CFD laboratory 1
Laminar flow development between two parallel plates

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December 4, 2021

Abstract

The first test case is the steady-state development of incompressible flow between two parallel plates in the laminar regime (Figure 1). The plates are considered infinite in the direction transversal to the flow. The flow develops from a condition of uniform velocity (rectangular profile) imposed at the inlet boundary, reaching a fully-developed state at a certain distance downstream of it. [1]

1 Introduction

The fully developed laminar flow between two parallel plates admits an analytical solution (plane Poiseuille flow):

$$\begin{cases} u(x,y,z) = u(y) = -\frac{\delta^2}{2\mu}\frac{dp_e}{dx}\frac{y}{\delta}(2-\frac{y}{\delta}) \; v(x,y,z) = 0 \; w(x,y,z) = 0 \\ \frac{dp_e}{dx} = const < 0 \\ \tau_{yx}(x,y,z) = -2\mu\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\mu\frac{du}{dy} = \frac{dp_e}{dx}(\delta - y) = \tau_{yx}(y) \end{cases} \tag{1}$$

Where μ is the dynamic viscosity of the fluid, δ is the half-distance between the plates, u, v, w are the velocity components along directions x, y, z (Figure 1), p_e is the excess pressure with respect to the hydrostatic component, and τ_{yx} is the only nonzero shear stress. [1]

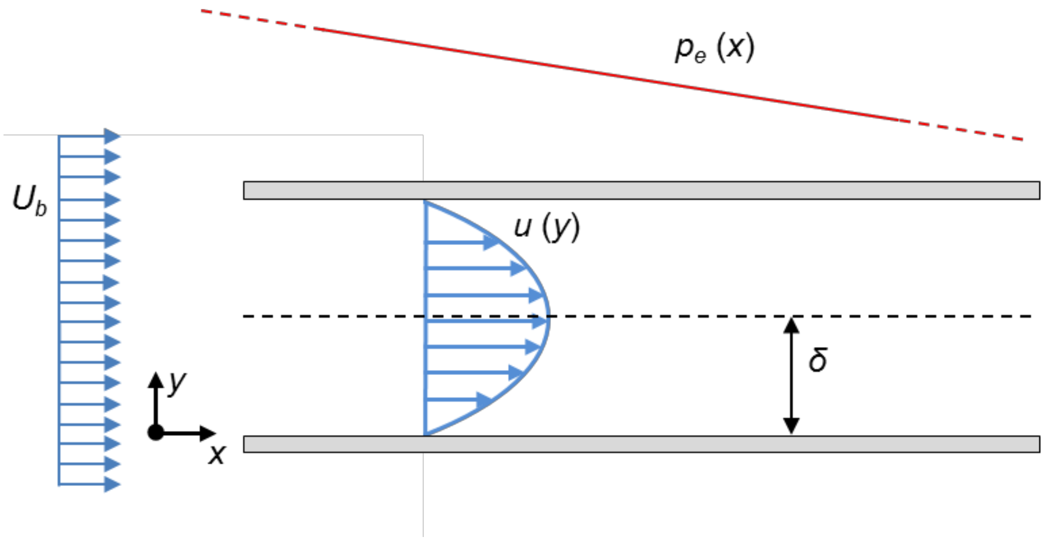

Figure 1: Sketch of the Case

The configuration of the problem is as follows:

- Length $L = 10\,cm$,
- Length $L_p = 9\,cm$,
- Half-channel height $\delta = 5\,mm$,
- Bulk velocity $U_b = 5\,mm/s$,
- Fluid: Water at $20°C$ ($\rho = 998.23\,kg/m^3$, Kinematic Viscosity $\nu = 1.006E-6\,m^2/s$).

