MATE 6551: Tarea 1

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Problem 0

If X is a topological space homeomorphic to D^n, then every continuous $f : X \to X$ has a fixed point.

Proof:

Let $\varphi : X \to D^n$ be a homeomorphism. Suppose, for a contradiction, that there exists a continuous $f : X \to X$ with no fixed point. Then for any $x \in X$, $f(x) \neq x \Longrightarrow \varphi(x) \neq \varphi(f(x))$ by the injectivity of φ. Then the points $\varphi(x), \varphi(f(x)) \in D^n$ define a unique ray ℓ_x starting at $\varphi(f(x))$ and passing through $\varphi(x)$.

Define $\psi : D^n \to S^{n-1}$ by $\psi(x) = \left(\ell_{\varphi^{-1}(x)} \setminus \{\varphi(f(x))\}\right) \cap S^{n-1}$. Note that this is well defined as $\ell_{\varphi^{-1}(x)} \setminus \{\varphi(f(x))\}$ is guaranteed to intersect S^{n-1} exactly once for all $x \in X$. An analytic geometry argument mentioned in class yields the continuity of ψ. Finally, note that $x \in S^{n-1} \Longrightarrow \psi(x) = x$, because the intersection of $\ell_{\varphi^{-1}(x)} \setminus \{\varphi(f(x))\}$ with S^{n-1} coincides precisely with $\varphi(x)$. Therefore, ψ is a retraction, thus making S^{n-1} a retract of D^n, which contradicts Lemma 0.2. ※

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Problem 1

Let $f \in \operatorname{Hom}(A, B)$ be a morphism in a category $\mathcal{C}$. If f has a left inverse g and a right inverse h, then $g = h$.

Proof:

By hypothesis, $g \circ f = 1_A$ and $f \circ h = 1_B$. Then:

$$g = g \circ 1_B = g \circ (f \circ h) = (g \circ f) \circ h = 1_A \circ h = h \tag{1}$$

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Problem 2

Let G be a group and let $\mathcal{C}$ be the one-object category it defines: obj $\mathcal{C} = \{*\}, \operatorname{Hom}(*, *) = G$, *and composition is the group operation. If H is a normal subgroup of G, define $x \sim y$ to mean $xy^{-1} \in H$. Show that $\sim$ is a congruence on $\mathcal{C}$ and that $[*, *] = G/H$ in the corresponding quotient category.*

Proof:

First, we prove that $\sim$ is an equivalence relation:

Let $x, y, z \in G$, then:

a) (Reflexivity):

Because H is a subgroup, it contains the identity element: $xx^{-1} = e \in H \Longrightarrow x \sim x$.

b) (Symmetry):

Suppose that $x \sim y$. Then $xy^{-1} \in H$. But H is a subgroup of G, so it follows that:
$\left(xy^{-1}\right)^{-1} = \left(y^{-1}\right)^{-1} x^{-1} = yx^{-1} \in H \Longrightarrow y \sim x$.

c) (Transitivity):

Suppose that $x \sim y$ and $y \sim z$. Then $xy^{-1}, yz^{-1} \in H$. But, because H is a subgroup:
$xy^{-1}yz^{-1} = xz^{-1} \in H \Longrightarrow x \sim z$.

To conclude that $\sim$ forms a congruence on $\mathcal{C}$, it suffices to show that, for all $x, x', y, y' \in G$, we have $x \sim x', y \sim y' \Longrightarrow xy \sim x'y'$. Note that, in general, the definition of a congruence in a category involves slightly different conditions, but the fact that $\mathcal{C}$ has a single hom-set allows us to reduce them to the condition shown above.

