MATE 6540: Topology Qualifying Exam

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Problem 0

A topological space $(X, \mathcal{T}_X)$ is pseudocompact $\iff$ every continuous function $f : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ is bounded. Here $\mathcal{T}_{\varepsilon^1}$ is the usual topology over $\mathbb{R}$.

(i) Show that pseudocompactness is a continuous invariant. Explain.

(ii) Show that if $(Y, \mathcal{T}_Y)$ is compact, then $(Y, \mathcal{T}_Y)$ is pseudocompact, but that the converse does not hold.

Proof (i):

Let $(X, \mathcal{T}_x)$ be a pseudocompact topological space, and let $\varphi : (X, \mathcal{T}_X) \to \left(\varphi(X), \mathcal{T}_{\varphi(X)}\right)$ be a continuous function. Now let $f : \left(\varphi(X), \mathcal{T}_{\varphi(X)}\right) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ be a continuous funcion. Then $(f \circ \varphi) : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ is continuous. But $(X, \mathcal{T}_X)$ is pseudocompact, so $f \circ \varphi$ is bounded $\implies f$ is bounded. Then $\left(\varphi(X), \mathcal{T}_{\varphi(X)}\right)$ is pseudocompact.

$\therefore$ pseudocompactness is a continuous invariant.

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Proof (ii):

We claim that $(Y, \mathcal{T}_Y)$ compact $\implies (Y, \mathcal{T}_Y)$ pseudocompact, we show the contrapositive.

Suppose that $(Y, \mathcal{T}_Y)$ is not pseudocompact, then there exists a continuous function $f : (Y, \mathcal{T}_Y) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ that is unbounded. Then $f(Y)$ extends infinitely in at least one direction. Suppose, without loss of generality, that it extends infinitely to the right. Then $f(Y) \subseteq (a, +\infty)$, for some $a \in \mathbb{R} \cup \{-\infty\}$. Now define the family $\{(a, i)\}_{n=1}^{\infty}$ and note that: $\bigcup_{i=1}^{\infty} (a, i) = (a, +\infty) \supseteq f(Y)$. But note that $(a, i) \in \mathcal{T}_{\varepsilon^1}, \forall i \in \mathbb{N} \implies f^{-1}((a, i)) \in \mathcal{T}_Y, \forall i \in \mathbb{N}$ because f is continuous. And:

$$\bigcup_{i=1}^{\infty} f^{-1}((a, i)) = f^{-1}\left(\bigcup_{i=1}^{\infty} (a, i)\right) = f^{-1}((a, +\infty)) \supseteq f^{-1}(f(Y)) \supseteq Y \tag{1}$$

This implies that the collection $C := \{f^{-1}((a, i)) \cap Y \mid i \in \mathbb{N}\}$ is an open cover for Y, but note that, for any finite subcollection $C' := \left\{f^{-1}\left((a, i_j)\right) \mid i_j \in \mathbb{N}, \quad \forall j \in \{1, ..., n\}\right\} \subseteq C$, there exists a natural number n such that i_n is an upper bound of C'. But $f(Y)$ extends infinitely to the right, so C' cannot be an open cover for Y.

$\therefore (Y, \mathcal{T}_Y)$ is not compact.

Now we show that the converse does not hold.

Consider the collection: $\mathcal{T} := \{U \subseteq \mathbb{R} \mid U = \emptyset \vee 0 \in U\}$. Note that $\emptyset \in \mathcal{T}$ by construction and $0 \in \mathbb{R} \implies \mathbb{R} \in \mathcal{T}$, and that arbitrary unions and finite intersections of sets containing 0 also contain 0. Then $\mathcal{T}$ is a topology over $\mathbb{R}$. Suppose $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ is a continuous function. Then, since $\mathbb{R} \setminus \{f(0)\} = (-\infty, 0) \cup (0, +\infty) \in \mathcal{T}_{\varepsilon^1}$, we have that $A := f^{-1}(\mathbb{R} \setminus f(\{0\})) \in \mathcal{T}$. But note that $A = f^{-1}(\mathbb{R} \setminus f(\{0\})) = f^{-1}(\mathbb{R}) \setminus f^{-1}(f(\{0\})) = \mathbb{R} \setminus f^{-1}(f(\{0\}))$, and $\{0\} \subseteq f^{-1}(f(\{0\}))$, it then follows that $0 \notin A$, but $A \in \mathcal{T}$, therefore $A = f^{-1}(\mathbb{R} \setminus f(\{0\})) = \emptyset$. This implies that f doesn't ever map to anything other than 0, so $f(x) = 0$, which is a bounded function.

