Introduction
The Poisson process

Stochastic Processes 2024/25

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Outline

- Introduction to the course
- Preliminaries: the exponential distribution
- The Poisson process: Introduction and motivation
- Interarrival and waiting time distributions
- Conditional distribution of the arrival times
- Extensions and applications
- Nonhomogeneous Poisson process
- Compound Poisson process
- Conditional Poisson process

Course organization

- Prof. José Niño Mora, jose.nino@uc3m.es, https://alum.mit.edu/www/jnimora

- **Office hours:** Thursdays 9:45–10:45 (before class, email before)

- **Focus:** course on **stochastic processes** for modeling and analysis of random phenomena, based on intuitive approach

- **Goal:** Mastering the process:
problem description $\rightarrow$ stochastic model formulation $\rightarrow$ analysis $\rightarrow$ computer solution $\rightarrow$ interpretation

- **Examples:** drawn from diverse application areas

- **Software:** R

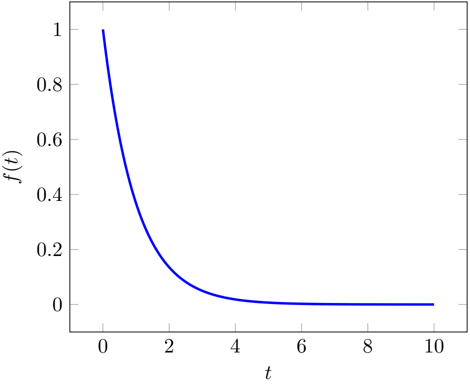
- **Grading:** Two partial exams (100%, no final exam)

Dates, Syllabus & References

- **Dates:** 7 sessions, November 7 – December 19
- **Topics:**
  1. Poisson process I–II
  2. Markov chains I–III
  3. Brownian motion I–II
- **References:**
  - Class lecture notes
  - S.M. Ross. Introduction to probability models. Academic Press. 2007
  - ...

Preliminaries: the exponential distribution

- $T \sim \mathrm{Exp}(\lambda)$ if $\mathbb{P}\{T \leqslant t\} = 1 - e^{-\lambda t}$, for $t \geqslant 0$
- Use: model of time elapsed until next occurrence of a certain event
- $\mathbb{E}[T] = 1/\lambda$, $\mathrm{Var}[T] = 1/\lambda^2$
- Density $f(t) = \lambda e^{-\lambda t}$. For $\lambda = 1$ event per unit time:



Preliminaries: the exponential distribution (cont.)

- **Memoryless property:** $\mathbb{P}\{T > t + s \mid T > t\} = \mathbb{P}\{T > s\}$

- **Exponential races:** Let $T_i \sim \mathrm{Exp}(\lambda_i)$ for $i = 1, \ldots, n$ be independent. Then $\min_i T_i \sim \mathrm{Exp}(\sum_i \lambda_i)$

- **Exponential races (cont.):** $\mathbb{P}\{T_j = \min_i T_i\} = \frac{\lambda_j}{\sum_i \lambda_i}$

- **Sums of exponentials:** Let $T_i \sim \mathrm{Exp}(\lambda)$ for $i = 1, \ldots, n$ be independent. Then $S_n = \sum_i T_i \sim \mathrm{Gamma}(n, \lambda)$, w/ density

$$f_{S_n} = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t \geqslant 0$$

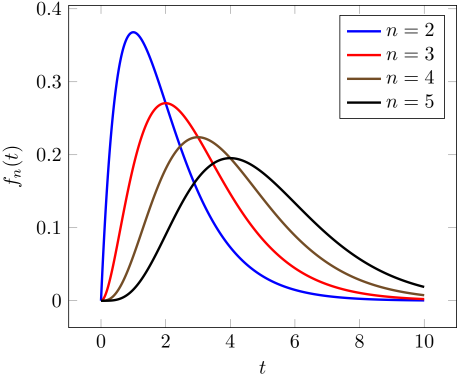
- Note: $\mathbb{E}[S_n] = \dfrac{n}{\lambda}, \quad \mathrm{Var}[S_n] = \dfrac{n}{\lambda^2}$

Preliminaries: the exponential distribution (cont.)

