Open Channel Hydrodynamics

Dr. Sergio Maldonado
CENV3066 Environmental Hydraulics
University of Southampton

Contents

1 Introduction

In the first week of this module we established the importance of having a solid understanding of open channel hydrodynamics – it underpins the study of other topics covered in this module (i.e. transport of pollutants and sediment). Now, open channel flow (e.g. a river) is, after all, a fluid flow. As such, it must have some similarities with, say, air flow through a fan or oceanic currents. However, if you have taken introductory Fluid Mechanics (or Thermofluids) and Hydraulics, these similarities may not be obvious to you. The first goal of these notes is to convince you that *all fluids* have certain commonalities, a common 'ancestor' theory from which concepts such as Bernoulli's equation or the Shallow Water Equations (which you will encounter later in these notes) may be derived. This theory is the Navier-Stokes equations. From these equations, we will explore, step by step, several simplifications that apply specifically to water flows in open channels. Another reason to do this – and the second goal of these notes – is to study several aspects of hydrodynamics which are often overlooked in introductory hydraulics without major consequences, but that become crucial when trying to understand environmental hydraulics, such as turbulence and its effect on the flow velocity profile and the transport of sediment and pollutants.

Starting from the general and moving towards the particular, we will start by reviewing the Navier-Stokes equations, then we will explore the great importance of turbulence (which is virtually always present in open channel flows), and finally we will appreciate how the adoption of certain assumptions, based on experimentation, can permit significant simplification of the mathematical models to be employed by the hydraulic engineer.

2 The fundamental hydrodynamic equations

Let us start by analysing what happens to a small element of *incompressible* fluid (i.e. for our purposes, water). First of all, mass is conserved. As you know from previous courses, this leads to the continuity equation:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity vector with components $u_{i=1,2,3} = (u_1, u_2, u_3) = (u, v, w)$ in the Cartesian frame of reference $\boldsymbol{x} = (x_1, x_2, x_3) = (x, y, z)$, where x, y and z point in the streamwise, transverse and vertical directions, respectively, as shown in fig. 1; and t denotes time[1]. Three different notations, discussed during the first week of the module, have been employed in eq. (1). We will change from one notation to another depending on convenience; for instance, in eq. (1) the first notation clearly shows that mass in the element is conserved because the divergence of the velocity field ($\nabla \cdot \boldsymbol{u}$) is zero, but the third notation may be more useful for algebraic purposes.

The element of water under consideration must also obey Newton's second law, meaning that it will experience an acceleration ($D\boldsymbol{u}/Dt$) proportional to the sum of the forces acting on it. This leads to the Navier-Stokes (NS) equations[2], which govern, to the best of our knowledge, the motion

[1]Note that each component of $\boldsymbol{u}(\boldsymbol{x}, t)$ also varies in time and space; i.e. $u_i = u_i(x, y, z, t)$

[2]Named after the 19th century scientists Claude-Louis Navier (France) and George Gabriel Stokes (University of Cambridge).

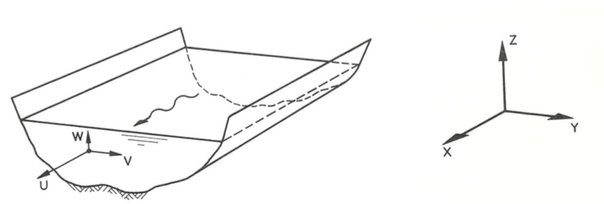

Figure 1: Sketch of an open channel flow (a river reach), showing the coordinate system adopted. The Navier-Stokes equations apply to a small element of fluid located anywhere in this reach (for instance, the point where the velocity vectors in the figure coincide) at any instance in time. [Taken from Jansen, P. P. *Principles of River Engineering: the Non-Tidal Alluvial River*. Delftse Uitgevers Maatschappij, 1994.]

of *all* fluids. The Navier-Stokes equations are (for incompressible flow):

$$\underbrace{\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}}_{\text{acceleration}} = \underbrace{\boldsymbol{g} - \frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u}}_{\text{forces (per mass)}}\,. \tag{2}$$

Let us take some time to examine this very important equation. On the left-most side of the equation, we have the *total acceleration* of the element, $D\boldsymbol{u}/Dt$, which is the actual acceleration that our element experiences. However, it is often convenient to write the total acceleration, in analogy with transport phenomena[3] (in this case, the property being transported is momentum), as being conformed by a *local acceleration*, $\partial\boldsymbol{u}/\partial t$, and a *convective acceleration*, $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$. The former is only zero when considering steady flows. The latter has to do with the element of water moving to a different zone within the flow field where velocity may be different, and thus the convective term may be non-zero even in steady flows[4], but vanishes under uniform flow conditions[5]. Since eq. (2) follows from Newton's second law, the right-most side of the equation naturally represents forces (per unit mass of the element under consideration). These forces can in turn be divided into two types: *volume and surface forces*. Volume forces act on the element of water itself and are of external origin. For our purposes, we will solely deal with the volume force caused on the element by the gravitational field, so $\boldsymbol{g}$ is the gravitational acceleration of magnitude g acting downwards; i.e. $\boldsymbol{g} = (0, 0, -g)$. Surface forces, on the other hand, act (obviously) on the surface of the element, and are caused by the fluid itself. The first force of this type has to do with the gradient of the pressure $(-\nabla p/\rho)$, whereas the second one $(\nu\nabla^2\boldsymbol{u})$ relates to the fluid's dynamic viscosity, μ (an intrinsic property of the fluid), via the kinematic viscosity $\nu \equiv \mu/\rho$, and is often neglected in introductory hydraulics under the assumption of inviscid flow – however, this term becomes important in the current context and we will be studying it further.

Complementary material: Many concepts have been introduced here, so complementary videos may be helpful. To this end, I invite you to stop here and watch the following 2-parts tutorial produced by a past UoS MEng student: https://youtu.be/ORhtfTH2E3Y and https://youtu.be/vszxLH-fUuI. These videos explain the Navier-Stokes equations both from an intuitive (Part I) and mathematical (Part II) perspectives.

[3] Transport phenomena will be covered in more detail later in the module.

[4] Consider steady flow through a nozzle. As an element of fluid passes through the nozzle, it will accelerate to conserve mass. In this case, local acceleration is zero (steady flow), so all acceleration must come from the convective term.

[5] **Q:** Why?

Together, eqs. (1) and (2) represent a set of four differential equations for the four unknowns: u, v, w and p. However, even though current computational resources do allow us to solve directly these equations for some cases[6], engineering problems are typically very complex due to large spatial and temporal scales (e.g. circulation in an estuary), which demands simplification of the NS equations in order to render them practical[7]. In the next sections, we explore certain simplifications of the NS equations for the particular problems that the hydraulic engineer is expected to face. But before that, we will talk about a very relevant topic commonly (almost always) encountered in environmental water flows: turbulence.

Problem: As you know well, in *fluids at rest* (i.e. hydrostatics): (i) pressure varies linearly with the vertical distance from the surface; and (ii) for a given vertical position, pressure does not vary in the horizontal plane. Show how these two facts can be recovered from eq. (2).

