MATE 6540: Topology Qualifying Exam

Due on 19 de mayo

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Problem 0

A topological space $(X, \mathcal{T}_X)$ is pseudocompact $\Longleftrightarrow$ every continuous function $f : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ is bounded. Here $\mathcal{T}_{\varepsilon^1}$ is the usual topology over $\mathbb{R}$.

(i) Show that pseudocompactness is a continuous invariant. Explain.

(ii) Show that if $(Y, \mathcal{T}_Y)$ is compact, then $(Y, \mathcal{T}_Y)$ is pseudocompact, but that the converse does not hold.

Problem 1

(Kuratowski's closure operator) Let X be a set, $\mathcal{P}(X)$ be its powerset, and $c : \mathcal{P}(X) \to \mathcal{P}(X)$ be a function that satisfies:

(i) $c(\emptyset) = \emptyset$

(ii) $A \subseteq c(A), \quad \forall A \in \mathcal{P}(X)$

(iii) $c(c(A)) = c(A), \quad \forall A \in \mathcal{P}(X)$

(iv) $c(A \cup B) = c(A) \cup c(B), \quad \forall A, B \in \mathcal{P}(X)$

Show that the collection $\mathcal{T} = \{X \setminus c(A) \mid A \in \mathcal{P}(X)\}$ is a topology over X, and that in this topology $\overline{A} = c(A), \quad \forall A \in \mathcal{P}(X)$. Here $\overline{A}$ is the closure of A in $(X, \mathcal{T})$.

Problem 2

Let A be a subset of a topological space $(X, \mathcal{T}_X)$. Show that the following are equivalent:

(i) $\mathtt{int}\left(\overline{A}\right) = \emptyset$.

(ii) $X \setminus \overline{A}$ is dense in X.

(iii) $X \setminus \overline{\left(X \setminus \overline{A}\right)} = \emptyset$.

(iv) $A \subseteq \overline{\left(X \setminus \overline{A}\right)}$.

Problem 3

A subset of a topological space is a G_δ-set if it is the intersection of countably many open sets. On the other hand, a subset of a topological space is an F_δ-set if it is the union of countably many closed sets.

(i) Let A be an F_δ-set of a topological space $(X, \mathcal{T}_X)$. Show that there is a nested sequence of closed sets $C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots$ such that $A = \bigcup_{i=1}^{\infty} C_i$.

(ii) Show that every closed set in a metric space (X, d) is a G_δ-set.

Problem 4

Let A, B be two non-empty subsets of $\mathbb{R}$ with the usual topology. Define:

$$C := \{x + y \mid x \in A \wedge y \in B\}. \tag{1}$$

(a) Show that, if A or B is open, then C is open.

(b) Show that, if A and B are compact, then C is compact.

Problem 5

(Intermediate value theorem) Let $f : (X, \mathcal{T}_X) \to (\mathbb{R}, \mathcal{T}_{\varepsilon^1})$ be a continuous function, where $(X, \mathcal{T}_X)$ is connected. Show that if a, b are two points in X and if r is a real number lying between $f(a)$ and $f(b)$, then there is a $c \in X$ such that $f(c) = r$.

Problem 6

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and suppose $X_1 \times X_2$ has the product topology. For each $i = 1, 2$, let $A_i \subseteq X_i$. Prove that:

(i) $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$.

(ii) $\mathtt{int}(A_1 \times A_2) = \mathtt{int}(A_1) \times \mathtt{int}(A_2)$.

Lemma 1:

Let C_1, C_2, D_1, D_2 be sets. Then $(C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \cap D_2)$.

Proof:

Note that:

$$\begin{aligned}
& (x, y) \in (C_1 \times C_2) \cap (D_1 \times D_2) \\
\iff & (x, y) \in (C_1 \times C_2) \wedge (x, y) \in (D_1 \times D_2) \\
\iff & (x \in C_1 \wedge y \in C_2) \wedge (x \in D_1 \wedge y \in D_2) \\
\iff & (x \in C_1 \wedge x \in D_1) \wedge (y \in C_2 \wedge y \in D_2) \\
\iff & x \in (C_1 \cap D_1) \wedge y \in (C_2 \cap D_2) \\
\iff & (x, y) \in (C_1 \cap D_1) \times (C_2 \cap D_2)
\end{aligned} \tag{2}$$

$\therefore (C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \cap D_2)$

Proof (i):

