Work and energy

Goals for Lecture

- To understand and calculate the work done by a force.

- To understand the meaning of kinetic energy.

- To learn how work changes the kinetic energy of a body and how to use this principle.

- To relate work and kinetic energy when the forces are not constant or the body follows a curved path.

- To solve problems involving power.

Work Done by a Constant Force

The work W done on a system by an agent exerting a constant force on the system is the product of the magnitude F of the force, the magnitude Δr of the displacement of the point of application of the force, and $\cos \alpha$, where α is the angle between the force and displacement vectors:

$$W = F \Delta r \cos \alpha$$

Or using definition of scalar product of two vectors:

$$W = \vec{F} \cdot \Delta \vec{r}$$
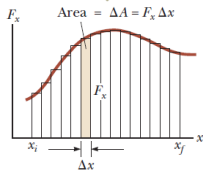
Work W is the SCALAR quantity!
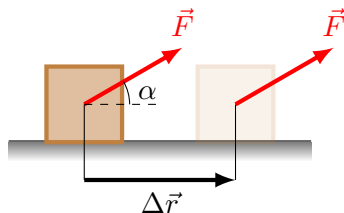
Work Done by a Varying Force

Consider a particle being displaced along the x axis under the action of a force that varies with position. The particle is displaced in the direction of increasing x from $x = x_i$ to $x = x_f$. In such a situation, we cannot use $W = F\Delta r \cos \alpha$ to calculate the work done by the force because this relationship applies only when $\vec{F}$ is constant in magnitude and direction. If, however, we imagine that the particle undergoes a very small displacement dx, the x component F_x of the force is approximately constant over this small interval; for this small displacement, we can approximate the work done on the particle by the force as
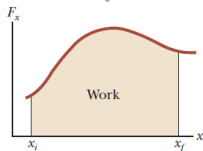
$$\delta W \approx F_x \Delta x$$

which is the area of the shaded rectangle.

The total work done for the displacement from x_i to x_f is approximately equal to the sum of the areas of all the rectangles.

a

The work done by the component F_x of the varying force as the particle moves from x_i to x_f is *exactly* equal to the area under the curve.
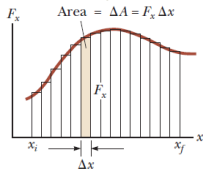
b

Work Done by a Varying Force

If we imagine the F_x versus x curve divided into a large number of such intervals, the total work done for the displacement from x_i to x_f is approximately equal to the sum of a large number of such terms:

$$W \approx \sum_{x_i}^{x_f} F_x \Delta x$$

If the size of the small displacements is allowed to approach zero, the number of terms in the sum increases without limit but the value of the sum approaches a definite value equal to the area bounded by the F_x curve and the x axis:
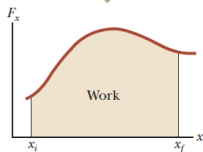
$$W = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x dx$$

The total work done for the displacement from x_i to x_f is approximately equal to the sum of the areas of all the rectangles.

a

The work done by the component F_x of the varying force as the particle moves from x_i to x_f is *exactly* equal to the area under the curve.
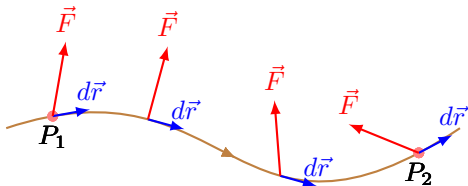


b

Work Done by a Varying Force

For the general case of a net force $\vec{F}$ whose magnitude and direction may vary, we use the scalar product:

$$W = \int\limits_{\text{point 1}}^{\text{point 2}} \vec{F}d\vec{r} = \int\limits_{\text{point 1}}^{\text{point 2}} F_x dx + F_y dy + F_z$$

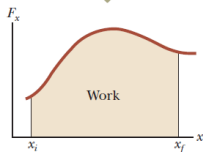where integration done between point 1 and point 2 of the body trajectory.

