MATE 6540: Topology Qualifying Exam

Due on 19 de mayo

Prof. Iván Cardona , C41, 19 de mayo

Sergio Rodríguez

Problem 0

A topological space $(X, \mathcal{T}_X)$ is pseudocompact $\iff$ every continuous function $f : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ is bounded. Here $\mathcal{T}_{\varepsilon^1}$ is the usual topology over $\mathbb{R}$.

(i) Show that pseudocompactness is a continuous invariant. Explain.

(ii) Show that if $(Y, \mathcal{T}_Y)$ is compact, then $(Y, \mathcal{T}_Y)$ is pseudocompact, but that the converse does not hold.

Problem 1

(Kuratowski's closure operator) Let X be a set, $\mathcal{P}(X)$ be its powerset, and $c : \mathcal{P}(X) \to \mathcal{P}(X)$ be a function that satisfies:

(i) $c(\emptyset) = \emptyset$

(ii) $A \subseteq c(A), \quad \forall A \in \mathcal{P}(X)$

(iii) $c(c(A)) = c(A), \quad \forall A \in \mathcal{P}(X)$

(iv) $c(A \cup B) = c(A) \cup c(B), \quad \forall A, B \in \mathcal{P}(X)$

Show that the collection $\mathcal{T} = \{X \setminus c(A) \mid A \in \mathcal{P}(X)\}$ is a topology over X, and that in this topology $\overline{A} = c(A), \quad \forall A \in \mathcal{P}(X)$. Here $\overline{A}$ is the closure of A in $(X, \mathcal{T})$.

Problem 2

Let A be a subset of a topological space $(X, \mathcal{T}_X)$. Show that the following are equivalent:

(i) $\mathtt{int}\left(\overline{A}\right) = \emptyset$.

(ii) $X \setminus \overline{A}$ *is dense in* X.

(iii) $X \setminus \overline{\left(X \setminus \overline{A}\right)} = \emptyset$.

(iv) $A \subseteq \overline{\left(X \setminus \overline{A}\right)}$.

Problem 3

A subset of a topological space is a G_δ-set if it is the intersection of countably many open sets. On the other hand, a subset of a topological space is an F_δ-set if it is the union of countably many closed sets.

(i) Let A be an F_δ-set of a topological space $(X, \mathcal{T}_X)$. Show that there is a nested sequence of closed sets $C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots$ such that $A = \bigcup_{i=1}^{\infty} C_i$.

(ii) Show that every closed set in a metric space (X, d) is a G_δ-set.

Problem 4

Let A, B be two non-empty subsets of $\mathbb{R}$ with the usual topology. Define:

$$C \coloneqq \{x + y \mid x \in A \wedge y \in B\}. \tag{1}$$

(a) Show that, if A or B is open, then C is open.

(b) Show that, if A and B are compact, then C is compact.

Problem 5

(Intermediate value theorem) Let $f : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ be a continuous function, where $(X, \mathcal{T}_X)$ is connected. Show that if a, b are two points in X and if r is a real number lying between $f(a)$ and $f(b)$, then there is a $c \in X$ such that $f(c) = r$.

Problem 6

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and suppose $X_1 \times X_2$ has the product topology. For each $i = 1, 2$, let $A_i \subseteq X_i$. Prove that:

(i) $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$.

(ii) $\mathtt{int}(A_1 \times A_2) = \mathtt{int}(A_1) \times \mathtt{int}(A_2)$.