Exam 1

Please work on the exam individually. Group discussions are now allowed. I trust that everyone will respect this requirement. If you need clarification, please reach out to me by email.

Some equations I will reference

Mannings equations

is used to determine normal depth conditions

$$Q = \frac{k}{n} R^{2/3} \cdot A \cdot S^{1/2}$$

- Q discharge
- R roughness (friction related)
- A cross sectional area
- S slope (it is friction slope (negative and zero slopes are problematic because this is supposed to measure energy resisting friction)
- n is the Gauckler–Manning coefficient. n is not dimensionless and therefore it changes when you change your metric system.
- k is a conversion factor between english ($k = 1.49$) and SI ($k = 1$) units.

Hydraulic depth/control

We use the froude number

$$Fr = \frac{V}{\sqrt{gy}} = \frac{Q/A}{\sqrt{gy}} = \frac{\text{inertia forces}}{\text{gravity forces}}$$

- V velocity
- g gravity

- y depth (which we think as a function of discharge $y = y(Q)$ on critical conditions)
- A cross sectional area
- Q discharge

Conservation of mass

$$Q_1 = Q_2$$

or

$$V_1 A_1 = V_2 A_2$$

Conservation of energy

$$H_1 = H_2$$

or

$$z_1 + y_1 + \frac{V_1^2}{2g} = z_2 + y_2 + \frac{V_2^2}{2g}$$

- z is the bed elevation with respect to a datum

Conservation of momentum

$$\frac{Q_1^2}{gA_1} + \bar{z}_1 A_1 = \frac{Q_2^2}{gA_2} + \bar{z}_2 A_2$$

Relationship between conjugate depths

$$\frac{y_2}{y_1} = \frac{1}{2}(\sqrt{1 + 8Fr_1^2} - 1)$$

$$\frac{y_1}{y_2} = \frac{1}{2}(\sqrt{1 + 8Fr_2^2} - 1)$$

bed elevation relationship with water surface

Under the assumption of no energy losses we have

$$\frac{dz}{dx} = (Fr^2 - 1)\frac{dy}{dx}$$

surface profile relationship with slow/gradually varied flow

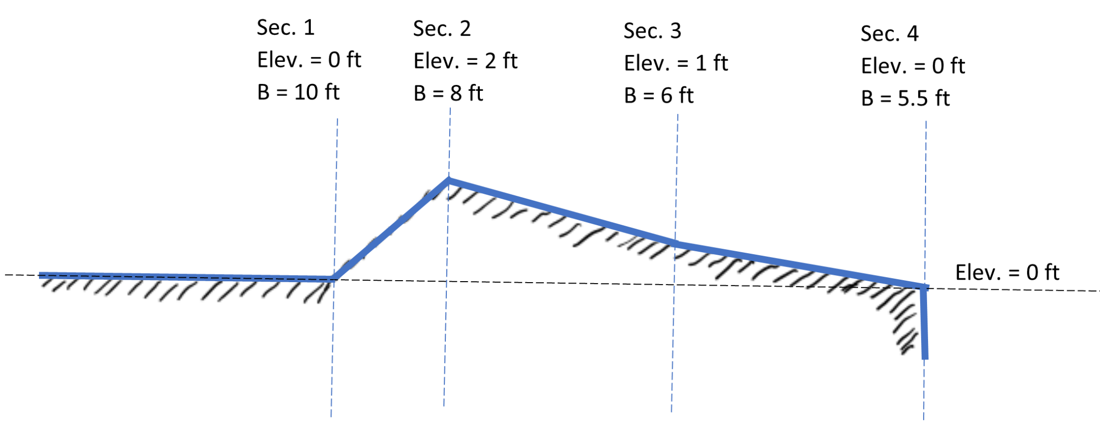
Under the assumptions that slopes are small, prismatic channel (probably rectangular too) , hydrostatic distribution, and the applicability of head losses

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2}$$

1.

A rectangular channel has variable width and bed level as shown below in the figure. A steady discharge of 500 cfs is passing through the channel.

Calculate the water level at sections 1 through 4. Neglect the effects of resistance (i.e. ignore friction effects), and ignore nonuniformity of velocity profile.



Solution

I'll proceed as we did during class. Since for water to go through every point it needs a minimum ammount of energy then we will calculate the minimum energy needed to make it through all while assuming there are no energy losses due to friction.

Since we know the channel is rectangular and we are solving for critical conditions I know that

$$A_c = By_c, \quad V_c A_c = Q, \quad V_c^2 = g y_c$$

and so solving y_c in terms of the discharge and the width I get

$$y_c = \sqrt[3]{\frac{Q^2}{gB^2}}$$

After that we can calculate the energy at every section

$$H_c = z + y_c + \frac{V_c^2}{2g}$$

and find the corresponding maximum.

