NP-completeness

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1 Overview

Polynomial-time algorithm
Polynomial-time algorithm: There exists a constant $c \in \mathbb{N}$ such that the algorithm has (worst-case) running-time $O(n^c)$, where n is the size of the input.

Example
Polynomial: n; $n^2 \log_2 n$; n^3; n^{20} Super-polynomial: $n^{\log_2 n}$; $2^{\sqrt{n}}$; 1.001^n; 2^n; $n!$

Central Question
Which computational problems have polynomial-time algorithms?

Million-dollar question
Intriguing class of problems: NP-complete problems.

NP-complete problems
It is unknown whether NP-complete problems have polynomial-time algorithms.

- A polynomial-time algorithm for one NP-complete problem would imply polynomial-time algorithms for all problems in NP.

Gerhard Woeginger's P vs NP page: `http://www.win.tue.nl/~gwoegi/P-versus-NP.htm`

Polynomial vs. NP-complete
Polynomial

- Shortest Path: Given a graph G, two vertices a and b of G, and an integer k, does G have a simple a–b-path of length at most k?
- Euler Tour: Given a graph G, does G have a cycle that traverses each edge of G exactly once?
- 2-CNF SAT: Given a propositional formula F in 2-CNF, is F satisfiable?*A k-CNF formula is a conjunction (AND) of clauses, and each clause is a disjunction (OR) of at most k literals, which are negated or unnegated Boolean variables.*

NP-complete

- Longest Path: Given a graph G and an integer k, does G have a simple path of length at least k?
- Hamiltonian Cycle: Given a graph G, does G have a simple cycle that visits each vertex of G?
- 3-CNF SAT: Given a propositional formula F in 3-CNF, is F satisfiable? *Example:* $(x \vee \neg y \vee z) \wedge (\neg x \vee z) \wedge (\neg y \vee \neg z)$.

Overview

What's next?

- Formally define P, NP, and NP-complete (NPC)
- (New) skill: show that a problem is NP-complete

2 Turing Machines, P, and NP

Decision problems and Encodings

<Name of Decision Problem>	
Input:	<What constitutes an instance>
Question:	<*Yes/No* question>

We want to know which decision problems can be solved in polynomial time – polynomial in the *size of the input* n.

- Assume a "reasonable" encoding of the input
- Many encodings are polynomial-time equivalent; i.e., one encoding can be computed from another in polynomial time.
- Important exception: unary versus binary encoding of integers.
  - An integer x takes $\lceil \log_2 x \rceil$ bits in binary and $x = 2^{\log_2 x}$ bits in unary.

Formal-language framework

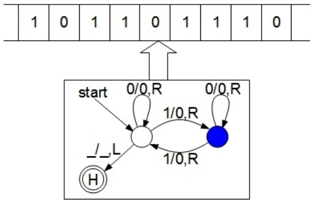
We can view decision problems as languages.

- Alphabet Σ: finite set of symbols. W.l.o.g., $\Sigma = \{0, 1\}$
- Language L over Σ: set of strings made with symbols from Σ: $L \subseteq \Sigma^*$
- Fix an encoding of instances of a decision problem Π into Σ
- Define the language $L_\Pi \subseteq \Sigma^*$ such that

$$x \in L_\Pi \Leftrightarrow x \text{ is a Yes-instance for } \Pi$$

Non-deterministic Turing Machine (NTM)

- *input word* $x \in \Sigma^*$ placed on an *infinite tape* (memory)
- read-write head initially placed on the first symbol of x
- computation step: if the machine is in state s and reads a, it can move into state s', writing b, and moving the head into direction $D \in \{L, R\}$ if $((s, a), (s', b, D)) \in \delta$.



