Definitions of modularity in Radalib and Radatools

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Abstract: Radatools is a set of freely distributed programs to analyze Complex Networks. In particular, it includes programs for Communities Detection, Mesoscales Determination, calculation of Network Properties, and general tools for the manipulation of Networks and Partitions. There are also several programs not strictly related with networks, standing out one for Agglomerative Hierarchical Clustering using Multidendrograms and Binary Dendrograms.

Radatools is just a set of binary executable programs whose source code is available in Radalib. Radalib is free software; you can redistribute it and/or modify it under the terms of the GNU Lesser General Public License version 2.1 as published by the Free Software Foundation.

Community detection and mesoscale determination in Radatools is based on the optimization of *modularity*. In this document we describe the different definitions of modularity that have been implemented in Radatools.

1 Definitions and notation

Let us suppose we have a complex network with N nodes and a partition C of the network in M communities.

1.1 Common to all kinds of networks

a_{ij} Adjacency matrix element, with value 1 if there is a link from node i to node j, 0 otherwise

A Adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{N \times N}$

w_{ij} Weight of the link from node i to node j, 0 if there is no link

W Weights matrix $W = (w_{ij}) \in \mathbb{R}^{N \times N}$

C_i Index of community to which node i belongs to

Unweighted networks may be analyzed using $w_{ij} = a_{ij}$.

1.2 Undirected networks

Undirected networks are characterized by symmetric adjacency and weights matrices: $a_{ij} = a_{ji}$ and $w_{ij} = w_{ji}$ for all pairs of nodes i and j.

k^i Degree, number of links ok node i

$$k^i = \sum_{j=1}^{N} a_{ij} \tag{1}$$

$2L$ Total degree

$$2L = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} = \sum_{i=1}^{N} k_i \tag{2}$$

w_i Strengh, sum of the weights of links of node i

$$w_i = \sum_{j=1}^{N} w_{ij} \tag{3}$$

$2w$ Total strength

$$2w = \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} = \sum_{i=1}^{N} w_i \tag{4}$$

In networks without self-loops, L is the number of links.

1.3 Directed networks

Directed networks have asymmetric adjacency and weights matrices, thus it is necessary to distinguish between links departing and arriving to nodes.

$k_i^{\mathbf{out}}$ Output degree, number of links from node i

$$k_i^{\text{out}} = \sum_{j=1}^{N} a_{ij} \tag{5}$$

$k_j^{\mathbf{in}}$ Input degree, number of links to node j

$$k_j^{\text{in}} = \sum_{i=1}^{N} a_{ij} \tag{6}$$

$2L$ Total degree

$$2L = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} = \sum_{i=1}^{N} k_i^{\text{out}} = \sum_{j=1}^{N} k_j^{\text{in}} \tag{7}$$

$w_i^{\mathbf{out}}$ Output strengh, sum of the weights of links from node i

$$w_i^{\text{out}} = \sum_{j=1}^{N} w_{ij} \tag{8}$$

$w_j^{\mathbf{out}}$ Input strengh, sum of the weights of links to node j

$$w_j^{\text{in}} = \sum_{i=1}^{N} w_{ij} \tag{9}$$

$2w$ Total strength

$$2w = \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} = \sum_{i=1}^{N} w_i^{\text{out}} = \sum_{j=1}^{N} w_j^{\text{in}} \tag{10}$$

Undirected networks are particular cases of the directed ones in which $k_i^{\text{out}} = k_j^{\text{in}} = k_i$ and $w_i^{\text{out}} = w_j^{\text{in}} = w_i$.

1.4 Signed networks

Signed networks are those with positive and negative weights.

