Kinematics of the particle

Goals for Lecture

- To describe straight-line motion in terms of velocity and acceleration
- To distinguish between average and instantaneous velocity and average and instantaneous acceleration
- To interpret graphs of position versus time, velocity versus time, and acceleration versus time for straight-line motion
- To understand straight-line motion with constant acceleration
- To examine freely falling bodies
- To analyze straight-line motion when the acceleration is not constant

Introduction

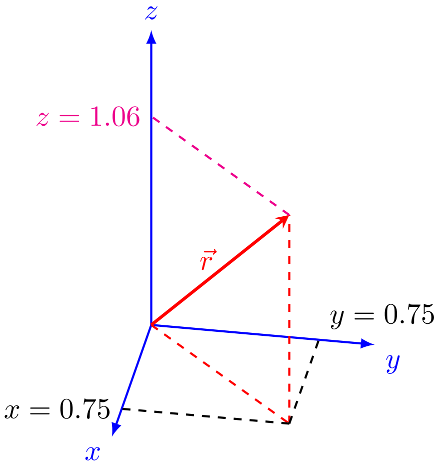
- Kinematics is the subdivision of mechanics treating ways of describing motion regardless of the causes inducing it.
- There are three ways to describe the motion of a point: the first employs vectors, the second coordinates, and the third is referred to as natural.

The vector method

Position Vector

One general way of locating a particle (or particle-like object) is with a position vector, which is a vector that extends from a reference point (usually the origin) to the particle. In the unit-vector notation of, can be written

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$



The vector method

The law of motion of the point

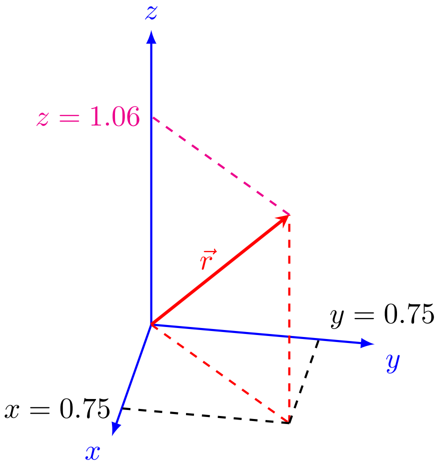
The dependences of

$$\vec{r} = \vec{r}(t)$$

or coordinates on time,

$$x = x(t)$$
$$y = y(t)$$
$$z = z(t)$$

are called the $\boxed{\text{law of motion}}$ of the point.

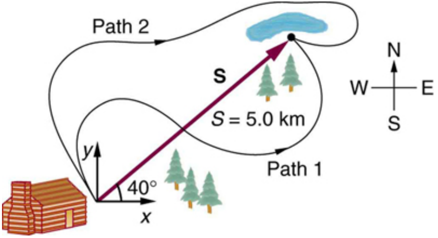


The vector method

Displacement vector

As a particle moves, its position vector changes in such a way that the vector always extends to the particle from the reference point (the origin). If the position vector changes—say, from $\vec{r}_1$ to $\vec{r}_2$ during a certain time interval—then the particle's $\boxed{\text{displacement } \Delta\vec{r}}$ during that time interval is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1.$$



The vector method

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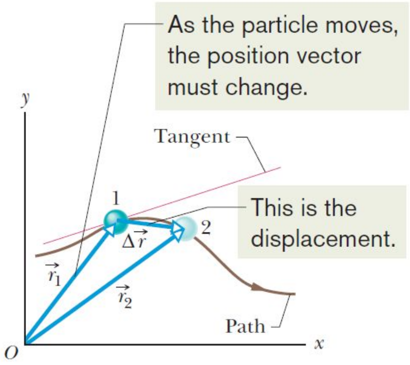