- **Sums of exponentials:** Let $T_i \sim \mathrm{Exp}(\lambda)$ for $i = 1, \ldots, n$ be independent. Then $S_n = \sum_i T_i \sim \mathrm{Gamma}(n, \lambda)$, w/ density

$$f_{S_n} = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t \geqslant 0$$

- For $\lambda = 1$ event per unit time:



Counting processes

- A stochastic process $\{N_t : t \geqslant 0\}$ is a **counting process** if N_t counts the number of events that occur by time t

- Ex1: # of people entering a store by time t

- Ex2: # of people born by time t

- Ex3: # of goals scored by time t by a given soccer player

Defining the Poisson process

- $\{N_t \colon t \geqslant 0\}$ has **independent increments** if the # of events in disjoint time intervals are independent
- $\{N_t \colon t \geqslant 0\}$ has **stationary increments** if the distribution of the # of events in a time interval depends only on its length
- Q: Are these assumptions reasonable for the above examples?
- **Definition:** N_t is a **Poisson process** w/ rate $\lambda > 0$ if:

  (i) $N_0 = 0$

  (ii) it has **independent increments**

  (iii) it has **stationary increments**, w/ $N_{s+t} - N_s \sim \mathrm{Poisson}(\lambda t)$:

$$\mathbb{P}\{N_{s+t} - N_s = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

- Consequence: $\mathbb{E}\big[N_{s+t} - N_s\big] = \lambda t$, $\mathrm{Var}\big[N_{s+t} - N_s\big] = \lambda t$

Example

Starting at 6am, customers arrive at a bakery according to a Poisson process with rate 30 customers/h. What is the probability that more than 65 customers arrive between 9 and 11am?

- Let $t = 0$ be 6am. We are asked for $\mathbb{P}\{N_5 - N_3 > 65\}$
- By stationary increments, with $\lambda = 30$,

$$\begin{aligned}\mathbb{P}\{N_5 - N_3 > 65\} &= \mathbb{P}\{N_2 > 65\} = 1 - \mathbb{P}\{N_2 \leqslant 65\} \\ &= 1 - \sum_{n=0}^{65} \mathbb{P}\{N_2 = n\} \\ &= 1 - \sum_{n=0}^{65} e^{-\lambda t} \frac{(\lambda t)^n}{n!} = 0.2355\end{aligned}$$

- In R: `1 - ppois(65, 2*30) = 0.2355`
- `ppois()` is used to calculate the CDF for a Poisson distribution. It takes two arguments: the value at which to evaluate the CDF and the mean of the Poisson distribution

Example

Joe receives whatsapp (WA) messages starting at 10am at the rate of 10 WAs/h according to a Poisson process. Find the probability that he will receive exactly 18 WAs by noon and 70 WAs by 5pm

- We are asked for $\mathbb{P}\{N_2 = 18, N_7 = 70\}$
- We have, with $\lambda = 10$,

$$\begin{aligned}
\mathbb{P}\{N_2 = 18, N_7 = 70\} &= \mathbb{P}\{N_2 = 18, N_7 - N_2 = 52\} \\
&= \mathbb{P}\{N_2 = 18\}\,\mathbb{P}\{N_7 - N_2 = 52\} \\
&= \mathbb{P}\{N_2 = 18\}\,\mathbb{P}\{N_5 = 52\} \\
&= \left(e^{-\lambda\times 2}\frac{(\lambda\times 2)^{18}}{18!}\right)\left(e^{-\lambda\times 5}\frac{(\lambda\times 5)^{52}}{52!}\right) \\
&= 0.0045
\end{aligned}$$

- In R: dpois(18, 2*10) * dpois(52, 5*10) = 0.00448
- dpois() returns the probability that the value of a variable that follows the Poisson distribution is equal to a given value

Another definition

- Note: A function $f(h)$ is $o(h)$ if $f(h)/h \to 0$ as $h \to 0$

- **Definition:** N_t is a **Poisson process** w/ rate $\lambda > 0$ if:
  (i) $N_0 = 0$
  (ii) it has stationary and independent increments
  (iii) $\mathbb{P}\{N_h = 1\} = \lambda h + o(h)$
  (iv) $\mathbb{P}\{N_h \geqslant 2\} = o(h)$

Interarrival time distribution

- Let N_t be a **Poisson process** w/ rate λ
- Let T_1: time of 1st event
- Let T_2: time from 1st to 2nd event
- Let T_n: time from $(n-1)$th to nth event
- $\{T_n : n \geqslant 1\}$: sequence of **interarrival times**
- **Q:** What is the distribution of T_n?