Suppose that $x \sim x'$ and $y \sim y'$. Then $x(x')^{-1}, y(y')^{-1} \in H$. Since H is normal in G, we know that $x\left(y(y')^{-1}\right)x^{-1} \in H$, but H is a subgroup, so $h := \left(x\left(y(y')^{-1}\right)x^{-1}\right)\left(x(x')^{-1}\right) \in H$. But:

$$\begin{aligned} h &= \left(x\left(y(y')^{-1}\right)x^{-1}\right)\left(x(x')^{-1}\right) \\ &= x\left(y(y')^{-1}\right)\left(x^{-1}x\right)(x')^{-1} \\ &= x\left(y(y')^{-1}\right)(x')^{-1} \\ &= (xy)\left((y')^{-1}(x')^{-1}\right) \\ &= (xy)(x'y')^{-1} \end{aligned} \tag{2}$$

$\therefore xy \sim x'y'$, which shows that $\sim$ is a congruence in $\mathcal{C}$.

By definition, $[*, *] = \{[f] \mid f \in \mathrm{Hom}_{\mathcal{C}}(*, *)\}$, where $[f]$ is the equivalence class of $f \in \mathrm{Hom}_{\mathcal{C}}(*, *) = G$ under $\sim$. This coincides exactly with the set G/H. Furthermore, composition in a quotient category is defined by $[g] \circ [f] = [g \circ f]$, which, in our case, translates to: $[f][g] = [fg]$. This coincides with the definition of the operation in a quotient group, which is well defined due to the normality of H.

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Problem 3

Let $x_0, x_1 \in X$ and let $f_i : X \to X$ for $i \in \{0, 1\}$ denote the constant map at x_i. Prove that $f_0 \simeq f_1$ if and only if there is a continuous $F : I \to X$ with $F(0) = x_0$ and $F(1) = x_1$.

Proof:

Suppose that $H : f_0 \simeq f_1$. Then define $F : I \to X$ by $F(t) = H(x_0, t)$. Note that F is the composition of continuous functions, and is thus continuous, and also note that:
$F(0) = H(x_0, 0) = f_0(x_0) = x_0$ and $F(1) = H(x_0, 1) = f_1(x_0) = x_1$.

Suppose now that there exists a continuous $F : I \to X$ with $F(0) = x_0$ and $F(1) = x_1$. Then define $H : X \times I \to X$ by $H(x, t) = F(t)$. Once again, note that H is the composition of continuous functions, and is thus continuous, and that for all $x \in X$:
$H(x, 0) = F(0) = x_0 = f_0(x)$ and $H(x, 1) = F(1) = x_1 = f_1(x)$.

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Problem 4

(i) Give an example of a continuous image of a contractible space that is not contractible.

(ii) Show that a retract of a contractible space is contractible.

Example (i):

Define $f : [0, 2\pi) \to S^1$ by $f(x) = (\cos(x), \sin(x))$. We know that sin and cos are continuous by analysis, then f is continuous. Also, since S^1 is the *unit* circle, we have that its circumference is 2π, then by the definition of sin and cos, f is surjective.

Now, note that $[0, 2\pi)$ is contractible via $H(x, t) = tx$, but $f([0, 2\pi)) = S^1$ is not contractible, because $\pi_1(S^1, (1, 0)) \simeq \mathbb{Z}$, which is non-trivial.

Proof (ii):

Let X be a contractible space with contraction $H : 1_X \simeq c_{x_0}$ for some fixed $x_0 \in X$ (where c_{x_0} is the constant function at x_0), and let $A \subseteq X$ be a retract of X with retraction $r : X \to A$. Then define $G : A \times I \to A$ by $G(a, t) = r(H(a, t))$. Because G is the composition of continuous functions, G is itself continuous, and the following hold:

$$G(a, 0) = r(H(a, 0)) = r(1_X(a)) = r(a) = a = 1_A(a) \tag{3}$$

and

$$G(a, 1) = r(H(a, 1)) = r\left(c_{x_0}(a)\right) = r(x_0) = c_{r(x_0)}(a) \tag{4}$$

which is a constant function.

$\therefore G : 1_A \simeq c_{r(x_0)}$.

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