The vector method

Average Velocity and Instantaneous Velocity

If a particle undergoes a displacement $\Delta\vec{r}$ in time interval Δt, its average velocity for that time interval is:

$$\langle\vec{v}\rangle = \frac{\Delta\vec{r}}{\Delta t}$$

When we speak of the velocity of a particle, we usually mean the particle's instantaneous velocity at some instant.This $\vec{v}$ is the value $\langle\vec{v}\rangle$ that approaches in the limit as we shrink the time interval t to 0 about that instant. Using the language of calculus, we may write as the derivative:

$$\vec{v} = \frac{d\vec{r}}{dt}$$

The coordinate method

Components of Instantaneous Velocity

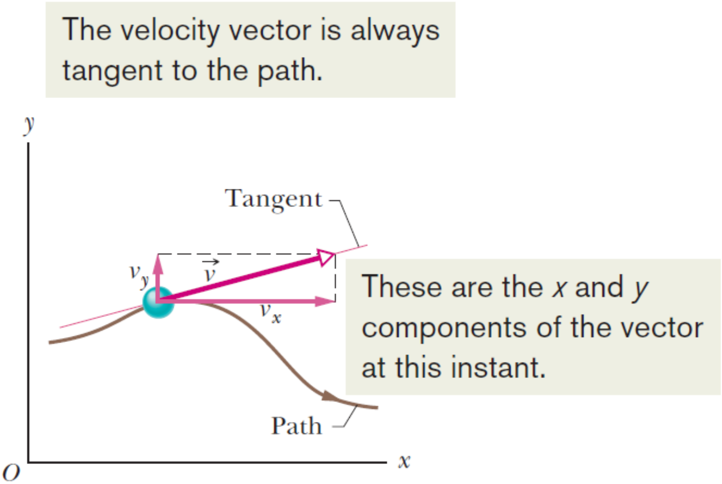
$$\vec{v} = \frac{d}{dt}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

So, the components of Instantaneous Velocity is:

$$v_x = \frac{dx}{dt}$$
$$v_y = \frac{dy}{dt}$$
$$v_z = \frac{dz}{dt}$$

The vector method

Direction of Instantaneous Velocity



The vector method

Average Acceleration and Instantaneous Acceleration

If a particle velocity changes from $\vec{v}_1$ to $\vec{v}_2$ in time interval Δt, its average acceleration for that time interval is:

$$\langle \vec{a} \rangle = \frac{\Delta \vec{v}}{\Delta t}$$

When we speak of the acceleration of a particle, we usually mean the particle's instantaneous acceleration at some instant. This $\vec{a}$ is the value $\langle \vec{a} \rangle$ that approaches in the limit as we shrink the time interval t to 0 about that instant. Using the language of calculus, we may write as the derivative:

$$\vec{a} = \frac{d\vec{v}}{dt}$$

The coordinate method

Components of Instantaneous Acceleration

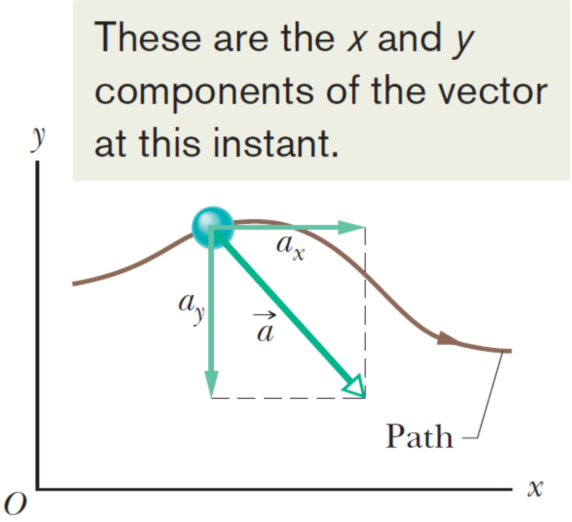
$$\vec{a} = \frac{d}{dt}(v_x\vec{i} + v_y\vec{j} + v_z\vec{k}) = \frac{dv_x}{dt}\vec{i} + \frac{dv_y}{dt}\vec{j} + \frac{dv_z}{dt}\vec{k}$$

So, the components of Instantaneous Velocity is:

$$a_x = \frac{dv_x}{dt}$$
$$a_y = \frac{dv_y}{dt}$$
$$a_z = \frac{dv_z}{dt}$$

The vector method

Direction of Instantaneous Acceleration



Example

Sample Problem 4.01 Two-dimensional position vector, rabbit run

A rabbit runs across a parking lot on which a set of coordinate axes has, strangely enough, been drawn. The coordinates (meters) of the rabbit's position as functions of time t (seconds) are given by

$$x = -0.31t^2 + 7.2t + 28 \quad (4\text{-}5)$$

and

$$y = 0.22t^2 - 9.1t + 30. \quad (4\text{-}6)$$

(a) At $t = 15$ s, what is the rabbit's position vector $\vec{r}$ in unit-vector notation and in magnitude-angle notation?