The total work done for the displacement from x_i to x_f is approximately equal to the sum of the areas of all the rectangles.

Area $= \Delta A = F_x \Delta x$

F_x

x_i Δx x_f x

The work done by the component F_x of the varying force as the particle moves from x_i to x_f is *exactly* equal to the area under the curve.

F_x

Work

x_i x_f x
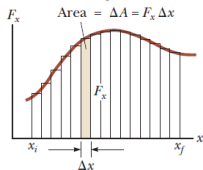
Power

To characterize intensity of the work performed, the quantity called (power) is introduced. Power is denned as the work performed by a force per unit time:

$$N = \frac{dW}{dt}.$$

If the force $\vec{F}$ performs the work $\vec{F} \cdot d\vec{r}$ during the time interval dt, the power developed by that force at a given moment of time is equal to $N = \vec{F} \cdot \frac{d\vec{r}}{dt}$; taking into account that $\frac{d\vec{r}}{dt} = \vec{v}$, we obtain

$$N = \vec{F} \cdot \vec{v}.$$

Thus, the power developed by the force $\vec{F}$ is equal to the scalar product of the force vector by the vector of velocity $\vec{v}$ with which the point moves under the action of the given force. Just like work, power is an scalar quantity.
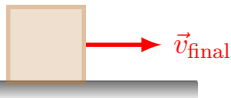
Kinetic energy

① The kinetic energy of a particle is $\boxed{K = \frac{mv^2}{2}}$.

② The net work on a body changes its velocity and therefore its kinetic energy.

$K_{\text{initial}} = \frac{mv_{\text{initial}}^2}{2}$ $\qquad\qquad$ $K_{\text{final}} = \frac{mv_{\text{final}}^2}{2}$

$\vec{v}_{\text{initial}}$ $\qquad\qquad$ $\vec{v}_{\text{final}}$

③ The work-energy theorem: The work done by the net force on a particle equals the change in the particle's kinetic energy. Mathematically, the $\boxed{\text{Work-kinetic energy theorem}}$ is expressed as:
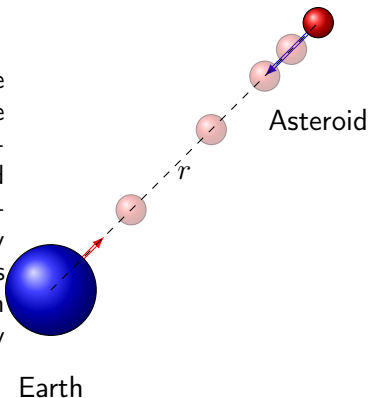
$$W = K_{\text{final}} - K_{\text{initial}}.$$

Potential Energy

One job of physics is to identify the different types of energy in the world, especially those that are of common importance. One general type of energy is $\boxed{\text{potential energy } U}$. Technically, potential energy is energy that can be associated with the configuration (arrangement) of a system of objects that exert forces on one another.

The system of objects consists of Earth and the Asteroid. The force between the objects is the gravitational force. The configuration of the system changes (the separation between the Asteroid and Earth changes). We can account for the Asteroid's motion and increase in kinetic energy by defining a gravitational potential energy U. This is the energy associated with the state of separation between two objects that attract each other by the gravitational force.


Asteroid

r

Earth

Potential Energy

One job of physics is to identify the different types of energy in the world, especially those that are of common importance. One general type of energy is [potential energy U]. Technically, potential energy is energy that can be associated with the configuration (arrangement) of a system of objects that exert forces on one another.