$w_{ij}^{\pm}$ Positive and negative weights

$$w_{ij} = w_{ij}^{+} - w_{ij}^{-} \tag{11}$$

$$w_{ij}^{+} = \max\{0, w_{ij}\} \tag{12}$$

$$w_{ij}^{-} = \max\{0, -w_{ij}\} \tag{13}$$

$w_i^{\pm}$ Positive and negative strengths

$$w_i^{\pm} = \sum_{j=1}^{N} w_{ij}^{\pm} \tag{14}$$

$w_i^{\pm}$ Positive and negative total strengths

$$2w^{\pm} = \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij}^{\pm} = \sum_{i=1}^{N} w_i^{\pm} \tag{15}$$

If the network is signed and directed, then it is also necessary to distinguish between links departing and arriving to nodes.

$w_i^{\pm,\mathbf{out}}$ Positive and negative output strengths

$$w_i^{\pm,\text{out}} = \sum_{j=1}^{N} w_{ij}^{\pm} \tag{16}$$

$w_j^{\pm,\mathbf{in}}$ Positive and negative input strengths

$$w_j^{\pm,\text{in}} = \sum_{i=1}^{N} w_{ij}^{\pm} \tag{17}$$

$2w^{\pm}$ Positive and negative total strengths

$$2w^{\pm} = \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij}^{\pm} = \sum_{i=1}^{N} w_i^{\pm,\text{out}} = \sum_{j=1}^{N} w_j^{\pm,\text{in}} \tag{18}$$

1.5 Bipartite networks

In bipartite networks, nodes can be of two different classes, that we will refer to as Individuals (I) and Teams (T).

b_{ia} Weight of the link between individual node $i \in \{1, \dots, N_I\}$, and team node $a \in \{1, \dots, N_T\}$.

B Matrix $B = (b_{ia}) \in \mathbb{R}^{N_I \times N_T}$

A Weighted adjacency block matrix $A \in \mathbb{R}^{(N_I+N_T)\times(N_I+N_T)}$ of a bipartite network

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \tag{19}$$

u_i Individual node strength

$$u_i = \sum_{a=1}^{N_T} b_{ia} \tag{20}$$

v_a Team node strength

$$v_a = \sum_{i=1}^{N_I} b_{ia} \tag{21}$$

u Individuals total strength

$$u = \sum_{i=1}^{N_I} u_i = \sum_{i=1}^{N_I} \sum_{a=1}^{N_T} b_{ia} \tag{22}$$

v Teams total strength

$$v = \sum_{a=1}^{N_T} v_a = \sum_{i=1}^{N_I} \sum_{a=1}^{N_T} b_{ia} \tag{23}$$

For these networks, $u = v$. Although not common, bipartite networks could also be directed.

B Matrix $B = (b_{ia}) \in \mathbb{R}^{N_I \times N_T}$

D Matrix $D = (d_{ai}) \in \mathbb{R}^{N_T \times N_I}$

A Weighted adjacency block matrix $A \in \mathbb{R}^{(N_I+N_T)\times(N_I+N_T)}$ of a directed bipartite network, with $D \neq B^T$

$$A = \begin{pmatrix} 0 & B \\ D & 0 \end{pmatrix} \tag{24}$$

$u_i^{\mathbf{out}}$ Individual node output strength

$$u_i^{\text{out}} = \sum_{a=1}^{N_T} b_{ia} \tag{25}$$

$u_i^{\mathbf{in}}$ Individual node input strength

$$u_i^{\text{in}} = \sum_{a=1}^{N_T} d_{ai} \tag{26}$$

$v_a^{\mathbf{out}}$ Team node output strength

$$v_a^{\text{out}} = \sum_{i=1}^{N_I} d_{ai} \tag{27}$$

$v_a^{\mathbf{in}}$ Team node input strength

$$v_a^{\text{in}} = \sum_{i=1}^{N_I} b_{ia} \tag{28}$$

$u^{\mathbf{out}}$ Individuals total output strength

$$u^{\text{out}} = \sum_{i=1}^{N_I} u_i^{\text{out}} = \sum_{i=1}^{N_I} \sum_{a=1}^{N_T} b_{ia} \tag{29}$$