KEY IDEA

The x and y coordinates of the rabbit's position, as given by Eqs. 4-5 and 4-6, are the scalar components of the rabbit's position vector $\vec{r}$. Let's evaluate those coordinates at the given time, and then we can use Eq. 3-6 to evaluate the magnitude and orientation of the position vector.

Calculations: We can write

$$\vec{r}(t) = x(t)\hat{\mathrm{i}} + y(t)\hat{\mathrm{j}}. \quad (4\text{-}7)$$

(We write $\vec{r}(t)$ rather than $\vec{r}$ because the components are functions of t, and thus $\vec{r}$ is also.)

At $t = 15$ s, the scalar components are

$$x = (-0.31)(15)^2 + (7.2)(15) + 28 = 66 \text{ m}$$

and

$$y = (0.22)(15)^2 - (9.1)(15) + 30 = -57 \text{ m},$$

so

$$\vec{r} = (66 \text{ m})\hat{\mathrm{i}} - (57 \text{ m})\hat{\mathrm{j}}, \quad \text{(Answer)}$$

Example

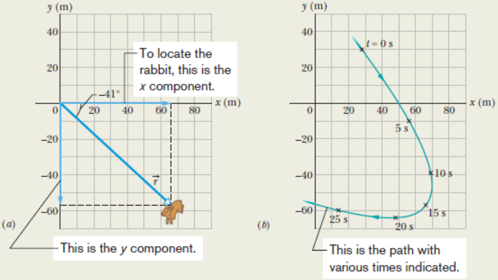


Figure 4-2 (*a*) A rabbit's position vector $\vec{r}$ at time $t = 15$ s. The scalar components of $\vec{r}$ are shown along the axes. (*b*) The rabbit's path and its position at six values of t.

which is drawn in Fig. 4-2*a*. To get the magnitude and angle of $\vec{r}$, notice that the components form the legs of a right triangle and r is the hypotenuse. So, we use Eq. 3-6:

$$r = \sqrt{x^2 + y^2} = \sqrt{(66\text{ m})^2 + (-57\text{ m})^2} = 87\text{ m}, \qquad \text{(Answer)}$$

and $$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-57\text{ m}}{66\text{ m}}\right) = -41°. \qquad \text{(Answer)}$$

Check: Although $\theta = 139°$ has the same tangent as $-41°$, the components of position vector $\vec{r}$ indicate that the desired angle is $139° - 180° = -41°$.

(b) Graph the rabbit's path for $t = 0$ to $t = 25$ s.

Graphing: We have located the rabbit at one instant, but to see its path we need a graph. So we repeat part (a) for several values of t and then plot the results. Figure 4-2*b* shows the plots for six values of t and the path connecting them.

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Example

Sample Problem 4.02 Two-dimensional velocity, rabbit run

For the rabbit in the preceding sample problem, find the velocity $\vec{v}$ at time $t = 15$ s.

KEY IDEA

We can find $\vec{v}$ by taking derivatives of the components of the rabbit's position vector.

Calculations: Applying the v_x part of Eq. 4-12 to Eq. 4-5, we find the x component of $\vec{v}$ to be

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(-0.31t^2 + 7.2t + 28)$$
$$= -0.62t + 7.2. \quad (4\text{-}13)$$

At $t = 15$ s, this gives $v_x = -2.1$ m/s. Similarly, applying the v_y part of Eq. 4-12 to Eq. 4-6, we find

$$v_y = \frac{dy}{dt} = \frac{d}{dt}(0.22t^2 - 9.1t + 30)$$
$$= 0.44t - 9.1. \quad (4\text{-}14)$$

At $t = 15$ s, this gives $v_y = -2.5$ m/s. Equation 4-11 then yields

$$\vec{v} = (-2.1 \text{ m/s})\hat{\imath} + (-2.5 \text{ m/s})\hat{\jmath}, \quad \text{(Answer)}$$

which is shown in Fig. 4-5, tangent to the rabbit's path and in the direction the rabbit is running at $t = 15$ s.

