Modelling Biological Systems Exam 2016

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2. Optimal fishing of an unstructured population.

Consider a fish population that grows according to

$$\frac{dn}{dt} = rn\left(1 - \left(\frac{n}{K}\right)^2\right),$$

where r and K are positive constants.

a) What is the (non-trivial) equilibrium population size?

Solve $f(n^*) = 0$

Trivial is $n = 0$

Non-trivial:

$$1 - \left(\frac{n}{K}\right)^2 = 0$$
$$n = K$$

since K is positive constant

b) Show that it is a stable equilibrium.

An equilibrium is stable if $f'(n^*) < 0$

$$f'(n) = f'(rn)\left(1 - \left(\frac{n}{K}\right)^2\right) + rnf'\left(1 - \left(\frac{n}{K}\right)^2\right) = r - \frac{rn^2}{K^2} - \frac{2rn^2}{K^2} = r - \frac{3rn^2}{K^2}$$

$$f'(n^*) = r - 3r = -2r < 0$$

since r is positive constant.

The same shows Growth Function graph based on our model - the slope is negative at $n = K$ point:

```
library(deSolve)

#define the drowth function

populationGrowth <- function(t, n, P) {
  dndt <- P$r0 * n * (1 - n**2/P$K**2)
  list(dndt)

}
```

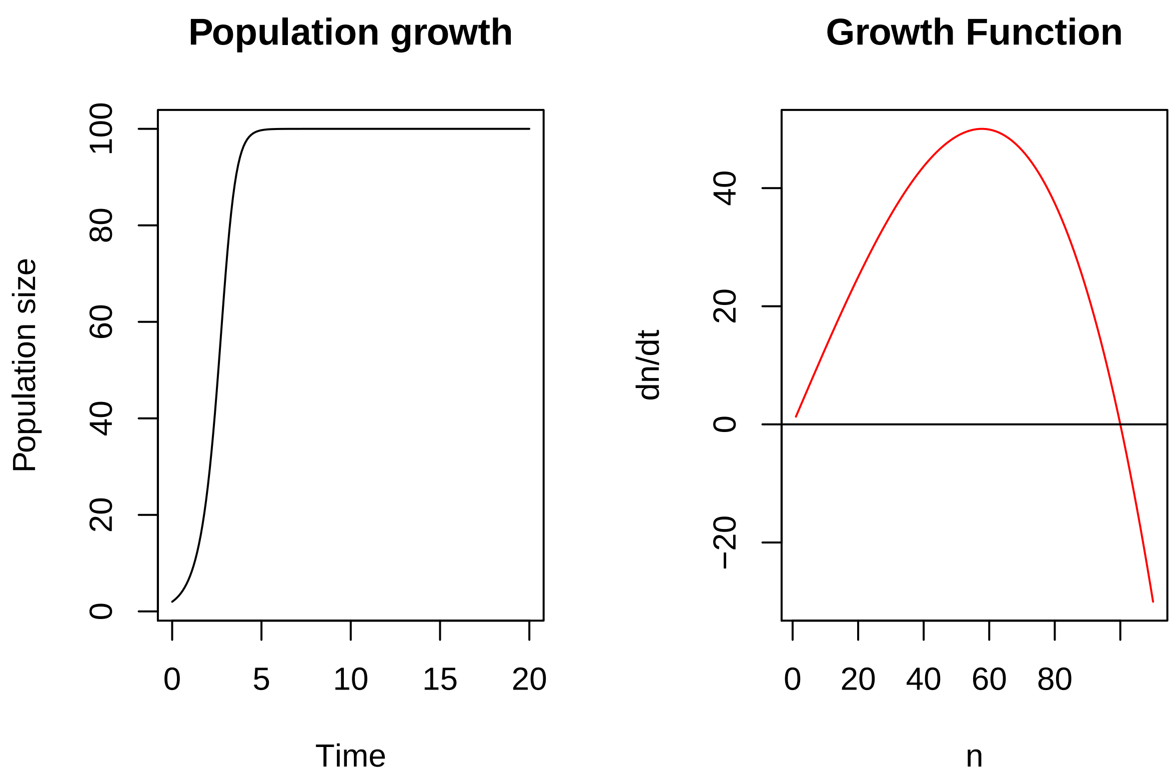
```r
# set the vector of time-points for the output
timevec <- seq(0, 20, by=0.1)

# list of parameters

P <- list(r0 = 1.3, K = 100)

n0 <- 2 #initial population size

# call the ode function
n <- 1:110
out <- ode(y = n0, func = populationGrowth, times = timevec, parms = P)
dndt <- unlist(populationGrowth(1, n, P))
par(mfrow = c(1,2))
plot(out[,1], out[,2], type='l', main = 'Population growth',
     xlab='Time', ylab='Population size')
plot(n, dndt, type='l', main='Growth Function', xlab='n', ylab='dn/dt',col='red')
abline(0,0)
```



Now assume the population is harvested, such that a proportion h is harvested per time unit.

c) Add the harvesting to the population dynamic model!

