

Introduction to solving intractable problems

Serge Gaspers

UNSW

Outline

- 1 Algorithms for NP-hard problems
- 2 Exponential Time Algorithms
- 3 Parameterized Complexity
 - FPT Algorithm for Vertex Cover
 - Algorithms for Vertex Cover
- 4 Further Reading

Outline

- 1 Algorithms for NP-hard problems
- 2 Exponential Time Algorithms
- 3 Parameterized Complexity
 - FPT Algorithm for Vertex Cover
 - Algorithms for Vertex Cover
- 4 Further Reading

Central question

P vs. NP

NP-hard problems

- no known polynomial time algorithm for any NP-hard problem
- belief: $P \neq NP$
- What to do when facing an NP-hard problem?

Example problem

Monitoring a power grid

Tammy is responsible for fault detection on the power grid of an energy company. She has access to k monitoring devices. Each one can be placed on a node of the electrical grid and can monitor the power lines that are connected to this node. Tammy's objective is to place the monitoring devices in such a way that each power line is monitored by at least one monitoring device.

Let us first give an abstraction of this problem and formulate it as a decision problem for graphs.

Example problem: VERTEX COVER

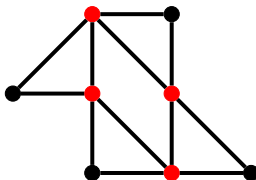
A **vertex cover** in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every edge of G has an endpoint in S .

VERTEX COVER

Input: Graph G , integer k

Question: Does G have a vertex cover of size k ?

Note: VERTEX COVER is **NP**-complete.



Coping with NP-hardness

- Approximation algorithms
 - There is a polynomial-time algorithm, which, given a graph G , finds a vertex cover of G of size at most $2 \cdot \text{OPT}$, where OPT is the size of a smallest vertex cover of G .
- Exact exponential time algorithms
 - There is an algorithm solving VERTEX COVER in time $O(1.1970^n)$, where $n = |V|$ (Xiao and Nagamochi, 2017).
- Fixed parameter algorithms
 - There is an algorithm solving VERTEX COVER in time $O(1.2738^k + kn)$ (Chen, Kanj, and Xia, 2010).
- Heuristics
 - The COVER heuristic (COVer Edges Randomly) finds a smaller vertex cover than state-of-the-art heuristics on a suite of hard benchmark instances (Richter, Helmert, and Gretton, 2007).
- Restricting the inputs
 - VERTEX COVER can be solved in polynomial time on bipartite graphs, trees, interval graphs, etc. (Golumbic, 2004).
- Quantum algorithms?
 - Not believed to solve NP-hard problems in polynomial time (Aaronson, 2005). Quadratic speedup possible in some cases.

Aims of this course

Design and analyze algorithms for NP-hard problems.

We focus on algorithms that solve NP-hard problems **exactly** and analyze their **worst case running time**.

Outline

- 1 Algorithms for NP-hard problems
- 2 Exponential Time Algorithms
- 3 Parameterized Complexity
 - FPT Algorithm for Vertex Cover
 - Algorithms for Vertex Cover
- 4 Further Reading

Running times

Worst case running time of an algorithm.

- An algorithm is **polynomial** if $\exists c \in \mathbb{N}$ such that the algorithm solves every instance in time $O(n^c)$, where n is the size of the instance.

Also: $n^{O(1)}$ or **poly**(n).

- **quasi-polynomial**: $2^{O(\log^c n)}$, $c \in O(1)$
- **sub-exponential**: $2^{o(n)}$
- **exponential**: $2^{\text{poly}(n)}$
- **double-exponential**: $2^{2^{\text{poly}(n)}}$

O^* -notation ignores polynomial factors in the input size:

$$O^*(f(n)) \equiv O(f(n) \cdot \text{poly}(n))$$

$$O^*(f(k)) \equiv O(f(k) \cdot \text{poly}(n))$$

Brute-force algorithms for NP-hard problems

Theorem 1

Every problem in NP can be solved in exponential time.

Brute-force algorithms for NP-hard problems

Theorem 1

Every problem in NP can be solved in exponential time.

For a proof, see the lecture on NP-completeness.

Three main categories for NP-complete problems

- Subset problems
- Permutation problems
- Partition problems

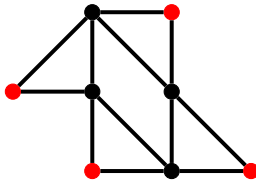
Subset Problem: INDEPENDENT SET

An **independent set** in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that the vertices in S are pairwise non-adjacent in G .

INDEPENDENT SET

Input: Graph G , integer k

Question: Does G have an independent set of size k ?