To get the magnitude and angle of $\vec{v}$, either we use a vector-capable calculator or we follow Eq. 3-6 to write

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-2.1 \text{ m/s})^2 + (-2.5 \text{ m/s})^2}$$
$$= 3.3 \text{ m/s} \quad \text{(Answer)}$$

and

$$\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\left(\frac{-2.5 \text{ m/s}}{-2.1 \text{ m/s}}\right)$$
$$= \tan^{-1} 1.19 = -130°. \quad \text{(Answer)}$$

Check: Is the angle $-130°$ or $-130° + 180° = 50°$?

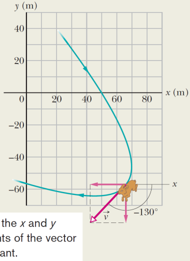

Figure 4-5 The rabbit's velocity $\vec{v}$ at $t = 15$ s.

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Example

Sample Problem 4.03 **Two-dimensional acceleration, rabbit run**

For the rabbit in the preceding two sample problems, find the acceleration $\vec{a}$ at time $t = 15$ s.

KEY IDEA

We can find $\vec{a}$ by taking derivatives of the rabbit's velocity components.

Calculations: Applying the a_x part of Eq. 4-18 to Eq. 4-13, we find the x component of $\vec{a}$ to be

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(-0.62t + 7.2) = -0.62 \text{ m/s}^2.$$

Similarly, applying the a_y part of Eq. 4-18 to Eq. 4-14 yields the y component as

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}(0.44t - 9.1) = 0.44 \text{ m/s}^2.$$

We see that the acceleration does not vary with time (it is a constant) because the time variable t does not appear in the expression for either acceleration component. Equation 4-17 then yields

$$\vec{a} = (-0.62 \text{ m/s}^2)\hat{\mathrm{i}} + (0.44 \text{ m/s}^2)\hat{\mathrm{j}}, \qquad \text{(Answer)}$$

which is superimposed on the rabbit's path in Fig. 4-7.

To get the magnitude and angle of $\vec{a}$, either we use a vector-capable calculator or we follow Eq. 3-6. For the magnitude we have

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.62 \text{ m/s}^2)^2 + (0.44 \text{ m/s}^2)^2}$$
$$= 0.76 \text{ m/s}^2. \qquad \text{(Answer)}$$

For the angle we have

$$\theta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\left(\frac{0.44 \text{ m/s}^2}{-0.62 \text{ m/s}^2}\right) = -35^\circ.$$

However, this angle, which is the one displayed on a calculator, indicates that $\vec{a}$ is directed to the right and downward in Fig. 4-7. Yet, we know from the components that $\vec{a}$ must be directed to the left and upward. To find the other angle that has the same tangent as -35° but is not displayed on a calculator, we add 180°:

$$-35^\circ + 180^\circ = 145^\circ. \qquad \text{(Answer)}$$

This *is* consistent with the components of $\vec{a}$ because it gives a vector that is to the left and upward. Note that $\vec{a}$ has the same magnitude and direction throughout the rabbit's run because the acceleration is constant. That means that we could draw the very same vector at any other point along the rabbit's path (just shift the vector to put its tail at some other point on the path without changing the length or orientation).

This has been the second sample problem in which we needed to take the derivative of a vector that is written in unit-vector notation. One common error is to neglect the unit vectors themselves, with a result of only a set of numbers and symbols. Keep in mind that a derivative of a vector is always another vector.

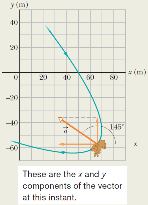

Figure 4-7 The acceleration $\vec{a}$ of the rabbit at $t = 15$ s. The rabbit happens to have this same acceleration at all points on its path.