$-hn$ should be added to starting model

$$\frac{dn}{dt} = rn\left(1 - \left(\frac{n}{K}\right)^2\right) - hn$$

d) Where is the new equilibrium population size?

Solve $f(n^*) = 0$

$$n\left(r\left(1 - \left(\frac{n}{K}\right)^2\right) - h\right) = 0$$

Trivial is $n = 0$

Non-trivial:

$$r\left(1 - \left(\frac{n}{K}\right)^2\right) - h = 0$$

$$r - \frac{rn^2}{K^2} - h = 0$$

$$\frac{r}{K^2}n^2 = r - h$$

$$n^2 = K^2\left(1 - \frac{h}{r}\right)$$

$$n = K\sqrt{1 - \frac{h}{r}}$$

Confirmation by running in R

```
library(deSolve)
#define the drowth function with fishing
populationGrowth <- function(t, n, P) {
  dndt <- P$r0 * n * (1 - n**2/P$K**2) - n*P$h
  list(dndt)
}
# set the vector of time-points for the output
timevec <- seq(0, 20, by=0.1)
# list of parameters
P <- list(r0 = 1.5, K = 100, h = 0.5)
n0 <- 50 #initial population size
# call the ode function
out <- ode(y = n0, func = populationGrowth, times = timevec, parms = P)
n <- 1:110
dndt <- unlist(populationGrowth(1, n, P))
par(mfrow = c(1,2))
```

```
plot(out[,1], out[,2], type='l', main = 'Population growth', xlab='Time', ylab='Population size')
plot(n, dndt, type='l', main='Growth Function', xlab='n', ylab='dn/dt',col='red')
abline(0,0)
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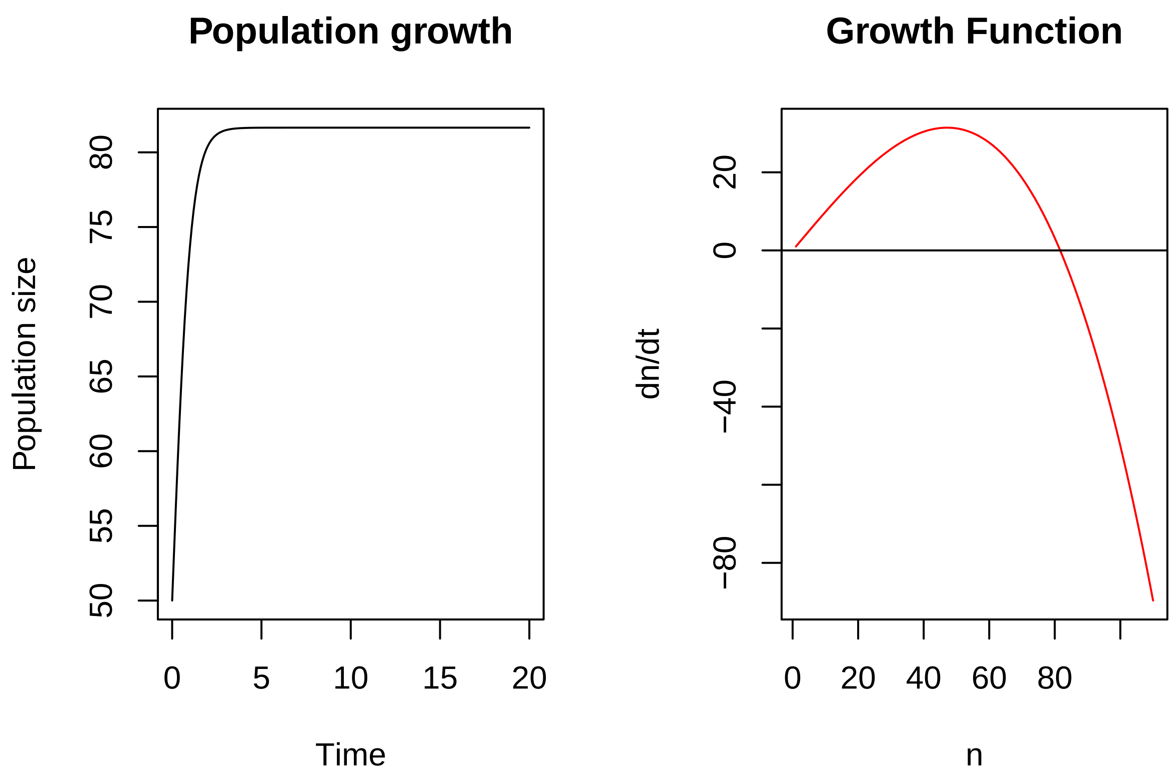

There is stable equilibrium point (lower than K) with negative slope.

e) At what harvest rate does the population go extinct?

If $h > r \Rightarrow n < 0$

This is confirmed by running same model with $h > r$

