Basics of Parameterized Complexity

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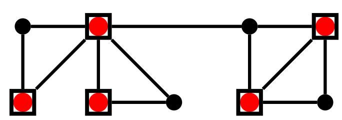
Contents

1 Introduction

1.1 Vertex Cover

A *vertex cover* in a graph $G = (V, E)$ is a subset of its vertices $S \subseteq V$ such that every edge of G has at least one endpoint in S.

VERTEX COVER	
Input:	A graph $G = (V, E)$ and an integer k
Parameter:	k
Question:	Does G have a vertex cover of size k?



Algorithms for Vertex Cover

- brute-force: $O^*(2^n)$
- brute-force: $O^*(n^k)$
- vc1: $O^*(2^k)$
- vc2: $O^*(1.4656^k)$
- (Chen, Kanj, and Xia, 2010): $O(1.2738^k + k \cdot n)$ (fastest known)

Running times in practice

$n = 1000$ vertices, $k = 20$ parameter

	Running Time	
Theoretical	Nb of Instructions	Real
2^n	$1.07 \cdot 10^{301}$	$4.941 \cdot 10^{282}$ years
n^k	10^{60}	$4.611 \cdot 10^{41}$ years
$2^k \cdot n$	$1.05 \cdot 10^9$	15.26 milliseconds
$1.4656^k \cdot n$	$2.10 \cdot 10^6$	0.31 milliseconds
$1.2738^k + k \cdot n$	$2.02 \cdot 10^4$	0.0003 milliseconds

Notes:
– We assume that 2^{36} instructions are carried out per second.
– The Big Bang happened roughly $13.5 \cdot 10^9$ years ago.

Goal of Parameterized Complexity
Confine the combinatorial explosion to a parameter k.

pc.jpg

(1) Which problem–parameter combinations are fixed-parameter tractable (FPT)? In other words, for which problem–parameter combinations are there algorithms with running times of the form

$$f(k) \cdot n^{O(1)},$$

where the f is a computable function independent of the input size n?
(2) How small can we make the $f(k)$?

Examples of Parameters

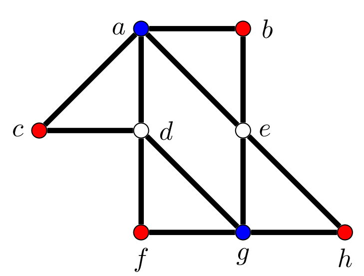
A Parameterized Problem	
Input:	an instance of the problem
Parameter:	a parameter
Question:	a Yes–No question about the instance and the parameter

- A parameter can be
  - solution size
  - input size (trivial parameterization)
  - related to the structure of the input (maximum degree, treewidth, branchwidth, genus, ...)
  - combinations of parameters
  - etc.

1.2 Coloring

A *k-coloring* of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

Coloring	
Input:	Graph G, integer k
Parameter:	k
Question:	Does G have a k-coloring?



Brute-force: $O^*(k^n)$, where $n = |V(G)|$. (Björklund, Husfeldt, and Koivisto, 2009): $O^*(2^n)$ by inclusion-exclusion (fastest known)

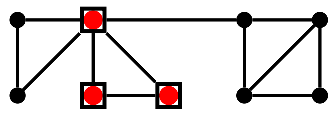
Coloring is probably not FPT

- Known: Coloring is NP-complete when $k = 3$
- Suppose there was a $O^*(f(k))$-time algorithm for Coloring
  - Then, 3-Coloring can be solved in $O^*(f(3)) \subseteq O^*(1)$ time
  - Therefore, P = NP
- Therefore, Coloring is not FPT unless P = NP

1.3 Clique

A *clique* in a graph $G = (V, E)$ is a subset of its vertices $S \subseteq V$ such that every two vertices from S are adjacent in G.

Clique	
Input:	Graph $G = (V, E)$, integer k
Parameter:	k
Question:	Does G have a clique of size k?



Is Clique NP-complete when k is a fixed constant? Is it FPT?

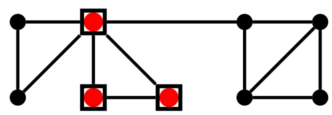
Algorithm for Clique

- For each subset $S \subseteq V$ of size k, check whether all vertices of S are adjacent
- Running time: $O^*\left(\binom{n}{k}\right) \subseteq O^*(n^k)$
- When $k \in O(1)$, this is polynomial
- But: we do not currently know an FPT algorithm for Clique
- Since Clique is W[1]-hard, we believe it is not FPT. (See lecture on W-hardness.)

1.4 Δ-Clique

A different parameter for Clique

Δ-Clique	
Input:	Graph $G = (V, E)$, integer k
Parameter:	$\Delta(G)$, i.e., the maximum degree of G
Question:	Does G have a clique of size k?



Is Δ-Clique FPT?

Algorithm for Δ-Clique

Input: A graph G and an integer k.
Output: YES if G has a clique of size k, and NO otherwise.

```
if k = 0 then
  return YES
else if k > Δ(G) + 1 then
  return NO
else
  /* A clique of size k contains at least one vertex v.
     For each v ∈ V, we check whether G has a k-clique S containing v (note that
     S ⊆ N_G[v] in this case).                                                    */
  foreach v ∈ V do
    foreach S ⊆ N_G[v] with |S| = k do
      if S is a clique in G then
        return YES
  return NO
```

Running time: $O^*((\Delta+1)^k) \subseteq O^*((\Delta+1)^\Delta)$. (FPT for parameter Δ)

2 Basic Definitions

Main Parameterized Complexity Classes

n: instance size
k: parameter

P: class of problems that can be solved in $n^{O(1)}$ time
FPT: class of parameterized problems that can be solved in $f(k) \cdot n^{O(1)}$ time
XP: class of parameterized problems that can be solved in $f(k) \cdot n^{g(k)}$ time ("polynomial when k is a constant")

$$\mathrm{P} \subseteq \mathrm{FPT} \subseteq \mathrm{W}[1] \subseteq \mathrm{W}[2] \cdots \subseteq \mathrm{W}[P] \subseteq \mathrm{XP}$$

Known: If FPT = W[1], then the Exponential Time Hypothesis fails, i.e. 3-SAT can be solved in $2^{o(n)}$ time, where n is the number of variables.

Note: We assume that f is *computable* and *non-decreasing*.

3 Further Reading

- Chapter 1, *Introduction* in (Cygan et al., 2015)
- Chapter 2, *The Basic Definitions* in (Downey and Fellows, 2013)
- Chapter I, *Foundations* in (Niedermeier, 2006)
- *Preface* in (Flum and Grohe, 2006)

References

Andreas Björklund, Thore Husfeldt, and Mikko Koivisto (2009). "Set Partitioning via Inclusion-Exclusion". In: *SIAM Journal on Computing* 39.2, pp. 546–563.

Jianer Chen, Iyad A. Kanj, and Ge Xia (2010). "Improved upper bounds for vertex cover". In: *Theoretical Computer Science* 411.40-42, pp. 3736–3756. DOI: 10.1016/j.tcs.2010.06.026.

Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh (2015). *Parameterized Algorithms*. Springer. DOI: 10.1007/978-3-319-21275-3.

Rodney G. Downey and Michael R. Fellows (2013). *Fundamentals of Parameterized Complexity*. Springer. DOI: 10.1007/978-1-4471-5559-1.

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