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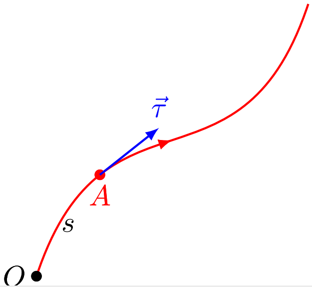
The «natural» method

Velocity of a point

This method is employed when the path of a point is known in advance. The location of a point A is defined by the $\boxed{\text{arc coordinate } s}$, that is, the distance from the chosen origin O measured along the path.



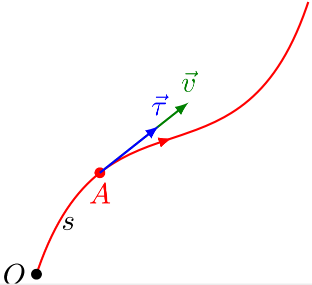
The «natural» method

Velocity of a point

The velocity vector $\vec{v}$ of the point A is oriented along a tangent to the path and therefore can be represented as follows:

$$\vec{v} = v_\tau \vec{\tau}$$

where $\boxed{v_\tau = \frac{ds}{dt}}$ is the projection of the vector $\vec{v}$ on the direction of the vector $\vec{\tau}$, with v_τ being an algebraic quantity.

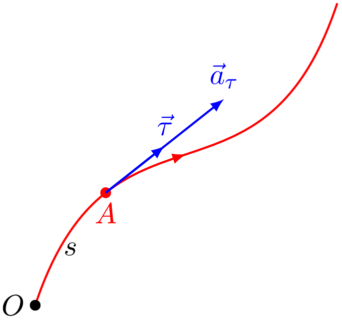


The «natural» method

Acceleration of a point

If us differentiate Eq. $\boxed{\vec{v} = v_\tau \vec{\tau}}$ with respect to time we get an acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_\tau}{dt}\vec{\tau} +$$



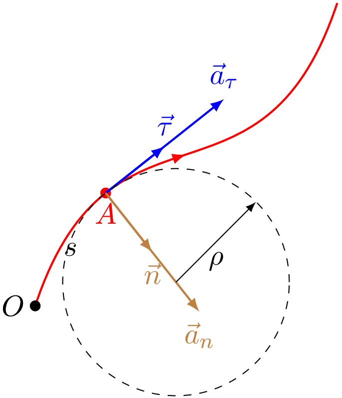
1. Tangential acceleration $\vec{a}_\tau = \frac{dv_\tau}{dt}\vec{\tau}$

The «natural» method

Acceleration of a point

If us differentiate Eq. $\boxed{\vec{v} = v_\tau \vec{\tau}}$ with respect to time we get an acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_\tau}{dt}\vec{\tau} + \frac{v^2}{\rho}\vec{n}$$



1. Tangential acceleration $\vec{a}_\tau = \frac{dv_\tau}{dt}\vec{\tau}$
2. Normal acceleration $\vec{a}_n = \frac{v^2}{\rho}\vec{n}$
   where ρ of the corresponding circle as the radius of curvature of the path at the same point.

The «natural» method

Acceleration of a point

If us differentiate Eq. $\boxed{\vec{v} = v_\tau \vec{\tau}}$ with respect to time we get an acceleration:

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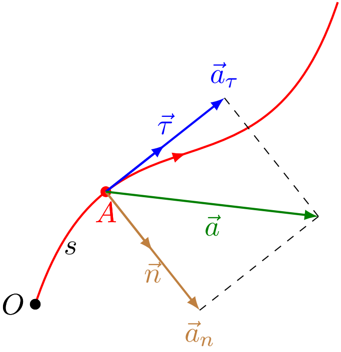

1. Tangential acceleration $\vec{a}_\tau = \frac{dv_\tau}{dt}\vec{\tau}$
2. Normal acceleration $\vec{a}_n = \frac{v^2}{\rho}\vec{n}$
   where ρ of the corresponding circle as the radius of curvature of the path at the same point.
3. Total acceleration $\vec{a} = \vec{a}_\tau + \vec{a}_n$
   The magnitude of $\vec{a}$ is $a = \sqrt{a_\tau^2 + a_n^2}$ as shown in picture.