3 Introduction to turbulence

Complementary material: I encourage you to watch the following video on turbulence before and after you read this section: https://youtu.be/AeBsiEYWZUY

3.1 Context

Turbulence is one of those concepts that are perhaps better grasped by the beginner using intuition rather than rigid definition. If asked *what is turbulence?* your response may at some point include the word *chaotic*, or a synonym (erratic, unpredictable, disorderly, etc.), in which case, you would be at least partially correct. Turbulent flows are characterised by significant and irregular spatial and temporal variations of the velocity and pressure fields (see black, continuous line in fig. 2). Terms 'chaotic' or 'random', however, do not do full justice to turbulent flows, which typically present an easy-to-identify pattern[8]: vortices (or **eddies**), which are responsible for mixing and transporting properties within the fluid. The latter is another important signature of turbulent flows: an enhanced ability to mix and transport. You are well aware of this when pouring milk into your tea - it will mix better and faster if you stir (induce turbulence). Note, however, that properties being mixed and transported are not always visible (like milk in your tea). Mixing of momentum and heat (both invisible to the eye), for example, are of tremendous importance in many fields of engineering and science. Another feature of turbulent flows, which is of particular significance to the hydraulic engineer, is that they are very effective at dissipating energy[9]. The mechanism of energy dissipation in turbulent flows involves the breaking of intrinsically unstable large eddies into

[6]This is called Direct Numerical Simulations, which are highly demanding from a computational perspective.

[7]However, note that computational power is not the only barrier to achieve Direct Numerical Simulations, but also e.g. accurate estimation of boundary conditions, numerical challenges and complexity associated with processing large volumes of data.

[8]This is a key complicating feature of turbulent flows: the coexistence of randomness and patterns, which usually do not go hand in hand. That being said, it is important to emphasise that from a strict statistical perspective, turbulent flows really are a random process.

[9]Consider, for instance, the Moody chart at the transition region. For a similar Reynolds number, a turbulent flow will dissipate more energy (higher friction factor) than a laminar one. Sometimes, the hydraulic engineer may wish to minimise this energy dissipation (e.g. in pipes), but some other times this is a welcome property of turbulent flows (e.g. in stilling basins).

progressively smaller eddies, which are eventually dissipated (converted into heat) by molecular viscosity. This process of energy dissipation is called **energy cascade**, and is well captured by undoubtedly the most cited verses in turbulence literature; namely (by L. F. Richardson):

Big whirls have little whirls,
which feed on their velocity;
and little whirls have lesser whirls,
and so on to viscosity.

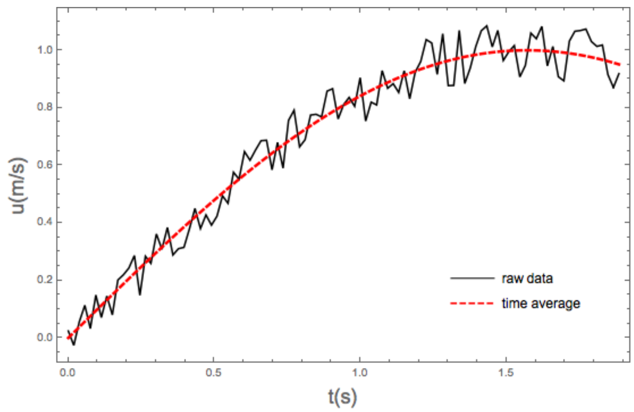

Figure 2: Plot showing the time series of the x-component of the flow velocity, u. Raw measurement in black, and time averaged velocity $\langle u \rangle$ shown with red, dashed line. At any point in time, the vertical distance between the two curves is the instantaneous fluctuation u'.

3.2 The Reynolds equations

With this context (and poem) in mind, let us proceed to a more mathematical treatment of turbulence. To this end, we begin with the methodology proposed by the prominent British fluid dynamicist Osborne Reynolds in the late 1800's. After Reynolds, we can decompose the instantaneous velocity and pressure into their mean (denoted by the angle brackets, $\langle \cdot \rangle$) and a fluctuation[10] (denoted by the prime symbol, $'$); namely,

$$\boldsymbol{u} = \langle \boldsymbol{u} \rangle + \boldsymbol{u'} \quad \text{and} \quad p = \langle p \rangle + p', \tag{3}$$

where the time-average operator $\langle \ \rangle$, applied to any component of $\boldsymbol{u}$ or p, is defined as:

$$\langle \ \rangle \equiv \frac{1}{\Delta t} \int_t^{t+\Delta t} (\ ) dt, \tag{4}$$

and **fluctuations satisfy the condition** $\langle \boldsymbol{u'} \rangle = 0$ and $\langle p' \rangle = 0$. See fig. 2.

Note: we carry out this decomposition in order to find the equations governing the evolution of the *mean variables*, which are typically of more practical importance than *instantaneous* values.

[10] In other words, we are simply saying that $\boldsymbol{u}$ and p fluctuate instantaneously about their mean value. Hence, you can also think of $\boldsymbol{u'}$ as the instantaneous deviation from the mean: $\boldsymbol{u'}(t) = \boldsymbol{u}(t) - \langle \boldsymbol{u} \rangle$ (the same goes for p).

If you are an observant student, you may argue that the value of, say, $\langle u_1 \rangle$, will depend on the selection of the averaging interval Δt. This would be a correct argument, which, moreover, automatically begs the question: what value of Δt should be chosen? There is no absolute answer to this, but in general, we want to set a Δt that is large enough to eliminate (or filter out) short-lived turbulent fluctuations, but small enough to retain important information such as variations on the mean flow due to e.g. a flood wave[11]. A good sense for what the value of Δt should be will naturally depend on the particular problem. In typical river flows, for instance, we can anticipate that Δt will usually be of the order of minutes. The time-average operator $\langle \cdot \rangle$ satisfies several rules; namely:[12]

- It is a linear operator, such that for any two variables a and b (could be e.g. u_1 and u_3), and a real constant α, we have $\langle a + b \rangle = \langle a \rangle + \langle b \rangle$, and $\langle \alpha b \rangle = \alpha \langle b \rangle$.
- Time and spatial derivatives commute with this operator; in other words, $\langle \partial b/\partial t \rangle = \partial \langle b \rangle / \partial t$, and $\langle \partial b/\partial x_i \rangle = \partial \langle b \rangle / \partial x_i$.
- Filtered quantities do not change with further filtering; i.e. $\langle \langle b \rangle \rangle = \langle b \rangle$.
- Also, $\langle a \langle b \rangle \rangle = \langle a \rangle \langle b \rangle$.

Armed with $\langle \; \rangle$ and the rules set above, as well as Reynolds decomposition (3), we can modify our original governing equations (1) and (2) to give us the evolution of the mean (and not instantaneous) quantities of interest (velocity components and pressure). For ease, let us start with the continuity equation (1). Use of Reynolds decomposition yields

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot (\langle \boldsymbol{u} \rangle + \boldsymbol{u'}) = \nabla \cdot \langle \boldsymbol{u} \rangle + \nabla \cdot \boldsymbol{u'} = 0.$$

Now we apply $\langle \; \rangle$ to the whole equation, obtaining

$$\langle \nabla \cdot \boldsymbol{u} \rangle = \nabla \cdot \langle \boldsymbol{u} \rangle = \nabla \cdot \langle \langle \boldsymbol{u} \rangle \rangle + \nabla \cdot \langle \boldsymbol{u'} \rangle = 0.$$

But we said before that $\langle \boldsymbol{u'} \rangle = 0$ and $\langle \langle \boldsymbol{u} \rangle \rangle = \langle \boldsymbol{u} \rangle$, so our continuity equation for the mean velocity field is simply $\nabla \cdot \langle \boldsymbol{u} \rangle = 0$, or using alternative notations:

$$\nabla \cdot \langle \boldsymbol{u} \rangle = \frac{\partial \langle u_i \rangle}{\partial x_i} = \frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y} + \frac{\partial \langle w \rangle}{\partial z} = 0, \tag{5}$$

which is analogous to (1)[13].

The momentum equations are, however, a more interesting matter. For convenience, let us rewrite (2) using Einstein notation for the momentum in the j direction (j could be x, y or z):

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = g_j - \frac{1}{\rho} \frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_i \partial x_i}, \tag{6}$$

[11]For example, go to fig. 2 and sketch how the time-averaged signal (red line) would look like if we chose $\Delta t = 2.0$ s. Also, what would you say is the approximate value of Δt in the red signal shown?