For either rise or fall, the change

$$\Delta U = U_2 - U_1$$

in gravitational potential energy is defined as being equal to the negative of the work done on the Asteroid by the gravitational force:

$$W = -\Delta U$$

Asteroid

r

Earth
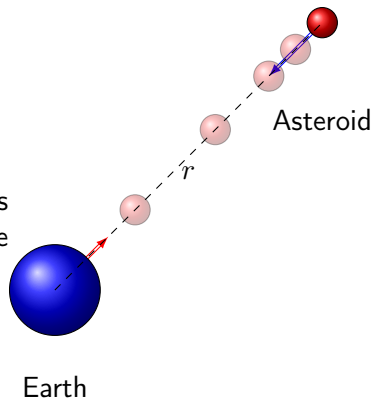
Conservative and Nonconservative Forces

1. When the system configuration changes, the force does work (call it W_1) on the body, transferring energy between the kinetic energy K of the body and some other type of energy of the system.
2. When the configuration change is reversed, the force reverses the energy transfer, doing work W_2 in the process.

In a situation in which $W_1 = -W_2$ is always true, the other type of energy is a potential energy and the force is said to be a conservative force. The gravitational force conservative.

Negative work done by the gravitational force

Positive work done by the gravitational force
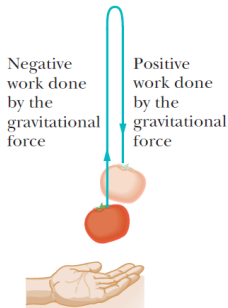
Forces for which this ratio is not satisfied are called a nonconservative force. The frictional force is nonconservative.

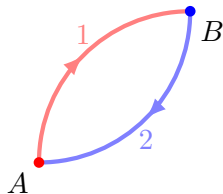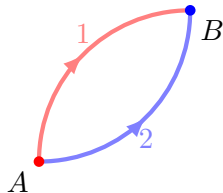# Path Independence of Conservative Forces

The primary test for determining whether a force is conservative or nonconservative is this:

Any choice of path between the points gives the same amount of work:

$$W_{\text{A}\to\text{B by path 1}} = W_{\text{A}\to\text{B by path 2}}$$

A round trip gives a total work of zero:

$$W_{\text{A}\to\text{B by path 1}} + W_{\text{B}\to\text{A by path 2}} = 0$$

The net work done by a conservative force on a particle moving around any closed path is zero.

Path Independence of Conservative Forces

An example

A block of cheese slides along a frictionless track from point a to point b. (b) Finding the work done on the cheese by the gravitational force is easier along the dashed path than along the actual path taken by the cheese; the result is the same for both paths.

The gravitational force is conservative. Any choice of path between the points gives the same amount of work.

(a)

(b)
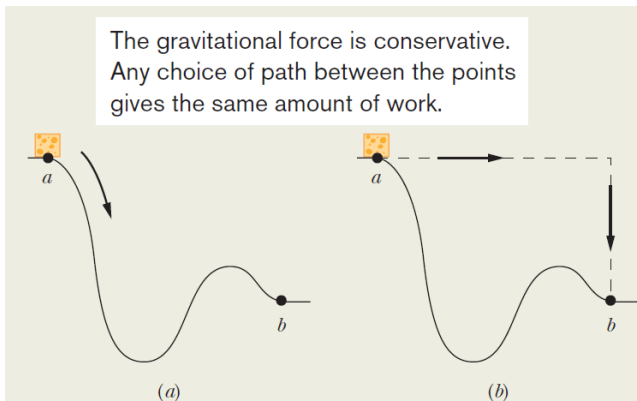
Determining Potential Energy Values

When that force does work W on the body, the change ΔU in the potential energy associated with the system is the negative of the work done:

$$W = -\Delta U$$

Thus, for calculating the value of potential energy we need to calculate the work of force work W on the body.