Brute-force:

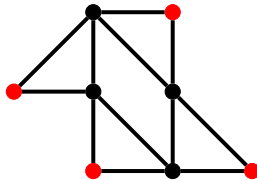
Subset Problem: INDEPENDENT SET

An **independent set** in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that the vertices in S are pairwise non-adjacent in G .

INDEPENDENT SET

Input: Graph G , integer k

Question: Does G have an independent set of size k ?



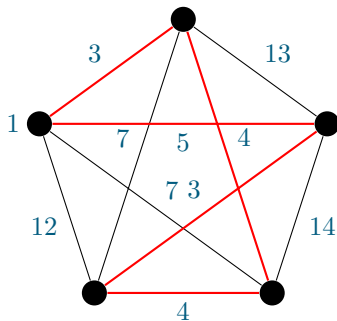
Brute-force: $O^*(2^n)$, where $n = |V(G)|$

Permutation Problem: TRAVELING SALESPERSON

TRAVELING SALESPERSON (TSP)

Input: a set of n cities, the distance $d(i, j) \in \mathbb{N}$ between every two cities i and j , integer k

Question: Is there a permutation of the cities (a **tour**) such that the total distance when traveling from city to city in the specified order, and returning back to the origin, is at most k ?



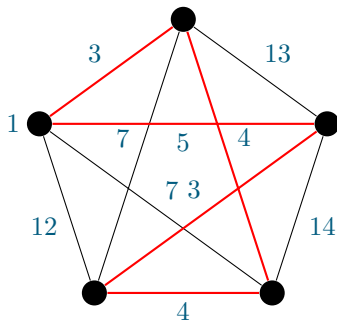
Brute-force:

Permutation Problem: TRAVELING SALESPERSON

TRAVELING SALESPERSON (TSP)

Input: a set of n cities, the distance $d(i, j) \in \mathbb{N}$ between every two cities i and j , integer k

Question: Is there a permutation of the cities (a **tour**) such that the total distance when traveling from city to city in the specified order, and returning back to the origin, is at most k ?



Brute-force: $O^*(n!) \subseteq 2^{O(n \log n)}$

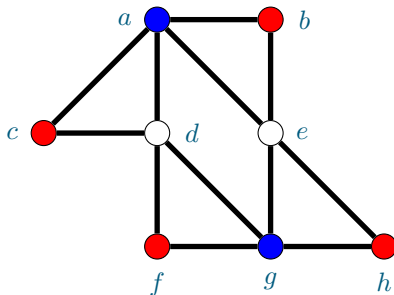
Partition Problem: COLORING

A k -coloring of a graph $G = (V, E)$ is a function $f : V \rightarrow \{1, 2, \dots, k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

COLORING

Input: Graph G , integer k

Question: Does G have a k -coloring?



Brute-force:

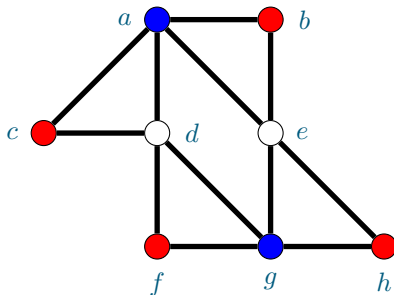
Partition Problem: COLORING

A k -coloring of a graph $G = (V, E)$ is a function $f : V \rightarrow \{1, 2, \dots, k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

COLORING

Input: Graph G , integer k

Question: Does G have a k -coloring?



Brute-force: $O^*(k^n)$, where $n = |V(G)|$

Exponential Time Algorithms

- natural question in Algorithms:
design faster (worst-case analysis) algorithms for problems
- might lead to practical algorithms
 - for small instances
 - you don't want to design software where your client/boss can find with better solutions *by hand* than your software
 - subroutines for
 - (sub)exponential time approximation algorithms
 - randomized algorithms with expected polynomial run time

Solve an NP-hard problem

- exhaustive search
 - trivial method
 - try all candidate solutions (certificates) for a ground set on n elements
 - running times for problems in NP
 - SUBSET PROBLEMS: $O^*(2^n)$
 - PERMUTATION PROBLEMS: $O^*(n!)$
 - PARTITION PROBLEMS: $O^*(c^{n \log n})$
- faster exact algorithms
 - for some problems, it is possible to obtain provably faster algorithms
 - running times $O(1.0836^n)$, $O(1.4689^n)$, $O(1.9977^n)$

Exponential Time Algorithms in Practice

- How large are the instances one can solve in practice?

Available time nb. of operations	1 s 2^{38}	1 min $\sim 2^{44}$	1 hour $\sim 2^{50}$	3 days $\sim 2^{56}$	6 months $\sim 2^{62}$
n^5	194	446	1,024	2,352	5,404
n^{10}	14	21	32	49	74
1.05^n	540	625	711	796	881
1.1^n	276	320	364	407	451
1.5^n	65	75	85	96	106
2^n	38	44	50	56	62
5^n	16	19	22	24	27
$n!$	14	16	17	19	20

Note: Intel Core i7-8086K executes $\sim 2^{38}$ instructions per second at 5 GHz.