Interarrival time distribution

- **Q:** What is the distribution of T_n?
- $\mathbb{P}\{T_1 > t\} = \mathbb{P}\{N_t = 0\} = e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t}$
- $\mathbb{P}\{T_2 > t \mid T_1 = s\} = \mathbb{P}\{N_{s+t} - N_s = 0\} = e^{-\lambda t}$

  $\Longrightarrow \mathbb{P}\{T_2 > t\} = e^{-\lambda t}$
- and so on: $\mathbb{P}\{T_n > t\} = e^{-\lambda t}$, i.e., $T_n \sim \mathrm{Exp}(\lambda)$
- Furthermore, the **interarrival times** T_n are **independent**
- Note: $\mathbb{E}[T_n] = 1/\lambda$
- Note: the process has **no memory**, hence **exponential distribution**

Waiting (Arrival) time distribution

- Let $S_n = \sum_{i=1}^{n} T_i$ be the **waiting time** until the nth event (i.e., its *arrival time*)
- **Q:** What is the distribution of S_n?
- $S_n \sim \mathrm{Gamma}(n, \lambda)$
- Note that $N_t \geqslant n \iff S_n \leqslant t$

Example

The times when goals are scored in hockey are modeled as a Poisson process, w/ mean time between goals of 15 m. (a) In a 60-min. game, find the prob. that a 4th goal occurs in the last 5 m

- We have $\lambda = 1/15$. (i) We are asked for ($n = 4$)

$$\mathbb{P}\{55 < S_4 \leqslant 60\} = \int_{55}^{60} \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}\, dt = 0.068$$

- In R: pgamma(60, 4, 1/15) - pgamma(55, 4, 1/15) = 0.06766216

- pgamma() is used to calculate the CDF for a gamma distribution

Example

The times when goals are scored in hockey are modeled as a Poisson process, w/ mean time between goals of 15 m. (b) If at least 3 goals are scored in a game, what is the mean time of the 3rd goal?

- We have $\lambda = 1/15$. (b) We are asked for (w/ $n = 3$)

$$\begin{aligned}\mathbb{E}[S_3 \mid S_3 < 60] &= \frac{1}{\mathbb{P}\{S_3 < 60\}} \int_0^{60} t f_{S_3}(t)\, dt \\ &= \frac{1}{\mathbb{P}\{S_3 < 60\}} \int_0^{60} t \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}\, dt \\ &= \frac{25.4938}{0.7619} = 33.461\text{m}\end{aligned}$$

Yet another definition of the Poisson process

- Let $S_n = \sum_{i=1}^{n} T_i$, with $T_1, T_2, \ldots$ i.i.d. w/ distribution $\mathrm{Exp}(\lambda)$

- The counting process $N_t \triangleq \max\{n \colon S_n \leqslant t\}$ is a **Poisson process**

- Useful for **Monte Carlo simulation** of a Poisson process

Thinning or splitting a Poisson process

- Let $\{N_t : t \geqslant 0\}$ be a Poisson process w/ rate λ

- Suppose each event is independently classified to be of type $k = 1, \ldots, K$ w/ probability p_k

- Let $\{N_t^k : t \geqslant 0\}$ be the counting process for type k events

- **Result:** $\{N_t^k : t \geqslant 0\}$ is a Poisson process w/ rate $\lambda_k \triangleq \lambda p_k$, and the processes $\{N_t^k : t \geqslant 0\}$, for $k = 1, \ldots, K$, are independent

Example

Assume births occur on a maternity ward as a Poisson process w/ rate 2 births/h. Each birth is male or female indep. w/ prob. $p = 0.519$ and $1 - p$ (a) On an 8-h shift, what is the mean and standard dev. of the number of female births?