Gravitational Potential Energy

We first consider a particle with mass m moving vertically along a y axis (the positive direction is upward). As the particle moves from point y_i to point y_f , the gravitational force does work on it. To find the corresponding change in the gravitational potential energy of the particle–Earth system, we use $W = -\Delta U$ with two changes:

1. We integrate along the y axis instead of the x axis, because the gravitational force acts vertically.
2. We substitute mg for the force symbol F, because has the magnitude mg and is directed down the y axis.
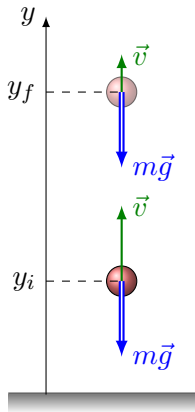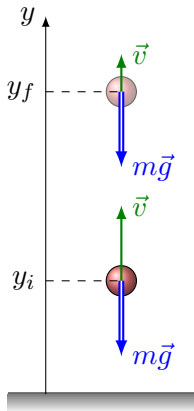
Gravitational Potential Energy

$$W = \int\limits_{y_i}^{y_f} -mg\,dy = -mg(y_f - y_i) = -\Delta U$$

Thus

$$\Delta U = mg(y_f - y_i)$$

Only changes ΔU in gravitational potential energy (or any other type of potential energy) are physically meaningful. However, to simplify a calculation or a discussion, we sometimes would like to say that a certain gravitational potential value U is associated with a certain particle–Earth system when the particle is at a certain height y. To do so, we rewrite equation as:

$$U - U_i = mg(y - y_i)$$

Gravitational Potential Energy

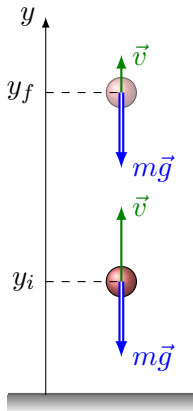$$W = \int\limits_{y_i}^{y_f} -mg\,dy = -mg(y_f - y_i) = -\Delta U$$

Thus

$$\Delta U = mg(y_f - y_i)$$

$$U - U_i = mg(y - y_i)$$

Then we take U_i to be the gravitational potential energy of the system when it is in a reference configuration in which the particle is at a reference point y_i. Usually we take $U_i = 0$ and $y_i = 0$:

$$\boxed{U = mgy}$$

Elastic Potential Energy

Consider the block–spring system, with the block moving on the end of a spring of spring constant k. As the block moves from point x_i to point x_f, the spring force $F_e = -kx$ does work on the block.
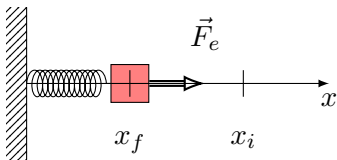
The corresponding change in the elastic potential energy:

$$\Delta U = -\int\limits_{x_i}^{x_f} (-kx)dx = \frac{kx_f^2}{2} - \frac{kx_i^2}{2}$$

To associate a potential energy value U with the block at position x, we choose the reference configuration to be when the spring is at its relaxed length and the block is at $x_i = 0$. Then the elastic potential energy $U_i = 0$:

$$\boxed{U = \frac{kx}{2}}$$

Conservation of Mechanical Energy

The mechanical energy E of a system is the sum of its potential energy U and the kinetic energy K of the objects within it:

$$E = K + U$$

When a conservative force does work W on an object within the system, that force transfers energy between kinetic energy K of the object and potential energy U of the system.

$$\Delta E = \Delta K + \Delta U = 0$$

Conservation of Mechanical Energy

The mechanical energy E of a system is the sum of its potential energy U and the kinetic energy K of the objects within it:

$$E = K + U$$

From Work-kinetic energy theorem, the change ΔK in kinetic energy is

$$\Delta K = W$$

and from the relationship between the change ΔU in potential energy and work of conservative force

$$-\Delta U = W$$

Combining this equations, we find that

$$\Delta E = \Delta K + \Delta U = 0$$

Conservation of Mechanical Energy

The mechanical energy E of a system is the sum of its potential energy U and the kinetic energy K of the objects within it:

$$E = K + U$$

$$\Delta E = \Delta K + \Delta U = 0$$

This result is called the principle of conservation of mechanical energy

In an isolated system where only conservative forces cause energy changes, the kinetic energy and potential energy can change, but their sum, the mechanical energy E of the system, cannot change.

