Center of mass

Goals for Lecture

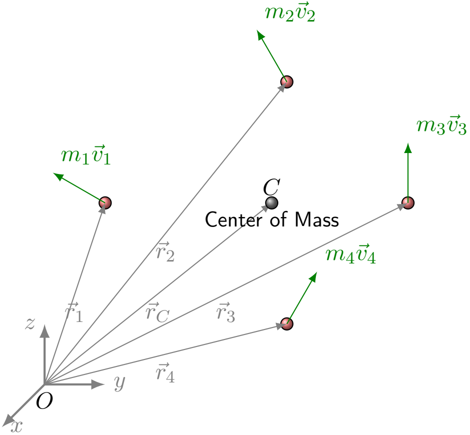
- To uanderstand what is Center of mass
- To formulate thet theorem on the motion of the center of mass.
- Reference frame of the center of mass.
- Equation of motion of a body with a variable mass

What is CENETER OF MASS

Any system of particles possesses one remarkable point C, the centre of inertia, or the centre of mass, displaying a number of interesting and significant properties. Its position relative to the origin O of a given reference frame is described by the radius vector $\vec{r}_C$ defined by the following formula:

$$\vec{r}_C = \frac{1}{M}\sum m_i \vec{r}_i$$

where m_i and $\vec{r}_i$ are the mass and the radius vector of the i-ih particle, and M is the mass of the whole system.

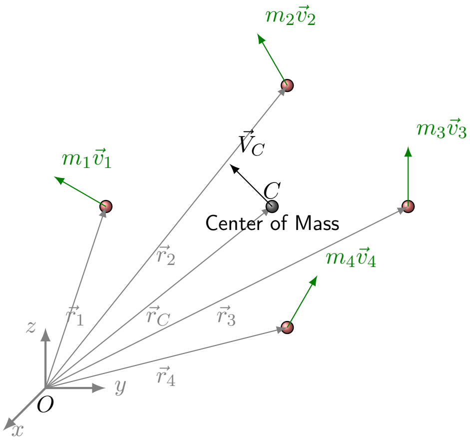


What is CENETER OF MASS

Differentiating last equation with respect to time, we get

$$\vec{V}_C = \frac{1}{M}\sum m_i \vec{v}_i = \frac{1}{M}\sum \vec{p}_i$$

If the velocity of the centre of mass is equal to zero, the system is said to be at rest as a whole. This provides a natural generalization of the concept of a motionless particle. Accordingly, the velocity $\vec{V}_C$ acquires the meaning of the velocity of the system moving as a whole.



From equation

$$\vec{V}_C = \frac{1}{M} \sum m_i \vec{v}_i = \frac{1}{M} \sum \vec{p}_i$$

we get

$$\vec{P} = M\vec{V}_C$$

i.e. the momentum of a system is equal to the product of the mass of the system by the velocity of its centre of mass.

The equation of motion for the centre of mass

The concept of a centre of mass allows equation of moution of particle system:

$$\frac{d\vec{P}}{dt} = \vec{F}^{ext}$$

to be rewritten in a more convenient form:

$$M\frac{d\vec{V}_C}{dt} = \vec{F}^{ext}$$

This is the equation of motion for the centre of inertia of a system. According to this equation, during the motion of any system of particles its centre of inertia moves as if all the mass of the system were concentrated at that point, and all external forces acting on the system were applied to it. In this case the acceleration of the centre of inertia is quite independent of the points to which the external forces are applied.

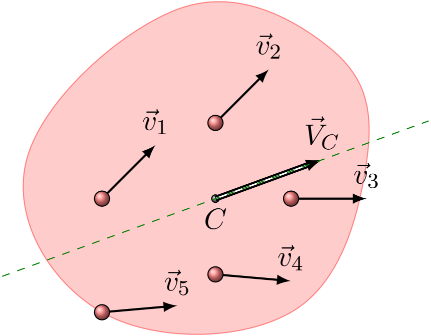
The equation of motion for the centre of mass

$$M\frac{d\vec{V}_C}{dt} = \vec{F}^{ext}$$

Next, it follows from this equation for the closed system $\vec{F}^{ext} = 0$, then $M\frac{d\vec{V}_C}{dt} = 0$, and therefore:

$$\vec{V}_C = \mathrm{const}$$

Thus, if the centre of inertia of a system moves uniformly and rectilinearly, the momentum of the system remains constant in the process of motion. Obviously, the reverse statement is also true.

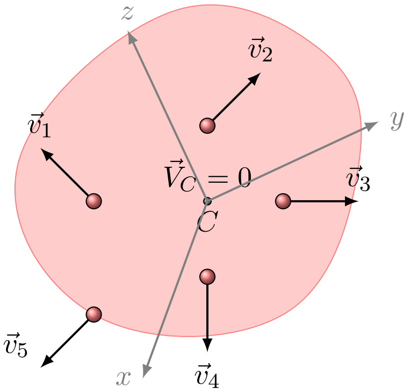


Reference frame of the center of mass

Reference frame of the center of mass is the reference frame in which the centre of inertia is at rest. Using this frame we can significantly simplify both the analysis of phenomena and the calculations.