[12]Verify all of these.

[13]**Q:** This means that both the *instantaneous* velocity field and the *mean* velocity field are divergence free. Can you see why this should be so?

which would be linear in u_j and p if it were not for the convective acceleration term $u_i \partial u_j/\partial x_i$. That all other terms are linear means that they are easily transformed upon application of the time-average operator; namely[14]:

$$\left\langle \frac{\partial u_j}{\partial t} \right\rangle = \frac{\partial \langle u_j \rangle}{\partial t} \quad ; \quad \left\langle \frac{\partial p}{\partial x_j} \right\rangle = \frac{\partial \langle p \rangle}{\partial x_j} \quad \text{and} \quad \left\langle \frac{\partial^2 u_j}{\partial x_i \partial x_i} \right\rangle = \frac{\partial^2 \langle u_j \rangle}{\partial x_i \partial x_i}. \tag{7}$$

But we will show that this is not the case for the convective acceleration $u_i \partial u_j/\partial x_i$. We start by writing $u_i \partial u_j/\partial x_i$ as $\partial(u_i u_j)/\partial x_i$ (called the conservative form), which we can do because

$$\partial(u_i u_j)/\partial x_i = u_i \partial u_j/\partial x_i + u_j \partial u_i/\partial x_i \quad \text{(chain rule)},$$

and we know form (1) that $\partial u_i/\partial x_i = 0$; hence:

$$u_i \partial u_j/\partial x_i = \partial(u_i u_j)/\partial x_i.$$

Making use of Reynolds decomposition for u_i and u_j, and applying the operator $\langle\ \rangle$ (recalling that $\langle u_i' \rangle = \langle u_j' \rangle = 0$), we have

$$\begin{aligned} \langle u_i u_j \rangle &= \big\langle (\langle u_i \rangle + u_i')(\langle u_j \rangle + u_j') \big\rangle \\ &= \big\langle \langle u_i \rangle \langle u_j \rangle + u_i' \langle u_j \rangle + u_j' \langle u_i \rangle + u_i' u_j' \big\rangle \\ &= \langle u_i \rangle \langle u_j \rangle + \langle u_i' u_j' \rangle, \end{aligned} \tag{8}$$

which allows us to write the convective acceleration as

$$\begin{aligned} \frac{\partial \langle u_i u_j \rangle}{\partial x_i} &= \frac{\partial \langle u_i \rangle \langle u_j \rangle}{\partial x_i} + \frac{\partial \langle u_i' u_j' \rangle}{\partial x_i} \\ &= \langle u_i \rangle \frac{\partial \langle u_j \rangle}{\partial x_i} + \frac{\partial \langle u_i' u_j' \rangle}{\partial x_i}, \end{aligned} \tag{9}$$

where we have used the fact that $\partial \langle u_i \rangle/\partial x_i = 0$ (eq. 5). Note that a new term has arisen from time-averaging of the convective acceleration (which is not true for the linear terms previously discussed). To see this more clearly, let us compile what we have done so far to write an equation for the mean momentum conservation in the j direction; namely:

$$\frac{\partial \langle u_j \rangle}{\partial t} + \langle u_i \rangle \frac{\partial \langle u_j \rangle}{\partial x_i} = g_j - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} + \nu \frac{\partial^2 \langle u_j \rangle}{\partial x_i \partial x_i} - \underbrace{\frac{\partial \langle u_i' u_j' \rangle}{\partial x_i}}_{\text{extra term}}. \tag{10}$$

This equation would be 'identical' to (6) if it were not for the last term in the right-hand side (r.h.s.). In fact, this is a very important term that intimately relates to turbulence. The averaged product $\langle u_i' u_j' \rangle$ (statistically speaking, a covariance) is called the **Reynolds stress**, and is central to the study of turbulence. Here, we will only address Reynolds stresses and their implications very briefly, but to the sufficient extent as to gain insight into important aspects of open channel hydrodynamics. The equations derived here (eqs. 5 and 10) are commonly called the Reynolds-Averaged Navier-Stokes (RANS) equations, or simply the Reynolds equations.

[14]Prove these equalities.

Practice: Taking the j direction as z (such that $u_j = w$), eq. (6) can be written as:

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).$$

Write in the same notation the x and y analogues of this equation. Then do the same for the three components of eq. (10).

3.3 Reynolds stresses

As pointed out above, if it were not for the Reynolds stresses, the instantaneous and mean momentum equations (eqs. 6 and 10, respectively) would be the same. So, what is the effect of Reynolds stresses on the mean flow? To answer this, it is illuminating to rewrite (10) in a different form; namely[15]:

$$\rho\frac{\mathrm{D}\langle u_j\rangle}{\mathrm{D}t} = \rho g_j + \frac{\partial}{\partial x_i}\left[2\mu\bar{S}_{ij} - \langle p\rangle\delta_{ij} - \rho\langle u_i' u_j'\rangle\right], \tag{11}$$

where δ_{ij} is the Kronecker delta, defined as $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ (i.e. the identity matrix[16]); and $\bar{S}_{ij}$ is the mean rate of strain, defined as

$$\bar{S}_{ij} \equiv \frac{1}{2}\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right). \tag{12}$$

Eq. (11) shows that the acceleration of a fluid element (left-hand side, or l.h.s. for short) is caused by volume forces (in our case, gravity) and forces due to the *gradient* of different types of stresses (i.e. $\partial(...)/\partial x_i$). The latter point is important. Let us focus on the term $-\partial\langle p\rangle/\partial x_j$, and consider as illustration $x_j = y$. If the mean pressure were uniform along y, it would not cause a net force (it would cancel itself out) – only a *change* of the mean pressure along y would cause a net force along this direction. Similarly, it is the gradient of viscous stresses ($2\mu\bar{S}_{ij}$) which causes net forces on the surface of the fluid element. From a physical perspective, viscous stresses have their origin in molecular dynamics. Without entering in detail, at the microscopic level molecules transport momentum to surrounding parts of the fluid due to their random motion, thus serving as a source of momentum *diffusion*. It is then easy to see that the *apparent* stress $-\rho\langle u_i' u_j'\rangle$ will have a similar effect, only that in this case it is the **velocity field random fluctuations that are responsible for the transport of momentum**.

Now that we know that fluctuations in the mean velocity field will affect the mean momentum by acting as an apparent additional stress, how do we quantify this stress?

3.4 The closure problem

Inspection of the equations we have derived for the mean flow, eqs. (5) and (10), should trigger an alarm: we have more unknowns than equations. We have four equations in total (the continuity

[15] This equivalent form of (10) is not immediately obvious. However, you can show it is true by considering: the continuity equation, the fact that ρ and μ are constants, and the symmetry of second derivatives.

[16] A diagonal matrix with non-zero elements equal to unity.

equation and three momentum equations associated with the Cartesian coordinates), corresponding to our four main variables $\langle u\rangle$, $\langle v\rangle$, $\langle w\rangle$ and $\langle p\rangle$... *plus* all the Reynolds stresses[17] $\langle u_i'u_j'\rangle$! This is called the **closure problem**, because our model is not complete, or closed, until we have as many equations as unknowns. This problem gives birth to a vast field of research dedicated exclusively to **turbulence closure models**. Given the scope of these lecture notes, we will only provide an overview of one of the simplest, and widely used, ways to solve the closure problem; namely: the concept of **eddy viscosity**.