$\therefore$ $(\mathbb{R}, \mathcal{T})$ is pseudocompact.

Consider the collection $D := \{(-i, i) \mid i \in \mathbb{N}\}$. Note that $\forall i \in \mathbb{N}, \quad 0 \in (-i, i) \Longrightarrow (-i, i) \in \mathcal{T}$. Also, $\bigcup_{i=1}^{\infty} (-i, i) = \mathbb{R}$. Therefore, D is an open cover for $\mathbb{R}$. Now take a finite subcollection $D' := \left\{\left(-i_j, i_j\right) \mid i_j \in \mathbb{N}, \quad \forall j \in \{1, ..., n\}\right\} \subseteq D$. But clearly D' does not cover $\mathbb{R}$.

$\therefore$ $(\mathbb{R}, \mathcal{T})$ is not compact.

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Problem 1

(Kuratowski's closure operator) Let X be a set, $\mathcal{P}(X)$ be its powerset, and $c : \mathcal{P}(X) \to \mathcal{P}(X)$ be a function that satisfies:

(i) $c(\emptyset) = \emptyset$

(ii) $A \subseteq c(A), \quad \forall A \in \mathcal{P}(X)$

(iii) $c(c(A)) = c(A), \quad \forall A \in \mathcal{P}(X)$

(iv) $c(A \cup B) = c(A) \cup c(B), \quad \forall A, B \in \mathcal{P}(X)$

Show that the collection $\mathcal{T} = \{X \setminus c(A) \mid A \in \mathcal{P}(X)\}$ is a topology over X, and that in this topology $\overline{A} = c(A), \quad \forall A \in \mathcal{P}(X)$. Here $\overline{A}$ is the closure of A in $(X, \mathcal{T})$.

Proof:

We claim that $\emptyset, X \in \mathcal{T}$.

Note that $c(X) \in \mathcal{P}(X)$ and $X \in c(X)$ imply that $c(X) = X$. Then $X \setminus c(X) = X \setminus X = \emptyset \in \mathcal{T}$. Similarly, note that $c(\emptyset) = \emptyset \Longrightarrow X \setminus c(\emptyset) = X \setminus \emptyset = X \in \mathcal{T}$.

$\therefore$ $\emptyset, X \in \mathcal{T}$.

We claim that $\mathcal{T}$ is closed under arbitrary unions.

Take $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{T}_X$. We show that $U := \bigcup\limits_{\alpha \in \Lambda} U_\alpha \in \mathcal{T}_X$. Note that $\forall \alpha \in \Lambda, \exists V_\alpha \in \mathcal{P}(X)$ such that $U_\alpha = X \setminus c(V_\alpha)$. Then $U = \bigcup\limits_{\alpha \in \Lambda} (X \setminus c(V_\alpha)) = X \setminus \bigcap\limits_{\alpha \in \Lambda} c(V_\alpha)$

We claim that $\mathcal{T}$ is closed under finite intersections.

Take $U, V \in \mathcal{T}$, then $\exists A, B \in \mathcal{P}(X)$ such that $U = X \setminus c(A)$ and $V = X \setminus c(B)$. Then:

$$\begin{aligned} U \cap V &= (X \setminus c(A)) \cap (X \setminus c(B)) \\ &= X \setminus (c(A) \cup c(B)) \\ &= X \setminus c(A \cup B) \in \mathcal{T} \end{aligned} \tag{2}$$

$\therefore U \cap V \in \mathcal{T}$.

$\therefore \mathcal{T}$ is closed under finite intersections, by induction.

We claim that $\overline{A} = c(A), \quad \forall A \in \mathcal{P}(X)$.

Problem 2

Let A be a subset of a topological space $(X, \mathcal{T}_X)$. Show that the following are equivalent:

(i) $\texttt{int}\left(\overline{A}\right) = \emptyset$.

(ii) $X \setminus \overline{A}$ *is dense in* X.

(iii) $X \setminus \overline{\left(X \setminus \overline{A}\right)} = \emptyset$.

(iv) $A \subseteq \overline{\left(X \setminus \overline{A}\right)}$.