In [1]:
```
import numpy as np

B = np.array([10, 8, 6, 5.5])
z = np.array([ 0, 2, 1, 0])
Q = 500
g = 32.17

yc = ( Q*Q / (g*B*B) )**(1/3)
print("Critical depths\n",yc)

Vc = (g*yc)**0.5
print("Critical velocities\n",Vc)

Hc = z + yc + Vc*Vc/(2*g)
print("Critical energies\n", Hc)

max = np.argmax(Hc)
print(max+1,"location with max",  Hc[max])

E_spec = Hc - z
print("Specific critical energies\n",E_spec)
```

```
Critical depths
 [4.26739636 4.95187483 5.99876915 6.35703383]
Critical velocities
 [11.71674618 12.62148221 13.89173867 14.30055168]
Critical energies
 [6.40109454 9.42781224 9.99815372 9.53555075]
3 location with max 9.998153717760873
Specific critical energies
 [6.40109454 7.42781224 8.99815372 9.53555075]
```

We can see from the calculations that the maximum energy $H = 9.9981$ happens at the third location and so I will assume that's the energy level across the channel.

Now I will attempt to find the water level on the different sections using this information. To do this we will use the specific energy formula substituting the velocity in terms of y, B, and Q namely

$$H - z = E = y + \frac{Q^2}{2gB^2y^2}$$

which we can rearrange as

$$y^3 - Ey^2 + \frac{Q^2}{2gB^2} = 0.$$

In [2]:
```
const = Q*Q/(2*g*B*B)
E = Hc[max] - z
print("Specific energies\n", E)


for i in range(4):
      coeff = [1, -E[i], 0, const[i]]
      roots = np.roots(coeff)
      roots = np.real(roots)
      print(f"{i+1}th location roots: ")
      print(roots)
```

```
Specific energies
 [9.99815372 7.99815372 8.99815372 9.99815372]
1th location roots:
[ 9.57426906  2.23760402 -1.81371937]
2th location roots:
[ 6.60757944  3.80523155 -2.41465727]
3th location roots:
[ 5.99876915  5.99876915 -2.99938457]
4th location roots:
[ 7.98212678  5.14422837 -3.12820144]
```

So for the different pairs of alternate depths at each location are (with the exception of the third since it's critical)

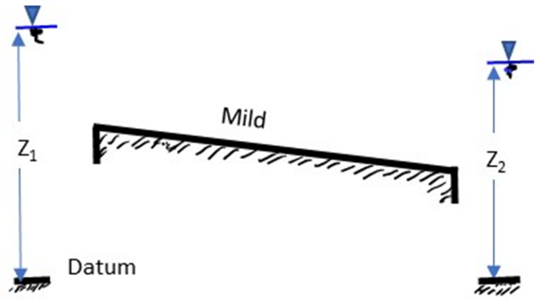
$$\begin{aligned} y_{1,1} &= 9.57ft, \qquad & y_{1,2} &= 2.23ft \\ y_{2,1} &= 6.60ft, \qquad & y_{2,2} &= 3.80ft \\ y_3 &= 5.99ft, \\ y_{4,1} &= 7.98ft, \qquad & y_{4,2} &= 5.14ft \end{aligned}$$

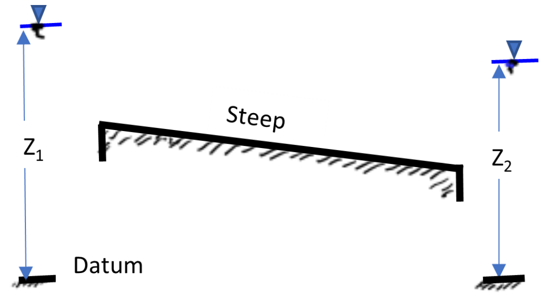
Since we don't have distance information, and therefore slope, we cannot choose one depth over another but it seems natural that the flow depth reduces as it goes on an adverse slope and for that condition we can only start at 9.57. Other than that I can't really draw any other conclusions. So the solutions are:

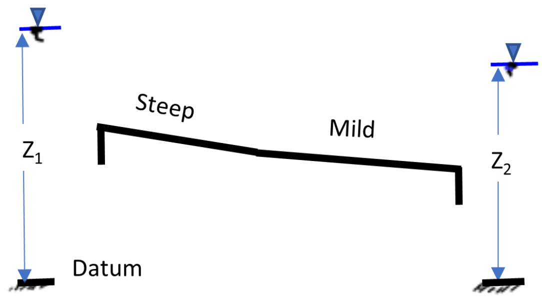
$$\begin{aligned} h_1 &= 9.57ft, \\ h_{2,1} &= 8.60ft, \qquad & h_{2,2} &= 5.80ft \\ h_3 &= 6.99ft, \\ h_{4,1} &= 7.98ft, \qquad & h_{4,2} &= 5.14ft \end{aligned}$$

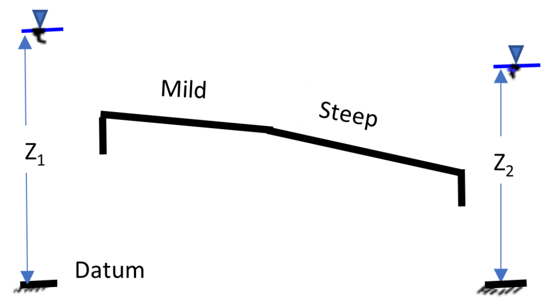
2.

