Departamento de Matemáticas
Recinto de Río Piedras

Topology Qualifying Exam — August 2018

Do exactly **five** *of the following problems. In order to obtain credit you must show all your work.*
The passing grade at the M.S. level is 2/3 and at the Ph.D. level is 3/4.

Problem 1. A topological space $(X, \mathcal{T}_X)$ is *pseudocompact* $\iff$ every continuous function $f : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\mathcal{E}^1})$ is bounded. Here $\mathcal{T}_{\mathcal{E}^1}$ is the usual topology over $\mathbb{R}$.

(i) (*8 points*) Show that pseudocompactness is a continuous invariant. Explain.

(ii) (*12 points*) Show that if $(Y, \mathcal{T}_Y)$ is compact then $(Y, \mathcal{T}_Y)$ is pseudocompact, but that the converse does not hold.

Problem 2. (*20 points*) (Kuratowski's closure operator) Let X be a set, $\mathcal{P}(X)$ be its power set and $c : \mathcal{P}(X) \to \mathcal{P}(X)$ be a function that satisfies:

(i) $c(\varnothing) = \varnothing$.

(ii) $A \subseteq c(A)$, for all $A \in \mathcal{P}(X)$.

(iii) $c(c(A)) = c(A)$, for all $A \in \mathcal{P}(X)$.

(iv) $c(A \cup B) = c(A) \cup c(B)$, for all $A, B \in \mathcal{P}(X)$.

Show that the collection $\mathcal{T} = \{X - c(A) \mid A \in \mathcal{P}(X)\}$ is a topology over X and that in this topology $\bar{A} = c(A)$, for all $A \in \mathcal{P}(X)$. Here $\bar{A}$ is the closure of A in $(X, \mathcal{T})$.

Problem 3. (*20 points*) Let A be a subset of the topological space $(X, \mathcal{T}_X)$. Show that the following are equivalent:

(i) $\mathtt{int}(\overline{A}) = \varnothing$.

(ii) $X - \overline{A}$ is dense in X.

(iii) $X - \overline{(X - \overline{A})} = \varnothing$.

(iv) $A \subseteq \overline{(X - \overline{A})}$.

Problem 4. (*20 points*) A subset of a topological space is a G_δ**-set**, if it is the intersection of countably many open sets. On the other hand, a subset of a topological space is an F_σ**-set**, if it is the union of countably many closed sets.

(i) Let A be an F_σ-set of the topological space $(X, \mathcal{T}_X)$. Show that there is a nested sequence of closed sets $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$ such that $A = \bigcup_{i=1}^{\infty} C_i$.

(ii) Show that every closed set in a metric space (X, d) is a G_δ-set.

Problem 5. (*20 points*) Let A, B be two non-empty subsets of $\mathbb{R}$, with the usual topology. Define

$$C = \{x + y : x \in A \text{ and } y \in B\}.$$

(a) Show that, if A or B is open, then C is open.
(b) Show that, if A and B are compact, then C is compact.

Problem 6. (*20 points*) (Intermediate value theorem) Let $f : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\mathcal{E}^1})$ be a continuous function, where $(X, \mathcal{T}_X)$ is connected. Show that if a, b are two points in X and if r is a real number lying between $f(a)$ and $f(b)$ then there is a $c \in X$ such that $f(c) = r$.

Problem 7. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and suppose $X_1 \times X_2$ has the product topology. For each $i = 1, 2$ let $A_i \subseteq X_i$. Prove that:

(i) (*10 points*) $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$.

(ii) (*10 points*) $\texttt{int}\,(A_1 \times A_2) = \texttt{int}\,(A_1) \times \texttt{int}\,(A_2)$. Here $\texttt{int}\,(S)$ is the interior of the set S.