The reference frame rigidly fixed to the centre of mass of a given system of particles and translating with respect to inertial frames is referred to as the frame of the centre of mass. The distinctive feature of this frame is that the total momentum of the system of particles is equal to zero:

$$\vec{P}_C = 0$$

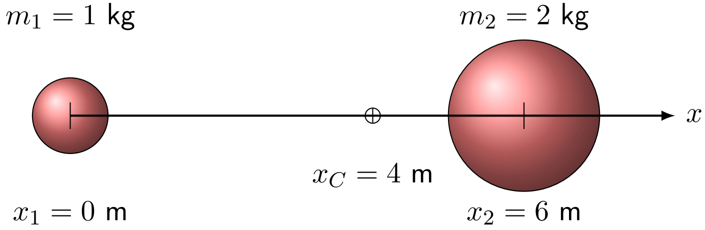


A system of two particles

Coordinates of the Center of mass

The coordinates of the centre of mass are:

$$x_C = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad y_C = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, \quad z_C = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}$$

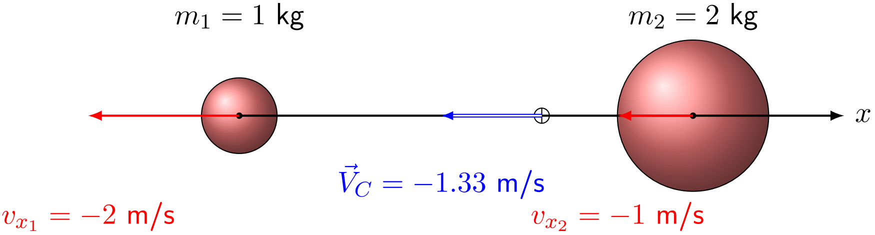


A system of two particles

Velocity of the Center of mass

The coordinates of the centre of mass are:

$$V_{x_C} = \frac{m_1 v_{x_1} + m_2 v_{x_2}}{m_1 + m_2}, \quad V_{y_C} = \frac{m_1 v_{y_1} + m_2 v_{y_2}}{m_1 + m_2}, \quad V_{z_C} = \frac{m_1 v_{z_1} + m_2 v_{z_2}}{m_1 + m_2}$$



A system of two particles

Momentum of the particles in the System of Center of Mass

Velocity of first particle relative to CM $\vec{\tilde{v}}_1 = \vec{v}_1 - \vec{V}_C = \frac{m_2(\vec{v}_1 - \vec{v}_2)}{m_1+m_2}$.
Velocity of second particle relative to CM $\vec{\tilde{v}}_2 = \vec{v}_2 - \vec{V}_C = \frac{m_1(\vec{v}_1 - \vec{v}_2)}{m_1+m_2}$.

Momentum of first particle relative to CM $\vec{\tilde{p}}_1 = m\vec{\tilde{v}}_1 = \frac{m_1 m_2}{m_1+m_2}(\vec{v}_1 - \vec{v}_2)$.
Momentum of second particle relative to CM
$\vec{\tilde{p}}_2 = -m\vec{\tilde{v}}_2 = \frac{m_1 m_2}{m_1+m_2}(\vec{v}_1 - \vec{v}_2)$.

In reference frame of the CM:

$$\vec{\tilde{p}}_1 = -\vec{\tilde{p}}_2$$

Or we can rewrite $\vec{\tilde{p}}_1$ or $\vec{\tilde{p}}_2$ in terms of relative velocity $\boxed{\vec{v}_{\mathrm{rel}} = \vec{v}_1 - \vec{v}_2}$ and reduced mass of the system $\boxed{\mu = \frac{m_1 m_2}{m_1+m_2}}$:

$$\vec{\tilde{p}}_1 = -\vec{\tilde{p}}_2 = \mu\vec{v}_{\mathrm{rel}}$$

A system of two particles

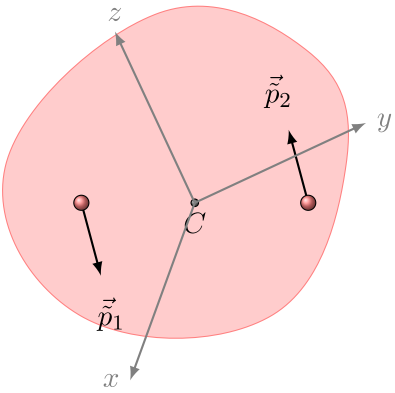
Momentum of the particles in the System of Center of Mass