Sketch TWO longitudinal profiles for EACH that can occur for possible values of the heights Z1 and Z2 in the figures below.



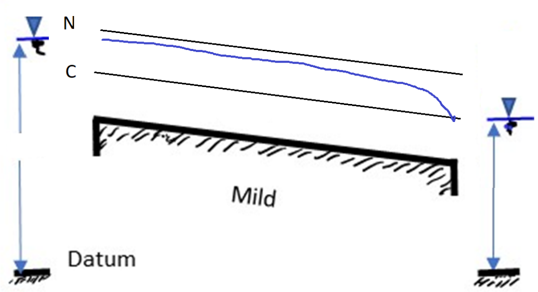


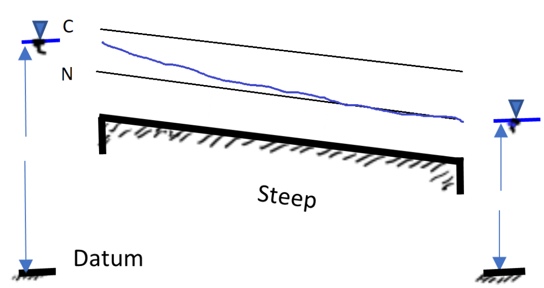


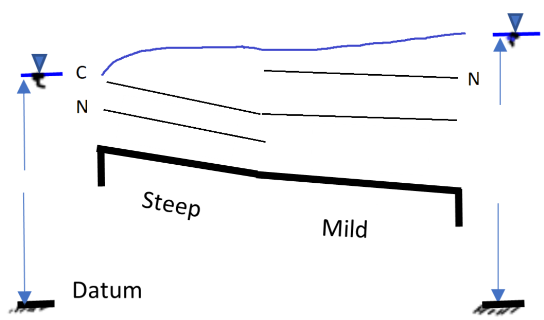


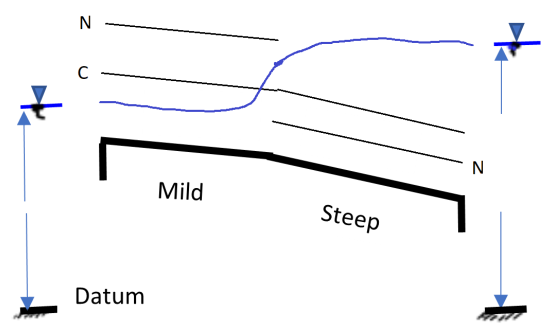
Solution

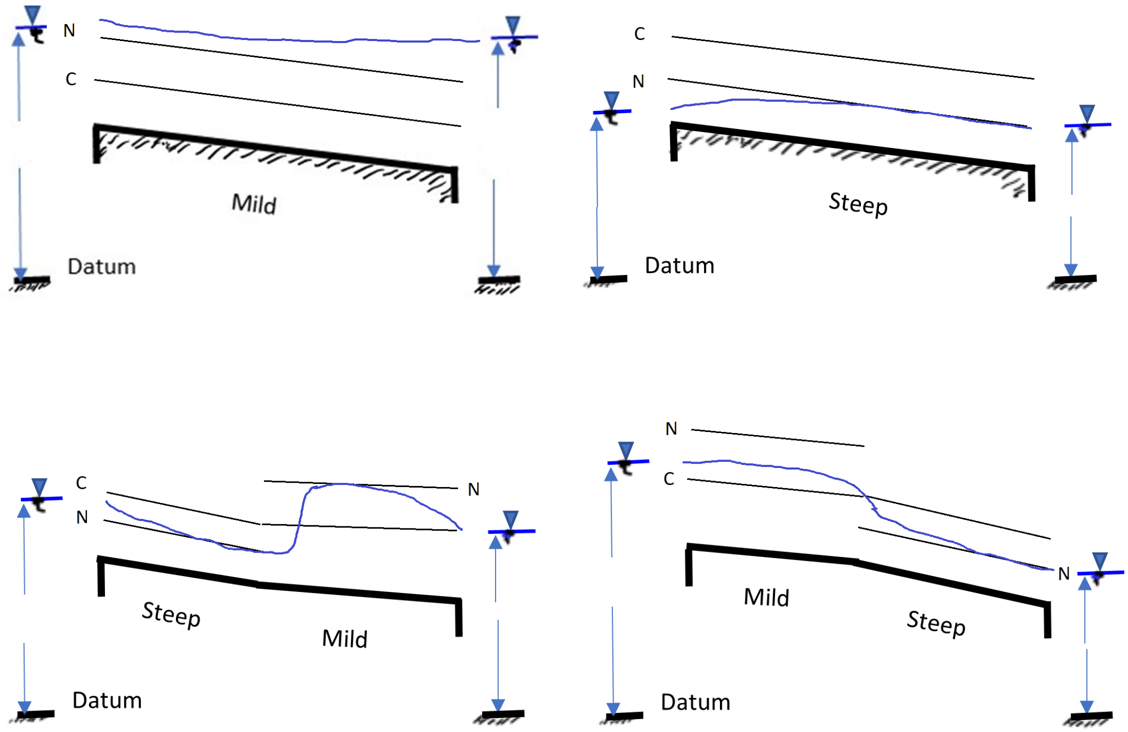
I drew all of this diagrams using the gradually varied flow equation. The two notable features are the tendency to increase or decrease in the corresponding zone and the sharpness as we leave or go in to the critical flow line.