Outline The remainder of the report is organized as follows: Section 2 provides some gross estimate of the required L_p to achieve a fully-developed flow in terms of channel height; Section 3 shows how a suitable configuration of the cartesian computational mesh was found; Section 4 compares the simulated solution with the analytical model both graphically and numerically;

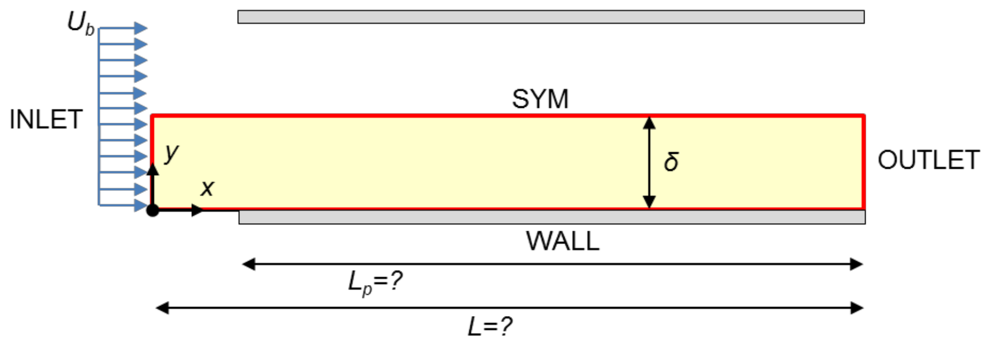

Figure 2: Domain and boundary conditions

Finally, Section 5 provides some analysis regarding the vorticity profile on both the developing and fully-developed region. It is worth noting that all simulations had a relative convergence tolerance of $1E-3$ for all variables taken into consideration.

2 Fully-developed flow conditions

The following plot was generated from a half-domain simulation with a 40-by-40 mesh, in a 12 cm long domain with a 2 cm margin, as per the lab document's recommendation. The observed profile was taken in

As we can see in Figure 3, the X-velocity Y-Profile stabilizes after roughly 4 deltas (2 cm), making our choice of an 18-delta channel with a 2-delta margin at the beginning (10 cm total) more than enough for the case of study. From now on, unless said otherwise, all X-specific data was taken after 12 deltas into the actual channel (7 cm from the origin of the X-axis), simulating the whole channel as opposed to just the lower half and using a 10 cm domain with a 1 cm margin in its stead.

3 Grid independence study

The following Grid-Independence study was performed by fixing 40 cells either along the X or Y axis, and then progressively increasing the amount of cells on the other axis until both X-velocity profile and Pressure-gradient convergence was observed.

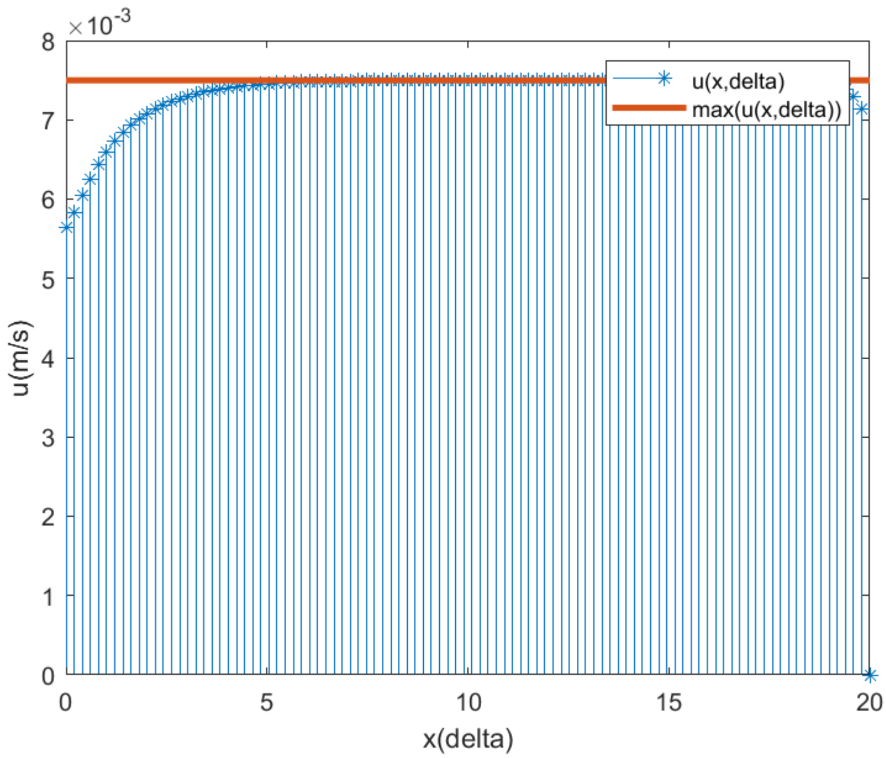

Figure 3: X-velocity Y-profile per delta-step

As we can observe in Figure 4, X-refinement is pretty much irrelevant for the X-velocity derivative profile; nevertheless, such is not the case for the Pressure-gradient which, as we can see in Figure 5, did not stabilize after at least 35 cells across the X-axis.

