

# **MATE 6540: Topology Qualifying Exam**

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**Problem 0**

A topological space  $(X, \mathcal{T}_X)$  is pseudocompact  $\iff$  every continuous function  $f : (X, \mathcal{T}_X) \rightarrow (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$  is bounded. Here  $\mathcal{T}_{\varepsilon^1}$  is the usual topology over  $\mathbb{R}$ .

(i) Show that pseudocompactness is a continuous invariant. Explain.

(ii) Show that if  $(Y, \mathcal{T}_Y)$  is compact, then  $(Y, \mathcal{T}_Y)$  is pseudocompact, but that the converse does not hold.

**Proof (i):**

Let  $(X, \mathcal{T}_X)$  be a pseudocompact topological space, and let  $\varphi : (X, \mathcal{T}_X) \rightarrow (\varphi(X), \mathcal{T}_{\varphi(X)})$  be a continuous function. Now let  $f : (\varphi(X), \mathcal{T}_{\varphi(X)}) \rightarrow (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$  be a continuous function. Then  $(f \circ \varphi) : (X, \mathcal{T}_X) \rightarrow (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$  is continuous. But  $(X, \mathcal{T}_X)$  is pseudocompact, so  $f \circ \varphi$  is bounded  $\implies f$  is bounded. Then  $(\varphi(X), \mathcal{T}_{\varphi(X)})$  is pseudocompact.

$\therefore$  pseudocompactness is a continuous invariant.

**MEP****Proof (ii):**

We claim that  $(Y, \mathcal{T}_Y)$  compact  $\implies (Y, \mathcal{T}_Y)$  pseudocompact, we show the contrapositive.

Suppose that  $(Y, \mathcal{T}_Y)$  is not pseudocompact, then there exists a continuous function  $f : (Y, \mathcal{T}_Y) \rightarrow (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$  that is unbounded. Then  $f(Y)$  extends infinitely in at least one direction. Suppose, without loss of generality, that it extends infinitely to the right. Then  $f(Y) \subseteq (a, +\infty)$ , for some  $a \in \mathbb{R} \cup \{-\infty\}$ . Now define the family  $\{(a, i)\}_{i=1}^{\infty}$  and note that:  $\bigcup_{i=1}^{\infty} (a, i) = (a, +\infty) \supseteq f(Y)$ . But note that  $(a, i) \in \mathcal{T}_{\varepsilon^1}, \forall i \in \mathbb{N} \implies f^{-1}((a, i)) \in \mathcal{T}_Y, \forall i \in \mathbb{N}$  because  $f$  is continuous. And:

$$\bigcup_{i=1}^{\infty} f^{-1}((a, i)) = f^{-1}\left(\bigcup_{i=1}^{\infty} (a, i)\right) = f^{-1}((a, +\infty)) \supseteq f^{-1}(f(Y)) \supseteq Y \quad (1)$$

This implies that the collection  $C := \{f^{-1}((a, i)) \cap Y \mid i \in \mathbb{N}\}$  is an open cover for  $Y$ , but note that, for any finite subcollection  $C' := \{f^{-1}((a, i_j)) \mid i_j \in \mathbb{N}, \forall j \in \{1, \dots, n\}\} \subseteq C$ , there exists a natural number  $n$  such that  $i_n$  is an upper bound of  $C'$ . But  $f(Y)$  extends infinitely to the right, so  $C'$  cannot be an open cover for  $Y$ .

$\therefore (Y, \mathcal{T}_Y)$  is not compact.

Now we show that the converse does not hold.

**Problem 1**

(Kuratowski's closure operator) Let  $X$  be a set,  $\mathcal{P}(X)$  be its powerset, and  $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  be a function that satisfies:

(i)  $c(\emptyset) = \emptyset$

(ii)  $A \subseteq c(A), \forall A \in \mathcal{P}(X)$

$$(iii) c(c(A)) = c(A), \quad \forall A \in \mathcal{P}(X)$$

$$(iv) c(A \cup B) = c(A) \cup c(B), \quad \forall A, B \in \mathcal{P}(X)$$

Show that the collection  $\mathcal{T} = \{X \setminus c(A) \mid A \in \mathcal{P}(X)\}$  is a topology over  $X$ , and that in this topology  $\overline{A} = c(A)$ ,  $\forall A \in \mathcal{P}(X)$ . Here  $\overline{A}$  is the closure of  $A$  in  $(X, \mathcal{T})$ .