- Q: finite, non-empty set of states
- Γ: finite, non-empty set of tape symbols
- $\_ \in \Gamma$: blank symbol (the only symbol allowed to occur on the tape infinitely often)
- $\Sigma \subseteq \Gamma \setminus \{b\}$: set of input symbols
- $q_0 \in Q$: start state
- $A \subseteq Q$: set of accepting (final) states
- $\delta \subseteq (Q \setminus A \times \Gamma) \times (Q \times \Gamma \times \{L, R\})$: transition relation, where L stands for a move to the left and R for a move to the right.

Accepted Language

Definition 1. A NTM *accepts* a word $x \in \Sigma^*$ if there exists a sequence of computation steps starting in the start state and ending in an accept state.

Definition 2. The language *accepted* by an NTM is the set of words it accepts.

Video

The LEGO Turing Machine `https://www.youtube.com/watch?v=cYw2ewoO6c4`

Acceptance in polynomial time

Definition 3. A language L is *accepted in polynomial time* by an NTM M if

- L is accepted by M, and
- there is a constant k such that for any word $x \in L$, the NTM M accepts x in $O(|x|^k)$ computation steps.

Deterministic Turing Machine

Definition 4. A *Deterministic Turing Machine (DTM)* is a Non-deterministic Turing Machine where the transition relation contains at most one tuple $((s, a), (\cdot, \cdot, \cdot))$ for each $s \in Q \setminus A$ and $a \in \Gamma$.

The transition relation δ can be viewed as a function $\delta : Q \setminus A \times \Gamma \to Q \times \Gamma \times \{L, R\}$. $\Rightarrow$ For a given input word $x \in \Sigma^*$, there is exactly one sequence of computation steps starting in the start state.

DTM equivalents

Many computational models are polynomial-time equivalent to DTMs:

- Random Access Machine (RAM, used for algorithms in the textbook)
- variants of Turing machines (multiple tapes, infinite only in one direction, ...)
- ...

P and NP

Definition 5 (P). $\mathrm{P} = \{L \subseteq \Sigma^* : \text{ there is a DTM accepting } L \text{ in polynomial time}\}$

Definition 6 (NP). $\mathrm{NP} = \{L \subseteq \Sigma^* : \text{ there is a NTM accepting } L \text{ in polynomial time}\}$

Definition 7 (coNP). $\mathrm{coNP} = \{L \subseteq \Sigma^* : \Sigma^* \setminus L \in \mathrm{NP}\}$

coP?

Theorem 8. *If $L \in \mathrm{P}$, then there is a polynomial-time DTM that halts in an accepting state on every word in L and it halts in a non-accepting state on every word not in L.*

Proof sketch. Suppose $L \in \mathrm{P}$. By the definition of P, there is a DTM M that accepts L in polynomial time. Idea: design a DTM M' that simulates M for $c \cdot n^k$ steps, where $c \cdot n^k$ is the running time of M and transitions to a non-accepting state if M does not halt in an accepting state. (Note that this proof is nonconstructive: we might not know the running time of M.) ☐

NP and certificates

Non-deterministic choices

A NTM for an NP-language L makes a polynomial number of non-deterministic choices on input $x \in L$. We can encode these non-deterministic choices into a *certificate* c, which is a polynomial-length word. Now, there exists a DTM, which, given x and c, verifies that $x \in L$ in polynomial time.

Thus, $L \in \mathrm{NP}$ iff there is a DTM V and for each $x \in L$ there exists a polynomial-length certificate c such that $V(x, c) = 1$, but $V(y, \cdot) = 0$ for each $y \notin L$.

CNF-SAT is in NP

- A *CNF formula* is a propositional formula in conjunctive normal form: a conjunction (AND) of clauses; each clause is a disjunction (OR) of literals; each literal is a negated or unnegated Boolean variable.
- An assignment $\alpha : \mathsf{var}(F) \to \{0, 1\}$ satisfies a clause C if it sets a literal of C to true, and it satisfies F if it satisfies all clauses in F.

CNF-SAT	
Input:	CNF formula F
Question:	Does F have a satisfying assignment?