• Let N_t, M_t and F_t be the overall, male and female birth processes

• N_t, M_t and F_t are Poisson processes w/ rates $\lambda = 2$, λp, $\lambda(1 - p)$

• The # of female births on an 8-h shift, F_8, follows a Poisson distribution w/ rate $\lambda(1 - p)8 = 7.696$

• Hence, $\mathbb{E}[F_8] = 7.696, \quad \sqrt{\mathrm{Var}[F_8]} = \sqrt{7.696} = 2.774$

Example

Assume births occur on a maternity ward as a Poisson process w/ rate 2 births/h. Each birth is male or female indep. w/ prob. $p = 0.519$ and $1 - p$. (b) Find the prob. that only girls were born between 2 and 5 pm

- We are asked for $\mathbb{P}\{M_3 = 0, F_3 > 0\}$. Have

$$
\begin{aligned}
\mathbb{P}\{M_3 = 0, F_3 > 0\} &= \mathbb{P}\{M_3 = 0\}\,\mathbb{P}\{F_3 > 0\} \\
&= e^{-\lambda_M \times 3}\,(1 - e^{-\lambda_F \times 3}) = 0.042
\end{aligned}
$$

Superposition of Poisson processes

- Let $\{N_t^k : t \geqslant 0\}$ be a Poisson process w/ rate λ_k, counting type k events, for $k = 1, \ldots, K$, and assume that such processes are independent
- Let $\{N_t : t \geqslant 0\}$ be the counting process for all events, w/ $N_t \triangleq \sum_k N_t^k$
- **Result:** $\{N_t : t \geqslant 0\}$ is a Poisson process w/ rate $\lambda \triangleq \sum_k \lambda_k$
- Furthermore, each event is of type k w/ probability $p_k \triangleq \lambda_k / \lambda$

Example

In Oz, sightings of lions, tigers, and bears each follow an indep. Poisson process w/ resp. rates λ_L, λ_T, and λ_B per h. (a) Find the prob. that Dorothy will not see any animal in the first 24 h. from arrival in Oz

• Let N_t^{lions}, N_t^{tigers}, N_t^{bears} be the sighting processes, the total sightings N_t are their superposition. We are asked for $\mathbb{P}\{N_{24} = 0\}$

• N_t is a Poisson process w/ rate $\lambda = \lambda_L + \lambda_T + \lambda_B$

• Hence, $\mathbb{P}\{N_{24} = 0\} = e^{-\lambda \times 24}$

Example

In Oz, sightings of lions, tigers, and bears each follow an indep. Poisson process w/ resp. rates λ_L, λ_T, and λ_B per h. (b) Dorothy saw 3 animals one day. Find the prob. that each species was seen

- Let N_t^{lions}, N_t^{tigers}, N_t^{bears} be the sighting processes, total sightings N_t
- We are asked for $\mathbb{P}\{N_t^{\text{lions}} = 1, N_t^{\text{tigers}} = 1, N_t^{\text{bears}} = 1 \mid N_{24} = 3\}$:

$$
\begin{aligned}
&= \frac{\mathbb{P}\{N_t^{\text{lions}} = 1, N_t^{\text{tigers}} = 1, N_t^{\text{bears}} = 1, N_{24} = 3\}}{\mathbb{P}\{N_{24} = 3\}} \\
&= \frac{\mathbb{P}\{N_t^{\text{lions}} = 1, N_t^{\text{tigers}} = 1, N_t^{\text{bears}} = 1\}}{\mathbb{P}\{N_{24} = 3\}} \\
&= \frac{\mathbb{P}\{N_t^{\text{lions}} = 1\}\,\mathbb{P}\{N_t^{\text{tigers}} = 1\}\,\mathbb{P}\{N_t^{\text{bears}} = 1\}}{\mathbb{P}\{N_{24} = 3\}} \\
&= \frac{\left(e^{-\lambda_L \times 24}\lambda_L \times 24\right)\left(e^{-\lambda_T \times 24}\lambda_T \times 24\right)\left(e^{-\lambda_B \times 24}\lambda_B \times 24\right)}{\left(e^{-\lambda \times 24}(\lambda \times 24)^3/3!\right)} \\
&= \frac{6\lambda_L\lambda_T\lambda_B}{\lambda^3}
\end{aligned}
$$

