Parameter Treewidth

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1 Algorithms for trees

Exercise

Recall: An *independent set* of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $G[S]$ has no edge.

#INDEPENDENT SETS ON TREES

Input:	A tree $T = (V, E)$
Output:	The number of independent sets of T.

- Design a polynomial time algorithm for #INDEPENDENT SETS ON TREES

Solution

- Select an arbitrary root r of T
- Bottom-up dynamic programming (starting at the leaves) to compute, for each subtree T_x rooted at x the values
  - $\#\mathsf{in}(x)$: the number of independent sets of T_x containing x, and
  - $\#\mathsf{out}(x)$: the number of independent sets of T_x not containing x.
- If x is a leaf, then $\#\mathsf{in}(x) = \#\mathsf{out}(x) = 1$
- Otherwise,

$$\#\mathsf{in}(x) = \Pi_{y \in \text{children}(x)} \#\mathsf{out}(y) \text{ and}$$
$$\#\mathsf{out}(x) = \Pi_{y \in \text{children}(x)} (\#\mathsf{in}(y) + \#\mathsf{out}(y))$$

- The final result is $\#\mathsf{in}(r) + \#\mathsf{out}(r)$

Exercise
Recall: A *dominating set* of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $N_G[S] = V$.

#DOMINATING SETS ON TREES	
Input:	A tree $T = (V, E)$
Output:	The number of dominating sets of T.

- Design a polynomial time algorithm for #DOMINATING SETS ON TREES

Solution

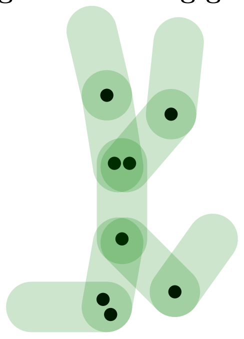
- Select an arbitrary root r of T
- Bottom-up dynamic programming (starting at the leaves) to compute, for each subtree T_x rooted at x the values
  - $\#\mathsf{in}(x)$: the number of dominating sets of T_x containing x,
  - $\#\mathsf{outD}(x)$: the number of dominating sets of T_x not containing x, and
  - $\#\mathsf{outND}(x)$: the number of vertex subsets of T_x dominating $V(T_x) \setminus \{x\}$.
- If x is a leaf, then $\#\mathsf{in}(x) = \#\mathsf{outND}(x) = 1$ and $\#\mathsf{outD}(x) = 0$.
- Otherwise,

$$\begin{aligned}
\#\mathsf{in}(x) &= \Pi_{y \in \text{children}(x)} \left(\#\mathsf{in}(y) + \#\mathsf{outD}(y) + \#\mathsf{outND}(y)\right), \\
\#\mathsf{outD}(x) &= \Pi_{y \in \text{children}(x)} \left(\#\mathsf{in}(y) + \#\mathsf{outD}(y)\right) \\
&\quad - \Pi_{y \in \text{children}(x)} \#\mathsf{outD}(y) \\
\#\mathsf{outND}(x) &= \Pi_{y \in \text{children}(x)} \#\mathsf{outD}(y)
\end{aligned}$$

- The final result is $\#\mathsf{in}(r) + \#\mathsf{outD}(r)$

2 Tree decompositions

Algorithms using graph decompositions

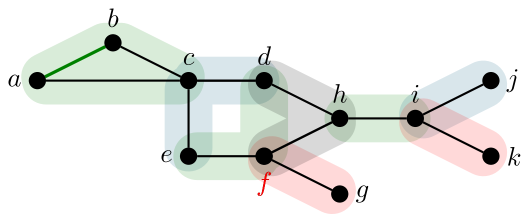


Idea: decompose the problem into subproblems and combine solutions to subproblems to a global solution.