**Proof:**

We claim that  $\emptyset, X \in \mathcal{T}$ .

Note that  $c(X) \in \mathcal{P}(X)$  and  $X \in c(X)$  imply that  $c(X) = X$ . Then  $X \setminus c(X) = X \setminus X = \emptyset \in \mathcal{T}$ . Similarly, note that  $c(\emptyset) = \emptyset \implies X \setminus c(\emptyset) = X \setminus \emptyset = X \in \mathcal{T}$ .

$\therefore \emptyset, X \in \mathcal{T}$ .

We claim that  $\mathcal{T}$  is closed under arbitrary unions.

Take  $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{T}_X$ . We show that  $U := \bigcup_{\alpha \in \Lambda} U_\alpha \in \mathcal{T}_X$ . Note that  $\forall \alpha \in \Lambda, \exists V_\alpha \in \mathcal{P}(X)$  such that  $U_\alpha = X \setminus c(V_\alpha)$ . Then  $U = \bigcup_{\alpha \in \Lambda} (X \setminus c(V_\alpha)) = X \setminus \bigcap_{\alpha \in \Lambda} c(V_\alpha)$

We claim that  $\mathcal{T}$  is closed under finite intersections.

Take  $U, V \in \mathcal{T}$ , then  $\exists A, B \in \mathcal{P}(X)$  such that  $U = X \setminus c(A)$  and  $V = X \setminus c(B)$ . Then:

$$\begin{aligned} U \cap V &= (X \setminus c(A)) \cap (X \setminus c(B)) \\ &= X \setminus (c(A) \cup c(B)) \\ &= X \setminus c(A \cup B) \in \mathcal{T} \end{aligned} \tag{2}$$

$\therefore U \cap V \in \mathcal{T}$ .

$\therefore \mathcal{T}$  is closed under finite intersections, by induction.

We claim that  $\overline{A} = c(A)$ ,  $\forall A \in \mathcal{P}(X)$ .

## Problem 2

Let  $A$  be a subset of a topological space  $(X, \mathcal{T}_X)$ . Show that the following are equivalent:

$$(i) \text{int}(\overline{A}) = \emptyset.$$

$$(ii) X \setminus \overline{A} \text{ is dense in } X.$$

$$(iii) X \setminus \overline{(X \setminus \overline{A})} = \emptyset.$$

$$(iv) A \subseteq \overline{(X \setminus \overline{A})}.$$

**Proof:**

$$((i) \Rightarrow (ii))$$

Suppose that  $\text{int}(\overline{A}) = \bigcup \{U \in \mathcal{T}_X \mid U \subseteq \overline{A}\} = \emptyset$ . Note that  $\overline{(X \setminus \overline{A})} \subseteq X$ . We claim that  $X \subseteq \overline{(X \setminus \overline{A})}$ . Take  $x \in X$  and  $U \in \mathcal{T}_X$  such that  $x \in U$ . Then  $U \neq \emptyset$ . Now suppose that  $U \subseteq \overline{A}$ , then  $U \subseteq \bigcup \{U \in \mathcal{T}_X \mid U \subseteq \overline{A}\} = \emptyset \Rightarrow U = \emptyset$ , which is a contradiction. ✖

$$\text{Then } U \cap (X \setminus \overline{A}) \neq \emptyset \Rightarrow x \in \overline{(X \setminus \overline{A})} \Rightarrow X \subseteq \overline{(X \setminus \overline{A})}.$$

$$\therefore \overline{(X \setminus \overline{A})} = X.$$

$$\therefore X \setminus \overline{A} \text{ is dense in } X.$$

$$((ii) \Rightarrow (iii))$$

Suppose that  $X \setminus \overline{A}$  is dense in  $X$ , then  $\overline{(X \setminus \overline{A})} = X$  by definition.

$$\text{Then } X \setminus \overline{(X \setminus \overline{A})} = X \setminus X = \emptyset.$$

$$\therefore X \setminus \overline{(X \setminus \overline{A})} = \emptyset.$$

$$((iii) \Rightarrow (iv))$$

Suppose that  $X \setminus \overline{(X \setminus \overline{A})} = \emptyset$ . Then  $\overline{(X \setminus \overline{A})} = X$ . But  $A \subseteq X$  by hypothesis.

