Introduction to solving intractable problems

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UNSW

Outline

1. Algorithms for NP-hard problems
2. Exponential Time Algorithms
3. Parameterized Complexity
   - FPT Algorithm for Vertex Cover
   - Algorithms for Vertex Cover
4. Further Reading

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Central question

P vs. NP

NP-hard problems

- no known polynomial time algorithm for any NP-hard problem
- belief: $\mathsf{P} \neq \mathsf{NP}$
- What to do when facing an NP-hard problem?

Example problem

Monitoring a power grid

Tammy is responsible for fault detection on the power grid of an energy company.

She has access to k monitoring devices. Each one can be placed on a node of the electrical grid and can monitor the power lines that are connected to this node.

Tammy's objective is to place the monitoring devices in such a way that each power line is monitored by at least one monitoring device.

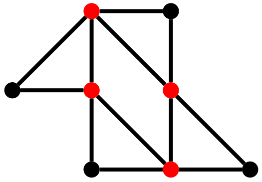
Let us first give an abstraction of this problem and formulate it as a decision problem for graphs.

Example problem: Vertex Cover

A vertex cover in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every edge of G has an endpoint in S.

> **Vertex Cover**
> Input: Graph G, integer k
> Question: Does G have a vertex cover of size k?

Note: Vertex Cover is NP-complete.



Coping with NP-hardness

- Approximation algorithms
  - There is a polynomial-time algorithm, which, given a graph G, finds a vertex cover of G of size at most $2 \cdot \mathsf{OPT}$, where OPT is the size of a smallest vertex cover of G.
- Exact exponential time algorithms
  - There is an algorithm solving Vertex Cover in time $O(1.1970^n)$, where $n = |V|$ (Xiao and Nagamochi, 2017).
- Fixed parameter algorithms
  - There is an algorithm solving Vertex Cover in time $O(1.2738^k + kn)$ (Chen, Kanj, and Xia, 2010).
- Heuristics
  - The COVER heuristic (COVer Edges Randomly) finds a smaller vertex cover than state-of-the-art heuristics on a suite of hard benchmark instances (Richter, Helmert, and Gretton, 2007).
- Restricting the inputs
  - Vertex Cover can be solved in polynomial time on bipartite graphs, trees, interval graphs, etc. (Golumbic, 2004).
- Quantum algorithms?
  - Not believed to solve NP-hard problems in polynomial time (Aaronson, 2005).

Aims of this course

Design and analyze algorithms for NP-hard problems.

We focus on algorithms that solve NP-hard problems exactly and analyze their worst case running time.

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Running times

Worst case running time of an algorithm.

- An algorithm is polynomial if $\exists c \in \mathbb{N}$ such that the algorithm solves every instance in time $O(n^c)$, where n is the size of the instance.
  Also: $n^{O(1)}$ or $\mathsf{poly}(n)$.
- quasi-polynomial: $2^{O(\log^c n)}$, $c \in O(1)$
- sub-exponential: $2^{o(n)}$
- exponential: $2^{\mathsf{poly}(n)}$
- double-exponential: $2^{2^{\mathsf{poly}(n)}}$

O^*-notation ignores polynomial factors in the input size:

$$O^*(f(n)) \equiv O(f(n) \cdot \mathsf{poly}(n))$$
$$O^*(f(k)) \equiv O(f(k) \cdot \mathsf{poly}(n))$$

Brute-force algorithms for NP-hard problems

Theorem 1

Every problem in NP can be solved in exponential time.

Brute-force algorithms for NP-hard problems

Theorem 1

Every problem in NP can be solved in exponential time.

For a proof, see the lecture on NP-completeness.

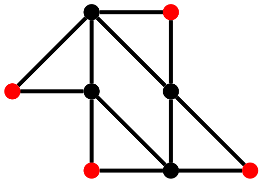
Three main categories for NP-complete problems

- Subset problems
- Permutation problems
- Partition problems

Subset Problem: Independent Set

An independent set in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that the vertices in S are pairwise non-adjacent in G.

Independent Set	
Input:	Graph G, integer k
Question:	Does G have an independent set of size k?