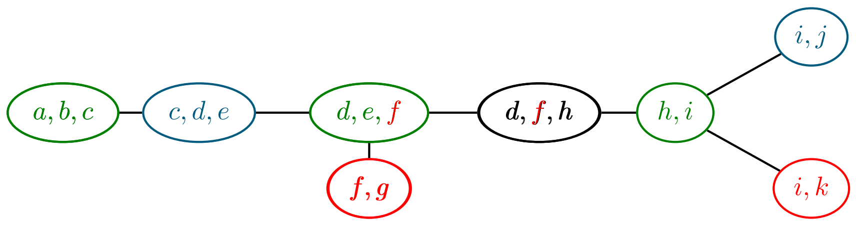
Parameter: overlap between subproblems.

Tree decompositions (by example)

- A graph G



- A *tree decomposition* of G



Conditions: covering and connectedness.

Tree decomposition (more formally)

- Let G be a graph, T a tree, and γ a labeling of the vertices of T by sets of vertices of G.
- We refer to the vertices of T as "nodes", and we call the sets $\gamma(t)$ "bags".
- The pair (T, γ) is a *tree decomposition* of G if the following three conditions hold:
  1. For every vertex v of G there exists a node t of T such that $v \in \gamma(t)$.
  2. For every edge vw of G there exists a node t of T such that $v, w \in \gamma(t)$ ("covering").
  3. For any three nodes t_1, t_2, t_3 of T, if t_2 lies on the unique path from t_1 to t_3, then $\gamma(t_1) \cap \gamma(t_3) \subseteq \gamma(t_2)$ ("connectedness").

Treewidth

- The *width* of a tree decomposition (T, γ) is defined as the maximum $|\gamma(t)| - 1$ taken over all nodes t of T.
- The *treewidth* $\mathsf{tw}(G)$ of a graph G is the minimum width taken over all its tree decompositions.

Basic Facts

- Trees have treewidth 1.
- Cycles have treewidth 2.
- Consider a tree decomposition (T, γ) of a graph G and two adjacent nodes i, j in T. Let T_i and T_j denote the two trees obtained from T by deleting the edge ij, such that T_i contains i and T_j contains j. Then, every vertex contained in both $\bigcup_{a \in V(T_i)} \gamma(a)$ and $\bigcup_{b \in V(T_j)} \gamma(b)$ is also contained in $\gamma(i) \cap \gamma(j)$.
- The complete graph on n vertices has treewidth $n - 1$.
- If a graph G contains a clique K_r, then every tree decomposition of G contains a node t such that $K_r \subseteq \gamma(t)$.

Complexity of Treewidth

Treewidth	
Input:	Graph $G = (V, E)$, integer k
Parameter:	k
Question:	Does G have treewidth at most k?

- Treewidth is NP-complete.
- Treewidth is FPT: there is a $k^{O(k^3)} \cdot |V|$ time algorithm (Bodlaender, 1996)

Easy problems for bounded treewidth

- Many graph problems that are polynomial time solvable on trees are FPT with parameter treewidth.
- Two general methods:
  - *Dynamic programming*: compute local information in a bottom-up fashion along a tree decomposition
  - *Monadic Second Order Logic*: express graph problem in some logic formalism and use a meta-algorithm

3 Monadic Second Order Logic

Monadic Second Order Logic

- **Monadic Second Order (MSO)** Logic is a powerful formalism for expressing graph properties. One can quantify over vertices, edges, vertex sets, and edge sets.
- **Courcelle's theorem** (Courcelle, 1990). Checking whether a graph G satisfies an MSO property is FPT parameterized by the treewidth of G plus the length of the MSO expression.
- **Arnborg et al.'s generalizations** (Arnborg, Lagergren, and Seese, 1991).
  - FPT algorithm for parameter $\mathsf{tw}(G) + |\phi(X)|$ that takes as input a graph G and an MSO sentence $\phi(X)$ where X is a free (non-quantified) vertex set variable, that computes a minimum-sized set of vertices X such that $\phi(X)$ is true in G.
  - Also, the input vertices and edges may be colored and their color can be tested.