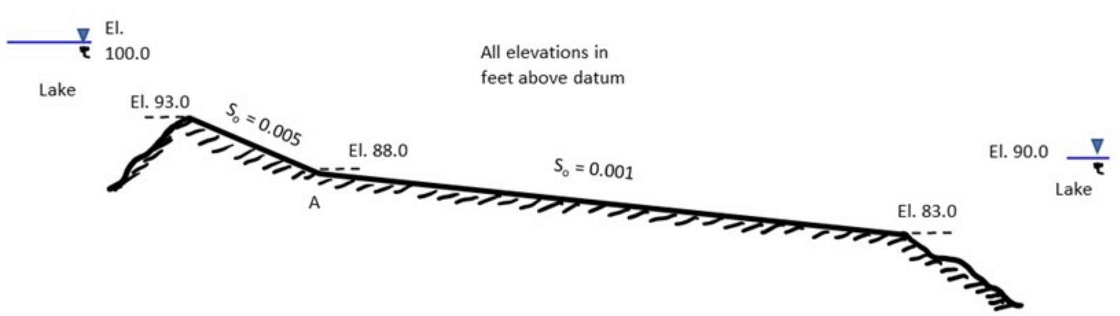




3.

The channel whose longitudinal section in shown in the figure is rectangular in section, 30 ft wide, with n = 0.014.

Find the discharge in the channel and sketch the longitudinal profile of the water surface. Determine whether a hydraulic jump occurs and if so whether it is upstream or downstream of Section A.



Solution

I approach this using the observations on section 5-6 "Discharge from a reservoir".

In [3]:
```
g=32.17;    B=30;    n=0.014;    k=1.49;
```

Assuming no losses on the channel entrance and that the lake water velocity is negligible the total energy near the entrance is $E = 7 + \frac{0^2}{2g} = 7$ and so the critical depth is $y_c = \frac{2}{3}E = 4.67ft$.

Let's first try to find the critical slope assuming that the slope after the lake is critical or steep. In this case critical flow happens near the entrance and so

$$V_c = \sqrt{gy_c} = 12.25\ ft$$

$$Q = AV_c = 1715.36\ cfs.$$

$$R = \frac{y_c B}{2y_c + B} = 3.56ft$$

So, from mannings equation I can find the critical slope as

$$S_c = \left(\frac{n}{k}V_c R^{-2/3}\right)^2 = 0.00243.$$

We conclude that the slope is in fact steep $S_0 = 0.005 > 0.00243 = S_c$ and so we keep the discharge and depth values at the entrance of the channel.

In [4]:
```
yc=2*7/3
P=2*yc+B
A=yc*B
R=A/P
vc=(g*yc)**0.5
Q=vc*A
Sc=( (n/k) * vc * (R**(-2/3)) )**2

print(yc)
print(vc)
print(Q)
print(R)
print(Sc)
```

```
4.666666666666667
12.252618767702955
1715.3666274784136
3.559322033898305
0.002438849487976162
```

After water passes critical depth at the entrance it becomes super critical and since we are on a steep slope we know that it goes into zone two above above normal and below critical. According to the gradually varying flow equation

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2} = \frac{+}{-} = - < 0$$

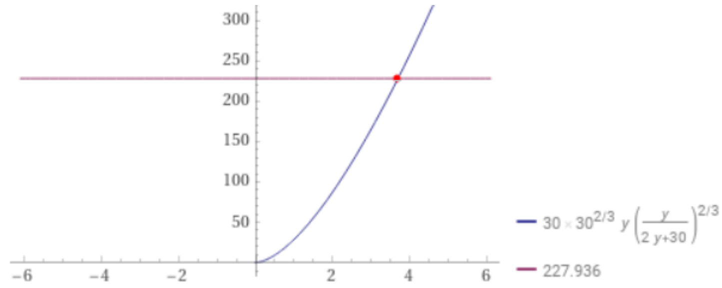
So we know the profile drops asymptotically towards normal depth. I will therefore find normal depth to sketch the profile over the first channel. Appplying Manning's equation to find the normal flow conditions we get

$$30y_n\left(\frac{30y_n}{2y_n + 30}\right)^{2/3} = AR^{2/3} = \frac{nQ}{kS_0^{1/2}} = 227.936$$

In [5]:
```
S_steep=0.005
righthandside=n*Q/(k*( S_steep**0.5 ))
print(righthandside)
```

```
227.93641937121052
```