Proof:

((i) $\Longrightarrow$ (ii))

Suppose that $\texttt{int}\left(\overline{A}\right) = \bigcup\left\{U \in \mathcal{T}_X \mid U \subseteq \overline{A}\right\} = \emptyset$. Note that $\overline{\left(X \setminus \overline{A}\right)} \subseteq X$. We claim that $X \subseteq \overline{\left(X \setminus \overline{A}\right)}$. Take $x \in X$ and $U \in \mathcal{T}_X$ such that $x \in U$. Then $U \neq \emptyset$. Now suppose that $U \subseteq \overline{A}$, then $U \subseteq \bigcup\left\{U \in \mathcal{T}_X \mid U \subseteq \overline{A}\right\} = \emptyset \Longrightarrow U = \emptyset$, which is a contradiction. ※

Then $U \cap \left(X \setminus \overline{A}\right) \neq \emptyset \Longrightarrow x \in \overline{\left(X \setminus \overline{A}\right)} \Longrightarrow X \subseteq \overline{\left(X \setminus \overline{A}\right)}$.

$\therefore \overline{\left(X \setminus \overline{A}\right)} = X$.

$\therefore X \setminus \overline{A}$ is dense in X.

((ii) $\Longrightarrow$ (iii))

Suppose that $X \setminus \overline{A}$ is dense in X, then $\overline{\left(X \setminus \overline{A}\right)} = X$ by definition.
Then $X \setminus \overline{\left(X \setminus \overline{A}\right)} = X \setminus X = \emptyset$.

$\therefore X \setminus \overline{\left(X \setminus \overline{A}\right)} = \emptyset$.

((iii) $\Longrightarrow$ (iv))

Suppose that $X \setminus \overline{\left(X \setminus \overline{A}\right)} = \emptyset$. Then $\overline{\left(X \setminus \overline{A}\right)} = X$. But $A \subseteq X$ by hypothesis.

$\therefore A \subseteq \overline{\left(X \setminus \overline{A}\right)}$

((iv) $\Longrightarrow$ (i))

Suppose that $A \subseteq \overline{\left(X \setminus \overline{A}\right)}$. Now suppose, by contradiction, that $\texttt{int}\left(\overline{A}\right) \neq \emptyset$. Then $\exists x \in \texttt{int}\left(\overline{A}\right) \Longrightarrow \exists U \in \mathcal{T}_X$ such that $x \in U \subseteq \overline{A}$. But $x \in \overline{A}$ and U neighborhood of x imply that $U \cap A \neq \emptyset$. So $\exists y \in U$ such that $y \in A \subseteq \overline{\left(X \setminus \overline{A}\right)}$. But now $y \in \overline{\left(X \setminus \overline{A}\right)}$ and U neighborhood of y imply that $U \cap \left(X \setminus \overline{A}\right) \neq \emptyset$. But this contradics the fact that $U \in \overline{A}$. ※

$\therefore \texttt{int}\left(\overline{A}\right) = \emptyset$.

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Problem 3

A subset of a topological space is a G_δ-set if it is the intersection of countably many open sets. On the other hand, a subset of a topological space is an F_δ-set if it is the union of countably many closed sets.

(i) Let A be an F_δ-set of a topological space $(X, \mathcal{T}_X)$. Show that there is a nested sequence of closed sets $C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots$ such that $A = \bigcup_{i=1}^{\infty} C_i$.

(ii) Show that every closed set in a metric space (X, d) is a G_δ-set.

Problem 4

Let A, B be two non-empty subsets of $\mathbb{R}$ with the usual topology. Define:

$$C := \{x + y \mid x \in A \land y \in B\}. \tag{3}$$

(a) Show that, if A or B is open, then C is open.

(b) Show that, if A and B are compact, then C is compact.

Proof (a):

Suppose, without loss of generality, that $A \in \mathcal{T}_{\varepsilon^1}$. Now take $(x + y) \in C$, then $x \in A$, which is open, so $\exists \delta \in (0, \infty)$ such that $(x - \delta, x + \delta) \subseteq A$.

We claim that $D := ((x + y) - \delta, (x + y) + \delta) \subseteq C$.

Take $d \in D$, then:

$$\begin{aligned}
&(x + y) - \delta < d < (x + y) + \delta \\
\Longrightarrow\ & x - \delta < d - y < x + \delta \\
\Longrightarrow\ & (d - y) \in (x - \delta, x + \delta) \subseteq A \\
\Longrightarrow\ & ((d - y) + y) \in C \\
\Longrightarrow\ & d \in C
\end{aligned} \tag{4}$$

$\therefore D \subseteq C$

$\therefore C$ is open.