Elements of MSO

An MSO formula has

- variables representing vertices $(u, v, \dots)$, edges $(a, b, \dots)$, vertex subsets $(X, Y, \dots)$, or edge subsets $(A, B, \dots)$ in the graph
- atomic operations
  - $u \in X$: testing set membership
  - $X = Y$: testing equality of objects
  - $\mathsf{inc}(u, a)$: incidence test "is vertex u an endpoint of the edge a?"
- propositional logic on subformulas: $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$, $\neg\phi_1$, $\phi_1 \Rightarrow \phi_2$
- Quantifiers: $\forall X \subseteq V$, $\exists A \subseteq E$, $\forall u \in V$, $\exists a \in E$, etc.

Shortcuts in MSO

We can define some shortcuts

- $u \neq v$ is $\neg(u = v)$
- $X \subseteq Y$ is $\forall v \in V.\ (v \in X) \Rightarrow (v \in Y)$
- $\forall v \in X\ \varphi$ is $\forall v \in V.\ (v \in X) \Rightarrow \varphi$
- $\exists v \in X\ \varphi$ is $\exists v \in V.\ (v \in X) \wedge \varphi$
- $\mathsf{adj}(u, v)$ is $(u \neq v) \wedge \exists a \in E.\ (\mathsf{inc}(u, a) \wedge \mathsf{inc}(v, a))$

MSO Logic Example

Example: 3-Coloring,

- "there are three independent sets in $G = (V, E)$ which form a partition of V"
- $$\begin{aligned}\mathsf{3COL} := {} & \exists R \subseteq V.\ \exists G \subseteq V.\ \exists B \subseteq V. \\ & \mathsf{partition}(R, G, B) \\ & \wedge \mathsf{independent}(R) \wedge \mathsf{independent}(G) \wedge \mathsf{independent}(B),\end{aligned}$$

  where

  $$\begin{aligned}\mathsf{partition}(R, G, B) := {} & \forall v \in V.\ ((v \in R \wedge v \notin G \wedge v \notin B) \\ & \vee (v \notin R \wedge v \in G \wedge v \notin B) \vee (v \notin R \wedge v \notin G \wedge v \in B))\end{aligned}$$

  and

  $$\mathsf{independent}(X) := \neg(\exists u \in X.\ \exists v \in X.\ \mathsf{adj}(u, v))$$

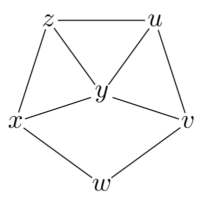
By Courcelle's theorem and our 3COL MSO formula, we have:

Theorem 1. 3-Coloring *is* FPT *with parameter treewidth.*

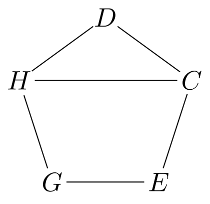
Treewidth only for graph problems?
Let us use treewidth to solve a Logic Problem

- associate a graph with the instance
- take the tree decomposition of the graph
- most widely used: primal graphs, incidence graphs, and dual graphs of formulas.

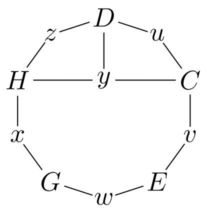
Three Treewidth Parameters
CNF Formula $F = C \wedge D \wedge E \wedge G \wedge H$ where $C = (u \vee v \vee \neg y)$, $D = (\neg u \vee z \vee y)$, $E = (\neg v \vee w)$, $G = (\neg w \vee x)$, $H = (x \vee y \vee \neg z)$.



primal graph



dual graph



incidence graph

This gives rise to parameters *primal treewidth*, *dual treewidth,* and *incidence treewidth.*

Definition 2. Let F be a CNF formula with variables $\mathsf{var}(F)$ and clauses $\mathsf{cla}(F)$. The *primal graph* of F is the graph with vertex set $\mathsf{var}(F)$ where two variables are adjacent if they appear together in a clause of F. The *dual graph* of F is the graph with vertex set $\mathsf{cla}(F)$ where two clauses are adjacent if they have a variable in common. The *incidence graph* of F is the bipartite graph with vertex set $\mathsf{var}(F) \cup \mathsf{cla}(F)$ where a variable and a clause are adjacent if the variable appears in the clause. The *primal treewidth*, *dual treewidth*, and *incidence treewidth* of F is the treewidth of the primal graph, the dual graph, and the incidence graph of F, respectively.