$$\therefore A \subseteq \overline{(X \setminus \overline{A})}$$

$$((iv) \Rightarrow (i))$$

Suppose that  $A \subseteq \overline{(X \setminus \overline{A})}$ . Take  $x \in A$ , then  $x \in \overline{(X \setminus \overline{A})}$ . Now take  $U \in \mathcal{T}_X$  with  $x \in U$ , then  $U \cap (X \setminus \overline{A}) \neq \emptyset \Rightarrow U \subseteq$

### Problem 3

A subset of a topological space is a  $G_\delta$ -set if it is the intersection of countably many open sets. On the other hand, a subset of a topological space is an  $F_\sigma$ -set if it is the union of countably many closed sets.

(i) Let  $A$  be an  $F_\sigma$ -set of a topological space  $(X, \mathcal{T}_X)$ . Show that there is a nested sequence of closed sets  $C_1 \subseteq C_2 \subseteq C_3 \subseteq \dots$  such that  $A = \bigcup_{i=1}^{\infty} C_i$ .

(ii) Show that every closed set in a metric space  $(X, d)$  is a  $G_\delta$ -set.

**Problem 4**

Let  $A, B$  be two non-empty subsets of  $\mathbb{R}$  with the usual topology. Define:

$$C := \{x + y \mid x \in A \wedge y \in B\}. \quad (3)$$

(a) Show that, if  $A$  or  $B$  is open, then  $C$  is open.

(b) Show that, if  $A$  and  $B$  are compact, then  $C$  is compact.

**Problem 5**

(Intermediate value theorem) Let  $f : (X, \mathcal{T}_X) \rightarrow (\mathbb{R}, \mathcal{T}_{\mathbb{R}})$  be a continuous function, where  $(X, \mathcal{T}_X)$  is connected. Show that if  $a, b$  are two points in  $X$  and if  $r$  is a real number lying between  $f(a)$  and  $f(b)$ , then there is a  $c \in X$  such that  $f(c) = r$ .

**Problem 6**

Let  $(X_1, \mathcal{T}_1)$  and  $(X_2, \mathcal{T}_2)$  be topological spaces and suppose  $X_1 \times X_2$  has the product topology. For each  $i = 1, 2$ , let  $A_i \subseteq X_i$ . Prove that:

$$(i) \overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}.$$

$$(ii) \text{int}(A_1 \times A_2) = \text{int}(A_1) \times \text{int}(A_2).$$

**Lemma 1:**

Let  $C_1, C_2, D_1, D_2$  be sets. Then  $(C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \cap D_2)$ .

**Proof:**

Note that:

$$\begin{aligned} (x, y) &\in (C_1 \times C_2) \cap (D_1 \times D_2) \\ \iff (x, y) &\in (C_1 \times C_2) \wedge (x, y) \in (D_1 \times D_2) \\ \iff (x \in C_1 \wedge y \in C_2) &\wedge (x \in D_1 \wedge y \in D_2) \\ \iff (x \in C_1 \wedge x \in D_1) &\wedge (y \in C_2 \wedge y \in D_2) \\ \iff x \in (C_1 \cap D_1) &\wedge y \in (C_2 \cap D_2) \\ \iff (x, y) &\in (C_1 \cap D_1) \times (C_2 \cap D_2) \end{aligned} \quad (4)$$

$$\therefore (C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \cap D_2)$$

**Proof (i):**

Take  $(x, y) \in \overline{A_1 \times A_2}$ , and take  $U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2$  such that  $x \in U_1$  and  $y \in U_2$ .