Solving this using wolfram we get $y_n = 3.6862$



Alternate form assuming y is real

Enlarge | Data | Customize

$$\frac{y^{5/3}}{(y+15)^{2/3}} = 1.2492$$

Alternate forms

More

$$y\left(\frac{y}{y+15}\right)^{2/3} = 1.2492$$

$$\frac{30\, y^{5/3}}{\left(\frac{y}{15}+1\right)^{2/3}} = 227.936$$

$$\frac{30\times 15^{2/3}\, y^{5/3}}{(y+15)^{2/3}} = 227.936$$

Alternate form assuming y>0

$$\frac{30\times 30^{2/3}\, y^{5/3}}{(2\,y+30)^{2/3}} = 227.936$$

Number line

Solution

Exact form | More digits

$y \approx 3.68627$

Since the slope on the second channel $S_0 = 0.002 < 0.00243 = S_c$ is smaller than the critical slope we know that the channel is mild and therefore the normal depth is above the critical depth and so there will be a hydraulic jump after section A as we go from zone 2 (subcritical) of a steep channel into zone 3 (super critical) of a mild channel which will push the profile up.

I will now find the normal depth at the second channel for sketching purposes. Using Manning's equation we get

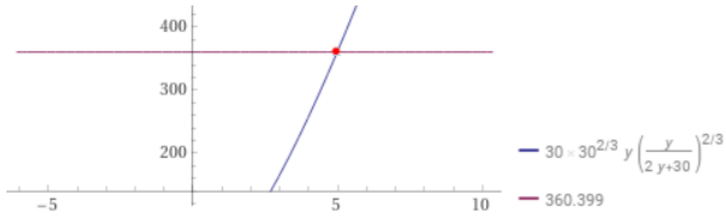
$$30y_c\left(\frac{30y_c}{2y_c+30}\right)^{2/3} = AR^{2/3} = \frac{nQ}{kS_0^{1/2}} = 360.399$$

In [6]:
```
S_mild=0.002
```

```
righthandside=n*Q/(k*( S_mild**0.5 ))
print(righthandside)
```

```
360.39912345817504
```

whose solution is (wolfram again) $y_n = 4.9847$.



Alternate form assuming y is real

Enlarge | Data | Customize

$$\frac{y^{5/3}}{(y+15)^{2/3}} = 1.97516$$

Alternate forms

More

$$y\left(\frac{y}{y+15}\right)^{2/3} = 1.97516$$

$$\frac{30\,y^{5/3}}{\left(\frac{y}{15}+1\right)^{2/3}} = 360.399$$

$$\frac{30 \times 15^{2/3}\,y^{5/3}}{(y+15)^{2/3}} = 360.399$$

Alternate form assuming y>0

$$\frac{30 \times 30^{2/3}\,y^{5/3}}{(2\,y+30)^{2/3}} = 360.399$$

Number line

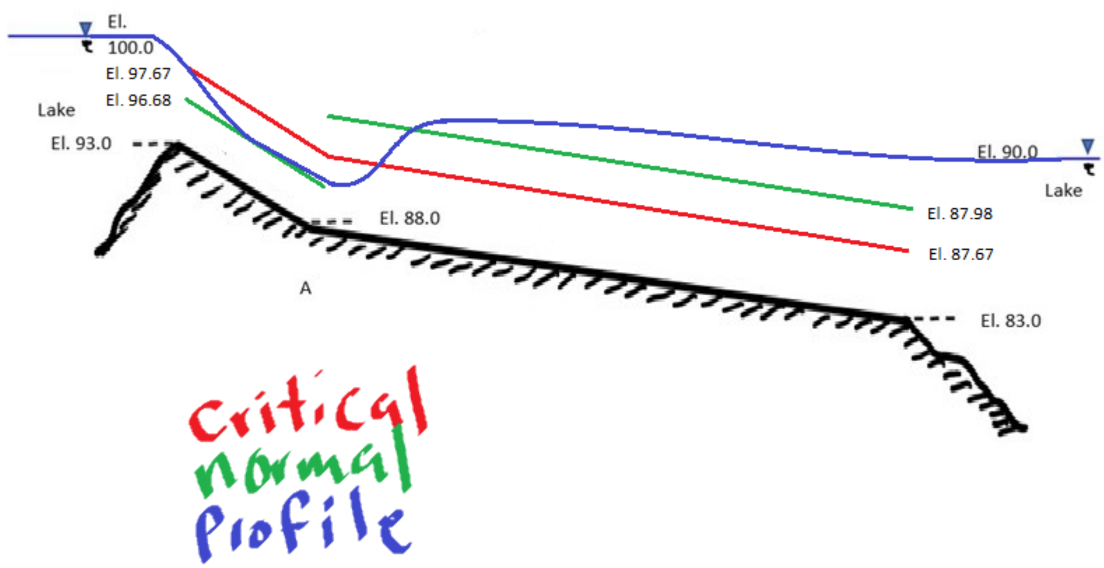
Solution

Exact form | More digits

$y \approx 4.98471$

Now notice that the water elevation when entering the lake downstream is $y = 7$ which is way above the normal height. We can use this information to infer that the hydraulic jump actually goes above the normal depth into zone 1 as otherwise we will have a decresing slope in zone 2. When in zone 1 on a mild channel the slope tends to 0 which also makes sense as we're entering a lake.