Brute-force:

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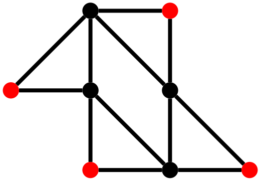
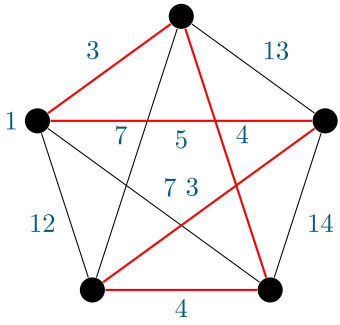


Brute-force: $O^*(2^n)$, where $n = |V(G)|$

Permutation Problem: TRAVELING SALESPERSON

TRAVELING SALESPERSON (TSP)

Input:	a set of n cities, the distance $d(i,j) \in \mathbb{N}$ between every two cities i and j, integer k
Question:	Is there a permutation of the cities (a tour) such that the total distance when traveling from city to city in the specified order, and returning back to the origin, is at most k?



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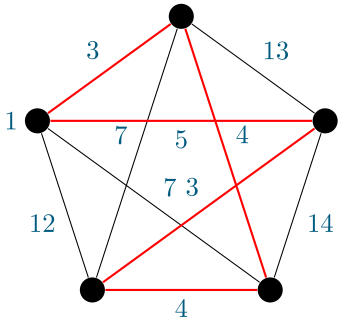
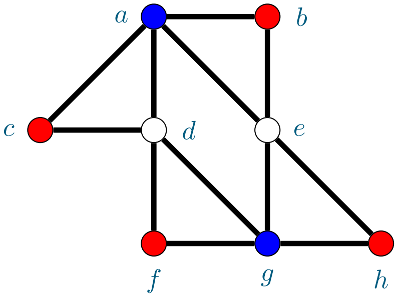


Brute-force: $O^*(n!) \subseteq 2^{O(n \log n)}$

Partition Problem: Coloring

A k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, \ldots, k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

Coloring
Input: Graph G, integer k
Question: Does G have a k-coloring?



Brute-force:

Partition Problem: Coloring

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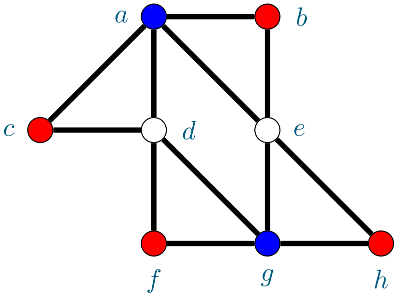

Brute-force: $O^*(k^n)$, where $n = |V(G)|$

Exponential Time Algorithms

- natural question in Algorithms:
  design faster (worst-case analysis) algorithms for problems
- might lead to practical algorithms
  - for small instances
    - you don't want to design software where your client/boss can find with better solutions *by hand* than your software
  - subroutines for
    - (sub)exponential time approximation algorithms
    - randomized algorithms with expected polynomial run time

Solve an NP-hard problem

- exhaustive search
  - trivial method
  - try all candidate solutions (certificates) for a ground set on n elements
  - running times for problems in NP
    - Subset Problems: $O^*(2^n)$
    - Permutation Problems: $O^*(n!)$
    - Partition Problems: $O^*(c^{n \log n})$
- faster exact algorithms
  - for some problems, it is possible to obtain provably faster algorithms
  - running times $O(1.0836^n), O(1.4689^n), O(1.9977^n)$

Exponential Time Algorithms in Practice

- How large are the instances one can solve in practice?

Available time nb. of operations	1 s 2^{38}	1 min $\sim 2^{44}$	1 hour $\sim 2^{50}$	3 days $\sim 2^{56}$	6 months $\sim 2^{62}$
n^5	194	446	1,024	2,352	5,404
n^{10}	14	21	32	49	74
1.05^n	540	625	711	796	881
1.1^n	276	320	364	407	451
1.5^n	65	75	85	96	106
2^n	38	44	50	56	62
5^n	16	19	22	24	27
$n!$	14	16	17	19	20

Note: Intel Core i7-8086K executes $\sim 2^{38}$ instructions per second at 5 GHz.

"For every polynomial-time algorithm you have, there is an exponential algorithm that I would rather run."

– Alan Perlis (1922-1990, programming languages, 1st recipient of Turing Award)

Hardware vs. Algorithms

- Suppose a 2^n algorithm enables us to solve instances up to size x
- Faster processors
  - processor speed doubles after 18–24 months (Moore's law)
  - can solve instances up to size $x + 1$
- Faster algorithm
  - design an $O^*(2^{n/2}) \subseteq O(1.4143^n)$ time algorithm
  - can solve instances up to size $2 \cdot x$

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A story

A computer scientist meets a biologist . . .

