MATE 6540: Topology Qualifying Exam

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Problem 0

A topological space $(X, \mathcal{T}_X)$ is pseudocompact $\iff$ every continuous function $f : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ is bounded. Here $\mathcal{T}_{\varepsilon^1}$ is the usual topology over $\mathbb{R}$.

(i) Show that pseudocompactness is a continuous invariant. Explain.

(ii) Show that if $(Y, \mathcal{T}_Y)$ is compact, then $(Y, \mathcal{T}_Y)$ is pseudocompact, but that the converse does not hold.

Proof (i):

Let $(X, \mathcal{T}_x)$ be a pseudocompact topological space, and let $\varphi : (X, \mathcal{T}_X) \to \left(\varphi(X), \mathcal{T}_{\varphi(X)}\right)$ be a continuous function. Now let $f : \left(\varphi(X), \mathcal{T}_{\varphi(X)}\right) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ be a continuous funcion. Then $(f \circ \varphi) : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ is continuous. But $(X, \mathcal{T}_X)$ is pseudocompact, so $f \circ \varphi$ is bounded $\implies f$ is bounded. Then $\left(\varphi(X), \mathcal{T}_{\varphi(X)}\right)$ is pseudocompact.

$\therefore$ pseudocompactness is a continuous invariant.

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Proof (ii):

We claim that $(Y, \mathcal{T}_Y)$ compact $\implies (Y, \mathcal{T}_Y)$ pseudocompact, we show the contrapositive.

Suppose that $(Y, \mathcal{T}_Y)$ is not pseudocompact, then there exists a continuous function $f : (Y, \mathcal{T}_Y) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ that is unbounded. Then $f(Y)$ extends infinitely in at least one direction. Suppose, without loss of generality, that it extends infinitely to the right. Then $f(Y) \subseteq (a, +\infty)$, for some $a \in \mathbb{R} \cup \{-\infty\}$. Now define the family $\{(a, i)\}_{n=1}^{\infty}$ and note that: $\bigcup_{i=1}^{\infty}(a, i) = (a, +\infty) \supseteq f(Y)$. But note that $(a, i) \in \mathcal{T}_{\varepsilon^1}, \forall i \in \mathbb{N} \implies f^{-1}((a, i)) \in \mathcal{T}_Y, \forall i \in \mathbb{N}$ because f is continuous. And:

$$\bigcup_{i=1}^{\infty} f^{-1}((a, i)) = f^{-1}\left(\bigcup_{i=1}^{\infty}(a, i)\right) = f^{-1}((a, +\infty)) \supseteq f^{-1}(f(Y)) \supseteq Y \tag{1}$$

This implies that the collection $C := \{f^{-1}((a, i)) \cap Y \mid i \in \mathbb{N}\}$ is an open cover for Y, but note that, for any finite subcollection $C' := \left\{f^{-1}\left((a, i_j)\right) \mid i_j \in \mathbb{N}, \quad \forall j \in \{1, ..., n\}\right\} \subseteq C$, there exists a natural number n such that i_n is an upper bound of C'. But $f(Y)$ extends infinitely to the right, so C' cannot be an open cover for Y.

$\therefore (Y, \mathcal{T}_Y)$ is not compact.

Now we show that the converse does not hold.

Problem 1

(Kuratowski's closure operator) Let X be a set, $\mathcal{P}(X)$ be its powerset, and $c : \mathcal{P}(X) \to \mathcal{P}(X)$ be a function that satisfies:

(i) $c(\emptyset) = \emptyset$

(ii) $A \subseteq c(A), \quad \forall A \in \mathcal{P}(X)$

(iii) $c(c(A)) = c(A), \quad \forall A \in \mathcal{P}(X)$

(iv) $c(A \cup B) = c(A) \cup c(B), \quad \forall A, B \in \mathcal{P}(X)$

Show that the collection $\mathcal{T} = \{X \setminus c(A) \mid A \in \mathcal{P}(X)\}$ *is a topology over* X*, and that in this topology* $\overline{A} = c(A), \quad \forall A \in \mathcal{P}(X)$*. Here* $\overline{A}$ *is the closure of* A *in* $(X, \mathcal{T})$*.*

Problem 2

Let A *be a subset of a topological space* $(X, \mathcal{T}_X)$*. Show that the following are equivalent:*

(i) $\mathtt{int}\left(\overline{A}\right) = \emptyset$.