As mentioned above, Reynolds stresses transfer momentum via random fluctuations of the mean velocity field, which is not too different from the way molecules transport momentum via their random motion leading to viscous stresses. It is then tempting to think that an analogy can be drawn between these two mechanisms. And this is precisely what the French mathematician and physicist, J. V. Boussinesq, did in 1877. Boussinesq *hypothesised* that, analogous to viscous stresses, the *deviatoric* Reynolds stresses could be mathematically *modelled* as being proportional to the mean rate of strain. First of all, let us define the deviatoric Reynolds stress[18]; namely: $(-\rho\langle u_i'u_j'\rangle + \frac{2}{3}\rho k\delta_{ij})$, where k is the *turbulent kinetic energy*[19], in turn defined as $k \equiv \frac{1}{2}\langle u_i'u_i'\rangle$. The Boussinesq hypothesis then tells us that

$$-\rho\langle u_i'u_j'\rangle + \frac{2}{3}\rho k\delta_{ij} = \rho\nu_T\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right), \tag{13}$$

where $\nu_T = \nu_T(\boldsymbol{x}, t)$ is a positive scalar coefficient called – in keeping with the analogy – the **turbulent or eddy viscosity**. As you would expect, the more turbulent the flow, the larger this coefficient should be. Eq. (11) may be rewritten as

$$\frac{\mathrm{D}\langle u_j\rangle}{\mathrm{D}t} = g_j + \frac{\partial}{\partial x_i}\left[\nu\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right) - \frac{\langle p\rangle\delta_{ij}}{\rho} - \langle u_i'u_j'\rangle\right], \tag{14}$$

which, after use of (13), becomes[20]

$$\frac{\mathrm{D}\langle u_j\rangle}{\mathrm{D}t} = g_j + \frac{\partial}{\partial x_i}\left[(\nu+\nu_T)\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right)\right] - \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\langle p\rangle + \frac{2}{3}\rho k\right). \tag{15}$$

The sum of ν and ν_T is sometimes called the *effective viscosity*, $\nu_{\text{eff}} \equiv \nu + \nu_T$. In practice, ν is usually neglected. This is because water has a small kinematic viscosity, ν ($\sim 10^{-6}$ m^{2}s^{-1}), which in turn means that it will almost always flow under turbulent regime (in natural open channels at least), thus leading to $\nu_T \gg \nu$ and hence $\nu_{\text{eff}} \approx \nu_T$. The above equation also shows that the effect of the turbulent kinetic energy k is to modify the mean pressure field, and even though it represents yet another unknown, it is often absorbed within a modified pressure[21], $\tilde{p} = \langle p\rangle + \frac{2}{3}\rho k$. The turbulent viscosity, ν_T, is often taken as a calibration parameter within numerical models. In the simplest case, a positive constant value of ν_T is employed. In any case, once ν_T is determined (and k dealt with, as discussed above), our model is finally closed since, by writing the Reynolds

[17]Considering that Reynolds stresses are the components of a symmetric tensor (matrix), such that $\langle u_i'u_j'\rangle = \langle u_j'u_i'\rangle$, how many (potentially) different components of this tensor do we have?

[18]The *why* and *how to* derive this is not of our immediate concern.

[19]i.e. the mean kinetic energy (per unit mass) contained in the fluctuations of the velocity field.

[20]Show this.

[21]In practice, k is often silently omitted, but the resulting pressure will be a pressure that accounts for its effect nonetheless.

stresses as functions of other existing variables – i.e. $\langle u_i' u_j' \rangle = f(\langle u_i \rangle, \langle u_j \rangle)$ – we achieve as many unknowns as equations.

The concept of turbulent viscosity is a simple and convenient way of solving the closure problem, but it is by no means the only nor the best one. It is not our objective to discuss turbulence closure models in great detail, but the curious student may wish to do some further reading on other methods, such as the mixing-length model or the $k - \epsilon$ model. Next, we discuss briefly the former in order to understand its importance in the development of a velocity profile.

3.5 The velocity profile

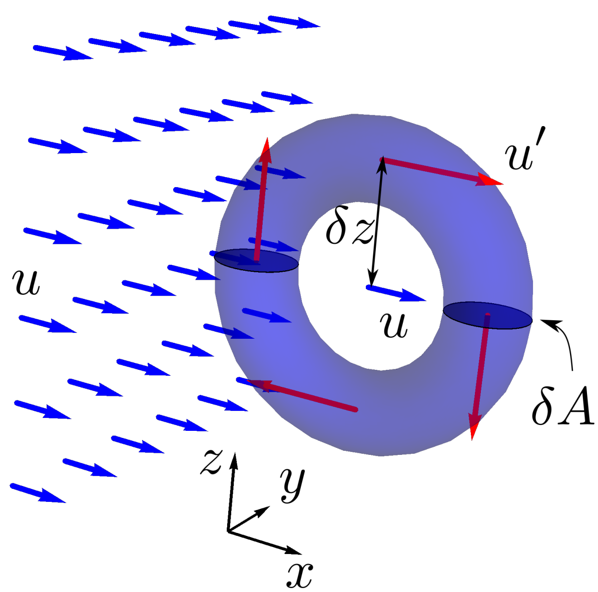

Figure 3: Sketch of Prandtl's eddy model. Eddy of radius δz, which rotates with tangential velocity u' (red arrows), is superimposed to a flow with steady, unidirectional velocity u (blue arrows).

We have said that the effect of turbulence is similar to that of viscosity, in that it transfers momentum between different parts of the fluid, thus giving raise to (apparent) shear stresses. We will now elaborate on this to understand how and why a *velocity profile* is developed. To exemplify this let us focus on the 2D problem involving only the $x - z$ plane and a perfectly *steady* flow with unidirectional velocity in the x-direction, which can nonetheless vary vertically (i.e. $\boldsymbol{u} = u(z)$)[22]. The German engineer and one of the most prominent fluid dynamicists of the past century, Ludwig Prandtl, developed a simple, yet insightful model to understand how turbulent eddies transport momentum. According to this model, the one-directional flow has a velocity u at a point (x_0, z_0), to which an eddy of radius δz is to be superimposed, as shown in fig. 3. This eddy has a toroid-shape (i.e. a doughnut) of cross-sectional area δA, which rotates at constant angular velocity, yielding a constant tangential velocity u'. Therefore, the two cross sections of the eddy highlighted in fig. 3 will be crossed by $\rho \delta A u'$ kilograms of water per second each. Thus, we can say that the eddy and the mean flow exchange mass at a rate of $2\rho\delta A u'$. But an element of mass which enters the eddy and flows upwards (within the eddy) will also experience a change in its velocity. Originally, at

[22]Since the flow is perfectly steady, we ignore time-average operators; or alternatively, we say that $\langle u \rangle = u$

(x_0, z_0) this element of mass had a velocity u, but at the top of the eddy $(x_0, z_0 + \delta z)$ it will have a different (higher) velocity, $u + \delta u$, where it is reasonable to assume that

$$\delta u = u' = \delta z \frac{du}{dz}. \tag{16}$$

Hence, the mean flow and the eddy are not only exchanging mass but, with it, also momentum, M, and they do so at a rate equal to $2\rho\delta A\delta u \times \delta u$ (i.e. [rate of mass exchange] $\times$ [change in velocity of this mass]), which can be expressed as

$$\frac{dM}{dt} = 2\rho\delta A(\delta u)^2 = 2\rho\delta A \left(\delta z \frac{du}{dz}\right)\left(\delta z \frac{du}{dz}\right). \tag{17}$$

But according to Newton's second law, any change of momentum must be due to a force (precisely equal to dM/dt). In this case, this force is due to a(n apparent) shear stress[23], τ, which, as any stress, must be equal to [some force]/[the area over which it acts]. In other words, if we divide the above equation, which is essentially a force ($= dM/dt$), by $2\delta A$ (the area over which our apparent shear stress acts), we will have an expression for τ; namely[24]

$$\tau = \rho\left(\delta z \frac{du}{dz}\right)\left(\delta z \frac{du}{dz}\right) = \rho(\delta z)^2 \left(\frac{du}{dz}\right)^2. \tag{18}$$

