

MATE 6540: Topology Qualifying Exam

Due on 19 de mayo

Prof. Iván Cardona , C41, 19 de mayo

Sergio Rodríguez

Problem 0

A topological space (X, \mathcal{T}_X) is pseudocompact \iff every continuous function $f : (X, \mathcal{T}_X) \rightarrow (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ is bounded. Here $\mathcal{T}_{\varepsilon^1}$ is the usual topology over \mathbb{R} .

(i) Show that pseudocompactness is a continuous invariant. Explain.

(ii) Show that if (Y, \mathcal{T}_Y) is compact, then (Y, \mathcal{T}_Y) is pseudocompact, but that the converse does not hold.

Proof (i):

Let (X, \mathcal{T}_x) be a pseudocompact topological space, and let $\varphi : (X, \mathcal{T}_X) \rightarrow (\varphi(X), \mathcal{T}_{\varphi(X)})$ be a continuous function. Now let $f : (\varphi(X), \mathcal{T}_{\varphi(X)}) \rightarrow (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ be a continuous function. Then $(f \circ \varphi) : (X, \mathcal{T}_X) \rightarrow (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ is continuous. But (X, \mathcal{T}_X) is pseudocompact, so $f \circ \varphi$ is bounded $\implies f$ is bounded. Then $(\varphi(X), \mathcal{T}_{\varphi(X)})$ is pseudocompact.

\therefore pseudocompactness is a continuous invariant.

MEP**Proof (ii):**

We claim that (Y, \mathcal{T}_Y) compact $\implies (Y, \mathcal{T}_Y)$ pseudocompact, we show the contrapositive.

Suppose that (Y, \mathcal{T}_Y) is not pseudocompact, then there exists a continuous function $f : (Y, \mathcal{T}_Y) \rightarrow (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ that is unbounded. Then $f(Y)$ extends infinitely in at least one direction. Suppose, without loss of generality, that it extends infinitely to the right. Then $f(Y) \subseteq (a, +\infty)$, for some $a \in \mathbb{R} \cup \{-\infty\}$. Now define the family $\{(a, i)\}_{i=1}^{\infty}$ and note that: $\bigcup_{i=1}^{\infty} (a, i) = (a, +\infty) \supseteq f(Y)$. But note that $(a, i) \in \mathcal{T}_{\varepsilon^1}, \forall i \in \mathbb{N} \implies f^{-1}((a, i)) \in \mathcal{T}_Y, \forall i \in \mathbb{N}$ because f is continuous. And:

$$\bigcup_{i=1}^{\infty} f^{-1}((a, i)) = f^{-1}\left(\bigcup_{i=1}^{\infty} (a, i)\right) = f^{-1}((a, +\infty)) \supseteq f^{-1}(f(Y)) \supseteq Y \quad (1)$$

This implies that the collection $C := \{f^{-1}((a, i)) \cap Y \mid i \in \mathbb{N}\}$ is an open cover for Y , but note that, for any finite subcollection $C' := \{f^{-1}((a, i_j)) \mid i_j \in \mathbb{N}, \forall j \in \{1, \dots, n\}\} \subseteq C$, there exists a natural number n such that i_n is an upper bound of C' . But $f(Y)$ extends infinitely to the right, so C' cannot be an open cover for Y .

$\therefore (Y, \mathcal{T}_Y)$ is not compact.

Now we show that the converse does not hold.

Consider the collection: $\mathcal{T} := \{U \subseteq \mathbb{R} \mid U = \emptyset \vee 0 \in U\}$. Note that $\emptyset \in \mathcal{T}$ by construction and $0 \in \mathbb{R} \implies \mathbb{R} \in \mathcal{T}$, and that arbitrary unions and finite intersections of sets containing 0 also contain 0. Then \mathcal{T} is a topology over \mathbb{R} . Suppose $f : (\mathbb{R}, \mathcal{T}) \rightarrow (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ is a continuous function. Then, since $\mathbb{R} \setminus \{f(0)\} = (-\infty, 0) \cup (0, +\infty) \in \mathcal{T}_{\varepsilon^1}$, we have that $A := f^{-1}(\mathbb{R} \setminus f(\{0\})) \in \mathcal{T}$. But note that $A = f^{-1}(\mathbb{R} \setminus f(\{0\})) = f^{-1}(\mathbb{R}) \setminus f^{-1}(f(\{0\})) = \mathbb{R} \setminus f^{-1}(f(\{0\}))$, and $\{0\} \subseteq f^{-1}(f(\{0\}))$, it then follows that $0 \notin A$, but $A \in \mathcal{T}$, therefore $A = f^{-1}(\mathbb{R} \setminus f(\{0\})) = \emptyset$. This implies that f doesn't ever map to anything other than 0, so $f(x) = 0$, which is a bounded function.

