edge clique cover for G, and vice-versa. □

Simplification rules for Edge Clique Cover II

The instance could have many degree-0 vertices.

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Since no edge is incident to v, a smallest edge clique cover for $G - v$ is a smallest edge clique cover for G, and vice-versa. □

(Isolated-Edge)

If $\exists uv \in E$ such that $d_G(u) = d_G(v) = 1$, then set $G \leftarrow G - \{u, v\}$ and $k \leftarrow k - 1$.

Simplification rules for EDGE CLIQUE COVER III

(Twins)

If $\exists u, v \in V$, $u \neq v$, such that $N_G[u] = N_G[v]$, then set $G \leftarrow G - v$.

Lemma 8

(Twins) is sound.

Simplification rules for EDGE CLIQUE COVER III

(Twins)

If $\exists u, v \in V$, $u \neq v$, such that $N_G[u] = N_G[v]$, then set $G \leftarrow G - v$.

Lemma 8

(Twins) is sound.

Proof.

We need to show that G has an edge clique cover of size at most k if and only if $G - v$ has an edge clique cover of size at most k.
$(\Rightarrow)$: If $\mathcal{C}$ is an edge clique cover of G of size at most k, then $\{S \setminus \{v\} : S \in \mathcal{C}\}$ is an edge clique cover of $G - v$ of size at most k.
$(\Leftarrow)$: Let $\mathcal{C}'$ be an edge clique cover of $G - v$ of size at most k. Partition $\mathcal{C}'$ into $\mathcal{C}'_u = \{S \in \mathcal{C}' : u \in S\}$ and $\mathcal{C}'_{\neg u} = \mathcal{C}' \setminus \mathcal{C}'_u$. Note that each set in $\mathcal{C}_u = \{S \cup \{v\} : S \in \mathcal{C}'_u\}$ is a clique in G since $N_G[u] = N_G[v]$ and that each edge incident to v is contained in at least one of these cliques. Now, $\mathcal{C}_u \cup \mathcal{C}'_{\neg u}$ is an edge clique cover of G of size at most k. □

Simplification rules for EDGE CLIQUE COVER IV

(Size-V)

If the previous simplification rules do not apply and $|V| > 2^k$, then return NO.

Lemma 9

(Size-V) is sound.

Simplification rules for Edge Clique Cover IV

(Size-V)

If the previous simplification rules do not apply and $|V| > 2^k$, then return No.

Lemma 9

(Size-V) is sound.

Proof.

For the sake of contradiction, assume neither (Isolated) nor (Twins) are applicable, $|V| > 2^k$, and G has an edge clique cover $\mathcal{C}$ of size at most k. Since $2^{\mathcal{C}}$ (the set of all subsets of $\mathcal{C}$) has size at most 2^k, and every vertex belongs to at least one clique in $\mathcal{C}$ by (Isolated), we have that there exists two vertices $u, v \in V$ such that $\{S \in \mathcal{C} : u \in S\} = \{S \in \mathcal{C} : v \in S\}$. But then, $N_G[u] = \bigcup_{S \in \mathcal{C} : u \in S} S = \bigcup_{S \in \mathcal{C} : v \in S} S = N_G[v]$, contradicting that (Twin) is not applicable. $\square$

Kernel for Edge Clique Cover

Theorem 10 ((Gramm et al., 2008))

Edge Clique Cover *has a kernel with* $O(2^k)$ *vertices and* $O(4^k)$ *edges.*

Corollary 11

Edge Clique Cover *is* FPT.

Outline

1. Vertex Cover
   - Simplification rules
   - Preprocessing algorithm
2. Kernelization algorithms
3. Kernel for HAMILTONIAN CYCLE
4. Kernel for EDGE CLIQUE COVER
5. Kernels and Fixed-parameter tractability
6. Further Reading

Kernels and Fixed-parameter tractability

Theorem 12

Let Π *be a decidable parameterized problem.*
Π *has a kernelization algorithm* $\Leftrightarrow$ Π *is* FPT*.*

Kernels and Fixed-parameter tractability

Theorem 12

Let Π be a decidable parameterized problem.
Π has a kernelization algorithm $\Leftrightarrow$ Π is FPT.

Proof.

($\Rightarrow$): An FPT algorithm is obtained by first running the kernelization, and then any brute-force algorithm on the resulting instance.
($\Leftarrow$): Let A be an FPT algorithm for Π with running time $O(f(k)n^c)$.
If $f(k) < n$, then A has running time $O(n^{c+1})$. In this case, the kernelization algorithm runs A and returns a trivial YES- or NO-instance depending on the answer of A.
Otherwise, $f(k) \geq n$. In this case, the kernelization algorithm outputs the input instance. $\square$

Outline

1. Vertex Cover
   - Simplification rules
   - Preprocessing algorithm
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3. Kernel for Hamiltonian Cycle
4. Kernel for Edge Clique Cover
5. Kernels and Fixed-parameter tractability
6. Further Reading

Further Reading

- Chapter 2, *Kernelization* in (Cygan et al., 2015)
- Chapter 4, *Kernelization* in (Downey and Fellows, 2013)
- Chapter 7, *Data Reduction and Problem Kernels* in (Niedermeier, 2006)
- Chapter 9, *Kernelization and Linear Programming Techniques* in (Flum and Grohe, 2006)
- the kernelization book (Fomin et al., 2019)

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