Conditional distribution of the arrival times

- Suppose we are told that exactly one event has taken place by time t
- **Q**: What is the distribution of the time at which the event occurred?

$$
\begin{aligned}
\mathbb{P}\{T_1 < s \mid N_t = 1\} &= \frac{\mathbb{P}\{T_1 < s, N_t = 1\}}{\mathbb{P}\{N_t = 1\}} \\
&= \frac{\mathbb{P}\{1 \text{ event in } [0, s), 0 \text{ events in } [s, t]\}}{\mathbb{P}\{N_t = 1\}} \\
&= \frac{\mathbb{P}\{1 \text{ event in } [0, s)\}\, \mathbb{P}\{0 \text{ events in } [s, t]\}}{\mathbb{P}\{N_t = 1\}} \\
&= \frac{\lambda s e^{-\lambda s}\, e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} \\
&= \frac{s}{t}
\end{aligned}
$$

- **Result:** $T_1 \mid N_t = 1 \sim \mathrm{Unif}[0, t]$

Conditional distribution of the arrival times (cont.)

- Suppose we are told that exactly n events have taken place by time t

- **Q**: What is the distribution of the times $S_1, \ldots, S_n$ at which the events occurred?

- **Result:** The conditional distribution of $S_1, \ldots, S_n$, given that $N_t = n$, is the same as the distribution of the **order statistics** $U_{(1)}, \ldots, U_{(n)}$ corresponding to n independent r.v. $U_1, \ldots, U_n$ w/ distribution $\mathrm{Unif}[0, t]$

- Their **joint conditional density** is

$$f(s_1, \ldots, s_n \mid n) = \frac{n!}{t^n}, \quad 0 < s_1 < \cdots < s_n < t$$

Example

Starting at time $t = 0$, customers arrive at a restaurant as a Poisson process w/ rate 20 customers/h. (a) Find the prob. that the 60th customer arrives in the interval $[2.9, 3]$

- The time of the 60th arrival, S_{60}, has a gamma distr. w/ parameters $n = 60$ and $\lambda = 20$. The requested prob. is $\mathbb{P}\{2.9 < S_{60} \leqslant 3\}$

- In R: `pgamma(3, 60, 20) - pgamma(2.9, 60, 20) = 0.1034`

Example

Starting at time $t = 0$, customers arrive at a restaurant as a Poisson process w/ rate 20 cust./h. (b) If 60 customers arrive by time $t = 3$, find the prob. that the 60th customer arrives in the interval $[2.9, 3]$

- The requested prob. is $\mathbb{P}\{2.9 < S_{60} \leqslant 3 \mid N_3 = 60\}$
- Given $N_3 = 60$, the time of the 60th arrival, S_{60}, has the same distribution as the maximum M of 60 i.i.d. r.v. w/ distribution $\mathrm{Unif}[0, 3]$
- Have:

$$
\begin{aligned}
\mathbb{P}\{2.9 < S_{60} \leqslant 3 \mid N_3 = 60\} &= \mathbb{P}\{2.9 < M \leqslant 3\} = \mathbb{P}\{M > 2.9\} \\
&= 1 - \mathbb{P}\{M \leqslant 2.9\} \\
&= 1 - \mathbb{P}\{U_1 \leqslant 2.9, \ldots, U_{60} \leqslant 2.9\} \\
&= 1 - \mathbb{P}\{U_1 \leqslant 2.9\}^{60} \\
&= 1 - \left(\frac{2.9}{3}\right)^{60} = 1 - 0.131 = 0.869
\end{aligned}
$$

Application to simulating a Poisson process

One way to simulate a Poisson process w/ rate λ on an interval $[0, t]$

1. Simulate the # of arrivals N in $[0, t]$ as $N \sim \mathrm{Poisson}(\lambda t)$
2. Generate N i.i.d. r.v. $U_1, \ldots, U_N \sim \mathrm{Unif}[0, t]$
3. Sort the U_n's in increasing order, $U_{(1)} < \cdots < U_{(N)}$, to obtain the Poisson process arrival times $S_n = U_{(n)}$