We see then that τ not only depends on the rate of strain du/dz, but also on δz. The latter variable is so central to this model that it receives its own name: the **mixing length** (typically denoted by l; i.e. $\delta z = l$). The mixing length, or eddy's size, is a measure of how far a particle deviates from its mean trajectory, thus contributing to mixing (in this case, of momentum) throughout the fluid. There is of course no easy way of measuring δz in practice, but a reasonable assumption is that it is proportional to the distance from the bed[25], z. In other words, $\delta z = \kappa z$ may be assumed, where κ is the coefficient of proportionality. Experiment shows that, interestingly, κ is very close to a 'universal' constant, namely $\kappa \approx 0.4$. For this important reason, κ receives its own name: Kármán, or **von Kármán constant**, after another great fluid dynamicist, Theodore von Kármán. By using $\delta z = \kappa z$, we can rewrite the above equation as

$$\frac{\tau}{\rho} = (\kappa z)^2 \left(\frac{du}{dz}\right)^2, \tag{19}$$

or, by defining a new variable $u_* \equiv \sqrt{\tau/\rho}$,

$$u_* = (\kappa z)\left(\frac{du}{dz}\right). \tag{20}$$

[23]Remember, we are talking here about the Reynolds *apparent* stress discussed previously, not an *actual* shear force acting on the fluid. This model (i.e. conceptualisation or idealisation) allows us to imagine, as a first approximation, how momentum is transported due to turbulent eddies, but reality is of course more complex than this.

[24]If this model for the turbulent stress, τ, is to agree with Boussinesq hypothesis, where τ depends on ν_T, what value should ν_T have?

[25]This is because in principle a particle far away from the bed is 'freer' to move vertically, whereas one near the bed is restricted by the latter.

Rewrite the above equation as

$$du = \frac{u_*}{\kappa}\frac{dz}{z}, \tag{21}$$

and integrate both sides to obtain

$$\frac{u}{u_*} = \frac{1}{\kappa}\ln z + C, \tag{22}$$

where C is some constant of integration. A clever selection of C (namely, $C = -(1/\kappa)\ln z_0$, where z_0 is the hypothetical distance from the bed where the velocity vanishes[26]) then enables us to rewrite the above equation as:

$$\frac{u}{u_*} = \frac{1}{\kappa}\ln\frac{z}{z_0}. \tag{23}$$

We see then that the consequence of Prandtl's eddy model, in combination with the (experimentally backed) assumption of $l = \delta z = \kappa z$, is that a logarithmic profile of u is developed in the vertical; i.e. u varies with the logarithm of the distance from the bed. This is called **the law of the wall**[27]. Some caveats aside, this approximation is actually quite good for the flow we have assumed; i.e. steady and unidirectional, which in turn is a good approximation to many open channel flows such as rivers (see fig. 4). The variable u_*, which we have arbitrarily defined is called, confusingly, the **shear velocity**. I say 'confusingly' because although it relates to the shear stress, τ, and it has units of velocity, it is not an actual, real velocity to be measured in the flow, nor it has a particularly useful physical interpretation[28].

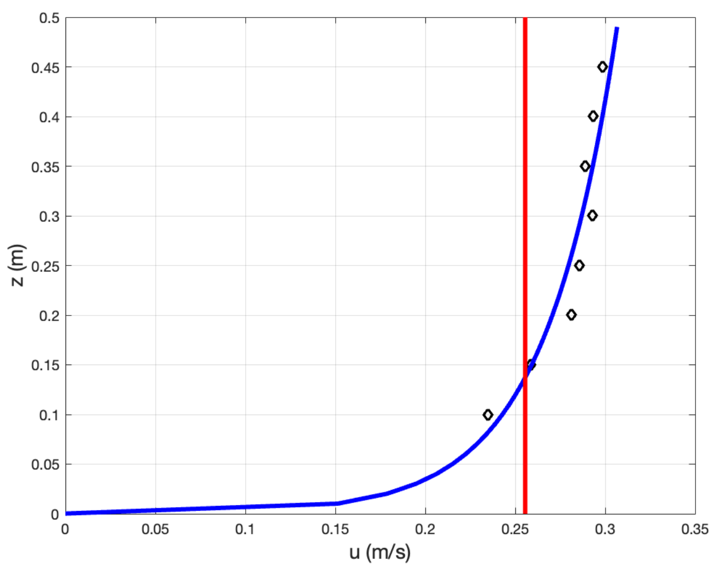

Figure 4: Log profile (eq. 23, blue line) superimposed to laboratory measurements (black symbols) in a 0.5 m deep flume (Boldrewood Campus, UoS) with a depth-averaged flow velocity of roughly 0.25 m/s (red line).

[26] At the bed surface the no-slip condition must hold true. The variable z_0 relates to the physical roughness of the bed. More on z_0 in Section 4.6.2.

[27] First published by von Kármán in 1930.

[28] A common misconception, not only held by students but even by more experienced engineers, is that u_* represents the flow velocity at or near the bed. Do not be persuaded by this tempting but erroneous thought.

4 Open channel flows (in particular)

4.1 Context

Everything we have discussed so far can be applied to the study of flow in open channels, but also to many other types of flow, such as breaking waves, pipe flows, air flow[29], etc. However, flows in natural open channels have certain features that separate them from other examples of fluid flows. For instance, **river and estuary flows occur predominantly in the horizontal plane**[30], such that vertical acceleration of the flow can be neglected. As will be shown, this simplifies significantly the mathematical treatment of open channels. Moreover, the river engineer is seldom interested in detailed 3-dimensional information of the velocity field – it frequently suffices to know what the *average* flow velocity or discharge is. But, *averaged* over what? We have talked before about averaging our equations with respect to time (the Reynolds equations), now we will see how something similar can be done with respect to different spatial coordinates (depth, width or both). Finally, a crucial aspect of open channel flows – especially in connection with sediment transport – is the concept of bed shear stress, with which we will finish these notes.

In the preceding sections, we have used different notations to: i) practice; and ii) safeguard scientific rigour. However, in what follows, rigorous notation may distract from the main point of these notes, which is to understand what the equations actually mean and what their limitations are. Therefore, with the reminder that **we are studying time-averaged quantities** (velocities and pressure), **let us simplify our notation by dropping the angle brackets**, such that **hereinafter** $\langle u \rangle$, $\langle v \rangle$, $\langle w \rangle$ and $\langle p \rangle$ will simply be written as u, v, w and p, respectively. We continue to work with our main governing equations for the evolution of time-averaged variables; namely: continuity and RANS equations. We start by writing said equations in the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{24}$$

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(\nu \nabla^2 u - \frac{\partial \langle u_i' u_1' \rangle}{\partial x_i}\right) \tag{25}$$

$$\frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (v^2)}{\partial y} + \frac{\partial (vw)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \left(\nu \nabla^2 v - \frac{\partial \langle u_i' u_2' \rangle}{\partial x_i}\right) \tag{26}$$

$$\frac{Dw}{Dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \left(\nu \nabla^2 w - \frac{\partial \langle u_i' u_3' \rangle}{\partial x_i}\right) \tag{27}$$

Our first main simplification for open channel flow concerns the terms in brackets in the r.h.s of the momentum equations, which relate to viscous and Reynolds stresses. In general, horizontal gradients (variations in x and y) of these stresses are small when compared to the vertical ones (in fact, viscous stresses are often neglected altogether). In the vertical-momentum equation, we

[29] In fact, many of the concepts put forward by Prandtl, von Kármán and others, for example, were developed in the context of aeronautics.