Magnitude of momentum are equal

$$\tilde{p}_1 = \tilde{p}_2 = \mu v_{\mathrm{rel}}$$

but directions are oposite:

$$\vec{\tilde{p}}_2 \uparrow\downarrow \vec{\tilde{p}}_1$$

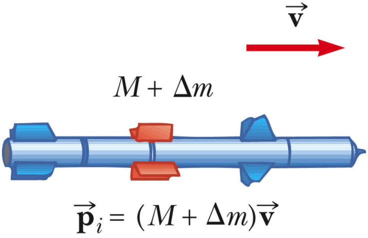


Motion of a Body with Variable Mass

Rocket Propulsion

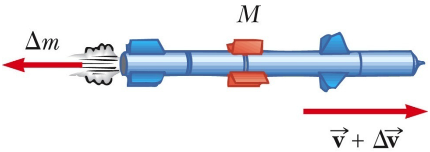
1. There are many cases when the mass of a body varies in the process of motion due to the continuous separation or addition of matter (a rocket, a jet, a flatcar being loaded in motion, etc.).
2. The operation of a rocket depends upon the law of conservation of linear momentum as applied to a system of particles, where the system is the rocket plus its ejected fuel.

1. The initial mass of the rocket plus all its fuel is $M + dm$ at time t_i and speed $\vec{v}$
2. The initial momentum of the system is $\vec{p}_i = (M + dm)\vec{v}$.



Rocket Propulsion

1. At some time $t + dt$, the rocket's mass has been reduced to M and an amount of fuel, dm has been ejected;
2. The rocket's speed has increased by $d\vec{v}$.



3. Because the gases are given some momentum when they are ejected out of the engine, the rocket receives a compensating momentum in the opposite direction
4. Therefore, the rocket is accelerated as a result of the «push» from the exhaust gases
5. In free space, the centre of mass of the system (rocket plus expelled gases) moves uniformly, independent of the propulsion process.

Equation of motion of a body with a variable mass

Change of momentum of system:

$$d\vec{p} = \vec{p}_f - \vec{p}_i = [M(\vec{v} + d\vec{v}) + dm_{\text{gas}} \cdot \vec{v}_{\text{gas}}] - (M + dm_{\text{gas}})\vec{v} =$$
$$= \cancel{M\vec{v}} + Md\vec{v} + dm_{\text{gas}} \cdot \vec{v}_{\text{gas}} - \cancel{M\vec{v}} - dm_{\text{gas}} \cdot \vec{v} =$$
$$= Md\vec{v} + dm_{\text{gas}}(\vec{v}_{\text{gas}} - \vec{v}) = Md\vec{v} + dm_{\text{gas}} \cdot \vec{u},$$

where $\vec{u} = \vec{v}_{\text{gas}} - \vec{v}$ is the velocity of the added (separated) matter with respect to the considered body, and $dm_{\text{rocket}} = dm = -dm_{\text{gas}}$. According to the law of particle system motion, the momentum of a system may vary only due to external forces $\frac{d\vec{p}}{dt} = \vec{F}^{ext}$, thus dividing this expression by dt, we obtain:

$$\boxed{M\frac{d\vec{v}}{dt} = \vec{F}^{ext} + \frac{dm}{dt}\vec{u}}$$

This is the fundamental equation of dynamics of a body with variable mass. It is referred to as the *Meshchersky equation*.

Equation of motion of a body with a variable mass

According to the law of particle system motion, the momentum of a system may vary only due to external forces $\frac{d\vec{p}}{dt} = \vec{F}^{ext}$, thus dividing this expression by dt, we obtain:

$$\boxed{M\frac{d\vec{v}}{dt} = \vec{F}^{ext} + \frac{dm}{dt}\vec{u}}$$

The last term equation is referred to as the reactive force or thrust:

$$\boxed{\vec{R} = \frac{dm}{dt}\vec{u}}.$$

This force appears as a result of the action that the added (separated) mass exerts on a given body. If mass is added, then $\frac{dm}{dt} > 0$ and the vector $\vec{R}$ coincides in direction with the vector $\vec{u}$; if mass is separated, $\frac{dm}{dt} < 0$ and the vector $\vec{R}$ is directed oppositely to the vector $\vec{u}$.

The basic equation for rocket propulsion

A rocket moves in the inertial reference frame in the absence of an external field of force $\vec{F}^{ext} = 0$, the gaseous jet escaping with the constant velocity $\vec{u}$ relative to the rocket. Lets find how the rocket velocity depends on its mass m if at the moment of launching the mass is equal to m_0.
In this case Equation of motion of a body with a variable mass yields $dv = udm/m$. Integrating this expression with allowance made for the initial conditions, we get

$$\boxed{v = -u \ln \frac{m_0}{m},}$$

this is the basic equation for rocket propulsion, where the minus sign shows that the vector $\vec{v}$ (the rocket velocity) is directed oppositely to the vector $\vec{u}$. It is seen that in this case ($\vec{u} = \mathrm{const}$) the rocket velocity does not depend on the fuel combustion time: $\vec{v}$ is determined only by the ratio of the initial rocket mass m_0 to the remaining mass m.