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Proof (b):

Suppose that A, B are compact. Then, by Tychonoff's Theorem, $A \times B$ is compact when given the product topology. Note that the product topology $\mathcal{T}_{A\times B}$ is a relative topology inherited from $\mathbb{R}^2$. Define a function $\varphi : (A \times B, \mathcal{T}_{A\times B}) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ by $\varphi(a, b) = a + b$. Note that this is a polynomial function, and thus continuous. Also note that:

$$\varphi(A \times B) = \{\varphi(a, b) \mid (a, b) \in A \times B\} = \{a + b \mid a \in A \wedge b \in B\} = C \tag{5}$$

But $A \times B$ is compact and compactness is a continuous invariant.

$\therefore C$ is compact.

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Problem 5

(Intermediate value theorem) Let $f : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ be a continuous function, where $(X, \mathcal{T}_X)$ is connected. Show that if a, b are two points in X and if r is a real number lying between $f(a)$ and $f(b)$, then there is a $c \in X$ such that $f(c) = r$.

Proof:

Note that $f^{-1}(\mathbb{R}) = X$. Now, suppose, by contradiction, that $\nexists c \in X$ such that $f(c) = d$. Then the previous equation still holds when you remove d from $\mathbb{R}$. That is:

$$f^{-1}(\mathbb{R} \setminus \{d\}) = f^{-1}((-\infty, d) \cup (d, +\infty)) = X \tag{6}$$

Now, let $A := f^{-1}((-\infty, d))$, and $B := f^{-1}((d, +\infty))$, and note that:

$$\begin{aligned}
&\text{(i)} && a \in A \implies A \neq \emptyset \wedge b \in B \implies B \neq \emptyset \\
&\text{(ii)} && A \cup B = f^{-1}((-\infty, d)) \cup f^{-1}((d, +\infty)) \\
& && \quad = f^{-1}((-\infty, d) \cup (d, +\infty)) = X \\
&\text{(iii)} && A \cap B = f^{-1}((-\infty, d)) \cap f^{-1}((d, +\infty)) \\
& && \quad = f^{-1}((-\infty, d) \cap (d, +\infty)) = f^{-1}(\emptyset) = \emptyset
\end{aligned} \tag{7}$$

Then $\{A, B\}$ forms a separation for X, which is a contradiction. ※

$\therefore \exists c \in \mathbb{R}$ such that $f(c) = d$.

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Problem 6

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and suppose $X_1 \times X_2$ has the product topology. For each $i = 1, 2$, let $A_i \subseteq X_i$. Prove that:

(i) $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$.

(ii) $\texttt{int}(A_1 \times A_2) = \texttt{int}(A_1) \times \texttt{int}(A_2)$.

Lemma 1:

Let C_1, C_2, D_1, D_2 be sets. Then $(C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \cap D_2)$.

Proof:

Note that:

$$\begin{aligned}
&(x, y) \in (C_1 \times C_2) \cap (D_1 \times D_2) \\
\iff &(x, y) \in (C_1 \times C_2) \land (x, y) \in (D_1 \times D_2) \\
\iff &(x \in C_1 \land y \in C_2) \land (x \in D_1 \land y \in D_2) \\
\iff &(x \in C_1 \land x \in D_1) \land (y \in C_2 \land y \in D_2) \\
\iff &x \in (C_1 \cap D_1) \land y \in (C_2 \cap D_2) \\
\iff &(x, y) \in (C_1 \cap D_1) \times (C_2 \cap D_2)
\end{aligned} \tag{8}$$

$\therefore (C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \cap D_2)$

Proof (i):

Take $(x, y) \in \overline{A_1 \times A_2}$, and take $U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2$ such that $x \in U_1$ and $y \in U_2$.
Then $(x, y) \in U_1 \times U_2$ and $U_1 \times U_2 \in \mathcal{T}_\Pi$ by the definition of the product topology (for finite products). But:

$$\begin{aligned}
(x, y) \in \overline{A_1 \times A_2} &\implies (U_1 \times U_2) \cap (A_1 \times A_2) \neq \emptyset \\
&\implies \exists (x', y') \in (U_1 \times U_2) \cap (A_1 \times A_2) \\
&\implies (x', y') \in (U_1 \cap A_1) \times (U_2 \cap A_2) \\
&\implies x' \in U_1 \cap A_1 \land y' \in U_2 \cap A_2 \\
&\implies U_1 \cap A_1 \neq \emptyset \land U_2 \cap A_2 \neq \emptyset \\
&\implies x \in \overline{A_1} \land y \in \overline{A_2} \\
&\implies (x, y) \in \overline{A_1} \times \overline{A_2}
\end{aligned} \tag{9}$$