(ii) $X \setminus \overline{A}$ *is dense in* X.

(iii) $X \setminus \overline{\left(X \setminus \overline{A}\right)} = \emptyset$.

(iv) $A \subseteq \overline{\left(X \setminus \overline{A}\right)}$.

Problem 3

A subset of a topological space is a G_δ*-set if it is the intersection of countably many open sets. On the other hand, a subset of a topological space is an* F_δ*-set if it is the union of countably many closed sets.*

(i) Let A *be an* F_δ*-set of a topological space* $(X, \mathcal{T}_X)$*. Show that there is a nested sequence of closed sets* $C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots$ *such that* $A = \bigcup_{i=1}^{\infty} C_i$*.*

(ii) Show that every closed set in a metric space (X, d) *is a* G_δ*-set.*

Problem 4

Let A, B *be two non-empty subsets of* $\mathbb{R}$ *with the usual topology. Define:*

$$C := \{x + y \mid x \in A \wedge y \in B\}. \tag{2}$$

(a) Show that, if A *or* B *is open, then* C *is open.*

(b) Show that, if A *and* B *are compact, then* C *is compact.*

Problem 5

(Intermediate value theorem) Let $f : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ *be a continuous function, where* $(X, \mathcal{T}_X)$ *is connected. Show that if* a, b *are two points in* X *and if* r *is a real number lying between* $f(a)$ *and* $f(b)$*, then there is a* $c \in X$ *such that* $f(c) = r$*.*

Problem 6

Let $(X_1, \mathcal{T}_1)$ *and* $(X_2, \mathcal{T}_2)$ *be topological spaces and suppose* $X_1 \times X_2$ *has the product topology. For each* $i = 1, 2$*, let* $A_i \subseteq X_i$*. Prove that:*

(i) $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$.

(ii) $\texttt{int}(A_1 \times A_2) = \texttt{int}(A_1) \times \texttt{int}(A_2)$.

Lemma 1:

Let C_1, C_2, D_1, D_2 be sets. Then $(C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \cap D_2)$.

Proof:

Note that:

$$\begin{aligned} & (x,y) \in (C_1 \times C_2) \cap (D_1 \times D_2) \\ \iff & (x,y) \in (C_1 \times C_2) \land (x,y) \in (D_1 \times D_2) \\ \iff & (x \in C_1 \land y \in C_2) \land (x \in D_1 \land y \in D_2) \\ \iff & (x \in C_1 \land x \in D_1) \land (y \in C_2 \land y \in D_2) \\ \iff & x \in (C_1 \cap D_1) \land y \in (C_2 \cap D_2) \\ \iff & (x,y) \in (C_1 \cap D_1) \times (C_2 \cap D_2) \end{aligned} \tag{3}$$

$\therefore (C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \cap D_2)$

Proof (i):

Take $(x,y) \in \overline{A_1 \times A_2}$, and take $U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2$ such that $x \in U_1$ and $y \in U_2$.
Then $(x,y) \in U_1 \times U_2$ and $U_1 \times U_2 \in \mathcal{T}_\Pi$ by the definition of the product topology (for finite products). But:

$$\begin{aligned} (x,y) \in \overline{A_1 \times A_2} & \implies (U_1 \times U_2) \cap (A_1 \times A_2) \neq \emptyset \\ & \implies \exists (x', y') \in (U_1 \times U_2) \cap (A_1 \times A_2) \\ & \implies (x', y') \in (U_1 \cap A_1) \times (U_2 \cap A_2) \\ & \implies x' \in U_1 \cap A_1 \land y' \in U_2 \cap A_2 \\ & \implies U_1 \cap A_1 \neq \emptyset \land U_2 \cap A_2 \neq \emptyset \\ & \implies x \in \overline{A_1} \land y \in \overline{A_2} \\ & \implies (x,y) \in \overline{A_1} \times \overline{A_2} \end{aligned} \tag{4}$$