$\therefore (\mathbb{R}, \mathcal{T})$ is pseudocompact.

Consider the collection $D := \{(-i, i) \mid i \in \mathbb{N}\}$. Note that $\forall i \in \mathbb{N}, 0 \in (-i, i) \implies (-i, i) \in \mathcal{T}$. Also, $\bigcup_{i=1}^{\infty} (-i, i) = \mathbb{R}$. Therefore, D is an open cover for \mathbb{R} . Now take a finite subcollection $D' := \{(-i_j, i_j) \mid i_j \in \mathbb{N}, \forall j \in \{1, \dots, n\}\} \subseteq D$. But clearly D' does not cover \mathbb{R} .

$\therefore (\mathbb{R}, \mathcal{T})$ is not compact.

MEP

Problem 1

(Kuratowski's closure operator) Let X be a set, $\mathcal{P}(X)$ be its powerset, and $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ be a function that satisfies:

(i) $c(\emptyset) = \emptyset$

(ii) $A \subseteq c(A), \forall A \in \mathcal{P}(X)$

(iii) $c(c(A)) = c(A), \forall A \in \mathcal{P}(X)$

(iv) $c(A \cup B) = c(A) \cup c(B), \forall A, B \in \mathcal{P}(X)$

Show that the collection $\mathcal{T} = \{X \setminus c(A) \mid A \in \mathcal{P}(X)\}$ is a topology over X , and that in this topology $\overline{A} = c(A), \forall A \in \mathcal{P}(X)$. Here \overline{A} is the closure of A in (X, \mathcal{T}) .

Proof:

We claim that $\emptyset, X \in \mathcal{T}$.

Note that $c(X) \in \mathcal{P}(X)$ and $X \in c(X)$ imply that $c(X) = X$. Then $X \setminus c(X) = X \setminus X = \emptyset \in \mathcal{T}$. Similarly, note that $c(\emptyset) = \emptyset \implies X \setminus c(\emptyset) = X \setminus \emptyset = X \in \mathcal{T}$.

$\therefore \emptyset, X \in \mathcal{T}$.

We claim that \mathcal{T} is closed under arbitrary unions.

Take $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{T}_X$. We show that $U := \bigcup_{\alpha \in \Lambda} U_\alpha \in \mathcal{T}_X$. Note that $\forall \alpha \in \Lambda, \exists V_\alpha \in \mathcal{P}(X)$ such that $U_\alpha = X \setminus c(V_\alpha)$. Then $U = \bigcup_{\alpha \in \Lambda} (X \setminus c(V_\alpha)) = X \setminus \bigcap_{\alpha \in \Lambda} c(V_\alpha)$

We claim that \mathcal{T} is closed under finite intersections.

Take $U, V \in \mathcal{T}$, then $\exists A, B \in \mathcal{P}(X)$ such that $U = X \setminus c(A)$ and $V = X \setminus c(B)$. Then:

$$\begin{aligned}
U \cap V &= (X \setminus c(A)) \cap (X \setminus c(B)) \\
&= X \setminus (c(A) \cup c(B)) \\
&= X \setminus c(A \cup B) \in \mathcal{T}
\end{aligned} \tag{2}$$

$\therefore U \cap V \in \mathcal{T}$.

$\therefore \mathcal{T}$ is closed under finite intersections, by induction.

We claim that $\overline{A} = c(A)$, $\forall A \in \mathcal{P}(X)$.