Potential Energy and Conservative Force

Once again we consider a particle that is part of a system in which a conservative force acts. This time suppose that the particle is constrained to move along an x axis while the conservative force does work on it. For one-dimensional motion, the work W done by a force that acts on a particle as the particle moves through a distance Δx is $F_x \Delta x$. From the relationship $W = -\Delta U$ we can then write:

$$\Delta U(x) = -F_x \Delta x$$

Solving for $F(x)$ and passing to the differential limit yield:

$$F_x = -\frac{dU}{dx}$$

In 3D case $F_x = -\frac{dU}{dx}$, $F_y = -\frac{dU}{dy}$ $F_z = -\frac{dU}{dz}$ we can generalize:

$$\vec{F} = -\left(\frac{dU}{dx}\vec{i} + \frac{dU}{dy}\vec{j} + \frac{dU}{dz}\vec{k} \right) = -\mathrm{grad}U$$

Gradient

$$\vec{F} = -\left(\frac{dU}{dx}\vec{i} + \frac{dU}{dy}\vec{j} + \frac{dU}{dz}\vec{k} \right) = -\text{grad}U$$

The meaning of a gradient becomes more obvious and descriptive as soon as we introduce the concept of an equipotential surface at all of whose points the potential energy U has the same magnitude. From equation follows that **the vector $\vec{F}$ is normal to the equipotential surface at a given point**. Next, let us consider the displacement $d\vec{r}$ in the direction of decreasing U values then the vector $\vec{F}$ is oriented in the direction of decreasing U values and vice versa, we may conclude that the gradient of U is a vector oriented along a normal to an equipotential surface in the direction of increasing values of potential energy U.

U_1 U_2 U_3 U_4 U_5

$\vec{F}$

$\text{grad}U$

$U_1 < U_2 < U_3 < U_4 < U_5$
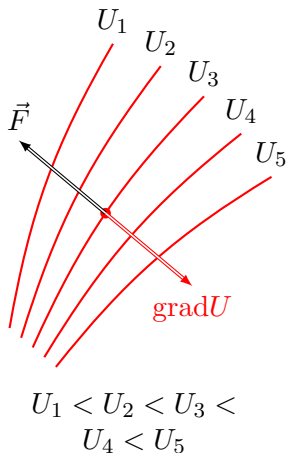
The Potential Energy Curve

In the absence of a nonconservative force, the mechanical energy E of a system has a constant value given by

$$U(x) + K(x) = E = \text{const}$$

Here $K(x)$ is the kinetic energy function kinetic energy as a function of the particle's location x. We may rewrite quation as

$$K(x) = E - U(x)$$

Since $K(x)$ can never be negative (because v^2 is always positive), the particle can never move to the in region where $E < U$.
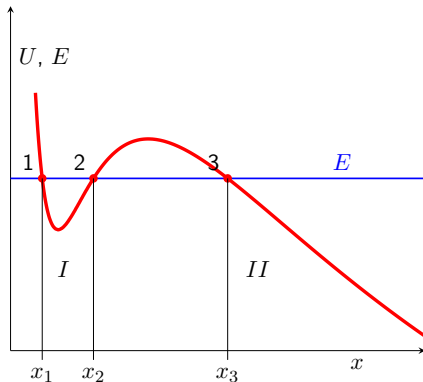
Particle can move either in the region confined by the x_1 and x_2 coordinates (oscillation) or to the right of the x_3 coordinate. The particle cannot however pass from the first I region to the second II one (or vice versa) due to the potential barrier dividing these regions.
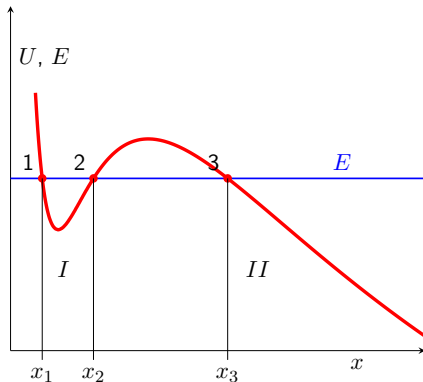
A place where $K = 0$ (because $U = E$) caled the turning point ,
because particle change direction of motion in such point.

x_1, x_2 and x_3 – are turning points .

