Kernelization

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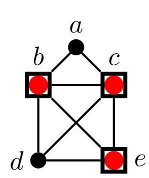
Contents

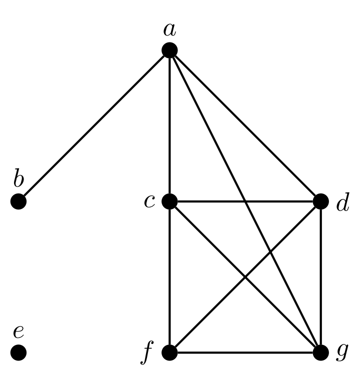
1 Vertex Cover

A *vertex cover* of a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that for each edge $\{u, v\} \in E$, we have $u \in S$ or $v \in S$.

VERTEX COVER	
Input:	A graph $G = (V, E)$ and an integer k
Parameter:	k
Question:	Does G have a vertex cover of size at most k?



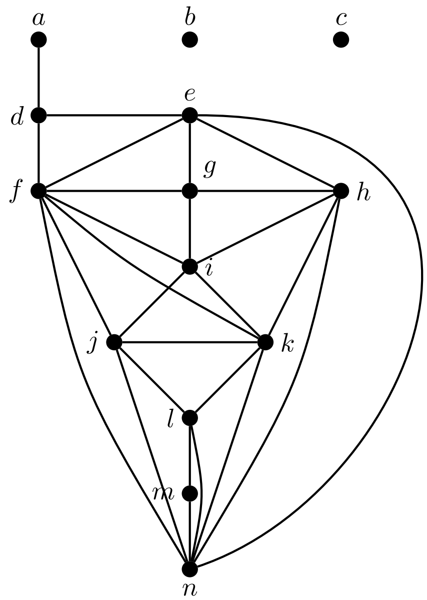
Exercise 1



$k = 4$

Is this a YES-instance for VERTEX COVER? (Is there $S \subseteq V$ with $|S| \leq 4$, such that $\forall\ uv \in E$, $u \in S$ or $v \in S$?)

Exercise 2



$k = 7$

1.1 Simplification rules

(Degree-0)
If $\exists v \in V$ such that $d_G(v) = 0$, then set $G \leftarrow G - v$.

Proving correctness. A simplification rule is *sound* if for every instance, it produces an equivalent instance. Two instances I, I' are *equivalent* if they are both YES-instances or they are both NO-instances.

Lemma 1. *(Degree-0) is sound.*

Proof. First, suppose $(G - v, k)$ is a YES-instance. Let S be a vertex cover for $G - v$ of size at most k. Then, S is also a vertex cover for G since no edge of G is incident to v. Thus, (G, k) is a YES-instance.

Now, suppose $(G - v, k)$ is a NO-instance. For the sake of contradiction, assume (G, k) is a YES-instance. Let S be a vertex cover for G of size at most k. But then, $S \setminus \{v\}$ is a vertex cover of size at most k for $G - v$; a contradiction. ☐

(Degree-1)
If $\exists v \in V$ such that $d_G(v) = 1$, then set $G \leftarrow G - N_G[v]$ and $k \leftarrow k - 1$.

Lemma 2. *(Degree-1) is sound.*

Proof. Let u be the neighbor of v in G. Thus, $N_G[v] = \{u, v\}$.

If S is a vertex cover of G of size at most k, then $S \setminus \{u, v\}$ is a vertex cover of $G - N_G[v]$ of size at most $k - 1$, because $u \in S$ or $v \in S$. If S' is a vertex cover of $G - N_G[v]$ of size at most $k - 1$, then $S' \cup \{u\}$ is a vertex cover of G of size at most k, since all edges that are in G but not in $G - N_G[v]$ are incident to u. ☐

(Large Degree)
If $\exists v \in V$ such that $d_G(v) > k$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

Lemma 3. *(Large Degree) is sound.*

Proof. Let S be a vertex cover of G of size at most k. If $v \notin S$, then $N_G(v) \subseteq S$, contradicting that $|S| \leq k$. ☐

(Number of Edges)
If $d_G(v) \leq k$ for each $v \in V$ and $|E| > k^2$ then return NO

Lemma 4. *(Number of Edges) is sound.*

Proof. Assume $d_G(v) \leq k$ for each $v \in V$ and $|E| > k^2$. Suppose $S \subseteq V$, $|S| \leq k$, is a vertex cover of G. We have that S covers at most k^2 edges. However, $|E| \geq k^2 + 1$. Thus, S is not a vertex cover of G. ☐

1.2 Preprocessing algorithm

VC-preprocess
Input: A graph G and an integer k.
Output: A graph G' and an integer k' such that G has a vertex cover of size at most k if and only if G' has a vertex cover of size at most k'.