Then  $(x, y) \in U_1 \times U_2$  and  $U_1 \times U_2 \in \mathcal{T}_{\Pi}$  by the definition of the product topology (for finite products). But:

$$\begin{aligned}
(x, y) \in \overline{A_1 \times A_2} &\implies (U_1 \times U_2) \cap (A_1 \times A_2) \neq \emptyset \\
&\implies \exists (x', y') \in (U_1 \times U_2) \cap (A_1 \times A_2) \\
&\implies (x', y') \in (U_1 \cap A_1) \times (U_2 \cap A_2) \\
&\implies x' \in U_1 \cap A_1 \wedge y' \in U_2 \cap A_2 \\
&\implies U_1 \cap A_1 \neq \emptyset \wedge U_2 \cap A_2 \neq \emptyset \\
&\implies x \in \overline{A_1} \wedge y \in \overline{A_2} \\
&\implies (x, y) \in \overline{A_1} \times \overline{A_2}
\end{aligned} \tag{5}$$

$$\therefore \overline{A_1 \times A_2} \subseteq \overline{A_1} \times \overline{A_2}$$

Similarly, take  $(a, b) \in \overline{A_1} \times \overline{A_2}$ , and take  $U_1 \times U_2 \in \mathcal{T}_\Pi$  such that  $(a, b) \in U_1 \times U_2$ .

Then  $a \in U_1 \in \mathcal{T}_1$  and  $b \in U_2 \in \mathcal{T}_2$  by the definition of the product topology (for finite products).

But:

$$\begin{aligned}
(a, b) \in \overline{A_1} \times \overline{A_2} &\implies a \in \overline{A_1} \wedge b \in \overline{A_2} \\
&\implies U_1 \cap A_1 \neq \emptyset \wedge U_2 \cap A_2 \neq \emptyset \\
&\implies \exists a' \in U_1 \cap A_1 \wedge \exists b' \in U_2 \cap A_2 \\
&\implies (a', b') \in (U_1 \cap A_1) \times (U_2 \cap A_2) \\
&\implies (a', b') \in (U_1 \times U_2) \cap (A_1 \times A_2) \\
&\implies (U_1 \times U_2) \cap (A_1 \times A_2) \neq \emptyset \\
&\implies (a, b) \in \overline{A_1 \times A_2}
\end{aligned} \tag{6}$$

$$\therefore \overline{A_1} \times \overline{A_2} \subseteq \overline{A_1 \times A_2}$$

$$\therefore \overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$$

**MEP**

**Proof (ii):**

Note that, for any subset  $B$  of a topological space  $(Y, \mathcal{T}_Y)$ , we have that

$z \in \text{int}(B) \iff \exists U \in \mathcal{T}_Y$  such that  $z \in U \subseteq B$ . Here's a brief proof:

Take  $z \in \text{int}(B) = \bigcup \{V \in \mathcal{T}_Y \mid V \subseteq B\} \subseteq B$ . This proves  $(\implies)$ . Now suppose  $\exists U \in \mathcal{T}_Y$  such that  $z \in U \subseteq B$ , then  $U \in \bigcup \{V \in \mathcal{T}_Y \mid V \subseteq B\} = \text{int}(B)$ . This proves  $(\impliedby)$ .

Now for the main proof. Take  $(x, y) \in \text{int}(A_1 \times A_2)$ , then  $\exists U_1 \times U_2 \in \mathcal{T}_\Pi$  such that  $(x, y) \in U_1 \times U_2 \subseteq A_1 \times A_2 \implies x \in U_1 \subseteq A_1$  and  $y \in U_2 \subseteq A_2 \implies x \in \text{int}(A_1)$  and  $y \in \text{int}(A_2)$ . Then  $(x, y) \in \text{int}(A_1) \times \text{int}(A_2)$ .

$$\therefore \text{int}(A_1 \times A_2) \subseteq \text{int}(A_1) \times \text{int}(A_2).$$

Now take  $(a, b) \in \text{int}(A_1) \times \text{int}(A_2)$ , then  $a \in \text{int}(A_1)$  and  $b \in \text{int}(A_2) \implies \exists U_1 \in \mathcal{T}_1$  such that  $a \in U_1 \subseteq A_1$  and  $\exists U_2 \in \mathcal{T}_2$  such that  $b \in U_2 \subseteq A_2$ . Then  $(a, b) \in U_1 \times U_2 \subseteq A_1 \times A_2$ , which implies that  $(a, b) \in \text{int}(A_1 \times A_2)$ .

$$\therefore \text{int}(A_1) \times \text{int}(A_2) \subseteq \text{int}(A_1 \times A_2).$$

$$\therefore \text{int}(A_1 \times A_2) = \text{int}(A_1) \times \text{int}(A_2).$$

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