Eliminating conflicts from experiments

$n = 1000$ experiments,
$k = 20$ experiments failed

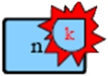
	Running Time	
Theoretical	Number of Instructions	Real
2^n	$1.07 \cdot 10^{301}$	$4.941 \cdot 10^{282}$ years
n^k	10^{60}	$4.611 \cdot 10^{41}$ years
$2^k \cdot n$	$1.05 \cdot 10^9$	0.01526 seconds

Notes

- We assume that 2^{36} instructions are carried out per second.
- The Big Bang happened roughly $13.5 \cdot 10^9$ years ago.

Goal of Parameterized Complexity

Confine the combinatorial explosion to a parameter k.



For which problem–parameter combinations can we find algorithms with running times of the form

$$f(k) \cdot n^{O(1)},$$

where the f is a computable function independent of the input size n?

Examples of Parameters

A Parameterized Problem

Input:	an instance of the problem
Parameter:	a parameter k
Question:	a YES/NO question about the instance and the parameter

- A parameter can be
  - input size (trivial parameterization)
  - solution size
  - related to the structure of the input (maximum degree, treewidth, branchwidth, genus, ...)
  - etc.

Main Complexity Classes

P: class of problems that can be solved in time $n^{O(1)}$
FPT: class of problems that can be solved in time $f(k) \cdot n^{O(1)}$
W[$\cdot$]: parameterized intractability classes
XP: class of problems that can be solved in time $f(k) \cdot n^{g(k)}$

$$\mathsf{P} \subseteq \mathsf{FPT} \subseteq \mathsf{W}[1] \subseteq \mathsf{W}[2] \cdots \subseteq \mathsf{W}[P] \subseteq \mathsf{XP}$$

Known: If FPT $=$ W[1], then the Exponential Time Hypothesis fails, i.e. 3-SAT can be solved in time $2^{o(n)}$.

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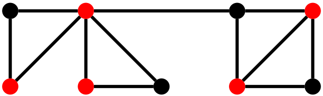
VERTEX COVER (VC)

Input:	A graph $G = (V, E)$ on n vertices, an integer k
Parameter:	k
Question:	Is there a set of vertices $C \subseteq V$ of size at most k such that every edge has at least one endpoint in C?



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Brute Force Algorithms

- $2^n \cdot n^{O(1)}$ not FPT
- $n^k \cdot n^{O(1)}$ not FPT

An FPT Algorithm

Algorithm vc1(G, k);

if $E = \emptyset$ **then** `// all edges are covered`
  return Yes
else if $k \leq 0$ **then** `// we cannot select any vertex`
  return No
else
  Select an edge $uv \in E$;
  return vc1$(G - u, k - 1)$ $\vee$ vc1$(G - v, k - 1)$

Running Time Analysis

- Let us look at an arbitrary execution of the algorithm.
- Recursive calls form a search tree T
  - with depth $\leq k$
  - where each node has ≤ 2 children
- $\Rightarrow T$ has $\leq 2^k$ leaves and $\leq 2^k - 1$ internal nodes
- at each node the algorithm spends time $n^{O(1)}$
- The running time is $O^*(2^k)$

A faster FPT Algorithm

A faster FPT Algorithm

Algorithm vc2(G, k);

```
if E = ∅ then                                  // all edges are covered
    return Yes
else if k ≤ 0 then                             // we used too many vertices
    return No
else if Δ(G) ≤ 2 then                          // G has maximum degree ≤ 2
    Solve the problem in polynomial time;
else
    Select a vertex v of maximum degree;
    return vc2(G − v, k − 1) ∨ vc2(G − N[v], k − d(v))
```

Running time analysis of vc2

- Number of leaves of the search tree:

$$\begin{aligned} T(k) &\leq T(k-1) + T(k-3) \\ x^k &\leq x^{k-1} + x^{k-3} \\ x^3 - x^2 - 1 &\leq 0 \end{aligned}$$

- The equation $x^3 - x^2 - 1 = 0$ has a unique positive real solution: $x \approx 1.4655\ldots$
- Running time: $1.4656^k \cdot n^{O(1)}$

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- Exponential-time algorithms
  - Chapter 1, *Introduction*, in (Fomin and Kratsch, 2010).
  - Survey on exponential-time algorithms (Woeginger, 2001).
  - Chapter 1, *Introduction*, in (Gaspers, 2010).
- Parameterized Complexity
  - Chapter 1, *Introduction*, in (Cygan et al., 2015)
  - Chapter 2, *The Basic Definitions*, in (Downey and Fellows, 2013)
  - Chapter I, *Foundations*, in (Niedermeier, 2006)
  - *Preface* in (Flum and Grohe, 2006)

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