$G' \leftarrow G$
$k' \leftarrow k$
repeat
| Execute simplification rules (Degree-0), (Degree-1), (Large Degree), and (Number of Edges) for (G', k')
until *no simplification rule applies*
return (G', k')

Effectiveness of preprocessing algorithms

- How effective is VC-preprocess?
- We would like to study preprocessing algorithms mathematically and quantify their effectiveness.

First try

- Say that a preprocessing algorithm for a problem Π is *nice* if it runs in polynomial time and for each instance for Π, it returns an instance for Π that is strictly smaller.
- $\rightarrow$ executing it a linear number of times reduces the instance to a single bit
- $\rightarrow$ such an algorithm would solve Π in polynomial time
- For NP-hard problems this is not possible unless P = NP
- We need a different measure of effectiveness

Measuring the effectiveness of preprocessing algorithms

- We will measure the effectiveness in terms of the *parameter*
- How large is the resulting instance in terms of the parameter?

Effectiveness of VC-preprocess

Lemma 5. *For any instance (G, k) for* Vertex Cover*, VC-preprocess produces an equivalent instance (G', k') of size $O(k^2)$.*

Proof. Since all simplification rules are sound, $(G = (V, E), k)$ and $(G' = (V', E'), k')$ are equivalent. By (Number of Edges), $|E'| \leq (k')^2 \leq k^2$. By (Degree-0) and (Degree-1), each vertex in V' has degree at least 2 in G'. Since $\sum_{v \in V'} d_{G'}(v) = 2|E'| \leq 2k^2$, this implies that $|V'| \leq k^2$. Thus, $|V'| + |E'| \subseteq O(k^2)$. □

2 Kernelization algorithms

Kernelization: definition

Definition 6. A *kernelization* for a parameterized problem Π is a **polynomial time** algorithm, which, for any instance I of Π with parameter k, produces an **equivalent** instance I' of Π with parameter k' such that $|I'| \leq f(k)$ and $k' \leq f(k)$ for a computable function f. We refer to the function f as the *size* of the kernel.

Note: We do not formally require that $k' \leq k$, but this will be the case for many kernelizations.

VC-preprocess is a quadratic kernelization

Theorem 7. *VC-preprocess is a $O(k^2)$ kernelization for* Vertex Cover.

3 Kernel for Hamiltonian Cycle

A *Hamiltonian cycle* of G is a subgraph of G that is a cycle on $|V(G)|$ vertices.

vc-Hamiltonian Cycle	
Input:	A graph $G = (V, E)$.
Parameter:	$k = vc(G)$, the size of a smallest vertex cover of G.
Question:	Does G have a Hamiltonian cycle?

Thought experiment: Imagine a very large instance where the parameter is tiny. How can you simplify such an instance?
Issue: We do not actually know a vertex cover of size k.We do not even know the value of k (it is not part of the input).

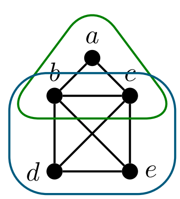
- Obtain a vertex cover using an approximation algorithm. We will use a 2-approximation algorithm, producing a vertex cover of size $\leq 2k$ in polynomial time.
- If C is a vertex cover of size $\leq 2k$, then $I = V \setminus C$ is an independent set of size $\geq |V| - 2k$.
- No two consecutive vertices in the Hamiltonian Cycle can be in I.
- A kernel with $\leq 4k$ vertices can now be obtained with the following simplification rule.

(Too-large)
Compute a vertex cover C of size $\leq 2k$ in polynomial time. If $2|C| < |V|$, then return No

4 Kernel for Edge Clique Cover

Definition 8. An *edge clique cover* of a graph $G = (V, E)$ is a set of cliques in G covering all its edges. In other words, if $\mathcal{C} \subseteq 2^V$ is an edge clique cover then each $S \in \mathcal{C}$ is a clique in G and for each $\{u, v\} \in E$ there exists an $S \in \mathcal{C}$ such that $u, v \in S$.

Example: $\{\{a, b, c\}, \{b, c, d, e\}\}$ is an edge clique cover for this graph.



Edge Clique Cover	
Input:	A graph $G = (V, E)$ and an integer k
Parameter:	k
Question:	Does G have an edge clique cover of size at most k?

The *size* of an edge clique cover $\mathcal{C}$ is the number of cliques contained in $\mathcal{C}$ and is denoted $|\mathcal{C}|$.

Helpful properties

Definition 9. A clique S in a graph G is a *maximal* clique if there is no other clique S' in G with $S \subset S'$.