We finally have all the pieces we need to make the sketch of the profile. (my artistic skills are not great, I know the 88 is below the 87's but I couldn't make a better drawing)



4.

Water is flowing at a velocity of 10 ft/sec and a depth of 10 ft in a channel of rectangular section. Find the change in depth and in absolute water level produced by

a) a smooth upward step of 1 ft;

b) a smooth downward stop of 1 ft, in the channel bed. Also,

c) find the maximum allowable size of the upward step for the upstream flow to be possible as specified.

Solution

For all the problems we will use the same setup. Consider section one before the step and section 2 after the step.

We start by writing down the relevant equations we will use. The conservation of mass

$$Q_1 = A_1 V_1 = A_2 V_2 = Q_2$$

can be rewritten as

$$100 = (10)(10) = y_1 V_1 = y_2 V_2$$

since the channel is rectangular and so the width is the same. Now we use the conservation of energy

$$y_1 + \frac{V_1^2}{2g} = \Delta z + y_2 + \frac{V_2^2}{2g}$$

and knowing the values of $y_1, V_1, \Delta z$ yields two equations in the two variables which we can solve.

We will be making use of the condition for critical flow $Fr = \frac{V}{\sqrt{gy}} = 1$

$$V = \sqrt{gy}.$$

In particular we know that we start on subcritical flow since

$$Fr_1^2 = \frac{100}{32.17 \cdot 10} = 0.31$$

In [7]:
```
10/32.17
```

Out[7]:
```
0.31084861672365555
```

a) upward step of 1ft

In this case $\Delta z = 1$ and so we are solving the system

$$10 + \frac{100}{2g} = 1 + y_2 + \frac{V_2^2}{2g}, \quad 100 = y_2 V_2$$

whose solutions acording to wolfram are

$$\begin{aligned} y &= -3.34 & V &= -29.90, \\ y &= 8.29 & V &= 12.05, \\ y &= 5.60 & V &= 17.84. \end{aligned}$$

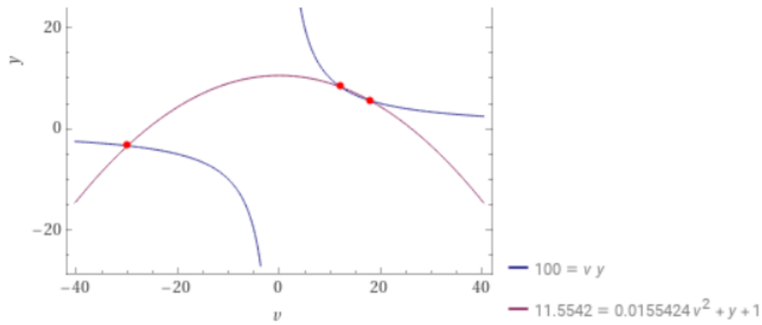
Since we assumed no energy losses and a smooth step we know that we will move continuously on the energy curve downwards since $\frac{dz}{dx} > 0$ and we are subcritical.

Therefore the solution has to be $y_2 = 8.29$ and $V_2 = 12.05$ which corresponds to a difference in depth of

$$y_2 - y_1 = -1.71 ft$$

and the absolute water level difference

$$z_2 - z_1 + y_2 - y_1 = 1 - 1.71 = -0.71 ft$$



Alternate forms

$\{v\, y = 100,\ y = 10.5542 - 0.0155424\, v^2\}$

$\{v\, y = 100,\ 0.0155424\, v^2 + y + 1 = 11.5542\}$

$\{100 = v\, y,\ 11.5542 = 0.0155424\,(v^2 + 64.34\, y + 64.34)\}$

Alternate form assuming v and y are real

$\{100 = v\, y,\ 11.5542 = 0.0155424\, v^2 + y + 1\}$

Solutions Exact forms More digits

$v \approx -29.9035 + 2.13163 \times 10^{-15}\, i,\quad y \approx -3.34409 + 1.98145 \times 10^{-15}\, i$

$v \approx 12.0542 - 5.68434 \times 10^{-15}\, i,\quad y \approx 8.29588 + 2.12994 \times 10^{-15}\, i$

$v \approx 17.8493 - 8.52651 \times 10^{-15}\, i,\quad y \approx 5.60246 + 4.73088 \times 10^{-15}\, i$

b) downward step of 1ft

In this case $\Delta z = -1$ and so we are solving the system

$$10 + \frac{100}{2g} = -1 + y_2 + \frac{V_2^2}{2g}, \quad 100 = y_2 V_2$$

whose solutions acording to wolfram are

$$\begin{aligned} y &= -3.14 \qquad & V &= -31.78, \\ y &= 11.34 \qquad & V &= 8.81, \\ y &= 4.35 \qquad & V &= 22.97. \end{aligned}$$