$\boldsymbol{u^{\mathrm{in}}}$ Individuals total input strength

$$u^{\mathrm{in}} = \sum_{i=1}^{N_I} u_i^{\mathrm{in}} = \sum_{i=1}^{N_I} \sum_{a=1}^{N_T} d_{ai} \tag{30}$$

$\boldsymbol{v^{\mathrm{out}}}$ Teams total output strength

$$v^{\mathrm{out}} = \sum_{a=1}^{N_T} v_a^{\mathrm{out}} = \sum_{i=1}^{N_I} \sum_{a=1}^{N_T} d_{ai} \tag{31}$$

$\boldsymbol{v^{\mathrm{out}}}$ Teams total input strength

$$v^{\mathrm{in}} = \sum_{a=1}^{N_T} v_a^{\mathrm{in}} = \sum_{i=1}^{N_I} \sum_{a=1}^{N_T} b_{ia} \tag{32}$$

Thus, $u^{\mathrm{out}} = v^{\mathrm{in}}$ and $u^{\mathrm{in}} = v^{\mathrm{out}}$. Finally, bipartite networks could also be signed, thus giving rise to the corresponding $u_i^{\pm}$, $v_a^{\pm}$, $u_i^{\pm,\mathrm{out}}$, $u_i^{\pm,\mathrm{in}}$, $v_a^{\pm,\mathrm{out}}$, $v_a^{\pm,\mathrm{in}}$, $u^{\pm}$, $v^{\pm}$, $u^{\pm,\mathrm{out}}$, $u^{\pm,\mathrm{in}}$, $v^{\pm,\mathrm{out}}$, $v^{\pm,\mathrm{in}}$.

2 Modularity types

The detection of communities in Radatools is currently performed by *modularity* optimization [1]. In Radatools it is possible to use different variants of *modularity*.

UN	Unweighted_Newman [1, 2]
UUN	Unweighted_Uniform_Nullcase [1, 3]
WN	Weighted_Newman [4, 2]
WS	Weighted_Signed [5]
WUN	Weighted_Uniform_Nullcase [4, 3]
WLA	Weighted_Local_Average
WULA	Weighted_Uniform_Local_Average
WLUN	Weighted_Links_Unweighted_Nullcase [1, 4, 2]
WNN	Weighted_No_Nullcase [6]
WLR	Weighted_Link_Rank [7]
WBPM	Weighted_Bipartite_Path_Motif [8]
WBPS	Weighted_Bipartite_Path_Signed [8]

2.1 [UN] Unweighted_Newman

$$Q = \frac{1}{2L} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(a_{ij} - \frac{k_i^{\text{out}} \, k_j^{\text{in}}}{2L} \right) \delta(C_i, C_j) \tag{33}$$

2.2 [UUN] Unweighted_Uniform_Nullcase

$$Q = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{a_{ij}}{2L} - \frac{1}{N^2} \right) \delta(C_i, C_j) \tag{34}$$

2.3 [WN] Weighted_Newman

$$Q = \frac{1}{2w} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(w_{ij} - \frac{w_i^{\text{out}} \, w_j^{\text{in}}}{2w} \right) \delta(C_i, C_j) \tag{35}$$

2.4 [WS] Weighted_Signed

$$Q = \frac{1}{2w^+ + 2w^-} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(w_{ij} - \left(\frac{w_i^{+,\text{out}} \, w_j^{+,\text{in}}}{2w^+} - \frac{w_i^{-,\text{out}} \, w_j^{-,\text{in}}}{2w^-} \right) \right) \delta(C_i, C_j) \tag{36}$$

2.5 [WUN] Weighted_Uniform_Nullcase

$$Q = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{w_{ij}}{2w} - \frac{1}{N^2} \right) \delta(C_i, C_j) \tag{37}$$