In the case of Y-refinement, 25 cells were enough to stabilize both the X-velocity derivative profile and average Pressure-gradient. Therefore, a 40-by-40 mesh kept being used for the remainder of the laboratory, since simulation times were low enough for a wide Y-refinement margin to not be a problem.

4 CFD solution validation

From Equation System 1 and Equation 2 (which describes Bulk velocity as a function of the Pressure-gradient magnitude), we derive the expressions in Equation System 3.

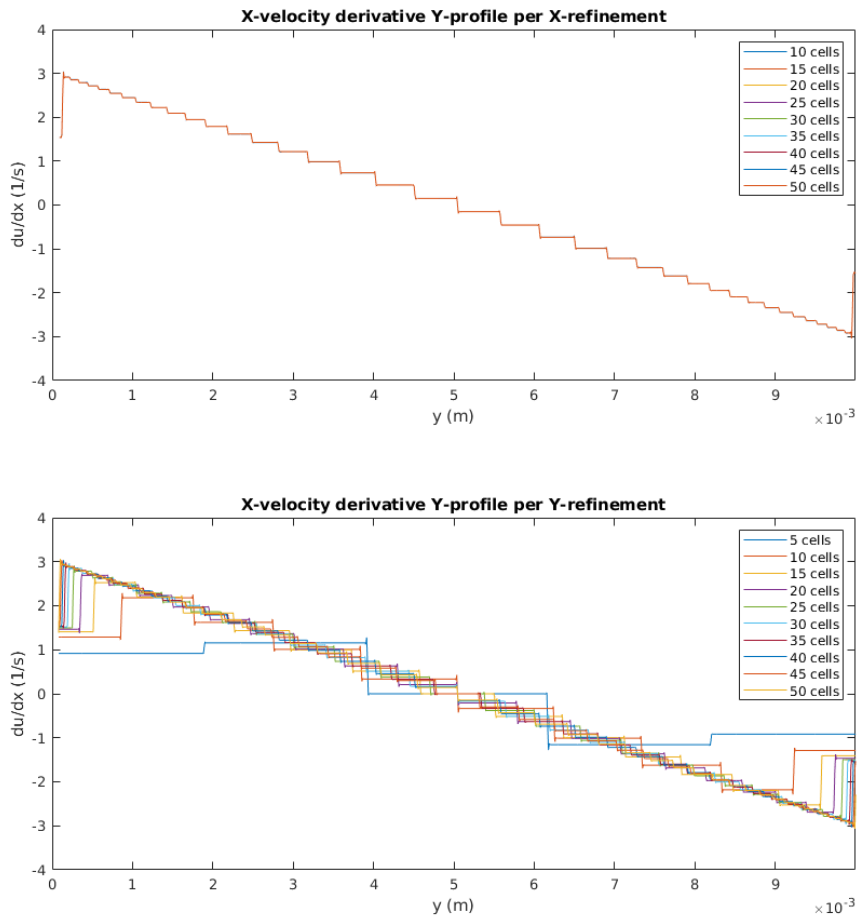

Figure 4: X-velocity derivative Y-profile per cell amount

$$U_b = -\frac{1}{3\mu}\frac{dp_e}{dx}\delta^2 \tag{2}$$

$$\begin{cases} \frac{dp_e}{dx} = -\frac{3\mu}{\delta^2}U_b \\ u(y) = -\frac{\delta}{2\mu}\frac{dp_e}{dx}y(2-\frac{y}{\delta}) \end{cases} \tag{3}$$

In order to objectively measure the simulation error with respect to the analytical solution, we use the expressions in Equation System 4. It is worth