Application to simulating a Poisson process (in R)

```
t <- 10

lambda <- 1/2

N <- rpois(1, lambda * t)

unifs <- runif(N, 0, t)

arrivals <- sort(unifs)

arrivals

1.82193882763386 2.30343171162531 3.2409208919853
3.66398209473118 4.03040210250765 5.1834518276155
6.33557516383007 8.13909047283232
```

The incidence paradox

Suppose buses arrive at a bus stop as a Poisson process with a mean time between buses of 10 m. You arrive at the bus stop at time t. How long can you expect to wait for a bus?

- Is the correct answer 5 m?

- Is the correct answer 10 m?

- Theoretical result: the expected length of the interarrival time that contains a fixed time t is not $1/\lambda$. It is

$$\mathbb{E}\left[S_{N_t+1} - S_{N_t}\right] = \frac{2 - e^{-\lambda t}}{\lambda} \to \frac{2}{\lambda} \quad \text{as } t \to \infty$$

- Hence you can expect to wait about $1/\lambda$ when t is large

The incidence paradox via simulation

Suppose buses arrive at a bus stop as a Poisson process with a mean time between buses of 10 m. You arrive at the bus stop at time t. How long can you expect to wait for a bus?

```
mytime <- 50
lambda <- 1/10
trials <- 10000
simlist <- numeric(trials)
for (i in 1:trials) {
    N <- rpois(1, 300*lambda)
    arrivals <- sort(runif(N, 0, 300))
    wait <- arrivals[arrivals > mytime][1] - mytime
    simlist[i] <- wait
}
mean(simlist)
10.116
```

Nonhomogeneous Poisson process

• **Definition:** $\{N_t : t \geqslant 0\}$ is a **nonhomogeneous Poisson process** w/ **intensity function** $\lambda(t)$ if:

(i) $N_0 = 0$

(ii) For $t > 0$, N_t has a Poisson distribution w/ mean $\mathbb{E}[N_t] = \int_0^t \lambda(s)\, ds$

(iii) For $0 \leqslant q < r \leqslant s < t$, $N_r - N_q$ and $N_t - N_s$ are independent r.v.

• Consequence: $\{N_t\}$ has independent increments, but not necessarily stationary increments

• Consequence: For $0 < s < t$, $N_t - N_s$ has a Poisson distribution with mean $\int_s^t \lambda(x)\, dx$

Example

Students arrive at a cafeteria as a nonhomogeneous Poisson process. The arrival rate grows linearly from 100 to 200 students/h between 11am and noon. The rate stays constant for the next two hours, and then decreases linearly to 100 from 2 to 3pm. Find the prob. that at least 400 people arrive in the cafeteria between 11:30am and 1:30pm

- The intensity function is (letting $t = 0$ correspond to 11am)

$$\lambda(t) = \begin{cases} 100 + 100t, & 0 \leqslant t \leqslant 1 \\ 200, & 1 < t \leqslant 3 \\ 500 - 100t, & 3 \leqslant t < 4 \end{cases}$$

- We are asked for $\mathbb{P}\{N_{2.5} - N_{0.5} \geqslant 400\}$
- Know that $N_{2.5} - N_{0.5}$ has a Poisson distr. w/ mean $\int_{0.5}^{2.5} \lambda(t)\,dt$

$\mathbb{E}[N_{2.5} - N_{0.5}] = \int_{0.5}^{2.5} \lambda(t)\,dt = \int_{0.5}^{1} (100 + 100t)\,dt + \int_{1}^{2.5} 200\,dt = 387.5$

$\mathbb{P}\{N_{2.5} - N_{0.5} \geqslant 400\} \approx 0.2693$ (`1 - ppois(399, 387.5)`)

Compound Poisson process

• **Definition:** A stochastic process $\{X_t \colon t \geqslant 0\}$ is a compound Poisson process if it can be represented as:

$$X_t = \sum_{n=1}^{N_t} Y_n,$$

where $\{N_t\}$ is a Poisson process, and $\{Y_n \colon n \geqslant 1\}$ is an i.i.d. sequence of r.v. that is also independent of $\{N_t\}$

• Note: X_t is said to be a compound Poisson r.v.