Lemma 10. *A graph G has an edge clique cover $\mathcal{C}$ of size at most k if and only if G has an edge clique cover $\mathcal{C}'$ of size at most k such that each $S \in \mathcal{C}'$ is a maximal clique.*

Proof sketch. ($\Rightarrow$): Replace each clique $S \in \mathcal{C}$ by a maximal clique S' with $S \subseteq S'$.
($\Leftarrow$): Trivial, since $\mathcal{C}'$ is an edge clique cover of size at most k. □

Simplification rules for Edge Clique Cover

Thought experiment: Imagine a very large instance where the parameter is tiny. How can you simplify such an instance?

The instance could have many degree-0 vertices.

(Isolated)

If there exists a vertex $v \in V$ with $d_G(v) = 0$, then set $G \leftarrow G - v$.

Lemma 11. *(Isolated) is sound.*

Proof sketch. Since no edge is incident to v, a smallest edge clique cover for $G - v$ is a smallest edge clique cover for G, and vice-versa. $\square$

(Isolated-Edge)

If $\exists uv \in E$ such that $d_G(u) = d_G(v) = 1$, then set $G \leftarrow G - \{u, v\}$ and $k \leftarrow k - 1$.

(Twins)

If $\exists u, v \in V$, $u \neq v$, such that $N_G[u] = N_G[v]$, then set $G \leftarrow G - v$.

Lemma 12. *(Twins) is sound.*

Proof. We need to show that G has an edge clique cover of size at most k if and only if $G - v$ has an edge clique cover of size at most k.

($\Rightarrow$): If $\mathcal{C}$ is an edge clique cover of G of size at most k, then $\{S \setminus \{v\} : S \in \mathcal{C}\}$ is an edge clique cover of $G - v$ of size at most k.

($\Leftarrow$): Let $\mathcal{C}'$ be an edge clique cover of $G - v$ of size at most k. Partition $\mathcal{C}'$ into $\mathcal{C}'_u = \{S \in \mathcal{C}' : u \in S\}$ and $\mathcal{C}'_{\neg u} = \mathcal{C}' \setminus \mathcal{C}'_u$. Note that each set in $\mathcal{C}_u = \{S \cup \{v\} : S \in \mathcal{C}'_u\}$ is a clique in G since $N_G[u] = N_G[v]$ and that each edge incident to v is contained in at least one of these cliques. Now, $\mathcal{C}_u \cup \mathcal{C}'_{\neg u}$ is an edge clique cover of G of size at most k. $\square$

(Size-V)

If the previous simplification rules do not apply and $|V| > 2^k$, then return No.

Lemma 13. *(Size-V) is sound.*

Proof. For the sake of contradiction, assume neither (Isolated) nor (Twins) are applicable, $|V| > 2^k$, and G has an edge clique cover $\mathcal{C}$ of size at most k. Since $2^{\mathcal{C}}$ (the set of all subsets of $\mathcal{C}$) has size at most 2^k, and every vertex belongs to at least one clique in $\mathcal{C}$ by (Isolated), we have that there exists two vertices $u, v \in V$ such that $\{S \in \mathcal{C} : u \in S\} = \{S \in \mathcal{C} : v \in S\}$. But then, $N_G[u] = \bigcup_{S \in \mathcal{C} : u \in S} S = \bigcup_{S \in \mathcal{C} : v \in S} S = N_G[v]$, contradicting that (Twin) is not applicable. $\square$

Kernel for Edge Clique Cover

Theorem 14 ((Gramm et al., 2008))**.** Edge Clique Cover *has a kernel with $O(2^k)$ vertices and $O(4^k)$ edges.*

Corollary 15. Edge Clique Cover *is* FPT.

5 Kernels and Fixed-parameter tractability

Theorem 16. *Let Π be a decidable parameterized problem. Π has a kernelization algorithm $\Leftrightarrow$ Π is* FPT.

Proof. ($\Rightarrow$): An FPT algorithm is obtained by first running the kernelization, and then any brute-force algorithm on the resulting instance.

($\Leftarrow$): Let A be an FPT algorithm for Π with running time $O(f(k)n^c)$. If $f(k) < n$, then A has running time $O(n^{c+1})$. In this case, the kernelization algorithm runs A and returns a trivial Yes- or No-instance depending on the answer of A. Otherwise, $f(k) \geq n$. In this case, the kernelization algorithm outputs the input instance. $\square$

6 Further Reading

- Chapter 2, *Kernelization* in (Cygan et al., 2015)
- Chapter 4, *Kernelization* in (Downey and Fellows, 2013)
- Chapter 7, *Data Reduction and Problem Kernels* in (Niedermeier, 2006)
- Chapter 9, *Kernelization and Linear Programming Techniques* in (Flum and Grohe, 2006)
- the kernelization book (Fomin et al., 2019)

References

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