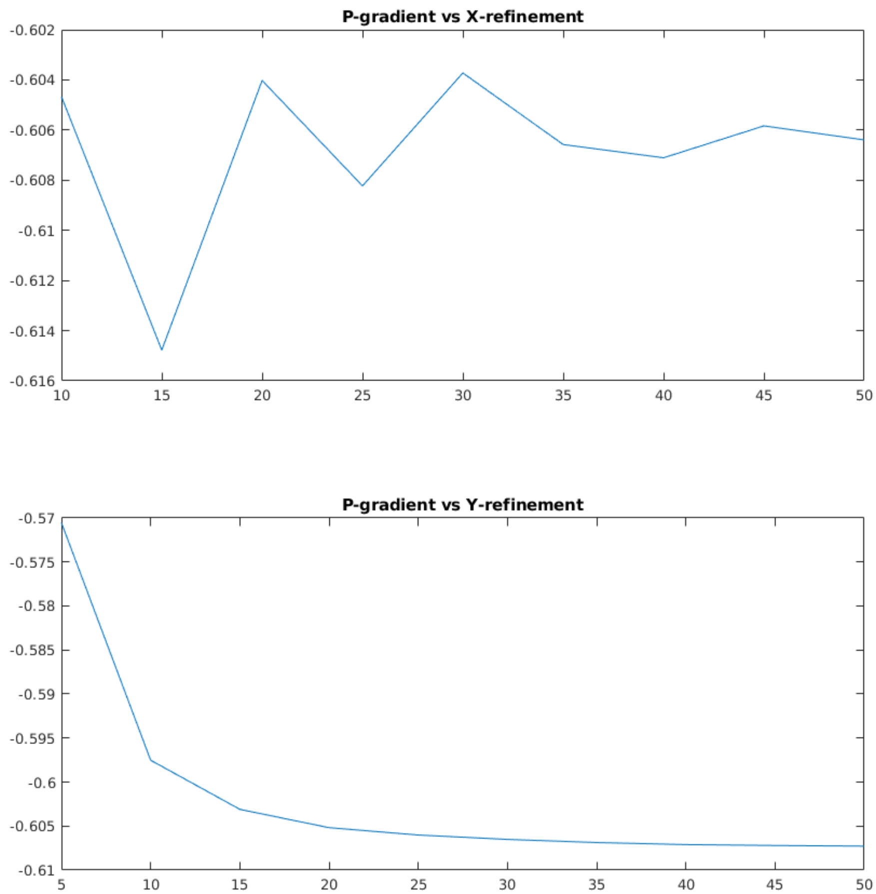

Figure 5: P-gradient vs cell amount

noting that the velocities inside the L^2-norm and L^∞-norm were evaluated for a fixed value of x inside the fully-developed region.

$$\begin{cases} \frac{dp}{dx}_{err} = \frac{\left|\frac{dp_e}{dx} - \frac{dp_e}{dx}_{sim}\right|}{\left|\frac{dp_e}{dx}\right|} \\ u_{err,L^2} = \frac{\left|\left|\frac{u-u_{sim}}{u}\right|\right|_{L^2}}{\sqrt{2\delta}} = \sqrt{\frac{\int_0^{2\delta}\left|\frac{u(y)-u_{sim}(y)}{u(y)}\right|^2 dy}{2\delta}} \\ u_{err,L^\infty} = \left|\left|\frac{u-u_{sim}}{u}\right|\right|_{L^\infty} = \sup_{y\in[0,2\delta]}\left|\frac{u(y)-u_{sim}(y)}{u(y)}\right| \end{cases} \tag{4}$$

The results were the following:

- Pressure-gradient relative error: $5.807746E - 03$.
- X-velocity relative Root-Mean-Squared error: $5.288724e - 03$.
- X-velocity relative Maximum error: $2.613112e - 02$.

Aditionally, as we can observe in Figure 6 and Figure 7, the profiles for both X-velocity and Shear-stress were coherent with the analytical model.