2.6 [WLA] Weighted_Local_Average

$$Q = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{w_{ij}}{2w} - \frac{1}{D_a} \, k_i^{\text{out}} \, k_j^{\text{in}} \, \frac{w_i^{\text{out}} + w_j^{\text{in}}}{k_i^{\text{out}} + k_j^{\text{in}}} \right) \delta(C_i, C_j) \tag{38}$$

where

$$D_a = \sum_{i=1}^{N} \sum_{j=1}^{N} k_i^{\text{out}} \, k_j^{\text{in}} \frac{w_i^{\text{out}} + w_j^{\text{in}}}{k_i^{\text{out}} + k_j^{\text{in}}} \tag{39}$$

2.7 [WULA] Weighted_Uniform_Local_Average

$$Q = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{w_{ij}}{2w} - \frac{1}{D_u} \, \frac{w_i^{\text{out}} + w_j^{\text{in}}}{k_i^{\text{out}} + k_j^{\text{in}}} \right) \delta(C_i, C_j) \tag{40}$$

where

$$D_u = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{w_i^{\text{out}} + w_j^{\text{in}}}{k_i^{\text{out}} + k_j^{\text{in}}} \tag{41}$$

2.8 [WLUN] Weighted_Links_Unweighted_Nullcase

$$Q = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{w_{ij}}{2w} - \frac{k_i^{\mathrm{out}}\, k_j^{\mathrm{in}}}{(2L)^2} \right) \delta(C_i, C_j) \tag{42}$$

2.9 [WNN] Weighted_No_Nullcase

$$Q = \frac{1}{2w} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \delta(C_i, C_j) \tag{43}$$

2.10 [WLR] Weighted_Link_Rank

$$Q = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\pi_i G_{ij} - \pi_i \pi_j \right) \delta(C_i, C_j) \tag{44}$$

where π_i are the components of the left leading eigenvector of G_{ij}, the random walk transition matrix with teleportation τ (the same as for PageRank):

$$\pi_j = \sum_{i=1}^{N} \pi_i G_{ij} \tag{45}$$

and

$$G_{ij} = \begin{cases} (1-\tau) \dfrac{w_{ij}}{w_i^{\mathrm{out}}} + \dfrac{\tau}{N} & \text{if } w_i^{\mathrm{out}} > 0 \\ \frac{1}{N} & \text{if } w_i^{\mathrm{out}} = 0 \end{cases} \tag{46}$$

The default value of the teleportation is $\tau = 0.15$.

2.11 [WBPM] Weighted_Bipartite_Path_Motif

For a bipartite network,

$$\begin{aligned} Q \quad &= \quad \sum_{i=1}^{N_I} \sum_{j=1}^{N_I} \left(\frac{(BD)_{ij}}{\Psi} - \frac{\alpha_I\, u_i^{\mathrm{out}}\, u_j^{\mathrm{in}}}{\Omega} \right) \delta(C_i, C_j) & (47) \\ &+ \quad \sum_{a=1}^{N_T} \sum_{b=1}^{N_T} \left(\frac{(DB)_{ab}}{\Psi} - \frac{\alpha_T\, v_a^{\mathrm{out}}\, v_b^{\mathrm{in}}}{\Omega} \right) \delta(C_a, C_b) & (48) \end{aligned}$$

where

$$(BD)_{ij} = \sum_{a=1}^{N_T} b_{ia}\, d_{aj} \tag{49}$$

$$(DB)_{ab} = \sum_{i=1}^{N_I} d_{ai}\, b_{ib} \tag{50}$$

and

$$\alpha_I = \frac{1}{u^{\text{out}}\, u^{\text{in}}\, v^{\text{in}}\, v^{\text{out}}} \sum_{a=1}^{N_T} v_a^{\text{in}}\, v_a^{\text{out}} \tag{51}$$

$$\alpha_T = \frac{1}{v^{\text{out}}\, v^{\text{in}}\, u^{\text{in}}\, u^{\text{out}}} \sum_{i=1}^{N_I} u_i^{\text{in}}\, u_i^{\text{out}} \tag{52}$$

