$$\mu(z_2) = \min\left(\mu_{A_2}(1.5),\ \mu_{B_2}(2.5)\right)$$
$$= \min(0.25,\ 0.75) = 0.25$$

Then by virtue of the above rules we have two values for z, i.e., $z_1 = 3.84$ and $z_2 = 3.28$. To infer the numerical value of z from both rules, the defuzzification relation (7.2.17b) can be used to find z = ((0.75)(3.84) + (0.25)(3.28) / (0.75 + 0.25) = 3.7.

7.2.5 The Inverted Pendulum Problem

In this section, the much discussed inverted pendulum problem will be first introduced and then two fuzzy control structures will be introduced for it. A third structure is given in Section 8.2.4. The first structure is based on the linguistic-type rules (see Equation (7.2.14), sometimes called the *Mamdani-type* controller (Mamdani and Assilian, 1975). The second structure is one in which the consequence of the rules is replaced by a dynamic relation such as a difference equation, or those in Equations (7.2.4) or (7.2.15) or simply a function. This structure is sometimes referred to as *Takagi-Sugeno-type* controller (Jamshidi *et al.*, 1993).

Consider an inverted pendulum or the cart pole system shown in Figure 7.8. This system is commonly treated as either a 4-state $\left(\theta,\ \dot{\theta},\ x, \text{ and } \dot{x}\right)$ or

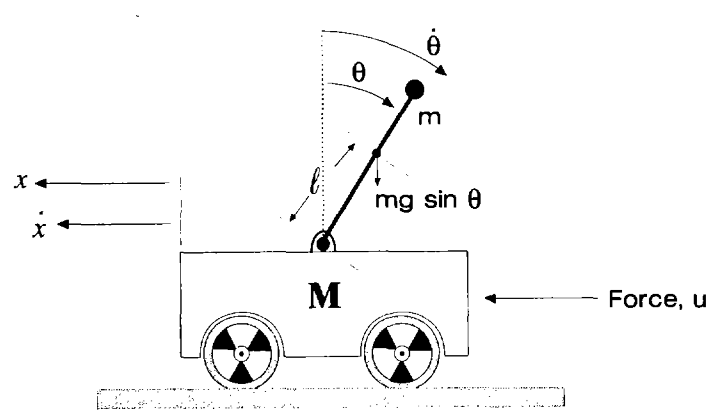


Figure 7.8 A schematic for an inverted pendulum.

a 2-state (θ and $\dot{\theta}$) space model. Below are both dynamic models of the inverted pendulum (Jamshidi *et al.*, 1992; Wang, 1994a).

Inverted Pendulum with 4 states:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{-\frac{7}{3}m^2\ell^3x_4^2\sin x_3 + (m\ell)^2 g\sin x_3\cos x_3 - \frac{7}{3}m\ell^2 u}{\left[m\ell\cos x_3\right]^2 - \left[\frac{7}{3}m\ell^2(M+m)\right]}$$

$$\dot{x}_3 = x_4 \tag{7.2.18}$$

$$\dot{x}_4 = \frac{(m\ell)^2 x_4^2\sin x_3\cos x_3 - mg\ell\sin x_3(M+m) + m\ell\cos x_3 u}{\left[m\ell\cos x_3\right]^2 - \left[\frac{7}{3}m\ell^2(M+m)\right]}$$

where $x^T = (\theta, \dot{\theta}, x, \dot{x})$, m and M are the masses of the pole and cart, respectively, usually assumed that $m << M$, ℓ is one-half of the length of the pole, and $g = 9.8$ m/sec^2 is the gravitational acceleration constant, and $u = F$ is the force acting on the cart.

Inverted pendulum with 2 states:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{g\sin x_1 - \frac{m\ell x_2^2\cos x_1\sin x_1}{m+M}}{\ell\left(\frac{4}{3} - \frac{m\cos^2 x_1}{m+M}\right)} + \frac{\frac{\cos x_1}{m+M}}{\ell\left(\frac{4}{3} - \frac{m\cos^2 x_1}{m+M}\right)}u \tag{7.2.19}$$

where the state vector $x^T = (\theta, \dot{\theta})$ and $u = F$. For the first numerical analysis of the system, the following values are used: $\ell = 0.5$ m, $m = 100$ grams, and $M = 1$ kgram (Wang, 1994a). Using these values, the two models will become

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{-0.003x_4^2 \sin x_3 + 0.025 \sin x_3 \cos x_3 - 0.06u}{0.0025 \cos^2 x_3 - 0.064} \tag{7.2.20}$$