• Note: $\mathbb{E}[X_t] = \lambda t \mathbb{E}[Y_1], \quad \mathrm{Var}[X_t] = \lambda t \mathbb{E}[Y_1^2]$

Examples of compound Poisson processes

• Suppose buses arrive at a sporting event as a Poisson process. Suppose that the # of fans in each bus are i.i.d. Then, if X_t is the # of fans that have arrived by time t, $\{X_t\}$ is a compound Poisson process

• Suppose customers leave a supermarket as a Poisson process. If the Y_n, the amount spent by customer $n = 1, 2, \ldots$, are i.i.d., then, if X_t is the total amount of money spent by time t, $\{X_t\}$ is a compound Poisson process

Example

Buses arrive at a sporting event as a Poisson process w/ rate 5 buses/h. Suppose that the # of fans in each bus are i.i.d., w/ mean 20 and standard deviation 10. (a) If X_t is the # of fans that have arrived by time t, what is the distribution of the process $\{X_t\}$? (b) What is the mean and variance of X_t?

(a) Let Y_n be the # of fans in bus $n = 1, 2, \ldots$. Then

$$X_t = \sum_{n=1}^{N_t} Y_n,$$

and hence $\{X_t\}$ is a compound Poisson process

(b) $\mathbb{E}[X_t] = \lambda t \mathbb{E}[Y_1] = 5 \times t \times 20 = 100t$,

$$\operatorname{Var}[X_t] = \lambda t \mathbb{E}[Y_1^2] = \lambda t \big(\operatorname{Var}[Y_1] + \mathbb{E}[Y_1]^2\big) = 5 \times t \times \left(10^2 + 20^2\right) = 2500t$$

Conditional Poisson process

• **Definition:** Let $\{N_t : t \geqslant 0\}$ be a counting process defined as follows. There is a positive r.v. L such that, conditional on $L = \lambda$, the counting process is a Poisson process w/ rate λ. We then say that $\{N_t : t \geqslant 0\}$ is a **conditional Poisson process**

• Suppose L is continuous with density g. We have

$$\begin{aligned}\mathbb{P}\{N_{s+t} - N_s = n\} &= \int_0^\infty \mathbb{P}\{N_{s+t} - N_s = n \mid L = \lambda\}\, g(\lambda)\, d\lambda \\ &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} g(\lambda)\, d\lambda\end{aligned}$$

• Hence, a conditional Poisson process has **stationary increments**

Conditional Poisson process (cont.)

- **Definition:** Let $\{N_t \colon t \geqslant 0\}$ be a counting process defined as follows. There is a positive r.v. L such that, conditional on $L = \lambda$, the counting process is a Poisson process w/ rate λ. We then say that $\{N_t \colon t \geqslant 0\}$ is a **conditional Poisson process**

- We have

$$\mathbb{E}[N_t] = \mathbb{E}\big[\mathbb{E}[N_t \mid L]\big] = \mathbb{E}[Lt] = t\mathbb{E}[L]$$

- Furthermore, using the conditional variance formula, we have

$$\begin{aligned} \mathrm{Var}[N_t] &= \mathbb{E}\big[\mathrm{Var}[N_t \mid L]\big] + \mathrm{Var}\big[\mathbb{E}[N_t \mid L]\big] \\ &= \mathbb{E}[Lt] + \mathrm{Var}[Lt] \\ &= t\mathbb{E}[L] + t^2\mathrm{Var}[L] \end{aligned}$$

Example

The number of whatsapp (WA) messages Joe receives follows a conditional Poisson process with random rate L, where $\mathbb{E}[L] = 10$ WA/h and $\mathrm{Var}[L] = 25$. Find the expected # of WA messages received during 8 h as well as its variance

(a) $\mathbb{E}[N_8] = 8\mathbb{E}[L] = 8 \times 10 = 80$

(b) $\mathrm{Var}[N_t] = 8\mathbb{E}[L] + 8^2\mathrm{Var}[L] = 8 \times 10 + 8^2 \times 25$