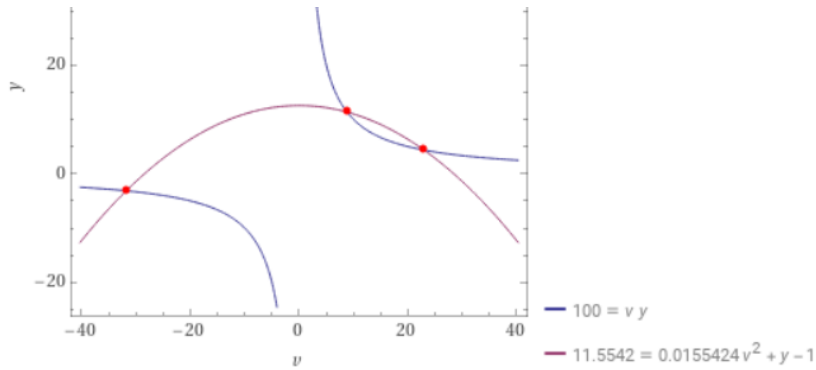
Since we assumed no energy losses and a smooth step we know that we will move continuously on the energy curve upwards since $\frac{dz}{dx} < 0$ and we are subcritical.

Therefore the solution has to be $y_2 = 11.34$ and $V_2 = 8.81$ which corresponds to a difference in depth of

$$y_2 - y_1 = 1.34ft$$

and the absolute water level difference

$$z_2 - z_1 + y_2 - y_1 = - + 1.34 = 0.34ft$$



Alternate forms

$\{v\, y = 100,\ y = 12.5542 - 0.0155424\, v^2\}$

$\{v\, y = 100,\ 0.0155424\, v^2 + y = 12.5542\}$

$\{100 = v\, y,\ 11.5542 = 0.0155424\,(v^2 + 64.34\, y - 64.34)\}$

Alternate form assuming v and y are real

$\{100 = v\, y,\ 11.5542 = 0.0155424\, v^2 + y - 1\}$

Solutions Exact forms More

$v \approx -31.7832 + 1.42109\times 10^{-15}\, i, \quad y \approx -3.14631 + 1.404\times 10^{-15}\, i$

$v \approx 8.8128 - 8.52651\times 10^{-15}\, i, \quad y \approx 11.3471 + 2.33579\times 10^{-15}\, i$

$v \approx 22.9704 - 1.42109\times 10^{-14}\, i, \quad y \approx 4.35342 + 1.0147\times 10^{-14}\, i$

c) maximum step height

Using the knowledge from the energy curve just like in the previous problems we know that a step up $\frac{dz}{dx}$ can only go as far as avaliable energy and so the minimum energy level will correspond highest step possible.

The minimum energy happens at critical flow, therefore we can find solve

$$y_2 V_2 = 100, \quad V_2^2 = g y_2$$

so $V_2^3 = 3217$ which yields

$$y_2 = 6.77407, \quad V_2 = 14.76$$

and the minimum energy will be $E_m = 10.1611$. Since the energy before the step is $E = 11.5542$ then the maximum step size without going below the energy level is

$$\Delta z = E - E_m = 1.3931 ft$$

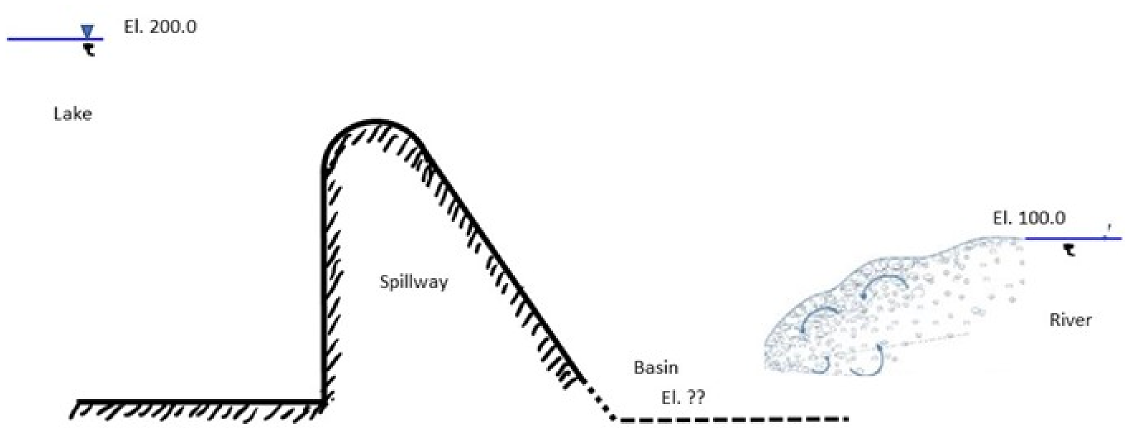
In [8]:
```
V = 3217**(1/3)
y = 100/V
print(y, V)
Em= y+V*V/(2*32.17)
print(Em)
E=10+100/(2*32.17)
print(E)
print(E-Em)
```

```
6.774069473458206 14.762175143289365
10.161104210187307
11.554243083618278
1.3931388734309706
```

5.