[30] By *horizontal*, we mean parallel to the bed, which is in turn assumed to have a small slope (for instance, natural rivers typically have slopes of about 1:1000).

neglect both viscous and turbulent stresses. Thus, we have:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{28}$$

$$\frac{\partial u}{\partial t}+\frac{\partial (u^2)}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z} \tag{29}$$

$$\frac{\partial v}{\partial t}+\frac{\partial (uv)}{\partial x}+\frac{\partial (v^2)}{\partial y}+\frac{\partial (vw)}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{1}{\rho}\frac{\partial \tau_{yz}}{\partial z} \tag{30}$$

$$\frac{Dw}{Dt}=-g-\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{31}$$

where we define the (sum of viscous and turbulent) stresses τ_{xz} and τ_{yz} as

$$\tau_{xz} \equiv \rho\nu\frac{\partial u}{\partial z}-\rho\langle u'w'\rangle \tag{32}$$

$$\tau_{yz} \equiv \rho\nu\frac{\partial v}{\partial z}-\rho\langle v'w'\rangle \tag{33}$$

The above set of equations serves as our starting point for the study of flow in open channels.

4.2 Hydrostatic pressure

As we have said, open channel flows occur mainly in the bed-parallel plane, such that $Dw/Dt = 0$. This assumption in turn simplifies (31) to $\partial p/\partial z = -\rho g$. Integration of this expression tells us that the vertical distribution of the pressure is hydrostatic; namely:

$$p = -\rho g z + \text{c.i.}, \tag{34}$$

where c.i. is some constant of integration, which is typically determined from the condition $p =$ [atmospheric pressure] at $z =$ [the free surface]. This condition is one of several *boundary conditions* that our problem needs to satisfy if it aspires to be physically realistic. Hence, before proceeding any further, let us explore all the relevant boundary conditions.

4.3 Boundary conditions

At the free surface, not only must pressure be atmospheric as we saw above, but as the water surface evolves in time, a particle of fluid located on said surface must remain on it. This may sound rather redundant or obvious, but it must be explicitly stated if our mathematical model is going to yield any sensible results. This condition is called a *kinematic boundary condition*, as it relates to motion[31] of the particles and not the forces causing said motion (which would instead be called a *dynamic* condition). In mathematical terms, we say that the free surface is described by a function z_s, which depends on x, y and t; i.e. $z_s = z_s(x, y, t)$. And to say that a particle with velocity $\boldsymbol{u} = (u, v, w)$ located at the surface (i.e. at $z = z_s$) must remain on it as the latter (the surface) evolves in time, is nothing but to state that:

$$w = \frac{Dz_s(x,y,t)}{Dt} \quad \text{at} \quad z = z_s; \tag{35}$$

[31] Kinematics comes from the Greek *kinema*, which means motion (thus *cinema* refers to motion of pictures).

in other words, the vertical component of the flow velocity *at the free surface* must be equal to the temporal variation of the surface. Applying the chain rule to the r.h.s., $Dz_s/Dt = \partial z_s/\partial t + (\partial z_s/\partial x)(dx/dt) + (\partial z_s/\partial y)(dy/dt) = \partial z_s/\partial t + u\partial z_s/\partial x + v\partial z_s/\partial y$, eventually leads to

$$\frac{\partial z_s}{\partial t} + u\frac{\partial z_s}{\partial x} + v\frac{\partial z_s}{\partial y} - w = 0 \quad \text{at} \quad z = z_s. \tag{36}$$

This will become useful soon. A similar condition can be applied to the bottom surface (or bed), described by a function z_b (i.e. the bathymetry). At any wall, real viscous fluids exhibit an important feature: their velocity vanishes[32], i.e. $\boldsymbol{u} = 0$ at $z = z_b$. However, when ignoring viscous effects (which is commonly done), some tangential velocity may exist at the bottom as a 'compensation', and so our condition for the bed is similar to the equation above, but replacing z_s by z_b. The difference arises from the fact that typically $\partial z_b/\partial t$ is either strictly zero (i.e. a fixed bed) or very small in comparison with other terms (i.e. the bed evolves in time very slowly), and so $\partial z_b/\partial t = 0$ is usually assumed[33], thus leading to the kinematic condition at the bottom[34]:

$$u\frac{\partial z_b}{\partial x} + v\frac{\partial z_b}{\partial y} - w = 0 \quad \text{at} \quad z = z_b. \tag{37}$$

Question: We just said that in real flows the velocity (with all its components) vanishes at the bottom. But is 'bed shear stress' not responsible for sediment motion*? How can there be bed shear stress if there is no velocity at the bed? And if there is no bed shear stress, how could the bed be eroded by the flow?

*we have not studied mechanics of sediment transport yet, but you may have an intuition for why this is 'true': if you apply a sufficiently large shear stress to a bed composed of loose material, the latter will tend to erode.

4.4 Depth-averaged equations

By integrating our equations over the vertical, from the channel bed, $z = z_b$, to the free surface, $z = z_s$, and dividing by the water depth, h (obviously, $h = z_s - z_b$)[35], we end up with the depth-averaged hydrodynamic equations also known as the **Shallow Water Equations**. These equations tell us what the 'instantaneous'[36] **(depth-)average, horizontal velocity** is at a given (x, y) point in the channel, without giving us any information about the velocity's vertical variation. In many applications, this information is enough for practical purposes. See fig. 5 for reference.

Let us begin by integrating the continuity equation (28) over the water depth:

$$\int_{z_b}^{z_s} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = 0. \tag{38}$$

[32] This is the no-slip condition referred to in footnote 26.

[33] And since $z_s = z_b + h$, one can also replace $\partial z_s/\partial t$ in eq. 36 with $\partial h/\partial t$ (for a fixed bed).

[34] This equation can also be derived from the fact that the velocity normal to the bed surface (which we consider to be fixed) is zero. Show this by recalling that a normal vector to $z = z_b(x, y)$ is given by $(\partial z_b/\partial x, \partial z_b/\partial y, -1)$ (or ∇z_b).

[35] In fig. 1, identify z_b, z_s and h.

[36] Remember, our starting point here is the (time-averaged) RANS equations, so *instantaneous* must be interpreted with care.

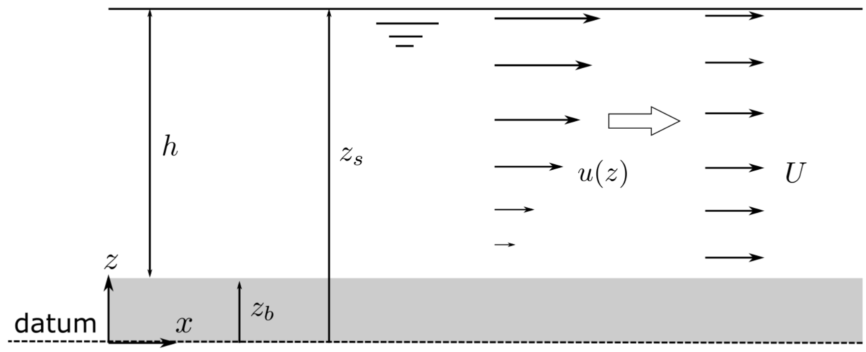

Figure 5: Sketch of an open channel flow showing relevant variables for the Shallow Water Equations (SWE). In the SWE, we work with the depth-average velocity U, rather than the actual $z-$dependent velocity $u(z)$ (the same is true for the $y-$component of the velocity). Only horizontal components of the flow velocity are considered.