Example: $(x \vee \neg y \vee z) \wedge (\neg x \vee z) \wedge (\neg y \vee \neg z)$.

Lemma 9. *CNF-SAT* $\in \mathrm{NP}$.

Proof. Certificate: assignment α to the variables. Given a certificate, it can be checked in polynomial time whether all clauses are satisfied. ☐

Brute-force algorithms for problems in NP

Theorem 10. *Every problem in* NP *can be solved in exponential time.*

Proof. Let Π be an arbitrary problem in NP. [Use certificate-based definition of NP] We know that $\exists$ a polynomial p and a polynomial-time verification algorithm V such that:

- for every $x \in \Pi$ (i.e., every YES-instance for Π) $\exists$ string $c \in \{0, 1\}^*$, $|c| \le p(|x|)$, such that $V(x, c) = 1$, and
- for every $x \notin \Pi$ (i.e., every NO-instance for Π) and every string $c \in \{0, 1\}^*$, $V(x, c) = 0$.

Now, we can prove that there exists an exponential-time algorithm for Π with input x:

- For each string $c \in \{0, 1\}^*$ with $|c| \le p(|x|)$, evaluate $V(x, c)$ and return YES if $V(x, c) = 1$.
- Return NO.

Running time: $2^{p(|x|)} \cdot n^{O(1)} \subseteq 2^{O(2 \cdot p(|x|))} = 2^{O(p(|x|))}$, but non-constructive. ☐

3 Reductions and NP-completeness

Polynomial-time reduction

Definition 11. A language L_1 is *polynomial-time reducible* to a language L_2, written $L_1 \leq_P L_2$, if there exists a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ such that for all $x \in \Sigma^*$,

$$x \in L_1 \Leftrightarrow f(x) \in L_2.$$

A polynomial time algorithm computing f is a *reduction algorithm*.

New polynomial-time algorithms via reductions

Lemma 12. *If $L_1, L_2 \in \Sigma^*$ are languages such that $L_1 \leq_P L_2$, then $L_2 \in \mathrm{P}$ implies $L_1 \in \mathrm{P}$.*

NP-completeness

Definition 13 (NP-hard)**.** A language $L \subseteq \Sigma^*$ is NP-*hard* if

$$L' \leq_P L \text{ for every } L' \in \mathrm{NP}.$$

Definition 14 (NP-complete)**.** A language $L \subseteq \Sigma^*$ is NP-*complete* (in NPC) if

1. $L \in \mathrm{NP}$, and
2. L is NP-hard.

A first NP-complete problem

Theorem 15. *CNF-SAT is* NP*-complete.*

Proved by encoding NTMs into SAT (Cook, 1971; Levin, 1973) and then CNF-SAT (Karp, 1972).

Proving NP-completeness

Lemma 16. *If L is a language such that $L' \leq_P L$ for some $L' \in \mathrm{NPC}$, then L is* NP*-hard. If, in addition, $L \in \mathrm{NP}$, then $L \in \mathrm{NPC}$.*

Proof. For all $L'' \in \mathrm{NP}$, we have $L'' \leq_P L' \leq_P L$. By transitivity, we have $L'' \leq_P L$. Thus, L is NP-hard. □

Proving NP-completeness (2)

Method to prove that a language L is NP-complete:

1. Prove $L \in \mathrm{NP}$
2. Prove L is NP-hard.
   - Select a known NP-complete language L'.
   - Describe an algorithm that computes a function f mapping every instance $x \in \Sigma^*$ of L' to an instance $f(x)$ of L.
   - Prove that $x \in L' \Leftrightarrow f(x) \in L$ for all $x \in \Sigma^*$.
   - Prove that the algorithm computing f runs in polynomial time.