$\therefore \overline{A_1 \times A_2} \subseteq \overline{A_1} \times \overline{A_2}$

Similarly, take $(a,b) \in \overline{A_1} \times \overline{A_2}$, and take $U_1 \times U_2 \in \mathcal{T}_\Pi$ such that $(a,b) \in U_1 \times U_2$.
Then $a \in U_1 \in \mathcal{T}_1$ and $b \in U_2 \in \mathcal{T}_2$ by the definition of the product topology (for finite products).
But:

$$
\begin{aligned}
(a,b) \in \overline{A_1} \times \overline{A_2} &\Longrightarrow a \in \overline{A_1} \land b \in \overline{A_2} \\
&\Longrightarrow U_1 \cap A_1 \neq \emptyset \land U_2 \cap A_2 \neq \emptyset \\
&\Longrightarrow \exists a' \in U_1 \cap A_1 \land \exists b' \in U_2 \cap A_2 \\
&\Longrightarrow (a', b') \in (U_1 \cap A_1) \times (U_2 \cap A_2) \\
&\Longrightarrow (a', b') \in (U_1 \times U_2) \cap (A_1 \times A_2) \\
&\Longrightarrow (U_1 \times U_2) \cap (A_1 \times A_2) \neq \emptyset \\
&\Longrightarrow (a,b) \in \overline{A_1 \times A_2}
\end{aligned} \tag{5}
$$

$\therefore \overline{A_1} \times \overline{A_2} \subseteq \overline{A_1 \times A_2}$

$\therefore \overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$

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Proof (ii):

Note that, for any subset B of a topological space $(Y, \mathcal{T}_Y)$, we have that $z \in \texttt{int}(B) \iff \exists U \in \mathcal{T}_Y$ such that $z \in U \subseteq B$. Here's a brief proof:
Take $z \in \texttt{int}(B) = \bigcup\{V \in \mathcal{T}_Y \mid V \subseteq B\} \subseteq B$. This proves ($\Longrightarrow$). Now suppose $\exists U \in \mathcal{T}_Y$ such that $z \in U \subseteq B$, then $U \in \bigcup\{V \in \mathcal{T}_Y \mid V \subseteq B\} = \texttt{int}(B)$. This proves ($\Longleftarrow$).

Now for the main proof. Take $(x, y) \in \texttt{int}(A_1 \times A_2)$, then $\exists U_1 \times U_2 \in \mathcal{T}_\Pi$ such that $(x, y) \in U_1 \times U_2 \subseteq A_1 \times A_2 \Longrightarrow x \in U_1 \subseteq A_1$ and $y \in U_2 \subseteq A_2 \Longrightarrow x \in \texttt{int}(A_1)$ and $y \in \texttt{int}(A_2)$. Then $(x, y) \in \texttt{int}(A_1) \times \texttt{int}(A_2)$.

$\therefore \texttt{int}(A_1 \times A_2) \subseteq \texttt{int}(A_1) \times \texttt{int}(A_2)$.

Now take $(a, b) \in \texttt{int}(A_1) \times \texttt{int}(A_2)$, then $a \in \texttt{int}(A_1)$ and $b \in \texttt{int}(A_2) \Longrightarrow \exists U_1 \in \mathcal{T}_1$ such that $a \in U_1 \subseteq A_1$ and $\exists U_2 \in \mathcal{T}_2$ such that $b \in U_2 \subseteq A_2$. Then $(a, b) \in U_1 \times U_2 \subseteq A_1 \times A_2$, which implies that $(a, b) \in \texttt{int}(A_1 \times A_2)$.

$\therefore \texttt{int}(A_1) \times \texttt{int}(A_2) \subseteq \texttt{int}(A_1 \times A_2)$.

$\therefore \texttt{int}(A_1 \times A_2) = \texttt{int}(A_1) \times \texttt{int}(A_2)$.

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