Use of Leibnitz integral rule yields[37]

$$
\begin{aligned}
\frac{\partial}{\partial x}\int_{z_b}^{z_s} udz + \frac{\partial}{\partial y}\int_{z_b}^{z_s} vdz \quad &- \quad \left.\left(u\frac{\partial z_s}{\partial x} + v\frac{\partial z_s}{\partial y} - w\right)\right|_{z=z_s} \\
&+ \quad \left.\left(u\frac{\partial z_b}{\partial x} + v\frac{\partial z_b}{\partial y} - w\right)\right|_{z=z_b} = 0,
\end{aligned} \tag{39}
$$

which, after invoking the boundary conditions (36) and (37), simplifies to

$$
\frac{\partial z_s}{\partial t} + \frac{\partial (hU)}{\partial x} + \frac{\partial (hV)}{\partial y} = 0, \tag{40}
$$

or for a fixed bed ($\partial z_b/\partial t = 0$),

$$
\frac{\partial h}{\partial t} + \frac{\partial (hU)}{\partial x} + \frac{\partial (hV)}{\partial y} = 0, \tag{41}
$$

where we have introduced our depth-averaged velocities, U and V, defined as:

$$
U \equiv \frac{1}{h}\int_{z_b}^{z_s} udz \tag{42}
$$

$$
V \equiv \frac{1}{h}\int_{z_b}^{z_s} vdz. \tag{43}
$$

The momentum equations (29) and (30) are treated in the exact same way, and though their treatment is a bit more laborious, it can be shown that averaging of said equations over the water depth yields[38]

$$
\frac{\partial (hU)}{\partial t} + \frac{\partial (\alpha_1 hU^2)}{\partial x} + \frac{\partial (\alpha_2 hUV)}{\partial y} = -gh\frac{\partial z_s}{\partial x} - \frac{1}{\rho}\tau_{xb} \tag{44}
$$

$$
\frac{\partial (hV)}{\partial t} + \frac{\partial (\alpha_2 hUV)}{\partial x} + \frac{\partial (\alpha_3 hV^2)}{\partial y} = -gh\frac{\partial z_s}{\partial y} - \frac{1}{\rho}\tau_{yb}. \tag{45}
$$

[37] Check this.

[38] Check this.

The coefficients α_1, α_2 and α_3 are correction coefficients that appear because *the mean of the product of two variables is not (in general) equal to the product of two mean variables*[39]. So, for instance, α_2 is defined as

$$\alpha_2 \equiv \frac{1}{hUV} \int_{z_b}^{z_s} uv dz, \tag{46}$$

with similar definitions for α_1 and α_3. These coefficients depend on the velocity vertical profile (information that is 'lost' after averaging over the depth) and typically vary between 1 and 1.1. Thus, in practice they are often neglected; i.e. $\alpha_1 = \alpha_2 = \alpha_3 = 1$ is commonly assumed. The terms τ_{xb} and τ_{yb} are the bottom stresses (combining turbulent and viscous effects) acting in the x and y directions, respectively, which are particularly relevant for sediment transport calculations. Similar terms could be added if stresses were considered at the free surface due to the effect of wind blowing.

Equations (41), (44) and (45) are called the Shallow Water Equations (SWE), and form the basis of numerical models extensively used by engineers and scientist studying e.g. river flows, circulation in lakes and estuaries, and coastal flooding[40].

Question: Inspect the Shallow Water Equations (eqs. 41, 44 and 45) and answer: (i) what are the variables of interest (those whose evolution is sought)? and (ii) is this a *closed* system of equations and/or what conditions are necessary to ensure it is?

4.5 Cross-section-mean equations

Even though the SWE represent a considerable simplification of the original Navier-Stokes equations, their numerical solution is not trivial and the computational and data requirements (e.g. bathymetry) may be prohibitively large in many field applications. Sometimes, the river engineer may be satisfied by an average flow velocity at a given cross section. This is achieved by carrying out a second integration, this time in the y direction, with the banks of the river as the integration limits. Consider the bed to be fixed and set the datum at $z = z_b$, such that $z_s = h$. The banks are located at $y = y_1$ and $y = y_2$, where the water depth is of course null; i.e. $h = 0$ at $y = y_1$ and $y = y_2$. Integrating the depth-averaged continuity equation (41) gives

$$\frac{\partial}{\partial t} \int_{y_1}^{y_2} h dy + \frac{\partial}{\partial x} \int_{y_1}^{y_2} hU dy + \left(hV - h\frac{\partial y}{\partial t} - hU\frac{\partial y}{\partial x} \right)\Bigg|_{y_1}^{y_2} = 0. \tag{47}$$

Note that $\int_{y_1}^{y_2} hdy$ is the cross-sectional area, A, and $\int_{y_1}^{y_2} hUdy$ is the flow volumetric discharge through that area, Q. Moreover, the term in brackets vanishes because $h = 0$ at the boundaries (the banks), thus leading to[41]

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0. \tag{48}$$

[39] That is, if, for instance, U and V are two (depth-)mean variables, their product UV is not necessarily equal to the depth-average of the product uv. Show this and state one particular case for which this equality does hold.

[40] If flood is due to storm surge, inclusion of a wind shear stress at $z = z_s$ is absolutely necessary.

[41] Note that if you consider steady flow (i.e. $\partial A/\partial t = 0$), the familiar relation $Q =$ constant or $Q = A_1U_1 = A_2U_2$, widely employed in introductory hydraulics, is recovered.

The cross sectional area can also be described as $A = Bz_s$, where B is the distance between the banks at the maximum water level. Thus, it is called the *storage width* because it determines the storage capacity of the river section. This means that the above equation can also be expressed as

$$B\frac{\partial z_s}{\partial t} + \frac{\partial Q}{\partial x} = 0. \tag{49}$$

As before, treatment of the (x component of the) momentum equation is more complicated, more so in this case because the velocity profile may significantly deviate from a uniform distribution for complex cross sections. Lateral integration of (44) yields

$$\frac{\partial}{\partial t}\int_{y_1}^{y_2} hU dy + \frac{\partial}{\partial x}\int_{y_1}^{y_2} \alpha_1 hU^2 dy = -g\int_{y_1}^{y_2} h\frac{\partial z_s}{\partial x}dy - \frac{1}{\rho}\int_{y_1}^{y_2} \tau_{xb} dy. \tag{50}$$

The first integral is equal to Q by definition. For the second integral, we must introduce another coefficient, α' defined as

$$\alpha' \equiv \frac{A}{Q^2}\int_{y_1}^{y_2} \alpha_1 hU^2 dy. \tag{51}$$

Finally, the third integral is evaluated by assuming that the free surface streamwise slope is constant over the width of the river (i.e. $\partial z_s/\partial x \neq f(y)$), yielding

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\alpha'\frac{Q^2}{A}\right) = -gA\frac{\partial z_s}{\partial x} - \frac{1}{\rho}P\tau_b, \tag{52}$$

where τ_b is the mean bed or bottom shear stress and P is the width over which it acts (i.e. the wetted perimeter)[42].

4.6 The bed shear stress

The bed shear stress or bed friction is a key aspect of open channel flows. As you may recall from introductory theory on open channel flow, to derive Chézy equation for the velocity of a steady, unidirectional and uniform flow, a balance between two forces is considered: pressure gradient due to the inclination of the channel and bed friction or resistance. Essentially, this means[43] $\tau_{xb} = -\rho g(A/P)\partial z_s/\partial x$. These two terms – bed resistance and channel inclination – are of paramount importance in the study of open channel flows[44]. A thorough study of the bed shear stress would have to take us to the theory of boundary layer and beyond. However, for our purposes, it is sufficient to understand it as the shear stress at the bottom boundary and accept, at least for the time being, that experiment shows that the bed shear stress scales well with U^2; i.e. $\tau_b \sim U^2$, to within a multiplicative factor, which is in turn related to Chézy or Manning's coefficients.

[42] Identify in fig. 1 the variables y_1, y_2 and P. From Part 2 Hydraulics, how do you call the ratio A/P?

[43] Derive this from eq. (52), assuming steady, uniform flow.

[44] Pretty much open channel flow is defined as a flow driven by gravity against bed resistance.