and

$$\Psi = \sum_{i=1}^{N_I} \sum_{j=1}^{N_I} (BD)_{ij} + \sum_{a=1}^{N_T} \sum_{b=1}^{N_T} (DB)_{ab} \tag{53}$$

$$\Omega = \frac{1}{u^{\text{in}}\, u^{\text{out}}} \sum_{i=1}^{N_I} u_i^{\text{in}}\, u_i^{\text{out}} + \frac{1}{v^{\text{in}}\, v^{\text{out}}} \sum_{a=1}^{N_T} v_a^{\text{in}}\, v_a^{\text{out}} \tag{54}$$

Note that

$$W = \begin{pmatrix} 0 & B \\ D & 0 \end{pmatrix} \qquad W^2 = \begin{pmatrix} BD & 0 \\ 0 & DB \end{pmatrix} \tag{55}$$

and

$$N = \begin{pmatrix} 0 & \dfrac{1}{u^{\text{out}}\, v^{\text{in}}} \boldsymbol{u}^{\text{out}}\, \boldsymbol{v}^{\text{in}T} \\ \dfrac{1}{v^{\text{out}}\, u^{\text{in}}} \boldsymbol{v}^{\text{out}}\, \boldsymbol{u}^{\text{in}T} & 0 \end{pmatrix} \tag{56}$$

$$N^2 = \begin{pmatrix} \alpha_I\, \boldsymbol{u}^{\text{out}}\, \boldsymbol{u}^{\text{in}T} & 0 \\ 0 & \alpha_T\, \boldsymbol{v}^{\text{out}}\, \boldsymbol{v}^{\text{in}T} \end{pmatrix} \tag{57}$$

If the network is not bipartite, the formulation is similar but with differences, and corresponds exactly to the path motif modularity (with only the extremes of the path inside the community) defined in [8]

$$Q = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{(W^2)_{ij}}{\Psi} - \frac{w_i^{\text{out}}\, w_j^{\text{in}}}{(2w)^2} \right) \delta(C_i, C_j) \tag{58}$$

where

$$(W^2)_{ij} = \sum_{r=1}^{N} w_{ir}\, w_{rj} \tag{59}$$

and

$$\Psi = \sum_{i=1}^{N} \sum_{j=1}^{N} (W^2)_{ij} \tag{60}$$

2.12 [WBPS] Weighted_Bipartite_Path_Signed

For a signed bipartite network,

$$Q = \sum_{i=1}^{N_I} \sum_{j=1}^{N_I} \left(\frac{(BD)_{ij}}{\Psi} - \frac{n_{ij}^I}{\Omega} \right) \delta(C_i, C_j) \tag{61}$$

$$+ \sum_{a=1}^{N_T} \sum_{b=1}^{N_T} \left(\frac{(DB)_{ab}}{\Psi} - \frac{n_{ab}^T}{\Omega} \right) \delta(C_a, C_b) \tag{62}$$

where

$$n^I_{ij} = \Big(\alpha_I^{++}\, u_i^{+,\text{out}}\, u_j^{+,\text{in}} + \alpha_I^{--}\, u_i^{-,\text{out}}\, u_j^{-,\text{in}}\Big) - \Big(\alpha_I^{+-}\, u_i^{+,\text{out}}\, u_j^{-,\text{in}} + \alpha_I^{-+}\, u_i^{-,\text{out}}\, u_j^{+,\text{in}}\Big) \tag{63}$$

$$n^T_{ab} = \Big(\alpha_T^{++}\, v_a^{+,\text{out}}\, v_b^{+,\text{in}} + \alpha_T^{--}\, v_a^{-,\text{out}}\, v_b^{-,\text{in}}\Big) - \Big(\alpha_T^{+-}\, v_a^{+,\text{out}}\, v_b^{-,\text{in}} + \alpha_T^{-+}\, v_a^{-,\text{out}}\, v_b^{+,\text{in}}\Big) \tag{64}$$