Problem 2

Let A be a subset of a topological space (X, \mathcal{T}_X) . Show that the following are equivalent:

- (i) $\text{int}(\overline{A}) = \emptyset$.
- (ii) $X \setminus \overline{A}$ is dense in X .
- (iii) $X \setminus \overline{(X \setminus \overline{A})} = \emptyset$.
- (iv) $A \subseteq \overline{(X \setminus \overline{A})}$.

Proof:

$((i) \implies (ii))$

Suppose that $\text{int}(\overline{A}) = \bigcup \{U \in \mathcal{T}_X \mid U \subseteq \overline{A}\} = \emptyset$. Note that $\overline{(X \setminus \overline{A})} \subseteq X$. We claim that $X \subseteq \overline{(X \setminus \overline{A})}$. Take $x \in X$ and $U \in \mathcal{T}_X$ such that $x \in U$. Then $U \neq \emptyset$. Now suppose that $U \subseteq \overline{A}$, then $U \subseteq \bigcup \{U \in \mathcal{T}_X \mid U \subseteq \overline{A}\} = \emptyset \implies U = \emptyset$, which is a contradiction. ✖

Then $U \cap (X \setminus \overline{A}) \neq \emptyset \implies x \in \overline{(X \setminus \overline{A})} \implies X \subseteq \overline{(X \setminus \overline{A})}$.

$\therefore \overline{(X \setminus \overline{A})} = X$.

$\therefore X \setminus \overline{A}$ is dense in X .

$((ii) \implies (iii))$

Suppose that $X \setminus \overline{A}$ is dense in X , then $\overline{(X \setminus \overline{A})} = X$ by definition.

Then $X \setminus \overline{(X \setminus \overline{A})} = X \setminus X = \emptyset$.

$\therefore X \setminus \overline{(X \setminus \overline{A})} = \emptyset$.

$((iii) \implies (iv))$

Suppose that $X \setminus \overline{(X \setminus \overline{A})} = \emptyset$. Then $\overline{(X \setminus \overline{A})} = X$. But $A \subseteq X$ by hypothesis.

$$\therefore A \subseteq \overline{(X \setminus \overline{A})}$$

((iv) \implies (i))

Suppose that $A \subseteq \overline{(X \setminus \overline{A})}$. Now suppose, by contradiction, that $\text{int}(\overline{A}) \neq \emptyset$. Then $\exists x \in \text{int}(\overline{A}) \implies \exists U \in \mathcal{T}_X$ such that $x \in U \subseteq \overline{A}$. But $x \in \overline{A}$ and U neighborhood of x imply that $U \cap A \neq \emptyset$. So $\exists y \in U$ such that $y \in A \subseteq \overline{(X \setminus \overline{A})}$. But now $y \in \overline{(X \setminus \overline{A})}$ and U neighborhood of y imply that $U \cap (X \setminus \overline{A}) \neq \emptyset$. But this contradicts the fact that $U \subseteq \overline{A}$. ✖

$$\therefore \text{int}(\overline{A}) = \emptyset.$$

MEP

Problem 3

A subset of a topological space is a G_δ -set if it is the intersection of countably many open sets. On the other hand, a subset of a topological space is an F_σ -set if it is the union of countably many closed sets.

(i) Let A be an F_σ -set of a topological space (X, \mathcal{T}_X) . Show that there is a nested sequence of closed sets $C_1 \subseteq C_2 \subseteq C_3 \subseteq \dots$ such that $A = \bigcup_{i=1}^{\infty} C_i$.

(ii) Show that every closed set in a metric space (X, d) is a G_δ -set.

Proof (i):

Problem 4

Let A, B be two non-empty subsets of \mathbb{R} with the usual topology. Define:

$$C := \{x + y \mid x \in A \wedge y \in B\}. \quad (3)$$

(a) Show that, if A or B is open, then C is open.

(b) Show that, if A and B are compact, then C is compact.

Proof (a):

Suppose, without loss of generality, that $A \in \mathcal{T}_{\mathbb{R}}$. Now take $(x + y) \in C$, then $x \in A$, which is open, so $\exists \delta \in (0, \infty)$ such that $(x - \delta, x + \delta) \subseteq A$.

We claim that $D := ((x + y) - \delta, (x + y) + \delta) \subseteq C$.