Incidence treewidth is most general

Lemma 3. *The incidence treewidth of F is at most the primal treewidth of F plus* 1.

Proof. Start from a tree decomposition (T, γ) of the primal graph with minimum width. For each clause C:

- There is a node t of T with $\mathsf{var}(C) \subseteq \gamma(t)$, since $\mathsf{var}(C)$ is a clique in the primal graph.
- Add to t a new neighbor t' with $\gamma(t') = \gamma(t) \cup \{C\}$.

$\square$

Lemma 4. *The incidence treewidth of F is at most the dual treewidth of F plus* 1.

Primal and dual treewidth are incomparable.

- One big clause alone gives large primal treewidth.
- $\{\{x, y_1\}, \{x, y_2\}, \ldots, \{x, y_n\}\}$ gives large dual treewidth.

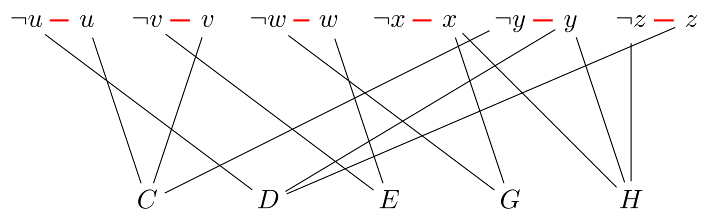
SAT parameterized by treewidth

Sat	
Input:	A CNF formula F
Question:	Is there an assignment of truth values to $\mathsf{var}(F)$ such that F evaluates to true?

Note: If Sat is FPT parameterized by incidence treewidth, then Sat is FPT parameterized by primal treewidth and by dual treewidth.

SAT is FPT for parameter incidence treewidth
CNF Formula $F = C \wedge D \wedge E \wedge G \wedge H$ where $C = (u \vee v \vee \neg y)$, $D = (\neg u \vee z \vee y)$, $E = (\neg v \vee w)$, $G = (\neg w \vee x)$, $H = (x \vee y \vee \neg z)$

Auxiliary graph:



- MSO Formula: *"There exists an independent set of literal vertices that dominates all the clause vertices."*
- The treewidth of the auxilary graph is at most twice the treewidth of the incidence graph plus one.

FPT via MSO

Theorem 5. SAT *is* FPT *for each of the following parameters: primal treewidth, dual treewidth, and incidence treewidth.*

4 Dynamic Programming over Tree Decompositions

Coucelle's theorem: discussion
Advantages of Courcelle's theorem:

- general, applies to many problems
- easy to obtain FPT results

Drawback of Courcelle's theorem

- the resulting running time depends non-elementarily on the treewidth t and the length ℓ of the MSO-sentence, i.e., a tower of 2's whose height is $\omega(1)$

$$2^{2^{2^{\cdot^{\cdot^{\cdot^{t+\ell}}}}}}$$

Dynamic progamming over tree decompositions
Idea: extend the algorithmic methods that work for trees to tree decompositions.