“For every polynomial-time algorithm you have, there is an exponential algorithm that I would rather run.”

– Alan Perlis (1922-1990, programming languages, 1st recipient of Turing Award)

Hardware vs. Algorithms

- Suppose a 2^n algorithm enables us to solve instances up to size x
- Faster processors
 - processor speed doubles after 18–24 months (Moore's law)
 - can solve instances up to size $x + 1$
- Faster algorithm
 - design an $O^*(2^{n/2}) \subseteq O(1.4143^n)$ time algorithm
 - can solve instances up to size $2 \cdot x$

Outline

- 1 Algorithms for NP-hard problems
- 2 Exponential Time Algorithms
- 3 Parameterized Complexity
 - FPT Algorithm for Vertex Cover
 - Algorithms for Vertex Cover
- 4 Further Reading

A story

A computer scientist meets a biologist . . .

Eliminating conflicts from experiments

$n = 1000$ experiments,
 $k = 20$ experiments failed

Theoretical	Running Time	
	Number of Instructions	Real
2^n	$1.07 \cdot 10^{301}$	$4.941 \cdot 10^{282}$ years
n^k	10^{60}	$4.611 \cdot 10^{41}$ years
$2^k \cdot n$	$1.05 \cdot 10^9$	0.01526 seconds

Notes

- We assume that 2^{36} instructions are carried out per second.
- The Big Bang happened roughly $13.5 \cdot 10^9$ years ago.

Goal of Parameterized Complexity

Confine the combinatorial explosion to a parameter k .



For which problem–parameter combinations can we find algorithms with running times of the form

$$f(k) \cdot n^{O(1)},$$

where the f is a computable function independent of the input size n ?

Examples of Parameters

A Parameterized Problem

Input: an instance of the problem

Parameter: a parameter k

Question: a YES/NO question about the instance and the parameter

- A parameter can be
 - input size (trivial parameterization)
 - solution size
 - related to the structure of the input (maximum degree, treewidth, branchwidth, genus, ...)
 - etc.

Main Complexity Classes

P: class of problems that can be solved in time $n^{O(1)}$

FPT: class of problems that can be solved in time $f(k) \cdot n^{O(1)}$

W[·]: parameterized intractability classes

XP: class of problems that can be solved in time $f(k) \cdot n^{g(k)}$

$$P \subseteq \text{FPT} \subseteq W[1] \subseteq W[2] \cdots \subseteq W[P] \subseteq \text{XP}$$

Known: If $\text{FPT} = W[1]$, then the Exponential Time Hypothesis fails, i.e. 3-SAT can be solved in time $2^{o(n)}$.

Outline

- 1 Algorithms for NP-hard problems
- 2 Exponential Time Algorithms
- 3 Parameterized Complexity
 - FPT Algorithm for Vertex Cover
 - Algorithms for Vertex Cover
- 4 Further Reading

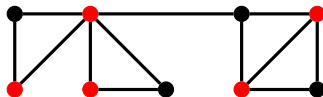
Vertex Cover

VERTEX COVER (VC)

Input: A graph $G = (V, E)$ on n vertices, an integer k

Parameter: k

Question: Is there a set of vertices $C \subseteq V$ of size at most k such that every edge has at least one endpoint in C ?



Outline

- 1 Algorithms for NP-hard problems
- 2 Exponential Time Algorithms
- 3 Parameterized Complexity
 - FPT Algorithm for Vertex Cover
 - Algorithms for Vertex Cover
- 4 Further Reading

Brute Force Algorithms

- $2^n \cdot n^{O(1)}$ not FPT
- $n^k \cdot n^{O(1)}$ not FPT

An FPT Algorithm

Algorithm $\text{vc1}(G, k)$;

```
1 if  $E = \emptyset$  then                // all edges are covered
2   | return Yes
3 else if  $k \leq 0$  then             // we cannot select any vertex
4   | return No
5 else
6   | Select an edge  $uv \in E$ ;
7   | return  $\text{vc1}(G - u, k - 1) \vee \text{vc1}(G - v, k - 1)$ 
```

Running Time Analysis

- Let us look at an arbitrary execution of the algorithm.
- Recursive calls form a **search tree** T
 - with depth $\leq k$
 - where each node has ≤ 2 children
- $\Rightarrow T$ has $\leq 2^k$ leaves and $\leq 2^k - 1$ internal nodes
- at each node the algorithm spends time $n^{O(1)}$
- The running time is $O^*(2^k)$