$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{0.0025x_4^2 \sin x_3 - 0.49 \sin x_3 + 0.05 \cos x_3}{0.0025 \cos^2 x_3 - 0.064} u$$

and for the 2-dimensional pendulum we have

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 1.58 \sin x_1 - 0.07x_2 \sin x_1 \cos x_1 + 1.5u$$

Since most classical control methods require a linear model, we can assume a set of nominal conditions $x_n = (0\ \ 0\ \ 0\ \ 0)^T$ or $x_n = (0\ \ 0)^T$, depending on which model we choose. Then, the linearized models will be given by

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 7.97 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0.97 \\ 0 \\ -0.8 \end{pmatrix} u \tag{7.2.21}$$

or

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 15.79 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1.46 \end{pmatrix} u \tag{7.2.22}$$

As one expects, these linearized models represent two unstable systems.

Example 7.2.3 Consider the 4th-order model of the inverted pendulum system of Equation (7.2.21). It is desired to stabilize it such that the new closed-loop poles are at −1, −2, −3, and −4.

SOLUTION: Since the states of the inverted pendulum represent position and velocity of both angular and linear positions, then a state feedback law

$$u = -kx = -k_1x_1 - k_2x_2 - k_3x_3 - k_4x_4$$

is effectively a PD controller and would suffice to stabilize the system provided that the system is controllable. A quick check on controllability shows that the system is, in fact, controllable. The application of any pole placement program would give a gain vector for k. This system's state feedback law will be used with its nonlinear model in Section 8.2 to design an adaptive fuzzy tuner controller.

One of the first steps in the design of any fuzzy controller is to develop a knowledge base for the system to eventually lead to an initial set of rules. There are at least five different methods to generate a fuzzy rule base:

i) simulate the closed-loop system through its mathematical model,
ii) interview an operator who has had many years of experience controlling the system,
iii) generate rules through an algorithm using numerical input/output data of the system,
iv) use learning or optimization methods such as neural networks (NN) or genetic algorithms (GA) to create the rules, and
v) in the absence of all of the above, if a system does exist, experiment with it in the laboratory or factory setting and gradually gain enough experience to create the initial set of rules.

In sequel, the second-order model of the inverted pendulum will be used to help the designer obtain a knowledge base leading to a set of linguistic rules.

Example 7.2.4 For the linearized model of the inverted pendulum described by (7.2.22), develop a knowledge base through simulation.

SOLUTION: Since the system (7.2.22) is unstable but controllable, we can design either a PD or a PID controller to first stabilize it, and then through variations of the system's desired poles, study it under different circumstances. We offer both approaches here.

PD Controller The application of a PD controller $u = \theta_d - k_p x_1 - k_d x_2$ to (7.2.22) leads to a closed-loop system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ a - bk_p & -bk_d \end{pmatrix} x + \begin{pmatrix} 0 \\ b \end{pmatrix} \theta_d \tag{7.2.23}$$

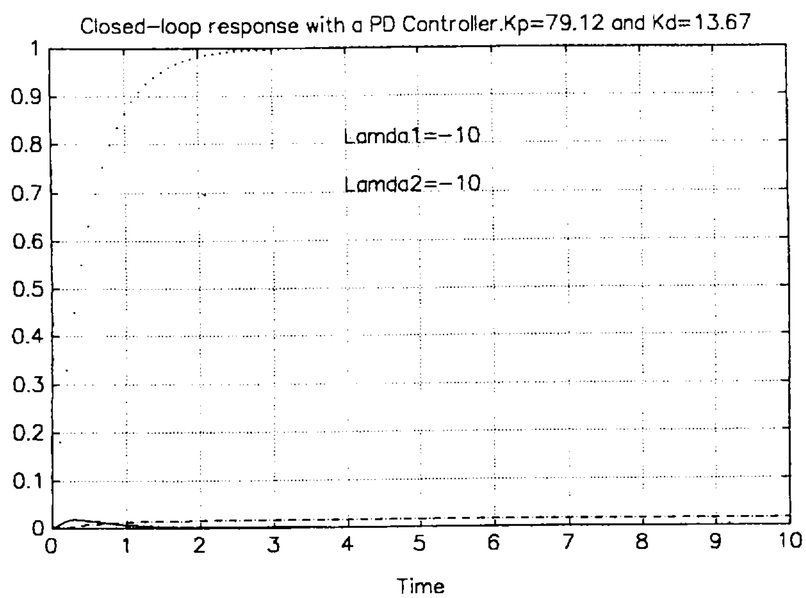
where $a = 15.79$, $b = 1.46$, and the desired angular position is $\theta_d(t) = 1 - e^{-2t}$. Assuming that the desired closed-loop poles are at λ_1 and λ_2, a desired characteristic polynomial $\Delta(\lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\,\lambda + \lambda_1\lambda_2$ is obtained. Equating the coefficients of this polynomial to the system's $\Delta(\lambda) = \det(\lambda I - A_c)$, where A_c is the closed-loop system matrix of (7.2.23), leads to:

$$k_p = \frac{\lambda_1\lambda_2 + a}{b}, \quad k_d = -\frac{(\lambda_1 + \lambda_2)}{b}$$

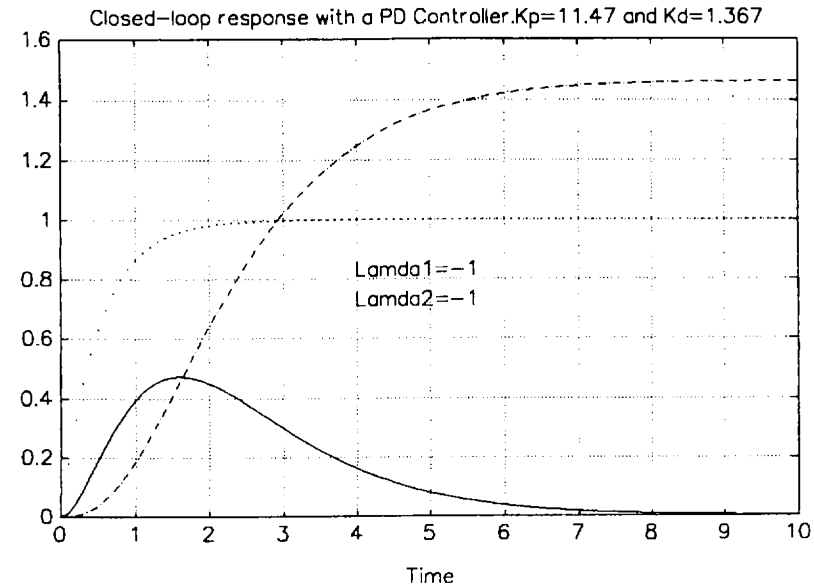
Using standard software environments such as Matlab (Jamshidi *et al.*, 1992), several simulation rules for different values of λ_1 and λ_2 were deduced. Figure 7.9 shows the simulation results.

As shown in this figure, the best results are obtained for low values of $\lambda_1 = \lambda_2 = -1.2$ or $k_p = 11.78$ and $k_d = 1.64$ with a 1% error. As k_p and k_d increase to very large values of $k_p = 79.12$ and $k_d = 13.67$, the error jumps up to 99%. This study gives the designer an insight into rules for the variations of k_p and k_d as a function of the system's responses. For example, to avoid large tracking errors, one can compose a rule such as:

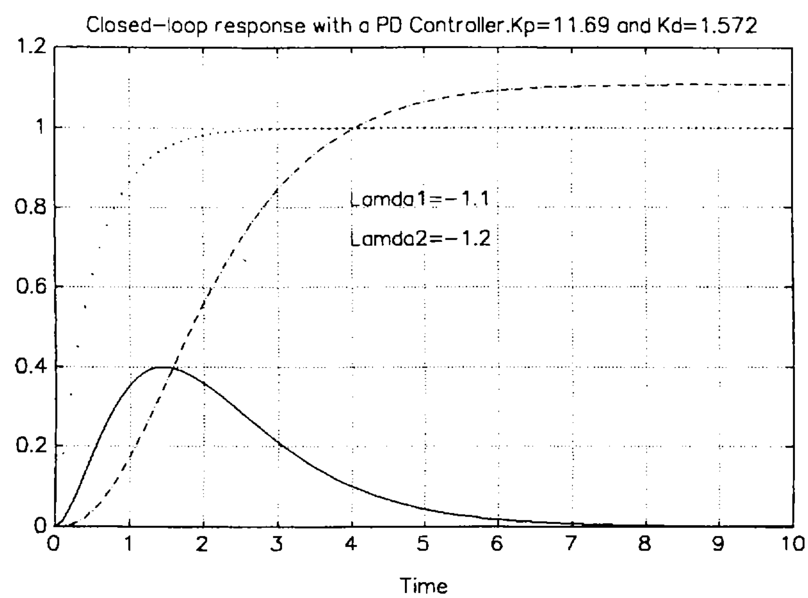
IF θ is small and $\dot{\theta}$ is very small, THEN k_p is medium and k_d is small.