4.6.1 Empirical determination of τ_b

There is no single way of determining the bed shear stress empirically. Several methods have been proposed, all with their corresponding advantages and disadvantages. However, it must be remarked that, with the exception of some nascent research, most methods estimate the bed shear stress *indirectly*, usually via the shear velocity (which we defined ourselves back in §3.5, remember?). For instance, a common method involves the log-law discussed in Section 3.5. Inspection of eq. (23) shows that a single measurement of $u(z)$, or data point (z, u), should be sufficient to obtain u_* via a best fit, if z_0 is known. Or alternatively, if several measurements of u are taken along z, then u_* can be computed without need of estimating z_0. This is done by noting that $u(z)$ varies linearly with $\ln z$, with the slope of the line being[45] u_*/κ. In other words, plot u vs $\ln z$ and fit a line to the plot; the slope of the line is u_*/κ (remember, $\kappa \approx 0.4$).

Then, the bed shear stress can be obtained readily from $\tau_b = \rho u_*^2$ (remember, we defined u_* as $u_* \equiv \sqrt{\tau/\rho}$). This value of τ_b can ultimately be related to the depth-averaged velocity[46] U via $\tau_b = \rho u_*^2 = \rho C_f U^2$. The proportionality coefficient C_f can then be linked to Manning's n or Chézy coefficient C_{ch} via

$$C_f = \frac{g}{C_{ch}^2} = \frac{gn^2}{R_h^{1/3}}, \tag{53}$$

where $R_h (\equiv A/P)$ is the hydraulic radius[47]. We will come back to the important concept of bed shear stress later in the module when we study the mechanics of sediment transport.

4.6.2 Hydraulically rough and smooth beds (a note on z_0)

In pipe flow, whether the pipe may be considered *rough* or *smooth* (Darcy friction factor) depends on the diameter of the pipe (relative roughness) and the Reynolds number[48]. The situation is similar for open channel flow: one cannot say whether a sand or gravel bed can be considered *rough* or *smooth* without reference to other parameters of the problem. Based on the relative importance of viscosity near the bottom, one can classify the bed as **hydraulically rough** or **hydraulically smooth**. But before defining these terms, let us revisit the variable we introduced back in Section 3.5 to arrive at the law of the wall; namely z_0.

We said that z_0 *represents some hypothetical distance from the bed at which the flow velocity vanishes*. It is hypothetical because the whole concept of the log profile is based on certain strong assumptions (such as the Prandtl eddy model) which do not necessarily reflect the complexity of the real problem. Now, the velocity does vanish at some point, but it is difficult to study the very-near-bed region with the simplified theory we have employed so far. However, with aid of empirical observations, it is remarked that there are two distinct regimes: one where viscosity dominates over the *physical* roughness of the bed, and vice versa. Without getting into the details of **boundary layer theory**, you can probably appreciate that turbulence decreases in importance

[45] Confirm that $\partial u / \partial(\ln z)$ is equal to a constant (which shows that the relationship between u and $\ln z$ is linear).

[46] which we can obtain since we know $u(z)$.

[47] Show that for a *very wide* channel, the hydraulic radius can be approximated simply as the water depth, h.

[48] We are of course referring to Moody's chart.

as once gets closer to the bed, because the latter represents a barrier (a wall) that constrains the fluctuating motion of the flow[49]. This region near the bed dominated by viscosity is called the **viscous sublayer**, but its importance is relative to how *physically rough* the bed is. By *physical roughness* we refer to the actual *unevenness* of the bed, which is naturally dictated by the grains of sediment composing it. A measure of said unevenness is given by **Nikuradse's equivalent sand roughness** (or simply Nikuradse's roughness) k_n, which in turn relates (empirically) to the bed sediment's diameter[50]. The thickness of the viscous sublayer (another abstract concept we have introduced), δ, is not determined through analytical means, but rather by experiments. One common empirical expression for δ is:

$$\delta = 11.6\frac{\nu}{u_*}. \tag{54}$$

One can then compare δ with k_n to conclude whether or not the unevenness of the bed is 'felt' by the turbulent flow above (see Fig. 6). In other words, if $\delta \gg k_n$, one can be confident that the turbulent channel flow is relatively 'unaware' of the material in the bed (which is 'buried' within the viscous sublayer), and we thus classify the latter as **hydraulically smooth**. The theoretical z_0 may then be related, empirically, to the thickness of the viscous sublayer, via, for example:

$$z_0 \approx \frac{\delta}{117} \quad \text{if} \quad k_n < 5\frac{\nu}{u_*}. \tag{55}$$

Similarly, it may be reasoned that if, instead, we have $k_n \gg \delta$, then the flow does 'feel' the roughness of the bed, and then the latter is considered to be **hydraulically rough**, as illustrated[51] in Fig. 6. In this case, z_0 will not depend on δ, but rather on the bed roughness itself, k_n. Again, empirical expressions are employed, such as

$$z_0 \approx \frac{k_n}{33} \quad \text{if} \quad k_n > 70\frac{\nu}{u_*}. \tag{56}$$

The range of validity in the previous two equations (the inequalities) should trigger an alert. We said previously that there were two *distinct* regimes (hydraulically smooth and rough), but clearly, a transition regime also exists (for $5 < k_n u_*/\nu < 70$), for which empirical expressions have also been derived to estimate z_0. But in trying to minimise the risk of confusing you with empirically-derived expressions and those stemming from first principles, I shall not include more of the former. In fact, this is a good opportunity to invite you to read carefully through these notes and highlight those equations that come from first principles (such as Navier-Stokes equations, which stem from Newton's second law and which have not changed since originally derived in the 1800's) and those derived from empirical findings (which are not *universal* and may vary from author to author and year to year). Bear in mind this warning on empiricism, which becomes the more important when studying sediment transport.

[49]Obviously, at the very wall, for example, the vertical fluctuation of the velocity field must be zero, otherwise the flow would *go through* the solid bed!

[50]As usual, there is no single relationship between k_n and the sediment diameter D, but in general $k_n \sim D$, with expressions such as $k_n \approx 2.5D$ being commonly utilised.

[51]Note that since $k_n \gg \delta$, a consequence is that the viscous sublayer is constantly destroyed by the roughness elements.

Problem: Defining Reynolds number as $\mathrm{Re} \equiv Uh/\nu$, and expressing the shear velocity either in terms of Manning's n or Chézy coefficient C_{ch}, rewrite eq. (54) to express δ as a function of Re and n or C_{ch}.

If you arrive at the right answer, you will see clearly that δ decreases with increasing Re (and hence increasingly intense turbulence), as would be expected. Another corollary is that z_0 also decreases with increasing Re for hydraulically smooth beds, but does not depend on Reynolds in hydraulically rough regime, in keeping with the analogy of pipe flow (see the Moody chart).

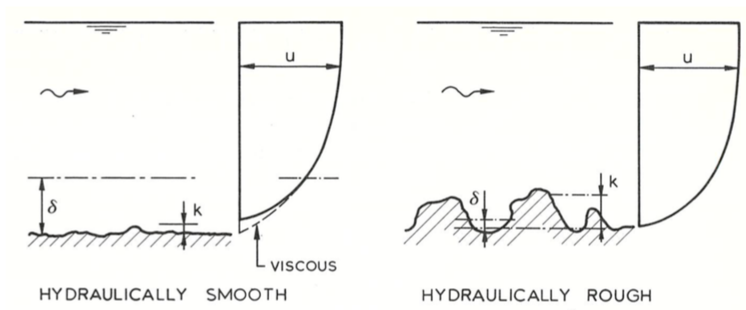

Figure 6: Hydraulically smooth and rough beds. [Taken from Jansen, P. P. *Principles of River Engineering: the Non-Tidal Alluvial River.* Delftse Uitgevers Maatschappij, 1994.]