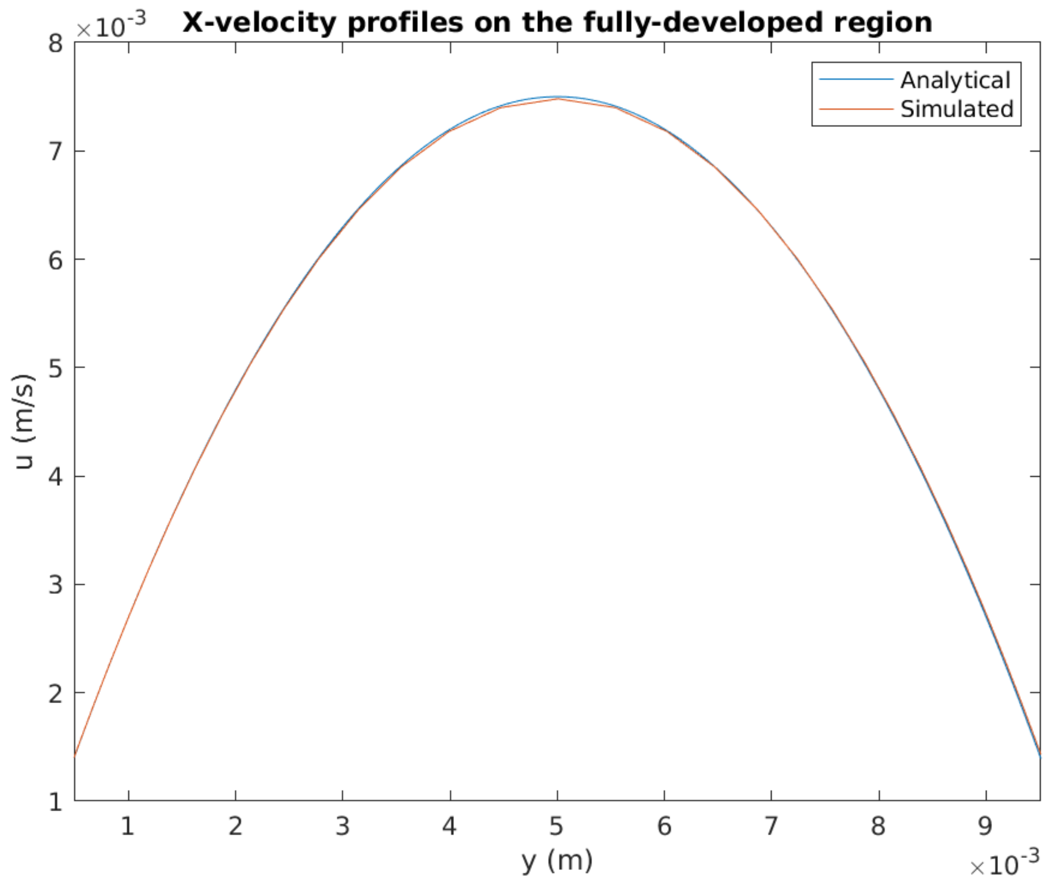

Figure 6: X-velocity profile comparison at $x = 0.07\, m$

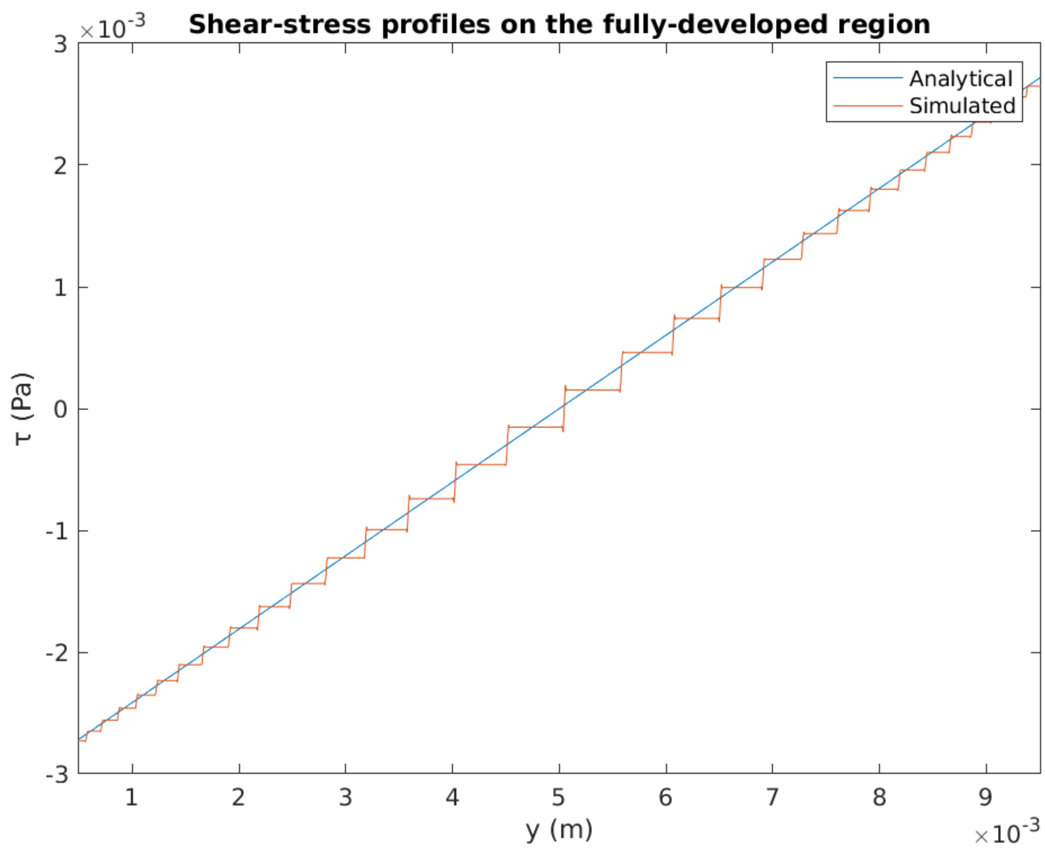

Figure 7: Shear-stress profile comparison at $x = 0.07\,m$

5 Vorticity

Using data from the original simulation (for visualization purposes) and a 100-by-40 mesh, we plotted vorticity across all channel regions and obtained Figure 8 as a result.

The main two differences between the developing and fully-developed regions are:

- Peak vorticity magnitude is higher on the developing region.
- The vorticity profile is completely linear on the fully-developed region, whereas it presents variable slope on the developing one.

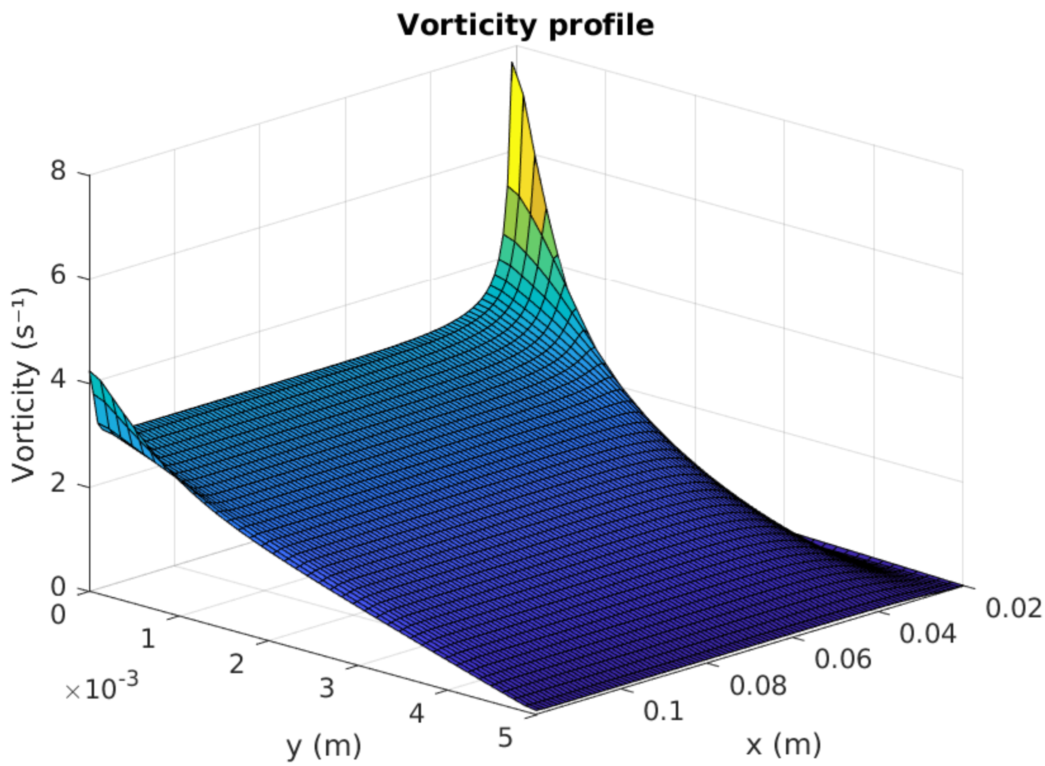

Figure 8: Vorticity surface-profile

References

[1] Prof. G. V. Messa and Dr. G. Ferrarese. Test case 1: Laminar flow development between two parallel plates. Lab Guide, Fluid Labs, 2021.