Take $(x, y) \in \overline{A_1 \times A_2}$, and take $U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2$ such that $x \in U_1$ and $y \in U_2$.
Then $(x, y) \in U_1 \times U_2$ and $U_1 \times U_2 \in \mathcal{T}_\Pi$ by the definition of the product topology (for finite products). But:

$$
\begin{aligned}
(x,y) \in \overline{A_1 \times A_2} &\Longrightarrow (U_1 \times U_2) \cap (A_1 \times A_2) \neq \emptyset \\
&\Longrightarrow \exists (x', y') \in (U_1 \times U_2) \cap (A_1 \times A_2) \\
&\Longrightarrow (x', y') \in (U_1 \cap A_1) \times (U_2 \cap A_2) \\
&\Longrightarrow x' \in U_1 \cap A_1 \wedge y' \in U_2 \cap A_2 \\
&\Longrightarrow U_1 \cap A_1 \neq \emptyset \wedge U_2 \cap A_2 \neq \emptyset \\
&\Longrightarrow x \in \overline{A_1} \wedge y \in \overline{A_2} \\
&\Longrightarrow (x,y) \in \overline{A_1} \times \overline{A_2}
\end{aligned} \tag{3}
$$

$\therefore \overline{A_1 \times A_2} \subseteq \overline{A_1} \times \overline{A_2}$

Similarly, take $(a,b) \in \overline{A_1} \times \overline{A_2}$, and take $U_1 \times U_2 \in \mathcal{T}_\Pi$ such that $(a,b) \in U_1 \times U_2$.
Then $a \in U_1 \in \mathcal{T}_1$ and $b \in U_2 \in \mathcal{T}_2$ by the definition of the product topology (for finite products).
But:

$$
\begin{aligned}
(a,b) \in \overline{A_1} \times \overline{A_2} &\Longrightarrow a \in \overline{A_1} \wedge b \in \overline{A_2} \\
&\Longrightarrow U_1 \cap A_1 \neq \emptyset \wedge U_2 \cap A_2 \neq \emptyset \\
&\Longrightarrow \exists a' \in U_1 \cap A_1 \wedge \exists b' \in U_2 \cap A_2 \\
&\Longrightarrow (a', b') \in (U_1 \cap A_1) \times (U_2 \cap A_2) \\
&\Longrightarrow (a', b') \in (U_1 \times U_2) \cap (A_1 \times A_2) \\
&\Longrightarrow (U_1 \times U_2) \cap (A_1 \times A_2) \neq \emptyset \\
&\Longrightarrow (a,b) \in \overline{A_1 \times A_2}
\end{aligned} \tag{4}
$$

$\therefore \overline{A_1} \times \overline{A_2} \subseteq \overline{A_1 \times A_2}$

$\therefore \overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$

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Proof (ii):

Note that, for any subset B of a topological space $(Y, \mathcal{T}_Y)$, we have that
$z \in \mathtt{int}(B) \iff \exists U \in \mathcal{T}_Y$ such that $z \in U \subseteq B$. Here's a brief proof:
Take $z \in \mathtt{int}(B) = \bigcup \{V \in \mathcal{T}_Y \mid V \subseteq B\} \subseteq B$. This proves ($\Longrightarrow$). Now suppose $\exists U \in \mathcal{T}_Y$ such that $z \in U \subseteq B$, then $U \in \bigcup \{V \in \mathcal{T}_Y \mid V \subseteq B\} = \mathtt{int}(B)$. This proves ($\Longleftarrow$).

Now for the main proof. Take $(x,y) \in \mathtt{int}(A_1 \times A_2)$, then $\exists U_1 \times U_2 \in \mathcal{T}_\Pi$ such that
$(x,y) \in U_1 \times U_2 \subseteq A_1 \times A_2 \Longrightarrow x \in U_1 \subseteq A_1$ and $y \in U_2 \subseteq A_2 \Longrightarrow x \in \mathtt{int}(A_1)$ and
$y \in \mathtt{int}(A_2)$. Then $(x,y) \in \mathtt{int}(A_1) \times \mathtt{int}(A_2)$.

$\therefore \mathtt{int}(A_1 \times A_2) \subseteq \mathtt{int}(A_1) \times \mathtt{int}(A_2)$.

Now take $(a,b) \in \mathtt{int}(A_1) \times \mathtt{int}(A_2)$, then $a \in \mathtt{int}(A_1)$ and $b \in \mathtt{int}(A_2) \Longrightarrow \exists U_1 \in \mathcal{T}_1$ such that $a \in U_1 \subseteq A_1$ and $\exists U_2 \in \mathcal{T}_2$ such that $b \in U_2 \subseteq A_2$. Then $(a,b) \in U_1 \times U_2 \subseteq A_1 \times A_2$, which implies that $(a,b) \in \mathtt{int}(A_1 \times A_2)$.

$\therefore \mathtt{int}(A_1) \times \mathtt{int}(A_2) \subseteq \mathtt{int}(A_1 \times A_2)$.

$\therefore \mathtt{int}(A_1 \times A_2) = \mathtt{int}(A_1) \times \mathtt{int}(A_2)$.

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