CÁLCULO III Matemáticas e Informática Curso 2018/2019	1$^{\text{er}}$ Apellido:	16/01/2019
	2º Apellido:	Tiempo: 3h
Dpto. Matemática Aplicada TIC ETSI Informáticos Universidad Politécnica de Madrid	Nombre:	Calificación:
	Número de matrícula:	

EXAMEN FINAL ENERO Y RECUPERACIONES

PRIMERA PARTE

1. **(1 punto)** Calcula la integral de la función $f(x,y) = xy$ sobre el recinto acotado del primer cuadrante limitado por la parábola $y = 2x^2$ y por la recta $y = 2$.

2. **(1 punto)** Calcula el volumen del recinto $D \subset \mathbb{R}^3$ interior al cilindro $x^2 + y^2 = 1$ y limitado por el paraboloide $z = 2 - x^2 - y^2$ y el plano $z = 0$.

3F. **(1,5 puntos)** Calcula el volumen del recinto acotado por la superficie $(x^2 + y^2 + z^2)^2 = a^3 x$, $a > 0$.

3R. **(1,5 puntos)** Calcula la integral de la función $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ sobre la esfera unidad $x^2 + y^2 + z^2 \leq 1$.

4. **(1,5 puntos)** Sea $\gamma \subset \mathbb{R}^3$ la curva parametrizada por $\alpha(t) = (t,\, 1 + \cos t,\, \sin t)$, $0 \leq t \leq 2\pi$.

(a) Halla la longitud de γ.

(b) Calcula su masa suponiendo densidad puntual $\rho(x,y,z) = x^2 + y^2 + z^2$.

SEGUNDA PARTE

1. **(1,5 puntos)** Se considera el campo vectorial $F(x,y) = (y - e^x \sin y)\mathbf{i} + (3x - e^x \cos y)\mathbf{j}$.
(a) Determina si F es o no conservativo y, en caso afirmativo, encuentra su función potencial.
(b) Calcula la integral de F sobre la circunferencia $x^2 + y^2 = 4$ orientada positivamente.

2. **(1 punto)** Calcula la integral del campo vectorial $F(x,y,z) = (-x,\, -y,\, z)$ sobre la superficie S parametrizada por $\Phi(u,v) = (u,\, v,\, u^2 + v^2)$, $(u,v) \in [0,\, 1]^2$.

3R. **(1,5 puntos)** Sea S la parte del paraboloide $z = 2 - x^2 - y^2$ que está dentro del cilindro $x^2 + y^2 = 1$.

(a) Calcula el área de la superficie S.

(b) Halla la integral curvilínea de $F(x,y,z) = (x,\, y,\, 2 - z)$ sobre la superficie S (indicando la orientación considerada).

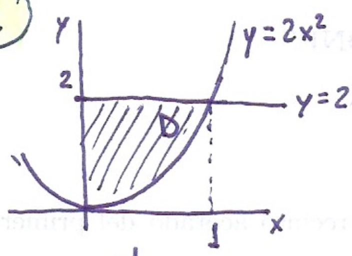
3F. **(1,5 punto)** Calcula la integral del rotacional de la función $F(x,y,z) = (y - z,\, z - x,\, x - y)$ sobre la superficie

$$S = \left\{(x,y,z) \,:\, x^2 + y^2 + z^2 = 1,\, x + y + z \geq 1\right\}$$

4. **(1 punto)** Calcula la serie de Fourier de senos de la función $f(x) = x$, $0 < x < \pi$, y dibuja la función suma en el intervalo $[-3\pi,\, 3\pi]$.

SOLUCIONES PRIMERA PARTE

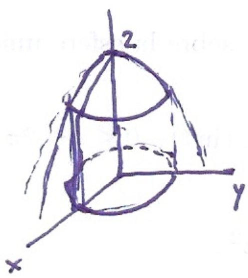
1



$$\iint_D xy\,dx\,dy = \int_0^1 dx \int_{2x^2}^{2} xy\,dy = \int_0^1 \left[\frac{xy^2}{2}\right]_{y=2x^2}^{y=2} dx =$$

$$= \int_0^1 (2x - 2x^5)\,dx = \left[x^2 - \frac{x^6}{3}\right]_{x=0}^{x=1} = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

2



$$D = \{(x,y,z) : x^2+y^2 \le 1,\ 0 \le z \le 2-x^2-y^2\}$$

$$V(D) = \iint_{x^2+y^2\le 1} [(2-x^2-y^2) - 0]\,dx\,dy = \int_0^{2\pi} d\theta \int_0^1 (2-\rho^2)\rho\,d\rho =$$

$$= 2\pi\left[\rho^2 - \frac{\rho^4}{4}\right]_{\rho=0}^{\rho=1} = 2\pi\left(1-\frac{1}{4}\right) = \boxed{\frac{3\pi}{2}}$$