Take $d \in D$, then:

$$\begin{aligned}
& (x + y) - \delta < d < (x + y) + \delta \\
& \implies x - \delta < d - y < x + \delta \\
& \implies (d - y) \in (x - \delta, x + \delta) \subseteq A \\
& \implies ((d - y) + y) \in C \\
& \implies d \in C
\end{aligned} \tag{4}$$

$\therefore D \subseteq C$

$\therefore C$ is open.

MEP

Problem 5

(Intermediate value theorem) Let $f : (X, \mathcal{T}_X) \rightarrow (\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ be a continuous function, where (X, \mathcal{T}_X) is connected. Show that if a, b are two points in X and if r is a real number lying between $f(a)$ and $f(b)$, then there is a $c \in X$ such that $f(c) = r$.

Proof:

Note that $f^{-1}(\mathbb{R}) = X$. Now, suppose, by contradiction, that $\nexists c \in X$ such that $f(c) = d$. Then the previous equation still holds when you remove d from \mathbb{R} . That is:

$$f^{-1}(\mathbb{R} \setminus \{d\}) = f^{-1}((-\infty, d) \cup (d, +\infty)) = X \tag{5}$$

Now, let $A := f^{-1}((-\infty, d))$, and $B := f^{-1}((d, +\infty))$, and note that:

$$\begin{aligned}
\text{(i)} \quad & a \in A \implies A \neq \emptyset \wedge b \in B \implies B \neq \emptyset \\
\text{(ii)} \quad & A \cup B = f^{-1}((-\infty, d)) \cup f^{-1}((d, +\infty)) \\
& = f^{-1}((-\infty, d) \cup (d, +\infty)) = X \\
\text{(iii)} \quad & A \cap B = f^{-1}((-\infty, d)) \cap f^{-1}((d, +\infty)) \\
& = f^{-1}((-\infty, d) \cap (d, +\infty)) = f^{-1}(\emptyset) = \emptyset
\end{aligned} \tag{6}$$

Then $\{A, B\}$ forms a separation for X , which is a contradiction. ✖

$\therefore \exists c \in \mathbb{R}$ such that $f(c) = d$.

MEP

Problem 6

Let (X_1, \mathcal{T}_1) and (X_2, \mathcal{T}_2) be topological spaces and suppose $X_1 \times X_2$ has the product topology. For each $i = 1, 2$, let $A_i \subseteq X_i$. Prove that:

$$(i) \quad \overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}.$$

$$(ii) \quad \text{int}(A_1 \times A_2) = \text{int}(A_1) \times \text{int}(A_2).$$

Lemma 1:

Let C_1, C_2, D_1, D_2 be sets. Then $(C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \cap D_2)$.

Proof:

Note that:

$$\begin{aligned}
 (x, y) &\in (C_1 \times C_2) \cap (D_1 \times D_2) \\
 \iff (x, y) &\in (C_1 \times C_2) \wedge (x, y) \in (D_1 \times D_2) \\
 \iff (x \in C_1 \wedge y \in C_2) &\wedge (x \in D_1 \wedge y \in D_2) \\
 \iff (x \in C_1 \wedge x \in D_1) &\wedge (y \in C_2 \wedge y \in D_2) \\
 \iff x \in (C_1 \cap D_1) &\wedge y \in (C_2 \cap D_2) \\
 \iff (x, y) &\in (C_1 \cap D_1) \times (C_2 \cap D_2)
 \end{aligned} \tag{7}$$

$$\therefore (C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \cap D_2)$$

Proof (i):

Take $(x, y) \in \overline{A_1 \times A_2}$, and take $U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2$ such that $x \in U_1$ and $y \in U_2$.