(a) Simulation Run No. 1.



(b) Simulation Run No. 2.



(c) Simulation Run No. 3.

Figure 7.9 (opposite page and above) Simulation results for PD control of the inverted pendulum; ------ represents θ (position) and —— represents $\dot{\theta}$ (velocity).

Such a rule would help avoid the drastic 99% error situation of Figure 7.9a. The results of examples like this will be used for adaptive fuzzy tuning of the standard crisp PD or PID controllers in Section 8.2.

PID Controller Here the control law is given by

$$\begin{aligned} u &= -k_p x_1 - k_i \int (x_1 - \theta_d)dt - k_d x_2 \\ &= \left(-k_p - k_d - k_i\right)x \end{aligned} \tag{7.2.24}$$

where $x = (x_1 \;\; x_2 \;\; x_3)^T$, $x_3 \triangleq \int (x_1 - \theta_d)dt$ is a third (pseudo) state variable as a result of the integral action of the controller, and $y = \theta$. Using the controller (7.2.24) for the system of Equation (7.2.22), one gets

$$\begin{aligned} \dot{x} &= A_c x + B\theta_d \\ &= \begin{pmatrix} 0 & 1 & 0 \\ a - bk_p & -bk_d & -bk_i \\ 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ b \\ -1 \end{pmatrix} \theta_d \end{aligned}$$

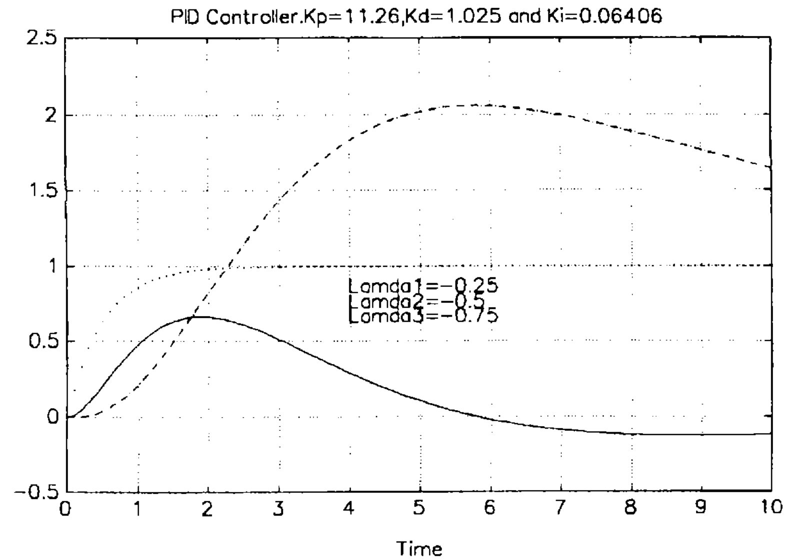
where, once again, $a = 15.79$ and $b = 1.46$. Assuming that the new poles of the system are located at λ_1, λ_2 and λ_3, equating the coefficients of the desired and system's characteristic polynomial, i.e., $\Delta_d(\lambda) = \Delta(\lambda) = \det(\lambda I - A_c)$, would, after some arithmetic operations, result in

$$\begin{aligned} k_p &= \left(a + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right)/b \\ k_i &= -\left(\lambda_1\lambda_2\lambda_3\right)/b \\ k_d &= -\left(\lambda_1 + \lambda_2 + \lambda_3\right)/b \end{aligned}$$