3F

$$(x^2+y^2+z^2)^2 = a^3 x \xrightarrow[\text{esfericas}]{} \rho^4 = a^3\rho\cos\theta\sin\varphi \Rightarrow \begin{cases} \rho = 0 \\ \rho = a\sqrt[3]{\cos\theta\sin\varphi} \end{cases}$$

$$\rho \ge 0 \iff \cos\theta \ge 0 \iff -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$V = \iiint_\Omega dx\,dy\,dz \underset{\substack{\uparrow \\ \text{esfericas}}}{=} \int_{-\pi/2}^{\pi/2} d\theta \int_0^{\pi} d\varphi \int_0^{a(\cos\theta\sin\varphi)^{1/3}} \rho^2 \sin\varphi\,d\rho = \frac{1}{3}\int_{-\pi/2}^{\pi/2} d\theta \int_0^{\pi} \sin\varphi \left[\rho^3\right]_{\rho=0}^{\rho=a(\cos\theta\sin\varphi)^{1/3}} d\varphi =$$

$$= \frac{a^3}{3}\int_{-\pi/2}^{\pi/2} d\theta \int_0^{\pi} \cos\theta\cdot\underbrace{\sin^2\varphi}_{\frac{1-\cos 2\varphi}{2}}\,d\varphi = \frac{a^3}{3}\left[\sin\theta\right]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \cdot \left[\frac{\varphi}{2} - \frac{\sin 2\varphi}{4}\right]_{\varphi=0}^{\varphi=\pi} =$$

$$= \frac{a^3}{3}\cdot 2\cdot\frac{\pi}{2} = \boxed{\frac{\pi a^3}{3}}$$

3R

$$\iiint_{x^2+y^2+z^2\le 1} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz \underset{\uparrow\ \text{esféricas}}{=} \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{1} \rho\cdot\rho^2 \sin\varphi\, d\rho = 2\pi\left[-\cos\varphi\right]_{\varphi=0}^{\varphi=\pi}\cdot\left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=1}$$

$$= 2\pi\cdot 2\cdot\frac{1}{4} = \boxed{\pi}$$

4

$\alpha(t) = (t,\ 1+\cos t,\ \sin t)$, $0\le t\le 2\pi$

$\alpha'(t) = (1, -\sin t, \cos t)$; $\|\alpha'(t)\| = \sqrt{1+(-\sin t)^2+\cos^2 t} = \sqrt{2}$

a) $$\ell(\gamma) = \int_0^{2\pi} \|\alpha'(t)\|\,dt = \int_0^{2\pi}\sqrt{2}\,dt = \boxed{2\sqrt{2}\,\pi}$$

b) $$m = \int_\gamma (x^2+y^2+z^2)\,ds = \int_0^{2\pi}\left[t^2+(1+\cos t)^2+\sin^2 t\right]\cdot\sqrt{2}\,dt =$$

$$= \int_0^{2\pi}\sqrt{2}\,(t^2+2\cos t+2)\,dt = \left[\sqrt{2}\left(\frac{t^3}{3}+2\sin t+2t\right)\right]_{t=0}^{t=2\pi} = \boxed{4\sqrt{2}\left(\frac{2\pi^2}{3}+1\right)\pi}$$

SOLUCIONES SEGUNDA PARTE

1'

$$F(x,y) = (\underbrace{y-e^x\sin y}_{P})\,\vec{\imath} + (\underbrace{3x-e^x\cos y}_{Q})\,\vec{\jmath}$$

a) $$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} = (3-e^x\cos y)-(1-e^x\cos y) = 2\ne 0 \Rightarrow \boxed{F \text{ no conservativo}}$$

b) $$\oint_{x^2+y^2=4} F\,ds \underset{\uparrow\ \text{Greene}}{=} \iint_{x^2+y^2\le 4}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy = \iint_{x^2+y^2\le 4} 2\,dx\,dy = 2\cdot(\text{area}\,(x^2+y^2\le 4)) = 2\cdot\pi\cdot 2^2 = \boxed{8\pi}$$

2'

$$\left.\begin{array}{l}\Phi(u,v) = (u,v,u^2+v^2)\\ (u,v)\in[0,1]^2\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\dfrac{\partial\Phi}{\partial u} = (1,0,2u)\\ \dfrac{\partial\Phi}{\partial v} = (0,1,2v)\end{array}\right. \Rightarrow \frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v} = (-2u,-2v,1)$$

$$\iint_S F\,d\sigma = \iint_{[0,1]^2} F(\Phi(u,v))\cdot\left(\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}\right)du\,dv = \iint_{[0,1]^2}(-u,-v,u^2+v^2)\cdot(-2u,-2v,1)\,du\,dv =$$

$$= \iint_{[0,1]^2} 3(u^2+v^2)\,du\,dv = \int_0^1 du\int_0^1 (3u^2+3v^2)\,dv = \int_0^1\left[3u^2v+v^3\right]_{v=0}^{v=1} du =$$

$$= \int_0^1 (3u^2+1)\,du = \left[u^3+u\right]_{u=0}^{u=1} = 1+1 = \boxed{2}$$

3R

$$z = 2-x^2-y^2 \Rightarrow \Phi(u,v) = (u, v, 2-u^2-v^2) \quad (u,v)\in D = \{(u,v) : u^2+v^2 \le 1\}$$