```
library(deSolve)
#define the drowth function with fishing
populationGrowth <- function(t, n, P) {
  dndt <- P$r0 * n * (1 - n**2/P$K**2) - n*P$h
  list(dndt)
}
# set the vector of time-points for the output
timevec <- seq(0, 20, by=0.1)
# list of parameters
P <- list(r0 = 1.5, K = 100, h = 1.7)
n0 <- 50 #initial population size
# call the ode function
out <- ode(y = n0, func = populationGrowth, times = timevec, parms = P)
```

```r
n <- 1:110
dndt <- unlist(populationGrowth(1, n, P))
par(mfrow = c(1,2))
plot(out[,1], out[,2], type='l', main = 'Population growth', xlab='Time', ylab='Population size')
plot(n, dndt, type='l', main='Growth Function', xlab='n', ylab='dn/dt',col='red')
abline(0,0)
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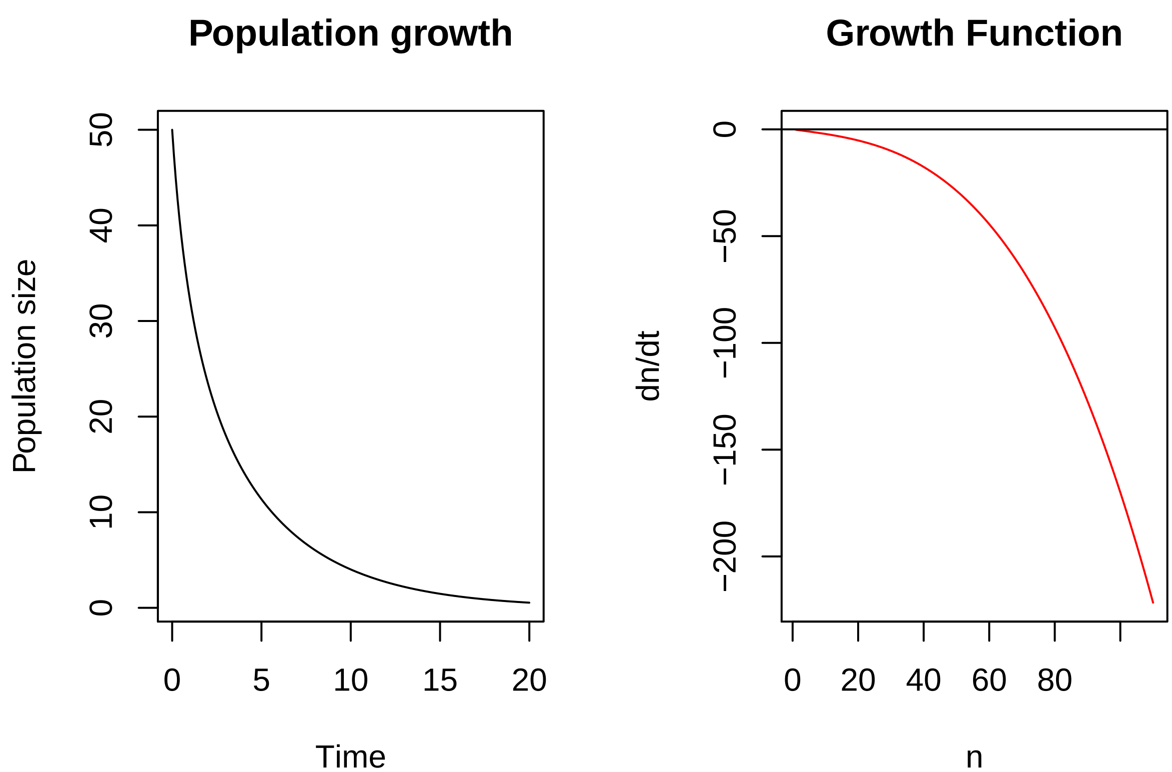

There is no second stable equilibrium point on Growth Function graph

f) At what harvest rate is the yield, i.e. the total number of harvested individuals, maximized?

Yield (hn) at equilibrium:

$$hK\sqrt{1-\frac{h}{r}}$$

Maximized yield can be found at 0 point for derivative of above function $f'(h^*) = 0$

$$f'(h) = f'(Kh)\sqrt{1-\frac{h}{r}} + khf'\left(\sqrt{1-\frac{h}{r}}\right) = K\sqrt{1-\frac{h}{r}} + Kh\frac{1}{2}\left(1-\frac{h}{r}\right)^{-\frac{1}{2}}\left(-\frac{1}{r}\right) =$$

$$= K\sqrt{1-\frac{h}{r}} - \frac{Kh}{2r\sqrt{1-\frac{h}{r}}} = \frac{2rK\left(1-\frac{h}{r}\right) - Kh}{2r\sqrt{1-\frac{h}{r}}} = K\frac{2r-3h}{2r\sqrt{1-\frac{h}{r}}}$$

So for $f'(h^*) = 0$ either $K = 0$ or

$$\frac{2r-3h}{2r\sqrt{1-\frac{h}{r}}} = 0$$

$$2r - 3h = 0$$

$$h = \frac{2}{3}r$$