$\therefore \overline{A_1 \times A_2} \subseteq \overline{A_1} \times \overline{A_2}$

Similarly, take $(a, b) \in \overline{A_1} \times \overline{A_2}$, and take $U_1 \times U_2 \in \mathcal{T}_\Pi$ such that $(a, b) \in U_1 \times U_2$.
Then $a \in U_1 \in \mathcal{T}_1$ and $b \in U_2 \in \mathcal{T}_2$ by the definition of the product topology (for finite products). But:

$$
\begin{aligned}
(a,b) \in \overline{A_1} \times \overline{A_2} &\Longrightarrow a \in \overline{A_1} \wedge b \in \overline{A_2} \\
&\Longrightarrow U_1 \cap A_1 \neq \emptyset \wedge U_2 \cap A_2 \neq \emptyset \\
&\Longrightarrow \exists a' \in U_1 \cap A_1 \wedge \exists b' \in U_2 \cap A_2 \\
&\Longrightarrow (a', b') \in (U_1 \cap A_1) \times (U_2 \cap A_2) \\
&\Longrightarrow (a', b') \in (U_1 \times U_2) \cap (A_1 \times A_2) \\
&\Longrightarrow (U_1 \times U_2) \cap (A_1 \times A_2) \neq \emptyset \\
&\Longrightarrow (a,b) \in \overline{A_1 \times A_2}
\end{aligned} \tag{10}
$$

$\therefore \overline{A_1} \times \overline{A_2} \subseteq \overline{A_1 \times A_2}$

$\therefore \overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$

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Proof (ii):

Note that, for any subset B of a topological space $(Y, \mathcal{T}_Y)$, we have that
$z \in \texttt{int}(B) \Longleftrightarrow \exists U \in \mathcal{T}_Y$ such that $z \in U \subseteq B$. Here's a brief proof:
Take $z \in \texttt{int}(B) = \bigcup\{V \in \mathcal{T}_Y \mid V \subseteq B\} \subseteq B$. This proves ($\Longrightarrow$). Now suppose $\exists U \in \mathcal{T}_Y$ such that $z \in U \subseteq B$, then $U \in \bigcup\{V \in \mathcal{T}_Y \mid V \subseteq B\} = \texttt{int}(B)$. This proves ($\Longleftarrow$).

Now for the main proof. Take $(x,y) \in \texttt{int}(A_1 \times A_2)$, then $\exists U_1 \times U_2 \in \mathcal{T}_\Pi$ such that $(x,y) \in U_1 \times U_2 \subseteq A_1 \times A_2 \Longrightarrow x \in U_1 \subseteq A_1$ and $y \in U_2 \subseteq A_2 \Longrightarrow x \in \texttt{int}(A_1)$ and $y \in \texttt{int}(A_2)$. Then $(x,y) \in \texttt{int}(A_1) \times \texttt{int}(A_2)$.

$\therefore \texttt{int}(A_1 \times A_2) \subseteq \texttt{int}(A_1) \times \texttt{int}(A_2)$.

Now take $(a,b) \in \texttt{int}(A_1) \times \texttt{int}(A_2)$, then $a \in \texttt{int}(A_1)$ and $b \in \texttt{int}(A_2) \Longrightarrow \exists U_1 \in \mathcal{T}_1$ such that $a \in U_1 \subseteq A_1$ and $\exists U_2 \in \mathcal{T}_2$ such that $b \in U_2 \subseteq A_2$. Then $(a,b) \in U_1 \times U_2 \subseteq A_1 \times A_2$, which implies that $(a,b) \in \texttt{int}(A_1 \times A_2)$.

$\therefore \texttt{int}(A_1) \times \texttt{int}(A_2) \subseteq \texttt{int}(A_1 \times A_2)$.

$\therefore \texttt{int}(A_1 \times A_2) = \texttt{int}(A_1) \times \texttt{int}(A_2)$.

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