and

$$\alpha_I^{++} = \frac{1}{u^{+,\text{out}}\, u^{+,\text{in}}\, v^{+,\text{in}}\, v^{+,\text{out}}} \sum_{a=1}^{N_T} v_a^{+,\text{in}}\, v_a^{+,\text{out}} \tag{65}$$

$$\alpha_I^{--} = \frac{1}{u^{-,\text{out}}\, u^{-,\text{in}}\, v^{-,\text{in}}\, v^{-,\text{out}}} \sum_{a=1}^{N_T} v_a^{-,\text{in}}\, v_a^{-,\text{out}} \tag{66}$$

$$\alpha_I^{+-} = \frac{1}{u^{+,\text{out}}\, u^{-,\text{in}}\, v^{+,\text{in}}\, v^{-,\text{out}}} \sum_{a=1}^{N_T} v_a^{+,\text{in}}\, v_a^{-,\text{out}} \tag{67}$$

$$\alpha_I^{-+} = \frac{1}{u^{-,\text{out}}\, u^{+,\text{in}}\, v^{-,\text{in}}\, v^{+,\text{out}}} \sum_{a=1}^{N_T} v_a^{-,\text{in}}\, v_a^{+,\text{out}} \tag{68}$$

$$\alpha_T^{++} = \frac{1}{v^{+,\text{out}}\, v^{+,\text{in}}\, u^{+,\text{in}}\, u^{+,\text{out}}} \sum_{i=1}^{N_I} u_i^{+,\text{in}}\, u_i^{+,\text{out}} \tag{69}$$

$$\alpha_T^{--} = \frac{1}{v^{-,\text{out}}\, v^{-,\text{in}}\, u^{-,\text{in}}\, u^{-,\text{out}}} \sum_{i=1}^{N_I} u_i^{-,\text{in}}\, u_i^{-,\text{out}} \tag{70}$$

$$\alpha_T^{+-} = \frac{1}{v^{+,\text{out}}\, v^{-,\text{in}}\, u^{+,\text{in}}\, u^{-,\text{out}}} \sum_{i=1}^{N_I} u_i^{+,\text{in}}\, u_i^{-,\text{out}} \tag{71}$$

$$\alpha_T^{-+} = \frac{1}{v^{-,\text{out}}\, v^{+,\text{in}}\, u^{-,\text{in}}\, u^{+,\text{out}}} \sum_{i=1}^{N_I} u_i^{-,\text{in}}\, v_i^{+,\text{out}} \tag{72}$$

and

$$\Psi = \sum_{i=1}^{N_I}\sum_{j=1}^{N_I} \big((B^+D^+)_{ij} + (B^-D^-)_{ij} + (B^+D^-)_{ij} + (B^-D^+)_{ij}\big) \tag{73}$$

$$+ \sum_{a=1}^{N_T}\sum_{b=1}^{N_T} \big((D^+B^+)_{ab} + (D^-B^-)_{ab} + (D^+B^-)_{ab} + (D^-B^+)_{ab}\big) \tag{74}$$