4 NP-complete problems

3-CNF SAT is NP-hard

Theorem 17. *3-CNF SAT is* NP*-complete.*

Proof. 3-CNF SAT is in NP, since it is a special case of CNF-SAT. To show that 3-CNF SAT is NP-hard, we give a polynomial reduction from CNF-SAT. Let F be a CNF formula. The reduction algorithm constructs a 3-CNF formula F' as follows. For each clause C in F:

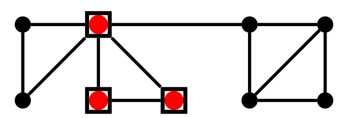
- If C has at most 3 literals, then copy C into F'.
- Otherwise, denote $C = (\ell_1 \vee \ell_2 \vee \cdots \vee \ell_k)$. Create $k-3$ new variables $y_1, \ldots, y_{k-3}$, and add the clauses $(\ell_1 \vee \ell_2 \vee y_1), (\neg y_1 \vee \ell_3 \vee y_2), (\neg y_2 \vee \ell_4 \vee y_3), \ldots, (\neg y_{k-3} \vee \ell_{k-1} \vee \ell_k)$.

Show that F is satisfiable $\Leftrightarrow$ F' is satisfiable. Show that F' can be computed in polynomial time (trivial; use a RAM). ☐

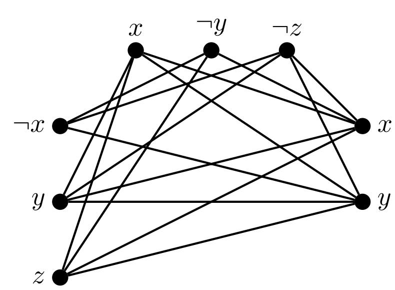
Clique

A *clique* in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every two vertices of S are adjacent in G.

CLIQUE	
Input:	Graph G, integer k
Question:	Does G have a clique of size k?



Theorem 18. CLIQUE *is* NP*-complete.*



$$(\neg x \vee y \vee z) \wedge (x \vee \neg y \vee \neg z) \wedge (x \vee y)$$

- CLIQUE is in NP
- Let $F = C_1 \wedge C_2 \wedge \ldots C_k$ be a 3-CNF formula
- Construct a graph G that has a clique of size k iff F is satisfiable
- For each clause $C_r = (\ell_1^r \vee \cdots \vee \ell_w^r)$, $1 \leq r \leq k$, create w new vertices $v_1^r, \ldots, v_w^r$
- Add an edge between v_i^r and v_j^s if

$$\begin{aligned} r &\neq s && \text{and} \\ \ell_i^r &\neq \neg \ell_j^s && \text{where } \neg\neg x = x. \end{aligned}$$

- Check correctness and polynomial running time
- Correctness: F has a satisfying assignment iff G has a clique of size k.

- ($\Rightarrow$): Let α be a sat. assignment for F. For each clause C_r, choose a literal ℓ_i^r with $\alpha(\ell_i^r) = 1$, and denote by s^r the corresponding vertex in G. Now, $\{s^r : 1 \leq r \leq k\}$ is a clique of size k in G since $\alpha(x) \neq \alpha(\neg x)$.
- ($\Leftarrow$): Let S be a clique of size k in G. Then, S contains exactly one vertex $s_r \in \{v_1^r, \dots, v_w^r\}$ for each $r \in \{1, \dots, k\}$. Denote by l^r the corresponding literal. Now, for any r, r', it is not the case that $l_r = \neg l_{r'}$. Therefore, there is an assignment α to $\mathsf{var}(F)$ such that $\alpha(l_r) = 1$ for each $r \in \{1, \dots, k\}$ and α satisfies F.

Vertex Cover

A *vertex cover* in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every edge of G has an endpoint in S.

Vertex Cover	
Input:	Graph G, integer k
Question:	Does G have a vertex cover of size k?

Theorem 19. Vertex Cover *is* NP*-complete.*

The proof is left as an exercise.

Hamiltonian Cycle

A *Hamiltonian Cycle* in a graph $G = (V, E)$ is a cycle visiting each vertex exactly once. (Alternatively, a permutation of V such that every two consecutive vertices are adjacent and the first and last vertex in the permutation are adjacent.)