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial u} &= (1,0,-2u) \\ \frac{\partial \Phi}{\partial v} &= (0,1,-2v) \end{aligned}\right\} \Rightarrow \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} = (2u, 2v, 1), \quad \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| = \sqrt{1+4(u^2+v^2)}$$

a) $$A(S) = \iint_D \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| du\,dv = \iint_D \sqrt{1+4(u^2+v^2)}\,du\,dv = \int_0^{2\pi} d\theta \int_0^1 \sqrt{1+4\rho^2}\cdot \rho\, d\rho =$$

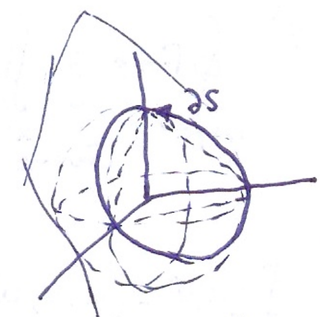
$$= 2\pi \left[\frac{(1+4\rho^2)^{3/2}}{\frac{3}{2}\cdot 8} \right]_{\rho=0}^{\rho=1} = \frac{\pi}{6}\left(5^{3/2}-1\right) = \boxed{\frac{5\sqrt{5}-1}{6}\pi}$$

b) $$\iint_S F\,d\sigma = \iint_D F(\Phi(u,v))\cdot\left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right) du\,dv = \iint_D (u, v, u^2+v^2)\cdot(2u, 2v, 1)\,du\,dv =$$

$$= \iint_D 3(u^2+v^2)\,du\,dv = \int_0^{2\pi} d\theta \int_0^1 3\rho^2\cdot\rho\,d\rho = 2\pi\left[\frac{3\rho^4}{4}\right]_{\rho=0}^{\rho=1} = \boxed{\frac{3\pi}{2}}$$

3F

$F(x,y,z) = (y-z, z-x, x-y)$



$S = \{(x,y,z) : x^2+y^2+z^2 = 1,\ x+y+z \ge 1\}$

$S_1 = \{(x,y,z) : x^2+y^2+z^2 \le 1,\ x+y+z = 1\}$

$S \equiv$ parte de la esfera exterior al plano $(x+y+z \ge 1$

$S_1 \equiv$ parte del plano $x+y+z=1$ interior a la esfera $\}$ comparten frontera $\partial S_1 = \partial S$

S_1 es la circunferencia que pasa por los puntos $(1,0,0)$, $(0,1,0)$ y $(0,0,1)$ que tiene por centro $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ y radio $d\left((1,0,0),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right) = \frac{\sqrt{6}}{3}$, por lo que su area es: $A(S_1) = \pi\cdot\left(\frac{\sqrt{6}}{3}\right)^2 = \frac{2\pi}{3}$ y $n = \frac{(1,1,1)}{\sqrt{3}}$

$$\operatorname{rot}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix} = (-2,-2,-2) = -2(1,1,1)$$

Entonces:

Integral de campo vect.

$$\iint_S \operatorname{rot}(F)\, d\sigma \underset{\text{Stokes}}{=} \oint_{\partial S} F\, ds = \oint_{\partial S_1} F\, ds \underset{\text{Stokes}}{=} \iint_{S_1} \operatorname{rot}(F)\, d\sigma = \cancel{\iint_{S_1}} \underbrace{\iint_{S_1} \operatorname{rot}(F)\cdot n\, d\sigma}_{\text{Int. de un campo escalar}} =$$

$$= \iint_{S_1} -2(1,1,1)\cdot\frac{(1,1,1)}{\sqrt{3}}\, d\sigma = -2\sqrt{3} \iint_{S_1} d\sigma = -2\sqrt{3}\cdot A(S_1) =$$

$$= -2\sqrt{3}\cdot\frac{2\pi}{3} = \boxed{\frac{-4\pi}{\sqrt{3}}}$$

4) $f(x) = x, \quad 0 < x < \pi \quad ; \quad f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\cdot\sin nx\, dx = \dots\dots = \frac{-2(-1)^n}{n}$$

$$\boxed{f(x) \sim 2\sum_{n=1}^{\infty} \frac{-(-1)^n \sin nx}{n} = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right)}$$