$$
\begin{aligned}
\Omega \;=\;& \frac{1}{u^{+,\mathrm{in}}\,u^{+,\mathrm{out}}}\sum_{i=1}^{N_I} u_i^{+,\mathrm{in}}\,u_i^{+,\mathrm{out}} \;+\; \frac{1}{u^{-,\mathrm{in}}\,u^{-,\mathrm{out}}}\sum_{i=1}^{N_I} u_i^{-,\mathrm{in}}\,u_i^{-,\mathrm{out}} && (75)\\
+\;& \frac{1}{u^{+,\mathrm{in}}\,u^{-,\mathrm{out}}}\sum_{i=1}^{N_I} u_i^{+,\mathrm{in}}\,u_i^{-,\mathrm{out}} \;+\; \frac{1}{u^{-,\mathrm{in}}\,u^{+,\mathrm{out}}}\sum_{i=1}^{N_I} u_i^{-,\mathrm{in}}\,u_i^{+,\mathrm{out}} && (76)\\
+\;& \frac{1}{v^{+,\mathrm{in}}\,v^{+,\mathrm{out}}}\sum_{a=1}^{N_T} v_a^{+,\mathrm{in}}\,v_a^{+,\mathrm{out}} \;+\; \frac{1}{v^{-,\mathrm{in}}\,v^{-,\mathrm{out}}}\sum_{a=1}^{N_T} v_a^{-,\mathrm{in}}\,v_a^{-,\mathrm{out}} && (77)\\
+\;& \frac{1}{v^{+,\mathrm{in}}\,v^{-,\mathrm{out}}}\sum_{a=1}^{N_T} v_a^{+,\mathrm{in}}\,v_a^{-,\mathrm{out}} \;+\; \frac{1}{v^{-,\mathrm{in}}\,v^{+,\mathrm{out}}}\sum_{a=1}^{N_T} v_a^{-,\mathrm{in}}\,v_a^{+,\mathrm{out}} && (78)
\end{aligned}
$$

Note that

$$W^2 = \left(W^+ - W^-\right)^2 = (W^2)^+ - (W^2)^- \tag{79}$$

$$N^2 = \left(W^+ - W^-\right)^2 = (N^2)^+ - (N^2)^- \tag{80}$$

where

$$W^+ = \begin{pmatrix} 0 & B^+ \\ D^+ & 0 \end{pmatrix} \tag{81}$$

$$W^- = \begin{pmatrix} 0 & B^- \\ D^- & 0 \end{pmatrix} \tag{82}$$

$$(W^2)^+ = \begin{pmatrix} B^+D^+ + B^-D^- & 0 \\ 0 & D^+B^+ + D^-B^- \end{pmatrix} \tag{83}$$

$$(W^2)^- = \begin{pmatrix} B^+D^- + B^-D^+ & 0 \\ 0 & D^+B^- + D^-B^+ \end{pmatrix} \tag{84}$$

and

$$
\begin{aligned}
(N^2)^+ \;=\;& \begin{pmatrix} \alpha_I^{++}\,\boldsymbol{u}^{+,\mathrm{out}}\,(\boldsymbol{u}^{+,\mathrm{in}})^T & 0 \\ 0 & \alpha_T^{++}\,\boldsymbol{v}^{+,\mathrm{out}}\,(\boldsymbol{v}^{+,\mathrm{in}})^T \end{pmatrix} && (85)\\
+\;& \begin{pmatrix} \alpha_I^{--}\,\boldsymbol{u}^{-,\mathrm{out}}\,(\boldsymbol{u}^{-,\mathrm{in}})^T & 0 \\ 0 & \alpha_T^{--}\,\boldsymbol{v}^{-,\mathrm{out}}\,(\boldsymbol{v}^{-,\mathrm{in}})^T \end{pmatrix} && (86)
\end{aligned}
$$

$$
\begin{aligned}
(N^2)^- \;=\;& \begin{pmatrix} \alpha_I^{+-}\,\boldsymbol{u}^{+,\mathrm{out}}\,(\boldsymbol{u}^{-,\mathrm{in}})^T & 0 \\ 0 & \alpha_T^{+-}\,\boldsymbol{v}^{+,\mathrm{out}}\,(\boldsymbol{v}^{-,\mathrm{in}})^T \end{pmatrix} && (87)\\
+\;& \begin{pmatrix} \alpha_I^{-+}\,\boldsymbol{u}^{-,\mathrm{out}}\,(\boldsymbol{u}^{+,\mathrm{in}})^T & 0 \\ 0 & \alpha_T^{-+}\,\boldsymbol{v}^{-,\mathrm{out}}\,(\boldsymbol{v}^{+,\mathrm{in}})^T \end{pmatrix} && (88)
\end{aligned}
$$