Water discharges at the rate of 10,000 cusecs (cfs) over a spillway 40 ft wide into a stilling basin of the same width. The lake level behind the spillway is 200 ft above datum; the river level downstream is 100 ft above datum.

Assuming no energy is dissipated in the flow down the spillway, find the basin invert level required to a hydraulic jump to form within the basin.



Solution

I consider section one to be before the hydraulic jump and section 2 to be after the hydraulic jump, both on the stilling basin. Since there's no energy loss we have

$$E_1 = 100 - E_2$$

further more, assuming steady state, the conservation of mass, and the fact that the witdh is the same, I know

$$\frac{Q}{B} = \frac{10000}{40} = 250 = V_1 y_1 = V_2 y_2$$

From where we can express the velocities as a function of the depths as

$$\frac{250}{y_1} = V_1 = V_1(y_1), \quad \frac{250}{y_2} = V_2 = V_2(y_2).$$

In [9]:
```
10000/40
```

Out[9]:
```
250.0
```

I also know that $y_2 = y_2(z) = 100 - z$ where z is the elevation of the stilling basin. Additionally, using the relationship between the conjugate depths we know that

$$y_1 = y_1(y_2(z)) = \frac{y_2}{2}(\sqrt{1 + 8Fr_2^2} - 1)$$

where

$$Fr_2 = Fr_2(y_2(z)) = \frac{V_2}{\sqrt{gy_2}}$$

Using the previous equations we can express all velocities and depths as functions of z, and from the expanded energy equation

$$y_1 + \frac{V_1^2}{2g} - 100 = y_2 + \frac{V_2^2}{2g}$$

and noting that $y_1 = \frac{y_2}{2}(\sqrt{1+8\frac{(250/y_2)^2}{gy_2}}-1) = \frac{y_2}{2}(\sqrt{1+\frac{500000}{gy_2^3}}-1)$

we get

$$\frac{y_2}{2}\left(\sqrt{1+\frac{500000}{gy_2^3}}-1\right)+\frac{1}{2g}\left(\frac{250}{\frac{y_2}{2}(\sqrt{1+\frac{500000}{gy_2^3}}-1)}\right)^2-100=y_2+$$

that is

In [10]:
```
print(250*250)
print(250*250*8)
```

```
62500
500000
```

I have no interest of solving that or doing algebra this time, so I'll just put it in wolfram. The variables mapping I used is

$$x=y_1,\quad y=y_2,\quad v=V_2,\quad w=V_1,\quad z=Fr_2$$

WolframAlpha computational intelligence.

g=38.17, z=v/sqrt(g*y), x= (y/2)*(sqrt(1+8*z^2)-1), v= 250/y, w = 250/x, x+w^2/(2*g)-100=y+v^2/(2*g)

NATURAL LANGUAGE MATH INPUT EXTENDED KEYBOARD EXAMPLES UPLOAD RANDOM

Input

$$\left\{g = 38.17,\ z = \frac{v}{\sqrt{g\,y}},\ x = \frac{y}{2}\left(\sqrt{1+8\,z^2}-1\right),\ v = \frac{250}{y},\ w = \frac{250}{x},\ x + \frac{w^2}{2\,g} - 100 = y + \frac{v^2}{2\,g}\right\}$$

Result

$$\left\{g = 38.17,\ z = \frac{v}{\sqrt{g\,y}},\ x = \frac{1}{2}\,y\left(\sqrt{8\,z^2+1}-1\right),\ v = \frac{250}{y},\ w = \frac{250}{x},\ \frac{w^2}{2\,g} + x - 100 = \frac{v^2}{2\,g} + y\right\}$$

Alternate form

$$\left\{g = 38.17,\ z = \frac{v}{\sqrt{g\,y}},\ x = y\left(\frac{1}{2}\sqrt{8\,z^2+1}-\frac{1}{2}\right),\ v = \frac{250}{y},\ w = \frac{250}{x},\ g\,y = -\frac{v^2}{2} + \frac{w^2}{2} - \frac{1}{2}\,g\,(200-2\,x)\right\}$$

Real solution Exact form More digits

$g \approx 38.17$, $v \approx 7.11548$, $w \approx 100.884$,
$x \approx 2.47809$, $y \approx 35.1347$, $z \approx 0.194301$

From where we conclude that

$$y_1 = 2.48\,ft,\quad y_2 = 35.13\,ft,\quad V_2 = 7.12\,ft/s,\quad V_1 = 100.88\,ft/s,\quad Fr_2$$

Finally, we find the elevation of the stilling basin $z = 100 - y_2 = 64.87\,ft$.

In [11]:
```
100-35.13
```
Out[11]:
```
64.87
```