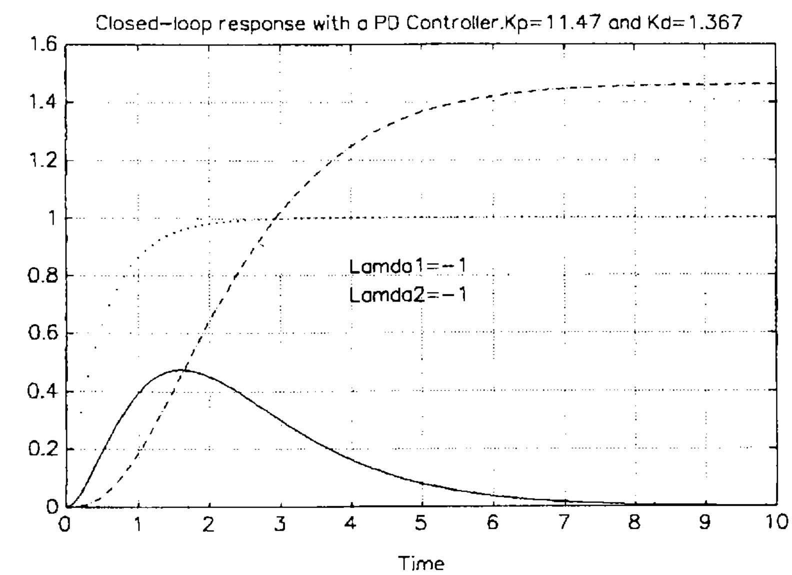
The simulation results are shown in Figure 7.10. Once again, four different sets of responses are shown here. Typical rules from such a study can be deduced as

> IF error is small and change in error is zero,
> THEN k_p is high, k_i is very high and k_d is medium.

This rule would characterize the response shown in Figure 7.10c. Note that the case in Figure 7.10d is essentially a proportional controller which

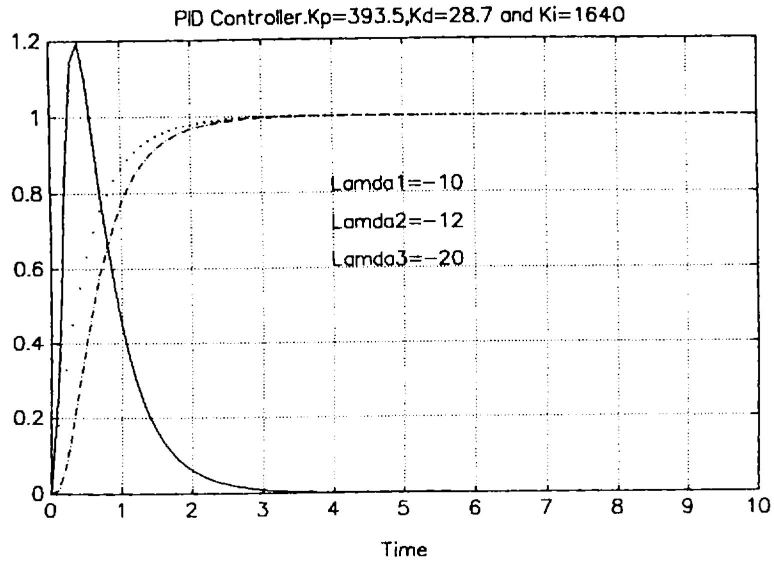


(a) Simulation Run No. 1.

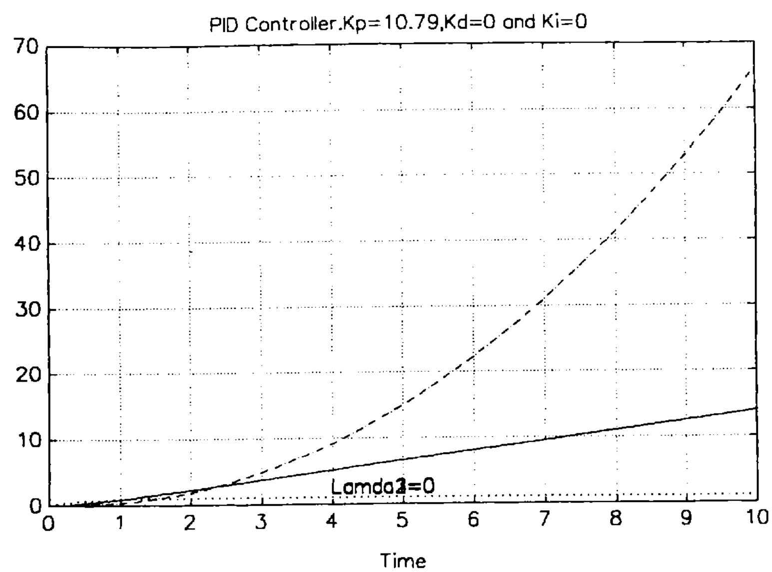


(b) Simulation Run No. 2.

Figure 7.10a & b Simulation results for PID control of the inverted pendulum; ------- represents θ (position) and —— represents $\dot{\theta}$ (velocity).



(c) Simulation Results No. 3



(d) Simulation Results No. 4

Figure 7.10c & d Simulation results for PID control of the inverted pendulum; ------ represents θ (position) and —— represents $\dot{\theta}$ (velocity).