If the network is not bipartite, the expressions become

$$Q = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{(W^2)_{ij}}{\Psi} - \frac{n_{ij}}{\Omega}\right)\delta(C_i, C_j) \tag{89}$$

where

$$n_{ij} = \left(\alpha^{++}\,w_i^{+,\mathrm{out}}\,w_j^{+,\mathrm{in}} + \alpha^{--}\,w_i^{-,\mathrm{out}}\,w_j^{-,\mathrm{in}}\right) - \left(\alpha^{+-}\,w_i^{+,\mathrm{out}}\,w_j^{-,\mathrm{in}} + \alpha^{-+}\,w_i^{-,\mathrm{out}}\,w_j^{+,\mathrm{in}}\right) \tag{90}$$

and

$$\alpha^{++} = \frac{1}{w^{+,\mathrm{out}}\, w^{+,\mathrm{in}}\, w^{+,\mathrm{in}}\, w^{+,\mathrm{out}}} \sum_{i=1}^{N} w_i^{+,\mathrm{in}}\, w_i^{+,\mathrm{out}} \tag{91}$$

$$\alpha^{--} = \frac{1}{w^{-,\mathrm{out}}\, w^{-,\mathrm{in}}\, w^{-,\mathrm{in}}\, w^{-,\mathrm{out}}} \sum_{i=1}^{N} w_i^{-,\mathrm{in}}\, w_i^{-,\mathrm{out}} \tag{92}$$

$$\alpha^{+-} = \frac{1}{w^{+,\mathrm{out}}\, w^{-,\mathrm{in}}\, w^{+,\mathrm{in}}\, w^{-,\mathrm{out}}} \sum_{i=1}^{N} w_i^{+,\mathrm{in}}\, w_i^{-,\mathrm{out}} \tag{93}$$

$$\alpha^{-+} = \frac{1}{w^{-,\mathrm{out}}\, w^{+,\mathrm{in}}\, w^{-,\mathrm{in}}\, w^{+,\mathrm{out}}} \sum_{i=1}^{N} w_i^{-,\mathrm{in}}\, w_i^{+,\mathrm{out}} \tag{94}$$

and

$$\Psi = \sum_{i=1}^{N_I} \sum_{j=1}^{N_I} \left((W^+W^+)_{ij} + (W^-W^-)_{ij} + (W^+W^-)_{ij} + (W^-W^+)_{ij} \right) \tag{95}$$

$$\Omega = \frac{1}{w^{+,\mathrm{in}}\, w^{+,\mathrm{out}}} \sum_{i=1}^{N_I} w_i^{+,\mathrm{in}}\, w_i^{+,\mathrm{out}} + \frac{1}{w^{-,\mathrm{in}}\, w^{-,\mathrm{out}}} \sum_{i=1}^{N_I} w_i^{-,\mathrm{in}}\, w_i^{-,\mathrm{out}} \tag{96}$$

$$+ \frac{1}{w^{+,\mathrm{in}}\, w^{-,\mathrm{out}}} \sum_{i=1}^{N_I} w_i^{+,\mathrm{in}}\, w_i^{-,\mathrm{out}} + \frac{1}{w^{-,\mathrm{in}}\, w^{+,\mathrm{out}}} \sum_{i=1}^{N_I} w_i^{-,\mathrm{in}}\, w_i^{+,\mathrm{out}} \tag{97}$$

3 Modularity optimization

We have implemented in RADATOOLS several optimization heuristics:

h Exhaustive search

t Tabu search [9]

e Extremal optimization [10]

s Spectral optimization [11]

f Fast algorithm [12]

l Louvain [13]

r Fine-tuning by reposition

b Fine-tuning by bootstrapping based on tabu search [9]

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