Region I – region of $\boxed{\text{finite}}$ motion; When the particle reaches x_1, the force on the particle is positive (because the slope dU/dx is negative). This means that the particle does not remain at x_1 but instead begins to move to the right, opposite its earlier motion. Also, when the particle reaches x_2, instead begins to move to the left.
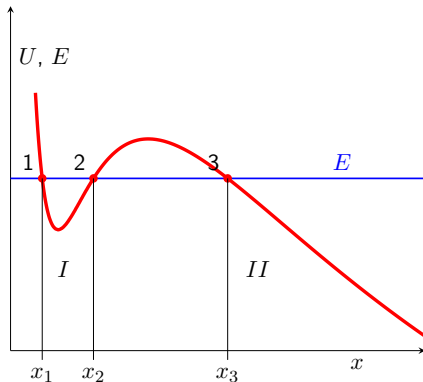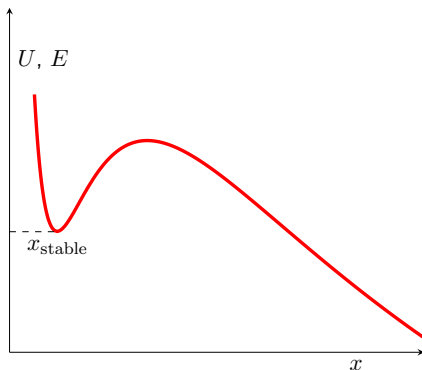
The Potential Energy Curve

Region II – region of infinite motion;

When the particle reaches x_3, it begins to move to the right, opposite its earlier motion. But, when the particle heads to the right, it will continue indefinitely.
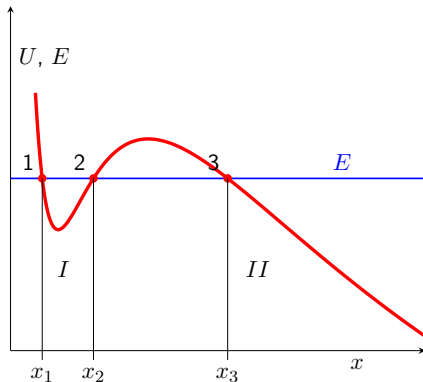
The Potential Energy Curve

If the particle is located exactly at the minimum in point x_{stable}, the force on it is zero, and the particle remains stationary. If we push it slightly left or right, a restoring force appears that moves it back to x_{stable}. A particle at such a position is said to be in stable equilibrium. (A marble placed at the bottom of a hemispherical bowl is an example.) If we place the particle in the cup-like potential well centered at x_{se}, it is between two turning points.

The Potential Energy Curve

If the particle is located exactly at the minimum in point x_{unstable}, the force on it is zero, and the particle remains stationary. However, if it is displaced even slightly in either direction, a nonzero force pushes it farther in the same direction, and the particle continues to move. A particle at such a position is said to be in unstable equilibrium . (A marble balanced on top of a bowling ball is an example.)
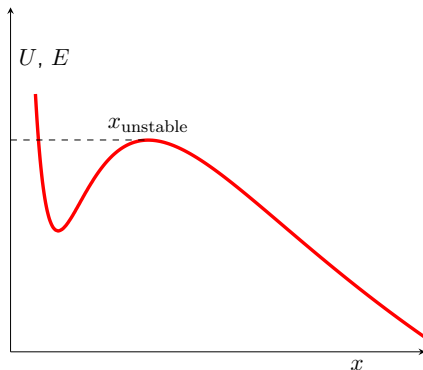
The Energy Conservation Law for a System

Until now we confined ourselves to treating the behaviour of a single particle in terms of its energy. Now we shall pass over to a system of particles. That may be any object, gas, any device, the solar system, etc.