Step 1 Compute a minimum width tree decomposition using Bodlaender's algorithm

Step 2 Transform it into a standard form making computations easier

Step 3 Bottom-up Dynamic Programming (from the leaves of the tree decomposition to the root)

Nice tree decomposition
A *nice* tree decomposition (T, γ) is rooted and has only 4 kinds of nodes:

- *leaf node*: leaf t in T and $|\gamma(t)| = 1$
- *introduce node*: node t with one child t' in T and $\gamma(t) = \gamma(t') \cup \{x\}$
- *forget node*: node t with one child t' in T and $\gamma(t) = \gamma(t') \setminus \{x\}$
- *join node*: node t with two children t_1, t_2 in T and $\gamma(t) = \gamma(t_1) = \gamma(t_2)$

Every tree decomposition of width w of a graph G on n vertices can be transformed into a nice tree decomposition of width w and $O(w \cdot n)$ nodes in polynomial time (Kloks, 1994).

4.1 SAT

Dynamic programming: primal treewidth

- Compute a nice tree decomposition (T,γ) of F's primal graph with minimum width rooted at some node r (Bodlaender, 1996; Kloks, 1994)
- Notation
  - T_t is the subtree of T rooted at node t
  - $\gamma_\downarrow(t) = \{x \in \gamma(t') : t' \in V(T_t)\}$ is the set of vertices/variables in T_t's bags
  - $F_\downarrow(t) = \{C \in \mathsf{cla}(F) : \mathsf{var}(C) \subseteq \gamma_\downarrow(t)\}$ is the set of clauses containing only variables from $\gamma_\downarrow$
  - For a clause $C \in \mathsf{cla}(F)$ and an assignment $\tau : S \to \{0,1\}$ to a subset of variables $S \subseteq \mathsf{var}(F)$, we can efficiently compute

$$\mathsf{falsifies}(\tau, C) = \begin{cases} 1 & \text{if } \tau \text{ sets each literal of } C \text{ to } 0 \\ 0 & \text{otherwise.} \end{cases}$$

- For each node t and each assignment $\tau : \gamma(t) \to \{0,1\}$, our DP algorithm will compute

$$\mathsf{sat}(t,\tau) = \begin{cases} 1 & \text{if } \tau \text{ can be extended to a} \\ & \text{satisfying assignment of } F_\downarrow(t) \\ 0 & \text{otherwise.} \end{cases}$$

- *leaf node*: $|\gamma(t)| = 1$

$$\mathsf{sat}(t,\tau) = \begin{cases} 0 & \text{if } \exists C \in \mathsf{cla}(F) \text{ s.t. } \mathsf{falsifies}(\tau, C) \\ 1 & \text{otherwise} \end{cases}$$

- *introduce node*: $\gamma(t) = \gamma(t') \cup \{x\}$.

$$\mathsf{sat}(t,\tau) = \mathsf{sat}(t', \tau|_{\gamma(t')}) \ \wedge \ (\nexists C \in F : \mathsf{falsifies}(\tau, C)).$$

- *forget node*: $\gamma(t) = \gamma(t') \setminus \{x\}$.

$$\mathsf{sat}(t,\tau) = \mathsf{sat}(t', \tau_{x=0}) \vee \mathsf{sat}(t', \tau_{x=1}),$$

$$\text{where } \tau_{x=a}(y) = \begin{cases} a & \text{if } y = x \\ \tau(y) & \text{otherwise} \end{cases}$$

- *join node*: $\gamma(t) = \gamma(t_1) = \gamma(t_2)$

$$\mathsf{sat}(t,\tau) = \mathsf{sat}(t_1,\tau) \wedge \mathsf{sat}(t_2,\tau).$$

- Finally: F is satisfiable iff $\exists \tau : \gamma(r) \to \{0,1\}$ such that $\mathsf{sat}(r,\tau) = 1$
- Running time: $O^*(2^k)$, where k is the primal treewidth of F
- Also extends to computing the number of satisfying assignments

Direct Algorithms
Known treewidth based algorithms for Sat:

k = primal tw	k = dual tw	k = incidence tw
$O^*(2^k)$	$O^*(2^k)$	$O^*(2^k)$

- These algorithms all count the number of satisfying assignments
- The algorithm for incidence treewidth (Slivovsky and Szeider, 2020) uses Fast Subset Convolution

4.2 CSP

Constraint Satisfaction Problem

CSP
Input: A set of variables X, a domain D, and a set of constraints C
Question: Is there an assignment $\tau : X \to D$ satisfying all the constraints in C?