A faster FPT Algorithm

A faster FPT Algorithm

Algorithm $\text{vc2}(G, k);$

```
1 if  $E = \emptyset$  then                                // all edges are covered
2   | return Yes
3 else if  $k \leq 0$  then                               // we used too many vertices
4   | return No
5 else if  $\Delta(G) \leq 2$  then                         //  $G$  has maximum degree  $\leq 2$ 
6   | Solve the problem in polynomial time;
7 else
8   | Select a vertex  $v$  of maximum degree;
9   | return  $\text{vc2}(G - v, k - 1) \vee \text{vc2}(G - N[v], k - d(v))$ 
```

Running time analysis of vc2

- Number of leaves of the search tree:

$$T(k) \leq T(k-1) + T(k-3)$$

$$x^k \leq x^{k-1} + x^{k-3}$$

$$x^3 - x^2 - 1 \leq 0$$

- The equation $x^3 - x^2 - 1 = 0$ has a unique positive real solution:
 $x \approx 1.4655 \dots$
- Running time: $1.4656^k \cdot n^{O(1)}$

Outline

- 1 Algorithms for NP-hard problems
- 2 Exponential Time Algorithms
- 3 Parameterized Complexity
 - FPT Algorithm for Vertex Cover
 - Algorithms for Vertex Cover
- 4 Further Reading

- Exponential-time algorithms
 - Chapter 1, *Introduction*, in (Fomin and Kratsch, 2010).
 - Survey on exponential-time algorithms (Woeginger, 2001).
 - Chapter 1, *Introduction*, in (Gaspers, 2010).
- Parameterized Complexity
 - Chapter 1, *Introduction*, in (Cygan et al., 2015)
 - Chapter 2, *The Basic Definitions*, in (Downey and Fellows, 2013)
 - Chapter I, *Foundations*, in (Niedermeier, 2006)
 - *Preface* in (Flum and Grohe, 2006)

References I

- Scott Aaronson (2005). “Guest Column: NP-complete problems and physical reality”. In: *SIGACT News* 36.1, pp. 30–52. DOI: [10.1145/1052796.1052804](https://doi.org/10.1145/1052796.1052804).
- Jianer Chen, Iyad A. Kanj, and Ge Xia (2010). “Improved upper bounds for vertex cover”. In: *Theoretical Computer Science* 411.40-42, pp. 3736–3756. DOI: [10.1016/j.tcs.2010.06.026](https://doi.org/10.1016/j.tcs.2010.06.026).
- Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh (2015). *Parameterized Algorithms*. Springer. DOI: [10.1007/978-3-319-21275-3](https://doi.org/10.1007/978-3-319-21275-3).
- Rodney G. Downey and Michael R. Fellows (2013). *Fundamentals of Parameterized Complexity*. Springer. DOI: [10.1007/978-1-4471-5559-1](https://doi.org/10.1007/978-1-4471-5559-1).
- Jörg Flum and Martin Grohe (2006). *Parameterized Complexity Theory*. Springer. DOI: [10.1007/3-540-29953-X](https://doi.org/10.1007/3-540-29953-X).
- Fedor V. Fomin and Dieter Kratsch (2010). *Exact Exponential Algorithms*. Springer. DOI: [10.1007/978-3-642-16533-7](https://doi.org/10.1007/978-3-642-16533-7).
- Serge Gaspers (2010). *Exponential Time Algorithms: Structures, Measures, and Bounds*. VDM Verlag Dr. Mueller.

References II

- Martin Charles Golumbic (2004). *Algorithmic Graph Theory and Perfect Graphs*. Elsevier.
- Rolf Niedermeier (2006). *Invitation to Fixed Parameter Algorithms*. Oxford University Press. DOI: [10.1093/ACPROF:OSO/9780198566076.001.0001](https://doi.org/10.1093/ACPROF:OSO/9780198566076.001.0001).
- Silvia Richter, Malte Helmert, and Charles Gretton (2007). “A Stochastic Local Search Approach to Vertex Cover”. In: *Proceedings of the 30th Annual German Conference on Artificial Intelligence (KI 2007)*. Vol. 4667. Lecture Notes in Computer Science. Springer, pp. 412–426. DOI: [10.1007/978-3-540-74565-5_31](https://doi.org/10.1007/978-3-540-74565-5_31).
- Gerhard J. Woeginger (2001). “Exact Algorithms for NP-Hard Problems: A Survey”. In: *Combinatorial Optimization*, pp. 185–208. DOI: [10.1007/3-540-36478-1_17](https://doi.org/10.1007/3-540-36478-1_17).
- Mingyu Xiao and Hiroshi Nagamochi (2017). “Exact algorithms for maximum independent set”. In: *Information and Computation* 255, pp. 126–146. DOI: [10.1016/j.ic.2017.06.001](https://doi.org/10.1016/j.ic.2017.06.001).