Hamiltonian Cycle	
Input:	Graph G
Question:	Does G have a Hamiltonian Cycle?

Theorem 20. Hamiltonian Cycle *is* NP*-complete.*

Proof sketch.

- Hamiltonian Cycle is in NP: the certificate is a Hamiltonian Cycle of G.
- Let us show: Vertex Cover $\leq_P$ Hamiltonian Cycle

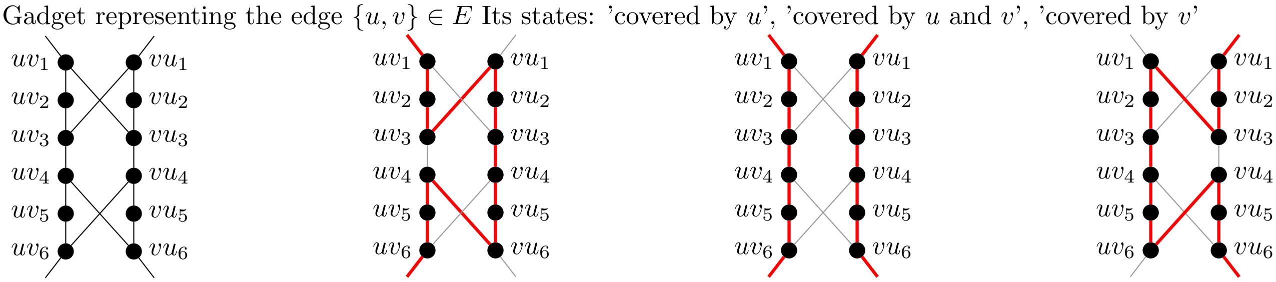
. • Let $(G = (V, E), k)$ be an instance for Vertex Cover (VC).

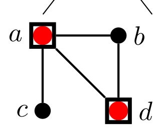
- We will construct an equivalent instance G' for Hamiltonian Cycle (HC).
- Intuition: Non-deterministic choices
  - for VC: which vertices to select in the vertex cover
  - for HC: which route the cycle takes

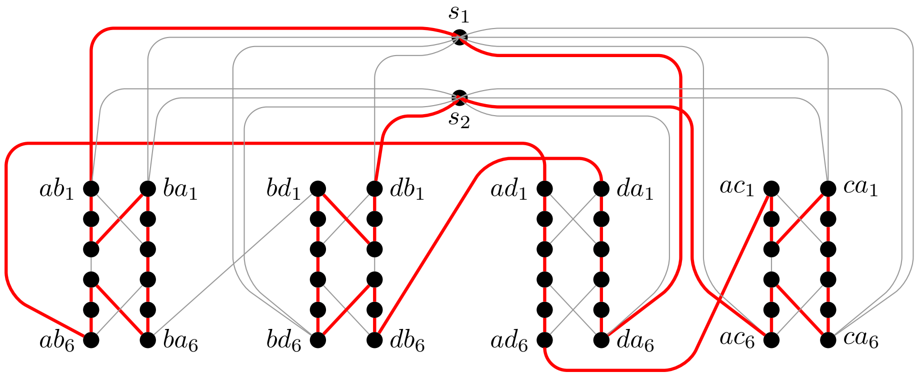
. • Add k vertices $s_1, \dots, s_k$ to G' (*selector vertices*)

- Each edge of G will be represented by a gadget (subgraph) of G'
- s.t. the set of edges covered by a vertex x in G corresponds to a partial cycle going through all gadgets of G' representing these edges.
- Attention: we need to allow for an edge to be covered by both endpoints

Gadget representing the edge $\{u, v\} \in E$ Its states: 'covered by u', 'covered by u and v', 'covered by v'







5 Further Reading

- Chapter 34, **NP-Completeness**, in (Cormen et al., 2009)
- Garey and Johnson's influential reference book (Garey and Johnson, 1979)

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