A *constraint* has a *scope* $S = (s_1, \dots, s_r)$ with $s_i \in X, i \in \{1, \dots, r\}$, and a *constraint relation* R consisting of r-tuples of values in D. An assignment $\tau : X \to D$ *satisfies* a constraint $c = (S, R)$ if there exists a tuple $(d_1, \dots, d_r)$ in R such that $\tau(s_i) = d_i$ for each $i \in \{1, \dots, r\}$.

Bounded Treewidth for Constraint Satisfaction

- Primal, dual, and incidence graphs are defined similarly as for SAT.

Theorem 6 ((Gottlob, Scarcello, and Sideri, 2002)). *CSP is* FPT *for parameter primal treewidth if* $|D| = O(1)$.

- What if domains are unbounded?

Unbounded domains

Theorem 7. *CSP is* W[1]*-hard for parameter primal treewidth.*

Proof Sketch. Parameterized reduction from CLIQUE. Let $(G = (V, E), k)$ be an instance of CLIQUE. Take k variables $x_1, \dots, x_k$, each with domain V. Add $\binom{k}{2}$ binary constraints $E_{i,j}$, $1 \le i < j \le k$. A constraint $E_{i,j}$ has scope (x_i, x_j) and its constraint relation contains the tuple (u, v) if $uv \in E$. The primal treewidth of this CSP instance is $k - 1$. □

5 Further Reading

- Chapter 7, *Treewidth* in (Cygan et al., 2015)
- Chapter 5, *Treewidth* in (Fomin and Kratsch, 2010)
- Chapter 10, *Tree Decompositions of Graphs* in (Niedermeier, 2006)
- Chapter 10, *Treewidth and Dynamic Programming* in (Downey and Fellows, 2013)
- Chapter 13, *Courcelle's Theorem* in (Downey and Fellows, 2013)

References

Stefan Arnborg, Jens Lagergren, and Detlef Seese (1991). "Easy problems for tree-decomposable graphs". In: *Journal of Algorithms* 12.2, pp. 308–340.

Hans L. Bodlaender (1996). "A linear-time algorithm for finding tree-decompositions of small treewidth". In: *SIAM Journal on Computing* 25.6, pp. 1305–1317.

Bruno Courcelle (1990). "The monadic second-order logic of graphs. I. Recognizable sets of finite graphs". In: *Information and Computation* 85.1, pp. 12–75.

Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh (2015). *Parameterized Algorithms*. Springer. DOI: 10.1007/978-3-319-21275-3.

Rodney G. Downey and Michael R. Fellows (2013). *Fundamentals of Parameterized Complexity*. Springer. DOI: 10.1007/978-1-4471-5559-1.

Fedor V. Fomin and Dieter Kratsch (2010). *Exact Exponential Algorithms*. Springer. DOI: 10.1007/978-3-642-16533-7.

Georg Gottlob, Francesco Scarcello, and Martha Sideri (2002). "Fixed-parameter complexity in AI and nonmonotonic reasoning". In: *Journal of Artificial Intelligence* 138.1-2, pp. 55–86.

Ton Kloks (1994). *Treewidth: Computations and Approximations*. Berlin: Springer.

Rolf Niedermeier (2006). *Invitation to Fixed Parameter Algorithms.* Oxford University Press. DOI: `10.1093/ACPROF:OSO/9780198566076.001.0001`.

Friedrich Slivovsky and Stefan Szeider (2020). “A Faster Algorithm for Propositional Model Counting Parameterized by Incidence Treewidth”. In: *Proceedings of the 23rd International Conference on Theory and Applications of Satisfiability Testing (SAT 2020).* Vol. 12178. Lecture Notes in Computer Science. Springer, pp. 267–276.