Then $(x, y) \in U_1 \times U_2$ and $U_1 \times U_2 \in \mathcal{T}_{\Pi}$ by the definition of the product topology (for finite products). But:

$$\begin{aligned}
 (x, y) \in \overline{A_1 \times A_2} &\implies (U_1 \times U_2) \cap (A_1 \times A_2) \neq \emptyset \\
 &\implies \exists (x', y') \in (U_1 \times U_2) \cap (A_1 \times A_2) \\
 &\implies (x', y') \in (U_1 \cap A_1) \times (U_2 \cap A_2) \\
 &\implies x' \in U_1 \cap A_1 \wedge y' \in U_2 \cap A_2 \\
 &\implies U_1 \cap A_1 \neq \emptyset \wedge U_2 \cap A_2 \neq \emptyset \\
 &\implies x \in \overline{A_1} \wedge y \in \overline{A_2} \\
 &\implies (x, y) \in \overline{A_1} \times \overline{A_2}
 \end{aligned} \tag{8}$$

$$\therefore \overline{A_1 \times A_2} \subseteq \overline{A_1} \times \overline{A_2}$$

Similarly, take $(a, b) \in \overline{A_1} \times \overline{A_2}$, and take $U_1 \times U_2 \in \mathcal{T}_{\Pi}$ such that $(a, b) \in U_1 \times U_2$.

Then $a \in U_1 \in \mathcal{T}_1$ and $b \in U_2 \in \mathcal{T}_2$ by the definition of the product topology (for finite products).

But:

$$\begin{aligned}
 (a, b) \in \overline{A_1} \times \overline{A_2} &\implies a \in \overline{A_1} \wedge b \in \overline{A_2} \\
 &\implies U_1 \cap A_1 \neq \emptyset \wedge U_2 \cap A_2 \neq \emptyset \\
 &\implies \exists a' \in U_1 \cap A_1 \wedge \exists b' \in U_2 \cap A_2 \\
 &\implies (a', b') \in (U_1 \cap A_1) \times (U_2 \cap A_2) \\
 &\implies (a', b') \in (U_1 \times U_2) \cap (A_1 \times A_2) \\
 &\implies (U_1 \times U_2) \cap (A_1 \times A_2) \neq \emptyset \\
 &\implies (a, b) \in \overline{A_1 \times A_2}
 \end{aligned} \tag{9}$$

$$\therefore \overline{A_1 \times A_2} \subseteq \overline{A_1} \times \overline{A_2}$$

$$\therefore \overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$$

MEP

Proof (ii):

Note that, for any subset B of a topological space (Y, \mathcal{T}_Y) , we have that

$z \in \text{int}(B) \iff \exists U \in \mathcal{T}_Y$ such that $z \in U \subseteq B$. Here's a brief proof:

Take $z \in \text{int}(B) = \bigcup \{V \in \mathcal{T}_Y \mid V \subseteq B\} \subseteq B$. This proves (\implies) . Now suppose $\exists U \in \mathcal{T}_Y$ such that $z \in U \subseteq B$, then $U \in \bigcup \{V \in \mathcal{T}_Y \mid V \subseteq B\} = \text{int}(B)$. This proves (\impliedby) .

Now for the main proof. Take $(x, y) \in \text{int}(A_1 \times A_2)$, then $\exists U_1 \times U_2 \in \mathcal{T}_\Pi$ such that $(x, y) \in U_1 \times U_2 \subseteq A_1 \times A_2 \implies x \in U_1 \subseteq A_1$ and $y \in U_2 \subseteq A_2 \implies x \in \text{int}(A_1)$ and $y \in \text{int}(A_2)$. Then $(x, y) \in \text{int}(A_1) \times \text{int}(A_2)$.

$$\therefore \text{int}(A_1 \times A_2) \subseteq \text{int}(A_1) \times \text{int}(A_2).$$

Now take $(a, b) \in \text{int}(A_1) \times \text{int}(A_2)$, then $a \in \text{int}(A_1)$ and $b \in \text{int}(A_2) \implies \exists U_1 \in \mathcal{T}_1$ such that $a \in U_1 \subseteq A_1$ and $\exists U_2 \in \mathcal{T}_2$ such that $b \in U_2 \subseteq A_2$. Then $(a, b) \in U_1 \times U_2 \subseteq A_1 \times A_2$, which implies that $(a, b) \in \text{int}(A_1 \times A_2)$.

$$\therefore \text{int}(A_1) \times \text{int}(A_2) \subseteq \text{int}(A_1 \times A_2).$$

$$\therefore \text{int}(A_1 \times A_2) = \text{int}(A_1) \times \text{int}(A_2).$$

MEP